CALCULUS

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MIT & University of Wisconsin-Stevens Point

MIT & University of Wisconsin-Stevens Point Calculus

Gilbert Strang & Edwin Herman

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TABLE OF CONTENTS

Licensing

1: Functions and Graphs

- 1.0: Prelude to Functions and Graphs
- 1.1: Review of Functions
  - 1.1E: Exercises for Section 1.1
- 1.2: Basic Classes of Functions
  - 1.2E: Exercises for Section 1.2
- 1.3: Trigonometric Functions
  - 1.3E: Exercises for Section 1.3
- 1.4: Inverse Functions
  - 1.4E: Exercises for Section 1.4
- 1.5: Exponential and Logarithmic Functions
  - 1.5E: Exercises for Section 1.5
- 1.R: Chapter 1 Review Exercises

2: Limits

- 2.0: Prelude to Limits
- 2.1: A Preview of Calculus
  - 2.1E: Exercises for Section 2.1
- 2.2: The Limit of a Function
  - 2.2E: Exercises for Section 2.2
- 2.3: The Limit Laws
  - 2.3E: Exercises for Section 2.3
- 2.4: Continuity
  - 2.4E: Exercises for Section 2.4
- 2.5: The Precise Definition of a Limit
  - 2.5E: Exercises for Section 2.5
- 2.R: Chapter 2 Review Exercises

3: Derivatives

- 3.0: Prelude to Derivatives
- 3.1: Defining the Derivative
  - 3.1E: Exercises for Section 3.1
- 3.2: The Derivative as a Function
  - 3.2E: Exercises for Section 3.2
- 3.3: Differentiation Rules
  - 3.3E: Exercises for Section 3.3
- 3.4: Derivatives as Rates of Change
  - 3.4E: Exercises for Section 3.4
- 3.5: Derivatives of Trigonometric Functions

- 3.5E: Exercises for Section 3.5
- 3.6: The Chain Rule
  - 3.6E: Exercises for Section 3.6
- 3.7: Derivatives of Inverse Functions
  - 3.7E: Exercises for Section 3.7
- 3.8: Implicit Differentiation
  - 3.8E: Exercises for Section 3.8
- 3.9: Derivatives of Exponential and Logarithmic Functions
  - 3.9E: Exercises for Section 3.9
- 3.R: Chapter 3 Review Exercises

4: Applications of Derivatives

- 4.0: Prelude to Applications of Derivatives
- 4.1: Related Rates
  - 4.1E: Exercises for Section 4.1
- 4.2: Linear Approximations and Differentials
  - 4.2E: Exercises for Section 4.2
- 4.3: Maxima and Minima
  - 4.3E: Exercises for Section 4.3
- 4.4: The Mean Value Theorem
  - 4.4E: Exercises for Section 4.4
- 4.5: Derivatives and the Shape of a Graph
  - 4.5E: Exercises for Section 4.5
- 4.6: Limits at Infinity and Asymptotes
  - 4.6E: Exercises for Section 4.6
- 4.7: Applied Optimization Problems
  - 4.7E: Exercises for Section 4.7
- 4.8: L'Hôpital's Rule
  - 4.8E: Exercises for Section 4.8
- 4.9: Newton's Method
  - 4.9E: Exercises for Section 4.9
- 4.10: Antiderivatives
  - 4.10E: Exercises for Section 4.10
- 4.R: Chapter 4 Review Exercises

5: Integration

- 5.0: Prelude to Integration
- 5.1: Approximating Areas
  - 5.1E: Exercises for Section 5.1
- 5.2: The Definite Integral
  - 5.2E: Exercises for Section 5.2
- 5.3: The Fundamental Theorem of Calculus
  - 5.3E: Exercises for Section 5.3
- 5.4: Integration Formulas and the Net Change Theorem

- 5.4E: Exercises for Section 5.4
- 5.5: Substitution
  - 5.5E: Exercises for Section 5.5
- 5.6: Integrals Involving Exponential and Logarithmic Functions
  - 5.6E: Exercises for Section 5.6
- 5.7: Integrals Resulting in Inverse Trigonometric Functions
  - 5.7E: Exercises for Section 5.7
- 5.7: Integrals Resulting in Inverse Trigonometric Functions
- 5.R: Chapter 5 Review Exercises

6: Applications of Integration

- 6.0: Prelude to Applications of Integration
- 6.1: Areas between Curves
  - 6.1E: Exercises for Section 6.1
- 6.2: Determining Volumes by Slicing
  - 6.2E: Exercises for Section 6.2
- 6.3: Volumes of Revolution - Cylindrical Shells
  - 6.3E: Exercises for Section 6.3
- 6.4: Arc Length of a Curve and Surface Area
  - 6.4E: Exercises for Section 6.4
- 6.5: Physical Applications of Integration
  - 6.5E: Exercises for Section 6.5
- 6.6: Moments and Centers of Mass
  - 6.6E: Exercises for Section 6.6
- 6.7: Integrals, Exponential Functions, and Logarithms
  - 6.7E: Exercises for Section 6.7
- 6.8: Exponential Growth and Decay
  - 6.8E: Exercises for Section 6.8
- 6.9: Calculus of the Hyperbolic Functions
  - 6.9E: Exercises for Section 6.9
- 6.R: Chapter 6 Review Exercises

7: Techniques of Integration

- 7.0: Prelude to Techniques of Integration
- 7.1: Integration by Parts
  - 7.1E: Exercises for Section 7.1
- 7.2: Trigonometric Integrals
  - 7.2E: Exercises for Section 7.2
- 7.3: Trigonometric Substitution
  - 7.3E: Exercises for Section 7.3
- 7.4: Partial Fractions
  - 7.4E: Exercises for Section 7.4
- 7.5: Other Strategies for Integration
  - 7.5E: Exercises for Section 7.5
- 7.6: Numerical Integration

- 7.6E: Exercises for Section 7.6
- 7.7: Improper Integrals
  - 7.7E: Exercises for Section 7.7
- 7.R: Chapter 7 Review Exercises

8: Introduction to Differential Equations

- 8.0: Prelude to Differential Equations
- 8.1: Basics of Differential Equations
  - 8.1E: Exercises for Section 8.1
- 8.2: Direction Fields and Numerical Methods
  - 8.2E: Exercises for Section 8.2
- 8.3: Separable Equations
  - 8.3E: Exercises for Section 8.3
- 8.4: The Logistic Equation
  - 8.4E: Exercises for Section 8.4
- 8.5: First-order Linear Equations
  - 8.5E: Exercises for Section 8.5
- 8.R: Chapter 8 Review Exercises

9: Sequences and Series

- 9.0: Prelude to Sequence and Series
- 9.1: Sequences
  - 9.1E: Exercises for Section 9.1
- 9.2: Infinite Series
  - 9.2E: Exercises for Section 9.2
- 9.3: The Divergence and Integral Tests
  - 9.3E: Exercises for Section 9.3
- 9.4: Comparison Tests
  - 9.4E: Exercises for Section 9.4
- 9.5: Alternating Series
  - 9.5E: Exercises for Section 9.5
- 9.6: Ratio and Root Tests
  - 9.6E: Exercises for Section 9.6
- 9.R: Chapter 9 Review Exercises

10: Power Series

- 10.0: Prelude to Power Series
- 10.1: Power Series and Functions
  - 10.1E: Exercises for Section 10.1
- 10.2: Properties of Power Series
  - 10.2E: Exercises for Section 10.2
- 10.3: Taylor and Maclaurin Series
  - 10.3E: Exercises for Section 10.3
- 10.4: Working with Taylor Series
  - 10.4E: Exercises for Section 10.4

- 10.R: Chapter 10 Review Exercises

11: Parametric Equations and Polar Coordinates

- 11.0: Prelude to Parametric Equations and Polar Coordinates
- 11.1: Parametric Equations
  - 11.1E: Exercises for Section 11.1
- 11.2: Calculus of Parametric Curves
  - 11.2E: Exercises for Section 11.2
- 11.3: Polar Coordinates
  - 11.3E: Exercises for Section 11.3
- 11.4: Area and Arc Length in Polar Coordinates
  - 11.4E: Exercises for Section 11.4
- 11.5: Conic Sections
  - 11.5E: Exercises for Section 11.5
- 11.R: Chapter 11 Review Exercises

12: Vectors in Space

- 12.0: Prelude to Vectors in Space
- 12.1: Vectors in the Plane
  - 12.1E: Exercises for Section 12.1
- 12.2: Vectors in Three Dimensions
  - 12.2E: Exercises for Section 12.2
- 12.3: The Dot Product
  - 12.3E: Exercises for Section 12.3
- 12.4: The Cross Product
  - 12.4E: Exercises for Section 12.4
- 12.5: Equations of Lines and Planes in Space
  - 12.5E: Exercises for Section 12.5
- 12.6: Quadric Surfaces
  - 12.6E: Exercises for Section 12.6
- 12.7: Cylindrical and Spherical Coordinates
  - 12.7E: Exercises for Section 12.7
- 12.R: Chapter 12 Review Exercises

13: Vector-Valued Functions

- 13.0: Prelude to Vector-Valued Functions
- 13.1: Vector-Valued Functions and Space Curves
  - 13.1E: Exercises for Section 13.1
- 13.2: Calculus of Vector-Valued Functions
  - 13.2E: Exercises for Section 13.2
- 13.3: Arc Length and Curvature
  - 13.3E: Exercises for Section 13.3
- 13.4: Motion in Space
  - 13.4E: Exercises for Section 13.4
- 13.R: Chapter 13 Review Exercises

14: Differentiation of Functions of Several Variables

- 14.0: Prelude to Differentiation of Functions of Several Variables
- 14.1: Functions of Several Variables
  - 14.1E: Exercises for Section 14.1
- 14.2: Limits and Continuity
  - 14.2E: Exercises for Section 14.2
- 14.3: Partial Derivatives
  - 14.3E: Exercises for Section 14.3
- 14.4: Tangent Planes and Linear Approximations
  - 14.4E: Exercises for Section 14.4
- 14.5: The Chain Rule for Multivariable Functions
  - 14.5E: Exercises for Section 14.5
- 14.6: Directional Derivatives and the Gradient
  - 14.6E: Exercises for Section 14.6
- 14.7: Maxima/Minima Problems
  - 14.7E: Exercises for Section 14.7
- 14.8: Lagrange Multipliers
  - 14.8E: Exercises for Section 14.8
- 14.R: Chapter 14 Review Exercises

15: Multiple Integration

- 15.0: Prelude to Multiple Integration
- 15.1: Double Integrals over Rectangular Regions
  - 15.1E: Exercises for Section 15.1
- 15.2: Double Integrals over General Regions
  - 15.2E: Exercises for Section 15.2
- 15.3: Double Integrals in Polar Coordinates
  - 15.3E: Exercises for Section 15.3
- 15.4: Triple Integrals
  - 15.4E: Exercises for Section 15.4
- 15.5: Triple Integrals in Cylindrical and Spherical Coordinates
  - 15.5E: Exercises for Section 15.5
- 15.6: Calculating Centers of Mass and Moments of Inertia
  - 15.6E: Exercises for Section 15.6
- 15.7: Change of Variables in Multiple Integrals
  - 15.7E: Exercises for Section 15.7
- 15.R: Chapter 15 Review Exercises

16: Vector Calculus

- 16.0: Prelude to Vector Calculus
- 16.1: Vector Fields
  - 16.1E: Exercises for Section 16.1
- 16.2: Line Integrals
  - 16.2E: Exercises for Section 16.2

- 16.3: Conservative Vector Fields
  - 16.3E: Exercises for Section 16.3
- 16.4: Green's Theorem
  - 16.4E: Exercises for Section 16.4
- 16.5: Divergence and Curl
  - 16.5E: Exercises for Section 16.5
- 16.6: Surface Integrals
  - 16.6E: Exercises for Section 16.6
- 16.7: Stokes' Theorem
  - 16.7E: Exercises for Section 16.7
- 16.8: The Divergence Theorem
  - 16.8E: Exercises for Section 16.8
- 16.R: Chapter 16 Review Exercises

17: Second-Order Differential Equations

- 17.0: Prelude to Second-Order Differential Equations
- 17.1: Second-Order Linear Equations
  - 17.1E: Exercises for Section 17.1
- 17.2: Nonhomogeneous Linear Equations
  - 17.2E: Exercises for Section 17.2
- 17.3: Applications of Second-Order Differential Equations
  - 17.3E: Exercises for Section 17.3
- 17.4: Series Solutions of Differential Equations
  - 17.4E: Exercises for Section 17.4
- 17.R: Chapter 17 Review Exercises

Appendices

- Appendix A: Table of Derivatives
- Appendix B: Table of Integrals

Index

Glossary

Detailed Licensing

Licensing

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CHAPTER OVERVIEW

1: Functions and Graphs

Calculus is the mathematics that describes changes in functions. In this chapter, we review all the functions necessary to study calculus. We define polynomial, rational, trigonometric, exponential, and logarithmic functions. We review how to evaluate these functions, and we show the properties of their graphs. We provide examples of equations with terms involving these functions and illustrate the algebraic techniques necessary to solve them. In short, this chapter provides the foundation for the material to come. It is essential to be familiar and comfortable with these ideas before proceeding to the formal introduction of calculus in the next chapter.

1.0: Prelude to Functions and Graphs
1.1: Review of Functions
1.1E: Exercises for Section 1.1
1.2: Basic Classes of Functions
1.2E: Exercises for Section 1.2
1.3: Trigonometric Functions
1.3E: Exercises for Section 1.3
1.4: Inverse Functions
1.4E: Exercises for Section 1.4
1.5: Exponential and Logarithmic Functions
1.5E: Exercises for Section 1.5
1.R: Chapter 1 Review Exercises

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1.0: Prelude to Functions and Graphs

In the past few years, major earthquakes have occurred in several countries around the world. In January 2010, an earthquake of magnitude 7.3 hit Haiti. A magnitude 9 earthquake shook northeastern Japan in March 2011. In April 2014, an 8.2-magnitude earthquake struck off the coast of northern Chile. What do these numbers mean? In particular, how does a magnitude 9 earthquake compare with an earthquake of magnitude 8.2? Or 7.3? Later in this chapter, we show how logarithmic functions are used to compare the relative intensity of two earthquakes based on the magnitude of each earthquake.

Figure 1.0.1: A portion of the San Andreas Fault in California. Major faults like this are the sites of most of the strongest earthquakes ever recorded. (credit: modification of work by Robb Hannawacker, NPS)

Calculus is the mathematics that describes changes in functions. In this chapter, we review all the functions necessary to study calculus. We define polynomial, rational, trigonometric, exponential, and logarithmic functions. We review how to evaluate these functions, and we show the properties of their graphs. We provide examples of equations with terms involving these functions and illustrate the algebraic techniques necessary to solve them. In short, this chapter provides the foundation for the material to come. It is essential to be familiar and comfortable with these ideas before proceeding to the formal introduction of calculus in the next chapter.

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1.1: Review of Functions

Learning Objectives

- Use functional notation to evaluate a function.
- Determine the domain and range of a function.
- Draw the graph of a function.
- Find the zeros of a function.
- Recognize a function from a table of values.
- Make new functions from two or more given functions.
- Describe the symmetry properties of a function.

In this section, we provide a formal definition of a function and examine several ways in which functions are represented—namely, through tables, formulas, and graphs. We study formal notation and terms related to functions. We also define composition of functions and symmetry properties. Most of this material will be a review for you, but it serves as a handy reference to remind you of some of the algebraic techniques useful for working with functions.

Functions

Given two sets A and B a set with elements that are ordered pairs (x, y) where x is an element of A and y is an element of B, is a relation from A to B. A relation from A to B defines a relationship between those two sets. A function is a special type of relation in which each element of the first set is related to exactly one element of the second set. The element of the first set is called the ***input***; the element of the second set is called the ***output***. Functions are used all the time in mathematics to describe relationships between two sets. For any function, when we know the input, the output is determined, so we say that the output is a function of the input. For example, the area of a square is determined by its side length, so we say that the area (the output) is a function of its side length (the input). The velocity of a ball thrown in the air can be described as a function of the amount of time the ball is in the air. The cost of mailing a package is a function of the weight of the package. Since functions have so many uses, it is important to have precise definitions and terminology to study them.

Figure 1.1.1: A function can be visualized as an input/output device

Definition: Functions

A *function* f consists of a set of inputs, a set of outputs, and a rule for assigning each input to exactly one output. The set of inputs is called the *domain* of the function. The set of outputs is called the *range* of the *function*.

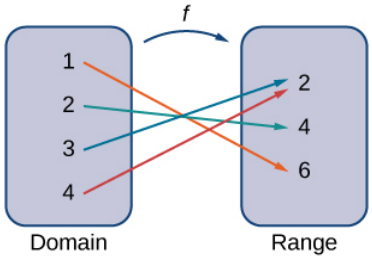


Figure 1.1.2: A function maps every element in the domain to exactly one element in the range. Although each input can be sent to only one output, two different inputs can be sent to the same output.

For example, consider the function f, where the domain is the set of all real numbers and the rule is to square the input. Then, the input $x = 3$ is assigned to the output $3^2 = 9$.

Since every nonnegative real number has a real-value square root, every nonnegative number is an element of the range of this function. Since there is no real number with a square that is negative, the negative real numbers are not elements of the range. We

conclude that the range is the set of nonnegative real numbers.

For a general function f with domain D, we often use x to denote the input and y to denote the output associated with x. When doing so, we refer to x as the **independent variable** and y as the **dependent variable**, because it depends on x. Using function notation, we write $y = f(x)$, and we read this equation as "y equals f of x." For the squaring function described earlier, we write $f(x) = x^2$.

The concept of a function can be visualized using Figures 1.1.1 - 1.1.3.

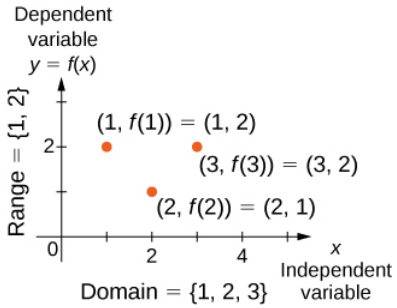


Figure 1.1.3: In this case, a graph of a function f has a domain of $\{1, 2, 3\}$ and a range of $\{1, 2\}$. The independent variable is x and the dependent variable is y.

We can also visualize a function by plotting points (x, y) in the coordinate plane where $y = f(x)$. The graph of a function is the set of all these points. For example, consider the function f, where the domain is the set $D = \{1, 2, 3\}$ and the rule is $f(x) = 3 - x$. In Figure 1.1.4, we plot a graph of this function.

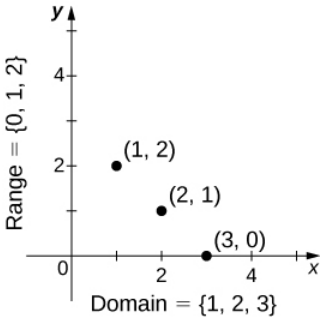


Figure 1.1.4: Here we see a graph of the function f with domain $\{1, 2, 3\}$ and rule $f(x) = 3 - x$. The graph consists of the points $(x, f(x))$ for all x in the domain.

Every function has a domain. However, sometimes a function is described by an equation, as in $f(x) = x^2$, with no specific domain given. In this case, the domain is taken to be the set of all real numbers x for which $f(x)$ is a real number. For example, since any real number can be squared, if no other domain is specified, we consider the domain of $f(x) = x^2$ to be the set of all real numbers. On the other hand, the square root function $f(x) = \sqrt{x}$ only gives a real output if x is nonnegative. Therefore, the domain of the function $f(x) = \sqrt{x}$ is the set of nonnegative real numbers, sometimes called the natural domain.

For the functions $f(x) = x^2$ and $f(x) = \sqrt{x}$, the domains are sets with an infinite number of elements. Clearly we cannot list all these elements. When describing a set with an infinite number of elements, it is often helpful to use set-builder or interval notation. When using set-builder notation to describe a subset of all real numbers, denoted R, we write

$$\{x \mid x \text{ has some property}\}.$$

We read this as the set of real numbers x such that x has some property. For example, if we were interested in the set of real numbers that are greater than one but less than five, we could denote this set using set-builder notation by writing

$$\{x \mid 1 < x < 5\}.$$

A set such as this, which contains all numbers greater than a and less than b, can also be denoted using the *interval notation* (a, b). Therefore,

$$(1, 5) = \{x \mid 1 < x < 5\}.$$

The numbers 1 and 5 are called the endpoints of this set. If we want to consider the set that includes the endpoints, we would denote this set by writing

$$[1,5] = \{x \mid 1 \leq x \leq 5\}.$$

We can use similar notation if we want to include one of the endpoints, but not the other. To denote the set of nonnegative real numbers, we would use the set-builder notation

$$\{x \mid x \geq 0\}.$$

The smallest number in this set is zero, but this set does not have a largest number. Using interval notation, we would use the symbol ∞, which refers to positive infinity, and we would write the set as

$$[0,\infty) = \{x \mid x \geq 0\}.$$

It is important to note that ∞ is not a real number. It is used symbolically here to indicate that this set includes all real numbers greater than or equal to zero. Similarly, if we wanted to describe the set of all nonpositive numbers, we could write

$$(-\infty, 0] = \{x \mid x \leq 0\}.$$

Here, the notation $-\infty$ refers to negative infinity, and it indicates that we are including all numbers less than or equal to zero, no matter how small. The set

$$(-\infty,\infty) = \{x \mid x \text{ is any real number}\}$$

refers to the set of all real numbers. Some functions are defined using different equations for different parts of their domain. These types of functions are known as piecewise-defined functions. For example, suppose we want to define a function f with a domain that is the set of all real numbers such that $f(x) = 3x + 1$ for $x \geq 2$ and $f(x) = x^2$ for $x < 2$. We denote this function by writing

$$f(x) = \begin{cases} 3x+1, & \text{if } x \geq 2 \\ x^2, & \text{if } x < 2 \end{cases}$$

When evaluating this function for an input x, the equation to use depends on whether $x \geq 2$ or $x < 2$. For example, since $5 > 2$, we use the fact that $f(x) = 3x + 1$ for $x \geq 2$ and see that $f(5) = 3(5) + 1 = 16$. On the other hand, for $x = -1$, we use the fact that $f(x) = x^2$ for $x < 2$ and see that $f(-1) = 1$.

Example 1.1.1: Evaluating Functions

For the function $f(x) = 3x^2 + 2x - 1$, evaluate:

a. $f(-2)$
b. $f(\sqrt{2})$
c. $f(a + h)$

Solution

Substitute the given value for x in the formula for $f(x)$.

a. $f(-2) = 3(-2)^2 + 2(-2) - 1 = 12 - 4 - 1 = 7$
b. $f(\sqrt{2}) = 3(\sqrt{2})^2 + 2\sqrt{2} - 1 = 6 + 2\sqrt{2} - 1 = 5 + 2\sqrt{2}$
c. $f(a + h) = 3(a + h)^2 + 2(a + h) - 1 = 3(a^2 + 2ah + h^2) + 2a + 2h - 1 = 3a^2 + 6ah + 3h^2 + 2a + 2h - 1$

Exercise 1.1.1

For $f(x) = x^2 - 3x + 5$, evaluate $f(1)$ and $f(a + h)$.

Hint

Substitute 1 and $a + h$ for x in the formula for $f(x)$.

Answer

$f(1) = 3$ and $f(a + h) = a^2 + 2ah + h^2 - 3a - 3h + 5$

Example 1.1.2: Finding Domain and Range

For each of the following functions, determine the i. domain and ii. range.

a. $f(x) = (x-4)^2 + 5$
b. $f(x) = \sqrt{3x+2} - 1$
c. $f(x) = \dfrac{3}{x-2}$

Solution

a. Consider $f(x) = (x-4)^2 + 5$.

1.Since $f(x) = (x-4)^2 + 5$ is a real number for any real number x, the domain of f is the interval $(-\infty, \infty)$.

2. Since $(x-4)^2 \geq 0$, we know $f(x) = (x-4)^2 + 5 \geq 5$. Therefore, the range must be a subset of $\{y \,|\, y \geq 5\}$. To show that every element in this set is in the range, we need to show that for a given y in that set, there is a real number x such that $f(x) = (x-4)^2 + 5 = y$. Solving this equation for x, we see that we need x such that

$$(x-4)^2 = y - 5.$$

This equation is satisfied as long as there exists a real number x such that

$$x - 4 = \pm\sqrt{y-5}$$

Since $y \geq 5$, the square root is well-defined. We conclude that for $x = 4 \pm \sqrt{y-5}$, $f(x) = y$, and therefore the range is $\{y \,|\, y \geq 5\}$.

b. Consider $f(x) = \sqrt{3x+2} - 1$.

1.To find the domain of f, we need the expression $3x + 2 \geq 0$. Solving this inequality, we conclude that the domain is $\{x \,|\, x \geq -2/3\}$.

2.To find the range of f, we note that since $\sqrt{3x+2} \geq 0$, $f(x) = \sqrt{3x+2} - 1 \geq -1$. Therefore, the range of f must be a subset of the set $\{y \,|\, y \geq -1\}$. To show that every element in this set is in the range of f, we need to show that for all y in this set, there exists a real number x in the domain such that $f(x) = y$. Let $y \geq -1$. Then, $f(x) = y$ if and only if

$$\sqrt{3x+2} - 1 = y.$$

Solving this equation for x, we see that x must solve the equation

$$\sqrt{3x+2} = y + 1.$$

Since $y \geq -1$, such an x could exist. Squaring both sides of this equation, we have $3x + 2 = (y+1)^2$.

Therefore, we need

$$3x = (y+1)^2 - 2,$$

which implies

$$x = \tfrac{1}{3}(y+1)^2 - \tfrac{2}{3}.$$

We just need to verify that x is in the domain of f. Since the domain of f consists of all real numbers greater than or equal to $\frac{-2}{3}$, and

$$\tfrac{1}{3}(y+1)^2 - \tfrac{2}{3} \geq -\tfrac{2}{3},$$

there does exist an x in the domain of f. We conclude that the range of f is $\{y \,|\, y \geq -1\}$.

c. Consider $f(x) = \dfrac{3}{x-2}$.

1.Since $3/(x-2)$ is defined when the denominator is nonzero, the domain is $\{x \,|\, x \neq 2\}$.

2.To find the range of f, we need to find the values of y such that there exists a real number x in the domain with the property that

$$\frac{3}{x-2} = y.$$

Solving this equation for x, we find that

$$x = \frac{3}{y} + 2.$$

Therefore, as long as $y \neq 0$, there exists a real number x in the domain such that $f(x) = y$. Thus, the range is $\{y \mid y \neq 0\}$.

? Exercise 1.1.2

Find the domain and range for $f(x) = \sqrt{4 - 2x} + 5$.

Hint

Use $4 - 2x \geq 0$.

Answer

Domain = $\{x \mid x \leq 2\}$ and range = $\{y \mid y \geq 5\}$

Representing Functions

Typically, a function is represented using one or more of the following tools:

- A table
- A graph
- A formula

We can identify a function in each form, but we can also use them together. For instance, we can plot on a graph the values from a table or create a table from a formula.

Tables

Functions described using a **table of values** arise frequently in real-world applications. Consider the following simple example. We can describe temperature on a given day as a function of time of day. Suppose we record the temperature every hour for a 24-hour period starting at midnight. We let our input variable x be the time after midnight, measured in hours, and the output variable y be the temperature x hours after midnight, measured in degrees Fahrenheit. We record our data in Table 1.1.1.

Table 1.1.1: Temperature as a Function of Time of Day

Hour After Midnight	Temperature(°F)	Hour After Midnight	Temperature(°F)
0	58	12	84
1	54	13	85
2	53	14	85
3	52	15	83
4	52	16	82
5	55	17	80
6	60	18	77
7	64	19	74
8	72	20	69
9	75	21	65
10	78	22	60
11	80	23	58

We can see from the table that temperature is a function of time, and the temperature decreases, then increases, and then decreases again. However, we cannot get a clear picture of the behavior of the function without graphing it.

Graphs

Given a function f described by a table, we can provide a visual picture of the function in the form of a graph. Graphing the temperatures listed in Table 1.1.1 can give us a better idea of their fluctuation throughout the day. Figure 1.1.5 shows the plot of the temperature function.

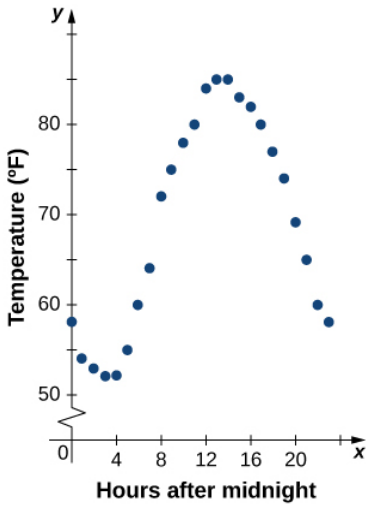


Figure 1.1.5: The graph of the data from Table 1.1.1 shows temperature as a function of time.

From the points plotted on the graph in Figure 1.1.5, we can visualize the general shape of the graph. It is often useful to connect the dots in the graph, which represent the data from the table. In this example, although we cannot make any definitive conclusion regarding what the temperature was at any time for which the temperature was not recorded, given the number of data points collected and the pattern in these points, it is reasonable to suspect that the temperatures at other times followed a similar pattern, as we can see in Figure 1.1.6.

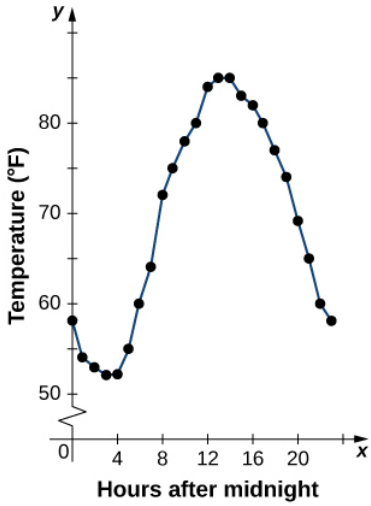


Figure 1.1.6: Connecting the dots in Figure 1.1.5 shows the general pattern of the data.

Algebraic Formulas

Sometimes we are not given the values of a function in table form, rather we are given the values in an explicit formula. Formulas arise in many applications. For example, the area of a circle of radius r is given by the formula $A(r) = \pi r^2$. When an object is thrown upward from the ground with an initial velocity v_0 ft/s, its height above the ground from the time it is thrown until it hits the ground is given by the formula $s(t) = -16t^2 + v_0 t$. When P dollars are invested in an account at an annual interest rate r compounded continuously, the amount of money after t years is given by the formula $A(t) = Pe^{rt}$. Algebraic formulas are important tools to calculate function values. Often we also represent these functions visually in graph form.

Given an algebraic formula for a function f, the graph of f is the set of points $(x, f(x))$, where x is in the domain of f and $f(x)$ is in the range. To graph a function given by a formula, it is helpful to begin by using the formula to create a table of inputs and outputs.

If the domain of f consists of an infinite number of values, we cannot list all of them, but because listing some of the inputs and outputs can be very useful, it is often a good way to begin.

When creating a table of inputs and outputs, we typically check to determine whether zero is an output. Those values of x where $f(x) = 0$ are called the zeros of a function. For example, the zeros of $f(x) = x^2 - 4$ are $x = \pm 2$. The zeros determine where the graph of f intersects the x-axis, which gives us more information about the shape of the graph of the function. The graph of a function may never intersect the x-axis, or it may intersect multiple (or even infinitely many) times.

Another point of interest is the y -intercept, if it exists. The y-intercept is given by $(0, f(0))$.

Since a function has exactly one output for each input, the graph of a function can have, at most, one y-intercept. If $x = 0$ is in the domain of a function f, then f has exactly one y-intercept. If $x = 0$ is not in the domain of f, then f has no y-intercept. Similarly, for any real number c, if c is in the domain of f, there is exactly one output $f(c)$, and the line $x = c$ intersects the graph of f exactly once. On the other hand, if c is not in the domain of f, $f(c)$ is not defined and the line $x = c$ does not intersect the graph of f. This property is summarized in the vertical line test.

Vertical Line Test

Given a function f, every vertical line that may be drawn intersects the graph of f no more than once. If any vertical line intersects a set of points more than once, the set of points does not represent a function.

We can use this test to determine whether a set of plotted points represents the graph of a function (Figure 1.1.7).

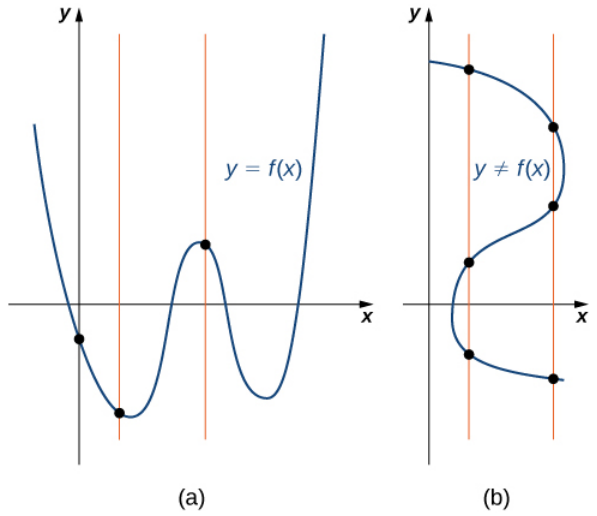


Figure 1.1.7: (a) The set of plotted points represents the graph of a function because every vertical line intersects the set of points, at most, once. (b) The set of plotted points does not represent the graph of a function because some vertical lines intersect the set of points more than once.

Example 1.1.3: Finding Zeros and y-Intercepts of a Function

Consider the function $f(x) = -4x + 2$.

a. Find all zeros of f.
b. Find the y-intercept (if any).
c. Sketch a graph of f.

Solution

1.To find the zeros, solve $f(x) = -4x + 2 = 0$. We discover that f has one zero at $x = 1/2$.

2. The y-intercept is given by $(0, f(0)) = (0, 2)$.

3. Given that f is a linear function of the form $f(x) = mx + b$ that passes through the points $(1/2, 0)$ and $(0, 2)$, we can sketch the graph of f (Figure 1.1.8).

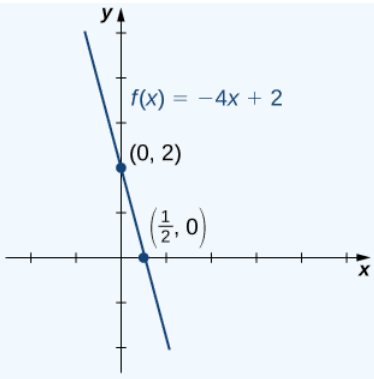


Figure 1.1.8: The function $f(x) = -4x + 2$ is a line with x-intercept $(1/2, 0)$ and y-intercept $(0, 2)$.

Example 1.1.4: Using Zeros and y-Intercepts to Sketch a Graph

Consider the function $f(x) = \sqrt{x+3} + 1$.

a. Find all zeros of f.
b. Find the y-intercept (if any).
c. Sketch a graph of f.

Solution

1.To find the zeros, solve $\sqrt{x+3} + 1 = 0$. This equation implies $\sqrt{x+3} = -1$. Since $\sqrt{x+3} \geq 0$ for all x, this equation has no solutions, and therefore f has no zeros.

2.The y-intercept is given by $(0, f(0)) = (0, \sqrt{3} + 1)$.

3.To graph this function, we make a table of values. Since we need $x + 3 \geq 0$, we need to choose values of $x \geq -3$. We choose values that make the square-root function easy to evaluate.

x	-3	-2	1
$f(x)$	1	2	3

Making use of the table and knowing that, since the function is a square root, the graph of f should be similar to the graph of $y = \sqrt{x}$, we sketch the graph (Figure 1.1.9).

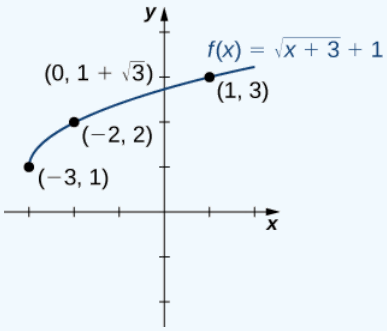


Figure 1.1.9: The graph of $f(x) = \sqrt{x+3} + 1$ has a y-intercept but no x-intercepts.

Exercise 1.1.4

Find the zeros of $f(x) = x^3 - 5x^2 + 6x$.

Hint

Factor the polynomial.

Answer

$x = 0, 2, 3$

Example 1.1.5: Finding the Height of a Free-Falling Object

If a ball is dropped from a height of 100 ft, its height s at time t is given by the function $s(t) = -16t^2 + 100$, where s is measured in feet and t is measured in seconds. The domain is restricted to the interval $[0, c]$, where $t = 0$ is the time when the ball is dropped and $t = c$ is the time when the ball hits the ground.

a. Create a table showing the height s(t) when $t = 0,\ 0.5,\ 1,\ 1.5,\ 2$, and 2.5. Using the data from the table, determine the domain for this function. That is, find the time c when the ball hits the ground.
b. Sketch a graph of s.

Solution

t	0	0.5	1	1.5	2	2.5
$s(t)$	100	96	84	64	36	0

Since the ball hits the ground when $t = 2.5$, the domain of this function is the interval $[0, 2.5]$.

2.

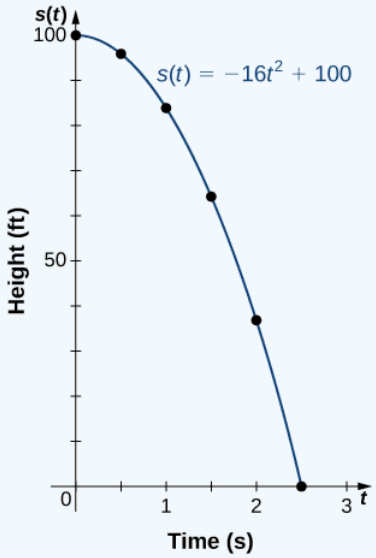


Figure 1.1.8, the values of $f(x)$ are getting smaller as x is getting larger. A function with this property is said to be decreasing. On the other hand, for the function $f(x) = \sqrt{x+3} + 1$ graphed in Figure 1.1.9, the values of $f(x)$ are getting larger as the values of x are getting larger. A function with this property is said to be increasing. It is important to note, however, that a function can be increasing on some interval or intervals and decreasing over a different interval or intervals. For example, using our temperature function plotted above, we can see that the function is decreasing on the interval $(0, 4)$, increasing on the interval $(4, 14)$, and then decreasing on the interval $(14, 23)$. We make the idea of a function increasing or decreasing over a particular interval more precise in the next definition.

Definition: Increasing and Decreasing on an Interval

We say that a function f is ***increasing on the interval*** I if for all $x_1,\ x_2 \in I$,

$$f(x_1) \leq f(x_2) \text{ when } x_1 < x_2.$$

We say f is strictly increasing on the interval I if for all $x_1, x_2 \in I$,

$$f(x_1) < f(x_2) \text{ when } x_1 < x_2.$$

We say that a function f is ***decreasing on the interval*** I if for all $x_1, x_2 \in I$,

$$f(x_1) \geq f(x_2) \text{ if } x_1 < x_2.$$

We say that a function f is strictly decreasing on the interval I if for all $x_1, x_2 \in I$,

$$f(x_1) > f(x_2) \text{ if } x_1 < x_2.$$

For example, the function $f(x) = 3x$ is increasing on the interval $(-\infty, \infty)$ because $3x_1 < 3x_2$ whenever $x_1 < x_2$. On the other hand, the function $f(x) = -x^3$ is decreasing on the interval $(-\infty, \infty)$ because $-x_1^3 > -x_2^3$ whenever $x_1 < x_2$ (Figure 1.1.10).

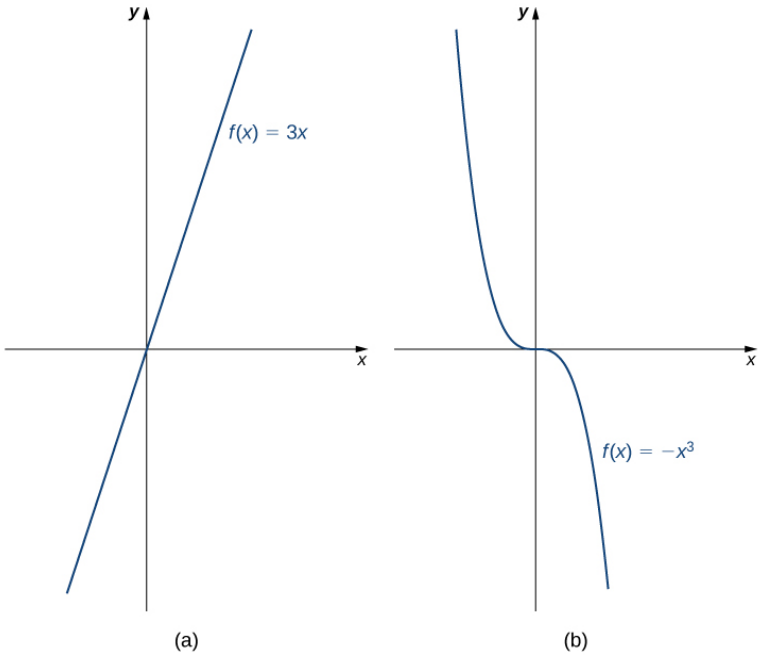


Figure 1.1.10: (a) The function $f(x) = 3x$ is increasing on the interval $(-\infty, \infty)$. (b) The function $f(x) = -x^3$ is decreasing on the interval $(-\infty, \infty)$.

Combining Functions

Now that we have reviewed the basic characteristics of functions, we can see what happens to these properties when we combine functions in different ways, using basic mathematical operations to create new functions. For example, if the cost for a company to manufacture x items is described by the function $C(x)$ and the revenue created by the sale of x items is described by the function $R(x)$, then the profit on the manufacture and sale of x items is defined as $P(x) = R(x) - C(x)$. Using the difference between two functions, we created a new function.

Alternatively, we can create a new function by composing two functions. For example, given the functions $f(x) = x^2$ and $g(x) = 3x + 1$, the composite function $f \circ g$ is defined such that

$$(f \circ g)(x) = f(g(x)) = (g(x))^2 = (3x + 1)^2.$$

The composite function $g \circ f$ is defined such that

$$(g \circ f)(x) = g(f(x)) = 3f(x) + 1 = 3x^2 + 1.$$

Note that these two new functions are different from each other.

Combining Functions with Mathematical Operators

To combine functions using mathematical operators, we simply write the functions with the operator and simplify. Given two functions f and g, we can define four new functions:

$(f + g)(x) = f(x) + g(x)$	Sum
$(f - g)(x) = f(x) - g(x)$	Difference
$(f \cdot g)(x) = f(x)g(x)$	Product
$(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$ for $g(x) \neq 0$	Quotient

Example 1.1.6: Combining Functions Using Mathematical Operations

Given the functions $f(x) = 2x - 3$ and $g(x) = x^2 - 1$, find each of the following functions and state its domain.

a. $(f + g)(x)$
b. $(f - g)(x)$
c. $(f \cdot g)(x)$
d. $\left(\frac{f}{g}\right)(x)$

Solution

1. $(f + g)(x) = (2x - 3) + (x^2 - 1) = x^2 + 2x - 4.$

The domain of this function is the interval $(-\infty, \infty)$.

2. $(f - g)(x) = (2x - 3) - (x^2 - 1) = -x^2 + 2x - 2.$

The domain of this function is the interval $(-\infty, \infty)$.

3. $(f \cdot g)(x) = (2x - 3)(x^2 - 1) = 2x^3 - 3x^2 - 2x + 3.$

The domain of this function is the interval $(-\infty, \infty)$.

4. $\left(\frac{f}{g}\right)(x) = \frac{2x - 3}{x^2 - 1}.$

The domain of this function is $\{x \mid x \neq \pm 1\}$.

Exercise 1.1.6

For $f(x) = x^2 + 3$ and $g(x) = 2x - 5$, find $(f/g)(x)$ and state its domain.

Hint

The new function $(f/g)(x)$ is a quotient of two functions. For what values of x is the denominator zero?

Answer

$\left(\frac{f}{g}\right)(x) = \frac{x^2+3}{2x-5}$. The domain is $\{x \mid x \neq \frac{5}{2}\}$.

Function Composition

When we compose functions, we take a function of a function. For example, suppose the temperature T on a given day is described as a function of time t (measured in hours after midnight) as in Table 1.1.1. Suppose the cost C, to heat or cool a building for 1 hour, can be described as a function of the temperature T. Combining these two functions, we can describe the cost of heating or cooling a building as a function of time by evaluating $C(T(t))$. We have defined a new function, denoted $C \circ T$, which is defined such that $(C \circ T)(t) = C(T(t))$ for all t in the domain of T. This new function is called a composite function. We note that since cost is a function of temperature and temperature is a function of time, it makes sense to define this new function $(C \circ T)(t)$. It does not make sense to consider $(T \circ C)(t)$, because temperature is not a function of cost.

Definition: Composite Functions

Consider the function f with domain A and range B, and the function g with domain D and range E. If B is a subset of D, then the composite function $(g \circ f)(x)$ is the function with domain A such that

$$(g \circ f)(x) = g(f(x))$$

A composite function $g \circ f$ can be viewed in two steps. First, the function f maps each input x in the domain of f to its output $f(x)$ in the range of f. Second, since the range of f is a subset of the domain of g, the output $f(x)$ is an element in the domain of g, and therefore it is mapped to an output $g(f(x))$ in the range of g. In Figure 1.1.11, we see a visual image of a composite function.

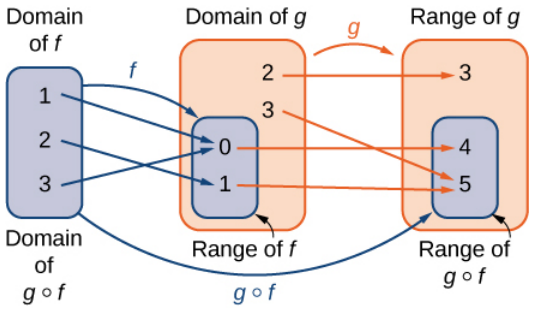


Figure 1.1.11: For the composite function $g \circ f$, we have $(g \circ f)(1) = 4$, $(g \circ f)(2) = 5$, and $(g \circ f)(3) = 4$.

Example 1.1.7: Compositions of Functions Defined by Formulas

Consider the functions $f(x) = x^2 + 1$ and $g(x) = 1/x$.

a. Find $(g \circ f)(x)$ and state its domain and range.
b. Evaluate $(g \circ f)(4)$, $(g \circ f)(-1/2)$.
c. Find $(f \circ g)(x)$ and state its domain and range.
d. Evaluate $(f \circ g)(4)$, $(f \circ g)(-1/2)$.

Solution

1. We can find the formula for $(g \circ f)(x)$ in two different ways. We could write

$$(g \circ f)(x) = g(f(x)) = g(x^2 + 1) = \frac{1}{x^2 + 1}.$$

Alternatively, we could write

$$(g \circ f)(x) = g(f(x)) = \frac{1}{f(x)} = \frac{1}{x^2 + 1}.$$

Since $x^2 + 1 \neq 0$ for all real numbers x, the domain of $(g \circ f)(x)$ is the set of all real numbers. Since $0 < 1/(x^2 + 1) \leq 1$, the range is, at most, the interval $(0, 1]$. To show that the range is this entire interval, we let $y = 1/(x^2 + 1)$ and solve this equation for x to show that for all y in the interval $(0, 1]$, there exists a real number x such that $y = 1/(x^2 + 1)$. Solving this equation for x, we see that $x^2 + 1 = 1/y$, which implies that

$$x = \pm\sqrt{\tfrac{1}{y} - 1}$$

If y is in the interval $(0, 1]$, the expression under the radical is nonnegative, and therefore there exists a real number x such that $1/(x^2 + 1) = y$. We conclude that the range of $g \circ f$ is the interval $(0, 1]$.

2. $(g \circ f)(4) = g(f(4)) = g(4^2 + 1) = g(17) = \frac{1}{17}$

$(g \circ f)(-\frac{1}{2}) = g(f(-\frac{1}{2})) = g((-\frac{1}{2})^2 + 1) = g(\frac{5}{4}) = \frac{4}{5}$

3. We can find a formula for $(f \circ g)(x)$ in two ways. First, we could write

$$(f \circ g)(x) = f(g(x)) = f(\tfrac{1}{x}) = (\tfrac{1}{x})^2 + 1.$$

Alternatively, we could write

$$(f \circ g)(x) = f(g(x)) = (g(x))^2 + 1 = (\tfrac{1}{x})^2 + 1.$$

The domain of $f \circ g$ is the set of all real numbers x such that $x \neq 0$. To find the range of f, we need to find all values y for which there exists a real number $x \neq 0$ such that

$$\left(\frac{1}{x}\right)^2 + 1 = y.$$

Solving this equation for x, we see that we need x to satisfy

$$\left(\frac{1}{x}\right)^2 = y - 1,$$

which simplifies to

$$\frac{1}{x} = \pm\sqrt{y - 1}$$

Finally, we obtain

$$x = \pm\frac{1}{\sqrt{y-1}}.$$

Since $1/\sqrt{y-1}$ is a real number if and only if $y > 1$, the range of f is the set $\{y \mid y \geq 1\}$.

4.$(f \circ g)(4) = f(g(4)) = f(\frac{1}{4}) = (\frac{1}{4})^2 + 1 = \frac{17}{16}$

$(f \circ g)(-\frac{1}{2}) = f(g(-\frac{1}{2})) = f(-2) = (-2)^2 + 1 = 5$

In Example 1.1.7, we can see that $(f \circ g)(x) \neq (g \circ f)(x)$. This tells us, in general terms, that the order in which we compose functions matters.

Exercise 1.1.7

Let $f(x) = 2 - 5x$. Let $g(x) = \sqrt{x}$. Find $(f \circ g)(x)$.

Solution

$(f \circ g)(x) = 2 - 5\sqrt{x}$.

Example 1.1.8: Composition of Functions Defined by Tables

Consider the functions f and g described by

x	-3	-2	-1	0	1	2	3	4
$f(x)$	0	4	2	4	-2	0	-2	4

x	-4	-2	0	2	4
$g(x)$	1	0	3	0	5

a. Evaluate $(g \circ f)(3)$,$(g \circ f)(0)$.
b. State the domain and range of $(g \circ f)(x)$.
c. Evaluate $(f \circ f)(3)$,$(f \circ f)(1)$.
d. State the domain and range of $(f \circ f)(x)$.

Solution:

1. $(g \circ f)(3) = g(f(3)) = g(-2) = 0$

$(g \circ f)(0) = g(4) = 5$

2.The domain of $g \circ f$ is the set $\{-3, -2, -1, 0, 1, 2, 3, 4\}$. Since the range of f is the set $\{-2, 0, 2, 4\}$, the range of $g \circ f$ is the set $\{0, 3, 5\}$.

3. $(f \circ f)(3) = f(f(3)) = f(-2) = 4$

$(f \circ f)(1) = f(f(1)) = f(-2) = 4$

4.The domain of $f \circ f$ is the set $\{-3, -2, -1, 0, 1, 2, 3, 4\}$. Since the range of f is the set $\{-2, 0, 2, 4\}$, the range of $f \circ f$ is the set $\{0, 4\}$.

Example 1.1.9: Application Involving a Composite Function

A store is advertising a sale of 20% off all merchandise. Caroline has a coupon that entitles her to an additional 15% off any item, including sale merchandise. If Caroline decides to purchase an item with an original price of x dollars, how much will she end up paying if she applies her coupon to the sale price? Solve this problem by using a composite function.

Solution

Since the sale price is 20% off the original price, if an item is x dollars, its sale price is given by $f(x) = 0.80x$. Since the coupon entitles an individual to 15% off the price of any item, if an item is y dollars, the price, after applying the coupon, is given by g(y)=0.85y. Therefore, if the price is originally x dollars, its sale price will be $f(x) = 0.80x$ and then its final price after the coupon will be $g(f(x)) = 0.85(0.80x) = 0.68x$.

? Exercise 1.1.9

If items are on sale for 10% off their original price, and a customer has a coupon for an additional 30% off, what will be the final price for an item that is originally x dollars, after applying the coupon to the sale price?

Hint

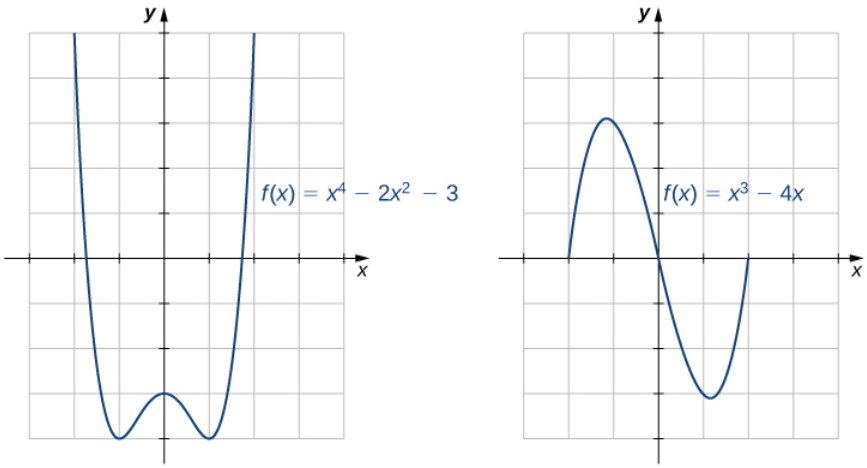
The sale price of an item with an original price of x dollars is $f(x) = 0.90x$. The coupon price for an item that is y dollars is $g(y) = 0.70y$.

Solution

$(g \circ f)(x) = 0.63x$

Symmetry of Functions

The graphs of certain functions have symmetry properties that help us understand the function and the shape of its graph. For example, consider the function $f(x) = x^4 - 2x^2 - 3$ shown in Figure 1.1.12a. If we take the part of the curve that lies to the right of the y-axis and flip it over the y-axis, it lays exactly on top of the curve to the left of the y-axis. In this case, we say the function has **symmetry about the y-axis**. On the other hand, consider the function $f(x) = x^3 - 4x$ shown in Figure 1.1.12b. If we take the graph and rotate it $180°$ about the origin, the new graph will look exactly the same. In this case, we say the function has **symmetry about the origin**.



(a) Symmetry about the y-axis (b) Symmetry about the origin

Figure 1.1.12: (a) A graph that is symmetric about the y-axis. (b) A graph that is symmetric about the origin.

If we are given the graph of a function, it is easy to see whether the graph has one of these symmetry properties. But without a graph, how can we determine algebraically whether a function f has symmetry? Looking at Figure 1.1.12a again, we see that since f is symmetric about the y-axis, if the point (x, y) is on the graph, the point $(-x, y)$ is on the graph. In other words, $f(-x) = f(x)$. If a function f has this property, we say f is an even function, which has symmetry about the y-axis. For example, $f(x) = x^2$ is even because

$$f(-x) = (-x)^2 = x^2 = f(x).$$

In contrast, looking at Figure 1.1.12b again, if a function f is symmetric about the origin, then whenever the point (x, y) is on the graph, the point $(-x, -y)$ is also on the graph. In other words, $f(-x) = -f(x)$. If f has this property, we say f is an odd function, which has symmetry about the origin. For example, $f(x) = x^3$ is odd because

$$f(-x) = (-x)^3 = -x^3 = -f(x).$$

Definition: Even and Odd Functions

- If $f(x) = f(-x)$ for all x in the domain of f, then f is an *even* function. An even function is symmetric about the y-axis.
- If $f(-x) = -f(x)$ for all x in the domain of f, then f is an *odd* function. An odd function is symmetric about the origin.

Example 1.1.10: Even and Odd Functions

Determine whether each of the following functions is even, odd, or neither.

a. $f(x) = -5x^4 + 7x^2 - 2$
b. $f(x) = 2x^5 - 4x + 5$
c. $f(x) = \frac{3x}{x^2+1}$

Solution

To determine whether a function is even or odd, we evaluate $f(-x)$ and compare it to $f(x)$ and $-f(x)$.

1. $f(-x) = -5(-x)^4 + 7(-x)^2 - 2 = -5x^4 + 7x^2 - 2 = f(x)$. Therefore, f is even.

2. $f(-x) = 2(-x)^5 - 4(-x) + 5 = -2x^5 + 4x + 5$. Now, $f(-x) \neq f(x)$. Furthermore, noting that $-f(x) = -2x^5 + 4x - 5$, we see that $f(-x) \neq -f(x)$. Therefore, f is neither even nor odd.

3. $f(-x) = 3(-x)/((-x)2 + 1) = -3x/(x^2 + 1) = -[3x/(x^2 + 1)] = -f(x)$. Therefore, f is odd.

Exercise 1.1.10

Determine whether $f(x) = 4x^3 - 5x$ is even, odd, or neither.

Hint

Compare $f(-x)$ with $f(x)$ and $-f(x)$.

Answer

$f(x)$ is odd.

One symmetric function that arises frequently is the **absolute value function**, written as $|x|$. The absolute value function is defined as

$$f(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Some students describe this function by stating that it "makes everything positive." By the definition of the absolute value function, we see that if $x < 0$, then $|x| = -x > 0$, and if $x > 0$, then $|x| = x > 0$. However, for $x = 0$, $|x| = 0$. Therefore, it is more accurate to say that for all nonzero inputs, the output is positive, but if $x = 0$, the output $|x| = 0$. We conclude that the range of the absolute value function is $\{y \mid y \geq 0\}$. In Figure 1.1.13, we see that the absolute value function is symmetric about the y-axis and is therefore an even function.

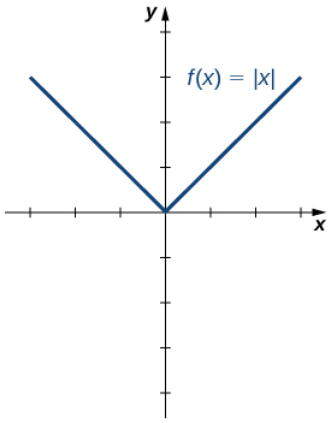


Figure 1.1.13: The graph of $f(x) = |x|$ is symmetric about the y-axis.

Example 1.1.11: Working with the Absolute Value Function

Find the domain and range of the function $f(x) = 2|x - 3| + 4$.

Solution

Since the absolute value function is defined for all real numbers, the domain of this function is $(-\infty, \infty)$. Since $|x - 3| \geq 0$ for all x, the function $f(x) = 2|x - 3| + 4 \geq 4$. Therefore, the range is, at most, the set $\{y \mid y \geq 4\}$. To see that the range is, in fact, this whole set, we need to show that for $y \geq 4$ there exists a real number x such that

$$2|x - 3| + 4 = y$$

A real number x satisfies this equation as long as

$$|x - 3| = \tfrac{1}{2}(y - 4)$$

Since $y \geq 4$, we know $y - 4 \geq 0$, and thus the right-hand side of the equation is nonnegative, so it is possible that there is a solution. Furthermore,

$$|x - 3| = \begin{cases} -(x - 3), & \text{if } x < 3 \\ x - 3, & \text{if } x \geq 3 \end{cases}$$

Therefore, we see there are two solutions:

$$x = \pm\tfrac{1}{2}(y - 4) + 3.$$

The range of this function is $\{y \mid y \geq 4\}$.

Exercise 1.1.11: Domain and Range

For the function $f(x) = |x + 2| - 4$, find the domain and range.

Hint

$|x + 2| \geq 0$ for all real numbers x.

Answer

Domain = $(-\infty, \infty)$, range = $\{y \mid y \geq -4\}$.

Key Concepts

- A function is a mapping from a set of inputs to a set of outputs with exactly one output for each input.
- If no domain is stated for a function $y = f(x)$, the domain is considered to be the set of all real numbers x for which the function is defined.
- When sketching the graph of a function f, each vertical line may intersect the graph, at most, once.
- A function may have any number of zeros, but it has, at most, one y-intercept.
- To define the composition $g \circ f$, the range of f must be contained in the domain of g.

- Even functions are symmetric about the y-axis whereas odd functions are symmetric about the origin.

Key Equations

- **Composition of two functions**

$$(g \circ f)(x) = g\big(f(x)\big)$$

- **Absolute value function**

$$f(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Glossary

absolute value function

$$f(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

composite function

given two functions f and g, a new function, denoted $g \circ f$, such that $(g \circ f)(x) = g(f(x))$

decreasing on the interval I

a function decreasing on the interval I if, for all $x_1,\ x_2 \in I,\ f(x_1) \geq f(x_2)$ if $x_1 < x_2$

dependent variable

the output variable for a function

domain

the set of inputs for a function

even function

a function is even if $f(-x) = f(x)$ for all x in the domain of f

function

a set of inputs, a set of outputs, and a rule for mapping each input to exactly one output

graph of a function

the set of points (x, y) such that x is in the domain of f and $y = f(x)$

increasing on the interval I

a function increasing on the interval I if for all $x_1,\ x_2 \in I,\ f(x_1) \leq f(x_2)$ if $x_1 < x_2$

independent variable

the input variable for a function

odd function

a function is odd if $f(-x) = -f(x)$ for all x in the domain of f

range

the set of outputs for a function

symmetry about the origin

the graph of a function f is symmetric about the origin if $(-x, -y)$ is on the graph of f whenever (x, y) is on the graph

symmetry about the y-axis

the graph of a function f is symmetric about the y-axis if $(-x, y)$ is on the graph of f whenever (x, y) is on the graph

table of values

a table containing a list of inputs and their corresponding outputs

vertical line test

given the graph of a function, every vertical line intersects the graph, at most, once

zeros of a function

when a real number x is a zero of a function f, $f(x) = 0$

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1.1E: Exercises for Section 1.1

For exercises 1 - 6, (a) determine the domain and the range of each relation, and (b) state whether the relation is a function.

1)

x	y	x	y
-3	9	1	1
-2	4	2	4
-1	1	3	9
0	0		

Answer

a. Domain = $\{-3,-2,-1,0,1,2,3\}$, Range = $\{0,1,4,9\}$

b. Yes, a function

2)

x	y	x	y
-3	-2	1	1
-2	-8	2	8
-1	-1	3	-2
0	0		

3)

x	y	x	y
1	-3	1	1
2	-2	2	2
3	-1	3	3
0	0		

Answer

a. Domain = $\{0,1,2,3\}$, Range = $\{-3,-2,-1,0,1,2,3\}$

b. No, not a function

4)

x	y	x	y
1	1	5	1
2	1	6	1
3	1	7	1
4	1		

5)

x	y	x	y
3	3	15	1
5	2	21	2
8	1	33	3
10	0		

Answer

a. Domain = $\{3, 5, 8, 10, 15, 21, 33\}$, Range = $\{0, 1, 2, 3\}$

b. Yes, a function

6)

x	y	x	y
-7	11	1	-2
-2	5	3	4
-2	1	6	11
0	-1		

For exercises 7 - 13, find the values for each function, if they exist, then simplify.

a. $f(0)$ b. $f(1)$ c. $f(3)$ d. $f(-x)$ e. $f(a)$ f. $f(a+h)$

7) $f(x) = 5x - 2$

Answer

a. -2 b. 3 c. 13 d. $-5x - 2$ e. $5a - 2$ f. $5a + 5h - 2$

8) $f(x) = 4x^2 - 3x + 1$

9) $f(x) = \dfrac{2}{x}$

Answer

a. Undefined b. 2 c. $\frac{2}{3}$ d. $-\dfrac{2}{x}$ e. $\dfrac{2}{a}$ f. $\dfrac{2}{a+h}$

10) $f(x) = |x - 7| + 8$

11) $f(x) = \sqrt{6x + 5}$

Answer

a. $\sqrt{5}$ b. $\sqrt{11}$ c. $\sqrt{23}$ d. $\sqrt{-6x + 5}$ e. $\sqrt{6a + 5}$ f. $\sqrt{6a + 6h + 5}$

12) $f(x) = \dfrac{x - 2}{3x + 7}$

13) $f(x) = 9$

Answer

a. 9 b. 9 c. 9 d. 9 e. 9 f. 9

For exercises 14 - 21, find the domain, range, and all zeros/intercepts, if any, of the functions.

14) $f(x) = \dfrac{x}{x^2 - 16}$

15) $g(x) = \sqrt{8x - 1}$

Answer

$x \geq \frac{1}{8}; \quad y \geq 0; \quad x = \frac{1}{8}$; no y-intercept

16) $h(x) = \dfrac{3}{x^2 + 4}$

17) $f(x) = -1 + \sqrt{x + 2}$

Answer

$x \geq -2; \quad y \geq -1; \quad x = -1; \quad y = -1 + \sqrt{2}$

18) $f(x) = 1x - \sqrt{9}$

19) $g(x) = \dfrac{3}{x - 4}$

Answer

$x \neq 4; \quad y \neq 0$; no x-intercept; $y = -\frac{3}{4}$

20) $f(x) = 4|x + 5|$

21) $g(x) = \sqrt{\dfrac{7}{x - 5}}$

Answer

$x > 5; \quad y > 0$; no intercepts

For exercises 22 - 27, set up a table to sketch the graph of each function using the following values: $x = -3, -2, -1, 0, 1, 2, 3.$

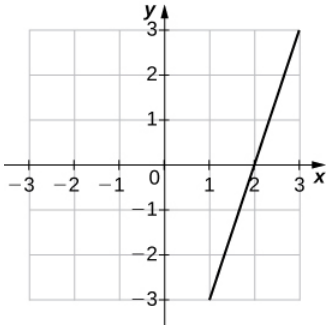
22) $f(x) = x^2 + 1$

x	y	x	y
-3	10	1	2
-2	5	2	5
-1	2	3	10
0	1		

23) $f(x) = 3x - 6$

x	y	x	y
-3	-15	1	-3
-2	-12	2	0
-1	-9	3	3
0	-6		

Answer



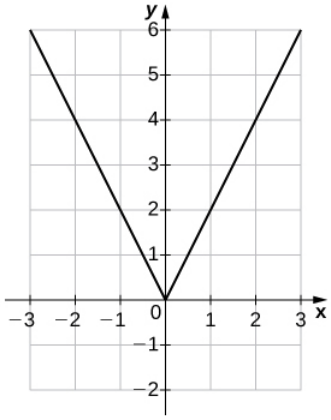
24) $f(x) = \frac{1}{2}x + 1$

x	y	x	y
-3	$-\frac{1}{2}$	1	$\frac{3}{2}$
-2	0	2	2
-1	$\frac{1}{2}$	3	$\frac{5}{2}$
0	1		

25) $f(x) = 2|x|$

x	y	x	y
-3	6	1	2
-2	4	2	4
-1	2	3	6
0	0		

Answer



26) $f(x) = -x^2$

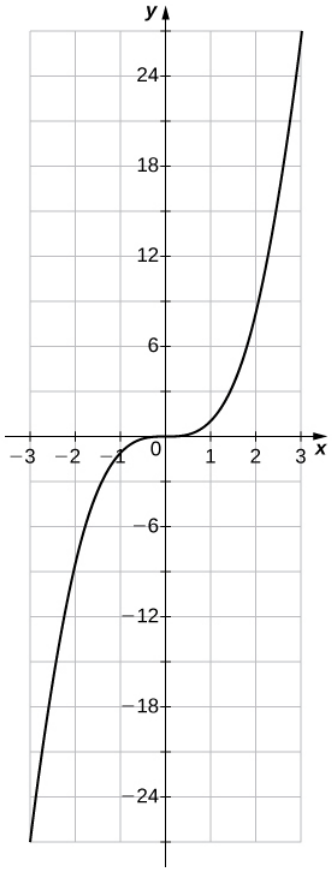
x	y	x	y
-3	-9	1	-1
-2	-4	2	-4
-1	-1	3	-9

x	y	x	y
0	0		

27) $f(x) = x^3$

x	y	x	y
-3	-27	1	1
-2	-8	2	8
-1	-1	3	27
0	0		

Answer



For exercises 28 - 35, use the vertical line test to determine whether each of the given graphs represents a function. Assume that a graph continues at both ends if it extends beyond the given grid. If the graph represents a function, then determine the following for each graph:

a. Domain and range

b. x -intercept, if any (estimate where necessary)

c. y-Intercept, if any (estimate where necessary)

d. The intervals for which the function is increasing

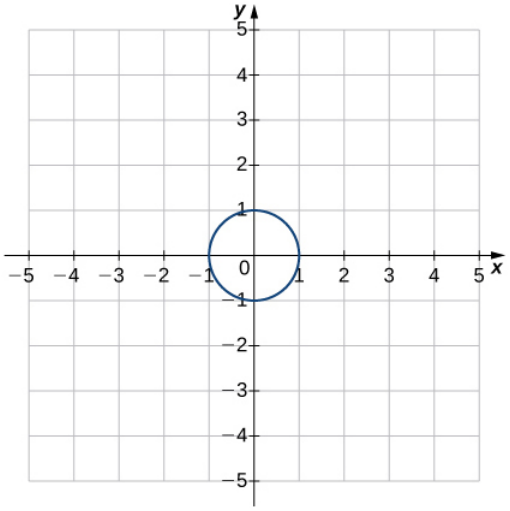
e. The intervals for which the function is decreasing

f. The intervals for which the function is constant

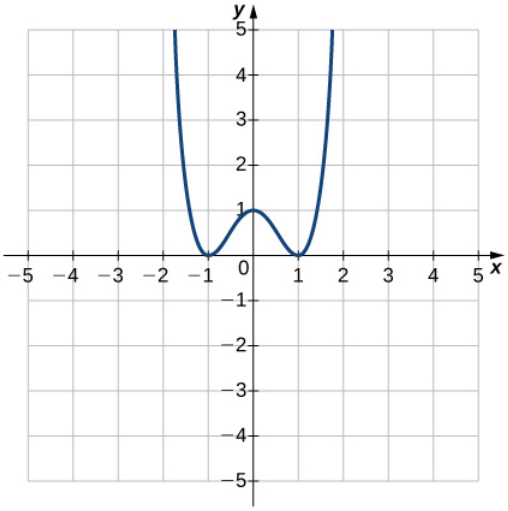
g. Symmetry about any axis and/or the origin

h. Whether the function is even, odd, or neither

28)



29)



Answer

Function;

a. Domain: all real numbers, range: $y \geq 0$

b. $x = \pm 1$

c. $y = 1$

d. $-1 < x < 0$ and $1 < x < \infty$

e. $-\infty < x < -1$ and $0 < x < 1$

f. Not constant

g. y-axis

h. Even

30)

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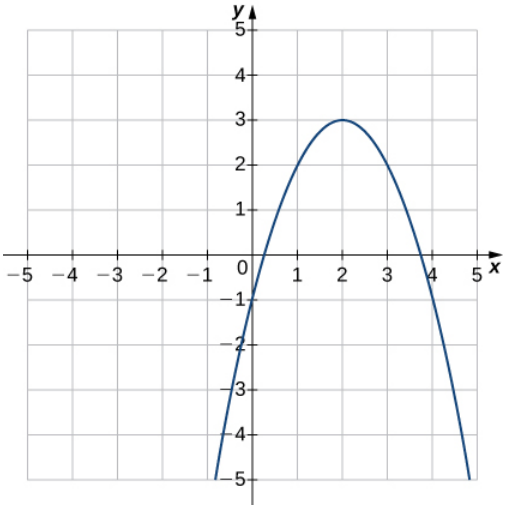

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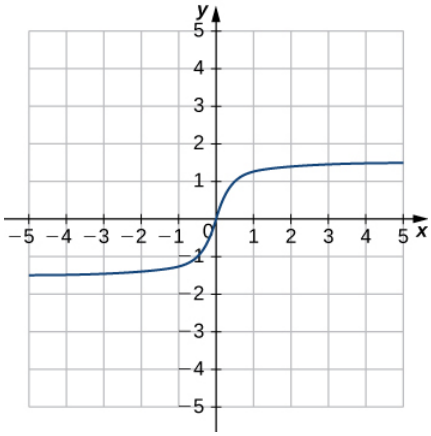


Answer

Function;

a. Domain: all real numbers, range: $-1.5 \leq y \leq 1.5$

b. $x = 0$

c. $y = 0$

d. all real numbers

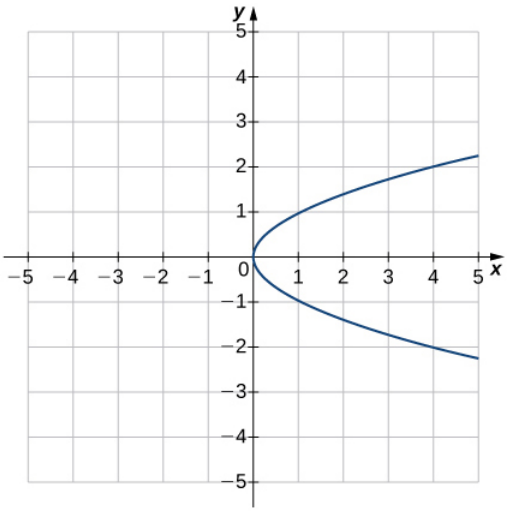
e. None

f. Not constant

g. Origin

h. Odd

32)



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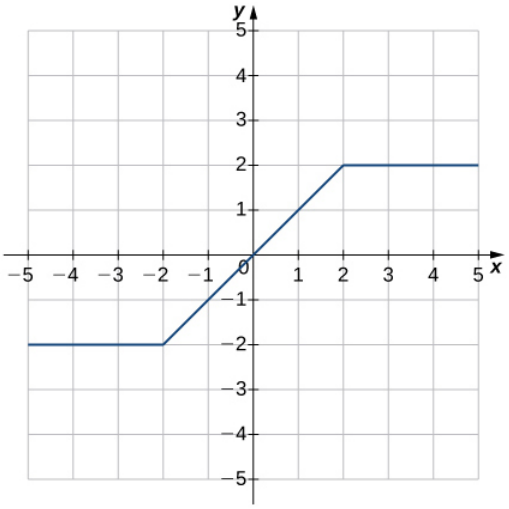

Answer

Function;

a. Domain: $-\infty < x < \infty$, range: $-2 \leq y \leq 2$

b. $x = 0$

c. $y = 0$

d. $-2 < x < 2$

e. Not decreasing

f. $-\infty < x < -2$ and $2 < x < \infty$

g. Origin

h. Odd

34)

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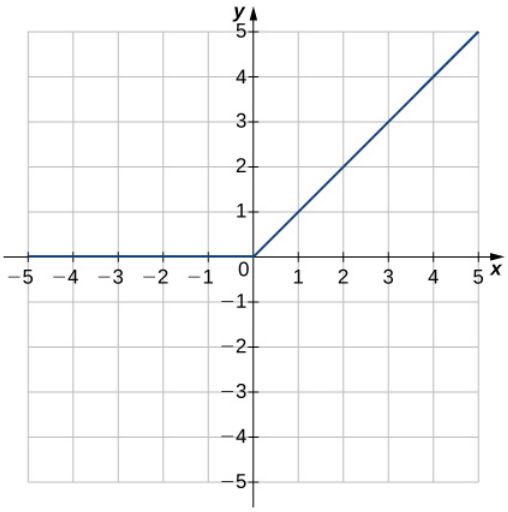
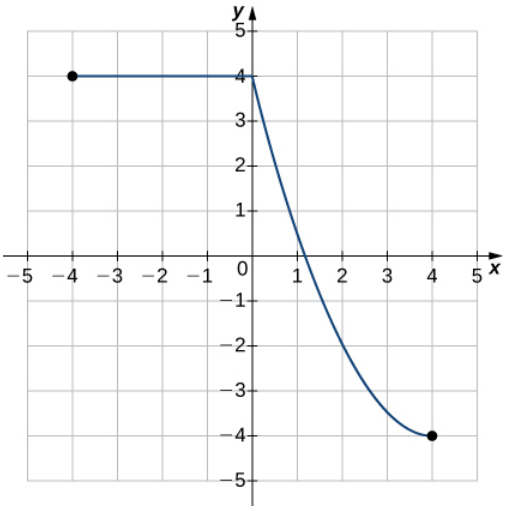


35)



Answer

Function;
a. Domain: $-4 \leq x \leq 4$, range: $-4 \leq y \leq 4$
b. $x = 1.2$
c. $y = 4$
d. Not increasing
e. $0 < x < 4$
f. $-4 < x < 0$
g. No Symmetry
h. Neither

For exercises 36 - 41, for each pair of functions, find a. $f + g$ b. $f - g$ c. $f \cdot g$ d. f/g. Determine the domain of each of these new functions.

36) $f(x) = 3x + 4, \quad g(x) = x - 2$

37) $f(x) = x - 8, \quad g(x) = 5x^2$

Answer

a. $5x^2 + x - 8$; all real numbers
b. $-5x^2 + x - 8$; all real numbers

c. $5x^3 - 40x^2$; all real numbers

d. $\frac{x-8}{5x^2}; \quad x \neq 0$

38) $f(x) = 3x^2 + 4x + 1, \quad g(x) = x + 1$

39) $f(x) = 9 - x^2, \quad g(x) = x^2 - 2x - 3$

Answer

a. $-2x + 6$; all real numbers

b. $-2x^2 + 2x + 12$; all real numbers

c. $-x^4 + 2x^3 + 12x^2 - 18x - 27$; all real numbers

d. $-\frac{x+3}{x+1}; \quad x \neq -1, 3$

40) $f(x) = \sqrt{x}, \quad g(x) = x - 2$

41) $f(x) = 6 + \frac{1}{x}, \quad g(x) = \frac{1}{x}$

Answer

a. $6 + \frac{2}{x}; \quad x \neq 0$

b. $6; \quad x \neq 0$

c. $6x + \frac{1}{x^2}; \quad x \neq 0$

d. $6x + 1; \quad x \neq 0$

For exercises 42 - 48, for each pair of functions, find a. $(f \circ g)(x)$ and b. $(g \circ f)(x)$ Simplify the results. Find the domain of each of the results.

42) $f(x) = 3x, \quad g(x) = x + 5$

43) $f(x) = x + 4, \quad g(x) = 4x - 1$

Answer

a. $4x + 3$; all real numbers

b. $4x + 15$; all real numbers

44) $f(x) = 2x + 4, \quad g(x) = x^2 - 2$

45) $f(x) = x^2 + 7, \quad g(x) = x^2 - 3$

Answer

a. $x^4 - 6x^2 + 16$; all real numbers

b. $x^4 + 14x^2 + 46$; all real numbers

46) $f(x) = \sqrt{x}, \quad g(x) = x + 9$

47) $f(x) = \frac{3}{2x+1}, \quad g(x) = \frac{2}{x}$

Answer

a. $\frac{3x}{4+x}; \quad x \neq 0, -4$

b. $\frac{4x+2}{3}; \quad x \neq -\frac{1}{2}$

48) $f(x) = |x + 1|, \quad g(x) = x^2 + x - 4$

49) The table below lists the NBA championship winners for the years 2001 to 2012.

Year	Winner
2001	LA Lakers
2002	LA Lakers
2003	Sam Antonio Spurs
2004	Detroit Pistons
2005	Sam Antonio Spurs
2006	Miami Heat
2007	Sam Antonio Spurs
2008	Boston Celtics
2009	LA Lakers
2010	LA Lakers
2011	Dallas Mavericks
2012	Miami Heat

a. Consider the relation in which the domain values are the years 2001 to 2012 and the range is the corresponding winner. Is this relation a function? Explain why or why not.

b. Consider the relation where the domain values are the winners and the range is the corresponding years. Is this relation a function? Explain why or why not.

Answer

a. Yes, because there is only one winner for each year.
b. No, because there are three teams that won more than once during the years 2001 to 2012.

50) [T] The area A of a square depends on the length of the side s.

a. Write a function $A(s)$ for the area of a square.

b. Find and interpret $A(6.5)$.

c. Find the exact and the two-significant-digit approximation to the length of the sides of a square with area 56 square units.

51) [T] The volume of a cube depends on the length of the sides s.

a. Write a function $V(s)$ for the area of a square.

b. Find and interpret $V(11.8)$.

Answer

a. $V(s) = s^3$
b. $V(11.8) \approx 1643$; a cube of side length 11.8 each has a volume of approximately 1643 cubic units.

52) **[T]** A rental car company rents cars for a flat fee of $20 and an hourly charge of $10.25. Therefore, the total cost C to rent a car is a function of the hours t the car is rented plus the flat fee.

a. Write the formula for the function that models this situation.

b. Find the total cost to rent a car for 2 days and 7 hours.

c. Determine how long the car was rented if the bill is $432.73.

53) [T] A vehicle has a 20-gal tank and gets 15 mpg. The number of miles N that can be driven depends on the amount of gas x in the tank.

a. Write a formula that models this situation.

b. Determine the number of miles the vehicle can travel on (i) a full tank of gas and (ii) 3/4 of a tank of gas.

c. Determine the domain and range of the function.

d. Determine how many times the driver had to stop for gas if she has driven a total of 578 mi.

Answer

a. $N(x) = 15x$
b. i. $N(20) = 15(20) = 300$; therefore, the vehicle can travel 300 mi on a full tank of gas.
ii. $N(15) = 225$; therefore, the vehicle can travel 225 mi on 3/4 of a tank of gas.
c. Domain: $0 \leq x \leq 20$; range: $[0, 300]$
d. The driver had to stop at least once, given that it takes approximately 39 gal of gas to drive a total of 578 mi.

54) [T] The volume V of a sphere depends on the length of its radius as $V = (4/3)\pi r^3$. Because Earth is not a perfect sphere, we can use the mean radius when measuring from the center to its surface. The mean radius is the average distance from the physical center to the surface, based on a large number of samples. Find the volume of Earth with mean radius 6.371×106 m.

55) [T] A certain bacterium grows in culture in a circular region. The radius of the circle, measured in centimeters, is given by $r(t) = 6 - \dfrac{5}{t^2+1}$, where t is time measured in hours since a circle of a 1-cm radius of the bacterium was put into the culture.

a. Express the area of the bacteria as a function of time.

b. Find the exact and approximate area of the bacterial culture in 3 hours.

c. Express the circumference of the bacteria as a function of time.

d. Find the exact and approximate circumference of the bacteria in 3 hours.

Answer

a. $A(t) = A(r(t)) = \pi \cdot (6 - \frac{5}{t^2+1})^2$
b. Exact: $\frac{121\pi}{4}$; approximately $95\ \text{cm}^2$
c. $C(t) = C(r(t)) = 2\pi(6 - \frac{5}{t^2+1})$
d. Exact: 11π; approximately 35 cm

56) [T] An American tourist visits Paris and must convert U.S. dollars to Euros, which can be done using the function $E(x) = 0.79x$, where x is the number of U.S. dollars and $E(x)$ is the equivalent number of Euros. Since conversion rates fluctuate, when the tourist returns to the United States 2 weeks later, the conversion from Euros to U.S. dollars is $D(x) = 1.245x$, where x is the number of Euros and $D(x)$ is the equivalent number of U.S. dollars.

a. Find the composite function that converts directly from U.S. dollars to U.S. dollars via Euros. Did this tourist lose value in the conversion process?

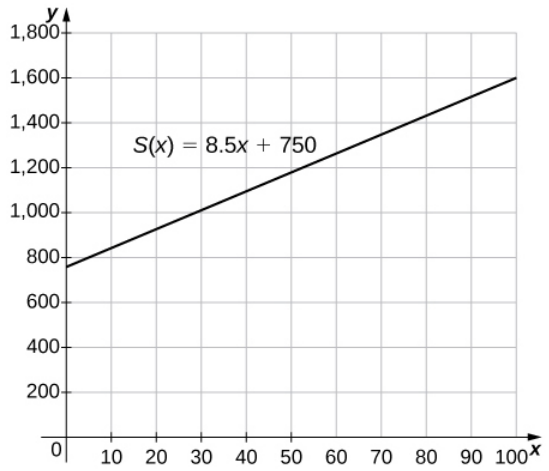
b. Use (a) to determine how many U.S. dollars the tourist would get back at the end of her trip if she converted an extra $200 when she arrived in Paris.

57) [T] The manager at a skateboard shop pays his workers a monthly salary S of $750 plus a commission of $8.50 for each skateboard they sell.

a. Write a function $y = S(x)$ that models a worker's monthly salary based on the number of skateboards x he or she sells.

b. Find the approximate monthly salary when a worker sells 25, 40, or 55 skateboards.

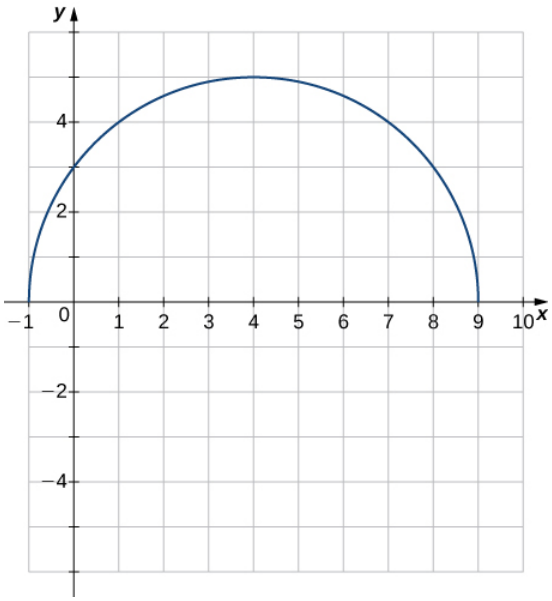
c. Use the INTERSECT feature on a graphing calculator to determine the number of skateboards that must be sold for a worker to earn a monthly income of $1400. (Hint: Find the intersection of the function and the line $y = 1400$.)



Answer

a. $S(x) = 8.5x + 750$ b. \$962.50, \$1090, \$1217.50 c. 77 skateboards

58) [T] Use a graphing calculator to graph the half-circle $y = \sqrt{25 - (x - 4)^2}$. Then, use the INTERCEPT feature to find the value of both the x- and y-intercepts.



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1.2: Basic Classes of Functions

Learning Objectives

- Calculate the slope of a linear function and interpret its meaning.
- Recognize the degree of a polynomial.
- Find the roots of a quadratic polynomial.
- Describe the graphs of basic odd and even polynomial functions.
- Identify a rational function.
- Describe the graphs of power and root functions.
- Explain the difference between algebraic and transcendental functions.
- Graph a piecewise-defined function.
- Sketch the graph of a function that has been shifted, stretched, or reflected from its initial graph position.

We have studied the general characteristics of functions, so now let's examine some specific classes of functions. We begin by reviewing the basic properties of linear and quadratic functions, and then generalize to include higher-degree polynomials. By combining root functions with polynomials, we can define general algebraic functions and distinguish them from the transcendental functions we examine later in this chapter. We finish the section with examples of piecewise-defined functions and take a look at how to sketch the graph of a function that has been shifted, stretched, or reflected from its initial form.

Linear Functions and Slope

The easiest type of function to consider is a **linear function**. Linear functions have the form $f(x) = ax + b$, where a and b are constants. In Figure 1.2.1, we see examples of linear functions when a is positive, negative, and zero. Note that if $a > 0$, the graph of the line rises as x increases. In other words, $f(x) = ax + b$ is increasing on $(-\infty, \infty)$. If $a < 0$, the graph of the line falls as x increases. In this case, $f(x) = ax + b$ is decreasing on $(-\infty, \infty)$. If $a = 0$, the line is horizontal.

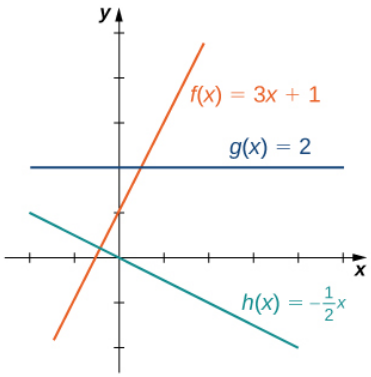


Figure 1.2.1: These linear functions are increasing or decreasing on (∞, ∞) and one function is a horizontal line.

As suggested by Figure 1.2.1, the graph of any linear function is a line. One of the distinguishing features of a line is its slope. The **slope** is the change in y for each unit change in x. The slope measures both the steepness and the direction of a line. If the slope is positive, the line points upward when moving from left to right. If the slope is negative, the line points downward when moving from left to right. If the slope is zero, the line is horizontal. To calculate the slope of a line, we need to determine the ratio of the change in y versus the change in x. To do so, we choose any two points (x_1, y_1) and (x_2, y_2) on the line and calculate $\frac{y_2 - y_1}{x_2 - x_1}$. In Figure 1.2.2, we see this ratio is independent of the points chosen.

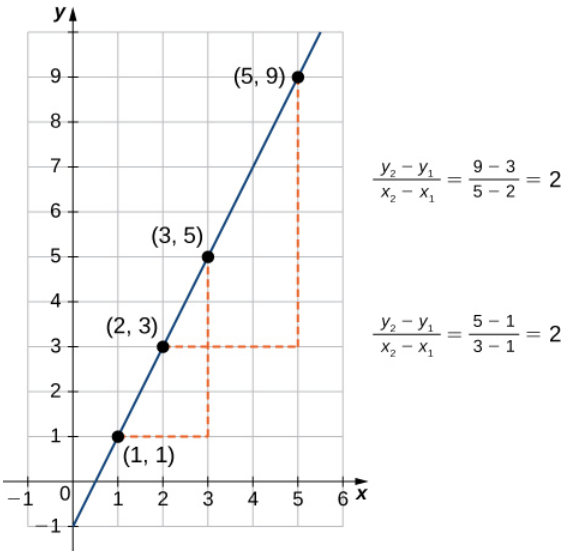


Figure 1.2.2: For any linear function, the slope $(y_2 - y_1)/(x_2 - x_1)$ is independent of the choice of points (x_1, y_1) and (x_2, y_2) on the line.

Definition: Slope of a Linear Function

Consider line L passing through points (x_1, y_1) and (x_2, y_2). Let $\Delta y = y_2 - y_1$ and $\Delta x = x_2 - x_1$ denote the changes in y and x,respectively. The slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

We now examine the relationship between slope and the formula for a linear function. Consider the linear function given by the formula $f(x) = ax + b$. As discussed earlier, we know the graph of a linear function is given by a line. We can use our definition of slope to calculate the slope of this line. As shown, we can determine the slope by calculating $(y_2 - y_1)/(x_2 - x_1)$ for any points (x_1, y_1) and (x_2, y_2) on the line. Evaluating the function f at $x = 0$, we see that $(0, b)$ is a point on this line. Evaluating this function at $x = 1$, we see that $(1, a + b)$ is also a point on this line. Therefore, the slope of this line is

$$\frac{(a + b) - b}{1 - 0} = a.$$

We have shown that the coefficient a is the slope of the line. We can conclude that the formula $f(x) = ax + b$ describes a line with slope a. Furthermore, because this line intersects the y-axis at the point $(0, b)$, we see that the y-intercept for this linear function is $(0, b)$. We conclude that the formula $f(x) = ax + b$ tells us the slope, a, and the y-intercept, $(0, b)$, for this line. Since we often use the symbol m to denote the slope of a line, we can write

$$\underbrace{f(x) = mx + b}_{\text{slope-intercept form}}$$

to denote the **slope-intercept form** of a linear function.

Sometimes it is convenient to express a linear function in different ways. For example, suppose the graph of a linear function passes through the point (x_1, y_1) and the slope of the line is m. Since any other point $(x, f(x))$ on the graph of f must satisfy the equation

$$m = \frac{f(x) - y_1}{x - x_1},$$

this linear function can be expressed by writing

$$\underbrace{f(x) - y_1 = m(x - x_1)}_{\text{point-slope equation}}.$$

We call this equation the **point-slope equation** for that linear function.

Since every nonvertical line is the graph of a linear function, the points on a nonvertical line can be described using the slope-intercept or point-slope equations. However, a vertical line does not represent the graph of a function and cannot be expressed in either of these forms. Instead, a vertical line is described by the equation $x = k$ for some constant k. Since neither the slope-intercept form nor the point-slope form allows for vertical lines, we use the notation

$$\underbrace{ax + by = c}_{\text{standard form}},$$

where a, b are both not zero, to denote the **standard form of a line.**

Definition: Point-Slope Equation, and the Slope-Intercept Form and Standard Form of the Equation of a Line

Consider a line passing through the point (x_1, y_1) with slope m. The equation

$$y - y_1 = m(x - x_1)$$

is the *point-slope equation* for that line.

Consider a line with slope m and y-intercept $(0, b)$. The equation

$$y = mx + b$$

is an equation for that line in *slope-intercept form.*

The *standard form of a line* is given by the equation

$$ax + by = c,$$

where a and b are both not zero. This form is more general because it allows for a vertical line, $x = k$.

Example 1.2.1: Finding the Slope and Equations of Lines

Consider the line passing through the points $(11, -4)$ and $(-4, 5)$, as shown in Figure 1.2.3.

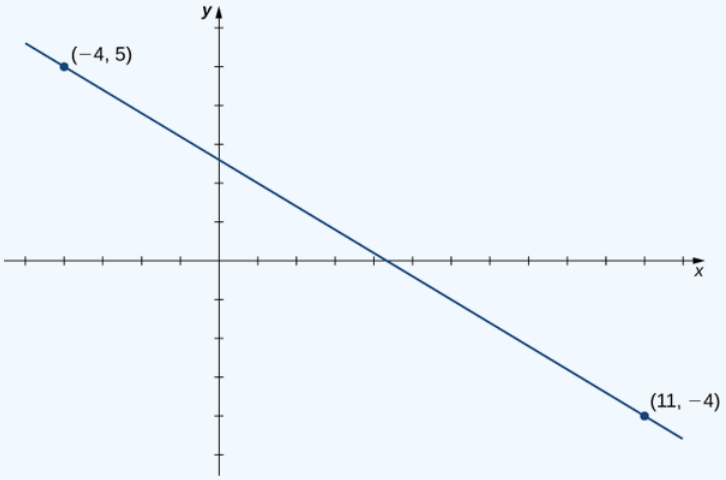


Figure 1.2.3: Finding the equation of a linear function with a graph that is a line between two given points.

1. Find the slope of the line.
2. Find an equation for this linear function in point-slope form.
3. Find an equation for this linear function in slope-intercept form.

Solution

1. The slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - (-4)}{-4 - 11} = -\frac{9}{15} = -\frac{3}{5}.$$

2. To find an equation for the linear function in point-slope form, use the slope $m = -3/5$ and choose any point on the line. If we choose the point $(11, -4)$, we get the equation

$$f(x) + 4 = -\frac{3}{5}(x - 11).$$

3. To find an equation for the linear function in slope-intercept form, solve the equation in part b. for $f(x)$. When we do this, we get the equation

$$f(x) = -\frac{3}{5}x + \frac{13}{5}.$$

Exercise 1.2.1

Consider the line passing through points $(-3, 2)$ and $(1, 4)$.

a. Find the slope of the line.
b. Find an equation of that line in point-slope form.
c. Find an equation of that line in slope-intercept form.

Hint

The slope $m = \Delta y/\Delta x$.

Answer a

$m = 1/2$.

Answer b

The point-slope form is $y - 4 = \frac{1}{2}(x - 1)$.

Answer c

The slope-intercept form is $y = \frac{1}{2}x + \frac{7}{2}$.

Example 1.2.2:

Jessica leaves her house at 5:50 a.m. and goes for a 9-mile run. She returns to her house at 7:08 a.m. Answer the following questions, assuming Jessica runs at a constant pace.

a. Describe the distance D (in miles) Jessica runs as a linear function of her run time t (in minutes).
b. Sketch a graph of D.
c. Interpret the meaning of the slope.

Solution

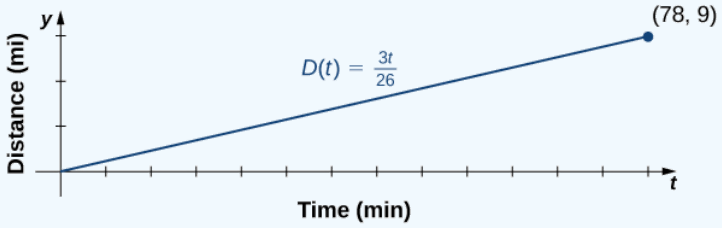
a. At time $t = 0$, Jessica is at her house, so $D(0) = 0$. At time $t = 78$ minutes, Jessica has finished running 9 mi, so $D(78) = 9$. The slope of the linear function is

$$m = \frac{9 - 0}{78 - 0} = \frac{3}{26}.$$

The y-intercept is $(0, 0)$, so the equation for this linear function is

$$D(t) = \frac{3}{26}t.$$

b. To graph D, use the fact that the graph passes through the origin and has slope $m = 3/26$.



c. The slope $m = 3/26 \approx 0.115$ describes the distance (in miles) Jessica runs per minute, or her average velocity.

Polynomials

A linear function is a special type of a more general class of functions: polynomials. A polynomial function is any function that can be written in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$$

for some integer $n \geq 0$ and constants $a_n, a_{n-1}, ..., a_0$, where $a_n \neq 0$. In the case when $n = 0$, we allow for $a_0 = 0$; if $a_0 = 0$, the function $f(x) = 0$ is called the **zero function**. The value n is called the **degree** of the polynomial; the constant a_n is called the **leading coefficient**. A linear function of the form $f(x) = mx + b$ is a polynomial of degree 1 if $m \neq 0$ and degree 0 if $m = 0$. A polynomial of degree 0 is also called a **constant function**. A polynomial function of degree 2 is called a **quadratic function**. In particular, a quadratic function has the form

$$f(x) = ax^2 + bx + c,$$

where $a \neq 0$. A polynomial function of degree 3 is called a **cubic function**.

Power Functions

Some polynomial functions are power functions. A **power function** is any function of the form $f(x) = ax^b$, where a and b are any real numbers. The exponent in a power function can be any real number, but here we consider the case when the exponent is a positive integer. (We consider other cases later.) If the exponent is a positive integer, then $f(x) = ax^n$ is a polynomial. If n is even, then $f(x) = ax^n$ is an even function because $f(-x) = a(-x)^n = ax^n$ if n is even. If n is odd, then $f(x) = ax^n$ is an odd function because $f(-x) = a(-x)^n = -ax^n$ if n is odd (Figure 1.2.4).

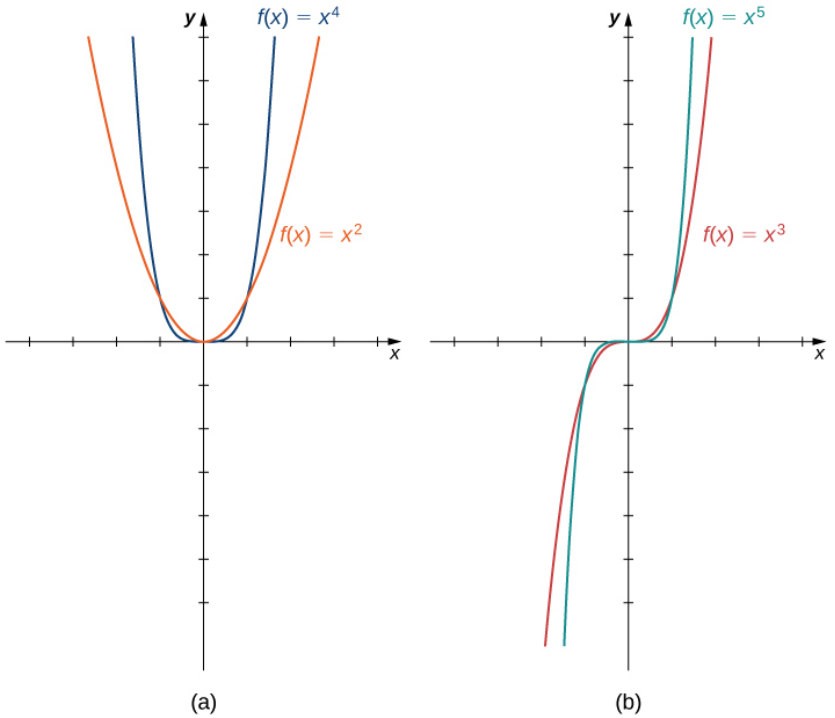


Figure 1.2.4: (a) For any even integer n, $f(x) = ax^n$ is an even function. (b) For any odd integer n, $f(x) = ax^n$ is an odd function.

Behavior at Infinity

To determine the behavior of a function f as the inputs approach infinity, we look at the values $f(x)$ as the inputs, x, become larger. For some functions, the values of $f(x)$ approach a finite number. For example, for the function $f(x) = 2 + 1/x$, the values $1/x$ become closer and closer to zero for all values of x as they get larger and larger. For this function, we say "$f(x)$ approaches two as x goes to infinity," and we write $f(x) \to 2$ as $x \to \infty$. The line $y = 2$ is a horizontal asymptote for the function $f(x) = 2 + 1/x$ because the graph of the function gets closer to the line as x gets larger.

For other functions, the values $f(x)$ may not approach a finite number but instead may become larger for all values of x as they get larger. In that case, we say "$f(x)$ approaches infinity as x approaches infinity," and we write $f(x) \to \infty$ as $x \to \infty$. For example, for the function $f(x) = 3x^2$, the outputs $f(x)$ become larger as the inputs x get larger. We can conclude that the function $f(x) = 3x^2$ approaches infinity as x approaches infinity, and we write $3x^2 \to \infty$ as $x \to \infty$. The behavior as $x \to -\infty$ and the meaning of $f(x) \to -\infty$ as $x \to \infty$ or $x \to -\infty$ can be defined similarly. We can describe what happens to the values of $f(x)$ as $x \to \infty$ and as $x \to -\infty$ as the **end behavior** of the function.

To understand the end behavior for polynomial functions, we can focus on quadratic and cubic functions. The behavior for higher-degree polynomials can be analyzed similarly. Consider a quadratic function $f(x) = ax^2 + bx + c$. If $a > 0$, the values $f(x) \to \infty$ as $x \to \pm\infty$. If $a < 0$, the values $f(x) \to -\infty$ as $x \to \pm\infty$. Since the graph of a quadratic function is a parabola, the parabola opens upward if $a > 0$.; the parabola opens downward if $a < 0$ (Figure 1.2.5a).

Now consider a cubic function $f(x) = ax^3 + bx^2 + cx + d$. If $a > 0$, then $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$. If $a < 0$, then $f(x) \to -\infty$ as $x \to \infty$ and $f(x) \to \infty$ as $x \to -\infty$. As we can see from both of these graphs, the leading term of the polynomial determines the end *behavior* (Figure 1.2.5b).

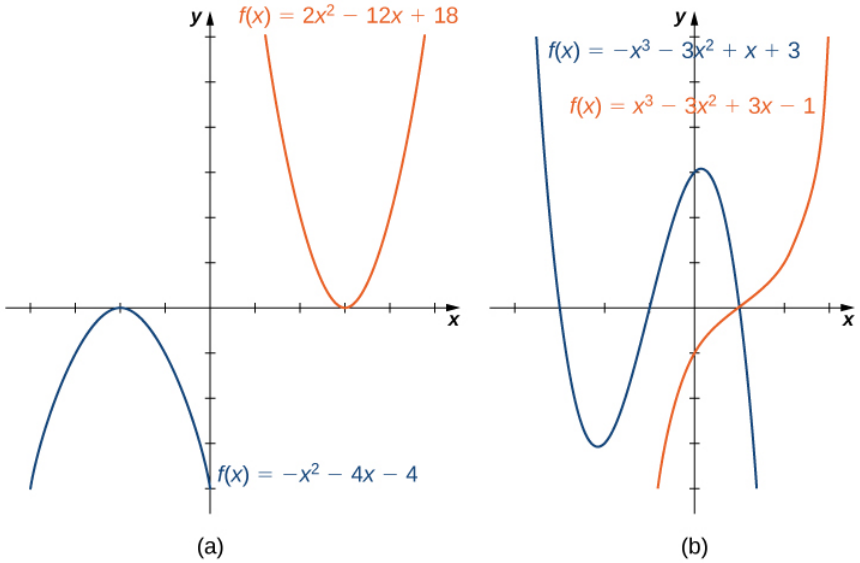


Figure 1.2.5: (a) For a quadratic function, if the leading coefficient $a > 0$,the parabola opens upward. If $a < 0$, the parabola opens downward. (b) For a cubic function f, if the leading coefficient $a > 0$, the values $f(x) \to \infty$ as $x \to \infty$ and the values $f(x) \to -\infty$ as $x \to -\infty$. If the leading coefficient $a < 0$, the opposite is true.

Zeros of Polynomial Functions

Another characteristic of the graph of a polynomial function is where it intersects the x-axis. To determine where a function f intersects the x-axis, we need to solve the equation $f(x) = 0$ for x. In the case of the linear function $f(x) = mx + b$, the x-intercept is given by solving the equation $mx + b = 0$. In this case, we see that the x-intercept is given by $(-b/m, 0)$. In the case of a quadratic function, finding the x-intercept(s) requires finding the zeros of a quadratic equation: $ax^2 + bx + c = 0$. In some cases, it is easy to factor the polynomial $ax^2 + bx + c$ to find the zeros. If not, we make use of the quadratic formula.

The Quadratic Formula

Consider the quadratic equation

$$ax^2 + bx + c = 0,$$

where $a \neq 0$. The solutions of this equation are given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{1.2.1}$$

If the discriminant $b^2 - 4ac > 0$, Equation 1.2.1 tells us there are two real numbers that satisfy the quadratic equation. If $b^2 - 4ac = 0$, this formula tells us there is only one solution, and it is a real number. If $b^2 - 4ac < 0$, no real numbers satisfy the quadratic equation.

In the case of higher-degree polynomials, it may be more complicated to determine where the graph intersects the x-axis. In some instances, it is possible to find the x-intercepts by factoring the polynomial to find its zeros. In other cases, it is impossible to calculate the exact values of the x-intercepts. However, as we see later in the text, in cases such as this, we can use analytical tools to approximate (to a very high degree) where the x-intercepts are located. Here we focus on the graphs of polynomials for which we can calculate their zeros explicitly.

Example 1.2.3: Graphing Polynomial Functions

For the following functions,

a. $f(x) = -2x^2 + 4x - 1$
b. $f(x) = x^3 - 3x^2 - 4x$

i. describe the behavior of $f(x)$ as $x \to \pm\infty$,
ii. find all zeros of f, and
iii. sketch a graph of f.

Solution

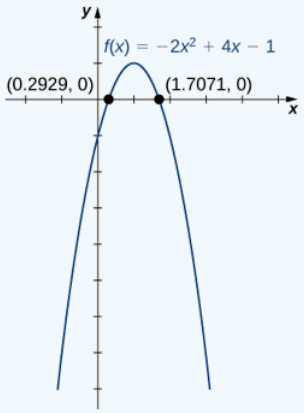
1. The function $f(x) = -2x^2 + 4x - 1$ is a quadratic function.

    1. Because $a = -2 < 0$, as $x \to \pm\infty, f(x) \to -\infty$.

    2. To find the zeros of f, use the quadratic formula. The zeros are

$$x = \frac{-4 \pm \sqrt{4^2 - 4(-2)(-1)}}{2(-2)} = \frac{-4 \pm \sqrt{8}}{-4} = \frac{-4 \pm 2\sqrt{2}}{-4} = \frac{2 \pm \sqrt{2}}{2}.$$

    3. To sketch the graph of f,use the information from your previous answers and combine it with the fact that the graph is a parabola opening downward.



2. The function $f(x) = x^3 - 3x^2 - 4x$ is a cubic function.

    1. Because $a = 1 > 0$, as $x \to \infty$, $f(x) \to \infty$. As $x \to -\infty$, $f(x) \to -\infty$.

    2. To find the zeros of f, we need to factor the polynomial. First, when we factor x out of all the terms, we find

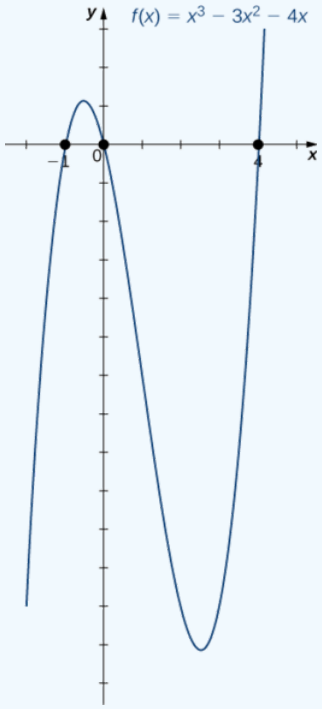
$$f(x) = x(x^2 - 3x - 4).$$

    Then, when we factor the quadratic function $x^2 - 3x - 4$, we find

$$f(x) = x(x - 4)(x + 1).$$

    Therefore, the zeros of f are $x = 0, 4, -1$.

3. Combining the results from parts i. and ii., draw a rough sketch of f.



? Exercise 1.2.2

Consider the quadratic function $f(x) = 3x^2 - 6x + 2$. Find the zeros of f. Does the parabola open upward or downward?

Hint

Use the quadratic formula.

Answer

The zeros are $x = 1 \pm \sqrt{3}/3$. The parabola opens upward.

Mathematical Models

A large variety of real-world situations can be described using **mathematical models**. A mathematical model is a method of simulating real-life situations with mathematical equations. Physicists, engineers, economists, and other researchers develop models by combining observation with quantitative data to develop equations, functions, graphs, and other mathematical tools to describe the behavior of various systems accurately. Models are useful because they help predict future outcomes. Examples of mathematical models include the study of population dynamics, investigations of weather patterns, and predictions of product sales.

As an example, let's consider a mathematical model that a company could use to describe its revenue for the sale of a particular item. The amount of revenue R a company receives for the sale of n items sold at a price of p dollars per item is described by the equation $R = p \cdot n$. The company is interested in how the sales change as the price of the item changes. Suppose the data in Table 1.2.1 show the number of units a company sells as a function of the price per item.

Table 1.2.1: Number of Units Sold n (in Thousands) as a Function of Price per Unit p (in Dollars)

p	6	8	10	12	14
n	19.4	18.5	16.2	13.8	12.2

In Figure 1.2.6, we see the graph the number of units sold (in thousands) as a function of price (in dollars). We note from the shape of the graph that the number of units sold is likely a linear function of price per item, and the data can be closely approximated by the linear function $n = -1.04p + 26$ for $0 \le p \le 25$, where n predicts the number of units sold in thousands. Using this linear function, the revenue (in thousands of dollars) can be estimated by the quadratic function

$$R(p) = p \cdot (-1.04p + 26) = -1.04p^2 + 26p \text{ for } 0 \leq p \leq 25.$$

In Example 1.2.4, we use this quadratic function to predict the amount of revenue the company receives depending on the price the company charges per item. Note that we cannot conclude definitively the actual number of units sold for values of p, for which no data are collected. However, given the other data values and the graph shown, it seems reasonable that the number of units sold (in thousands) if the price charged is p dollars may be close to the values predicted by the linear function $n = -1.04p + 26$.

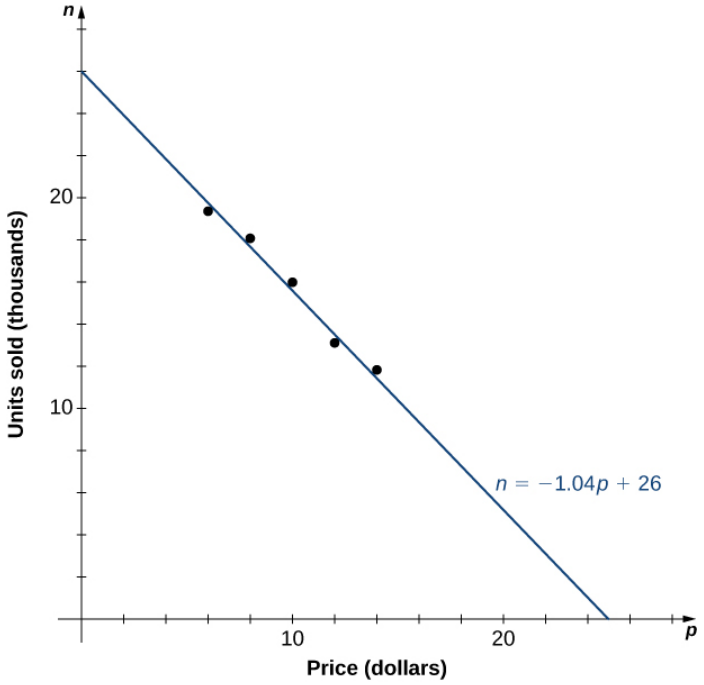


Figure 1.2.6: The data collected for the number of items sold as a function of price is roughly linear. We use the linear function $n = -1.04p + 26$ to estimate this function.

Example 1.2.4: Maximizing Revenue

A company is interested in predicting the amount of revenue it will receive depending on the price it charges for a particular item. Using the data from Table 1.2.1, the company arrives at the following quadratic function to model revenue R as a function of price per item p:

$$R(p) = p \cdot (-1.04p + 26) = -1.04p^2 + 26p$$

for $0 \leq p \leq 25$.

a. Predict the revenue if the company sells the item at a price of $p = \$5$ and $p = \$17$.
b. Find the zeros of this function and interpret the meaning of the zeros.
c. Sketch a graph of R.
d. Use the graph to determine the value of p that maximizes revenue. Find the maximum revenue.

Solution

a. Evaluating the revenue function at $p = 5$ and $p = 17$, we can conclude that

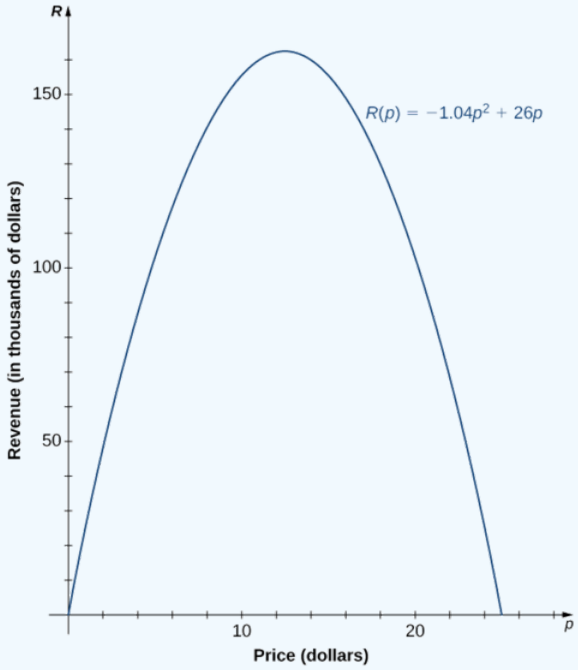
$R(5) = -1.04(5)^2 + 26(5) = 104$, so revenue $= \$104,000$;

$R(17) = -1.04(17)^2 + 26(17) = 141.44$, so revenue $= \$141,440$.

b. The zeros of this function can be found by solving the equation $-1.04p^2 + 26p = 0$. When we factor the quadratic expression, we get $p(-1.04p + 26) = 0$. The solutions to this equation are given by $p = 0, 25$. For these values of p, the revenue is zero. When $p = \$0$, the revenue is zero because the company is giving away its merchandise for free. When $p = \$25$,the revenue is zero because the price is too high, and no one will buy any items.

c. Knowing the fact that the function is quadratic, we also know the graph is a parabola. Since the leading coefficient is negative, the parabola opens downward. One property of parabolas is that they are symmetric about the axis of

symmetry, so since the zeros are at $p = 0$ and $p = 25$, the parabola must be symmetric about the line halfway between them, or $p = 12.5$.



d. The function is a parabola with zeros at $p = 0$ and $p = 25$, and it is symmetric about the line $p = 12.5$, so the maximum revenue occurs at a price of $p = \$12.50$ per item. At that price, the revenue is $R(p) = -1.04(12.5)^2 + 26(12.5) = \$162,500$.

Algebraic Functions

By allowing for quotients and fractional powers in polynomial functions, we create a larger class of functions. An **algebraic function** is one that involves addition, subtraction, multiplication, division, rational powers, and roots. Two types of algebraic functions are rational functions and root functions.

Just as rational numbers are quotients of integers, rational functions are quotients of polynomials. In particular, a **rational function** is any function of the form $f(x) = p(x)/q(x)$,where $p(x)$ and $q(x)$ are polynomials. For example,

$$f(x) = \frac{3x-1}{5x+2} \text{ and } g(x) = \frac{4}{x^2+1}$$

are rational functions. A **root function** is a power function of the form $f(x) = x^{1/n}$, where n is a positive integer greater than one. For example, $f(x) = x^{1/2} = \sqrt{x}$ is the square-root function and $g(x) = x^{1/3} = \sqrt[3]{x}$ is the cube-root function. By allowing for compositions of root functions and rational functions, we can create other algebraic functions. For example, $f(x) = \sqrt{4-x^2}$ is an algebraic function.

Example 1.2.5: Finding Domain and Range for Algebraic Functions

For each of the following functions, find the domain and range.

a. $f(x) = \dfrac{3x-1}{5x+2}$
b. $f(x) = \sqrt{4-x^2}$

Solution

1. It is not possible to divide by zero, so the domain is the set of real numbers x such that $x \neq -2/5$. To find the range, we need to find the values y for which there exists a real number x such that

$$y = \frac{3x-1}{5x+2}$$

When we multiply both sides of this equation by $5x + 2$, we see that x must satisfy the equation

$$5xy + 2y = 3x - 1.$$

From this equation, we can see that x must satisfy

$$2y + 1 = x(3 - 5y).$$

If y=$3/5$, this equation has no solution. On the other hand, as long as $y \neq 3/5$,

$$x = \frac{2y + 1}{3 - 5y}$$

satisfies this equation. We can conclude that the range of f is $\{y \mid y \neq 3/5\}$.

2. To find the domain of f, we need $4 - x^2 \geq 0$. When we factor, we write $4 - x^2 = (2 - x)(2 + x) \geq 0$. This inequality holds if and only if both terms are positive or both terms are negative. For both terms to be positive, we need to find x such that

$$2 - x \geq 0 \text{ and } 2 + x \geq 0.$$

These two inequalities reduce to $2 \geq x$ and $x \geq -2$. Therefore, the set $\{x \mid -2 \leq x \leq 2\}$ must be part of the domain. For both terms to be negative, we need

$$2 - x \leq 0 \text{ and } 2 + x \leq 0.$$

These two inequalities also reduce to $2 \leq x$ and $x \leq -2$. There are no values of x that satisfy both of these inequalities. Thus, we can conclude the domain of this function is $\{x \mid -2 \leq x \leq 2\}$.

If $-2 \leq x \leq 2$, then $0 \leq 4 - x^2 \leq 4$. Therefore, $0 \leq \sqrt{4 - x2} \leq 2$, and the range of f is $\{y \mid 0 \leq y \leq 2\}$.

? Exercise 1.2.3

Find the domain and range for the function $f(x) = (5x + 2)/(2x - 1)$.

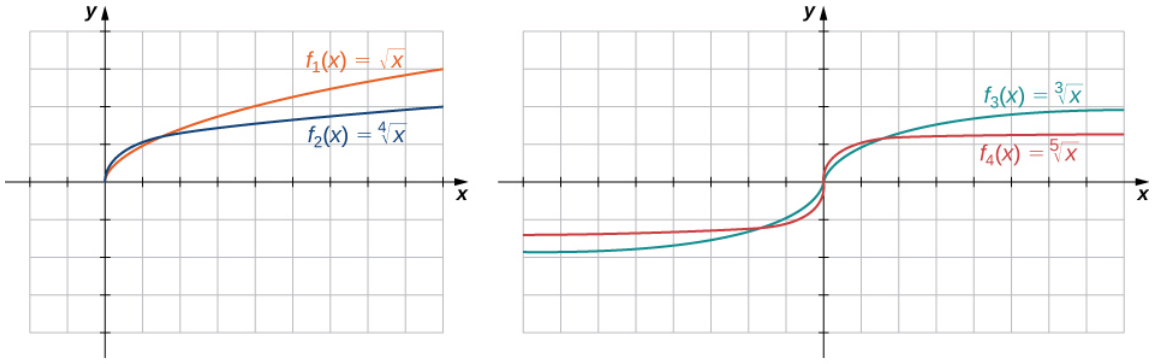
Hint

The denominator cannot be zero. Solve the equation $y = (5x + 2)/(2x - 1)$ for x to find the range.

Answer

The domain is the set of real numbers x such that $x \neq 1/2$. The range is the set $\{y \mid y \neq 5/2\}$.

The root functions $f(x) = x^{1/n}$ have defining characteristics depending on whether n is odd or even. For all even integers $n \geq 2$, the domain of $f(x) = x^{1/n}$ is the interval $[0, \infty)$. For all odd integers $n \geq 1$, the domain of $f(x) = x^{1/n}$ is the set of all real numbers. Since $x^{1/n} = (-x)^{1/n}$ for odd integers n,$f(x) = x^{1/n}$ is an odd function if$n$ is odd. See the graphs of root functions for different values of n in Figure 1.2.7.



(a) (b)

Figure 1.2.7: (a) If n is even, the domain of $f(x) = \sqrt[n]{x}$ is $[0, \infty)$. (b) If n is odd, the domain of $f(x) = \sqrt[n]{x}$ is $(-\infty, \infty)$ and the function $f(x) = \sqrt[n]{x}$ is an odd function.

Example 1.2.6: Finding Domains for Algebraic Functions

For each of the following functions, determine the domain of the function.

a. $f(x) = \dfrac{3}{x^2 - 1}$
b. $f(x) = \dfrac{2x + 5}{3x^2 + 4}$
c. $f(x) = \sqrt{4 - 3x}$
d. $f(x) = \sqrt[3]{2x - 1}$

Solution

a. You cannot divide by zero, so the domain is the set of values x such that $x^2 - 1 \neq 0$. Therefore, the domain is $\{x \mid x \neq \pm 1\}$.
b. You need to determine the values of x for which the denominator is zero. Since $3x^2 + 4 \geq 4$ for all real numbers x, the denominator is never zero. Therefore, the domain is $(-\infty, \infty)$.
c. Since the square root of a negative number is not a real number, the domain is the set of values x for which $4 - 3x \geq 0$. Therefore, the domain is $\{x \mid x \leq 4/3\}$.
d. The cube root is defined for all real numbers, so the domain is the interval $(-\infty, \infty)$.

Exercise 1.2.4

Find the domain for each of the following functions: $f(x) = (5 - 2x)/(x^2 + 2)$ and $g(x) = \sqrt{5x - 1}$.

Hint

Determine the values of x when the expression in the denominator of f is nonzero, and find the values of x when the expression inside the radical of g is nonnegative.

Answer

The domain of f is $(-\infty, \infty)$. The domain of g is $\{x \mid x \geq 1/5\}$.

Transcendental Functions

Thus far, we have discussed algebraic functions. Some functions, however, cannot be described by basic algebraic operations. These functions are known as **transcendental functions** because they are said to "transcend," or go beyond, algebra. The most common transcendental functions are trigonometric, exponential, and logarithmic functions. A trigonometric function relates the ratios of two sides of a right triangle. They are $\sin x, \cos x, \tan x, \cot x, \sec x,$ and $\csc x$. (We discuss trigonometric functions later in the chapter.) An exponential function is a function of the form $f(x) = b^x$, where the base $b > 0,\ b \neq 1$. A logarithmic function is a function of the form $f(x) = \log_b(x)$ for some constant $b > 0,\ b \neq 1$, where $\log_b(x) = y$ if and only if $b^y = x$. (We also discuss exponential and logarithmic functions later in the chapter.)

Example 1.2.7: Classifying Algebraic and Transcendental Functions

Classify each of the following functions, a. through c., as algebraic or transcendental.

a. $f(x) = \dfrac{\sqrt{x^3 + 1}}{4x + 2}$
b. $f(x) = 2^{x^2}$
c. $f(x) = \sin(2x)$

Solution

a. Since this function involves basic algebraic operations only, it is an algebraic function.
b. This function cannot be written as a formula that involves only basic algebraic operations, so it is transcendental. (Note that algebraic functions can only have powers that are rational numbers.)
c. As in part b, this function cannot be written using a formula involving basic algebraic operations only; therefore, this function is transcendental.

Exercise 1.2.5:

Is $f(x) = x/2$ an algebraic or a transcendental function?

Answer

Algebraic

Piecewise-Defined Functions

Sometimes a function is defined by different formulas on different parts of its domain. A function with this property is known as a piecewise-defined function. The absolute value function is an example of a piecewise-defined function because the formula changes with the sign of x:

$$f(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}.$$

Other piecewise-defined functions may be represented by completely different formulas, depending on the part of the domain in which a point falls. To graph a piecewise-defined function, we graph each part of the function in its respective domain, on the same coordinate system. If the formula for a function is different for $x < a$ and $x > a$, we need to pay special attention to what happens at $x = a$ when we graph the function. Sometimes the graph needs to include an open or closed circle to indicate the value of the function at $x = a$. We examine this in the next example.

Example 1.2.8: Graphing a Piecewise-Defined Function

Sketch a graph of the following piecewise-defined function:

$$f(x) = \begin{cases} x + 3, & \text{if } x < 1 \\ (x-2)^2, & \text{if } x \geq 1 \end{cases}$$

Solution

Graph the linear function $y = x + 3$ on the interval $(-\infty, 1)$ and graph the quadratic function $y = (x-2)^2$ on the interval $[1, \infty)$. Since the value of the function at $x = 1$ is given by the formula $f(x) = (x-2)^2$, we see that $f(1) = 1$. To indicate this on the graph, we draw a closed circle at the point $(1, 1)$. The value of the function is given by $f(x) = x + 3$ for all $x < 1$, but not at $x = 1$. To indicate this on the graph, we draw an open circle at $(1, 4)$.

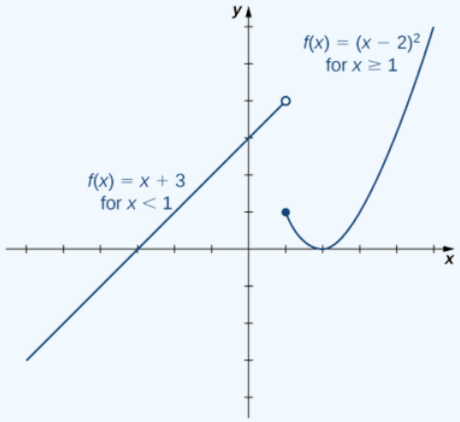
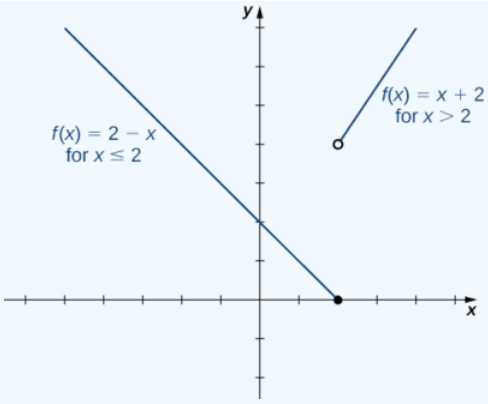


Figure 1.2.8: This piecewise-defined function is linear for $x < 1$ and quadratic for $x \geq 1$.

2) Sketch a graph of the function

$$f(x) = \begin{cases} 2 - x, & \text{if } x \leq 2 \\ x + 2, & \text{if } x > 2 \end{cases}.$$

Solution:



Example 1.2.9: Parking Fees Described by a Piecewise-Defined Function

In a big city, drivers are charged variable rates for parking in a parking garage. They are charged \$10 for the first hour or any part of the first hour and an additional \$2 for each hour or part thereof up to a maximum of \$30 for the day. The parking garage is open from 6 a.m. to 12 midnight.

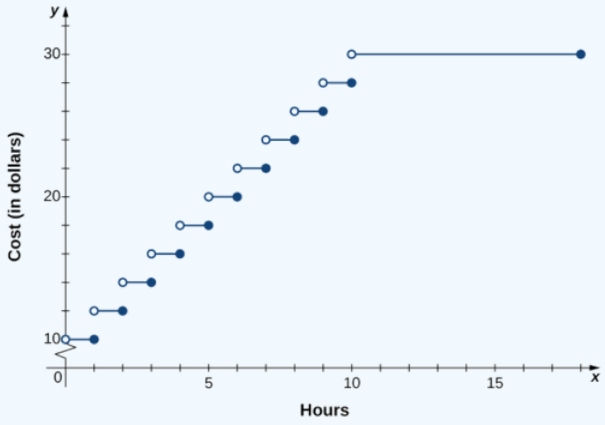
a. Write a piecewise-defined function that describes the cost C to park in the parking garage as a function of hours parked x.
b. Sketch a graph of this function $C(x)$.

Solution

1.Since the parking garage is open 18 hours each day, the domain for this function is $\{x \mid 0 < x \leq 18\}$. The cost to park a car at this parking garage can be described piecewise by the function

$$C(x) = \begin{cases} 10, & \text{for } 0 < x \leq 1 \\ 12, & \text{for } 1 < x \leq 2 \\ 14, & \text{for } 2 < x \leq 3 \\ 16, & \text{for } 3 < x \leq 4 \\ \vdots & \\ 30, & \text{for } 10 < x \leq 18 \end{cases}.$$

2.The graph of the function consists of several horizontal line segments.



Exercise 1.2.6

The cost of mailing a letter is a function of the weight of the letter. Suppose the cost of mailing a letter is $49¢$ for the first ounce and $21¢$ for each additional ounce. Write a piecewise-defined function describing the cost C as a function of the weight x for $0 < x \leq 3$, where C is measured in cents and x is measured in ounces.

Hint

The piecewise-defined function is constant on the intervals $(0, 1]$, $(1, 2]$, ...

Answer

$$C(x) = \begin{cases} 49, 0 < x \leq 1 \\ 70, 1 < x \leq 2 \\ 91, 2 < x \leq 3 \end{cases}$$

Transformations of Functions

We have seen several cases in which we have added, subtracted, or multiplied constants to form variations of simple functions. In the previous example, for instance, we subtracted 2 from the argument of the function $y = x^2$ to get the function $f(x) = (x - 2)^2$. This subtraction represents a shift of the function $y = x^2$ two units to the right. A shift, horizontally or vertically, is a type of **transformation of a function**. Other transformations include horizontal and vertical scalings, and reflections about the axes.

A vertical shift of a function occurs if we add or subtract the same constant to each output y. For $c > 0$, the graph of $f(x) + c$ is a shift of the graph of $f(x)$ up c units, whereas the graph of $f(x) - c$ is a shift of the graph of $f(x)$ down c units. For example, the graph of the function $f(x) = x^3 + 4$ is the graph of $y = x^3$ shifted up 4 units; the graph of the function $f(x) = x^3 - 4$ is the graph of $y = x^3$ shifted down 4 units (Figure 1.2.9).

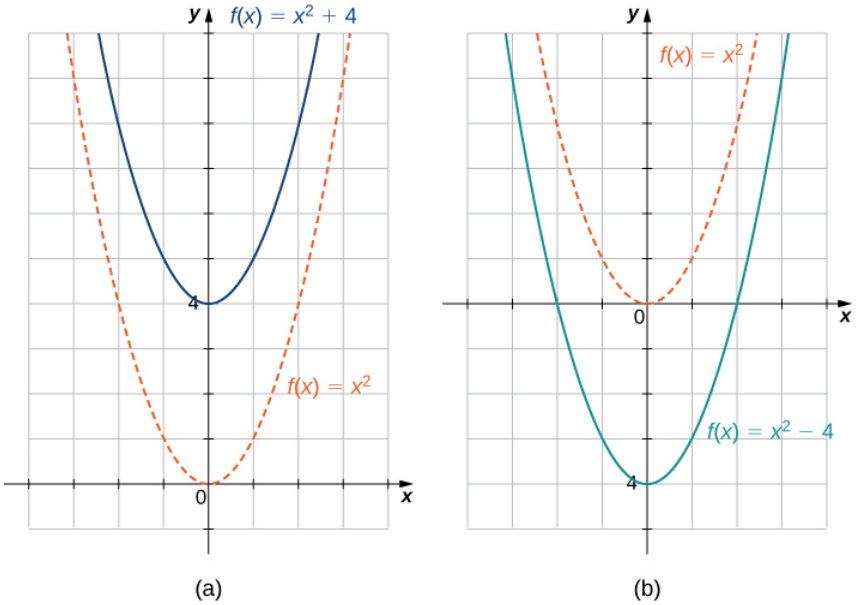


Figure 1.2.9: (a) For $c > 0$, the graph of $y = f(x) + c$ is a vertical shift up c units of the graph of $y = f(x)$. (b) For $c > 0$, the graph of $y = f(x) - c$ is a vertical shift down c units of the graph of $y = f(x)$.

A horizontal shift of a function occurs if we add or subtract the same constant to each input x. For $c > 0$, the graph of $f(x + c)$ is a shift of the graph of $f(x)$ to the left c units; the graph of $f(x - c)$ is a shift of the graph of $f(x)$ to the right c units. Why does the graph shift left when adding a constant and shift right when subtracting a constant? To answer this question, let's look at an example.

Consider the function $f(x) = |x + 3|$ and evaluate this function at $x - 3$. Since $f(x - 3) = |x|$ and $x - 3 < x$, the graph of $f(x) = |x + 3|$ is the graph of $y = |x|$ shifted left 3 units. Similarly, the graph of $f(x) = |x - 3|$ is the graph of $y = |x|$ shifted right 3 units (Figure 1.2.10).

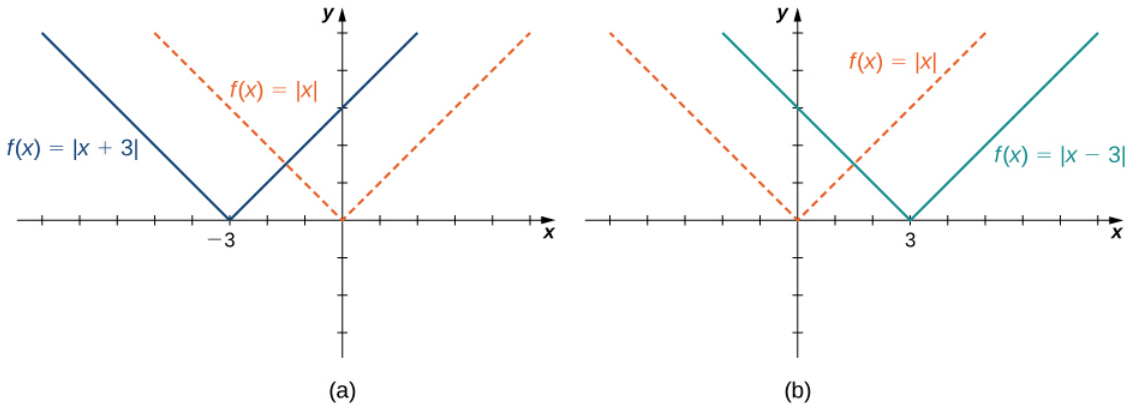


Figure 1.2.10: (a) For $c > 0$, the graph of $y = f(x + c)$ is a horizontal shift left c units of the graph of $y = f(x)$. (b) For $c > 0$, the graph of $y = f(x - c)$ is a horizontal shift right c units of the graph of $y = f(x)$.

A vertical scaling of a graph occurs if we multiply all outputs y of a function by the same positive constant. For $c > 0$, the graph of the function $cf(x)$ is the graph of $f(x)$ scaled vertically by a factor of c. If $c > 1$, the values of the outputs for the function $cf(x)$ are larger than the values of the outputs for the function $f(x)$; therefore, the graph has been stretched vertically. If $0 < c < 1$, then the outputs of the function $cf(x)$ are smaller, so the graph has been compressed. For example, the graph of the function $f(x) = 3x^2$ is the graph of $y = x^2$ stretched vertically by a factor of 3, whereas the graph of $f(x) = x^2/3$ is the graph of $y = x^2$ compressed vertically by a factor of 3 (Figure 1.2.11b).

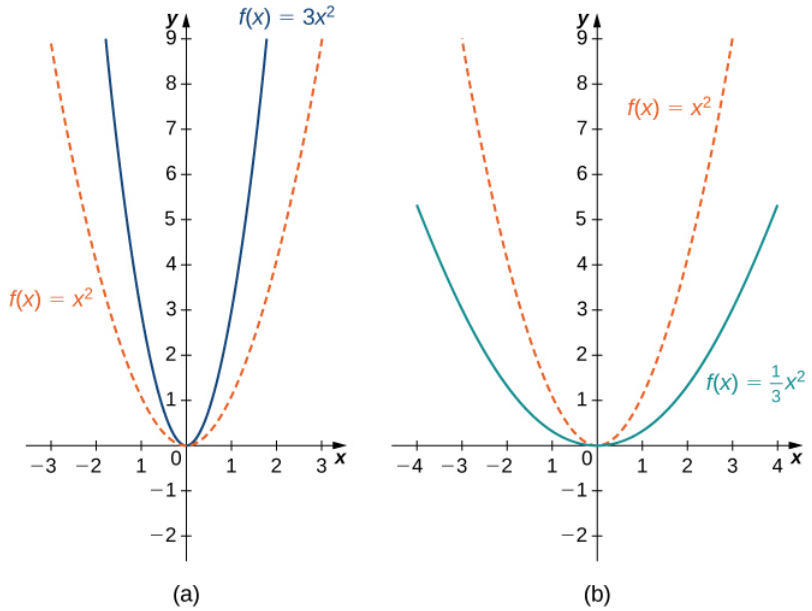


Figure 1.2.11: (a) If $c > 1$, the graph of $y = cf(x)$ is a vertical stretch of the graph of $y = f(x)$. (b) If $0 < c < 1$, the graph of $y = cf(x)$ is a vertical compression of the graph of $y = f(x)$.

The horizontal scaling of a function occurs if we multiply the inputs x by the same positive constant. For $c > 0$, the graph of the function $f(cx)$ is the graph of $f(x)$ scaled horizontally by a factor of c. If $c > 1$, the graph of $f(cx)$ is the graph of $f(x)$ compressed horizontally. If $0 < c < 1$, the graph of $f(cx)$ is the graph of $f(x)$ stretched horizontally. For example, consider the function $f(x) = \sqrt{2x}$ and evaluate f at $x/2$. Since $f(x/2) = \sqrt{x}$, the graph of $f(x) = \sqrt{2x}$ is the graph of $y = \sqrt{x}$ compressed horizontally. The graph of $y = \sqrt{x/2}$ is a horizontal stretch of the graph of $y = \sqrt{x}$ (Figure 1.2.12).

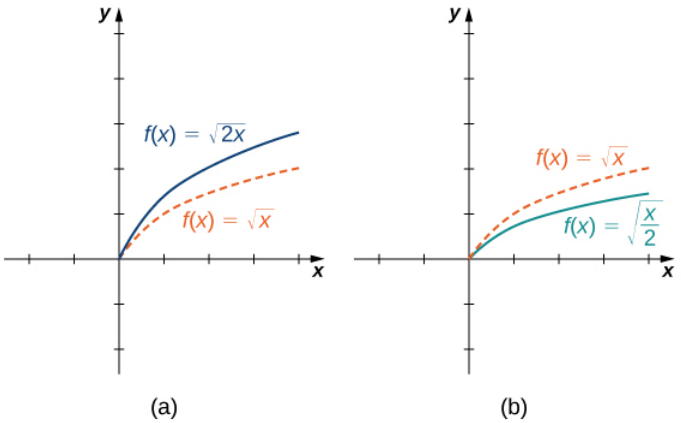


Figure 1.2.12: (a) If $c > 1$, the graph of $y = f(cx)$ is a horizontal compression of the graph of $y = f(x)$. (b) If $0 < c < 1$, the graph of $y = f(cx)$ is a horizontal stretch of the graph of $y = f(x)$.

We have explored what happens to the graph of a function f when we multiply f by a constant $c > 0$ to get a new function $cf(x)$. We have also discussed what happens to the graph of a function $f$when we multiply the independent variable x by $c > 0$ to get a new function $f(cx)$. However, we have not addressed what happens to the graph of the function if the constant c is negative. If we have a constant $c < 0$, we can write c as a positive number multiplied by -1; but, what kind of transformation do we get when we multiply the function or its argument by -1? When we multiply all the outputs by -1, we get a reflection about the x-axis. When we multiply all inputs by -1, we get a reflection about the y-axis. For example, the graph of $f(x) = -(x^3 + 1)$ is the graph of $y = (x^3 + 1)$ reflected about the x-axis. The graph of $f(x) = (-x)^3 + 1$ is the graph of $y = x^3 + 1$ reflected about the y-axis (Figure 1.2.13).

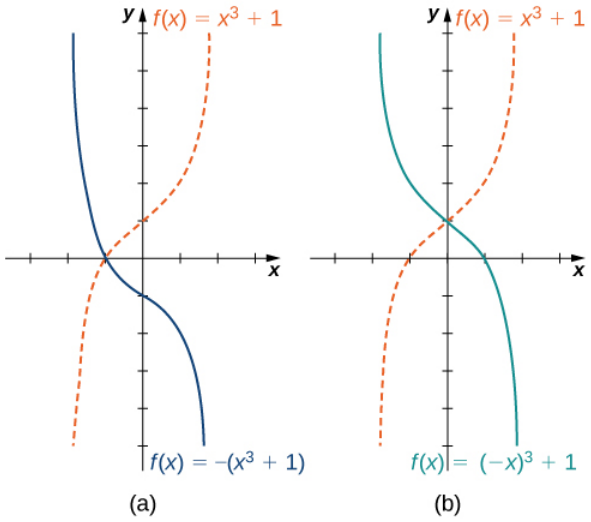


Figure 1.2.13: (a) The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected about the x-axis. (b) The graph of $y = f(-x)$ is the graph of $y = f(x)$ reflected about the y-axis.

If the graph of a function consists of more than one transformation of another graph, it is important to transform the graph in the correct order. Given a function $f(x)$, the graph of the related function $y = cf(a(x + b)) + d$ can be obtained from the graph of $y = f(x)$by performing the transformations in the following order.

- Horizontal shift of the graph of $y = f(x)$. If $b > 0$, shift left. If $b < 0$ shift right.
- Horizontal scaling of the graph of $y = f(x + b)$ by a factor of $|a|$. If $a < 0$, reflect the graph about the y-axis.
- Vertical scaling of the graph of $y = f(a(x + b))$ by a factor of $|c|$. If $c < 0$, reflect the graph about the x -axis.
- Vertical shift of the graph of $y = cf(a(x + b))$. If $d > 0$, shift up. If $d < 0$, shift down.

We can summarize the different transformations and their related effects on the graph of a function in the following table.

Transformation of $f(c > 0)$	**Effect of the graph of** f
$f(x) + c$	Vertical shift up c units

Transformation of $f(c > 0)$	Effect of the graph of f
$f(x) - c$	Vertical shift down c units
$f(x + c)$	Shift left by c units
$f(x - c)$	Shift right by c units
$cf(x)$	Vertical stretch if $c > 1$; vertical compression if $0 < c < 1$
$f(cx)$	Horizontal stretch if $0 < c < 1$; horizontal compression if $c > 1$
$-f(x)$	Reflection about the x-axis
$f(-x)$	Reflection about the y-axis

Example 1.2.10: Transforming a Function

For each of the following functions, a. and b., sketch a graph by using a sequence of transformations of a well-known function.

a. $f(x) = -|x + 2| - 3$
b. $f(x) = 3\sqrt{-x} + 1$

Solution

1.Starting with the graph of $y = |x|$, shift 2 units to the left, reflect about the x-axis, and then shift down 3 units.

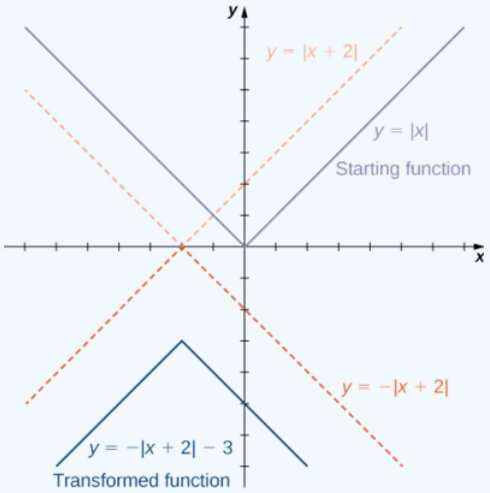


Figure 1.2.14: The function $f(x) = -|x + 2| - 3$ can be viewed as a sequence of three transformations of the function $y = |x|$.

2. Starting with the graph of $y = \sqrt{x}$, reflect about the y-axis, stretch the graph vertically by a factor of 3, and move up 1 unit.

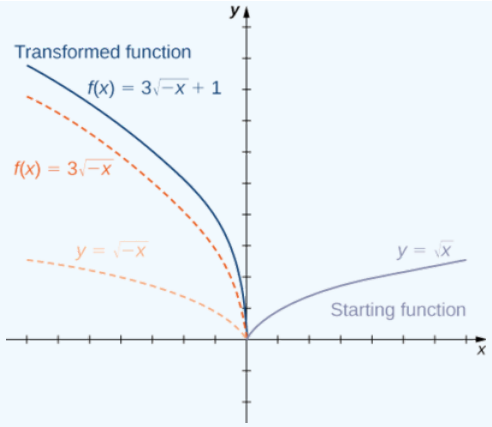


Figure 1.2.15: The function $f(x) = 3\sqrt{-x} + 1$can be viewed as a sequence of three transformations of the function $y = \sqrt{x}$.

? Exercise 1.2.7

Describe how the function $f(x) = -(x+1)^2 - 4$ can be graphed using the graph of $y = x^2$ and a sequence of transformations

Answer

Shift the graph $y = x^2$ to the left 1 unit, reflect about the x-axis, then shift down 4 units.

Key Concepts

- The power function $f(x) = x^n$ is an even function if n is even and $n \neq 0$, and it is an odd function if n is odd.
- The root function $f(x) = x^{1/n}$ has the domain $[0, \infty)$ if n is even and the domain $(-\infty, \infty)$ if n is odd. If n is odd, then $f(x) = x^{1/n}$ is an odd function.
- The domain of the rational function $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomial functions, is the set of x such that $q(x) \neq 0$.
- Functions that involve the basic operations of addition, subtraction, multiplication, division, and powers are algebraic functions. All other functions are transcendental. Trigonometric, exponential, and logarithmic functions are examples of transcendental functions.
- A polynomial function f with degree $n \geq 1$ satisfies $f(x) \to \pm\infty$ as $x \to \pm\infty$. The sign of the output as $x \to \infty$ depends on the sign of the leading coefficient only and on whether n is even or odd.
- Vertical and horizontal shifts, vertical and horizontal scalings, and reflections about the x- and y-axes are examples of transformations of functions.

Key Equations

- **Point-slope equation of a line**

$$y - y_1 = m(x - x_1)$$

- **Slope-intercept form of a line**

$$y = mx + b$$

- **Standard form of a line**

$$ax + by = c$$

- **Polynomial function**

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Glossary

algebraic function

a function involving any combination of only the basic operations of addition, subtraction, multiplication, division, powers, and roots applied to an input variable x

cubic function
a polynomial of degree 3; that is, a function of the form $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$

degree
for a polynomial function, the value of the largest exponent of any term

linear function
a function that can be written in the form $f(x) = mx + b$

logarithmic function
a function of the form $f(x) = \log_b(x)$ for some base $b > 0$, $b \neq 1$ such that $y = \log_b(x)$ if and only if $b^y = x$

mathematical model
A method of simulating real-life situations with mathematical equations

piecewise-defined function
a function that is defined differently on different parts of its domain

point-slope equation
equation of a linear function indicating its slope and a point on the graph of the function

polynomial function
a function of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$

power function
a function of the form $f(x) = x^n$ for any positive integer $n \geq 1$

quadratic function
a polynomial of degree 2; that is, a function of the form $f(x) = ax^2 + bx + c$ where $a \neq 0$

rational function
a function of the form $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials

root function
a function of the form $f(x) = x^{1/n}$ for any integer $n \geq 2$

slope
the change in y for each unit change in x

slope-intercept form
equation of a linear function indicating its slope and y-intercept

transcendental function
a function that cannot be expressed by a combination of basic arithmetic operations

transformation of a function
a shift, scaling, or reflection of a function

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1.2E: Exercises for Section 1.2

In exercises 1 - 8, for each pair of points,

a. find the slope of the line passing through the points and

b. indicate whether the line is increasing, decreasing, horizontal, or vertical.

1) $(-2, 4)$ and $(1, 1)$

Answer

a. $m = -1$
b. Decreasing

2) $(-1, 4)$ and $(3, -1)$

3) $(3, 5)$ and $(-1, 2)$

Answer

a. $m = 3/4$
b. Increasing

4) $(6, 4)$ and $(4, -3)$

5) $(2, 3)$ and $(5, 7)$

Answer

a. $m = 4/3$
b. Increasing

6) $(1, 9)$ and $(-8, 5)$

7) $(2, 4)$ and $(1, 4)$

Answer

a. $m = 0$
b. Horizontal

8) $(1, 4)$ and $(1, 0)$

In exercises 9 - 16, write the equation of the line satisfying the given conditions in slope-intercept form.

9) Slope $=-6$, passes through $(1, 3)$

Answer

$y = -6x + 9$

10) Slope $=3$, passes through $(-3, 2)$

11) Slope $=\frac{1}{3}$, passes through $(0, 4)$

Answer

$y = \frac{1}{3}x + 4$

12) Slope $=\frac{2}{5}$, x-intercept $=8$

13) Passing through $(2, 1)$ and $(-2, -1)$

Answer

$y = \frac{1}{2}x$

14) Passing through $(-3, 7)$ and $(1, 2)$

15) x-intercept $=5$ and y-intercept $=-3$

Answer

$y = \frac{3}{5}x - 3$

16) x-Intercept $=-6$ and y-intercept $=9$

In exercises 17 - 24, for each linear equation,

a. give the slope m and y-intercept b, if any, and

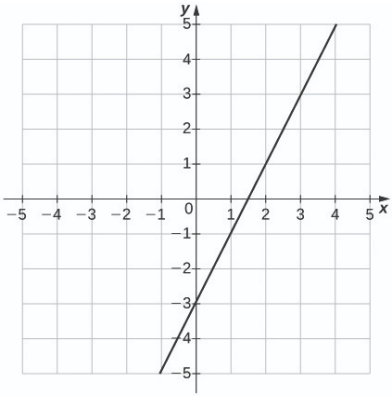
b. graph the line.

17) $y = 2x - 3$

Answer

a. $m = 2,\ b = -3$

b.



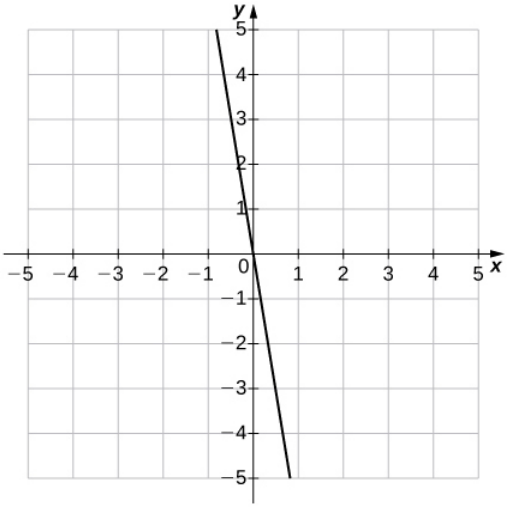
18) $y = -\frac{1}{7}x + 1$

19) $f(x) = -6x$

Answer

a. $m = -6,\ b = 0$

b.



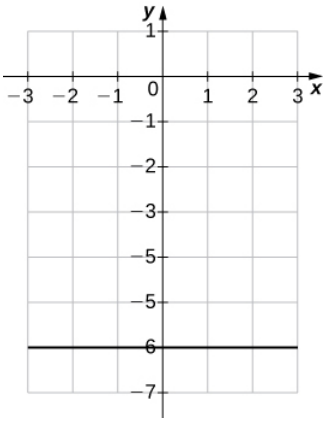
20) $f(x) = -5x + 4$

21) $4y + 24 = 0$

Answer

a. $m = 0,\ b = -6$

b.



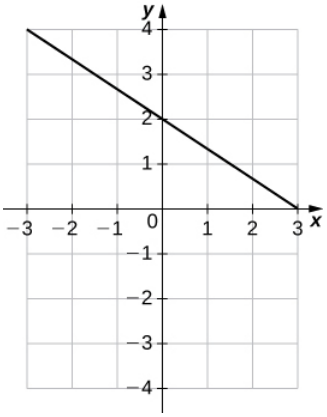
22) $8x - 4 = 0$

23) $2x + 3y = 6$

Answer

a. $m = -\frac{2}{3},\ b = 2$

b.



24) $6x - 5y + 15 = 0$

In exercises 25 - 29, for each polynomial,

a. find the degree;

b. find the zeros, if any;

c. find the y-intercept(s), if any;

d. use the leading coefficient to determine the graph's end behavior; and

e. determine algebraically whether the polynomial is even, odd, or neither.

25) $f(x) = 2x^2 - 3x - 5$

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Answer

a. 2
b. $\frac{5}{2},\ -1$;
c. -5
d. Both ends rise
e. Neither

26) $f(x) = -3x^2 + 6x$

27) $f(x) = \frac{1}{2}x^2 - 1$

Answer

a. 2
b. $\pm\sqrt{2}$
c. -1
d. Both ends rise
e. Even

28) $f(x) = x^3 + 3x^2 - x - 3$

29) $f(x) = 3x - x^3$

Answer

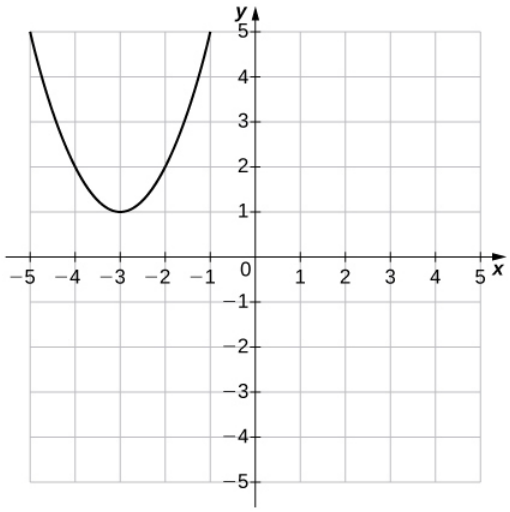
a. 3
b. $0, \pm\sqrt{3}$
c. 0
d. Left end rises, right end falls
e. Odd

For exercises 30 - 31, use the graph of $f(x) = x^2$ to graph each transformed function g.

30) $g(x) = x^2 - 1$

31) $g(x) = (x + 3)^2 + 1$

Answer

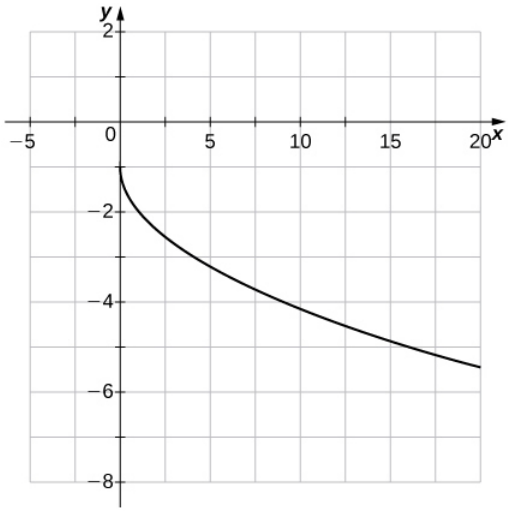


For exercises 32 - 33, use the graph of $f(x) = \sqrt{x}$ to graph each transformed function g.

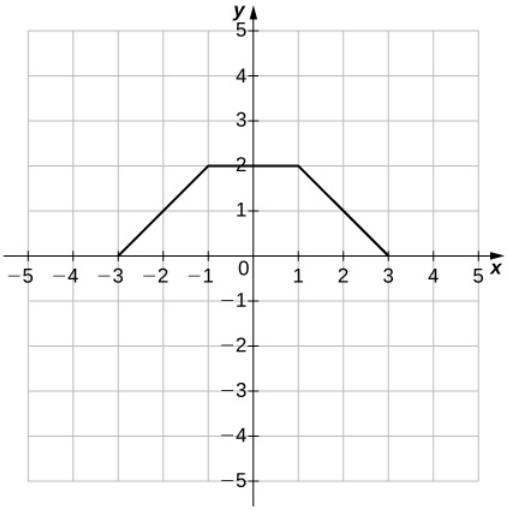
32) $g(x) = \sqrt{x + 2}$

33) $g(x) = -\sqrt{x} - 1$

Answer



For exercises 34 - 35, use the graph of $y = f(x)$ to graph each transformed function g.



34) $g(x) = f(x) + 1$

35) $g(x) = f(x - 1) + 2$

Answer

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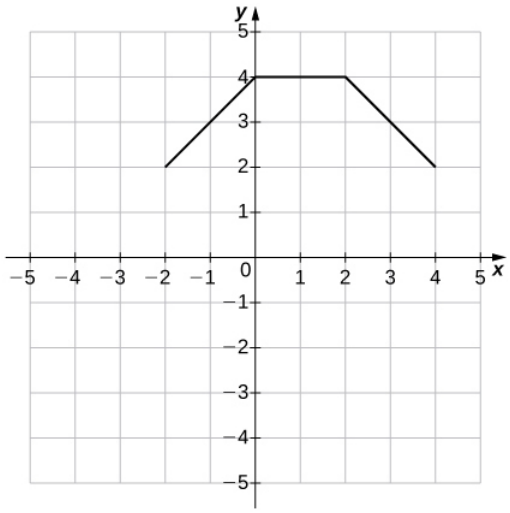


In exercises 36 - 39, for each of the piecewise-defined functions,

a. evaluate at the given values of the independent variable, and

b. sketch the graph.

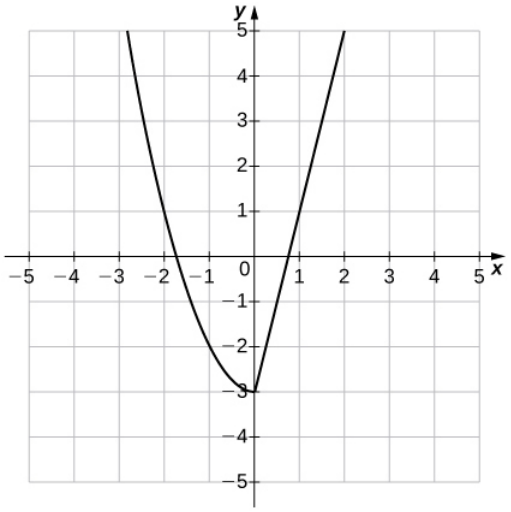
36) $f(x) = \begin{cases} 4x+3, & \text{if } x \le 0 \\ -x+1, & \text{if } x > 0 \end{cases}$; $f(-3);\ f(0);\ f(2)$

37) $f(x) = \begin{cases} x^2-3, & \text{if } x \le 0 \\ 4x-3, & \text{if } x > 0 \end{cases}$; $f(-4);\ f(0);\ f(2)$

Answer

a. $f(-4) = 13, \quad f(0) = -3, \quad f(2) = 5$

b.



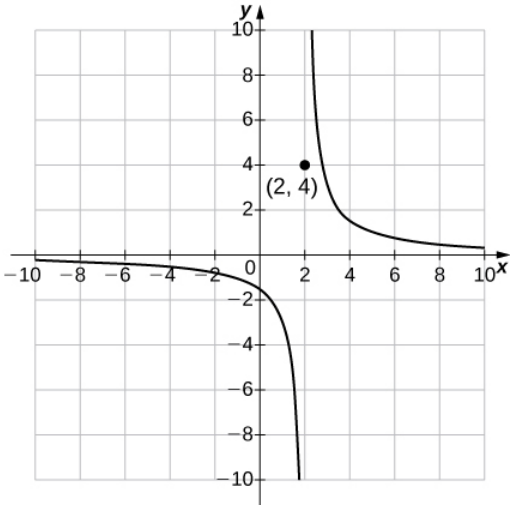
38) $h(x) = \begin{cases} x+1, & \text{if } x \le 5 \\ 4, & \text{if } x > 5 \end{cases}$; $h(0);\ h(\pi);\ h(5)$

39) $g(x) = \begin{cases} \dfrac{3}{x-2}, & \text{if } x \ne 2 \\ 4, & \text{if } x = 2 \end{cases}$; $g(0);\ g(-4);\ g(2)$

Answer

a. $g(0) = -\frac{3}{2},\ g(-4) = -\frac{1}{2},\ g(2) = 4$

b.



In exercises 40 - 44, determine whether the statement is *true or false*. Explain why.

40) $f(x) = \dfrac{4x+1}{7x-2}$ is a transcendental function.

41) $g(x) = \sqrt[3]{x}$ is an odd root function.

Answer

True; $n = 3$

42) A logarithmic function is an algebraic function.

43) A function of the form $f(x) = x^b$, where b is a real valued constant, is an exponential function.

Answer

False; $f(x) = x^b$, where b is a real-valued constant, is a power function

44) The domain of an even root function is all real numbers.

45) [T] A company purchases some computer equipment for \$20,500. At the end of a 3-year period, the value of the equipment has decreased linearly to \$12,301.

a. Find a function $y = V(t)$ that determines the value V of the equipment at the end of t years.

b. Find and interpret the meaning of the x- and y-intercepts for this situation.

c. What is the value of the equipment at the end of 5 years?

d. When will the value of the equipment be \$3000?

Answer

a. $V(t) = -2733t + 20500$
b. $(0, 20,500)$ means that the initial purchase price of the equipment is \$20,500; $(7.5, 0)$ means that in 7.5 years the computer equipment has no value.
c. \$6835
d. In approximately 6.4 years

46) [T] Total online shopping during the Christmas holidays has increased dramatically during the past 5 years. In 2012 $(t = 0)$,total online holiday sales were \$42.3 billion, whereas in 2013 they were \$48.1 billion.

a. Find a linear function S that estimates the total online holiday sales in the year t.

b. Interpret the slope of the graph of S.

c. Use part a. to predict the year when online shopping during Christmas will reach \$60 billion.

47) [T] A family bakery makes cupcakes and sells them at local outdoor festivals. For a music festival, there is a fixed cost of $125 to set up a cupcake stand. The owner estimates that it costs $0.75 to make each cupcake. The owner is interested in determining the total cost C as a function of number of cupcakes made.

a. Find a linear function that relates cost C to x, the number of cupcakes made.

b. Find the cost to bake 160 cupcakes.

c. If the owner sells the cupcakes for $1.50 apiece, how many cupcakes does she need to sell to start making profit? (Hint: Use the INTERSECTION function on a calculator to find this number.)

Answer

a. $C = 0.75x + 125$
b. $245
c. 167 cupcakes

48) [T] A house purchased for $250,000 is expected to be worth twice its purchase price in 18 years.

a. Find a linear function that models the price P of the house versus the number of years t since the original purchase.

b. Interpret the slope of the graph of P.

c. Find the price of the house 15 years from when it was originally purchased.

49) [T] A car was purchased for $26,000. The value of the car depreciates by $1500 per year.

a. Find a linear function that models the value V of the car after t years.

b. Find and interpret $V(4)$.

Answer

a. $V(t) = -1500t + 26,000$
b. In 4 years, the value of the car is $20,000.

50) [T] A condominium in an upscale part of the city was purchased for $432,000. In 35 years it is worth $60,500. Find the rate of depreciation.

51) [T] The total cost C (in thousands of dollars) to produce a certain item is modeled by the function $C(x) = 10.50x + 28,500$, where x is the number of items produced. Determine the cost to produce 175 items.

Answer

$30,337,500

52) [T] A professor asks her class to report the amount of time t they spent writing two assignments. Most students report that it takes them about 45 minutes to type a four-page assignment and about 1.5 hours to type a nine-page assignment.

a. Find the linear function $y = N(t)$ that models this situation, where N is the number of pages typed and t is the time in minutes.

b. Use part a. to determine how many pages can be typed in 2 hours.

c. Use part a. to determine how long it takes to type a 20-page assignment.

53) [T] The output (as a percent of total capacity) of nuclear power plants in the United States can be modeled by the function $P(t) = 1.8576t + 68.052$, where t is time in years and $t = 0$ corresponds to the beginning of 2000. Use the model to predict the percentage output in 2015.

Answer

96% of the total capacity

54) [T] The admissions office at a public university estimates that 65% of the students offered admission to the class of 2019 will actually enroll.

a. Find the linear function $y = N(x)$, where N is the number of students that actually enroll and x is the number of all students offered admission to the class of 2019.

b. If the university wants the 2019 freshman class size to be 1350, determine how many students should be admitted.

Contributors

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1.3: Trigonometric Functions

Learning Objectives

- Convert angle measures between degrees and radians.
- Recognize the triangular and circular definitions of the basic trigonometric functions.
- Write the basic trigonometric identities.
- Identify the graphs and periods of the trigonometric functions.
- Describe the shift of a sine or cosine graph from the equation of the function.

Trigonometric functions are used to model many phenomena, including sound waves, vibrations of strings, alternating electrical current, and the motion of pendulums. In fact, almost any repetitive, or cyclical, motion can be modeled by some combination of trigonometric functions. In this section, we define the six basic trigonometric functions and look at some of the main identities involving these functions.

Radian Measure

To use trigonometric functions, we first must understand how to measure the angles. Although we can use both radians and degrees, **radians** are a more natural measurement because they are related directly to the unit circle, a circle with radius 1. The radian measure of an angle θ, let s be the length of the corresponding arc on the unit circle (Figure 1.3.1). We say the angle corresponding to the arc of length 1 has radian measure 1.

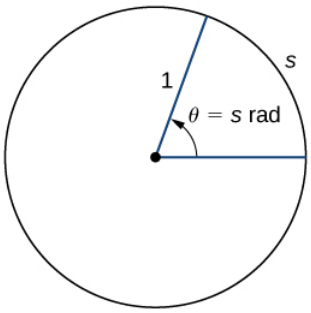


Figure 1.3.1: The radian measure of an angle θ is the arc length s of the associated arc on the unit circle.

Since an angle of $360°$ corresponds to the circumference of a circle, or an arc of length 2π, we conclude that an angle with a degree measure of $360°$ has a radian measure of 2π. Similarly, we see that $180°$ is equivalent to π radians. Table 1.3.1 shows the relationship between common degree and radian values.

Table 1.3.1: Common Angles Expressed in Degrees and Radians

Degrees	Radians	Degrees	Radians
0	0	120	$2\pi/3$
30	$\pi/6$	135	$3\pi/4$
45	$\pi/4$	150	$5\pi/6$
60	$\pi/3$	180	π
90	$\pi/2$		

Converting between Radians and Degrees

a. Express $225°$ using radians.
b. Express $5\pi/3$ rad using degrees.

Solution

Use the fact that $180°$ is equivalent to π radians as a **conversion factor** (Table 1.3.1):

$$\frac{\pi \text{ rad}}{} \quad \frac{180°}{}$$

$$1 = \frac{\pi \text{ rad}}{180°} = \frac{180°}{\pi \text{ rad}}.$$

a. $225° = 225° \cdot \left(\frac{\pi}{180°}\right) = \left(\frac{5\pi}{4}\right)$ rad

b. $\frac{5\pi}{3}$ rad $= \frac{5\pi}{3} \cdot \frac{180°}{\pi} = 300°$

Exercise 1.3.1

a. Express $210°$ using radians.

b. Express $11\pi/6$ rad using degrees.

Hint

π radians is equal to $180°$

Answer

a. $7\pi/6$

b. $330°$

The Six Basic Trigonometric Functions

Trigonometric functions allow us to use angle measures, in radians or degrees, to find the coordinates of a point on any circle—not only on a unit circle—or to find an angle given a point on a circle. They also define the relationship between the sides and angles of a triangle.

To define the trigonometric functions, first consider the unit circle centered at the origin and a point $P = (x, y)$ on the unit circle. Let θ be an angle with an initial side that lies along the positive x-axis and with a terminal side that is the line segment OP. An angle in this position is said to be in standard position (Figure 1.3.2). We can then define the values of the six trigonometric functions for θ in terms of the coordinates x and y.

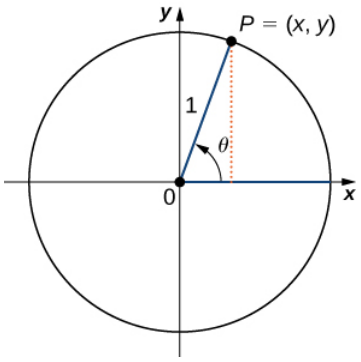


Figure 1.3.2: The angle θ is in standard position. The values of the trigonometric functions for θ are defined in terms of the coordinates x and y.

Definition: Trigonometric functions

Let $P = (x, y)$ be a point on the unit circle centered at the origin O. Let θ be an angle with an initial side along the positive x-axis and a terminal side given by the line segment OP. The trigonometric functions are then defined as

$\sin\theta = y$	$\csc\theta = \frac{1}{y}$
$\cos\theta = x$	$\sec\theta = \frac{1}{x}$
$\tan\theta = \frac{y}{x}$	$\cot\theta = \frac{x}{y}$

If $x = 0, \sec\theta$ and $\tan\theta$ are undefined. If $y = 0$, then $\cot\theta$ and $\csc\theta$ are undefined.

We can see that for a point $P = (x, y)$ on a circle of radius r with a corresponding angle θ, the coordinates x and y satisfy

$$\cos\theta = \frac{x}{r} \tag{1.3.1}$$

$$x = r\cos\theta \tag{1.3.2}$$

and

$$\sin\theta = \frac{y}{r} \tag{1.3.3}$$

$$y = r\sin\theta. \tag{1.3.4}$$

The values of the other trigonometric functions can be expressed in terms of x, y, and r (Figure 1.3.3).

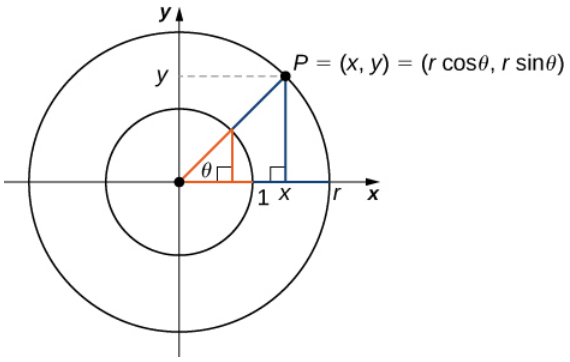


Figure 1.3.3: For a point $P = (x, y)$ on a circle of radius r, the coordinates x and y satisfy $x = r\cos\theta$ and $y = r\sin\theta$.

Table 1.3.2 shows the values of sine and cosine at the major angles in the first quadrant. From this table, we can determine the values of sine and cosine at the corresponding angles in the other quadrants. The values of the other trigonometric functions are calculated easily from the values of $\sin\theta$ and $\cos\theta$.

Table 1.3.2: Values of $\sin\theta$ and $\cos\theta$ at Major Angles θ in the First Quadrant

θ	$\sin\theta$	$\cos\theta$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0

Example 1.3.2: Evaluating Trigonometric Functions

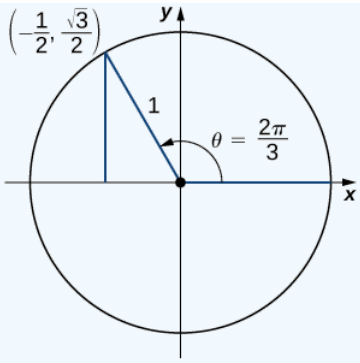
Evaluate each of the following expressions.

a. $\sin\left(\frac{2\pi}{3}\right)$
b. $\cos\left(-\frac{5\pi}{6}\right)$
c. $\tan\left(\frac{15\pi}{4}\right)$

Solution:

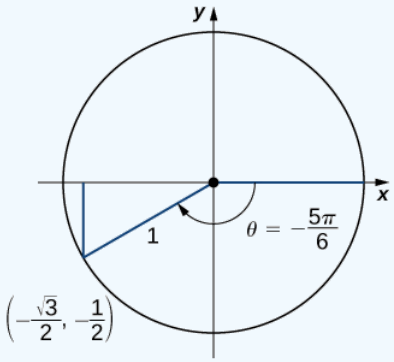
a) On the unit circle, the angle $\theta = \frac{2\pi}{3}$ corresponds to the point $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Therefore,

$$\sin\left(\frac{2\pi}{3}\right) = y = \left(\frac{\sqrt{3}}{2}\right).$$



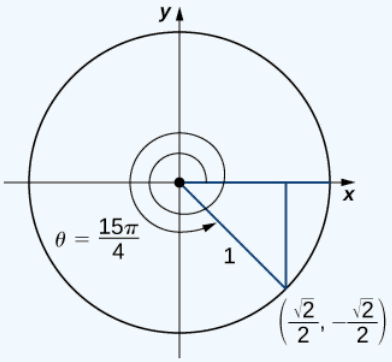
b) An angle $\theta = -\frac{5\pi}{6}$ corresponds to a revolution in the negative direction, as shown. Therefore,

$$\cos\left(-\frac{5\pi}{6}\right) = x = -\frac{\sqrt{3}}{2}.$$



c) An angle $\theta = \frac{15\pi}{4} = 2\pi + \frac{7\pi}{4}$. Therefore, this angle corresponds to more than one revolution, as shown. Knowing the fact that an angle of $\frac{7\pi}{4}$ corresponds to the point $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$, we can conclude that

$$\tan\left(\frac{15\pi}{4}\right) = \frac{y}{x} = -1.$$



? Exercise 1.3.2

Evaluate $\cos(3\pi/4)$ and $\sin(-\pi/6)$.

Hint

Look at angles on the unit circle.

Answer

$\cos(3\pi/4) = -\sqrt{2}/2$

$\cos(5\pi/4) = -\sqrt{2}/2$

$\sin(-\pi/6) = -1/2$

As mentioned earlier, the ratios of the side lengths of a right triangle can be expressed in terms of the trigonometric functions evaluated at either of the acute angles of the triangle. Let θ be one of the acute angles. Let A be the length of the adjacent leg, O be the length of the opposite leg, and H be the length of the hypotenuse. By inscribing the triangle into a circle of radius H, as shown in Figure 1.3.4, we see that A, H, and O satisfy the following relationships with θ:

$\sin\theta = \frac{O}{H}$	$\csc\theta = \frac{H}{O}$
$\cos\theta = \frac{A}{H}$	$\sec\theta = \frac{H}{A}$
$\tan\theta = \frac{O}{A}$	$\cot\theta = \frac{A}{O}$

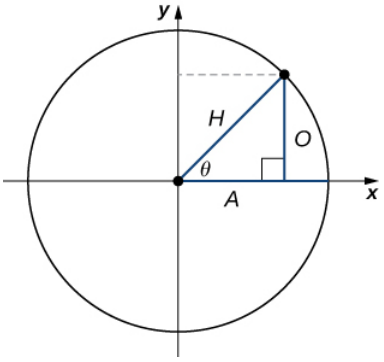


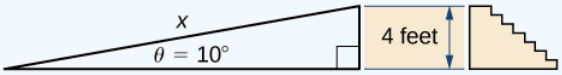
Figure 1.3.4: By inscribing a right triangle in a circle, we can express the ratios of the side lengths in terms of the trigonometric functions evaluated at θ.

Example 1.3.3: Constructing a Wooden Ramp

A wooden ramp is to be built with one end on the ground and the other end at the top of a short staircase. If the top of the staircase is 4 ft from the ground and the angle between the ground and the ramp is to be 10°, how long does the ramp need to be?

Solution

Let x denote the length of the ramp. In the following image, we see that x needs to satisfy the equation $\sin(10°) = 4/x$. Solving this equation for x, we see that $x = 4/\sin(10°) \approx 23.035$ ft.



Exercise 1.3.3

A house painter wants to lean a 20-ft ladder against a house. If the angle between the base of the ladder and the ground is to be 60°, how far from the house should she place the base of the ladder?

Hint

Draw a right triangle with hypotenuse 20.

Answer

10 ft

Trigonometric Identities

A **trigonometric identity** is an equation involving trigonometric functions that is true for all angles θ for which the functions are defined. We can use the identities to help us solve or simplify equations. The main trigonometric identities are listed next.

Trigonometric Identities

Reciprocal identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

Pythagorean identities

$$\sin^2\theta + \cos^2\theta = 1 \tag{1.3.5}$$

$$1 + \tan^2\theta = \sec^2\theta \tag{1.3.6}$$

$$1 + \cot^2\theta = \csc^2\theta \tag{1.3.7}$$

Addition and subtraction formulas

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Double-angle formulas

$$\sin(2\theta) = 2\sin\theta\cos\theta \tag{1.3.8}$$

$$\cos(2\theta) = 2\cos^2\theta - 1 \tag{1.3.9}$$

$$= 1 - 2\sin^2\theta \tag{1.3.10}$$

$$= \cos^2\theta - \sin^2\theta \tag{1.3.11}$$

Example 1.3.4: Solving Trigonometric Equations

For each of the following equations, use a trigonometric identity to find all solutions.

a. $1 + \cos(2\theta) = \cos\theta$
b. $\sin(2\theta) = \tan\theta$

Solution

a) Using the double-angle formula for $\cos(2\theta)$, we see that θ is a solution of

$$1 + \cos(2\theta) = \cos\theta$$

if and only if

$$1 + 2\cos^2\theta - 1 = \cos\theta,$$

which is true if and only if

$$2\cos^2\theta - \cos\theta = 0.$$

To solve this equation, it is important to note that we need to factor the left-hand side and not divide both sides of the equation by $\cos\theta$. The problem with dividing by $\cos\theta$ is that it is possible that $\cos\theta$ is zero. In fact, if we did divide both

sides of the equation by $\cos\theta$, we would miss some of the solutions of the original equation. Factoring the left-hand side of the equation, we see that θ is a solution of this equation if and only if

$$\cos\theta(2\cos\theta - 1) = 0.$$

Since $\cos\theta = 0$ when

$$\theta = \frac{\pi}{2}, \frac{\pi}{2} \pm \pi, \frac{\pi}{2} \pm 2\pi, \ldots,$$

and $\cos\theta = 1/2$ when

$$\theta = \frac{\pi}{3}, \frac{\pi}{3} \pm 2\pi, \ldots \text{ or } \theta = -\frac{\pi}{3}, -\frac{\pi}{3} \pm 2\pi, \ldots,$$

we conclude that the set of solutions to this equation is

$$\theta = \frac{\pi}{2} + n\pi, \ \theta = \frac{\pi}{3} + 2n\pi$$

and

$$\theta = -\frac{\pi}{3} + 2n\pi, \ n = 0, \pm 1, \pm 2, \ldots.$$

b) Using the double-angle formula for $\sin(2\theta)$ and the reciprocal identity for $\tan(\theta)$, the equation can be written as

$$2\sin\theta\cos\theta = \frac{\sin\theta}{\cos\theta}.$$

To solve this equation, we multiply both sides by $\cos\theta$ to eliminate the denominator, and say that if θ satisfies this equation, then θ satisfies the equation

$$2\sin\theta\cos^2\theta - \sin\theta = 0.$$

However, we need to be a little careful here. Even if θ satisfies this new equation, it may not satisfy the original equation because, to satisfy the original equation, we would need to be able to divide both sides of the equation by $\cos\theta$. However, if $\cos\theta = 0$, we cannot divide both sides of the equation by $\cos\theta$. Therefore, it is possible that we may arrive at extraneous solutions. So, at the end, it is important to check for extraneous solutions. Returning to the equation, it is important that we factor $\sin\theta$ out of both terms on the left-hand side instead of dividing both sides of the equation by $\sin\theta$. Factoring the left-hand side of the equation, we can rewrite this equation as

$$\sin\theta(2\cos^2\theta - 1) = 0.$$

Therefore, the solutions are given by the angles θ such that $\sin\theta = 0$ or $\cos^2\theta = 1/2$. The solutions of the first equation are $\theta = 0, \pm\pi, \pm 2\pi, \ldots$. The solutions of the second equation are $\theta = \pi/4, (\pi/4) \pm (\pi/2), (\pi/4) \pm \pi, \ldots$. After checking for extraneous solutions, the set of solutions to the equation is

$$\theta = n\pi$$

and

$$\theta = \frac{\pi}{4} + \frac{n\pi}{2}$$

with $n = 0, \pm 1, \pm 2, \ldots$.

? Exercise 1.3.4

Find all solutions to the equation $\cos(2\theta) = \sin\theta$.

Hint

Use the double-angle formula for cosine (Equation 1.3.8).

Answer

$\theta = \frac{3\pi}{2} + 2n\pi, \frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi$

for $n = 0, \pm 1, \pm 2, \ldots$.

Example 1.3.5: Proving a Trigonometric Identity

Prove the trigonometric identity $1 + \tan^2\theta = \sec^2\theta$.

Solution:

We start with the Pythagorean identity (Equation 1.3.5)

$$\sin^2\theta + \cos^2\theta = 1.$$

Dividing both sides of this equation by $\cos^2\theta$, we obtain

$$\frac{\sin^2\theta}{\cos^2\theta} + 1 = \frac{1}{\cos^2\theta}.$$

Since $\sin\theta/\cos\theta = \tan\theta$ and $1/\cos\theta = \sec\theta$, we conclude that

$$\tan^2\theta + 1 = \sec^2\theta.$$

Exercise 1.3.5

Prove the trigonometric identity $1 + \cot^2\theta = \csc^2\theta$.

Answer

Divide both sides of the identity $\sin^2\theta + \cos^2\theta = 1$ by $\sin^2\theta$.

Graphs and Periods of the Trigonometric Functions

We have seen that as we travel around the unit circle, the values of the trigonometric functions repeat. We can see this pattern in the graphs of the functions. Let $P = (x, y)$ be a point on the unit circle and let θ be the corresponding angle . Since the angle θ and $\theta + 2\pi$ correspond to the same point P, the values of the trigonometric functions at θ and at $\theta + 2\pi$ are the same. Consequently, the trigonometric functions are **periodic functions**. The period of a function f is defined to be the smallest positive value p such that $f(x + p) = f(x)$ for all values x in the domain of f. The sine, cosine, secant, and cosecant functions have a period of 2π. Since the tangent and cotangent functions repeat on an interval of length π, their period is π (Figure 1.3.5).

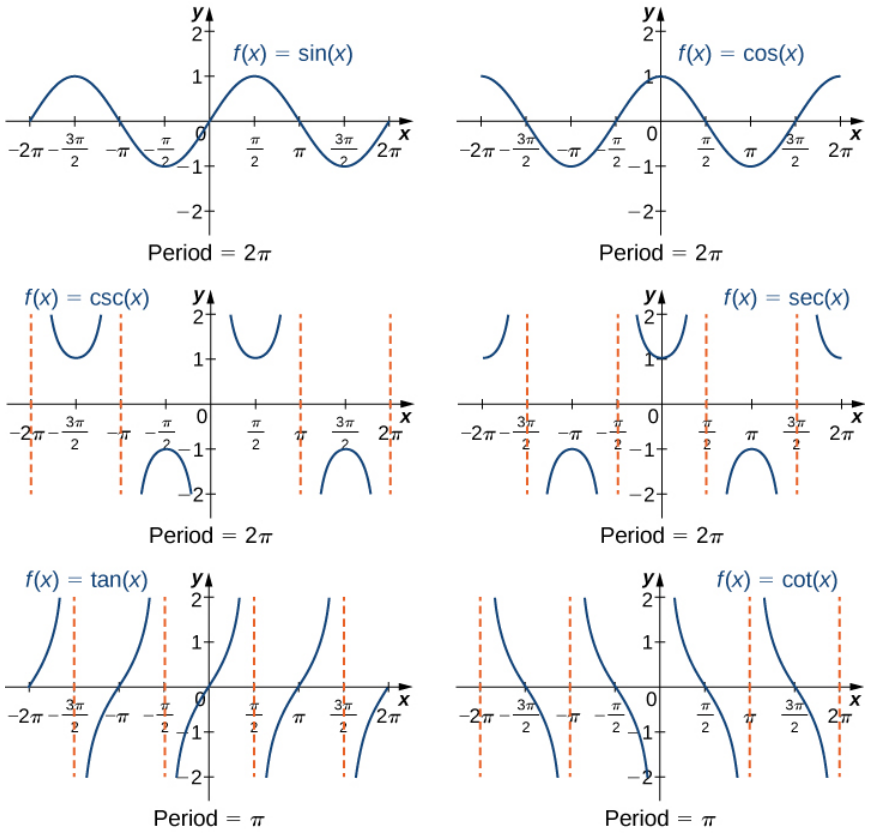


Figure 1.3.5: The six trigonometric functions are periodic.

Just as with algebraic functions, we can apply transformations to trigonometric functions. In particular, consider the following function:

$$f(x) = A\sin(B(x-\alpha)) + C.$$

In Figure 1.3.6, the constant α causes a horizontal or phase shift. The factor B changes the period. This transformed sine function will have a period $2\pi/|B|$. The factor A results in a vertical stretch by a factor of $|A|$. We say $|A|$ is the "amplitude of f." The constant C causes a vertical shift.

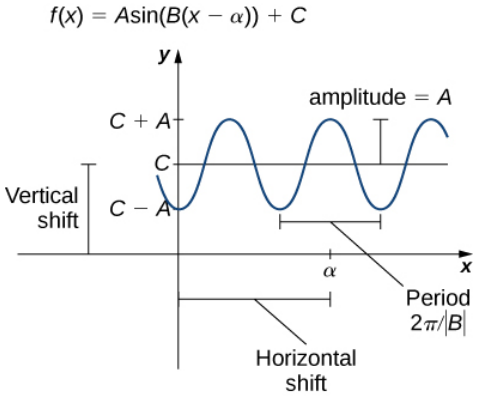


Figure 1.3.6: A graph of a general sine function.

Notice in Figure 1.3.6 that the graph of $y = \cos x$ is the graph of $y = \sin x$ shifted to the left $\pi/2$ units. Therefore, we can write

$$\cos x = \sin(x + \pi/2).$$

Similarly, we can view the graph of $y = \sin x$ as the graph of $y = \cos x$ shifted right $\pi/2$ units, and state that $\sin x = \cos(x - \pi/2)$.

A shifted sine curve arises naturally when graphing the number of hours of daylight in a given location as a function of the day of the year. For example, suppose a city reports that June 21 is the longest day of the year with 15.7 hours and December 21 is the

shortest day of the year with 8.3 hours. It can be shown that the function

$$h(t) = 3.7 \sin\left(\frac{2\pi}{365}(x - 80.5)\right) + 12$$

is a model for the number of hours of daylight h as a function of day of the year t (Figure 1.3.7).

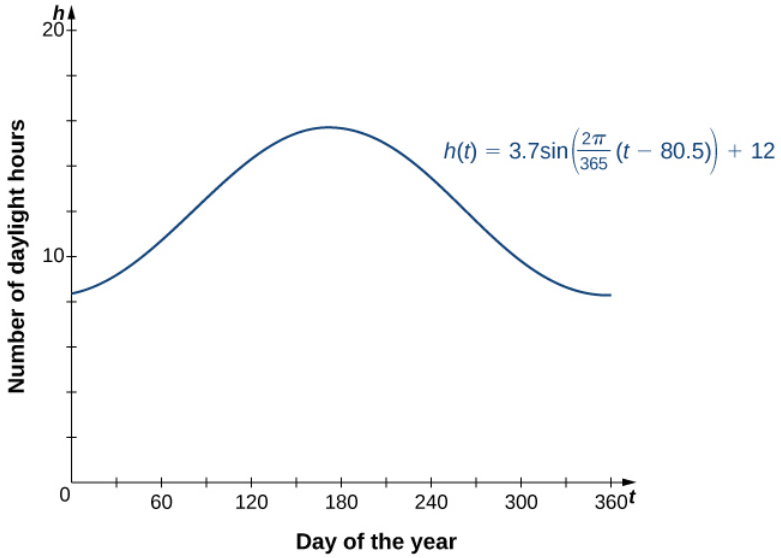


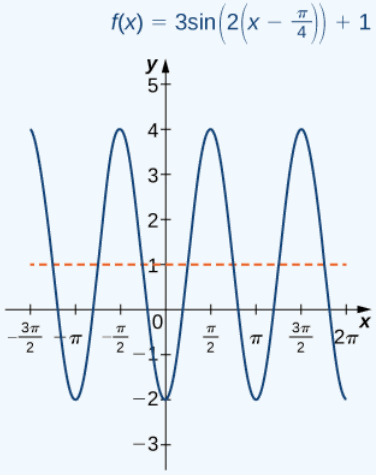
Figure 1.3.7: The hours of daylight as a function of day of the year can be modeled by a shifted sine curve.

Example 1.3.6: Sketching the Graph of a Transformed Sine Curve

Sketch a graph of $f(x) = 3\sin(2(x - \frac{\pi}{4})) + 1$.

Solution

This graph is a phase shift of $y = \sin(x)$ to the right by $\pi/4$ units, followed by a horizontal compression by a factor of 2, a vertical stretch by a factor of 3, and then a vertical shift by 1 unit. The period of f is π.



Exercise 1.3.6

Describe the relationship between the graph of $f(x) = 3\sin(4x) - 5$ and the graph of $y = \sin(x)$.

Hint

The graph of f can be sketched using the graph of $y = \sin(x)$ and a sequence of three transformations.

Answer

To graph $f(x) = 3\sin(4x) - 5$, the graph of $y = \sin(x)$ needs to be compressed horizontally by a factor of 4, then stretched vertically by a factor of 3, then shifted down 5 units. The function f will have a period of $\pi/2$ and an amplitude of 3.

Key Concepts

- Radian measure is defined such that the angle associated with the arc of length 1 on the unit circle has radian measure 1. An angle with a degree measure of $180°$ has a radian measure of π rad.
- For acute angles θ,the values of the trigonometric functions are defined as ratios of two sides of a right triangle in which one of the acute angles is θ.
- For a general angle θ, let (x, y) be a point on a circle of radius r corresponding to this angle θ. The trigonometric functions can be written as ratios involving x, y, and r.
- The trigonometric functions are periodic. The sine, cosine, secant, and cosecant functions have period 2π. The tangent and cotangent functions have period π.

Key Equations

- **Generalized sine function**

$$f(x) = A\sin(B(x - \alpha)) + C$$

Glossary

periodic function

a function is periodic if it has a repeating pattern as the values of x move from left to right

radians

for a circular arc of length s on a circle of radius 1, the radian measure of the associated angle θ is s

trigonometric functions

functions of an angle defined as ratios of the lengths of the sides of a right triangle

trigonometric identity

an equation involving trigonometric functions that is true for all angles θ for which the functions in the equation are defined

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1.3E: Exercises for Section 1.3

In exercises 1 - 5, convert each angle in degrees to radians. Write the answer as a multiple of π.

1) $240°$

Answer

$\frac{4\pi}{3}$ rad

2) $15°$

3) $60°$

Answer

$\frac{\pi}{3}$ rad

4) $-225°$

5) $330°$

Answer

$\frac{11\pi}{6}$ rad

In exercises 6 - 10, convert each angle in radians to degrees.

6) $\frac{\pi}{2}$ rad

7) $\frac{7\pi}{6}$ rad

Answer

$210°$

8) $\frac{11\pi}{2}$ rad

9) -3π rad

Answer

$-540°$

10) $\frac{5\pi}{12}$ rad

In exercises 11 - 16, evaluate the functional values.

11) $\cos\frac{4\pi}{3}$

Answer

$\cos\frac{4\pi}{3} = -0.5$

12) $\tan\frac{19\pi}{4}$

13) $\sin\left(-\frac{3\pi}{4}\right)$

Answer

$\sin\left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

14) $\sec\left(-\frac{\pi}{6}\right)$

15) $\sin\left(-\frac{\pi}{12}\right)$

Answer

$\sin\left(-\frac{\pi}{12}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}}$

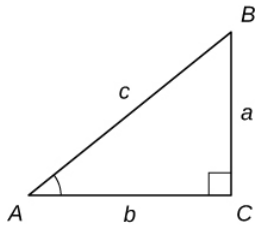
16) $\cos\left(-\frac{5\pi}{12}\right)$

In exercises 17 - 22, consider triangle ABC, a right triangle with a right angle at C.

a. Find the missing side of the triangle.

b. Find the six trigonometric function values for the angle at A.

Where necessary, round to one decimal place.



17) $a = 4,\ c = 7$

Answer

a. $b = 5.7$

b. $\sin A = \frac{4}{7},\ \cos A = \frac{5.7}{7},\ \tan A = \frac{4}{5.7},\ \csc A = \frac{7}{4},\ \sec A = \frac{7}{5.7},\ \cot A = \frac{5.7}{4}$

18) $a = 21,\ c = 29$

19) $a = 85.3,\ b = 125.5$

Answer

a. $c = 151.7$

b. $\sin A = 0.5623,\ \cos A = 0.8273,\ \tan A = 0.6797,\ \csc A = 1.778,\ \sec A = 1.209,\ \cot A = 1.471$

20) $b = 40,\ c = 41$

21) $a = 84,\ b = 13$

Answer

a. $c = 85$

b. $\sin A = \frac{84}{85},\ \cos A = \frac{13}{85},\ \tan A = \frac{84}{13},\ \csc A = \frac{85}{84},\ \sec A = \frac{85}{13},\ \cot A = \frac{13}{84}$

22) $b = 28,\ c = 35$

In exercises 23 - 26, P is a point on the unit circle.

a. Find the (exact) missing coordinate value of each point and

b. find the values of the six trigonometric functions for the angle θ with a terminal side that passes through point P.

Rationalize denominators.

23) $P\left(\frac{7}{25}, y\right),\quad y > 0$

Answer

a. $y = \frac{24}{25}$

b. $\sin\theta = \frac{24}{25},\ \cos\theta = \frac{7}{25},\ \tan\theta = \frac{24}{7},\ \csc\theta = \frac{25}{24},\ \sec\theta = \frac{25}{7},\ \cot\theta = \frac{7}{24}$

24) $P\left(-\frac{15}{17}, y\right),\quad y > 0$

25) $P\left(x, \frac{\sqrt{7}}{3}\right),\quad x > 0$

Answer

a. $x = -\frac{\sqrt{2}}{3}$

b. $\sin\theta = \frac{\sqrt{7}}{3},\ \cos\theta = -\frac{\sqrt{2}}{3},\ \tan\theta = -\frac{\sqrt{14}}{2},\ \csc\theta = \frac{3\sqrt{7}}{7},\ \sec\theta = -\frac{3\sqrt{2}}{2},\ \cot\theta = -\frac{\sqrt{14}}{7}$

26) $P\left(x, -\frac{\sqrt{15}}{4}\right), \quad y > 0$

In exercises 27 - 34, simplify each expression by writing it in terms of sines and cosines, then simplify. The final answer does not have to be in terms of sine and cosine only.

27) $\tan^2 x + \sin x \csc x$

Answer

$\sec^2 x$

28) $\sec x \sin x \cot x$

29) $\frac{\tan^2 x}{\sec^2 x}$

Answer

$\sin^2 x$

30) $\sec x - \cos x$

31) $(1 + \tan\theta)^2 - 2\tan\theta$

Answer

$\sec^2\theta$

32) $(\sin x)(\csc x - \sin x)$

33) $\frac{\cos t}{\sin t} + \frac{\sin t}{1 + \cos t}$

Answer

$\frac{1}{\sin t} = \csc t$

34) $\frac{1 + \tan^2\alpha}{1 + \cot^2\alpha}$

In exercises 35 - 42, verify that each equation is an identity.

35) $\frac{\tan\theta\cot\theta}{\csc\theta} = \sin\theta$

36) $\frac{\sec^2\theta}{\tan\theta} = \sec\theta\csc\theta$

37) $\frac{\sin t}{\csc t} + \frac{\cos t}{\sec t} = 1$

38) $\frac{\sin x}{\cos x + 1} + \frac{\cos x - 1}{\sin x} = 0$

39) $\cot\gamma + \tan\gamma = \sec\gamma\csc\gamma$

40) $\sin^2\beta + \tan^2\beta + \cos^2\beta = \sec^2\beta$

41) $\frac{1}{1 - \sin\alpha} + \frac{1}{1 + \sin\alpha} = 2\sec^2\alpha$

42) $\frac{\tan\theta - \cot\theta}{\sin\theta\cos\theta} = \sec^2\theta - \csc^2\theta$

In exercises 43 - 50, solve the trigonometric equations on the interval $0 \le \theta < 2\pi$.

43) $2\sin\theta - 1 = 0$

Answer

$\left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$

44) $1 + \cos\theta = \frac{1}{2}$

45) $2\tan^2\theta = 2$

Answer

$\left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$

46) $4\sin^2\theta - 2 = 0$

47) $\sqrt{3}\cot\theta + 1 = 0$

Answer

$\left\{\frac{2\pi}{3}, \frac{5\pi}{3}\right\}$

48) $3\sec\theta - 2\sqrt{3} = 0$

49) $2\cos\theta\sin\theta = \sin\theta$

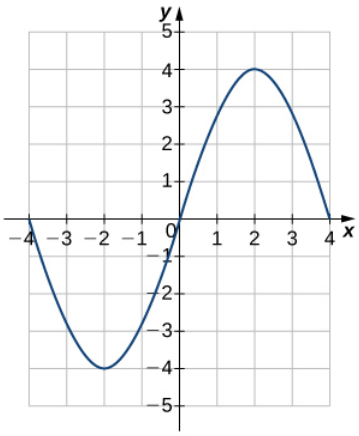
Answer

$\left\{0, \pi, \frac{\pi}{3}, \frac{5\pi}{3}\right\}$

50) $\csc^2\theta + 2\csc\theta + 1 = 0$

In exercises 51 - 54, each graph is of the form $y = A\sin Bx$ or $y = A\cos Bx$, where $B > 0$. Write the equation of the graph.

51)



Answer

$y = 4\sin\left(\frac{\pi}{4}x\right)$

52)

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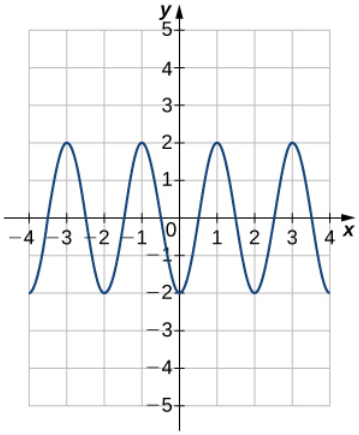
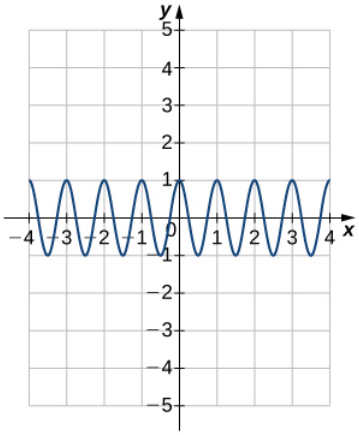


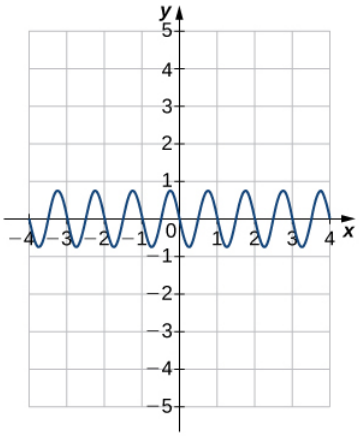
53)



Answer

$y = \cos 2\pi x$

54)



In exercises 55 - 60, find

a. the amplitude,

b. the period, and

c. the phase shift with direction for each function.

55) $y = \sin\left(x - \frac{\pi}{4}\right)$

Answer

a. 1
b. 2π
c. $\frac{\pi}{4}$ units to the right

56) $y = 3\cos(2x + 3)$

57) $y = -\frac{1}{2}\sin\left(\frac{1}{4}x\right)$

Answer

a. $\frac{1}{2}$
b. 8π
c. No phase shift

58) $y = 2\cos\left(x - \frac{\pi}{3}\right)$

59) $y = -3\sin(\pi x + 2)$

Answer

a. 3
b. 2
c. $\frac{2}{\pi}$ units to the left

60) $y = 4\cos\left(2x - \frac{\pi}{2}\right)$

61) [T] The diameter of a wheel rolling on the ground is 40 in. If the wheel rotates through an angle of $120°$, how many inches does it move? Approximate to the nearest whole inch.

Answer

Approximately 42 in.

62) [T] Find the length of the arc intercepted by central angle θ in a circle of radius r. Round to the nearest hundredth.

a. $r = 12.8$ cm, $\theta = \frac{5\pi}{6}$ rad b. $r = 4.378$ cm, $\theta = \frac{7\pi}{6}$ rad c. $r = 0.964$ cm, $\theta = 50°$ d. $r = 8.55$ cm, $\theta = 325°$

63) [T] As a point P moves around a circle, the measure of the angle changes. The measure of how fast the angle is changing is called angular speed, ω, and is given by $\omega = \theta/t$, where θ is in radians and t is time. Find the angular speed for the given data. Round to the nearest thousandth.

a. $\theta = \frac{7\pi}{4}$ rad, $t = 10$ sec b. $\theta = \frac{3\pi}{5}$ rad, $t = 8$ sec c. $\theta = \frac{2\pi}{9}$ rad, $t = 1$ min d. $\theta = 23.76$ rad, $t = 14$ min

Answer

a. 0.550 rad/sec
b. 0.236 rad/sec
c. 0.698 rad/min
d. 1.697 rad/min

64) [T] A total of $250,000\ \text{m}^2$ of land is needed to build a nuclear power plant. Suppose it is decided that the area on which the power plant is to be built should be circular.

a)Find the radius of the circular land area.

b)If the land area is to form a $45°$ sector of a circle instead of a whole circle, find the length of the curved side.

65) [T] The area of an isosceles triangle with equal sides of length x is $\frac{1}{2}x^2\sin\theta$,

where θ is the angle formed by the two sides. Find the area of an isosceles triangle with equal sides of length 8 in. and angle $\theta = \frac{5\pi}{12}$ rad.

Answer

$\approx 30.9 \text{ in}^2$

66) [T] A particle travels in a circular path at a constant angular speed ω. The angular speed is modeled by the function $\omega = 9|\cos(\pi t - \pi/12)|$. Determine the angular speed at $t = 9$ sec.

67) [T] An alternating current for outlets in a home has voltage given by the function $V(t) = 150\cos 368t$,

where V is the voltage in volts at time t in seconds.

a) Find the period of the function and interpret its meaning.

b) Determine the number of periods that occur when 1 sec has passed.

Answer

a. $\frac{\pi}{184}$; the voltage repeats every $\frac{\pi}{184}$ sec
b. Approximately 59 periods

68) [T] The number of hours of daylight in a northeast city is modeled by the function

$$N(t) = 12 + 3\sin\left[\frac{2\pi}{365}(t - 79)\right],$$

where t is the number of days after January 1.

a) Find the amplitude and period.

b) Determine the number of hours of daylight on the longest day of the year.

c) Determine the number of hours of daylight on the shortest day of the year.

d) Determine the number of hours of daylight 90 days after January 1.

e) Sketch the graph of the function for one period starting on January 1.

69) [T] Suppose that $T = 50 + 10\sin\left[\frac{\pi}{12}(t - 8)\right]$ is a mathematical model of the temperature (in degrees Fahrenheit) at t hours after midnight on a certain day of the week.

a) Determine the amplitude and period.

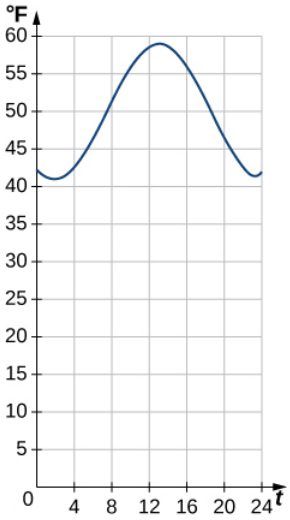
b) Find the temperature 7 hours after midnight.

c) At what time does $T = 60°$?

d) Sketch the graph of T over $0 \le t \le 24$.

Answer

a. Amplitude = 10; Period=24
b. $47.4°$F
c. 14 hours later, or 2 p.m.
d.



70) [T] The function $H(t) = 8\sin\left(\frac{\pi}{6}t\right)$ models the height H (in feet) of the tide t hours after midnight. Assume that $t = 0$ is midnight.

a) Find the amplitude and period.

b) Graph the function over one period.

c) What is the height of the tide at 4:30 a.m.?

Contributors

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1.4: Inverse Functions

Learning Objectives

- Determine the conditions for when a function has an inverse.
- Use the horizontal line test to recognize when a function is one-to-one.
- Find the inverse of a given function.
- Draw the graph of an inverse function.
- Evaluate inverse trigonometric functions.

An inverse function reverses the operation done by a particular function. In other words, whatever a function does, the inverse function undoes it. In this section, we define an inverse function formally and state the necessary conditions for an inverse function to exist. We examine how to find an inverse function and study the relationship between the graph of a function and the graph of its inverse. Then we apply these ideas to define and discuss properties of the inverse trigonometric functions.

Existence of an Inverse Function

We begin with an example. Given a function f and an output $y = f(x)$, we are often interested in finding what value or values x were mapped to y by f. For example, consider the function $f(x) = x^3 + 4$. Since any output $y = x^3 + 4$, we can solve this equation for x to find that the input is $x = \sqrt[3]{y-4}$. This equation defines x as a function of y. Denoting this function as f^{-1}, and writing $x = f^{-1}(y) = \sqrt[3]{y-4}$, we see that for any x in the domain of f, $f^{-1}f(x)) = f^{-1}(x^3 + 4) = x$. Thus, this new function, f^{-1}, "undid" what the original function f did. A function with this property is called the inverse function of the original function.

Definition: Inverse Functions

Given a function f with domain D and range R, its *inverse function* (if it exists) is the function f^{-1} with domain R and range D such that $f^{-1}(y) = x$ if and only if $f(x) = y$. In other words, for a function f and its inverse f^{-1},

$$f^{-1}(f(x)) = x$$

for all x in D and

$$f(f^{-1}(y)) = y$$

for all y in R.

Note that f^{-1} is read as "f inverse." Here, the -1 is not used as an exponent so

$$f^{-1}(x) \neq \frac{1}{f(x)}.$$

Figure 1.4.1shows the relationship between the domain and range of f and the domain and range of f^{-1}.

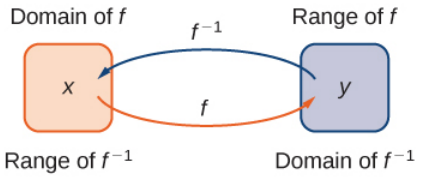


Figure 1.4.1: Given a function f and its inverse f^{-1}, $f^{-1}(y) = x$ if and only if $f(x) = y$. The range of f becomes the domain of f^{-1} and the domain of f becomes the range of f^{-1}.

Recall that a function has exactly one output for each input. Therefore, to define an inverse function, we need to map each input to exactly one output. For example, let's try to find the inverse function for $f(x) = x^2$. Solving the equation $y = x^2$ for x, we arrive at the equation $x = \pm\sqrt{y}$. This equation does not describe x as a function of y because there are two solutions to this equation for every $y > 0$. The problem with trying to find an inverse function for $f(x) = x^2$ is that two inputs are sent to the same output for each output $y > 0$. The function $f(x) = x^3 + 4$ discussed earlier did not have this problem. For that function, each input was sent to a different output. A function that sends each input to a different output is called a one-to-one function.

Definition: One-to-One functions

We say a function f is a one-to-one function if $f(x_1) \neq f(x_2)$ when $x_1 \neq x_2$.

One way to determine whether a function is one-to-one is by looking at its graph. If a function is one-to-one, then no two inputs can be sent to the same output. Therefore, if we draw a horizontal line anywhere in the xy-plane, according to the **horizontal line test**, it cannot intersect the graph more than once. We note that the horizontal line test is different from the vertical line test. The vertical line test determines whether a graph is the graph of a function. The horizontal line test determines whether a function is one-to-one (Figure 1.4.2).

Horizontal Line Test

A function f is one-to-one if and only if every horizontal line intersects the graph of f no more than once.

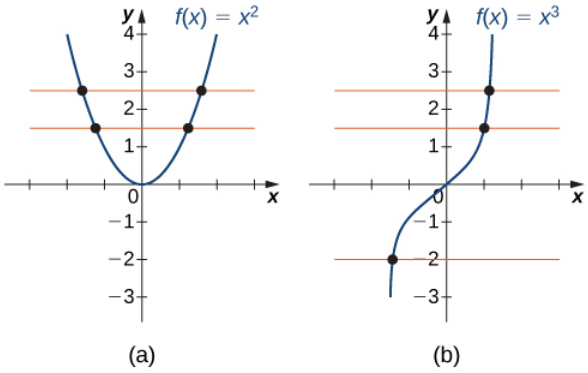
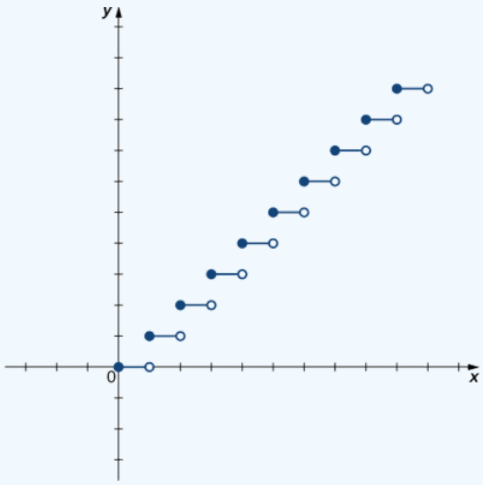


Figure 1.4.2: (a) The function $f(x) = x^2$ is not one-to-one because it fails the horizontal line test. (b) The function $f(x) = x^3$ is one-to-one because it passes the horizontal line test.

Example 1.4.1: Determining Whether a Function Is One-to-One

For each of the following functions, use the horizontal line test to determine whether it is one-to-one.

a)



b)

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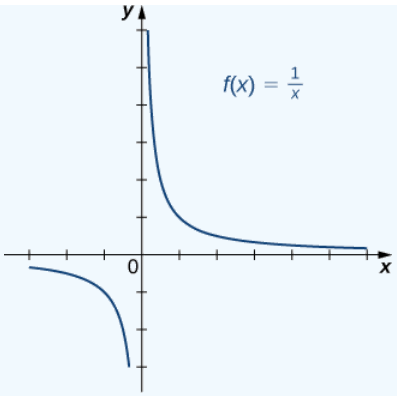
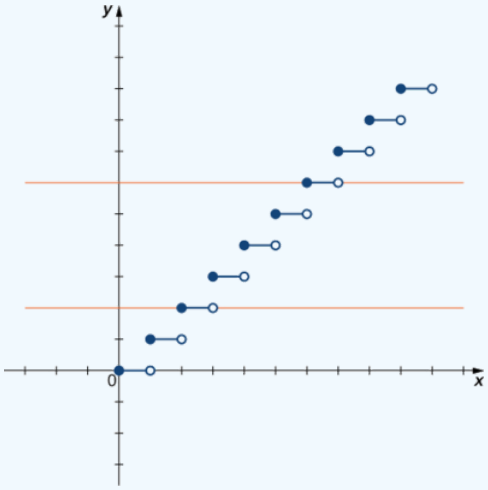
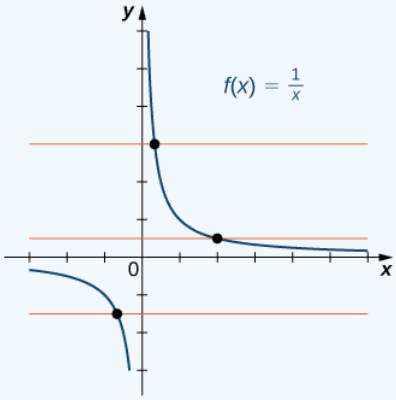


Solution

a) Since the horizontal line $y = n$ for any integer $n \geq 0$ intersects the graph more than once, this function is not one-to-one.

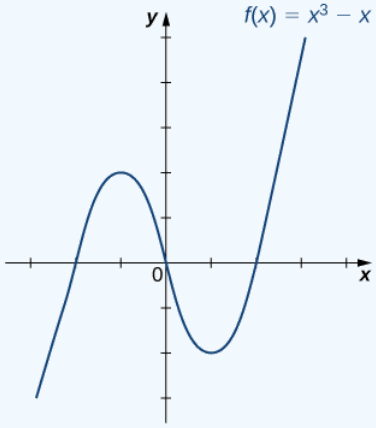


b) Since every horizontal line intersects the graph once (at most), this function is one-to-one.



? Exercise 1.4.1

Is the function f graphed in the following image one-to-one?



Solution

Use the horizontal line test.

Answer

No

Finding a Function's Inverse

We can now consider one-to-one functions and show how to find their inverses. Recall that a function maps elements in the domain of f to elements in the range of f. The inverse function maps each element from the range of f back to its corresponding element from the domain of f. Therefore, to find the inverse function of a one-to-one function f, given any y in the range of f, we need to determine which x in the domain of f satisfies $f(x) = y$. Since f is one-to-one, there is exactly one such value x. We can find that value x by solving the equation $f(x) = y$ for x. Doing so, we are able to write x as a function of y where the domain of this function is the range of f and the range of this new function is the domain of f. Consequently, this function is the inverse of f, and we write $x = f^{-1}(y)$. Since we typically use the variable x to denote the independent variable and y to denote the dependent variable, we often interchange the roles of x and y, and write $y = f^{-1}(x)$. Representing the inverse function in this way is also helpful later when we graph a function f and its inverse f^{-1} on the same axes.

Problem-Solving Strategy: Finding an Inverse Function

1. Solve the equation $y = f(x)$ for x.
2. Interchange the variables x and y and write $y = f^{-1}(x)$.

Example 1.4.2: Finding an Inverse Function

Find the inverse for the function $f(x) = 3x - 4$. State the domain and range of the inverse function. Verify that $f^{-1}(f(x)) = x$.

Solution

Follow the steps outlined in the strategy.

Step 1. If $y = 3x - 4$, then $3x = y + 4$ and $x = \frac{1}{3}y + \frac{4}{3}$.

Step 2. Rewrite as $y = \frac{1}{3}x + \frac{4}{3}$ and let $y = f^{-1}(x)$.Therefore, $f^{-1}(x) = \frac{1}{3}x + \frac{4}{3}$.

Since the domain of f is $(-\infty, \infty)$, the range of f^{-1} is $(-\infty, \infty)$. Since the range of f is $(-\infty, \infty)$, the domain of f^{-1} is $(-\infty, \infty)$.

You can verify that $f^{-1}(f(x)) = x$ by writing

$$f^{-1}(f(x)) = f^{-1}(3x - 4) = \tfrac{1}{3}(3x - 4) + \tfrac{4}{3} = x - \tfrac{4}{3} + \tfrac{4}{3} = x.$$

Note that for $f^{-1}(x)$ to be the inverse of $f(x)$, both $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$ for all x in the domain of the inside function.

Exercise 1.4.2

Find the inverse of the function $f(x) = 3x/(x-2)$. State the domain and range of the inverse function.

Hint

Use the Problem-Solving Strategy for finding inverse functions.

Answer

$f^{-1}(x) = \dfrac{2x}{x-3}$. The domain of f^{-1} is $\{x \mid x \neq 3\}$. The range of f^{-1} is $\{y \mid y \neq 2\}$.

Graphing Inverse Functions

Let's consider the relationship between the graph of a function f and the graph of its inverse. Consider the graph of f shown in Figure 1.4.3 and a point (a, b) on the graph. Since $b = f(a)$, then $f^{-1}(b) = a$. Therefore, when we graph f^{-1}, the point (b, a) is on the graph. As a result, the graph of f^{-1} is a reflection of the graph of f about the line $y = x$.

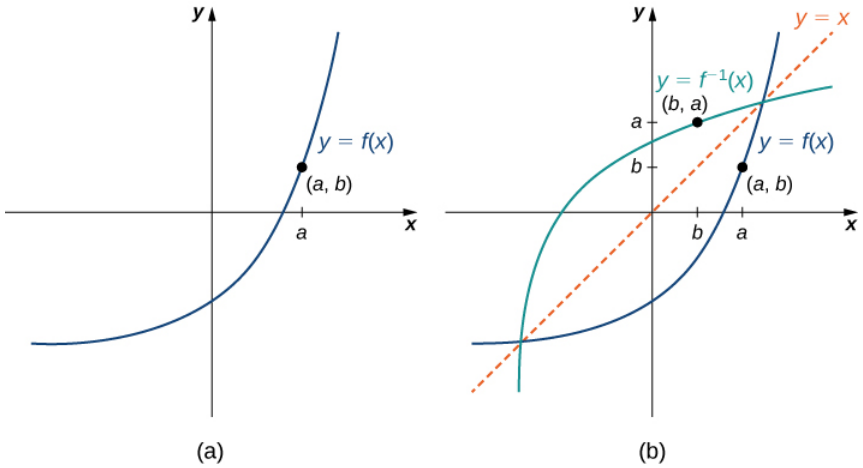
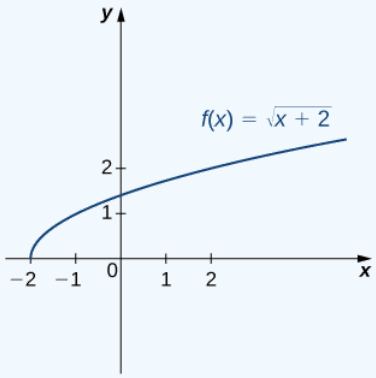


Figure 1.4.3: (a) The graph of this function f shows point (a, b) on the graph of f. (b) Since (a, b) is on the graph of f, the point (b, a) is on the graph of f^{-1}. The graph of f^{-1} is a reflection of the graph of f about the line $y = x$.

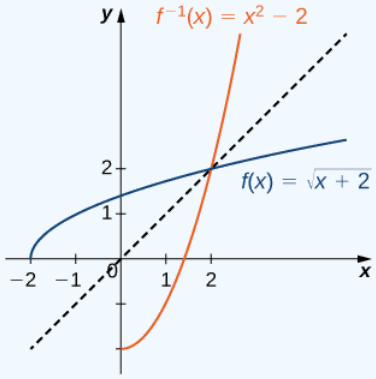
Example 1.4.3: Sketching Graphs of Inverse Functions

For the graph of f in the following image, sketch a graph of f^{-1} by sketching the line $y = x$ and using symmetry. Identify the domain and range of f^{-1}.



Solution

Reflect the graph about the line $y = x$. The domain of f^{-1} is $[0, \infty)$. The range of f^{-1} is $[-2, \infty)$. By using the preceding strategy for finding inverse functions, we can verify that the inverse function is $f^{-1}(x) = x^2 - 2$, as shown in the graph.



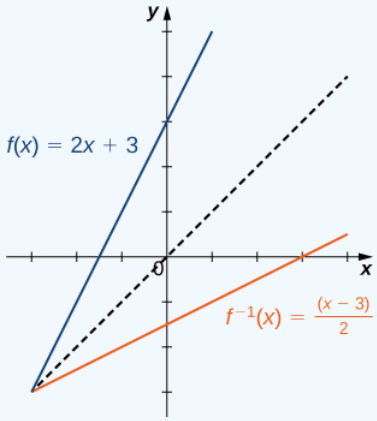
Exercise 1.4.3

Sketch the graph of $f(x) = 2x + 3$ and the graph of its inverse using the symmetry property of inverse functions.

Hint

The graphs are symmetric about the line $y = x$

Answer



Restricting Domains

As we have seen, $f(x) = x^2$ does not have an inverse function because it is not one-to-one. However, we can choose a subset of the domain of f such that the function is one-to-one. This subset is called a **restricted domain**. By restricting the domain of f, we can define a new function g such that the domain of g is the restricted domain of f and $g(x) = f(x)$ for all x in the domain of g. Then we can define an inverse function for g on that domain. For example, since $f(x) = x^2$ is one-to-one on the interval $[0, \infty)$, we can define a new function g such that the domain of g is $[0, \infty)$ and $g(x) = x^2$ for all x in its domain. Since g is a one-to-one function, it has an inverse function, given by the formula $g^{-1}(x) = \sqrt{x}$. On the other hand, the function $f(x) = x^2$ is also one-to-one on the domain $(-\infty, 0]$. Therefore, we could also define a new function h such that the domain of h is $(-\infty, 0]$ and $h(x) = x^2$ for all x in the domain of h. Then h is a one-to-one function and must also have an inverse. Its inverse is given by the formula $h^{-1}(x) = -\sqrt{x}$ (Figure 1.4.4).

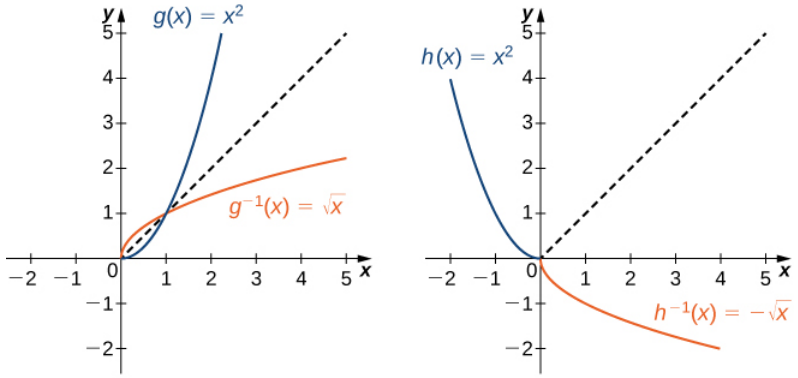


Figure 1.4.4: (a) For $g(x) = x^2$ restricted to $[0, \infty)$, $g^{-1}(x) = \sqrt{x}$. (b) For $h(x) = x^2$ restricted to $(-\infty, 0]$, $h^{-1}(x) = -\sqrt{x}$.

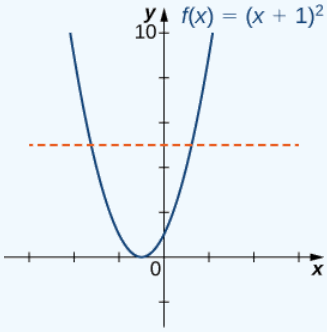
Example 1.4.4: Restricting the Domain

Consider the function $f(x) = (x + 1)^2$.

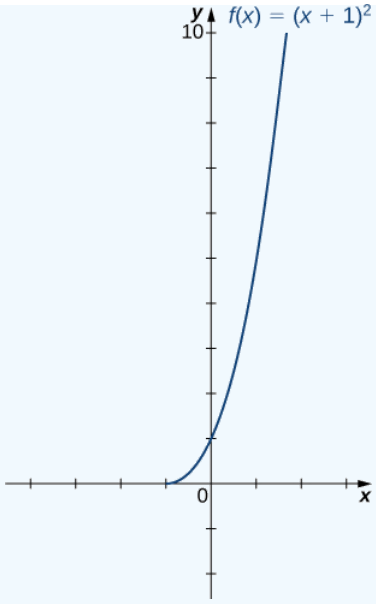
a. Sketch the graph of f and use the horizontal line test to show that f is not one-to-one.
b. Show that f is one-to-one on the restricted domain $[-1, \infty)$. Determine the domain and range for the inverse of f on this restricted domain and find a formula for f^{-1}.

Solution

a) The graph of f is the graph of $y = x^2$ shifted left 1 unit. Since there exists a horizontal line intersecting the graph more than once, f is not one-to-one.



b) On the interval $[-1, \infty)$, f is one-to-one.



The domain and range of f^{-1} are given by the range and domain of f, respectively. Therefore, the domain of f^{-1} is $[0,\infty)$ and the range of f^{-1} is $[-1,\infty)$. To find a formula for f^{-1}, solve the equation $y=(x+1)^2$ for x. If $y=(x+1)^2$, then $x=-1\pm\sqrt{y}$. Since we are restricting the domain to the interval where $x\geq -1$, we need $\pm\sqrt{y}\geq 0$. Therefore, $x=-1+\sqrt{y}$. Interchanging x and y, we write $y=-1+\sqrt{x}$ and conclude that $f^{-1}(x)=-1+\sqrt{x}$.

Exercise 1.4.4

Consider $f(x)=1/x^2$ restricted to the domain $(-\infty,0)$. Verify that f is one-to-one on this domain. Determine the domain and range of the inverse of f and find a formula for f^{-1}.

Hint

The domain and range of f^{-1} is given by the range and domain of f, respectively. To find f^{-1}, solve $y=1/x^2$ for x.

Answer

The domain of f^{-1} is $(0,\infty)$. The range of f^{-1} is $(-\infty,0)$. The inverse function is given by the formula $f^{-1}(x)=-1/\sqrt{x}$.

Inverse Trigonometric Functions

The six basic trigonometric functions are periodic, and therefore they are not one-to-one. However, if we restrict the domain of a trigonometric function to an interval where it is one-to-one, we can define its inverse. Consider the sine function. The sine function is one-to-one on an infinite number of intervals, but the standard convention is to restrict the domain to the interval $[-\frac{\pi}{2},\frac{\pi}{2}]$. By doing so, we define the inverse sine function on the domain $[-1,1]$ such that for any x in the interval $[-1,1]$, the inverse sine function tells us which angle θ in the interval $[-\frac{\pi}{2},\frac{\pi}{2}]$ satisfies $\sin\theta=x$. Similarly, we can restrict the domains of the other trigonometric functions to define **inverse trigonometric functions**, which are functions that tell us which angle in a certain interval has a specified trigonometric value.

Definition: inverse trigonometric functions

The inverse sine function, denoted $\sin^{-1}$ or $\arcsin$, and the inverse cosine function, denoted $\cos^{-1}$ or $\arccos$, are defined on the domain $D=\{x|-1\leq x\leq 1\}$ as follows:

$$\sin^{-1}(x)=y$$

- if and only if $\sin(y)=x$ and $-\frac{\pi}{2}\leq y\leq\frac{\pi}{2}$;

$$\cos^{-1}(x)=y$$

- if and only if $\cos(y) = x$ and $0 \le y \le \pi$.

The inverse tangent function, denoted $\tan^{-1}$ or $\arctan$, and inverse cotangent function, denoted $\cot^{-1}$ or arccot, are defined on the domain $D = \{x | -\infty < x < \infty\}$ as follows:

$$\tan^{-1}(x) = y$$

- if and only if $\tan(y) = x$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$;

$$\cot^{-1}(x) = y$$

- if and only if $\cot(y) = x$ and $0 < y < \pi$.

The inverse cosecant function, denoted $\csc^{-1}$ or arccsc, and inverse secant function, denoted $\sec^{-1}$ or arcsec, are defined on the domain $D = \{x \,|\, |x| \ge 1\}$ as follows:

$$\csc^{-1}(x) = y$$

- if and only if $\csc(y) = x$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $y \ne 0$;

$$\sec^{-1}(x) = y$$

- if and only if $\sec(y) = x$ and$0 \le y \le \pi$, $y \ne \pi/2$.

To graph the inverse trigonometric functions, we use the graphs of the trigonometric functions restricted to the domains defined earlier and reflect the graphs about the line $y = x$ (Figure 1.4.5).

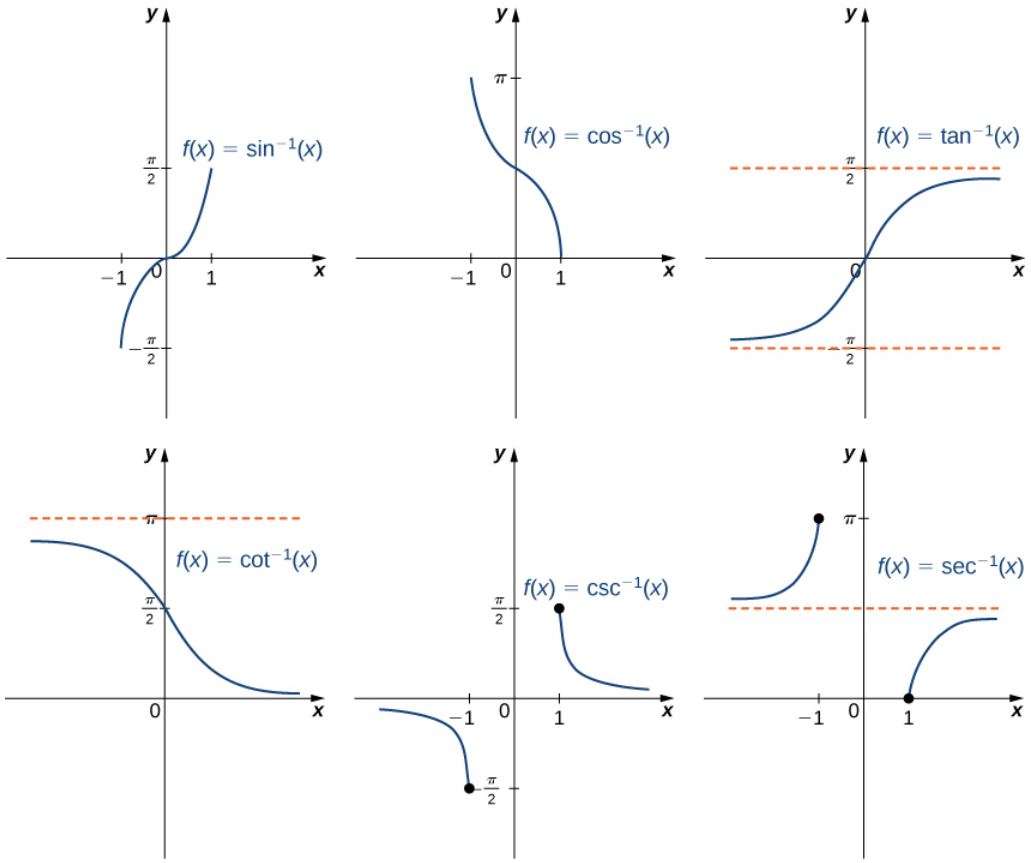


Figure 1.4.5: The graph of each of the inverse trigonometric functions is a reflection about the line $y = x$ of the corresponding restricted trigonometric function.

When evaluating an inverse trigonometric function, the output is an angle. For example, to evaluate $\cos^{-1}\left(\frac{1}{2}\right)$, we need to find an angle θ such that $\cos\theta = \frac{1}{2}$. Clearly, many angles have this property. However, given the definition of $\cos^{-1}$, we need the angle θ that not only solves this equation, but also lies in the interval $[0, \pi]$. We conclude that $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

We now consider a composition of a trigonometric function and its inverse. For example, consider the two expressions $\sin\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$ and $\sin^{-1}(\sin(\pi))$.

For the first one, we simplify as follows:

$$\sin\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

For the second one, we have

$$\sin^{-1}(\sin(\pi)) = \sin^{-1}(0) = 0.$$

The inverse function is supposed to "undo" the original function, so why isn't $\sin^{-1}(\sin(\pi)) = \pi$? Recalling our definition of inverse functions, a function f and its inverse f^{-1} satisfy the conditions $f(f^{-1}(y)) = y$ for all y in the domain of f^{-1} and $f^{-1}(f(x)) = x$ for all x in the domain of f, so what happened here? The issue is that the inverse sine function, $\sin^{-1}$, is the inverse of the restricted sine function defined on the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Therefore, for x in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, it is true that $\sin^{-1}(\sin x) = x$. However, for values of x outside this interval, the equation does not hold, even though $\sin^{-1}(\sin x)$ is defined for all real numbers x.

What about $\sin(\sin^{-1} y)$? Does that have a similar issue? The answer is no. Since the domain of $\sin^{-1}$ is the interval $[-1, 1]$, we conclude that $\sin\left(\sin^{-1} y\right) = y$ if $-1 \leq y \leq 1$ and the expression is not defined for other values of y. To summarize,

$$\sin(\sin^{-1} y) = y \text{ if } -1 \leq y \leq 1$$

and

$$\sin^{-1}(\sin x) = x \text{ if } -\tfrac{\pi}{2} \leq x \leq \tfrac{\pi}{2}.$$

Similarly, for the cosine function,

$$\cos(\cos^{-1} y) = y \text{ if } -1 \leq y \leq 1$$

and

$$\cos^{-1}(\cos x) = x \text{ if } 0 \leq x \leq \pi.$$

Similar properties hold for the other trigonometric f*unctions and their inverses.*

Example 1.4.5: Evaluating Expressions Involving Inverse Trigonometric Functions

Evaluate each of the following expressions.

a. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
b. $\tan\left(\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right)$
c. $\cos^{-1}\left(\cos\left(\frac{5\pi}{4}\right)\right)$
d. $\sin^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right)$

Solution

a. Evaluating $\sin^{-1}(-\sqrt{3}/2)$ is equivalent to finding the angle θ such that $\sin\theta = -\sqrt{3}/2$ and $-\pi/2 \leq \theta \leq \pi/2$. The angle $\theta = -\pi/3$ satisfies these two conditions. Therefore, $\sin^{-1}(-\sqrt{3}/2) = -\pi/3$.
b. First we use the fact that $\tan^{-1}(-1/\sqrt{3}) = -\pi/6$. Then $\tan(-\pi/6) = -1/\sqrt{3}$. Therefore, $\tan(\tan^{-1}(-1/\sqrt{3})) = -1/\sqrt{3}$.
c. To evaluate $\cos^{-1}(\cos(5\pi/4))$,first use the fact that $\cos(5\pi/4) = -\sqrt{2}/2$. Then we need to find the angle θ such that $\cos(\theta) = -\sqrt{2}/2$ and $0 \leq \theta \leq \pi$. Since $3\pi/4$ satisfies both these conditions, we have $\cos^{-1}(\cos(5\pi/4)) = \cos^{-1}(-\sqrt{2}/2)) = 3\pi/4$.
d. Since $\cos(2\pi/3) = -1/2$, we need to evaluate $\sin^{-1}(-1/2)$. That is, we need to find the angle θ such that $\sin(\theta) = -1/2$ and $-\pi/2 \leq \theta \leq \pi/2$. Since $-\pi/6$ satisfies both these conditions, we can conclude that $\sin^{-1}(\cos(2\pi/3)) = \sin^{-1}(-1/2) = -\pi/6$.

The Maximum Value of a Function

In many areas of science, engineering, and mathematics, it is useful to know the maximum value a function can obtain, even if we don't know its exact value at a given instant. For instance, if we have a function describing the strength of a roof beam, we would want to know the maximum weight the beam can support without breaking. If we have a function that describes the

speed of a train, we would want to know its maximum speed before it jumps off the rails. Safe design often depends on knowing maximum values.

This project describes a simple example of a function with a maximum value that depends on two equation coefficients. We will see that maximum values can depend on several factors other than the independent variable x.

1. Consider the graph in Figure 1.4.6 of the function $y = \sin x + \cos x$. Describe its overall shape. Is it periodic? How do you know?

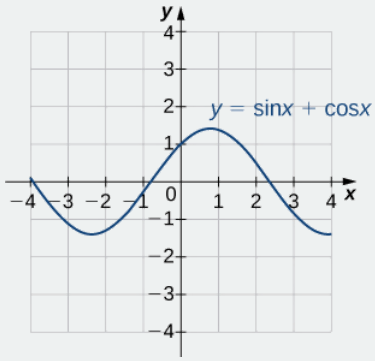


Figure 1.4.6: The graph of $y = \sin x + \cos x$.

Using a graphing calculator or other graphing device, estimate the x- and y-values of the maximum point for the graph (the first such point where $x > 0$). It may be helpful to express the x-value as a multiple of π.

2. Now consider other graphs of the form $y = A\sin x + B\cos x$ for various values of A and B. Sketch the graph when $A = 2$ and $B = 1$, and find the x- and y-values for the maximum point. (Remember to express the x-value as a multiple of π, if possible.) Has it moved?

3. Repeat for $A = 1,\ B = 2$. Is there any relationship to what you found in part (2)?

4. Complete the following table, adding a few choices of your own for A and B :

A	B	x	y	A	B	x	y
0	1			3	4		
1	0			4	3		
1	1			$\sqrt{3}$	1		
1	2			1	$\sqrt{3}$		
2	1			12	5		
2	2			5	12		

5. Try to figure out the formula for the y-values.

6. The formula for the x-values is a little harder. The most helpful points from the table are $(1, 1)$, $(1, \sqrt{3})$, $(\sqrt{3}, 1)$. (Hint: Consider inverse trigonometric functions.)

7. If you found formulas for parts (5) and (6), show that they work together. That is, substitute the x-value formula you found into $y = A\sin x + B\cos x$ and simplify it to arrive at the y-value formula you found.

Key Concepts

- For a function to have an inverse, the function must be one-to-one. Given the graph of a function, we can determine whether the function is one-to-one by using the horizontal line test.
- If a function is not one-to-one, we can restrict the domain to a smaller domain where the function is one-to-one and then define the inverse of the function on the smaller domain.
- For a function f and its inverse f^{-1}, $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1} and $f^{-1}(f(x)) = x$ for all x in the domain of f.

- Since the trigonometric functions are periodic, we need to restrict their domains to define the inverse trigonometric functions.
- The graph of a function f and its inverse f^{-1} are symmetric about the line $y = x$.

Key Equations

- **Inverse function**

  $f^{-1}(f(x)) = x$ for all x in D, and $f(f^{-1}(y)) = y$ for all y in R.

Glossary

horizontal line test
a function f is one-to-one if and only if every horizontal line intersects the graph of f, at most, once

inverse function
for a function f, the inverse function f^{-1} satisfies $f^{-1}(y) = x$ if $f(x) = y$

inverse trigonometric functions
the inverses of the trigonometric functions are defined on restricted domains where they are one-to-one functions

one-to-one function
a function f is one-to-one if $f(x_1) \neq f(x_2)$ if $x_1 \neq x_2$

restricted domain
a subset of the domain of a function f

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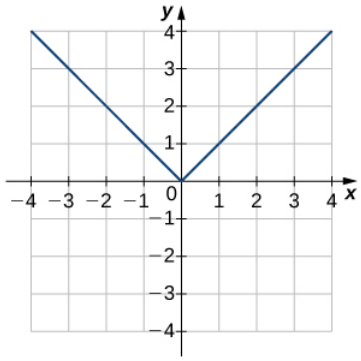
1.4E: Exercises for Section 1.4

In exercises 1 - 6, use the horizontal line test to determine whether each of the given graphs is one-to-one.

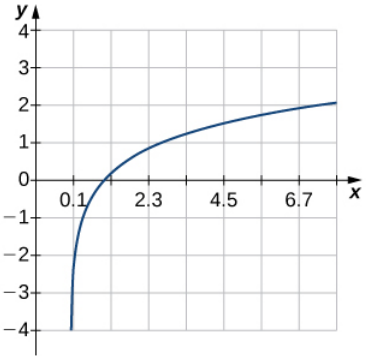
1)



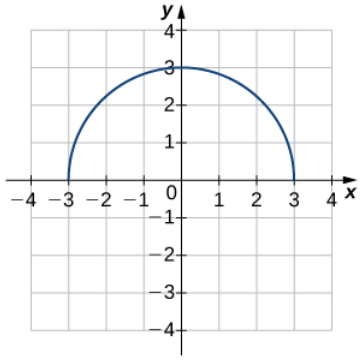
Answer

Not one-to-one

2)



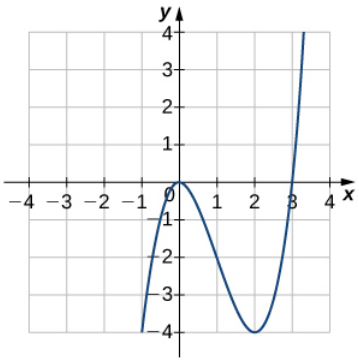
3)



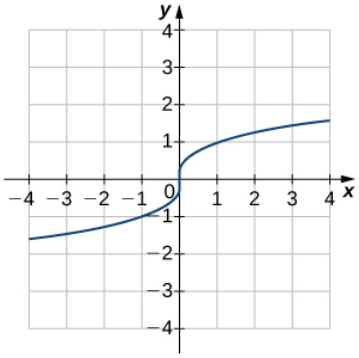
Answer

Not one-to-one

4)



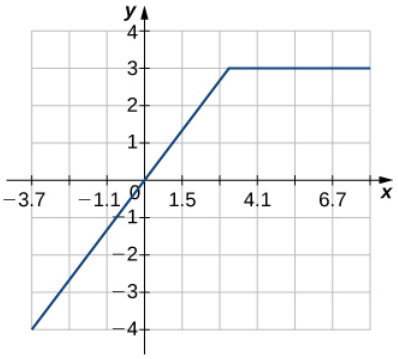
5)



Answer

One-to-one

6)



In exercises 7 - 12,

a. find the inverse function, and

b. find the domain and range of the inverse function.

7) $f(x) = x^2 - 4, \quad x \geq 0$

Answer

a. $f^{-1}(x) = \sqrt{x+4}$

b. Domain: $x \geq -4$, Range: $y \geq 0$

8) $f(x) = \sqrt[3]{x-4}$

9) $f(x) = x^3 + 1$

Answer

a. $f^{-1}(x) = \sqrt[3]{x-1}$

b. Domain: all real numbers, Range: all real numbers

10) $f(x) = (x-1)^2, \quad x \leq 1$

11) $f(x) = \sqrt{x-1}$

Answer

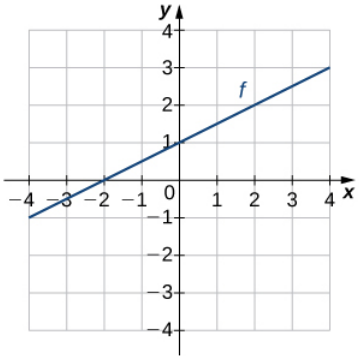
a. $f^{-1}(x) = x^2 + 1$,

b. Domain: $x \geq 0$, Range: $y \geq 1$

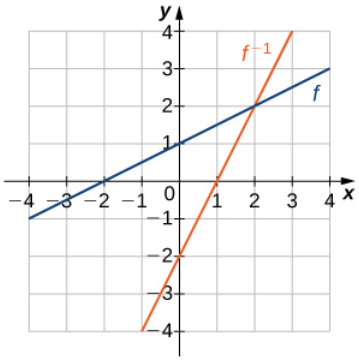
12) $f(x) = \dfrac{1}{x+2}$

In exercises 13 - 16, use the graph of f to sketch the graph of its inverse function.

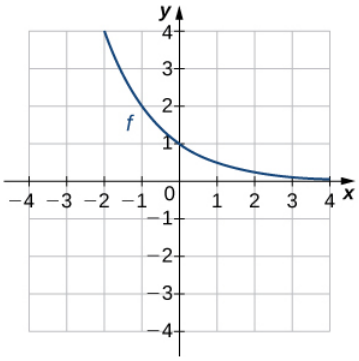
13)



Answer



14)



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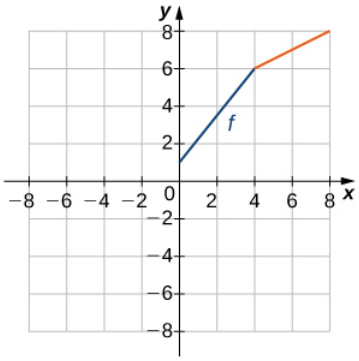
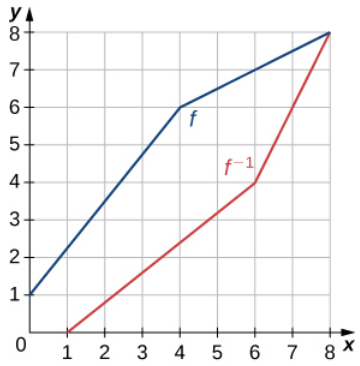
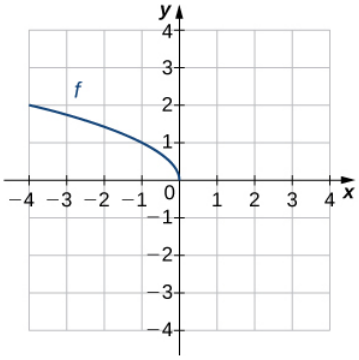
15)



Answer



16)



In exercises 17 - 24, use composition to determine which pairs of functions are inverses.

17) $f(x) = 8x, \quad g(x) = \frac{x}{8}$

Answer

These are inverses.

18) $f(x) = 8x + 3, \quad g(x) = \frac{x - 3}{8}$

19) $f(x) = 5x - 7, \quad g(x) = \frac{x + 5}{7}$

Answer

These are not inverses.

20) $f(x) = \frac{2}{3}x + 2, \quad g(x) = \frac{3}{2}x + 3$

21) $f(x) = \dfrac{1}{x-1},\ x \neq 1, \quad g(x) = \dfrac{1}{x} + 1,\ x \neq 0$

Answer

These are inverses.

22) $f(x) = x^3 + 1, \quad g(x) = (x-1)^{1/3}$

23) $f(x) = x^2 + 2x + 1,\ x \geq -1, \quad g(x) = -1 + \sqrt{x},\ x \geq 0$

Answer

These are inverses.

24) $f(x) = \sqrt{4 - x^2},\ 0 \leq x \leq 2, \quad g(x) = \sqrt{4 - x^2},\ 0 \leq x \leq 2$

In exercises 25 - 33, evaluate the functions. Give the exact value.

25) $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$

Answer

$\frac{\pi}{6}$

26) $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

27) $\cot^{-1}(1)$

Answer

$\frac{\pi}{4}$

28) $\sin^{-1}(-1)$

29) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Answer

$\frac{\pi}{6}$

30) $\cos\left(\tan^{-1}(\sqrt{3})\right)$

31) $\sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$

Answer

$\frac{\sqrt{2}}{2}$

32) $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$

33) $\tan^{-1}\left(\tan\left(-\frac{\pi}{6}\right)\right)$

Answer

$-\frac{\pi}{6}$

34) The function $C = T(F) = (5/9)(F - 32)$ converts degrees Fahrenheit to degrees Celsius.

a) Find the inverse function $F = T^{-1}(C)$

b) What is the inverse function used for?

35) [T] The velocity V (in centimeters per second) of blood in an artery at a distance x cm from the center of the artery can be modeled by the function $V = f(x) = 500(0.04 - x^2)$ for $0 \leq x \leq 0.2$.

a) Find $x = f^{-1}(V)$.

b) Interpret what the inverse function is used for.

c) Find the distance from the center of an artery with a velocity of 15 cm/sec, 10 cm/sec, and 5 cm/sec.

Answer

a. $x = f^{-1}(V) = \sqrt{0.04 - \frac{V}{500}}$

b. The inverse function determines the distance from the center of the artery at which blood is flowing with velocity V.

c. 0.1 cm; 0.14 cm; 0.17 cm

36) A function that converts dress sizes in the United States to those in Europe is given by $D(x) = 2x + 24$.

a) Find the European dress sizes that correspond to sizes 6, 8, 10, and 12 in the United States.

b) Find the function that converts European dress sizes to U.S. dress sizes.

c) Use part b. to find the dress sizes in the United States that correspond to 46, 52, 62, and 70.

37) [T] The cost to remove a toxin from a lake is modeled by the function $C(p) = \frac{75p}{85 - p}$, where C is the cost (in thousands of dollars) and p is the amount of toxin in a small lake (measured in parts per billion [ppb]). This model is valid only when the amount of toxin is less than 85 ppb.

a) Find the cost to remove 25 ppb, 40 ppb, and 50 ppb of the toxin from the lake.

b) Find the inverse function.

c) Use part b. to determine how much of the toxin is removed for $50,000.

Answer

a. $31,250, $66,667, $107,143

b. $p = \frac{85C}{C + 75}$

c. 34 ppb

38) [T] A race car is accelerating at a velocity given by $v(t) = \frac{25}{4}t + 54$,

where v is the velocity (in feet per second) at time t.

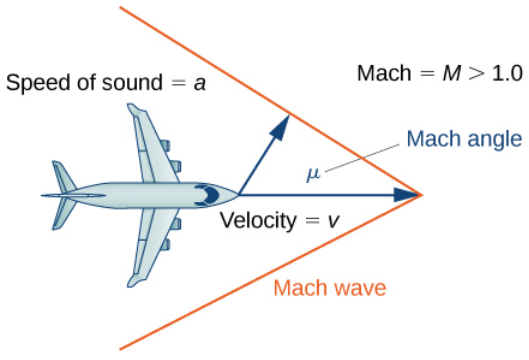
a) Find the velocity of the car at 10 sec.

b) Find the inverse function.

c) Use part b. to determine how long it takes for the car to reach a speed of 150 ft/sec.

39) [T] An airplane's Mach number M is the ratio of its speed to the speed of sound. When a plane is flying at a constant altitude, then its Mach angle is given by $\mu = 2\sin^{-1}\left(\frac{1}{M}\right)$.

Find the Mach angle (to the nearest degree) for the following Mach numbers.



a. $\mu = 1.4$

b. $\mu = 2.8$

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c. $\mu = 4.3$

Answer

a. $\sim 92°$ b. $\sim 42°$ c. $\sim 27°$

40) [T] Using $\mu = 2\sin^{-1}\left(\frac{1}{M}\right)$, find the Mach number M for the following angles.

a. $\mu = \frac{\pi}{6}$

b. $\mu = \frac{2\pi}{7}$

c. $\mu = \frac{3\pi}{8}$

41) [T] The temperature (in degrees Celsius) of a city in the northern United States can be modeled by the function

$$T(x) = 5 + 18\sin\left[\tfrac{\pi}{6}(x - 4.6)\right],$$

where x is time in months and $x = 1.00$ corresponds to January 1. Determine the month and day when the temperature is $21°C$.

Answer

$x \approx 6.69, 8.51$; so, the temperature occurs on June 21 and August 15

42) [T] The depth (in feet) of water at a dock changes with the rise and fall of tides. It is modeled by the function $D(t) = 5\sin\left(\frac{\pi}{6}t - \frac{7\pi}{6}\right) + 8$, where t is the number of hours after midnight. Determine the first time after midnight when the depth is 11.75 ft.

43) [T] An object moving in simple harmonic motion is modeled by the function $s(t) = -6\cos\left(\dfrac{\pi t}{2}\right)$, where s is measured in inches and t is measured in seconds. Determine the first time when the distance moved is 4.5 in.

Answer

~ 1.5 sec

44) [T] A local art gallery has a portrait 3 ft in height that is hung 2.5 ft above the eye level of an average person. The viewing angle θ can be modeled by the function $\theta = \tan^{-1}\frac{5.5}{x} - \tan^{-1}\frac{2.5}{x}$, where x is the distance (in feet) from the portrait. Find the viewing angle when a person is 4 ft from the portrait.

45) [T] Use a calculator to evaluate $\tan^{-1}(\tan(2.1))$ and $\cos^{-1}(\cos(2.1))$. Explain the results of each.

Answer

$\tan^{-1}(\tan(2.1)) \approx -1.0416$; the expression does not equal 2.1 since $2.1 > 1.57 = \frac{\pi}{2}$—in other words, it is not in the restricted domain of $\tan x$. $\cos^{-1}(\cos(2.1)) = 2.1$, since 2.1 is in the restricted domain of $\cos x$.

46) [T] Use a calculator to evaluate $\sin(\sin^{-1}(-2))$ and $\tan(\tan^{-1}(-2))$. Explain the results of each.

Contributors

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1.5: Exponential and Logarithmic Functions

Learning Objectives

- Identify the form of an exponential function.
- Explain the difference between the graphs of x^b and b^x.
- Recognize the significance of the number e.
- Identify the form of a logarithmic function.
- Explain the relationship between exponential and logarithmic functions.
- Describe how to calculate a logarithm to a different base.
- Identify the hyperbolic functions, their graphs, and basic identities.

In this section we examine exponential and logarithmic functions. We use the properties of these functions to solve equations involving exponential or logarithmic terms, and we study the meaning and importance of the number e. We also define hyperbolic and inverse hyperbolic functions, which involve combinations of exponential and logarithmic functions. (Note that we present alternative definitions of exponential and logarithmic functions in the chapter Applications of Integrations, and prove that the functions have the same properties with either definition.)

Exponential Functions

Exponential functions arise in many applications. One common example is **population growth.** For example, if a population starts with P_0 individuals and then grows at an annual rate of 2%, its population after 1 year is

$$P(1) = P_0 + 0.02P_0 = P_0(1 + 0.02) = P_0(1.02).$$

Its population after 2 years is

$$P(2) = P(1) + 0.02P(1) = P(1)(1.02) = P_0(1.02)^2.$$

In general, its population after t years is

$$P(t) = P_0(1.02)^t,$$

which is an exponential function. More generally, any function of the form $f(x) = b^x$, where $b > 0$, $b \neq 1$, is an exponential function with **base** b and **exponent** x. Exponential functions have constant bases and variable exponents. Note that a function of the form $f(x) = x^b$ for some constant b is not an exponential function but a power function.

To see the difference between an exponential function and a power function, we compare the functions $y = x^2$ and $y = 2^x$. In Table 1.5.1, we see that both 2^x and x^2 approach infinity as $x \to \infty$. Eventually, however, 2^x becomes larger than x^2 and grows more rapidly as $x \to \infty$. In the opposite direction, as $x \to -\infty$, $x^2 \to \infty$, whereas $2^x \to 0$. The line $y = 0$ is a horizontal asymptote for $y = 2^x$.

Table 1.5.1

x	-3	-2	-1	0	1	2	3	4	5	6
x^2	9	4	1	0	1	4	9	16	25	36
2^x	1/8	1/4	1/2	1	2	4	8	16	32	64

In Figure 1.5.1, we graph both $y = x^2$ and $y = 2^x$ to show how the graphs differ.

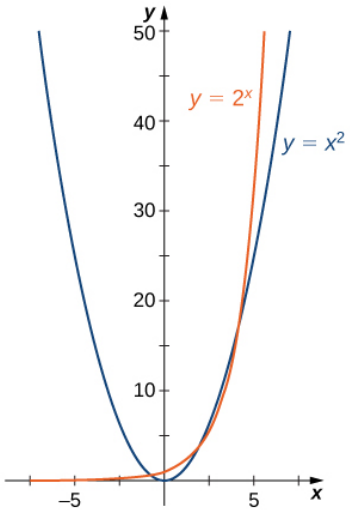

Figure 1.5.1: Both 2^x and x^2 approach infinity as $x \to \infty$, but 2^x grows more rapidly than x^2. As $x \to -\infty$, $x^2 \to \infty$, whereas $2^x \to 0$.

Evaluating Exponential Functions

Recall the properties of exponents: If x is a positive integer, then we define $b^x = b \cdot b \cdots b$ (with x factors of b). If x is a negative integer, then $x = -y$ for some positive integer y, and we define $b^x = b^{-y} = 1/b^y$. Also, b^0 is defined to be 1. If x is a rational number, then $x = p/q$, where p and q are integers and $b^x = b^{p/q} = \sqrt[q]{b^p}$. For example, $9^{3/2} = \sqrt{9^3} = \left(\sqrt{9}\right)^3 = 27$. However, how is b^x defined if x is an irrational number? For example, what do we mean by $2^{\sqrt{2}}$? This is too complex a question for us to answer fully right now; however, we can make an approximation.

Table 1.5.2: Values of 2^x for a List of Rational Numbers Approximating $\sqrt{2}$

x	1.4	1.41	1.414	1.4142	1.41421	1.414213
2^x	2.639	2.65737	2.66475	2.665119	2.665138	2.665143

In Table 1.5.2, we list some rational numbers approaching $\sqrt{2}$, and the values of 2^x for each rational number x are presented as well. We claim that if we choose rational numbers x getting closer and closer to $\sqrt{2}$, the values of 2^x get closer and closer to some number L. We define that number L to be $2^{\sqrt{2}}$.

Example 1.5.1: Bacterial Growth

Suppose a particular population of bacteria is known to double in size every 4 hours. If a culture starts with 1000 bacteria, the number of bacteria after 4 hours is $n(4) = 1000 \cdot 2$. The number of bacteria after 8 hours is $n(8) = n(4) \cdot 2 = 1000 \cdot 2^2$. In general, the number of bacteria after $4m$ hours is $n(4m) = 1000 \cdot 2^m$. Letting $t = 4m$, we see that the number of bacteria after t hours is $n(t) = 1000 \cdot 2^{t/4}$. Find the number of bacteria after 6 hours, 10 hours, and 24 hours.

Solution

The number of bacteria after 6 hours is given by

$$n(6) = 1000 \cdot 2^{6/4} \approx 2828 \text{ bacteria.}$$

The number of bacteria after 10 hours is given by

$$n(10) = 1000 \cdot 2^{10/4} \approx 5657 \text{ bacteria.}$$

The number of bacteria after 24 hours is given by $n(24) = 1000 \cdot 2^6 = 64,000$ bacteria.

Exercise 1.5.1

Given the exponential function $f(x) = 100 \cdot 3^{x/2}$, evaluate $f(4)$ and $f(10)$.

Answer

$f(4) = 900$

$f(10) = 24,300.$

Graphing Exponential Functions

For any base $b > 0$, $b \neq 1$, the exponential function $f(x) = b^x$ is defined for all real numbers x and $b^x > 0$. Therefore, the domain of $f(x) = b^x$ is $(-\infty, \infty)$ and the range is $(0, \infty)$. To graph b^x, we note that for $b > 1$, b^x is increasing on $(-\infty, \infty)$ and $b^x \to \infty$ as $x \to \infty$, whereas $b^x \to 0$ as $x \to -\infty$. On the other hand, if $0 < b < 1$, $f(x) = b^x$ is decreasing on $(-\infty, \infty)$ and $b^x \to 0$ as $x \to \infty$ whereas $b^x \to \infty$ as $x \to -\infty$ (Figure 1.5.2).

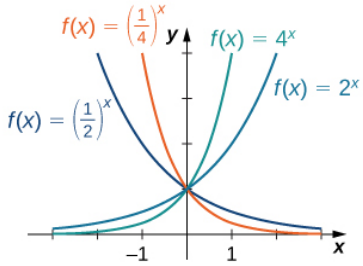

Figure 1.5.2: If $b > 1$, then b^x is increasing on $(-\infty, \infty)$. If $0 < b < 1$, then b^x is decreasing on $(-\infty, \infty)$.

Note that exponential functions satisfy the general laws of exponents. To remind you of these laws, we state them as rules.

Laws of Exponents

For any constants $a > 0$, $b > 0$, and for all x and y,

1. $b^x \cdot b^y = b^{x+y}$
2. $\dfrac{b^x}{b^y} = b^{x-y}$
3. $(b^x)^y = b^{xy}$
4. $(ab)^x = a^x b^x$
5. $\dfrac{a^x}{b^x} = \left(\dfrac{a}{b}\right)^x$

Example 1.5.2: Using the Laws of Exponents

Use the laws of exponents to simplify each of the following expressions.

a. $\dfrac{(2x^{2/3})^3}{(4x^{-1/3})^2}$

b. $\dfrac{(x^3y^{-1})^2}{(xy^2)^{-2}}$

Soution

a. We can simplify as follows:

$$\frac{(2x^{2/3})^3}{(4x^{-1/3})^2} = \frac{2^3(x^{2/3})^3}{4^2(x^{-1/3})^2} = \frac{8x^2}{16x^{-2/3}} = \frac{x^2x^{2/3}}{2} = \frac{x^{8/3}}{2}.$$

b. We can simplify as follows:

$$\frac{(x^3y^{-1})^2}{(xy^2)^{-2}} = \frac{(x^3)^2(y^{-1})^2}{x^{-2}(y^2)^{-2}} = \frac{x^6y^{-2}}{x^{-2}y^{-4}} = x^6x^2y^{-2}y^4 = x^8y^2.$$

https://math.libretexts.org/@go/page/2480

Exercise 1.5.2

Use the laws of exponents to simplify $\dfrac{6x^{-3}y^2}{12x^{-4}y^5}$.

Hint

$x^a/x^b = x^{a-b}$

Answer

$x/(2y^3)$

The Number e

A special type of exponential function appears frequently in real-world applications. To describe it, consider the following example of exponential growth, which arises from **compounding interest** in a savings account. Suppose a person invests P dollars in a savings account with an annual interest rate r, compounded annually. The amount of money after 1 year is

$$A(1) = P + rP = P(1 + r).$$

The amount of money after 2 years is

$$A(2) = A(1) + rA(1) = P(1 + r) + rP(1 + r) = P(1 + r)^2.$$

More generally, the amount after t years is

$$A(t) = P(1 + r)^t.$$

If the money is compounded 2 times per year, the amount of money after half a year is

$$A\left(\frac{1}{2}\right) = P + \left(\frac{r}{2}\right)P = P\left(1 + \left(\frac{r}{2}\right)\right).$$

The amount of money after 1 year is

$$A(1) = A\left(\frac{1}{2}\right) + \left(\frac{r}{2}\right)A\left(\frac{1}{2}\right) = P\left(1 + \frac{r}{2}\right) + \frac{r}{2}\left(\left(P(1 + \frac{r}{2}\right)\right) = P\left(1 + \frac{r}{2}\right)^2.$$

After t years, the amount of money in the account is

$$A(t) = P\left(1 + \frac{r}{2}\right)^{2t}.$$

More generally, if the money is compounded n times per year, the amount of money in the account after t years is given by the function

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}.$$

What happens as $n \to \infty$? To answer this question, we let $m = n/r$ and write

$$\left(1 + \frac{r}{n}\right)^{nt} = \left(1 + \frac{1}{m}\right)^{mrt},$$

and examine the behavior of $(1 + 1/m)^m$ as $m \to \infty$, using a table of values (Table 1.5.3).

Table 1.5.3: Values of $\left(1 + \dfrac{1}{m}\right)^m$ as $m \to \infty$

m	10	100	1000	10,000	100,000	1,000,000
$\left(1 + \dfrac{1}{m}\right)^m$	2.5937	2.7048	2.71692	2.71815	2.718268	2.718280

Looking at this table, it appears that $(1 + 1/m)^m$ is approaching a number between 2.7 and 2.8 as $m \to \infty$. In fact, $(1 + 1/m)^m$ does approach some number as $m \to \infty$. We call this number e. To six decimal places of accuracy,

$$e \approx 2.718282.$$

Leonhard Euler

The letter e was first used to represent this number by the Swiss mathematician Leonhard Euler during the 1720s. Although Euler did not discover the number, he showed many important connections between e and logarithmic functions. We still use the notation e today to honor Euler's work because it appears in many areas of mathematics and because we can use it in many practical applications.

Returning to our savings account example, we can conclude that if a person puts P dollars in an account at an annual interest rate r, compounded continuously, then $A(t) = Pe^{rt}$. This function may be familiar. Since functions involving base e arise often in applications, we call the function $f(x) = e^x$ the **natural exponential function**. Not only is this function interesting because of the definition of the number e, but also, as discussed next, its graph has an important property.

Since $e > 1$, we know $f(x) = e^x$ is increasing on $(-\infty, \infty)$. In Figure 1.5.3, we show a graph of $f(x) = e^x$ along with a tangent line to the graph of f at $x = 0$. We give a precise definition of tangent line in the next chapter; but, informally, we say a tangent line to a graph of f at $x = a$ is a line that passes through the point $(a, f(a))$ and has the same "slope" as f at that point . The function $f(x) = e^x$ is the only exponential function b^x with tangent line at $x = 0$ that has a slope of 1. As we see later in the text, having this property makes the natural exponential function the most simple exponential function to use in many instances.

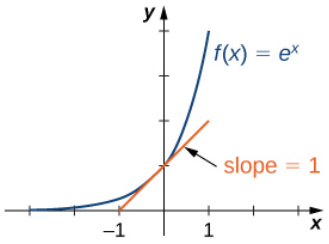

Figure 1.5.3: The graph of $f(x) = e^x$ has a tangent line with slope 1 at $x = 0$.

Example 1.5.3: Compounding Interest

Suppose \$500 is invested in an account at an annual interest rate of $r = 5.5\%$, compounded continuously.

a. Let t denote the number of years after the initial investment and $A(t)$ denote the amount of money in the account at time t. Find a formula for $A(t)$.
b. Find the amount of money in the account after 10 years and after 20 years.

Solution

a. If P dollars are invested in an account at an annual interest rate r, compounded continuously, then $A(t) = Pe^{rt}$. Here $P = \$500$ and $r = 0.055$. Therefore, $A(t) = 500e^{0.055t}$.

b. After 10 years, the amount of money in the account is

$$A(10) = 500e^{0.055 \cdot 10} = 500e^{0.55} \approx \$866.63.$$

After 20 years, the amount of money in the account is

$$A(20) = 500e^{0.055 \cdot 20} = 500e^{1.1} \approx \$1,502.08.$$

Exercise 1.5.3

If \$750 is invested in an account at an annual interest rate of 4%, compounded continuously, find a formula for the amount of money in the account after t years. Find the amount of money after 30 years.

Hint

$A(t) = Pe^{rt}$

Answer

$A(t) = 750e^{0.04t}$. After 30 years, there will be approximately $\$2,490.09$.

Logarithmic Functions

Using our understanding of exponential functions, we can discuss their inverses, which are the logarithmic functions. These come in handy when we need to consider any phenomenon that varies over a wide range of values, such as the pH scale in chemistry or decibels in sound levels.

The exponential function $f(x) = b^x$ is *one-to-one*, with domain $(-\infty, \infty)$ and range $(0, \infty)$. Therefore, it has an inverse function, called the logarithmic function with base b. For any $b > 0,\ b \neq 1$, the logarithmic function with base b, denoted $\log_b$, has domain $(0, \infty)$ and range $(-\infty, \infty)$,and satisfies

$$\log_b(x) = y$$

if and only if $b^y = x$.

For example,

$$\log_2(8) = 3$$

since $2^3 = 8$,

$$\log_{10}\left(\frac{1}{100}\right) = -2$$

since $10^{-2} = \dfrac{1}{10^2} = \dfrac{1}{100}$,

$$\log_b(1) = 0$$

since $b^0 = 1$ for any base $b > 0$.

Furthermore, since $y = \log_b(x)$ and $y = b^x$ are inverse functions,

$$\log_b(b^x) = x$$

and

$$b^{\log_b(x)} = x.$$

The most commonly used logarithmic function is the function $\log_e$. Since this function uses natural e as its base, it is called the **natural logarithm**. Here we use the notation $\ln(x)$ or $\ln x$ to mean $\log_e(x)$. For example,

$$\ln(e) = \log_e(e) = 1$$
$$\ln(e^3) = \log_e(e^3) = 3$$
$$\ln(1) = \log_e(1) = 0.$$

Since the functions $f(x) = e^x$ and $g(x) = \ln(x)$ are inverses of each other,

$$\ln(e^x) = x \text{ and } e^{\ln x} = x,$$

and their graphs are symmetric about the line $y = x$ (Figure 1.5.4).

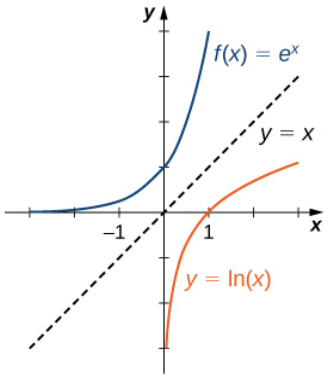

Figure 1.5.4: The functions $y = e^x$ and $y = \ln(x)$ are inverses of each other, so their graphs are symmetric about the line $y = x$.

In general, for any base $b > 0$, $b \neq 1$, the function $g(x) = \log_b(x)$ is symmetric about the line $y = x$ with the function $f(x) = b^x$. Using this fact and the graphs of the exponential functions, we graph functions $\log_b$ for several values of $b > 1$ (Figure 1.5.5).

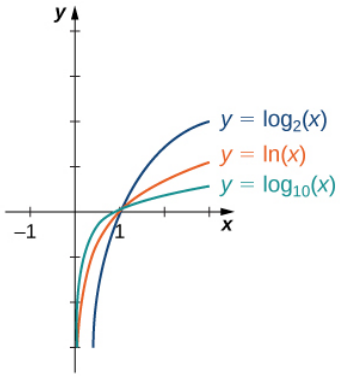

Figure 1.5.5: Graphs of $y = \log_b(x)$ are depicted for $b = 2,\ e,\ 10$.

Before solving some equations involving exponential and logarithmic functions, let's review the basic properties of logarithms.

Properties of Logarithms

If $a,\ b,\ c > 0$, $b \neq 1$, and r is any real number, then

- **Product property**

$$\log_b(ac) = \log_b(a) + \log_b(c) \tag{1.5.1}$$

- **Quotient property**

$$\log_b\left(\frac{a}{c}\right) = \log_b(a) - \log_b(c) \tag{1.5.2}$$

- **Power property**

$$\log_b(a^r) = r\log_b(a) \tag{1.5.3}$$

Example 1.5.4: Solving Equations Involving Exponential Functions

Solve each of the following equations for x.

a. $5^x = 2$
b. $e^x + 6e^{-x} = 5$

Solution

a. Applying the natural logarithm function to both sides of the equation, we have

$$\ln 5^x = \ln 2.$$

Using the power property of logarithms,

$$x \ln 5 = \ln 2.$$

Therefore,

$$x = \frac{\ln 2}{\ln 5}.$$

b. Multiplying both sides of the equation by e^x,we arrive at the equation

$$e^{2x} + 6 = 5e^x.$$

Rewriting this equation as

$$e^{2x} - 5e^x + 6 = 0,$$

we can then rewrite it as a quadratic equation in e^x:

$$(e^x)^2 - 5(e^x) + 6 = 0.$$

Now we can solve the quadratic equation. Factoring this equation, we obtain

$$(e^x - 3)(e^x - 2) = 0.$$

Therefore, the solutions satisfy $e^x = 3$ and $e^x = 2$. Taking the natural logarithm of both sides gives us the solutions $x = \ln 3, \ln 2$.

Exercise 1.5.4

Solve

$$e^{2x}/(3 + e^{2x}) = 1/2.$$

Hint

First solve the equation for e^{2x}

Answer

$x = \dfrac{\ln 3}{2}$.

Example 1.5.5: Solving Equations Involving Logarithmic Functions

Solve each of the following equations for x.

a. $\ln\left(\dfrac{1}{x}\right) = 4$
b. $\log_{10}\sqrt{x} + \log_{10} x = 2$
c. $\ln(2x) - 3\ln(x^2) = 0$

Solution

a. By the definition of the natural logarithm function,

$$\ln\left(\frac{1}{x}\right) = 4$$

- if and only if $e^4 = \dfrac{1}{x}$.

Therefore, the solution is $x = 1/e^4$.

b. Using the product (Equation 1.5.1) and power (Equation 1.5.3) properties of logarithmic functions, rewrite the left-hand side of the equation as

$$\begin{aligned}\log_{10}\sqrt{x} + \log_{10} x &= \log_{10} x\sqrt{x} \\ &= \log_{10} x^{3/2} \\ &= \frac{3}{2}\log_{10} x.\end{aligned}$$

Therefore, the equation can be rewritten as

$$\frac{3}{2}\log_{10} x = 2$$

or

$$\log_{10} x = \frac{4}{3}.$$

The solution is $x = 10^{4/3} = 10\sqrt[3]{10}$.

c. Using the power property (Equation 1.5.3) of logarithmic functions, we can rewrite the equation as $\ln(2x) - \ln(x^6) = 0$.

Using the quotient property (Equation 1.5.2), this becomes

$$\ln\left(\frac{2}{x^5}\right) = 0$$

Therefore, $2/x^5 = 1$, which implies $x = \sqrt[5]{2}$. We should then check for any extraneous solutions.

Exercise 1.5.5

Solve $\ln(x^3) - 4\ln(x) = 1$.

Hint

First use the power property, then use the product property of logarithms.

Answer

$x = \dfrac{1}{e}$

When evaluating a logarithmic function with a calculator, you may have noticed that the only options are $\log_{10}$ or $\log$, called the **common logarithm**, or $\ln$, which is the natural logarithm. However, exponential functions and logarithm functions can be expressed in terms of any desired base b. If you need to use a calculator to evaluate an expression with a different base, you can apply the change-of-base formulas first. Using this change of base, we typically write a given exponential or logarithmic function in terms of the natural exponential and natural logarithmic functions.

Rule: Change-of-Base Formulas

Let $a > 0$, $b > 0$, and $a \neq 1$, $b \neq 1$.

1. $a^x = b^{x \log_b a}$ for any real number x.

   If $b = e$, this equation reduces to $a^x = e^{x \log_e a} = e^{x \ln a}$.

2. $\log_a x = \dfrac{\log_b x}{\log_b a}$ for any real number $x > 0$.

   If $b = e$, this equation reduces to $\log_a x = \dfrac{\ln x}{\ln a}$.

Proof

For the first change-of-base formula, we begin by making use of the power property of logarithmic functions. We know that for any base $b > 0$, $b \neq 1$, $\log_b(a^x) = x \log_b a$. Therefore,

$$b^{\log_b(a^x)} = b^{x \log_b a}.$$

In addition, we know that b^x and $\log_b(x)$ are inverse functions. Therefore,

$$b^{\log_b(a^x)} = a^x.$$

Combining these last two equalities, we conclude that $a^x = b^{x \log_b a}$.

To prove the second property, we show that

$$(\log_b a) \cdot (\log_a x) = \log_b x.$$

Let $u = \log_b a, v = \log_a x$, and $w = \log_b x$. We will show that $u \cdot v = w$. By the definition of logarithmic functions, we know that $b^u = a$, $a^v = x$, and $b^w = x$. From the previous equations, we see that

$$b^{uv} = (b^u)^v = a^v = x = b^w.$$

Therefore, $b^{uv} = b^w$. Since exponential functions are one-to-one, we can conclude that $u \cdot v = w$.

□

Example 1.5.6: Changing Bases

Use a calculating utility to evaluate $\log_3 7$ with the change-of-base formula presented earlier.

Solution

Use the second equation with $a = 3$ and $b = e$: $\log_3 7 = \dfrac{\ln 7}{\ln 3} \approx 1.77124$.

Exercise 1.5.6

Use the change-of-base formula and a calculating utility to evaluate $\log_4 6$.

Hint

Use the change of base to rewrite this expression in terms of expressions involving the natural logarithm function.

Answer

$\log_4 6 = \dfrac{\ln 6}{\ln 4} \approx 1.29248$

Example 1.5.7: The Richter Scale for Earthquakes

In 1935, Charles Richter developed a scale (now known as the Richter scale) to measure the magnitude of an **earthquake**. The scale is a base-10 logarithmic scale, and it can be described as follows: Consider one earthquake with magnitude R_1 on the Richter scale and a second earthquake with magnitude R_2 on the Richter scale. Suppose $R_1 > R_2$, which means the earthquake of magnitude R_1 is stronger, but how much stronger is it than the other earthquake?

Figure 1.5.6: (credit: modification of work by Robb Hannawacker, NPS)

A way of measuring the intensity of an earthquake is by using a seismograph to measure the amplitude of the earthquake waves. If A_1 is the amplitude measured for the first earthquake and A_2 is the amplitude measured for the second earthquake, then the amplitudes and magnitudes of the two earthquakes satisfy the following equation:

$$R_1 - R_2 = \log_{10}\left(\frac{A1}{A2}\right).$$

Consider an earthquake that measures 8 on the Richter scale and an earthquake that measures 7 on the Richter scale. Then,

$$8 - 7 = \log_{10}\left(\frac{A1}{A2}\right).$$

Therefore,

$$\log_{10}\left(\frac{A1}{A2}\right) = 1,$$

which implies $A_1/A_2 = 10$ or $A_1 = 10A_2$. Since A_1 is 10 times the size of A_2, we say that the first earthquake is 10 times as intense as the second earthquake. On the other hand, if one earthquake measures 8 on the Richter scale and another measures 6, then the relative intensity of the two earthquakes satisfies the equation

$$\log_{10}\left(\frac{A1}{A2}\right) = 8 - 6 = 2.$$

Therefore, $A_1 = 100A_2$.That is, the first earthquake is 100 times more intense than the second earthquake.

How can we use logarithmic functions to compare the relative severity of the magnitude 9 earthquake in Japan in 2011 with the magnitude 7.3 earthquake in Haiti in 2010?

Solution

To compare the Japan and Haiti earthquakes, we can use an equation presented earlier:

$$9 - 7.3 = \log_{10}\left(\frac{A1}{A2}\right).$$

Therefore, $A_1/A_2 = 10^{1.7}$, and we conclude that the earthquake in Japan was approximately 50 times more intense than the earthquake in Haiti.

Exercise 1.5.7

Compare the relative severity of a magnitude 8.4 earthquake with a magnitude 7.4 earthquake.

Hint

$R_1 - R_2 = \log_{10}(A1/A2)$.

Answer

The magnitude 8.4 earthquake is roughly 10 times as severe as the magnitude 7.4 earthquake.

Hyperbolic Functions

The hyperbolic functions are defined in terms of certain combinations of e^x and e^{-x}. These functions arise naturally in various engineering and physics applications, including the study of water waves and vibrations of elastic membranes. Another common use for a hyperbolic function is the representation of a hanging chain or cable, also known as a *catenary* (Figure 1.5.7). If we introduce a coordinate system so that the low point of the chain lies along the y-axis, we can describe the height of the chain in terms of a hyperbolic function. First, we define the **hyperbolic functions.**

Figure 1.5.7:The shape of a strand of silk in a spider's web can be described in terms of a hyperbolic function. The same shape applies to a chain or cable hanging from two supports with only its own weight. (credit: "Mtpaley", Wikimedia Commons)

Definitions: hyperbolic functions

Hyperbolic cosine

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Hyperbolic sine

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Hyperbolic tangent

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cosecant

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

Hyperbolic secant

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

Hyperbolic cotangent

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

The name $\cosh$ rhymes with "gosh," whereas the name $\sinh$ is pronounced "cinch." $\tanh$, sech, csch, and $\coth$ are pronounced "tanch," "seech," "coseech," and "cotanch," respectively.

Using the definition of $\cosh(x)$ and principles of physics, it can be shown that the height of a hanging chain, such as the one in Figure 1.5.8, can be described by the function $h(x) = a\cosh(x/a) + c$ for certain constants a and c.

But why are these functions called hyperbolic functions? To answer this question, consider the quantity $\cosh^2 t - \sinh^2 t$. Using the definition of $\cosh$ and $\sinh$, we see that

$$\cosh^2 t - \sinh^2 t = \frac{e^{2t} + 2 + e^{-2t}}{4} - \frac{e^{2t} - 2 + e^{-2t}}{4} = 1.$$

This identity is the analog of the trigonometric identity $\cos^2 t + \sin^2 t = 1$. Here, given a value t, the point $(x, y) = (\cosh t, \sinh t)$ lies on the unit hyperbola $x^2 - y^2 = 1$ (Figure 1.5.8).

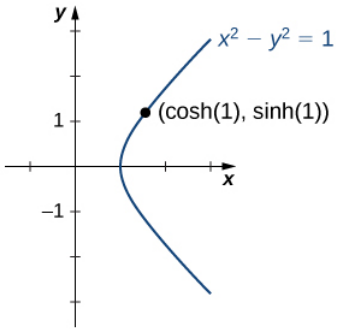

Figure 1.5.8: The unit hyperbola $\cosh^2 t - \sinh^2 t = 1$.

Graphs of Hyperbolic Functions

To graph $\cosh x$ and $\sinh x$, we make use of the fact that both functions approach $(1/2)e^x$ as $x \to \infty$, since $e^{-x} \to 0$ as $x \to \infty$. As $x \to -\infty$, $\cosh x$ approaches $1/2e^{-x}$, whereas $\sinh x$ approaches $-1/2e^{-x}$. Therefore, using the graphs of $1/2e^x, 1/2e^{-x}$, and $-1/2e^{-x}$ as guides, we graph $\cosh x$ and $\sinh x$. To graph $\tanh x$, we use the fact that $\tanh(0) = 0$, $-1 < \tanh(x) < 1$ for all x, $\tanh x \to 1$ as $x \to \infty$, and $\tanh x \to -1$ as $x \to -\infty$. The graphs of the other three hyperbolic functions can be sketched using the graphs of $\cosh x$, $\sinh x$, and $\tanh x$ (Figure 1.5.9).

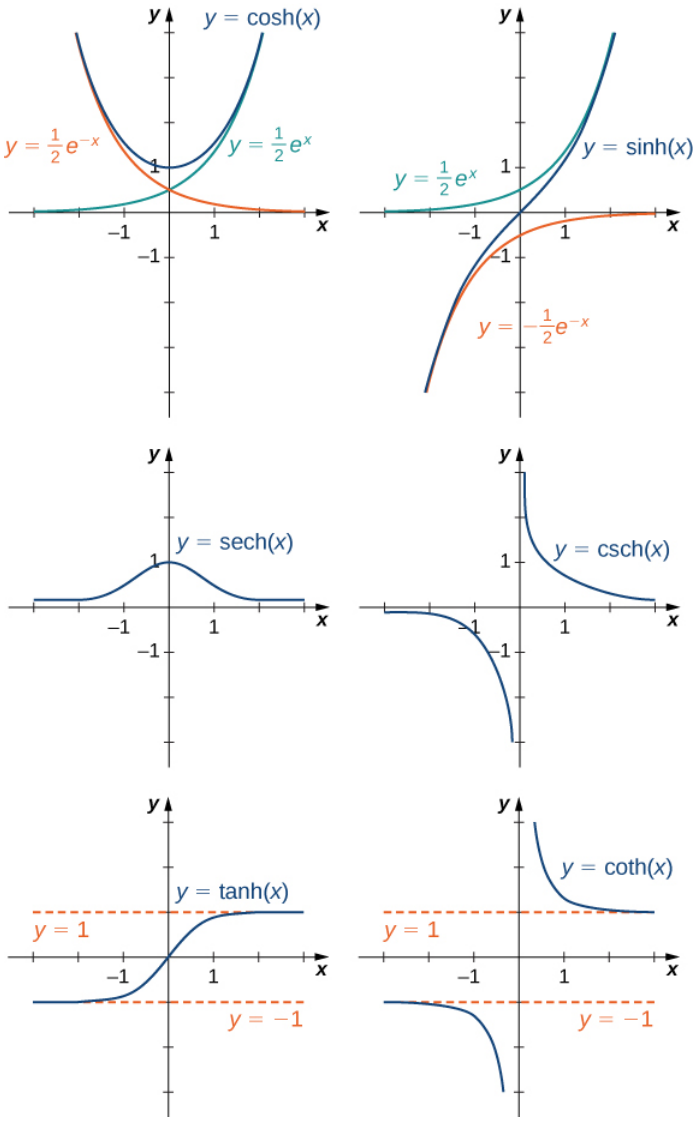

Figure 1.5.9: The hyperbolic functions involve combinations of e^x and e^{-x}.

Identities Involving Hyperbolic Functions

The identity $\cosh^2 t - \sinh^2 t = 1$, shown in Figure 1.5.8, is one of several identities involving the hyperbolic functions, some of which are listed next. The first four properties follow easily from the definitions of hyperbolic sine and hyperbolic cosine. Except for some differences in signs, most of these properties are analogous to identities for trigonometric functions.

Identities Involving Hyperbolic Functions

1. $\cosh(-x) = \cosh x$
2. $\sinh(-x) = -\sinh x$
3. $\cosh x + \sinh x = e^x$
4. $\cosh x - \sinh x = e^{-x}$
5. $\cosh^2 x - \sinh^2 x = 1$
6. $1 - \tanh^2 x = \operatorname{sech}^2 x$
7. $\coth^2 x - 1 = \operatorname{csch}^2 x$
8. $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
9. $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

Example 1.5.8: Evaluating Hyperbolic Functions

a. Simplify $\sinh(5\ln x)$.

b. If $\sinh x = 3/4$, find the values of the remaining five hyperbolic functions.

Solution:

a. Using the definition of the $\sinh$ function, we write

$$\sinh(5\ln x) = \frac{e^{5\ln x} - e^{-5\ln x}}{2} = \frac{e^{\ln(x^5)} - e^{\ln(x^{-5})}}{2} = \frac{x^5 - x^{-5}}{2}.$$

b. Using the identity $\cosh^2 x - \sinh^2 x = 1$,we see that

$$\cosh^2 x = 1 + \left(\tfrac{3}{4}\right)^2 = \frac{25}{16}.$$

Since $\cosh x \geq 1$ for all x, we must have $\cosh x = 5/4$. Then, using the definitions for the other hyperbolic functions, we conclude that $\tanh x = 3/5, \operatorname{csch} x = 4/3, \operatorname{sech} x = 4/5$, and $\coth x = 5/3$.

Exercise 1.5.8

Simplify $\cosh(2\ln x)$.

Hint

Use the definition of the $\cosh$ function and the power property of logarithm functions.

Answer

$(x^2 + x^{-2})/2$

Inverse Hyperbolic Functions

From the graphs of the hyperbolic functions, we see that all of them are one-to-one except $\cosh x$ and $\operatorname{sech} x$. If we restrict the domains of these two functions to the interval $[0, \infty)$, then all the hyperbolic functions are one-to-one, and we can define the **inverse hyperbolic functions**. Since the hyperbolic functions themselves involve exponential functions, the inverse hyperbolic functions involve logarithmic functions.

Definitions: Inverse Hyperbolic Functions

$$\sinh^{-1} x = \operatorname{arcsinh} x = \ln\left(x + \sqrt{x^2+1}\right) \qquad \cosh^{-1} x = \operatorname{arccosh} x = \ln\left(x + \sqrt{x^2-1}\right)$$

$$\tanh^{-1} x = \operatorname{arctanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad \coth^{-1} x = \operatorname{arccot} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$$

$$\operatorname{sech}^{-1} x = \operatorname{arcsech} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) \qquad \operatorname{csch}^{-1} x = \operatorname{arccsch} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)$$

Let's look at how to derive the first equation. The others follow similarly. Suppose $y = \sinh^{-1} x$. Then, $x = \sinh y$ and, by the definition of the hyperbolic sine function, $x = \dfrac{e^y - e^{-y}}{2}$. Therefore,

$$e^y - 2x - e^{-y} = 0.$$

Multiplying this equation by e^y, we obtain

$$e^{2y} - 2xe^y - 1 = 0.$$

This can be solved like a quadratic equation, with the solution

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}.$$

Since $e^y > 0$,the only solution is the one with the positive sign. Applying the natural logarithm to both sides of the equation, we conclude that

$$y = \ln(x + \sqrt{x^2 + 1}).$$

Example 1.5.9: Evaluating Inverse Hyperbolic Functions

Evaluate each of the following expressions.

$$\sinh^{-1}(2)$$

$$\tanh^{-1}(1/4)$$

Solution:

$$\sinh^{-1}(2) = \ln(2 + \sqrt{2^2 + 1}) = \ln(2 + \sqrt{5}) \approx 1.4436$$

$$\tanh^{-1}(1/4) = \frac{1}{2}\ln\left(\frac{1 + 1/4}{1 - 1/4}\right) = \frac{1}{2}\ln\left(\frac{5/4}{3/4}\right) = \frac{1}{2}\ln\left(\frac{5}{3}\right) \approx 0.2554$$

Exercise 1.5.9

Evaluate $\tanh^{-1}(1/2)$.

Hint

Use the definition of $\tanh^{-1} x$ and simplify.

Answer

$\frac{1}{2}\ln(3) \approx 0.5493$.

Key Concepts

- The exponential function $y = b^x$ is increasing if $b > 1$ and decreasing if $0 < b < 1$. Its domain is $(-\infty, \infty)$ and its range is $(0, \infty)$.
- The logarithmic function $y = \log_b(x)$ is the inverse of $y = b^x$. Its domain is $(0, \infty)$ and its range is $(-\infty, \infty)$.
- The natural exponential function is $y = e^x$ and the natural logarithmic function is $y = \ln x = \log_e x$.
- Given an exponential function or logarithmic function in base a, we can make a change of base to convert this function to any base $b > 0$, $b \neq 1$. We typically convert to base e.
- The hyperbolic functions involve combinations of the exponential functions e^x and e^{-x}. As a result, the inverse hyperbolic functions involve the natural logarithm.

Glossary

base

the number b in the exponential function $f(x) = b^x$ and the logarithmic function $f(x) = \log_b x$

exponent

the value x in the expression b^x

hyperbolic functions

the functions denoted $\sinh$, $\cosh$, $\tanh$, csch, sech, and $\coth$, which involve certain combinations of e^x and e^{-x}

inverse hyperbolic functions

the inverses of the hyperbolic functions where $\cosh$ and sech are restricted to the domain $[0, \infty)$;each of these functions can be expressed in terms of a composition of the natural logarithm function and an algebraic function

natural exponential function

the function $f(x) = e^x$

natural logarithm

the function $\ln x = \log_e x$

number e

as m gets larger, the quantity $(1 + (1/m)^m$ gets closer to some real number; we define that real number to be e; the value of e is approximately 2.718282

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1.5E: Exercises for Section 1.5

In exercises 1 - 4, evaluate the given exponential functions as indicated, accurate to two significant digits after the decimal.

1) $f(x) = 5^x$

a. $x = 3$

b. $x = \frac{1}{2}$

c. $x = \sqrt{2}$

Answer

a. 125
b. 2.24
c. 9.74

2) $f(x) = (0.3)^x$

a. $x = -1$

b. $x = 4$

c. $x = -1.5$

3) $f(x) = 10^x$

a. $x = -2$

b. $x = 4$

c. $x = \frac{5}{3}$

Answer

a. 0.01
b. $10,000$
c. 46.42

4) $f(x) = e^x$

a. $x = 2$

b. $x = -3.2$

c. $x = \pi$

In exercises 5 - 10, match the exponential equation to the correct graph.

a. $y = 4^{-x}$

b. $y = 3^{x-1}$

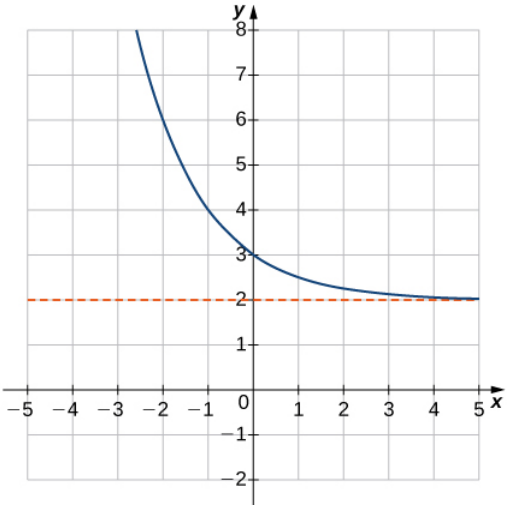
c. $y = 2^{x+1}$

d. $y = \left(\frac{1}{2}\right)^x + 2$

e. $y = -3^{-x}$

f. $y = 1 - 5^x$

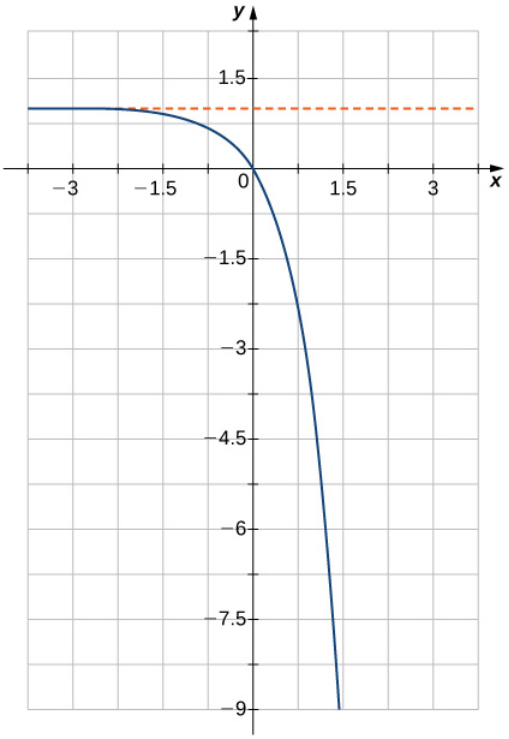
5)



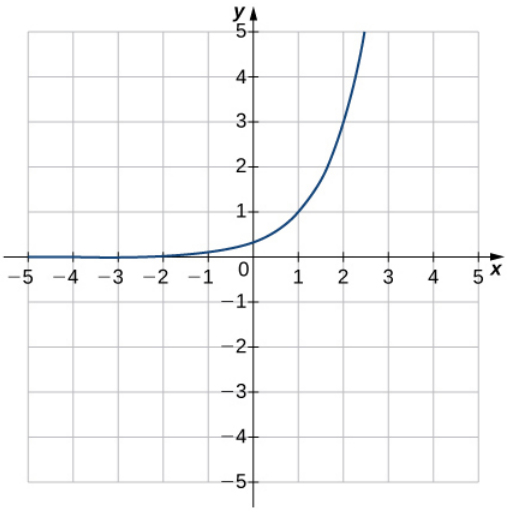
Answer

d

6)



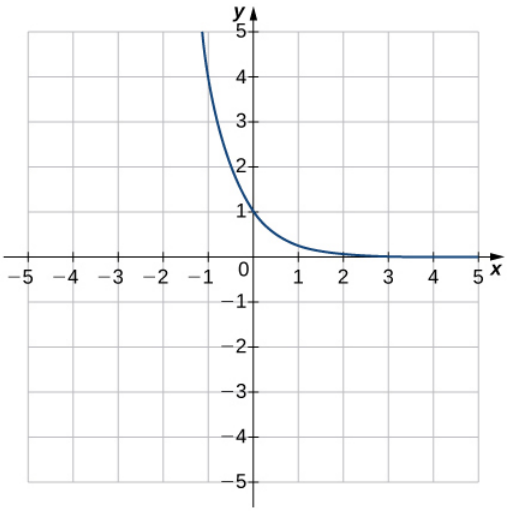
7)


y
x
−5 −4 −3 −2 −1 0 1 2 3 4 5
5 4 3 2 1 −1 −2 −3 −4 −5

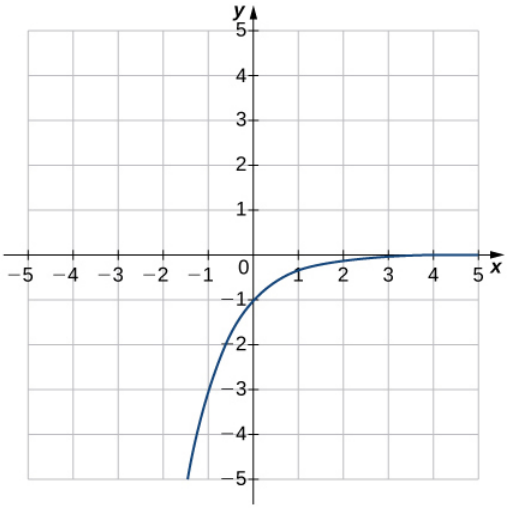
Answer

b

8)


y
x
−5 −4 −3 −2 −1 0 1 2 3 4 5
5 4 3 2 1 −1 −2 −3 −4 −5

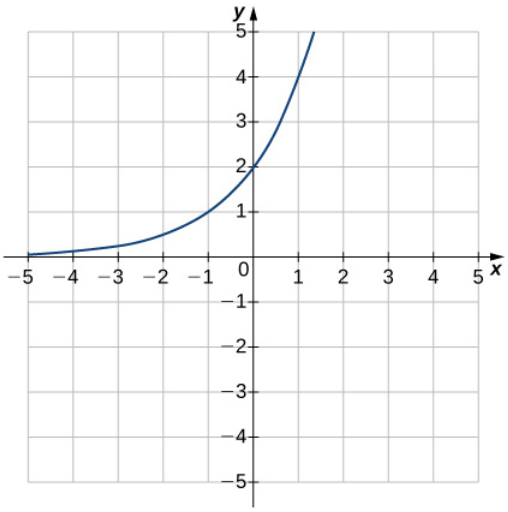
9)


y
x
−5 −4 −3 −2 −1 0 1 2 3 4 5
5 4 3 2 1 −1 −2 −3 −4 −5

Answer

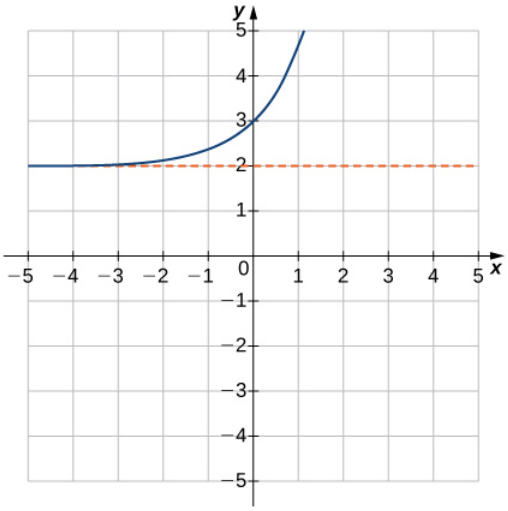
e

10)



In exercises 11 - 17, sketch the graph of the exponential function. Determine the domain, range, and horizontal asymptote.

11) $f(x) = e^x + 2$



Answer

Domain: all real numbers, Range: $(2, \infty)$, $y = 2$

12) $f(x) = -2^x$

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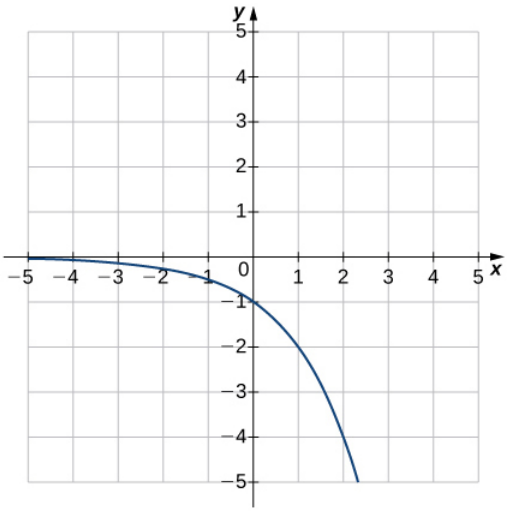
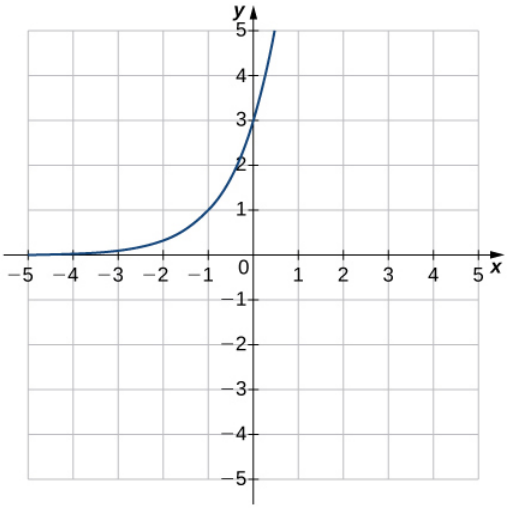


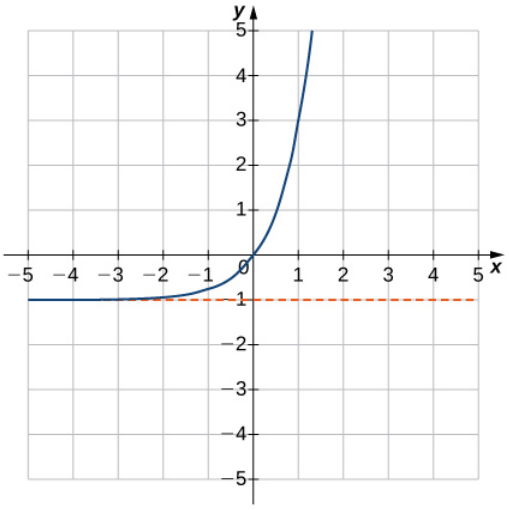
13) $f(x) = 3^{x+1}$



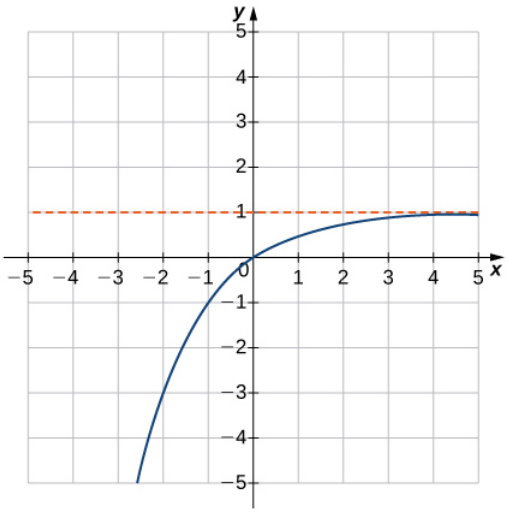
Answer

Domain: all real numbers, Range: $(0, \infty),\ y = 0$

14) $f(x) = 4^x - 1$



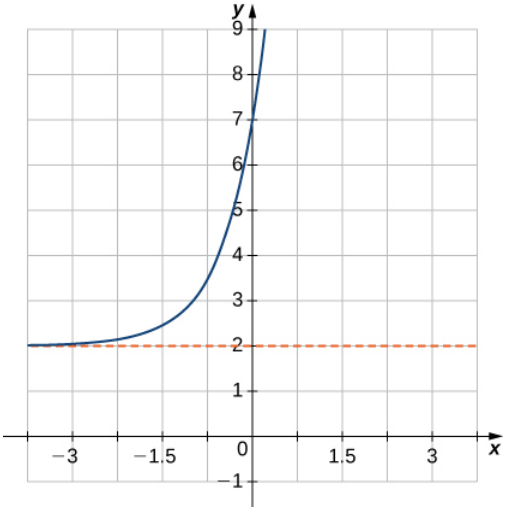
15) $f(x) = 1 - 2^{-x}$



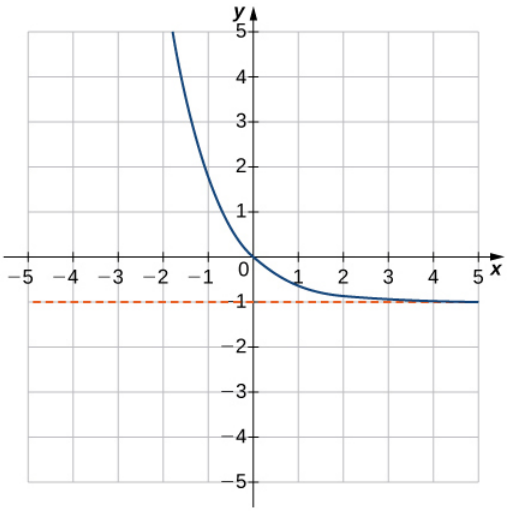
Answer

Domain: all real numbers, Range: $(-\infty, 1),\ y = 1$

16) $f(x) = 5^{x+1} + 2$



17) $f(x) = e^{-x} - 1$



Answer

Domain: all real numbers, Range: $(-1, \infty),\ y = -1$

In exercises 18 - 25, write the equation in equivalent exponential form.

18) $\log_3 81 = 4$

19) $\log_8 2 = \frac{1}{3}$

Answer

$8^{1/3} = 2$

20) $\log_5 1 = 0$

21) $\log_5 25 = 2$

Answer

$5^2 = 25$

22) $\log 0.1 = -1$

23) $\ln\left(\frac{1}{e^3}\right) = -3$

Answer

$e^{-3} = \frac{1}{e^3}$

24) $\log_9 3 = 0.5$

25) $\ln 1 = 0$

Answer

$e^0 = 1$

In exercises 26 - 35, write the equation in equivalent logarithmic form.

26) $2^3 = 8$

27) $4^{-2} = \frac{1}{16}$

Answer

$\log_4\left(\frac{1}{16}\right) = -2$

28) $10^2 = 100$

29) $9^0 = 1$

Answer

$\log_9 1 = 0$

30) $\left(\frac{1}{3}\right)^3 = \frac{1}{27}$

31) $\sqrt[3]{64} = 4$

Answer

$\log_{64} 4 = \frac{1}{3}$

32) $e^x = y$

33) $9^y = 150$

Answer

$\log_9 150 = y$

34) $b^3 = 45$

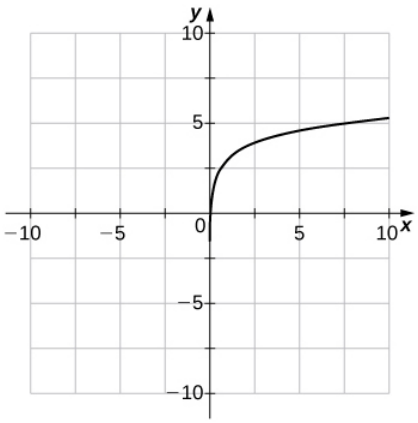
35) $4^{-3/2} = 0.125$

Answer

$\log_4 0.125 = -\frac{3}{2}$

In exercises 36 - 41, sketch the graph of the logarithmic function. Determine the domain, range, and vertical asymptote.

36) $f(x) = 3 + \ln x$



37) $f(x) = \ln(x - 1)$

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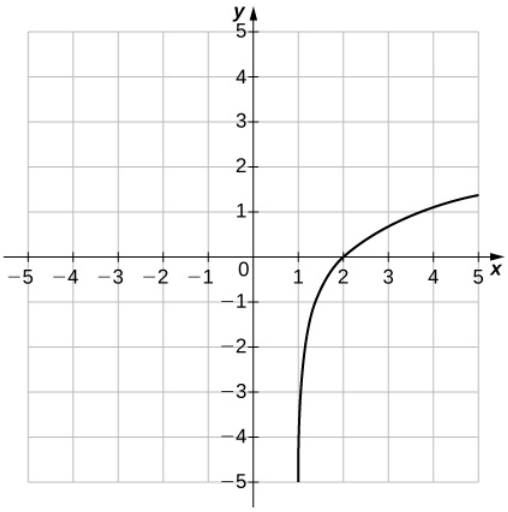
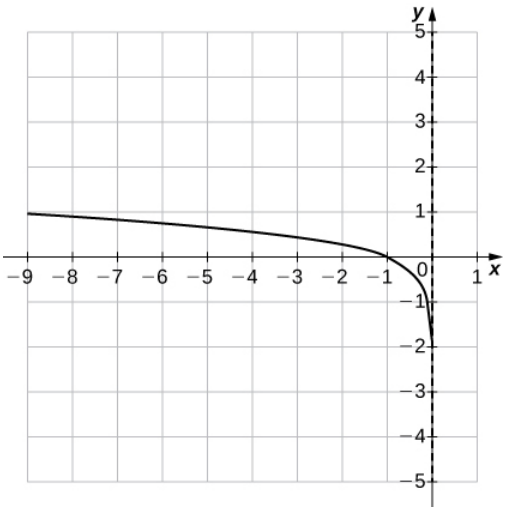


Answer

Domain: $(1, \infty)$, Range: $(-\infty, \infty),\ x = 1$

38) $f(x) = \ln(-x)$



39) $f(x) = 1 - \ln x$

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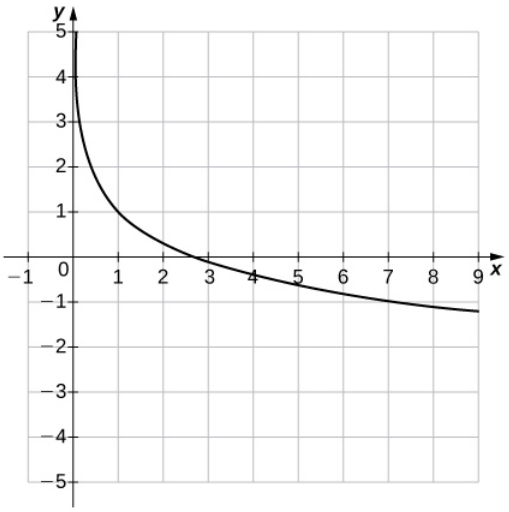
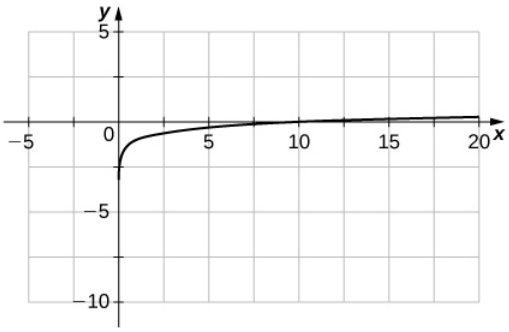


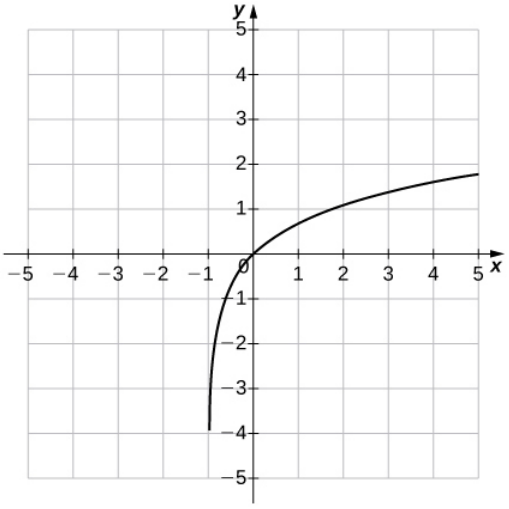
Answer

Domain: $(0, \infty)$, Range: $(-\infty, \infty)$, $x = 0$

40) $f(x) = \log x - 1$



41) $f(x) = \ln(x + 1)$



Answer

Domain: $(-1, \infty)$, Range: $(-\infty, \infty)$, $x = -1$

In exercises 42 - 47, use properties of logarithms to write the expressions as a sum, difference, and/or product of logarithms.

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42) $\log x^4 y$

43) $\log_3 \frac{9a^3}{b}$

Answer

$2 + 3\log_3 a - \log_3 b$

44) $\ln a\sqrt[3]{b}$

45) $\log_5 \sqrt{125xy^3}$

Answer

$\frac{3}{2} + \frac{1}{2}\log_5 x + \frac{3}{2}\log_5 y$

46) $\log_4 \frac{\sqrt[3]{xy}}{64}$

47) $\ln\left(\frac{6}{\sqrt{e^3}}\right)$

Answer

$-\frac{3}{2} + \ln 6$

In exercises 48 - 55, solve the exponential equation exactly.

48) $5^x = 125$

49) $e^{3x} - 15 = 0$

Answer

$\frac{\ln 15}{3}$

50) $8^x = 4$

51) $4^{x+1} - 32 = 0$

Answer

$\frac{3}{2}$

52) $3^{x/14} = \frac{1}{10}$

53) $10^x = 7.21$

Answer

$\log 7.21$

54) $4 \cdot 2^{3x} - 20 = 0$

55) $7^{3x-2} = 11$

Answer

$\frac{2}{3} + \frac{\log 11}{3\log 7}$

In exercises 56 - 63, solve the logarithmic equation exactly, if possible.

56) $\log_3 x = 0$

57) $\log_5 x = -2$

Answer

$x = \frac{1}{25}$

58) $\log_4(x + 5) = 0$

59) $\log(2x-7)=0$

Answer

$x=4$

60) $\ln\sqrt{x+3}=2$

61) $\log_6(x+9)+\log_6 x=2$

Answer

$x=3$

62) $\log_4(x+2)-\log_4(x-1)=0$

63) $\ln x+\ln(x-2)=\ln 4$

Answer

$1+\sqrt{5}$

In exercises 64 - 69, use the change-of-base formula and either base 10 or base e to evaluate the given expressions. Answer in exact form and in approximate form, rounding to four decimal places.

64) $\log_5 47$

65) $\log_7 82$

Answer

$\frac{\log 82}{\log 7}\approx 2.2646$

66) $\log_6 103$

67) $\log_{0.5} 211$

Answer

$\frac{\log 211}{\log 0.5}\approx -7.7211$

68) $\log_2 \pi$

69) $\log_{0.2} 0.452$

Answer

$\frac{\log 0.452}{\log 0.2}\approx 0.4934$

70) Rewrite the following expressions in terms of exponentials and simplify.

a. $2\cosh(\ln x)$ b. $\cosh 4x+\sinh 4x$ c. $\cosh 2x-\sinh 2x$ d. $\ln(\cosh x+\sinh x)+\ln(\cosh x-\sinh x)$

71) [T] The number of bacteria N in a culture after t days can be modeled by the function $N(t)=1300\cdot(2)^{t/4}$. Find the number of bacteria present after 15 days.

Answer

$\sim 17,491$

72) [T] The demand D (in millions of barrels) for oil in an oil-rich country is given by the function $D(p)=150\cdot(2.7)^{-0.25p}$, where p is the price (in dollars) of a barrel of oil. Find the amount of oil demanded (to the nearest million barrels) when the price is between $15 and $20.

73) [T] The amount A of a $100,000 investment paying continuously and compounded for t years is given by $A(t)=100,000\cdot e^{0.055t}$. Find the amount A accumulated in 5 years.

Answer

Approximately $131,653 is accumulated in 5 years.

74) [T] An investment is compounded monthly, quarterly, or yearly and is given by the function $A = P\left(1 + \frac{j}{n}\right)^{nt}$, where A is the value of the investment at time t, P is the initial principle that was invested, j is the annual interest rate, and n is the number of time the interest is compounded per year. Given a yearly interest rate of 3.5% and an initial principle of $100,000, find the amount A accumulated in 5 years for interest that is compounded a. daily, b., monthly, c. quarterly, and d. yearly.

75) [T] The concentration of hydrogen ions in a substance is denoted by $[H+]$, measured in moles per liter. The pH of a substance is defined by the logarithmic function $pH = -\log[H+]$. This function is used to measure the acidity of a substance. The pH of water is 7. A substance with a pH less than 7 is an acid, whereas one that has a pH of more than 7 is a base.

a. Find the pH of the following substances. Round answers to one digit.

b. Determine whether the substance is an acid or a base.

i. Eggs: $[H+] = 1.6 \times 10^{-8}$ mol/L

ii. Beer: $[H+] = 3.16 \times 10^{-3}$ mol/L

iii. Tomato Juice: $[H+] = 7.94 \times 10^{-5}$ mol/L

Answer

i. a. pH = 8 b. Base
ii. a. pH = 3 b. Acid
iii. a. pH = 4 b. Acid

76) [T] Iodine-131 is a radioactive substance that decays according to the function $Q(t) = Q_0 \cdot e^{-0.08664t}$, where Q_0 is the initial quantity of a sample of the substance and t is in days. Determine how long it takes (to the nearest day) for 95% of a quantity to decay.

77) [T] According to the World Bank, at the end of 2013 $(t = 0)$ the U.S. population was 316 million and was increasing according to the following model:

$$P(t) = 316e^{0.0074t},$$

where P is measured in millions of people and t is measured in years after 2013.

a. Based on this model, what will be the population of the United States in 2020?

b. Determine when the U.S. population will be twice what it is in 2013.

Answer

a. ~ 333 million
b. 94 years from 2013, or in 2107

78) [T] The amount A accumulated after 1000 dollars is invested for t years at an interest rate of 4% is modeled by the function $A(t) = 1000(1.04)^t$.

a. Find the amount accumulated after 5 years and 10 years.

b. Determine how long it takes for the original investment to triple.

79) [T] A bacterial colony grown in a lab is known to double in number in 12 hours. Suppose, initially, there are 1000 bacteria present.

a. Use the exponential function $Q = Q_0 e^{kt}$ to determine the value k, which is the growth rate of the bacteria. Round to four decimal places.

b. Determine approximately how long it takes for $200,000$ bacteria to grow.

Answer

a. $k \approx 0.0578$
b. ≈ 92 hours

80) [T] The rabbit population on a game reserve doubles every 6 months. Suppose there were 120 rabbits initially.

a. Use the exponential function $P = P_0 a^t$ to determine the growth rate constant a. Round to four decimal places.

b. Use the function in part a. to determine approximately how long it takes for the rabbit population to reach 3500.

81) [T] The 1906 earthquake in San Francisco had a magnitude of 8.3 on the Richter scale. At the same time, in Japan, an earthquake with magnitude 4.9 caused only minor damage. Approximately how much more energy was released by the San Francisco earthquake than by the Japanese earthquake?

Answer

The San Francisco earthquake had $10^{3.4}$ or ~ 2512 times more energy than the Japan earthquake.

Contributors

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1.R: Chapter 1 Review Exercises

True or False? Justify your answer with a proof or a counterexample.

1) A function is always one-to-one.

2) $f \circ g = g \circ f$, assuming f and g are functions.

Answer

False

3) A relation that passes the horizontal and vertical line tests is a one-to-one function.

4) A relation passing the horizontal line test is a function.

Answer

False

State the domain and range of the given functions:

$f = x^2 + 2x - 3$, $g = \ln(x - 5)$, $h = \dfrac{1}{x + 4}$

5) h

6) g

Answer

Domain: $x > 5$, Range: all real numbers

7) $h \circ f$

8) $g \circ f$

Answer

Domain: $x > 2$ and $x < -4$, Range: all real numbers

Find the degree, y-intercept, and zeros for the following polynomial functions.

9) $f(x) = 2x^2 + 9x - 5$

10) $f(x) = x^3 + 2x^2 - 2x$

Answer

Degree of 3, y-intercept: $(0, 0)$, Zeros: $0,\ \sqrt{3} - 1,\ -1 - \sqrt{3}$

Simplify the following trigonometric expressions.

11) $\dfrac{\tan^2 x}{\sec^2 x} + \cos^2 x$

12) $\cos^2 x - \sin^2 x$

Answer

$\cos(2x)$

Solve the following trigonometric equations on the interval $\theta = [-2\pi, 2\pi]$ exactly.

13) $6 \cos 2x - 3 = 0$

14) $\sec^2 x - 2 \sec x + 1 = 0$

Answer

$0, \pm 2\pi$

Solve the following logarithmic equations.

15) $5^x = 16$

16) $\log_2(x+4) = 3$

Answer

4

Are the following functions one-to-one over their domain of existence? Does the function have an inverse? If so, find the inverse $f^{-1}(x)$ of the function. Justify your answer.

17) $f(x) = x^2 + 2x + 1$

18) $f(x) = \dfrac{1}{x}$

Answer

One-to-one; yes, the function has an inverse; inverse: $f^{-1}(x) = \dfrac{1}{x}$

For the following problems, determine the largest domain on which the function is one-to-one and find the inverse on that domain.

19) $f(x) = \sqrt{9-x}$

20) $f(x) = x^2 + 3x + 4$

Answer

$x \geq -\frac{3}{2}, \quad f^{-1}(x) = -\frac{3}{2} + \frac{1}{2}\sqrt{4x-7}$

21) A car is racing along a circular track with diameter of 1 mi. A trainer standing in the center of the circle marks his progress every 5 sec. After 5 sec, the trainer has to turn 55° to keep up with the car. How fast is the car traveling?

For the following problems, consider a restaurant owner who wants to sell T-shirts advertising his brand. He recalls that there is a fixed cost and variable cost, although he does not remember the values. He does know that the T-shirt printing company charges $440 for 20 shirts and $1000 for 100 shirts.

22) a. Find the equation $C = f(x)$ that describes the total cost as a function of number of shirts and

b. determine how many shirts he must sell to break even if he sells the shirts for $10 each.

Answer

a. $C(x) = 300 + 7x$
b. 100 shirts

23) a. Find the inverse function $x = f^{-1}(C)$ and describe the meaning of this function.

b. Determine how many shirts the owner can buy if he has $8000 to spend.

For the following problems, consider the population of Ocean City, New Jersey, which is cyclical by season.

24) The population can be modeled by $P(t) = 82.5 - 67.5\cos[(\pi/6)t]$, where t is time in months ($t = 0$ represents January 1) and P is population (in thousands). During a year, in what intervals is the population less than 20,000? During what intervals is the population more than 140,000?

Answer

The population is less than 20,000 from December 8 through January 23 and more than 140,000 from May 29 through August 2

25) In reality, the overall population is most likely increasing or decreasing throughout each year. Let's reformulate the model as $P(t) = 82.5 - 67.5\cos[(\pi/6)t] + t$, where t is time in months ($t = 0$ represents January 1) and P is population (in thousands). When is the first time the population reaches 200,000?

For the following problems, consider radioactive dating. A human skeleton is found in an archeological dig. Carbon dating is implemented to determine how old the skeleton is by using the equation $y = e^{rt}$, where y is the percentage of radiocarbon still present in the material, t is the number of years passed, and $r = -0.0001210$ is the decay rate of radiocarbon.

26) If the skeleton is expected to be 2000 years old, what percentage of radiocarbon should be present?

Answer

78.51%

27) Find the inverse of the carbon-dating equation. What does it mean? If there is 25% radiocarbon, how old is the skeleton?

Contributors

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CHAPTER OVERVIEW

2: Limits

The idea of a limit is central to all of calculus. We begin this chapter by examining why limits are so important. Then, we go on to describe how to find the limit of a function at a given point. Not all functions have limits at all points, and we discuss what this means and how we can tell if a function does or does not have a limit at a particular value. This chapter has been created in an informal, intuitive fashion, but this is not always enough if we need to prove a mathematical statement involving limits. The last section of this chapter presents the more precise definition of a limit and shows how to prove whether a function has a limit.

2.0: Prelude to Limits
2.1: A Preview of Calculus
2.1E: Exercises for Section 2.1
2.2: The Limit of a Function
2.2E: Exercises for Section 2.2
2.3: The Limit Laws
2.3E: Exercises for Section 2.3
2.4: Continuity
2.4E: Exercises for Section 2.4
2.5: The Precise Definition of a Limit
2.5E: Exercises for Section 2.5
2.R: Chapter 2 Review Exercises

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2.0: Prelude to Limits

Science fiction writers often imagine spaceships that can travel to far-off planets in distant galaxies. However, back in 1905, Albert Einstein showed that a limit exists to how fast any object can travel. The problem is that the faster an object moves, the more mass it attains (in the form of energy), according to the equation

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where m_0 is the object's mass at rest, v is its speed, and c is the speed of light. What is this speed limit? (We explore this problem further in the chapter)

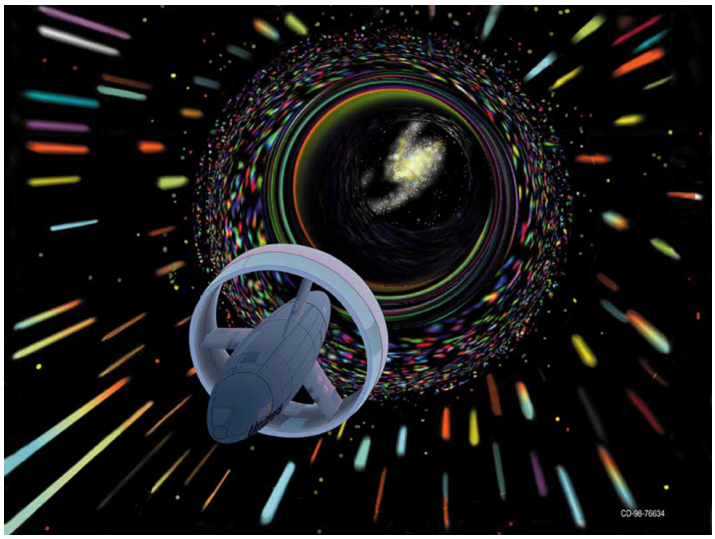

Figure 2.0.1:The vision of human exploration by the National Aeronautics and Space Administration (NASA) to distant parts of the universe illustrates the idea of space travel at high speeds. But, is there a limit to how fast a spacecraft can go? (credit: NASA)

The idea of a limit is central to all of calculus. We begin this chapter by examining why limits are so important. Then, we go on to describe how to find the limit of a function at a given point. Not all functions have limits at all points, and we discuss what this means and how we can tell if a function does or does not have a limit at a particular value. This chapter has been created in an informal, intuitive fashion, but this is not always enough if we need to prove a mathematical statement involving limits. The last section of this chapter presents the more precise definition of a limit and shows how to prove whether a function has a limit.

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2.1: A Preview of Calculus

Learning Objectives

- Describe the tangent problem and how it led to the idea of a derivative.
- Explain how the idea of a limit is involved in solving the tangent problem.
- Recognize a tangent to a curve at a point as the limit of secant lines.
- Identify instantaneous velocity as the limit of average velocity over a small time interval.
- Describe the area problem and how it was solved by the integral.
- Explain how the idea of a limit is involved in solving the area problem.
- Recognize how the ideas of limit, derivative, and integral led to the studies of infinite series and multivariable calculus.

As we embark on our study of calculus, we shall see how its development arose from common solutions to practical problems in areas such as engineering physics—like the space travel problem posed in the chapter opener. Two key problems led to the initial formulation of calculus: (1) the tangent problem, or how to determine the slope of a line tangent to a curve at a point; and (2) the area problem, or how to determine the area under a curve.

The Tangent Problem and Differential Calculus

Rate of change is one of the most critical concepts in calculus. We begin our investigation of rates of change by looking at the graphs of the three lines $f(x) = -2x - 3,\ g(x) = \frac{1}{2}x + 1$, and $h(x) = 2$, shown in Figure 2.1.1.

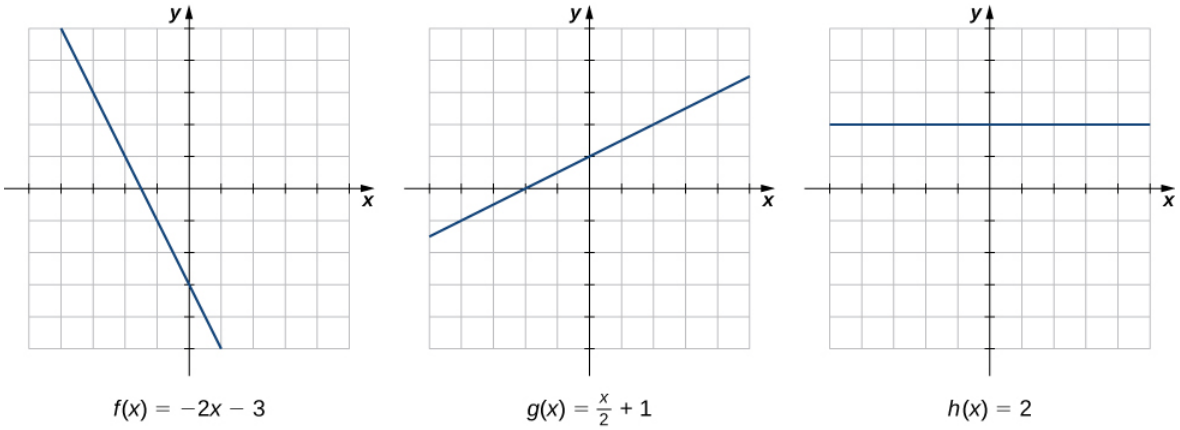

Figure 2.1.1: The rate of change of a linear function is constant in each of these three graphs, with the constant determined by the slope.

As we move from left to right along the graph of $f(x) = -2x - 3$, we see that the graph decreases at a constant rate. For every 1 unit we move to the right along the x-axis, the y-coordinate decreases by 2 units. This **rate of change** is determined by the slope (-2) of the line. Similarly, the slope of $1/2$ in the function $g(x)$ tells us that for every change in x of 1 unit there is a corresponding change in y of $1/2$ unit. The function $h(x) = 2$ has a slope of zero, indicating that the values of the function remain constant. We see that the slope of each linear function indicates the rate of change of the function.

Compare the graphs of these three functions with the graph of $k(x) = x^2$ (Figure 2.1.2). The graph of $k(x) = x^2$ starts from the left by decreasing rapidly, then begins to decrease more slowly and level off, and then finally begins to increase—slowly at first, followed by an increasing rate of increase as it moves toward the right. Unlike a linear function, no single number represents the rate of change for this function. We quite naturally ask: How do we measure the rate of change of a nonlinear function?

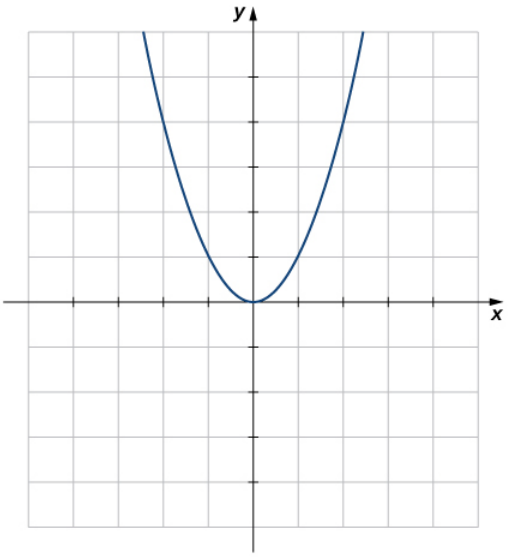

Figure 2.1.2: The function $k(x) = x^2$ does not have a constant rate of change.

We can approximate the rate of change of a function $f(x)$ at a point $(a, f(a))$ on its graph by taking another point $(x, f(x))$ on the graph of $f(x)$, drawing a line through the two points, and calculating the slope of the resulting line. Such a line is called a secant line. Figure 2.1.3 shows a *secant line* to a function $f(x)$ at a point $(a, f(a))$.

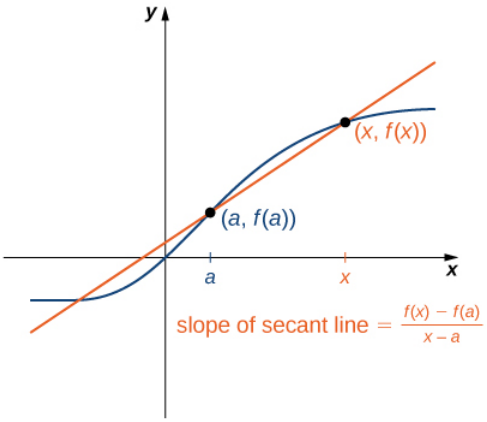

Figure 2.1.3: The slope of a secant line through a point $(a, f(a))$ estimates the rate of change of the function at the point $(a, f(a))$.

We formally define a secant line as follows:

Definition: Secant Line

The *secant* to the function $f(x)$ through the points $(a, f(a))$ and $(x, f(x))$ is the line passing through these points. Its slope is given by

$$m_{sec} = \frac{f(x) - f(a)}{x - a}. \tag{2.1.1}$$

The accuracy of approximating the rate of change of the function with a secant line depends on how close x is to a. As we see in Figure 2.1.4, if x is closer to a, the slope of the secant line is a better measure of the rate of change of $f(x)$ at a.

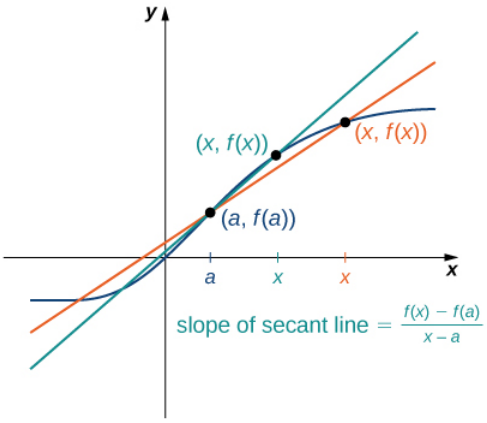

Figure 2.1.4: As x gets closer to a, the slope of the secant line becomes a better approximation to the rate of change of the function $f(x)$ at a.

The secant lines themselves approach a line that is called the **tangent** to the function $f(x)$ at a (Figure 2.1.5). The slope of the tangent line to the graph at a measures the rate of change of the function at a. This value also represents the derivative of the function $f(x)$ at a, or the rate of change of the function at a. This derivative is denoted by $f'(a)$. **Differential calculus** is the field of calculus concerned with the study of derivatives and their applications.

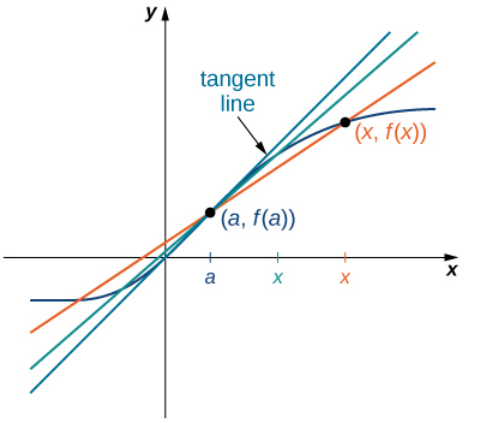

Figure 2.1.5: Solving the Tangent Problem: As x approaches a, the secant lines approach the tangent line.

Example 2.1.1 illustrates how to find slopes of secant lines. These slopes estimate the slope of the tangent line or, equivalently, the rate of change of the function at the point at which the slopes are calculated.

Example 2.1.1: Finding Slopes of Secant Lines

Estimate the slope of the tangent line (rate of change) to $f(x) = x^2$ at $x = 1$ by finding slopes of secant lines through $(1, 1)$ and each of the following points on the graph of $f(x) = x^2$.

a. $(2, 4)$
b. $\left(\frac{3}{2}, \frac{9}{4}\right)$

Solution:

Use the formula for the slope of a secant line (Equation 2.1.1).

a. $m_{sec} = \frac{4-1}{2-1} = 3$

b. $m_{sec} = \frac{\frac{9}{4}-1}{\frac{3}{2}-1} = \frac{5}{2} = 2.5$

The point in part b. is closer to the point $(1, 1)$, so the slope of 2.5 is closer to the slope of the tangent line. A good estimate for the slope of the tangent would be in the range of 2 to 2.5 (Figure 2.1.6).

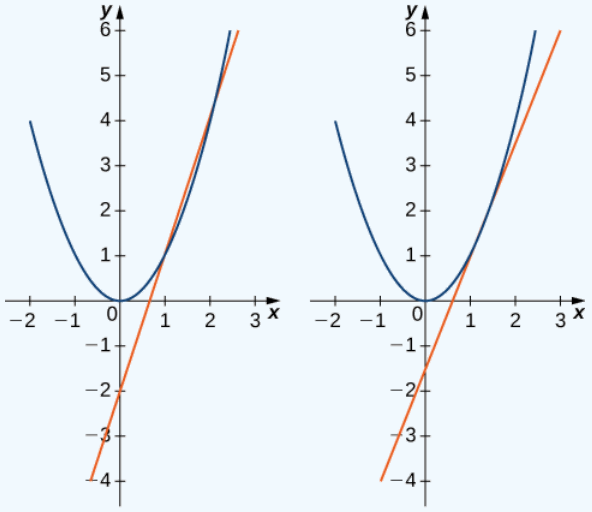

Figure 2.1.6: The secant lines to $f(x) = x^2$ at $(1, 1)$ through (a) $(2, 4)$ and (b) $(\frac{3}{2}, \frac{9}{4})$ provide successively closer approximations to the tangent line to $f(x) = x^2$ at $(1, 1)$.

Exercise 2.1.1

Estimate the slope of the tangent line (rate of change) to $f(x) = x^2$ at $x = 1$ by finding slopes of secant lines through $(1, 1)$ and the point $(\frac{5}{4}, \frac{25}{16})$ on the graph of $f(x) = x^2$.

Answer

2.25

We continue our investigation by exploring a related question. Keeping in mind that velocity may be thought of as the rate of change of position, suppose that we have a function, $s(t)$, that gives the position of an object along a coordinate axis at any given time t. Can we use these same ideas to create a reasonable definition of the instantaneous velocity at a given time $t = a$? We start by approximating the instantaneous velocity with an average velocity. First, recall that the speed of an object traveling at a constant rate is the ratio of the distance traveled to the length of time it has traveled. We define the *average velocity* of an object over a time period to be the change in its position divided by the length of the time period.

Definition: Average Velocity

Let $s(t)$ be the position of an object moving along a coordinate axis at time t. The *average velocity* of the object over a time interval $[a, t]$ where $a < t$ (or $[t, a]$ if $t < a$) is

$$v_{ave} = \frac{s(t) - s(a)}{t - a}. \tag{2.1.2}$$

As t is chosen closer to a, the average velocity becomes closer to the instantaneous velocity. Note that finding the average velocity of a position function over a time interval is essentially the same as finding the slope of a secant line to a function. Furthermore, to find the slope of a tangent line at a point a, we let the x-values approach a in the slope of the secant line. Similarly, to find the instantaneous velocity at time a, we let the t-values approach a in the average velocity. This process of letting x or t approach a in an expression is called taking a **limit**. Thus, we may define the *instantaneous velocity* as follows.

Definition: Instantaneous Velocity

For a position function $s(t)$, the *instantaneous velocity* at a time $t = a$ is the value that the average velocities approach on intervals of the form $[a, t]$ and $[t, a]$ as the values of t become closer to a, provided such a value exists.

Example 2.1.2 illustrates this concept of limits and average velocity.

Example 2.1.2: Finding Average Velocity

A rock is dropped from a height of 64 ft. It is determined that its height (in feet) above ground t seconds later (for $0 \le t \le 2$) is given by $s(t) = -16t^2 + 64$. Find the average velocity of the rock over each of the given time intervals. Use this information to guess the instantaneous velocity of the rock at time $t = 0.5$.

a. $[0.49, 0.5]$
b. $[0.5, 0.51]$

Solution

Substitute the data into Equation 2.1.2 for the definition of average velocity.

a. $$v_{ave} = \frac{s(0.49) - s(0.5)}{0.49 - 0.5} = -15.84$$

b. $$v_{ave} = \frac{s(0.51) - s(0.5)}{0.51 - 0.5} = -16.016$$

The instantaneous velocity is somewhere between −15.84 and −16.16 ft/sec. A good guess might be −16 ft/sec.

Exercise 2.1.2

An object moves along a coordinate axis so that its position at time t is given by $s(t) = t^3$. Estimate its instantaneous velocity at time $t = 2$ by computing its average velocity over the time interval $[2, 2.001]$.

Hint

Use Equation 2.1.2 with $v_{ave} = \frac{s(2.001) - s(2)}{2.001 - 2}$.

Answer

12.006001

The Area Problem and Integral Calculus

We now turn our attention to a classic question from calculus. Many quantities in physics—for example, quantities of work—may be interpreted as the area under a curve. This leads us to ask the question: How can we find the area between the graph of a function and the x-axis over an interval (Figure 2.1.7)?

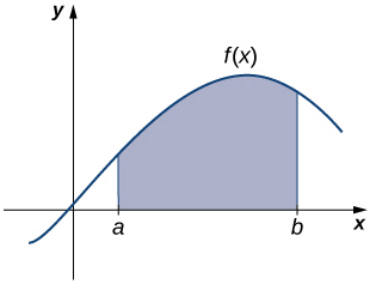

Figure 2.1.7: The Area Problem: How do we find the area of the shaded region?

As in the answer to our previous questions on velocity, we first try to approximate the solution. We approximate the area by dividing up the interval $[a, b]$ into smaller intervals in the shape of rectangles. The approximation of the area comes from adding up

the areas of these rectangles (Figure 2.1.8).

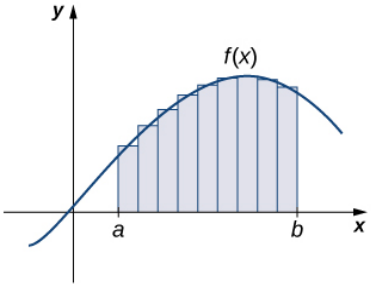

Figure 2.1.8: The area of the region under the curve is approximated by summing the areas of thin rectangles.

As the widths of the rectangles become smaller (approach zero), the sums of the areas of the rectangles approach the area between the graph of $f(x)$ and the x-axis over the interval $[a, b]$. Once again, we find ourselves taking a limit. Limits of this type serve as a basis for the definition of the definite integral. **Integral calculus** is the study of integrals and their applications.

Example 2.1.3: Estimation Using Rectangles

Estimate the area between the x-axis and the graph of $f(x) = x^2 + 1$ over the interval $[0, 3]$ by using the three rectangles shown in Figure 2.1.9.

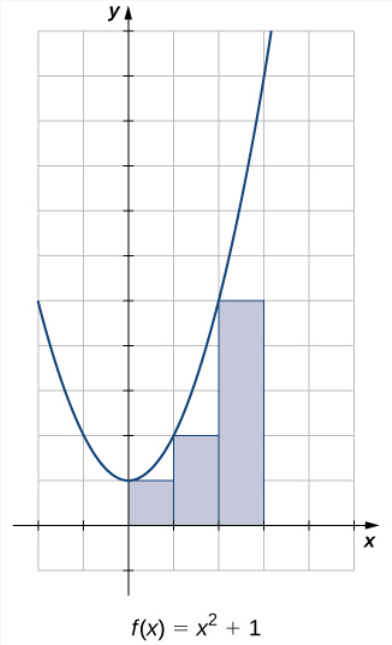

Figure 2.1.9: The area of the region under the curve of $f(x) = x^2 + 1$ can be estimated using rectangles.

Solution

The areas of the three rectangles are 1 unit^2, 2 unit^2, and 5 unit^2. Using these rectangles, our area estimate is 8 unit^2.

Exercise 2.1.3

Estimate the area between the x-axis and the graph of $f(x) = x^2 + 1$ over the interval $[0, 3]$ by using the three rectangles shown in Figure 2.1.10.

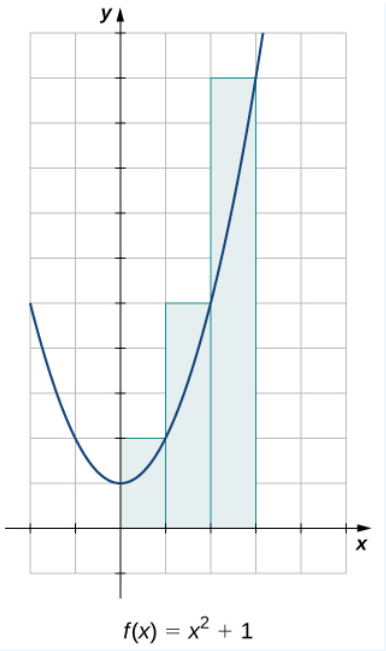

Figure 2.1.10: The area of the region under the curve of $f(x) = x^2 + 1$ can be estimated using rectangles.

Hint

Use Example 2.1.3 as a guide

Answer

17 unit^2

Other Aspects of Calculus

So far, we have studied functions of one variable only. Such functions can be represented visually using graphs in two dimensions; however, there is no good reason to restrict our investigation to two dimensions. Suppose, for example, that instead of determining the velocity of an object moving along a coordinate axis, we want to determine the velocity of a rock fired from a catapult at a given time, or of an airplane moving in three dimensions. We might want to graph real-value functions of two variables or determine volumes of solids of the type shown in Figure 2.1.11. These are only a few of the types of questions that can be asked and answered using *multivariable calculus*. Informally, multivariable calculus can be characterized as the study of the calculus of functions of two or more variables. However, before exploring these and other ideas, we must first lay a foundation for the study of calculus in one variable by exploring the concept of a limit.

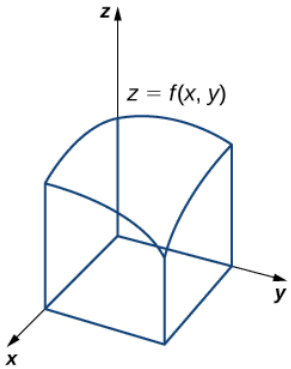

Figure 2.1.11: We can use multivariable calculus to find the volume between a surface defined by a function of two variables and a plane.

Key Concepts

- Differential calculus arose from trying to solve the problem of determining the slope of a line tangent to a curve at a point. The slope of the tangent line indicates the rate of change of the function, also called the *derivative*. Calculating a derivative requires finding a limit.
- Integral calculus arose from trying to solve the problem of finding the area of a region between the graph of a function and the x-axis. We can approximate the area by dividing it into thin rectangles and summing the areas of these rectangles. This summation leads to the value of a function called the *integral*. The integral is also calculated by finding a limit and, in fact, is related to the derivative of a function.
- Multivariable calculus enables us to solve problems in three-dimensional space, including determining motion in space and finding volumes of solids.

Key Equations

- **Slope of a Secant Line**

$$m_{sec} = \frac{f(x) - f(a)}{x - a}$$

- **Average Velocity over Interval [a,t]**

$$v_{ave} = \frac{s(t) - s(a)}{t - a}$$

Glossary

average velocity
the change in an object's position divided by the length of a time period; the average velocity of an object over a time interval $[t, a]$ (if $t < a$ or $[a, t]$ if $t > a$), with a position given by $s(t)$, that is $v_{ave} = \frac{s(t) - s(a)}{t - a}$

differential calculus
the field of calculus concerned with the study of derivatives and their applications

instantaneous velocity
The instantaneous velocity of an object with a position function that is given by $s(t)$ is the value that the average velocities on intervals of the form $[t, a]$ and $[a, t]$ approach as the values of t move closer to a, provided such a value exists

integral calculus
the study of integrals and their applications

limit
the process of letting x or t approach a in an expression; the limit of a function $f(x)$ as x approaches a is the value that $f(x)$ approaches as x approaches a

multivariable calculus
the study of the calculus of functions of two or more variables

secant
A secant line to a function $f(x)$ at a is a line through the point $(a, f(a))$ and another point on the function; the slope of the secant line is given by $m_{sec} = \frac{f(x) - f(a)}{x - a}$

tangent
A tangent line to the graph of a function at a point $(a, f(a))$ is the line that secant lines through $(a, f(a))$ approach as they are taken through points on the function with x-values that approach a; the slope of the tangent line to a graph at a measures the rate of change of the function at a

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2.1E: Exercises for Section 2.1

For exercises 1 - 3 , points $P(1,2)$ and $Q(x,y)$ are on the graph of the function $f(x)=x^2+1$.

1) [T] Complete the following table with the appropriate values: y-coordinate of Q, the point $Q(x,y)$, and the slope of the secant line passing through points P and Q. Round your answer to eight significant digits.

x	y	$Q(x,y)$	m_{sec}
1.1	a.	e.	i.
1.01	b.	f.	j.
1.001	c.	g.	k.
1.0001	d.	h.	l.

Answer

a. 2.2100000
b. 2.0201000
c. 2.0020010
d. 2.0002000
e. (1.1000000, 2.2100000)
f. (1.0100000, 2.0201000)
g. (1.0010000, 2.0020010)
h. (1.0001000, 2.0002000)
i. 2.1000000
j. 2.0100000
k. 2.0010000
l. 2.0001000

2) Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the line tangent to f at $x=1$.

3) Use the value in the preceding exercise to find the equation of the tangent line at point P. Graph $f(x)$ and the tangent line.

Answer

$y=2x$

For the exercises 4-6, points $P(1,1)$ and $Q(x,y)$ are on the graph of the function $f(x)=x^3$.

4) [T] Complete the following table with the appropriate values: y-coordinate of Q, the point $Q(x,y)$, and the slope of the secant line passing through points P and Q. Round your answer to eight significant digits.

x	y	$Q(x,y)$	m_{sec}
1.1	a.	e.	i.
1.01	b.	f.	j.
1.001	c.	g.	k.
1.0001	d.	h.	l.2

5) Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the tangent line to f at $x=1$.

Answer

3

6) Use the value in the preceding exercise to find the equation of the tangent line at point P. Graph $f(x)$ and the tangent line.

For the exercises 7 - 9, points $P(4,2)$ and $Q(x,y)$ are on the graph of the function $f(x)=\sqrt{x}$.

7) [T] Complete the following table with the appropriate values: y-coordinate of Q, the point $Q(x,y)$, and the slope of the secant line passing through points P and Q. Round your answer to eight significant digits.

x	y	$Q(x,y)$	m_{sec}
4.1	a.	e.	i.
4.01	b.	f.	j.
4.001	c.	g.	k.
4.0001	d.	h.	l.

Answer

a. 2.0248457
b. 2.0024984
c. 2.0002500
d. 2.0000250
e. (4.1000000,2.0248457)
f. (4.0100000,2.0024984)
g. (4.0010000,2.0002500)
h. (4.00010000,2.0000250)
i. 0.24845673
j. 0.24984395
k. 0.24998438
l. 0.24999844

8) Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the tangent line to f at $x=4$.

9) Use the value in the preceding exercise to find the equation of the tangent line at point P.

Answer

$y=\frac{x}{4}+1$

For exercises 10 - 12, points $P(1.5,0)$ and $Q(\phi,y)$ are on the graph of the function $f(\phi)=\cos(\pi\phi)$.

10) [T] Complete the following table with the appropriate values: y-coordinate of Q, the point $Q(\phi,y)$, and the slope of the secant line passing through points P and Q. Round your answer to eight significant digits.

x	y	$Q(\phi,y)$	m_{sec}
1.4	a.	e.	i.
1.49	b.	f.	j.
1.499	c.	g.	k.
1.4999	d.	h.	l.

11) Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the tangent line to f at $\phi=1.5$.

Answer

π

12) Use the value in the preceding exercise to find the equation of the tangent line at point P.

For exercises 13 - 15, points $P(-1,-1)$ and $Q(x,y)$ are on the graph of the function $f(x)=\frac{1}{x}$.

13) [T] Complete the following table with the appropriate values: y-coordinate of Q, the point $Q(x,y)$, and the slope of the secant line passing through points P and Q. Round your answer to eight significant digits.

x	y	$Q(x,y)$	m_{sec}
-1.05	a.	e.	i.
-1.01	b.	f.	j.
-1.005	c.	g.	k.
-1.001	d.	h.	l.

Answer

a. −0.95238095
b. −0.99009901
c. −0.99502488
d. −0.99900100
e. (−1;.0500000,−0;.95238095)
f. (−1;.0100000,−0;.9909901)
g. (−1;.0050000,−0;.99502488)
h. (1.0010000,−0;.99900100)
i. −0.95238095
j. −0.99009901
k. −0.99502488
l. −0.99900100

14) Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the line tangent to f at $x=-1$.

15) Use the value in the preceding exercise to find the equation of the tangent line at point P.

Answer

$y=-x-2$

For exercises 16 - 17, the position function of a ball dropped from the top of a 200-meter tall building is given by $s(t)=200-4.9t^2$, where position s is measured in meters and time t is measured in seconds. Round your answer to eight significant digits.

16) [T] Compute the average velocity of the ball over the given time intervals.

a. [4.99,5]

b. [5,5.01]

c. [4.999,5]

d. [5,5.001]

17) Use the preceding exercise to guess the instantaneous velocity of the ball at $t=5$ sec.

Answer

-49 m/sec (velocity of the ball is 49 m/sec downward)

For exercises 18 - 19, consider a stone tossed into the air from ground level with an initial velocity of 15 m/sec. Its height in meters at time t seconds is $h(t)=15t-4.9t^2$.

18) [T] Compute the average velocity of the stone over the given time intervals.

a. [1,1.05]

b. [1,1.01]

c. [1,1.005]

d. [1,1.001]

19) Use the preceding exercise to guess the instantaneous velocity of the stone at $t = 1$ sec.

Answer

5.2 m/sec

For exercises 20 - 21, consider a rocket shot into the air that then returns to Earth. The height of the rocket in meters is given by $h(t) = 600 + 78.4t - 4.9t^2$, where t is measured in seconds.

20) [T] Compute the average velocity of the rocket over the given time intervals.

a. [9,9.01]

b. [8.99,9]

c. [9,9.001]

d. [8.999,9]

21) Use the preceding exercise to guess the instantaneous velocity of the rocket at $t = 9$ sec.

Answer

-9.8 m/sec

For exercises, consider an athlete running a 40-m dash. The position of the athlete is given by $d(t) = \frac{t^3}{6} + 4t$, where d is the position in meters and t is the time elapsed, measured in seconds.

22) [T] Compute the average velocity of the runner over the given time intervals.

a. [1.95,2.05]

b. [1.995,2.005]

c. [1.9995,2.0005]

d. [2,2.00001]

23) Use the preceding exercise to guess the instantaneous velocity of the runner at $t = 2$ sec.

Answer

6 m/sec

For exercises 24 - 25, consider the function $f(x) = |x|$.

24) Sketch the graph of f over the interval $[-1, 2]$ and shade the region above the x-axis.

25) Use the preceding exercise to find the exact value of the area between the x-axis and the graph of f over the interval $[-1, 2]$ using rectangles. For the rectangles, use the square units, and approximate both above and below the lines. Use geometry to find the exact answer.

Answer

Under, 1 $unit^2$; over: 4 $unit^2$.
The exact area of the two triangles is $\frac{1}{2}(1)(1) + \frac{1}{2}(2)(2) = 2.5 units^2$.

For exercises 26 - 27, consider the function $f(x) = \sqrt{1 - x^2}$. (Hint: This is the upper half of a circle of radius 1 positioned at $(0, 0)$.)

26) Sketch the graph of f over the interval $[-1, 1]$.

27) Use the preceding exercise to find the exact area between the x-axis and the graph of f over the interval $[-1, 1]$ using rectangles. For the rectangles, use squares 0.4 by 0.4 units, and approximate both above and below the lines. Use geometry to find the exact answer.

Answer

Under, 0.96 units^2; over, 1.92 units^2).
The exact area of the semicircle with radius 1 is $\frac{\pi(1)^2}{2} = \frac{\pi}{2} \text{ units}^2$

For exercises 28 - 29, consider the function $f(x) = -x^2 + 1$.

28) Sketch the graph of f over the interval $[-1, 1]$.

29) Approximate the area of the region between the x-axis and the graph of f over the interval $[-1, 1]$.

Answer

Approximately $1.3333333 \text{ units}^2$

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2.2: The Limit of a Function

Learning Objectives

- Using correct notation, describe the limit of a function.
- Use a table of values to estimate the limit of a function or to identify when the limit does not exist.
- Use a graph to estimate the limit of a function or to identify when the limit does not exist.
- Define one-sided limits and provide examples.
- Explain the relationship between one-sided and two-sided limits.
- Using correct notation, describe an infinite limit.
- Define a vertical asymptote.

The concept of a limit or limiting process, essential to the understanding of calculus, has been around for thousands of years. In fact, early mathematicians used a limiting process to obtain better and better approximations of areas of circles. Yet, the formal definition of a limit—as we know and understand it today—did not appear until the late 19th century. We therefore begin our quest to understand limits, as our mathematical ancestors did, by using an intuitive approach. At the end of this chapter, armed with a conceptual understanding of limits, we examine the formal definition of a limit.

We begin our exploration of limits by taking a look at the graphs of the functions

- $f(x) = \dfrac{x^2 - 4}{x - 2}$,
- $g(x) = \dfrac{|x - 2|}{x - 2}$, and
- $h(x) = \dfrac{1}{(x - 2)^2}$,

which are shown in Figure 2.2.1. In particular, let's focus our attention on the behavior of each graph at and around $x = 2$.

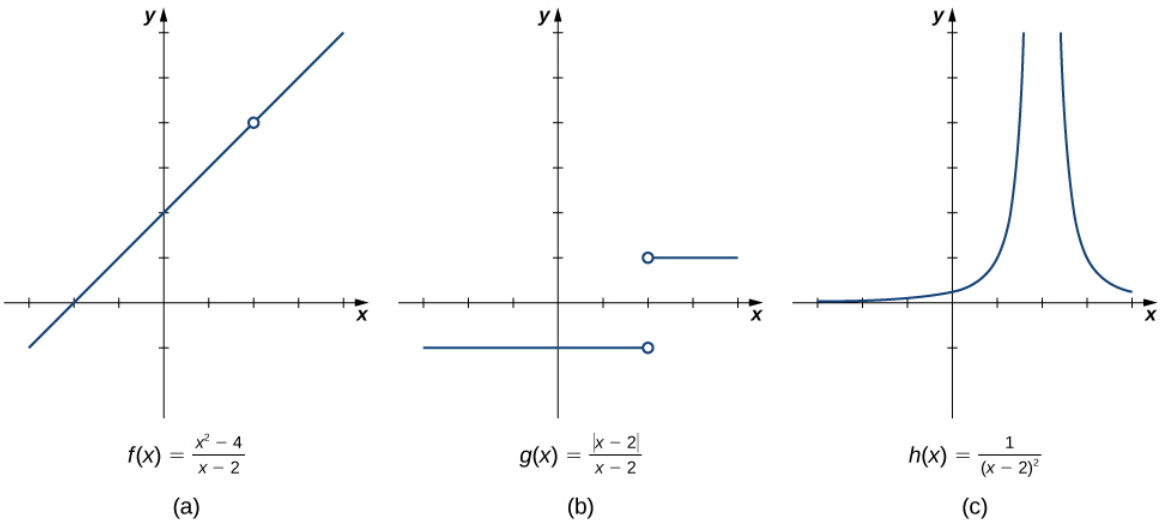

Figure 2.2.1: These graphs show the behavior of three different functions around $x = 2$.

Each of the three functions is undefined at $x = 2$, but if we make this statement and no other, we give a very incomplete picture of how each function behaves in the vicinity of $x = 2$. To express the behavior of each graph in the vicinity of 2 more completely, we need to introduce the concept of a limit.

Intuitive Definition of a Limit

Let's first take a closer look at how the function $f(x) = (x^2 - 4)/(x - 2)$ behaves around $x = 2$ in Figure 2.2.1. As the values of x approach 2 from either side of 2, the values of $y = f(x)$ approach 4. Mathematically, we say that the limit of $f(x)$ as x approaches 2 is 4. Symbolically, we express this limit as

$$\lim_{x \to 2} f(x) = 4$$

From this very brief informal look at one limit, let's start to develop an *intuitive definition of the limit*. We can think of the limit of a function at a number a as being the one real number L that the functional values approach as the x-values approach a, provided such a real number L exists. Stated more carefully, we have the following definition:

Definition (Intuitive): Limit

Let $f(x)$ be a function defined at all values in an open interval containing a, with the possible exception of a itself, and let L be a real number. If all values of the function $f(x)$ approach the real number L as the values of $x(\neq a)$ approach the number a, then we say that the limit of $f(x)$ as x approaches a is L. (More succinctly, as x gets closer to a, $f(x)$ gets closer and stays close to L.) Symbolically, we express this idea as

$$\lim_{x\to a} f(x) = L. \tag{2.2.1}$$

We can estimate limits by constructing tables of functional values and by looking at their graphs. This process is described in the following Problem-Solving Strategy.

Problem-Solving Strategy: Evaluating a Limit Using a Table of Functional Values

1. To evaluate $\lim_{x\to a} f(x)$, we begin by completing a table of functional values. We should choose two sets of x-values—one set of values approaching a and less than a, and another set of values approaching a and greater than a. Table 2.2.1 demonstrates what your tables might look like.

Table 2.2.1

x	$f(x)$	x	$f(x)$
$a-0.1$	$f(a-0.1)$	$a+0.1$	$f(a+0.1)$
$a-0.01$	$f(a-0.01)$	$a+0.01$	$f(a+0.01)$
$a-0.001$	$f(a-0.001)$	$a+0.001$	$f(a+0.001)$
$a-0.0001$	$f(a-0.0001)$	$a+0.0001$	$f(a+0.0001)$
Use additional values as necessary.		Use additional values as necessary.	

2. Next, let's look at the values in each of the $f(x)$ columns and determine whether the values seem to be approaching a single value as we move down each column. In our columns, we look at the sequence $f(a-0.1)$, $f(a-0.01)$, $f(a-0.001)$, $f(a-0.0001)$, and so on, and $f(a+0.1)$, $f(a+0.01)$, $f(a+0.001)$, $f(a+0.0001)$, and so on. (Note: Although we have chosen the x-values $a\pm 0.1$, $a\pm 0.01$, $a\pm 0.001$, $a\pm 0.0001$, and so forth, and these values will probably work nearly every time, on very rare occasions we may need to modify our choices.)

3. If both columns approach a common y-value L, we state $\lim_{x\to a} f(x) = L$. We can use the following strategy to confirm the result obtained from the table or as an alternative method for estimating a limit.

4. Using a graphing calculator or computer software that allows us graph functions, we can plot the function $f(x)$, making sure the functional values of $f(x)$ for x-values near a are in our window. We can use the trace feature to move along the graph of the function and watch the y-value readout as the x-values approach a. If the y-values approach L as our x-values approach a from both directions, then $\lim_{x\to a} f(x) = L$. We may need to zoom in on our graph and repeat this process several times.

We apply this Problem-Solving Strategy to compute a limit in Examples 2.2.1A and 2.2.1B.

Example 2.2.1A: Evaluating a Limit Using a Table of Functional Values

Evaluate $\lim_{x\to 0} \dfrac{\sin x}{x}$ using a table of functional values.

Solution

We have calculated the values of $f(x) = \dfrac{\sin x}{x}$ for the values of x listed in Table 2.2.2.

Table 2.2.2

x	$\frac{\sin x}{x}$	x	$\frac{\sin x}{x}$
-0.1	0.998334166468	0.1	0.998334166468
-0.01	0.999983333417	0.01	0.999983333417
-0.001	0.999999833333	0.001	0.999999833333
-0.0001	0.999999998333	0.0001	0.999999998333

Note: The values in this table were obtained using a calculator and using all the places given in the calculator output.

As we read down each $\frac{\sin x}{x}$ column, we see that the values in each column appear to be approaching one. Thus, it is fairly reasonable to conclude that $\lim_{x\to 0}\frac{\sin x}{x} = 1$. A calculator-or computer-generated graph of $f(x) = \frac{\sin x}{x}$ would be similar to that shown in Figure 2.2.2, and it confirms our estimate.

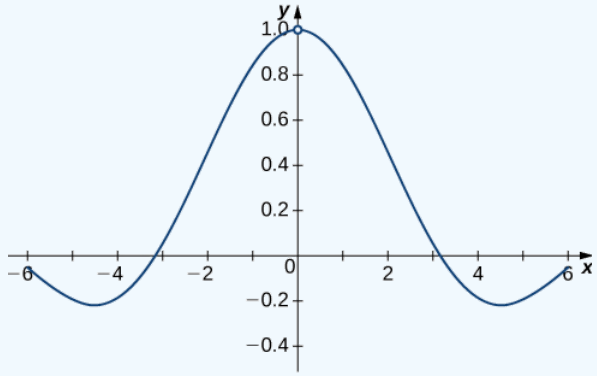

Figure 2.2.2: The graph of $f(x) = (\sin x)/x$ confirms the estimate from Table 2.2.2.

Example 2.2.1B: Evaluating a Limit Using a Table of Functional Values

Evaluate $\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}$ using a table of functional values.

Solution

As before, we use a table—in this case, Table 2.2.3—to list the values of the function for the given values of x.

Table 2.2.3

x	$\frac{\sqrt{x}-2}{x-4}$	x	$\frac{\sqrt{x}-2}{x-4}$
3.9	0.251582341869	4.1	0.248456731317
3.99	0.25015644562	4.01	0.24984394501
3.999	0.250015627	4.001	0.249984377
3.9999	0.250001563	4.0001	0.249998438
3.99999	0.25000016	4.00001	0.24999984

After inspecting this table, we see that the functional values less than 4 appear to be decreasing toward 0.25 whereas the functional values greater than 4 appear to be increasing toward 0.25. We conclude that $\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4} = 0.25$. We confirm this estimate using the graph of $f(x) = \frac{\sqrt{x}-2}{x-4}$ shown in Figure 2.2.3.

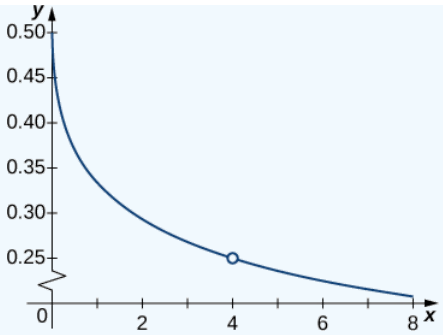

Figure 2.2.3: The graph of $\frac{\sqrt{x}-2}{x-4}$ confirms the estimate from Table 2.2.3.

Exercise 2.2.1

Estimate $\lim_{x\to 1} \dfrac{\frac{1}{x}-1}{x-1}$ using a table of functional values. Use a graph to confirm your estimate.

Hint

Use 0.9, 0.99, 0.999, 0.9999, 0.99999 and 1.1, 1.01, 1.001, 1.0001, 1.00001 as your table values.

Answer

$$\lim_{x\to 1} \frac{\frac{1}{x}-1}{x-1} = -1$$

At this point, we see from Examples 2.2.1A and 2.2.1B that it may be just as easy, if not easier, to estimate a limit of a function by inspecting its graph as it is to estimate the limit by using a table of functional values. In Example 2.2.2, we evaluate a limit exclusively by looking at a graph rather than by using a table of functional values.

Example 2.2.2: Evaluating a Limit Using a Graph

For $g(x)$ shown in Figure 2.2.4, evaluate $\lim_{x\to -1} g(x)$.

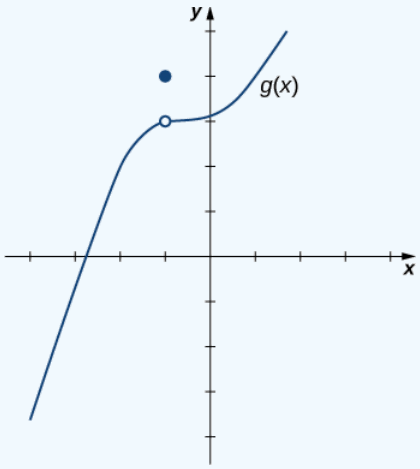

Figure 2.2.4: The graph of $g(x)$ includes one value not on a smooth curve.

Solution:

Despite the fact that $g(-1) = 4$, as the x-values approach -1 from either side, the $g(x)$ values approach 3. Therefore, $\lim_{x\to -1} g(x) = 3$. Note that we can determine this limit without even knowing the algebraic expression of the function.

Based on Example 2.2.2, we make the following observation: It is possible for the limit of a function to exist at a point, and for the function to be defined at this point, but the limit of the function and the value of the function at the point may be different.

Exercise 2.2.2

Use the graph of $h(x)$ in Figure 2.2.5 to evaluate $\lim_{x\to 2} h(x)$, if possible.

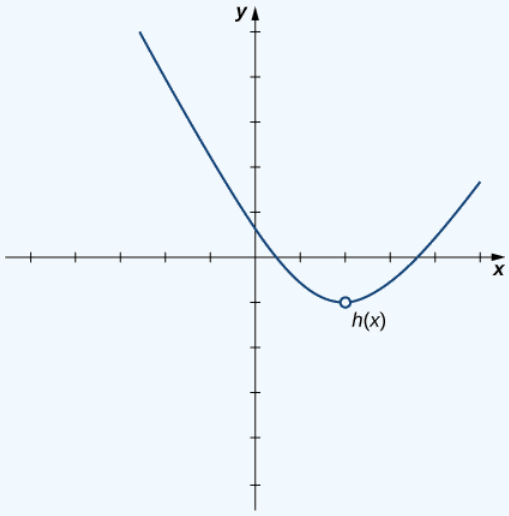

Figure 2.2.5:

Hint

What y-value does the function approach as the x-values approach 2?

Solution

$\lim_{x\to 2} h(x) = -1$.

Looking at a table of functional values or looking at the graph of a function provides us with useful insight into the value of the limit of a function at a given point. However, these techniques rely too much on guesswork. We eventually need to develop alternative methods of evaluating limits. These new methods are more algebraic in nature and we explore them in the next section; however, at this point we introduce two special limits that are foundational to the techniques to come.

Two Important Limits

Let a be a real number and c be a constant.

i. $\lim_{x\to a} x = a$
ii. $\lim_{x\to a} c = c$

We can make the following observations about these two limits.

i. For the first limit, observe that as x approaches a, so does $f(x)$, because $f(x) = x$. Consequently, $\lim_{x\to a} x = a$.
ii. For the second limit, consider Table 2.2.4.

Table 2.2.4

x	$f(x) = c$	x	$f(x) = c$
$a - 0.1$	c	$a + 0.1$	c
$a - 0.01$	c	$a + 0.01$	c
$a - 0.001$	c	$a + 0.001$	c
$a - 0.0001$	c	$a + 0.0001$	c

Observe that for all values of x (regardless of whether they are approaching a), the values $f(x)$ remain constant at c. We have no choice but to conclude $\lim_{x\to a} c = c$.

The Existence of a Limit

As we consider the limit in the next example, keep in mind that for the limit of a function to exist at a point, the functional values must approach a single real-number value at that point. If the functional values do not approach a single value, then the limit does not exist.

Example 2.2.3: Evaluating a Limit That Fails to Exist

Evaluate $\lim_{x\to 0} \sin(1/x)$ using a table of values.

Solution

Table 2.2.5 lists values for the function $\sin(1/x)$ for the given values of x.

Table 2.2.5

x	$\sin(1/x)$	x	$\sin(1/x)$
-0.1	0.544021110889	0.1	−0.544021110889
-0.01	0.50636564111	0.01	−0.50636564111
-0.001	−0.8268795405312	0.001	0.8268795405312
-0.0001	0.305614388888	0.0001	−0.305614388888
-0.00001	−0.035748797987	0.00001	0.035748797987
-0.000001	0.349993504187	0.000001	−0.349993504187

After examining the table of functional values, we can see that the y-values do not seem to approach any one single value. It appears the limit does not exist. Before drawing this conclusion, let's take a more systematic approach. Take the following sequence of x-values approaching 0:

$$\frac{2}{\pi}, \frac{2}{3\pi}, \frac{2}{5\pi}, \frac{2}{7\pi}, \frac{2}{9\pi}, \frac{2}{11\pi}, \ldots.$$

The corresponding y-values are

$$1, -1, 1, -1, 1, -1, \ldots.$$

At this point we can indeed conclude that $\lim_{x\to 0} \sin(1/x)$ does not exist. (Mathematicians frequently abbreviate "does not exist" as DNE. Thus, we would write $\lim_{x\to 0} \sin(1/x)$ DNE.) The graph of $f(x) = \sin(1/x)$ is shown in Figure 2.2.6 and it gives a clearer picture of the behavior of $\sin(1/x)$ as x approaches 0. You can see that $\sin(1/x)$ oscillates ever more wildly between -1 and 1 as x approaches 0.

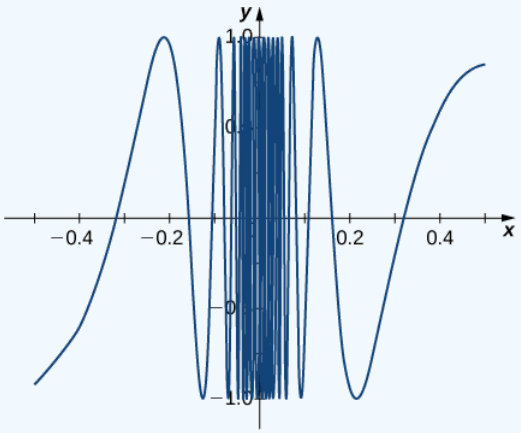

Figure 2.2.6: The graph of $f(x) = \sin(1/x)$ oscillates rapidly between -1 and 1 as x approaches 0.

Exercise 2.2.3

Use a table of functional values to evaluate $\lim\limits_{x\to 2}\dfrac{|\,x^2-4\,|}{x-2}$, if possible.

Hint

Use x-values 1.9, 1.99, 1.999, 1.9999, 1.99999 and 2.1, 2.01, 2.001, 2.0001, 2.00001 in your table.

Answer

$\lim\limits_{x\to 2}\dfrac{|\,x^2-4\,|}{x-2}$ does not exist.

One-Sided Limits

Sometimes indicating that the limit of a function fails to exist at a point does not provide us with enough information about the behavior of the function at that particular point. To see this, we now revisit the function $g(x)=|x-2|/(x-2)$ introduced at the beginning of the section (see Figure 2.2.1(b)). As we pick values of x close to 2, $g(x)$ does not approach a single value, so the limit as x approaches 2 does not exist—that is, $\lim\limits_{x\to 2} g(x)$ DNE. However, this statement alone does not give us a complete picture of the behavior of the function around the x-value 2. To provide a more accurate description, we introduce the idea of a **one-sided limit**. For all values to the left of 2 (or the negative side of 2), $g(x)=-1$. Thus, as x approaches 2 from the left, $g(x)$ approaches -1. Mathematically, we say that the limit as x approaches 2 from the left is -1. Symbolically, we express this idea as

$$\lim_{x\to 2^-} g(x)=-1.$$

Similarly, as x approaches 2 from the right (or from the positive side), $g(x)$ approaches 1. Symbolically, we express this idea as

$$\lim_{x\to 2^+} g(x)=1.$$

We can now present an informal definition of one-sided limits.

Definition: One-sided Limits

We define two types of one-sided limits.

Limit from the left:

Let $f(x)$ be a function defined at all values in an open interval of the form (z,a), and let L be a real number. If the values of the function $f(x)$ approach the real number L as the values of x (where $x<a$) approach the number a, then we say that L is the limit of $f(x)$ as x approaches a from the left. Symbolically, we express this idea as

$$\lim_{x\to a^-} f(x)=L.$$

Limit from the right:

Let $f(x)$ be a function defined at all values in an open interval of the form (a,c), and let L be a real number. If the values of the function $f(x)$ approach the real number L as the values of x (where $x>a$) approach the number a, then we say that L is the limit of $f(x)$ as x approaches a from the right. Symbolically, we express this idea as

$$\lim_{x\to a^+} f(x)=L.$$

Example 2.2.4: Evaluating One-Sided Limits

For the function $f(x)=\begin{cases}x+1, & \text{if } x<2\\ x^2-4, & \text{if } x\geq 2\end{cases}$, evaluate each of the following limits.

a. $\lim\limits_{x\to 2^-} f(x)$

b. $\lim\limits_{x\to 2^+} f(x)$

Solution

We can use tables of functional values again. Observe in Table 2.2.6 that for values of x less than 2, we use $f(x) = x + 1$ and for values of x greater than 2, we use $f(x) = x^2 - 4$.

Table 2.2.6

x	$f(x) = x + 1$	x	$f(x) = x^2 - 4$
1.9	2.9	2.1	0.41
1.99	2.99	2.01	0.0401
1.999	2.999	2.001	0.004001
1.9999	2.9999	2.0001	0.00040001
1.99999	2.99999	2.00001	0.0000400001

Based on this table, we can conclude that a. $\lim_{x \to 2^-} f(x) = 3$ and b. $\lim_{x \to 2^+} f(x) = 0$. Therefore, the (two-sided) limit of $f(x)$ does not exist at $x = 2$. Figure 2.2.7 shows a graph of $f(x)$ and reinforces our conclusion about these limits.

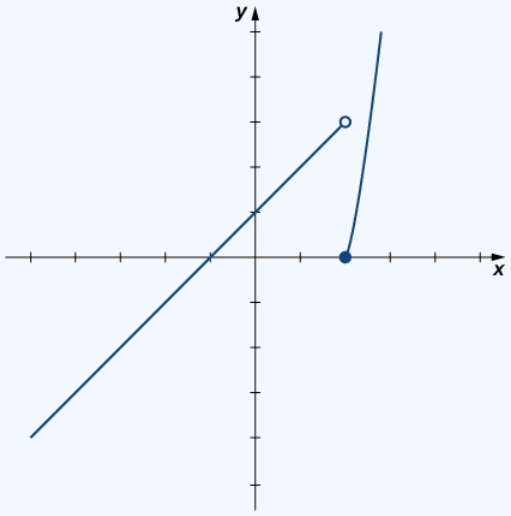

Figure 2.2.7: The graph of $f(x) = \begin{cases} x + 1, & \text{if } x < 2 \\ x^2 - 4, & \text{if } x \geq 2 \end{cases}$ has a break at $x = 2$.

Exercise 2.2.4

Use a table of functional values to estimate the following limits, if possible.

a. $\lim_{x \to 2^-} \dfrac{|\, x^2 - 4 \,|}{x - 2}$

b. $\lim_{x \to 2^+} \dfrac{|\, x^2 - 4 \,|}{x - 2}$

Hint

Use x-values 1.9, 1.99, 1.999, 1.9999, 1.99999 to estimate $\lim_{x \to 2^-} \dfrac{|\, x^2 - 4 \,|}{x - 2}$.

Use x-values 2.1, 2.01, 2.001, 2.0001, 2.00001 to estimate $\lim_{x \to 2^+} \dfrac{|\, x^2 - 4 \,|}{x - 2}$.

(These tables are available from a previous Checkpoint problem.)

Solution a

a. $\lim_{x \to 2^-} \dfrac{|\, x^2 - 4 \,|}{x - 2} = -4$

Solution b

$\lim_{x \to 2^+} \dfrac{|\, x^2 - 4 \,|}{x - 2} = 4$

Let us now consider the relationship between the limit of a function at a point and the limits from the right and left at that point. It seems clear that if the limit from the right and the limit from the left have a common value, then that common value is the limit of the function at that point. Similarly, if the limit from the left and the limit from the right take on different values, the limit of the function does not exist. These conclusions are summarized in Note.

Relating One-Sided and Two-Sided Limits

Let $f(x)$ be a function defined at all values in an open interval containing a, with the possible exception of a itself, and let L be a real number. Then,

$$\lim_{x \to a} f(x) = L$$

if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$.

Infinite Limits

Evaluating the limit of a function at a point or evaluating the limit of a function from the right and left at a point helps us to characterize the behavior of a function around a given value. As we shall see, we can also describe the behavior of functions that do not have finite limits.

We now turn our attention to $h(x) = 1/(x-2)^2$, the third and final function introduced at the beginning of this section (see Figure 2.2.1(c)). From its graph we see that as the values of x approach 2, the values of $h(x) = 1/(x-2)^2$ become larger and larger and, in fact, become infinite. Mathematically, we say that the limit of $h(x)$ as x approaches 2 is positive infinity. Symbolically, we express this idea as

$$\lim_{x \to 2} h(x) = +\infty.$$

More generally, we define *infinite limits* as follows:

Definitions: Infinite Limits

We define three types of *infinite limits*.

Infinite limits from the left: Let $f(x)$ be a function defined at all values in an open interval of the form (b, a).

i. If the values of $f(x)$ increase without bound as the values of x (where $x < a$) approach the number a, then we say that the limit as x approaches a from the left is positive infinity and we write

$$\lim_{x \to a^-} f(x) = +\infty.$$

ii. If the values of $f(x)$ decrease without bound as the values of x (where $x < a$) approach the number a, then we say that the limit as x approaches a from the left is negative infinity and we write

$$\lim_{x \to a^-} f(x) = -\infty.$$

Infinite limits from the right: Let $f(x)$ be a function defined at all values in an open interval of the form (a, c).

i. If the values of $f(x)$ increase without bound as the values of x (where $x > a$) approach the number a, then we say that the limit as x approaches a from the right is positive infinity and we write

$$\lim_{x \to a^+} f(x) = +\infty.$$

ii. If the values of $f(x)$ decrease without bound as the values of x (where $x > a$) approach the number a, then we say that the limit as x approaches a from the right is negative infinity and we write

$$\lim_{x \to a^+} f(x) = -\infty.$$

Two-sided infinite limit: Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a

i. If the values of $f(x)$ increase without bound as the values of x (where $x \neq a$) approach the number a, then we say that the limit as x approaches a is positive infinity and we write

$$\lim_{x \to a} f(x) = +\infty.$$

ii. If the values of $f(x)$ decrease without bound as the values of x (where $x \neq a$) approach the number a, then we say that the limit as x approaches a is negative infinity and we write

$$\lim_{x \to a} f(x) = -\infty.$$

It is important to understand that when we write statements such as $\lim_{x \to a} f(x) = +\infty$ or $\lim_{x \to a} f(x) = -\infty$ we are describing the behavior of the function, as we have just defined it. We are not asserting that a limit exists. For the limit of a function $f(x)$ to exist at a, it must approach a real number L as x approaches a. That said, if, for example, $\lim_{x \to a} f(x) = +\infty$, we always write $\lim_{x \to a} f(x) = +\infty$ rather than $\lim_{x \to a} f(x)$ DNE.

Example 2.2.5: Recognizing an Infinite Limit

Evaluate each of the following limits, if possible. Use a table of functional values and graph $f(x) = 1/x$ to confirm your conclusion.

a. $\lim_{x \to 0^-} \frac{1}{x}$
b. $\lim_{x \to 0^+} \frac{1}{x}$
c. $\lim_{x \to 0} \frac{1}{x}$

Solution

Begin by constructing a table of functional values.

Table 2.2.7

x	$\frac{1}{x}$	x	$\frac{1}{x}$
-0.1	-10	0.1	10
-0.01	-100	0.01	100
-0.001	-1000	0.001	1000
-0.0001	-10,000	0.0001	10,000
-0.00001	-100,000	0.00001	100,000
-0.000001	-1,000,000	0.000001	1,000,000

a. The values of $1/x$ decrease without bound as x approaches 0 from the left. We conclude that

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

b. The values of $1/x$ increase without bound as x approaches 0 from the right. We conclude that

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty.$$

c. Since $\lim_{x \to 0^-} \frac{1}{x} = -\infty$ and $\lim_{x \to 0^+} \frac{1}{x} = +\infty$ have different values, we conclude that

$$\lim_{x \to 0} \frac{1}{x} \quad \text{DNE.}$$

The graph of $f(x) = 1/x$ in Figure 2.2.8 confirms these conclusions.

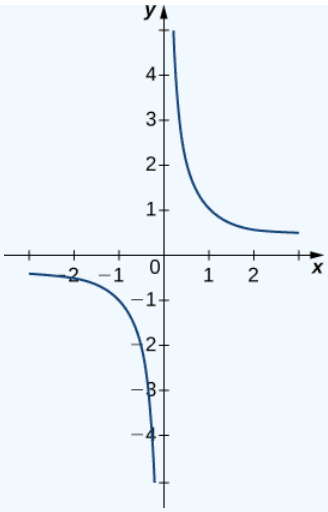

Figure 2.2.8: The graph of $f(x) = 1/x$ confirms that the limit as x approaches 0 does not exist.

Exercise 2.2.5

Evaluate each of the following limits, if possible. Use a table of functional values and graph $f(x) = 1/x^2$ to confirm your conclusion.

a. $\lim\limits_{x \to 0^-} \dfrac{1}{x^2}$
b. $\lim\limits_{x \to 0^+} \dfrac{1}{x^2}$
c. $\lim\limits_{x \to 0} \dfrac{1}{x^2}$

Hint

Follow the procedures from Example 2.2.5.

Answer

a. $\lim\limits_{x \to 0^-} \dfrac{1}{x^2} = +\infty$;

b. $\lim\limits_{x \to 0^+} \dfrac{1}{x^2} = +\infty$;

c. $\lim\limits_{x \to 0} \dfrac{1}{x^2} = +\infty$

It is useful to point out that functions of the form $f(x) = 1/(x-a)^n$, where n is a positive integer, have infinite limits as x approaches a from either the left or right (Figure 2.2.9). These limits are summarized in the above definitions.

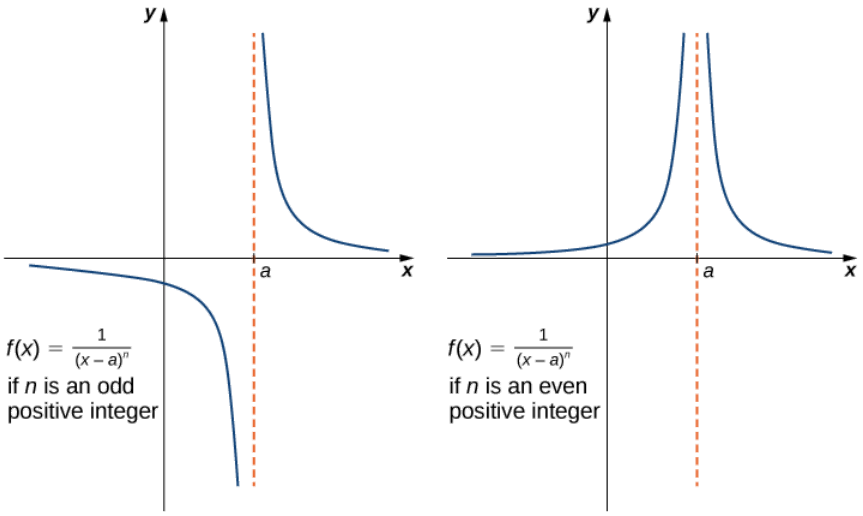

Figure 2.2.9: The function $f(x) = 1/(x-a)^n$ has infinite limits at a.

Infinite Limits from Positive Integers

If n is a positive even integer, then

$$\lim_{x \to a} \frac{1}{(x-a)^n} = +\infty. \tag{2.2.2}$$

If n is a positive odd integer, then

$$\lim_{x \to a^+} \frac{1}{(x-a)^n} = +\infty \tag{2.2.3}$$

and

$$\lim_{x \to a^-} \frac{1}{(x-a)^n} = -\infty. \tag{2.2.4}$$

We should also point out that in the graphs of $f(x) = 1/(x-a)^n$, points on the graph having x-coordinates very near to a are very close to the vertical line $x = a$. That is, as x approaches a, the points on the graph of $f(x)$ are closer to the line $x = a$. The line $x = a$ is called a vertical asymptote of the graph. We formally define a *vertical asymptote* as follows:

Definition: Vertical Asymptotes

Let $f(x)$ be a function. If any of the following conditions hold, then the line $x = a$ is a vertical asymptote of $f(x)$.

$$\lim_{x \to a^-} f(x) = +\infty$$

$$\lim_{x \to a^-} f(x) = -\infty$$

$$\lim_{x \to a^+} f(x) = +\infty$$

$$\lim_{x \to a^+} f(x) = -\infty$$

$$\lim_{x \to a} f(x) = +\infty$$

$$\lim_{x \to a} f(x) = -\infty$$

Example 2.2.6: Finding a Vertical Asymptote

Evaluate each of the following limits using Equations 2.2.2, 2.2.3, and 2.2.4 above. Identify any vertical asymptotes of the function $f(x) = 1/(x+3)^4$.

a. $\lim\limits_{x \to -3^-} \dfrac{1}{(x+3)^4}$
b. $\lim\limits_{x \to -3^+} \dfrac{1}{(x+3)^4}$
c. $\lim\limits_{x \to -3} \dfrac{1}{(x+3)^4}$

Solution

We can use the above equations directly.

a. $\lim\limits_{x \to -3^-} \dfrac{1}{(x+3)^4} = +\infty$
b. $\lim\limits_{x \to -3^+} \dfrac{1}{(x+3)^4} = +\infty$
c. $\lim\limits_{x \to -3} \dfrac{1}{(x+3)^4} = +\infty$

The function $f(x) = 1/(x+3)^4$ has a vertical asymptote of $x = -3$.

Exercise 2.2.6

Evaluate each of the following limits. Identify any vertical asymptotes of the function $f(x) = \dfrac{1}{(x-2)^3}$.

a. $\lim\limits_{x \to 2^-} \dfrac{1}{(x-2)^3}$
b. $\lim\limits_{x \to 2^+} \dfrac{1}{(x-2)^3}$
c. $\lim\limits_{x \to 2} \dfrac{1}{(x-2)^3}$

Answer a

$\lim\limits_{x \to 2^-} \dfrac{1}{(x-2)^3} = -\infty$

Answer b

$\lim\limits_{x \to 2^+} \dfrac{1}{(x-2)^3} = +\infty$

Answer c

$\lim\limits_{x \to 2} \dfrac{1}{(x-2)^3}$ DNE. The line $x = 2$ is the vertical asymptote of $f(x) = 1/(x-2)^3$.

In the next example we put our knowledge of various types of limits to use to analyze the behavior of a function at several different points.

Example 2.2.7: Behavior of a Function at Different Points

Use the graph of $f(x)$ in Figure 2.2.10 to determine each of the following values:

a. $\lim\limits_{x \to -4^-} f(x)$; $\lim\limits_{x \to -4^+} f(x)$; $\lim\limits_{x \to -4} f(x)$; $f(-4)$
b. $\lim\limits_{x \to -2^-} f(x)$; $\lim\limits_{x \to -2^+} f(x)$; $\lim\limits_{x \to -2} f(x)$; $f(-2)$
c. $\lim\limits_{x \to 1^-} f(x)$; $\lim\limits_{x \to 1^+} f(x)$; $\lim\limits_{x \to 1} f(x)$; $f(1)$
d. $\lim\limits_{x \to 3^-} f(x)$; $\lim\limits_{x \to 3^+} f(x)$; $\lim\limits_{x \to 3} f(x)$; $f(3)$

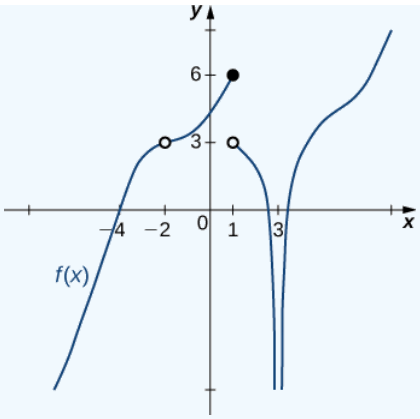

Figure 2.2.10: The graph shows $f(x)$.

Solution

Using the definitions above and the graph for reference, we arrive at the following values:

a. $\lim\limits_{x\to -4^-} f(x) = 0$; $\lim\limits_{x\to -4^+} f(x) = 0$; $\lim\limits_{x\to -4} f(x) = 0$; $f(-4) = 0$
b. $\lim\limits_{x\to -2^-} f(x) = 3$; $\lim\limits_{x\to -2^+} f(x) = 3$; $\lim\limits_{x\to -2} f(x) = 3$; $f(-2)$ is undefined
c. $\lim\limits_{x\to 1^-} f(x) = 6$; $\lim\limits_{x\to 1^+} f(x) = 3$; $\lim\limits_{x\to 1} f(x)$ DNE; $f(1) = 6$
d. $\lim\limits_{x\to 3^-} f(x) = -\infty$; $\lim\limits_{x\to 3^+} f(x) = -\infty$; $\lim\limits_{x\to 3} f(x) = -\infty$; $f(3)$ is undefined

Exercise 2.2.7

Evaluate $\lim\limits_{x\to 1} f(x)$ for $f(x)$ shown here:

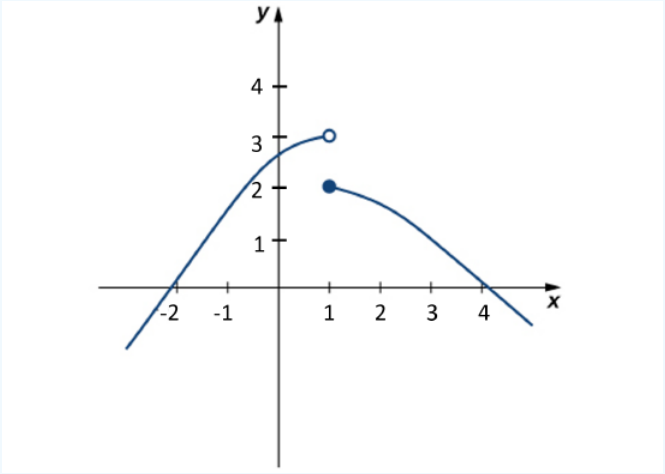

Figure 2.2.11. The graph of a piecewise function f.

Hint

Compare the limit from the right with the limit from the left.

Answer

$\lim\limits_{x\to 1} f(x)$ does not exist

Example 2.2.8: Einstein's Equation

In the Chapter opener we mentioned briefly how Albert Einstein showed that a limit exists to how fast any object can travel. Given Einstein's equation for the mass of a moving object

$$m = \frac{m_0}{}$$

$$\sqrt{1-\frac{v^2}{c^2}},$$

what is the value of this bound?

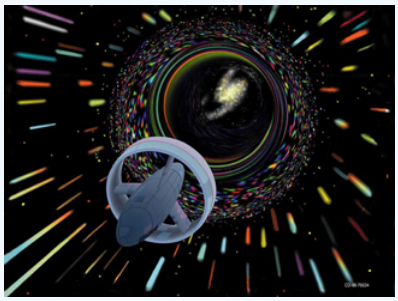

Figure 2.2.12. (Credit:NASA)

Solution

Our starting point is Einstein's equation for the mass of a moving object,

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}},$$

where m_0 is the object's mass at rest, v is its speed, and c is the speed of light. To see how the mass changes at high speeds, we can graph the ratio of masses m/m_0 as a function of the ratio of speeds, v/c (Figure 2.2.13).

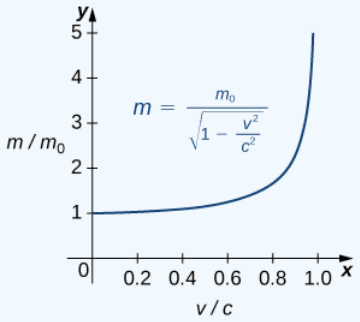

Figure 2.2.13: This graph shows the ratio of masses as a function of the ratio of speeds in Einstein's equation for the mass of a moving object.

We can see that as the ratio of speeds approaches 1—that is, as the speed of the object approaches the speed of light—the ratio of masses increases without bound. In other words, the function has a vertical asymptote at $v/c = 1$. We can try a few values of this ratio to test this idea.

Table 2.2.8

v/c	$\sqrt{1-\frac{v^2}{c^2}}$	m/m_o
0.99	0.1411	7.089
0.999	0.0447	22.37
0.9999	0.0141	70.7

Thus, according to Table 2.2.8:, if an object with mass 100 kg is traveling at 0.9999c, its mass becomes 7071 kg. Since no object can have an infinite mass, we conclude that no object can travel at or more than the speed of light.

Key Concepts

- A table of values or graph may be used to estimate a limit.
- If the limit of a function at a point does not exist, it is still possible that the limits from the left and right at that point may exist.
- If the limits of a function from the left and right exist and are equal, then the limit of the function is that common value.
- We may use limits to describe infinite behavior of a function at a point.

Key Equations

- **Intuitive Definition of the Limit**

$$\lim_{x \to a} f(x) = L$$

- **Two Important Limits**

$$\lim_{x \to a} x = a \qquad \lim_{x \to a} c = c$$

- **One-Sided Limits**

$$\lim_{x \to a^-} f(x) = L \qquad \lim_{x \to a^+} f(x) = L$$

- **Infinite Limits from the Left**

$$\lim_{x \to a^-} f(x) = +\infty \qquad \lim_{x \to a^-} f(x) = -\infty$$

- **Infinite Limits from the Right**

$$\lim_{x \to a^+} f(x) = +\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

- **Two-Sided Infinite Limits**

$\lim_{x \to a} f(x) = +\infty$: $\lim_{x \to a^-} f(x) = +\infty$ and $\lim_{x \to a^+} f(x) = +\infty$

$\lim_{x \to a} f(x) = -\infty$: $\lim_{x \to a^-} f(x) = -\infty$ and $\lim_{x \to a^+} f(x) = -\infty$

Glossary

infinite limit

A function has an infinite limit at a point a if it either increases or decreases without bound as it approaches a

intuitive definition of the limit

If all values of the function $f(x)$ approach the real number L as the values of $x(\neq a)$ approach a, $f(x)$ approaches L

one-sided limit

A one-sided limit of a function is a limit taken from either the left or the right

vertical asymptote

A function has a vertical asymptote at $x = a$ if the limit as x approaches a from the right or left is infinite

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2.2E: Exercises for Section 2.2

Intuitive Definition of Limits

For exercises 1 - 2, consider the function $f(x) = \dfrac{x^2 - 1}{|x - 1|}$.

1) [T] Complete the following table for the function. Round your solutions to four decimal places.

x	$f(x)$	x	$f(x)$
0.9	a.	1.1	e.
0.99	b.	1.01	f.
0.999	c.	1.001	g.
0.9999	d.	1.0001	h.

2) What do your results in the preceding exercise indicate about the two-sided limit $\lim\limits_{x \to 1} f(x)$? Explain your response.

Answer

$\lim\limits_{x \to 1} f(x)$ does not exist because $\lim\limits_{x \to 1^-} f(x) = -2 \neq \lim\limits_{x \to 1^+} f(x) = 2$.

For exercises 3 - 5, consider the function $f(x) = (1 + x)^{1/x}$.

3) [T] Make a table showing the values of f for $x = -0.01,\ -0.001,\ -0.0001,\ -0.00001$ and for $x = 0.01,\ 0.001,\ 0.0001,\ 0.00001$. Round your solutions to five decimal places.

x	$f(x)$	x	$f(x)$
-0.01	a.	0.01	e.
-0.001	b.	0.001	f.
-0.0001	c.	0.0001	g.
-0.00001	d.	0.00001	h.

4) What does the table of values in the preceding exercise indicate about the function $f(x) = (1 + x)^{1/x}$?

Answer

$\lim\limits_{x \to 0} (1 + x)^{1/x} \approx 2.7183$.

5) To which mathematical constant do the values in the preceding exercise appear to be approaching? This is the actual limit here.

In exercises 6 - 8, use the given values to set up a table to evaluate the limits. Round your solutions to eight decimal places.

6) [T] $\lim\limits_{x \to 0} \dfrac{\sin 2x}{x}$; $\pm 0.1,\ \pm 0.01,\ \pm 0.001,\ \pm .0001$

x	$\frac{\sin 2x}{x}$	x	$\frac{\sin 2x}{x}$
-0.1	a.	0.1	e.
-0.01	b.	0.01	f.
-0.001	c.	0.001	g.
-0.0001	d.	0.0001	h.

Answer

a. 1.98669331; b. 1.99986667; c. 1.99999867; d. 1.99999999; e. 1.98669331; f. 1.99986667; g. 1.99999867; h. 1.99999999;

$\lim_{x\to 0}\frac{\sin 2x}{x}=2$

7) [T] $\lim_{x\to 0}\frac{\sin 3x}{x}$ $\pm 0.1,\ \pm 0.01,\ \pm 0.001,\ \pm 0.0001$

x	$\frac{\sin 3x}{x}$	x	$\frac{\sin 3x}{x}$
-0.1	a.	0.1	e.
-0.01	b.	0.01	f.
-0.001	c.	0.001	g.
-0.0001	d.	0.0001	h.

8) Use the preceding two exercises to conjecture (guess) the value of the following limit: $\lim_{x\to 0}\frac{\sin ax}{x}$ for a, a positive real value.

Answer

$\lim_{x\to 0}\frac{\sin ax}{x}=a$

[T] In exercises 9 - 14, set up a table of values to find the indicated limit. Round to eight significant digits.

9) $\lim_{x\to 2}\frac{x^2-4}{x^2+x-6}$

x	$\frac{x^2-4}{x^2+x-6}$	x	$\frac{x^2-4}{x^2+x-6}$
1.9	a.	2.1	e.
1.99	b.	2.01	f.
1.999	c.	2.001	g.
1.9999	d.	2.0001	h.

10) $\lim_{x\to 1}(1-2x)$

x	$1-2x$	x	$1-2x$
0.9	a.	1.1	e.
0.99	b.	1.01	f.
0.999	c.	1.001	g.
0.9999	d.	1.0001	h.

Answer

a. −0.80000000; b. −0.98000000; c. −0.99800000; d. −0.99980000; e. −1.2000000; f. −1.0200000; g. −1.0020000; h. −1.0002000;

$\lim_{x\to 1}(1-2x)=-1$

11) $\lim_{x\to 0}\frac{5}{1-e^{1/x}}$

x	$\frac{5}{1-e^{1/x}}$	x	$\frac{5}{1-e^{1/x}}$
-0.1	a.	0.1	e.
-0.01	b.	0.01	f.

x	$\frac{5}{1-e^{1/x}}$	x	$\frac{5}{1-e^{1/x}}$
-0.001	c.	0.001	g.
-0.0001	d.	0.0001	h.

12) $\lim_{z\to 0} \dfrac{z-1}{z^2(z+3)}$

z	$\frac{z-1}{z^2(z+3)}$	z	$\frac{z-1}{z^2(z+3)}$
-0.1	a.	0.1	e.
-0.01	b.	0.01	f.
-0.001	c.	0.001	g.
-0.0001	d.	0.0001	h.

Answer

a. −37.931034; b. −3377.9264; c. −333,777.93; d. −33,337,778; e. −29.032258; f. −3289.0365; g. −332,889.04; h. −33,328,889

$\lim_{x\to 0} \dfrac{z-1}{z^2(z+3)} = -\infty$

13) $\lim_{t\to 0^+} \dfrac{\cos t}{t}$

t	$\frac{\cos t}{t}$
0.1	a.
0.01	b.
0.001	c.
0.0001	d.

14) $\lim_{x\to 2} \dfrac{1-\frac{2}{x}}{x^2-4}$

x	$\frac{1-\frac{2}{x}}{x^2-4}$	x	$\frac{1-\frac{2}{x}}{x^2-4}$
1.9	a.	2.1	e.
1.99	b.	2.01	f.
1.999	c.	2.001	g.
1.9999	d.	2.0001	h.

Answer

a. 0.13495277; b. 0.12594300; c. 0.12509381; d. 0.12500938; e. 0.11614402; f. 0.12406794; g. 0.12490631; h. 0.12499063;

$\therefore \lim_{x\to 2} \dfrac{1-\frac{2}{x}}{x^2-4} = 0.1250 = \dfrac{1}{8}$

[T] In exercises 15 - 16, set up a table of values and round to eight significant digits. Based on the table of values, make a guess about what the limit is. Then, use a calculator to graph the function and determine the limit. Was the conjecture correct? If not, why does the method of tables fail?

15) $\lim_{\theta\to 0} \sin\left(\dfrac{\pi}{\theta}\right)$

θ	$\sin\left(\frac{\pi}{\theta}\right)$	θ	$\sin\left(\frac{\pi}{\theta}\right)$
-0.1	a.	0.1	e.
-0.01	b.	0.01	f.
-0.001	c.	0.001	g.
-0.0001	d.	0.0001	h.

16) $\lim_{\alpha \to 0^+} \frac{1}{\alpha} \cos\left(\frac{\pi}{\alpha}\right)$

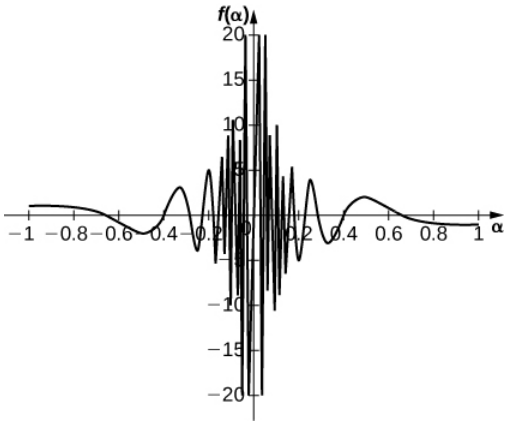
a	$\frac{1}{\alpha}\cos\left(\frac{\pi}{\alpha}\right)$
0.1	a.
0.01	b.
0.001	c.
0.0001	d.

Answer

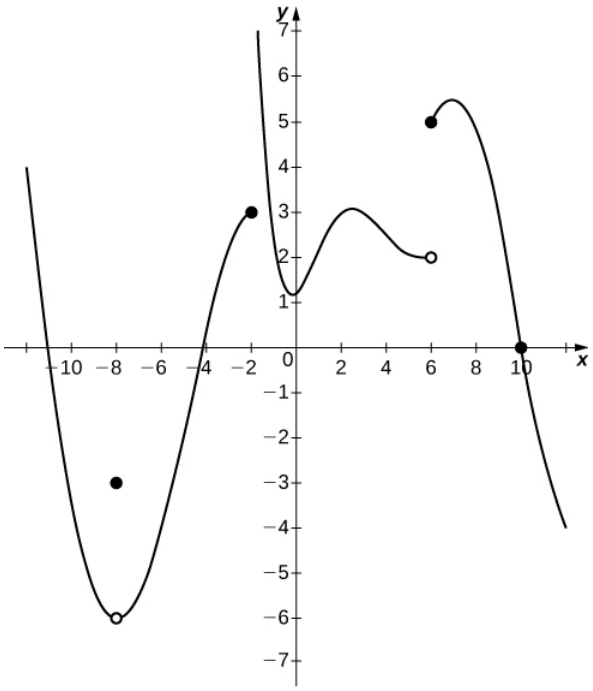
a. 10.00000; b. 100.00000; c. 1000.0000; d. 10,000.000;

Guess: $\lim_{\alpha \to 0^+} \frac{1}{\alpha} \cos\left(\frac{\pi}{\alpha}\right) = \infty$;

Actual: DNE , since the graph shows the function oscillates wildly between values approaching positive infinity and values approaching negative infinity, as the value of α approaches 0 from the positive side.



In exercises 17 - 20, consider the graph of the function $y = f(x)$ shown here. Which of the statements about $y = f(x)$ are true and which are false? Explain why a statement is false.



17) $\lim_{x\to 10} f(x) = 0$

18) $\lim_{x\to -2^+} f(x) = 3$

Answer

False; $\lim_{x\to -2^+} f(x) = +\infty$

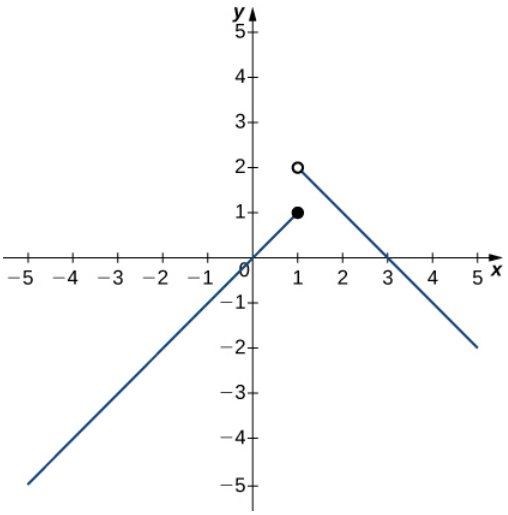
19) $\lim_{x\to -8} f(x) = f(-8)$

20) $\lim_{x\to 6} f(x) = 5$

Answer

False; $\lim_{x\to 6} f(x)$ DNE since $\lim_{x\to 6^-} f(x) = 2$ and $\lim_{x\to 6^+} f(x) = 5$.

In exercises 21 - 25, use the following graph of the function $y = f(x)$ to find the values, if possible. Estimate when necessary.



21) $\lim_{x\to 1^-} f(x)$

22) $\lim_{x \to 1^+} f(x)$

Answer

2

23) $\lim_{x \to 1} f(x)$

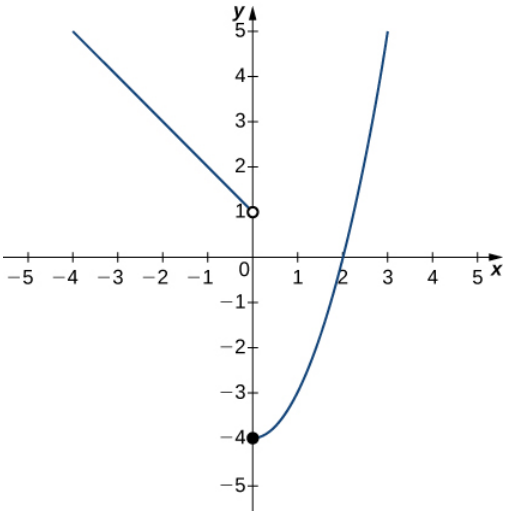
24) $\lim_{x \to 2} f(x)$

Answer

1

25) $f(1)$

In exercises 26 - 29, use the graph of the function $y = f(x)$ shown here to find the values, if possible. Estimate when necessary.



26) $\lim_{x \to 0^-} f(x)$

Answer

1

27) $\lim_{x \to 0^+} f(x)$

28) $\lim_{x \to 0} f(x)$

Answer

DNE

29) $\lim_{x \to 2} f(x)$

In exercises 30 - 35, use the graph of the function $y = f(x)$ shown here to find the values, if possible. Estimate when necessary.

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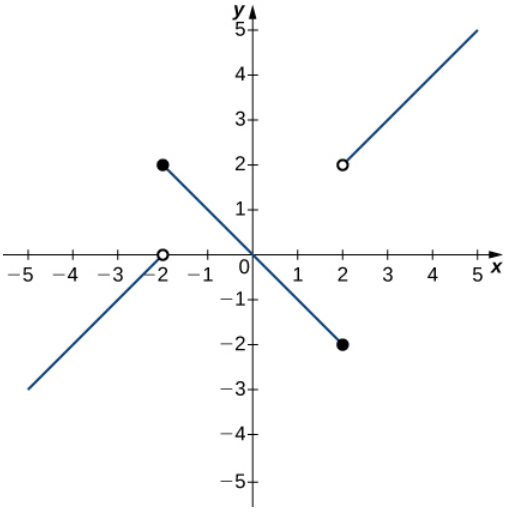


30) $\lim_{x \to -2^-} f(x)$

Answer

0

31) $\lim_{x \to -2^+} f(x)$

32) $\lim_{x \to -2} f(x)$

Answer

DNE

33) $\lim_{x \to 2^-} f(x)$

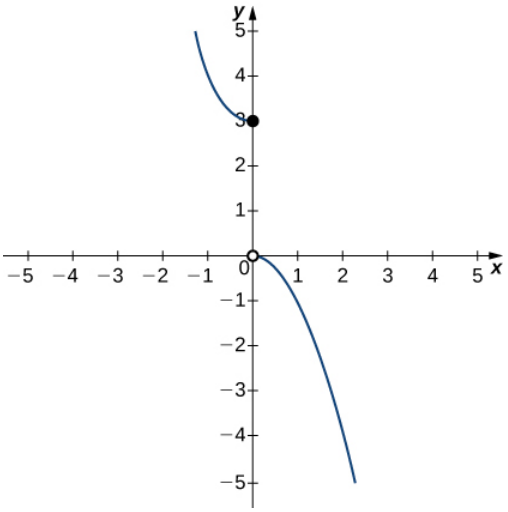
34) $\lim_{x \to 2^+} f(x)$

Answer

2

35) $\lim_{x \to 2} f(x)$

In exercises 36 - 38, use the graph of the function $y = g(x)$ shown here to find the values, if possible. Estimate when necessary.



36) $\lim_{x \to 0^-} g(x)$

Answer

3

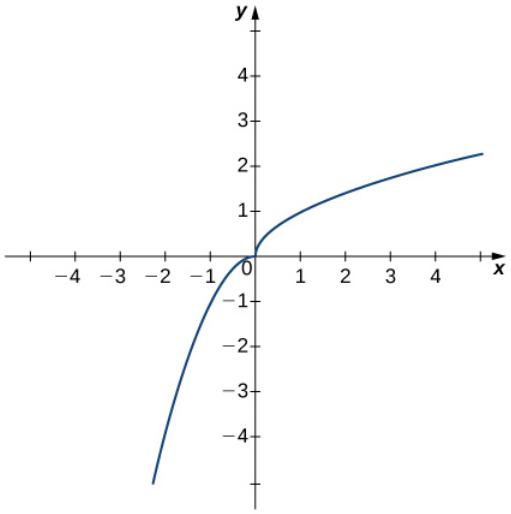
37) $\lim_{x \to 0^+} g(x)$

38) $\lim_{x \to 0} g(x)$

Answer

DNE

In exercises 39 - 41, use the graph of the function $y = h(x)$ shown here to find the values, if possible. Estimate when necessary.



39) $\lim_{x \to 0^-} h(x)$

40) $\lim_{x \to 0^+} h(x)$

Answer

0

41) $\lim_{x \to 0} h(x)$

In exercises 42 - 46, use the graph of the function $y = f(x)$ shown here to find the values, if possible. Estimate when necessary.

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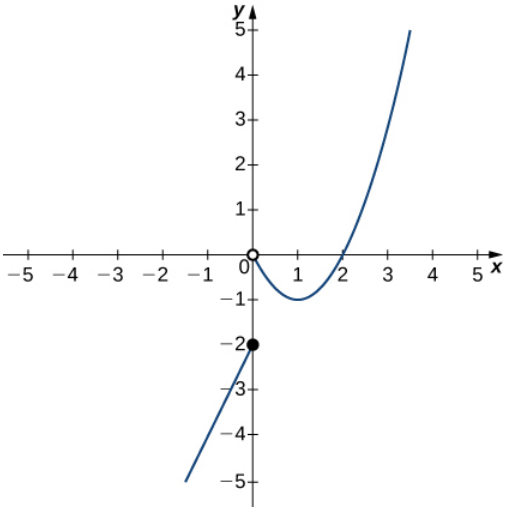


42) $\lim_{x\to 0^-} f(x)$

Answer

-2

43) $\lim_{x\to 0^+} f(x)$

44) $\lim_{x\to 0} f(x)$

Answer

DNE

45) $\lim_{x\to 1} f(x)$

46) $\lim_{x\to 2} f(x)$

Answer

0

Infinite Limits

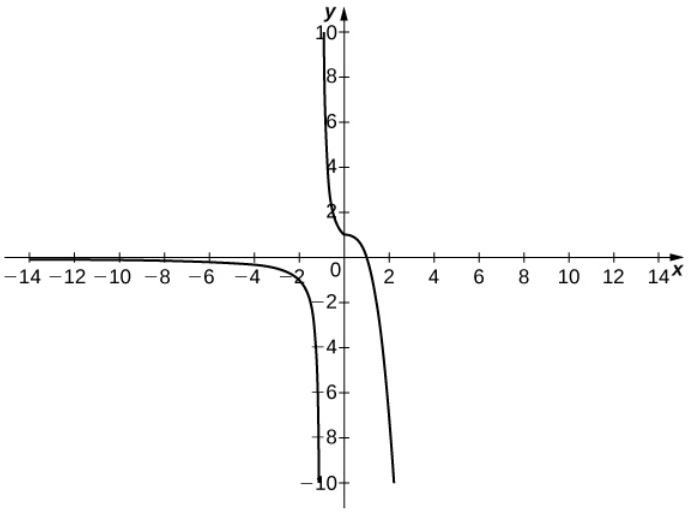
In exercises 47 - 51, sketch the graph of a function with the given properties.

47) $\lim_{x\to 2} f(x) = 1, \quad \lim_{x\to 4^-} f(x) = 3, \quad \lim_{x\to 4^+} f(x) = 6, \quad x = 4$ is not defined.

48) $\lim_{x\to -\infty} f(x) = 0, \quad \lim_{x\to -1^-} f(x) = -\infty, \quad \lim_{x\to -1^+} f(x) = \infty, \quad \lim_{x\to 0} f(x) = f(0), \quad f(0) = 1, \quad \lim_{x\to \infty} f(x) = -\infty$

Answer

Answers may vary

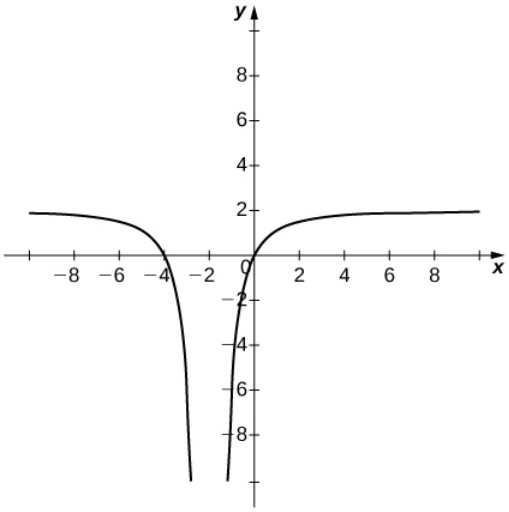


49) $\lim_{x\to-\infty} f(x) = 2, \quad \lim_{x\to3^-} f(x) = -\infty, \quad \lim_{x\to3^+} f(x) = \infty, \quad \lim_{x\to\infty} f(x) = 2, \quad f(0) = -\frac{1}{3}$

50) $\lim_{x\to-\infty} f(x) = 2, \quad \lim_{x\to-2} f(x) = -\infty, \quad \lim_{x\to\infty} f(x) = 2, \quad f(0) = 0$

Answer

Answer may vary



51) $\lim_{x\to-\infty} f(x) = 0, \quad \lim_{x\to-1^-} f(x) = \infty, \quad \lim_{x\to-1^+} f(x) = -\infty, \quad f(0) = -1, \quad \lim_{x\to1^-} f(x) = -\infty, \quad \lim_{x\to1^+} f(x) = \infty, \quad \lim_{x\to\infty} f(x) = 0$

52) Shock waves arise in many physical applications, ranging from supernovas to detonation waves. A graph of the density of a shock wave with respect to distance, x, is shown here. We are mainly interested in the location of the front of the shock, labeled X_{SF} in the diagram.

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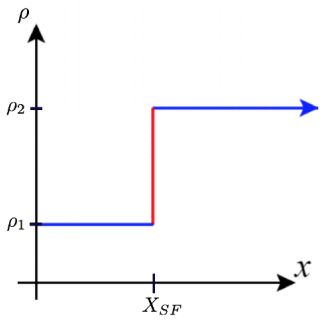

a. Evaluate $\lim_{x \to X_{SF}^{+}} \rho(x)$.

b. Evaluate $\lim_{x \to X_{SF}^{-}} \rho(x)$.

c. Evaluate $\lim_{x \to X_{SF}} \rho(x)$. Explain the physical meanings behind your answers.

Answer

a. ρ_2 b. ρ_1 c. DNE unless $\rho_1 = \rho_2$. As you approach X_{SF} from the right, you are in the high-density area of the shock. When you approach from the left, you have not experienced the "shock" yet and are at a lower density.

53) A track coach uses a camera with a fast shutter to estimate the position of a runner with respect to time. A table of the values of position of the athlete versus time is given here, where x is the position in meters of the runner and t is time in seconds. What is $\lim_{t \to 2} x(t)$? What does it mean physically?

$t(sec)$	$x(m)$
1.75	4.5
1.95	6.1
1.99	6.42
2.01	6.58
2.05	6.9
2.25	8.5

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2.3: The Limit Laws

Learning Objectives

- Recognize the basic limit laws.
- Use the limit laws to evaluate the limit of a function.
- Evaluate the limit of a function by factoring.
- Use the limit laws to evaluate the limit of a polynomial or rational function.
- Evaluate the limit of a function by factoring or by using conjugates.
- Evaluate the limit of a function by using the squeeze theorem.

In the previous section, we evaluated limits by looking at graphs or by constructing a table of values. In this section, we establish laws for calculating limits and learn how to apply these laws. In the Student Project at the end of this section, you have the opportunity to apply these limit laws to derive the formula for the area of a circle by adapting a method devised by the Greek mathematician Archimedes. We begin by restating two useful limit results from the previous section. These two results, together with the limit laws, serve as a foundation for calculating many limits.

Evaluating Limits with the Limit Laws

The first two limit laws were stated previously and we repeat them here. These basic results, together with the other limit laws, allow us to evaluate limits of many algebraic functions.

Theorem 2.3.1: Basic Limit Results

For any real number a and any constant c,

I. $\lim\limits_{x \to a} x = a$
II. $\lim\limits_{x \to a} c = c$

Example 2.3.1: Evaluating a Basic Limit

Evaluate each of the following limits using "Basic Limit Results."

a. $\lim\limits_{x \to 2} x$
b. $\lim\limits_{x \to 2} 5$

Solution

a. The limit of x as x approaches a is a: $\lim\limits_{x \to 2} x = 2$.
b. The limit of a constant is that constant: $\lim\limits_{x \to 2} 5 = 5$.

We now take a look at the **limit laws**, the individual properties of limits. The proofs that these laws hold are omitted here.

Theorem 2.3.2: Limit Laws

Let $f(x)$ and $g(x)$ be defined for all $x \neq a$ over some open interval containing a. Assume that L and M are real numbers such that $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$. Let c be a constant. Then, each of the following statements holds:

- **Sum law for limits**:

$$\lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$$

- **Difference law for limits**:

$$\lim_{x \to a}(f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L - M$$

- **Constant multiple law for limits**:

$$\lim cf(x) = c \cdot \lim f(x) = cL$$

- **Product law for limits**:

$$\lim_{x\to a}(f(x)\cdot g(x)) = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x) = L\cdot M$$

- **Quotient law for limits**:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \frac{L}{M}$$

for $M \neq 0$.

- **Power law for limits**:

$$\lim_{x\to a}\big(f(x)\big)^n = \big(\lim_{x\to a} f(x)\big)^n = L^n$$

for every positive integer n.

- **Root law for limits**:

$$\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L}$$

for all L if n is odd and for $L \geq 0$ if n is even.

We now practice applying these limit laws to evaluate a limit.

Example $2.3.2A$: Evaluating a Limit Using Limit Laws

Use the limit laws to evaluate

$$\lim_{x\to -3}(4x+2).$$

Solution

Let's apply the limit laws one step at a time to be sure we understand how they work. We need to keep in mind the requirement that, at each application of a limit law, the new limits must exist for the limit law to be applied.

$$\begin{aligned}
\lim_{x\to -3}(4x+2) &= \lim_{x\to -3} 4x + \lim_{x\to -3} 2 && \text{Apply the sum law.}\\
&= 4\cdot \lim_{x\to -3} x + \lim_{x\to -3} 2 && \text{Apply the constant multiple law.}\\
&= 4\cdot(-3)+2 = -10. && \text{Apply the basic limit results and simplify.}
\end{aligned}$$

Example $2.3.2B$: Using Limit Laws Repeatedly

Use the limit laws to evaluate

$$\lim_{x\to 2}\frac{2x^2-3x+1}{x^3+4}.$$

Solution

To find this limit, we need to apply the limit laws several times. Again, we need to keep in mind that as we rewrite the limit in terms of other limits, each new limit must exist for the limit law to be applied.

$$\lim_{x\to 2}\frac{2x^2-3x+1}{x^3+4}=\frac{\lim_{x\to 2}(2x^2-3x+1)}{\lim_{x\to 2}(x^3+4)}$$ Apply the quotient law, make sure that $(2)^3+4\neq 0$.

$$=\frac{2\cdot\lim_{x\to 2}x^2-3\cdot\lim_{x\to 2}x+\lim_{x\to 2}1}{\lim_{x\to 2}x^3+\lim_{x\to 2}4}$$ Apply the sum law and constant multiple law.

$$=\frac{2\cdot\left(\lim_{x\to 2}x\right)^2-3\cdot\lim_{x\to 2}x+\lim_{x\to 2}1}{\left(\lim_{x\to 2}x\right)^3+\lim_{x\to 2}4}$$ Apply the power law.

$$=\frac{2(4)-3(2)+1}{(2)^3+4}=\frac{1}{4}.$$ Apply the basic limit laws and simplify.

? Exercise 2.3.2

Use the limit laws to evaluate $\lim_{x\to 6}(2x-1)\sqrt{x+4}$. In each step, indicate the limit law applied.

Hint

Begin by applying the product law.

Answer

$11\sqrt{10}$

Limits of Polynomial and Rational Functions

By now you have probably noticed that, in each of the previous examples, it has been the case that $\lim_{x\to a}f(x)=f(a)$. This is not always true, but it does hold for all polynomials for any choice of a and for all rational functions at all values of a for which the rational function is defined.

Theorem 2.3.3: Limits of Polynomial and Rational Functions

Let $p(x)$ and $q(x)$ be polynomial functions. Let a be a real number. Then,

$$\lim_{x\to a}p(x)=p(a)$$

$$\lim_{x\to a}\frac{p(x)}{q(x)}=\frac{p(a)}{q(a)}$$

when $q(a)\neq 0$.

To see that this theorem holds, consider the polynomial

$$p(x)=c_nx^n+c_{n-1}x^{n-1}+\cdots+c_1x+c_0.$$

By applying the sum, constant multiple, and power laws, we end up with

$$\begin{aligned}\lim_{x\to a}p(x)&=\lim_{x\to a}(c_nx^n+c_{n-1}x^{n-1}+\cdots+c_1x+c_0)\\&=c_n\left(\lim_{x\to a}x\right)^n+c_{n-1}\left(\lim_{x\to a}x\right)^{n-1}+\cdots+c_1\left(\lim_{x\to a}x\right)+\lim_{x\to a}c_0\\&=c_na^n+c_{n-1}a^{n-1}+\cdots+c_1a+c_0\\&=p(a)\end{aligned}$$

It now follows from the quotient law that if $p(x)$ and $q(x)$ are polynomials for which $q(a)\neq 0$,

then

$$\lim\frac{p(x)}{}-\frac{p(a)}{}$$

$$\lim_{x\to a} q(x) - q(a).$$

Example 2.3.3: Evaluating a Limit of a Rational Function

Evaluate the $\lim_{x\to 3} \dfrac{2x^2-3x+1}{5x+4}$.

Solution

Since 3 is in the domain of the rational function $f(x)=\dfrac{2x^2-3x+1}{5x+4}$, we can calculate the limit by substituting 3 for x into the function. Thus,

$$\lim_{x\to 3} \frac{2x^2-3x+1}{5x+4} = \frac{10}{19}.$$

Exercise 2.3.3

Evaluate $\lim_{x\to -2}(3x^3-2x+7)$.

Hint

Use LIMITS OF POLYNOMIAL AND RATIONAL FUNCTIONS as reference

Answer

-13

Additional Limit Evaluation Techniques

As we have seen, we may evaluate easily the limits of polynomials and limits of some (but not all) rational functions by direct substitution. However, as we saw in the introductory section on limits, it is certainly possible for $\lim_{x\to a} f(x)$ to exist when $f(a)$ is undefined. The following observation allows us to evaluate many limits of this type:

If for all $x \neq a,\ f(x)=g(x)$ over some open interval containing a, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x).$$

To understand this idea better, consider the limit $\lim_{x\to 1} \dfrac{x^2-1}{x-1}$.

The function

$$f(x)=\frac{x^2-1}{x-1}=\frac{(x-1)(x+1)}{x-1}$$

and the function $g(x)=x+1$ are identical for all values of $x \neq 1$. The graphs of these two functions are shown in Figure 2.3.1.

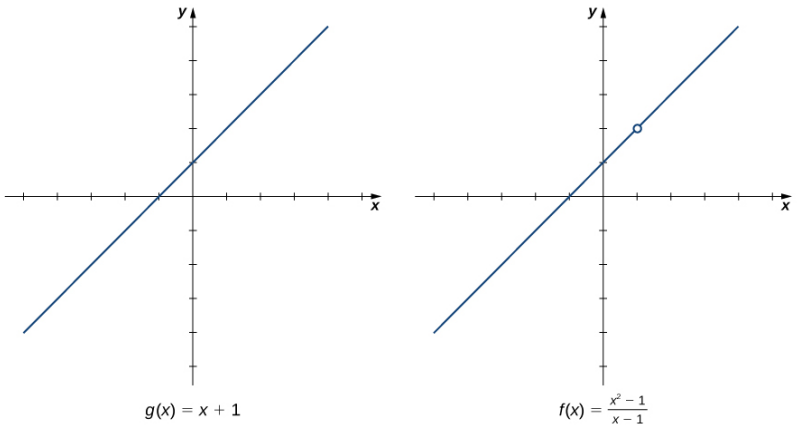


Figure 2.3.1: The graphs of $f(x)$ and $g(x)$ are identical for all $x \neq 1$. Their limits at 1 are equal.

We see that

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{x\to 1}\frac{(x-1)(x+1)}{x-1}=\lim_{x\to 1}(x+1)=2.$$

The limit has the form $\lim_{x\to a} f(x)/g(x)$, where $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$. (In this case, we say that $f(x)/g(x)$ has the indeterminate form $0/0$.) The following Problem-Solving Strategy provides a general outline for evaluating limits of this type.

Problem-Solving Strategy: Calculating a Limit When $f(x)/g(x)$ has the Indeterminate Form $0/0$

1. First, we need to make sure that our function has the appropriate form and cannot be evaluated immediately using the limit laws.
2. We then need to find a function that is equal to $h(x)=f(x)/g(x)$ for all $x\neq a$ over some interval containing a. To do this, we may need to try one or more of the following steps:
   a. If $f(x)$ and $g(x)$ are polynomials, we should factor each function and cancel out any common factors.
   b. If the numerator or denominator contains a difference involving a square root, we should try multiplying the numerator and denominator by the conjugate of the expression involving the square root.
   c. If $f(x)/g(x)$ is a complex fraction, we begin by simplifying it.
3. Last, we apply the limit laws.

The next examples demonstrate the use of this Problem-Solving Strategy. Example 2.3.4 illustrates the factor-and-cancel technique; Example 2.3.5 shows multiplying by a conjugate. In Example 2.3.6, we look at simplifying a complex fraction.

Example 2.3.4: Evaluating a Limit by Factoring and Canceling

Evaluate $\lim_{x\to 3}\dfrac{x^2-3x}{2x^2-5x-3}$.

Solution

Step 1. The function $f(x)=\dfrac{x^2-3x}{2x^2-5x-3}$ is undefined for $x=3$. In fact, if we substitute 3 into the function we get $0/0$, which is undefined. Factoring and canceling is a good strategy:

$$\lim_{x\to 3}\frac{x^2-3x}{2x^2-5x-3}=\lim_{x\to 3}\frac{x(x-3)}{(x-3)(2x+1)}$$

Step 2. For all $x\neq 3$, $\dfrac{x^2-3x}{2x^2-5x-3}=\dfrac{x}{2x+1}$. Therefore,

$$\lim_{x\to 3}\frac{x(x-3)}{(x-3)(2x+1)}=\lim_{x\to 3}\frac{x}{2x+1}.$$

Step 3. Evaluate using the limit laws:

$$\lim_{x\to 3}\frac{x}{2x+1}=\frac{3}{7}.$$

Exercise 2.3.4

Evaluate $\lim_{x\to -3}\dfrac{x^2+4x+3}{x^2-9}$.

Hint

Follow the steps in the Problem-Solving Strategy

Answer

$\dfrac{1}{3}$

Example 2.3.5: Evaluating a Limit by Multiplying by a Conjugate

Evaluate $\lim\limits_{x\to -1}\dfrac{\sqrt{x+2}-1}{x+1}$.

Solution

Step 1. $\dfrac{\sqrt{x+2}-1}{x+1}$ has the form $0/0$ at -1. Let's begin by multiplying by $\sqrt{x+2}+1$, the conjugate of $\sqrt{x+2}-1$, on the numerator and denominator:

$$\lim_{x\to -1}\frac{\sqrt{x+2}-1}{x+1}=\lim_{x\to -1}\frac{\sqrt{x+2}-1}{x+1}\cdot\frac{\sqrt{x+2}+1}{\sqrt{x+2}+1}.$$

Step 2. We then multiply out the numerator. We don't multiply out the denominator because we are hoping that the $(x+1)$ in the denominator cancels out in the end:

$$=\lim_{x\to -1}\frac{x+1}{(x+1)(\sqrt{x+2}+1)}.$$

Step 3. Then we cancel:

$$=\lim_{x\to -1}\frac{1}{\sqrt{x+2}+1}.$$

Step 4. Last, we apply the limit laws:

$$\lim_{x\to -1}\frac{1}{\sqrt{x+2}+1}=\frac{1}{2}.$$

Exercise 2.3.5

Evaluate $\lim\limits_{x\to 5}\dfrac{\sqrt{x-1}-2}{x-5}$.

Hint

Follow the steps in the Problem-Solving Strategy

Answer

$\dfrac{1}{4}$

Example 2.3.6: Evaluating a Limit by Simplifying a Complex Fraction

Evaluate $\lim\limits_{x\to 1}\dfrac{\dfrac{1}{x+1}-\dfrac{1}{2}}{x-1}$.

Solution

Step 1. $\dfrac{\dfrac{1}{x+1}-\dfrac{1}{2}}{x-1}$ has the form $0/0$ at 1. We simplify the algebraic fraction by multiplying by $2(x+1)/2(x+1)$:

$$\lim_{x\to 1}\frac{\dfrac{1}{x+1}-\dfrac{1}{2}}{x-1}=\lim_{x\to 1}\frac{\dfrac{1}{x+1}-\dfrac{1}{2}}{x-1}\cdot\frac{2(x+1)}{2(x+1)}.$$

Step 2. Next, we multiply through the numerators. Do not multiply the denominators because we want to be able to cancel the factor $(x-1)$:

$$=\lim_{x\to 1}\frac{2-(x+1)}{2(x-1)(x+1)}.$$

Step 3. Then, we simplify the numerator:

$$= \lim_{x \to 1} \frac{-x+1}{2(x-1)(x+1)}.$$

Step 4. Now we factor out -1 from the numerator:

$$= \lim_{x \to 1} \frac{-(x-1)}{2(x-1)(x+1)}.$$

Step 5. Then, we cancel the common factors of $(x-1)$:

$$= \lim_{x \to 1} \frac{-1}{2(x+1)}.$$

Step 6. Last, we evaluate using the limit laws:

$$\lim_{x \to 1} \frac{-1}{2(x+1)} = -\frac{1}{4}.$$

? Exercise 2.3.6

Evaluate $\lim_{x \to -3} \dfrac{\dfrac{1}{x+2}+1}{x+3}$.

Hint

Follow the steps in the Problem-Solving Strategy

Answer

-1

Example 2.3.7 does not fall neatly into any of the patterns established in the previous examples. However, with a little creativity, we can still use these same techniques.

✓ Example 2.3.7: Evaluating a Limit When the Limit Laws Do Not Apply

Evaluate $\lim_{x \to 0} \left(\dfrac{1}{x} + \dfrac{5}{x(x-5)} \right)$.

Solution:

Both $1/x$ and $5/x(x-5)$ fail to have a limit at zero. Since neither of the two functions has a limit at zero, we cannot apply the sum law for limits; we must use a different strategy. In this case, we find the limit by performing addition and then applying one of our previous strategies. Observe that

$$\frac{1}{x} + \frac{5}{x(x-5)} = \frac{x-5+5}{x(x-5)} = \frac{x}{x(x-5)}.$$

Thus,

$$\lim_{x \to 0} \left(\frac{1}{x} + \frac{5}{x(x-5)} \right) = \lim_{x \to 0} \frac{x}{x(x-5)} = \lim_{x \to 0} \frac{1}{x-5} = -\frac{1}{5}.$$

? Exercise 2.3.7

Evaluate $\lim_{x \to 3} \left(\dfrac{1}{x-3} - \dfrac{4}{x^2-2x-3} \right)$.

Hint

Use the same technique as Example 2.3.7. Don't forget to factor x^2-2x-3 before getting a common denominator.

Answer

$\frac{1}{4}$

Let's now revisit one-sided limits. Simple modifications in the limit laws allow us to apply them to one-sided limits. For example, to apply the limit laws to a limit of the form $\lim_{x \to a^-} h(x)$, we require the function $h(x)$ to be defined over an open interval of the form (b, a); for a limit of the form $\lim_{x \to a^+} h(x)$, we require the function $h(x)$ to be defined over an open interval of the form (a, c). Example 2.3.8A illustrates this point.

Example 2.3.8A: Evaluating a One-Sided Limit Using the Limit Laws

Evaluate each of the following limits, if possible.

a. $\lim_{x \to 3^-} \sqrt{x-3}$
b. $\lim_{x \to 3^+} \sqrt{x-3}$

Solution

Figure 2.3.2 illustrates the function $f(x) = \sqrt{x-3}$ and aids in our understanding of these limits.

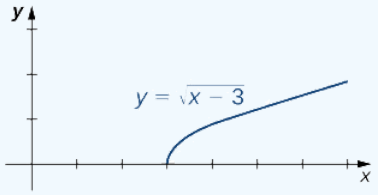


Figure 2.3.2: The graph shows the function $f(x) = \sqrt{x-3}$.

a. The function $f(x) = \sqrt{x-3}$ is defined over the interval $[3, +\infty)$. Since this function is not defined to the left of 3, we cannot apply the limit laws to compute $\lim_{x \to 3^-} \sqrt{x-3}$. In fact, since $f(x) = \sqrt{x-3}$ is undefined to the left of 3, $\lim_{x \to 3^-} \sqrt{x-3}$ does not exist.

b. Since $f(x) = \sqrt{x-3}$ is defined to the right of 3, the limit laws do apply to $\lim_{x \to 3^+} \sqrt{x-3}$. By applying these limit laws we obtain $\lim_{x \to 3^+} \sqrt{x-3} = 0$.

In Example 2.3.8B we look at one-sided limits of a piecewise-defined function and use these limits to draw a conclusion about a two-sided limit of the same function.

Example 2.3.8B: Evaluating a Two-Sided Limit Using the Limit Laws

For $f(x) = \begin{cases} 4x-3, & \text{if } x < 2 \\ (x-3)^2, & \text{if } x \geq 2 \end{cases}$, evaluate each of the following limits:

a. $\lim_{x \to 2^-} f(x)$
b. $\lim_{x \to 2^+} f(x)$
c. $\lim_{x \to 2} f(x)$

Solution

Figure 2.3.3 illustrates the function $f(x)$ and aids in our understanding of these limits.

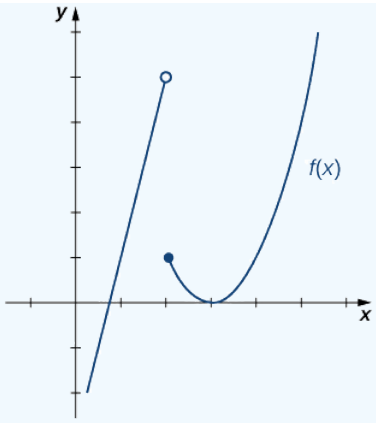


Figure 2.3.3: This graph shows a function $f(x)$.

a. Since $f(x) = 4x - 3$ for all x in $(-\infty, 2)$, replace $f(x)$ in the limit with $4x - 3$ and apply the limit laws:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (4x - 3) = 5$$

b. Since $f(x) = (x - 3)^2$for all x in $(2, +\infty)$, replace $f(x)$ in the limit with $(x - 3)^2$ and apply the limit laws:

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} (x - 3)^2 = 1.$$

c. Since $\lim_{x \to 2^-} f(x) = 5$ and $\lim_{x \to 2^+} f(x) = 1$, we conclude that $\lim_{x \to 2} f(x)$ does not exist.

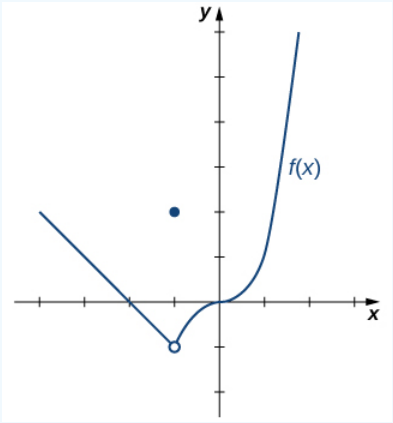
Exercise 2.3.8

Graph $f(x) = \begin{cases} -x - 2, & \text{if } x < -1 \\ 2, & \text{if } x = -1 \\ x^3, & \text{if } x > -1 \end{cases}$ and evaluate $\lim_{x \to -1^-} f(x)$.

Hint

Use the method in Example 2.3.8B to evaluate the limit.

Answer



$$\lim_{x \to -1^-} f(x) = -1$$

We now turn our attention to evaluating a limit of the form $\lim_{x \to a} \frac{f(x)}{g(x)}$, where $\lim_{x \to a} f(x) = K$, where $K \neq 0$ and $\lim_{x \to a} g(x) = 0$. That is, $f(x)/g(x)$ has the form $K/0, K \neq 0$ at a.

Example 2.3.9: Evaluating a Limit of the Form $K/0$, $K \neq 0$ Using the Limit Laws

Evaluate $\lim\limits_{x \to 2^-} \dfrac{x-3}{x^2-2x}$.

Solution

Step 1. After substituting in $x = 2$, we see that this limit has the form $-1/0$. That is, as x approaches 2 from the left, the numerator approaches -1; and the denominator approaches 0. Consequently, the magnitude of $\dfrac{x-3}{x(x-2)}$ becomes infinite. To get a better idea of what the limit is, we need to factor the denominator:

$$\lim_{x \to 2^-} \frac{x-3}{x^2-2x} = \lim_{x \to 2^-} \frac{x-3}{x(x-2)}$$

Step 2. Since $x-2$ is the only part of the denominator that is zero when 2 is substituted, we then separate $1/(x-2)$ from the rest of the function:

$$= \lim_{x \to 2^-} \frac{x-3}{x} \cdot \frac{1}{x-2}$$

Step 3. Using the Limit Laws, we can write:

$$= \left(\lim_{x \to 2^-} \frac{x-3}{x} \right) \cdot \left(\lim_{x \to 2^-} \frac{1}{x-2} \right).$$

Step 4. $\lim\limits_{x \to 2^-} \dfrac{x-3}{x} = -\dfrac{1}{2}$ and $\lim\limits_{x \to 2^-} \dfrac{1}{x-2} = -\infty$. Therefore, the product of $(x-3)/x$ and $1/(x-2)$ has a limit of $+\infty$:

$$\lim_{x \to 2^-} \frac{x-3}{x^2-2x} = +\infty.$$

Exercise 2.3.9

Evaluate $\lim\limits_{x \to 1} \dfrac{x+2}{(x-1)^2}$.

Solution

Use the methods from Example 2.3.9.

Answer

$+\infty$

The Squeeze Theorem

The techniques we have developed thus far work very well for algebraic functions, but we are still unable to evaluate limits of very basic trigonometric functions. The next theorem, called the **squeeze theorem,** proves very useful for establishing basic trigonometric limits. This theorem allows us to calculate limits by "squeezing" a function, with a limit at a point a that is unknown, between two functions having a common known limit at a. Figure 2.3.4 illustrates this idea.

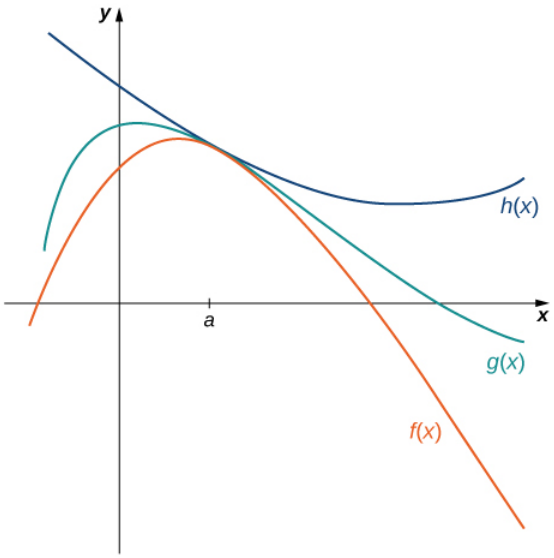


Figure 2.3.4: The Squeeze Theorem applies when $f(x) \le g(x) \le h(x)$ and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x)$.

Theorem 2.3.4: The Squeeze Theorem

Let $f(x), g(x)$, and $h(x)$ be defined for all $x \neq a$ over an open interval containing a. If

$$f(x) \le g(x) \le h(x)$$

for all $x \neq a$ in an open interval containing a and

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$$

where L is a real number, then $\lim_{x \to a} g(x) = L$.

Example 2.3.10: Applying the Squeeze Theorem

Apply the squeeze theorem to evaluate $\lim_{x \to 0} x \cos x$.

Solution

Because $-1 \le \cos x \le 1$ for all x, we have $-x \le x \cos x \le x$ for $x \ge 0$ and $-x \ge x \cos x \ge x$ for $x \le 0$ (if x is negative the direction of the inequalities changes when we multiply). Since $\lim_{x \to 0}(-x) = 0 = \lim_{x \to 0} x$, from the squeeze theorem, we obtain $\lim_{x \to 0} x \cos x = 0$. The graphs of $f(x) = -x,\ g(x) = x \cos x$, and $h(x) = x$ are shown in Figure 2.3.5.

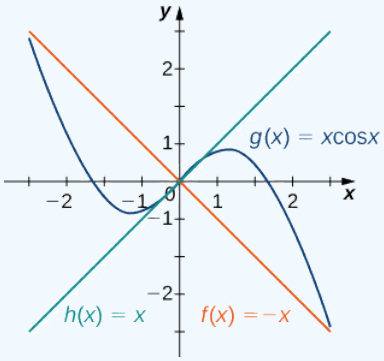


Figure 2.3.5: The graphs of $f(x)$, $g(x)$, and $h(x)$ are shown around the point $x = 0$.

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Exercise 2.3.10

Use the squeeze theorem to evaluate $\lim_{x\to 0} x^2 \sin\frac{1}{x}$.

Hint

Use the fact that $-x^2 \le x^2 \sin(1/x) \le x^2$ to help you find two functions such that $x^2 \sin(1/x)$ is squeezed between them.

Answer

0

We now use the squeeze theorem to tackle several very important limits. Although this discussion is somewhat lengthy, these limits prove invaluable for the development of the material in both the next section and the next chapter. The first of these limits is $\lim_{\theta\to 0} \sin\theta$. Consider the unit circle shown in Figure 2.3.6. In the figure, we see that $\sin\theta$ is the y-coordinate on the unit circle and it corresponds to the line segment shown in blue. The radian measure of angle θ is the length of the arc it subtends on the unit circle. Therefore, we see that for $0 < \theta < \frac{\pi}{2}$, we have $0 < \sin\theta < \theta$.

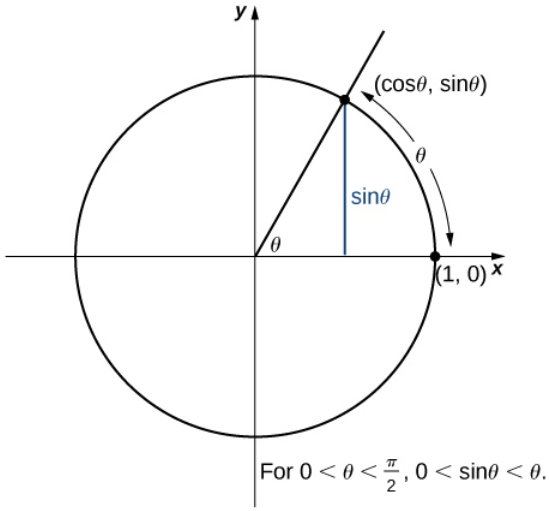

Figure 2.3.6: The sine function is shown as a line on the unit circle.

Because $\lim_{\theta\to 0^+} 0 = 0$ and $\lim_{\theta\to 0^+} \theta = 0$, by using the squeeze theorem we conclude that

$$\lim_{\theta\to 0^+} \sin\theta = 0.$$

To see that $\lim_{\theta\to 0^-} \sin\theta = 0$ as well, observe that for $-\frac{\pi}{2} < \theta < 0, 0 < -\theta < \frac{\pi}{2}$ and hence, $0 < \sin(-\theta) < -\theta$. Consequently, $0 < -\sin\theta < -\theta$. It follows that $0 > \sin\theta > \theta$. An application of the squeeze theorem produces the desired limit. Thus, since $\lim_{\theta\to 0^+} \sin\theta = 0$ and $\lim_{\theta\to 0^-} \sin\theta = 0$,

$$\lim_{\theta\to 0} \sin\theta = 0$$

Next, using the identity $\cos\theta = \sqrt{1-\sin^2\theta}$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, we see that

$$\lim_{\theta\to 0} \cos\theta = \lim_{\theta\to 0} \sqrt{1-\sin^2\theta} = 1.$$

We now take a look at a limit that plays an important role in later chapters—namely, $\lim_{\theta\to 0} \frac{\sin\theta}{\theta}$. To evaluate this limit, we use the unit circle in Figure 2.3.7. Notice that this figure adds one additional triangle to Figure 2.3.6. We see that the length of the side opposite angle θ in this new triangle is $\tan\theta$. Thus, we see that for $0 < \theta < \frac{\pi}{2}$, we have $\sin\theta < \theta < \tan\theta$.

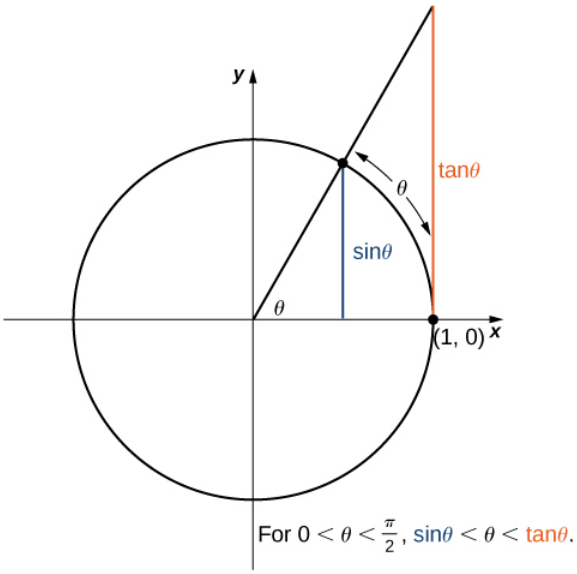

Figure 2.3.7: The sine and tangent functions are shown as lines on the unit circle.

By dividing by $\sin\theta$ in all parts of the inequality, we obtain

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Equivalently, we have

$$1 > \frac{\sin\theta}{\theta} > \cos\theta.$$

Since $\lim\limits_{\theta\to 0^+} 1 = 1 = \lim\limits_{\theta\to 0^+} \cos\theta$, we conclude that $\lim\limits_{\theta\to 0^+} \dfrac{\sin\theta}{\theta} = 1$, by the squeeze theorem. By applying a manipulation similar to that used in demonstrating that $\lim\limits_{\theta\to 0^-} \sin\theta = 0$, we can show that $\lim\limits_{\theta\to 0^-} \dfrac{\sin\theta}{\theta} = 1$. Thus,

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$$

In Example 2.3.11, we use this limit to establish $\lim\limits_{\theta\to 0} \dfrac{1-\cos\theta}{\theta} = 0$. This limit also proves useful in later chapters.

Example 2.3.11: Evaluating an Important Trigonometric Limit

Evaluate $\lim\limits_{\theta\to 0} \dfrac{1-\cos\theta}{\theta}$.

Solution

In the first step, we multiply by the conjugate so that we can use a trigonometric identity to convert the cosine in the numerator to a sine:

$$\begin{aligned}
\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} &= \lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} \cdot \frac{1+\cos\theta}{1+\cos\theta} \\
&= \lim_{\theta\to 0} \frac{1-\cos^2\theta}{\theta(1+\cos\theta)} \\
&= \lim_{\theta\to 0} \frac{\sin^2\theta}{\theta(1+\cos\theta)} \\
&= \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \cdot \frac{\sin\theta}{1+\cos\theta} \\
&= \left(\lim_{\theta\to 0} \frac{\sin\theta}{\theta}\right) \cdot \left(\lim_{\theta\to 0} \frac{\sin\theta}{1+\cos\theta}\right) \\
&= 1 \cdot \frac{0}{2} = 0.
\end{aligned}$$

Therefore,

$$\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0.$$

Exercise 2.3.11

Evaluate $\lim_{\theta \to 0} \frac{1 - \cos \theta}{\sin \theta}$.

Hint

Multiply numerator and denominator by $1 + \cos \theta$.

Answer

0

Deriving the Formula for the Area of a Circle

Some of the geometric formulas we take for granted today were first derived by methods that anticipate some of the methods of calculus. The Greek mathematician *Archimedes* (ca. 287−212; BCE) was particularly inventive, using polygons inscribed within circles to approximate the area of the circle as the number of sides of the polygon increased. He never came up with the idea of a limit, but we can use this idea to see what his geometric constructions could have predicted about the limit.

We can estimate the area of a circle by computing the area of an inscribed regular polygon. Think of the regular polygon as being made up of n triangles. By taking the limit as the vertex angle of these triangles goes to zero, you can obtain the area of the circle. To see this, carry out the following steps:

1.Express the height h and the base b of the isosceles triangle in Figure 2.3.8 in terms of θ and r.

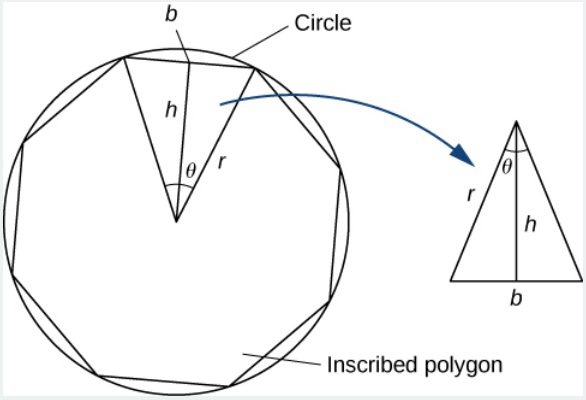

Figure 2.3.8

2. Using the expressions that you obtained in step 1, express the area of the isosceles triangle in terms of θ and r.

(Substitute $\frac{1}{2}\sin\theta$ for $\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$ in your expression.)

3. If an n-sided regular polygon is inscribed in a circle of radius r, find a relationship between θ and n. Solve this for n. Keep in mind there are 2π radians in a circle. (Use radians, not degrees.)

4. Find an expression for the area of the n-sided polygon in terms of r and θ.

5. To find a formula for the area of the circle, find the limit of the expression in step 4 as θ goes to zero. (Hint: $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$).

The technique of estimating areas of regions by using polygons is revisited in Introduction to Integration.

Key Concepts

- The limit laws allow us to evaluate limits of functions without having to go through step-by-step processes each time.
- For polynomials and rational functions,

$$\lim_{x \to a} f(x) = f(a).$$

- You can evaluate the limit of a function by factoring and canceling, by multiplying by a conjugate, or by simplifying a complex fraction.
- The squeeze theorem allows you to find the limit of a function if the function is always greater than one function and less than another function with limits that are known.

Key Equations

- **Basic Limit Results**

$$\lim_{x \to a} x = a \qquad \lim_{x \to a} c = c$$

- **Important Limits**

$$\lim_{\theta \to 0} \sin \theta = 0$$

$$\lim_{\theta \to 0} \cos \theta = 1$$

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

$$\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0$$

Glossary

constant multiple law for limits
the limit law

$$\lim_{x \to a} cf(x) = c \cdot \lim_{x \to a} f(x) = cL$$

difference law for limits
the limit law

$$\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L - M$$

limit laws
the individual properties of limits; for each of the individual laws, let $f(x)$ and $g(x)$ be defined for all $x \neq a$ over some open interval containing a; assume that L and M are real numbers so that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$; let c be a constant

power law for limits
the limit law

$$\lim_{x \to a} (f(x))^n = (\lim_{x \to a} f(x))^n = L^n$$

for every positive integer n

product law for limits
the limit law

$$\lim_{x \to a} (f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = L \cdot M$$

quotient law for limits

the limit law $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \dfrac{L}{M}$ for M≠0

root law for limits

the limit law $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L}$ for all L if n is odd and for $L \geq 0$ if n is even

squeeze theorem

states that if $f(x) \leq g(x) \leq h(x)$ for all $x \neq a$ over an open interval containing a and $\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$ where L is a real number, then $\lim_{x \to a} g(x) = L$

sum law for limits

The limit law $\lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$

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2.3E: EXERCISES FOR SECTION 2.3

In exercises 1 - 4, use the limit laws to evaluate each limit. Justify each step by indicating the appropriate limit law(s).

1) $\lim\limits_{x\to 0}(4x^2-2x+3)$

Answer

Use constant multiple law and difference law:

$\lim\limits_{x\to 0}(4x^2-2x+3)=4\lim\limits_{x\to 0}x^2-2\lim\limits_{x\to 0}x+\lim\limits_{x\to 0}3=0+0+3=3$

2) $\lim\limits_{x\to 1}\dfrac{x^3+3x^2+5}{4-7x}$

3) $\lim\limits_{x\to -2}\sqrt{x^2-6x+3}$

Answer

Use root law: $\lim\limits_{x\to -2}\sqrt{x^2-6x+3}=\sqrt{\lim\limits_{x\to -2}(x^2-6x+3)}=\sqrt{19}$

4) $\lim\limits_{x\to -1}(9x+1)^2$

In exercises 5 - 10, use direct substitution to evaluate the limit of each continuous function.

5) $\lim\limits_{x\to 7}x^2$

Answer

$\lim\limits_{x\to 7}x^2 \;=\; 49$

6) $\lim\limits_{x\to -2}(4x^2-1)$

7) $\lim\limits_{x\to 0}\dfrac{1}{1+\sin x}$

Answer

$\lim\limits_{x\to 0}\dfrac{1}{1+\sin x} \;=\; 1$

8) $\lim\limits_{x\to 2}e^{2x-x^2}$

9) $\lim\limits_{x\to 1}\dfrac{2-7x}{x+6}$

Answer

$\lim\limits_{x\to 1}\dfrac{2-7x}{x+6} \;=\; -\dfrac{5}{7}$

10) $\lim\limits_{x\to 3}\ln e^{3x}$

In exercises 11 - 20, use direct substitution to show that each limit leads to the indeterminate form $0/0$. Then, evaluate the limit analytically.

11) $\lim\limits_{x\to 4}\dfrac{x^2-16}{x-4}$

Answer

When $x=4$, $\dfrac{x^2-16}{x-4}=\dfrac{16-16}{4-4}=\dfrac{0}{0}$;

then, $\lim\limits_{x\to 4}\dfrac{x^2-16}{x-4}=\lim\limits_{x\to 4}\dfrac{(x+4)(x-4)}{x-4}=\lim\limits_{x\to 4}(x+4)=4+4=8$

12) $\lim\limits_{x\to 2}\dfrac{x-2}{x^2-2x}$

13) $\lim\limits_{x\to 6}\dfrac{3x-18}{2x-12}$

Answer

When $x=6$, $\dfrac{3x-18}{2x-12}=\dfrac{18-18}{12-18}=\dfrac{0}{0}$;

then, $\lim\limits_{x\to 6}\dfrac{3x-18}{2x-12}=\lim\limits_{x\to 6}\dfrac{3(x-6)}{2(x-6)}=\lim\limits_{x\to 6}\dfrac{3}{2}=\dfrac{3}{2}$

14) $\lim\limits_{h\to 0}\dfrac{(1+h)^2-1}{h}$

15) $\lim\limits_{t\to 9}\dfrac{t-9}{\sqrt{t}-3}$

Answer

When $t=9$, $\dfrac{t-9}{\sqrt{t}-3}=\dfrac{9-9}{3-3}=\dfrac{0}{0}$;

then, $\lim\limits_{t\to 9}\dfrac{t-9}{\sqrt{t}-3}=\lim\limits_{t\to 9}\dfrac{t-9}{\sqrt{t}-3}\dfrac{\sqrt{t}+3}{\sqrt{t}+3}=\lim\limits_{t\to 9}\dfrac{(t-9)(\sqrt{t}+3)}{t-9}=\lim\limits_{t\to 9}(\sqrt{t}+3)=\sqrt{9}+3=6$

16) $\lim\limits_{h\to 0}\dfrac{\dfrac{1}{a+h}-\dfrac{1}{a}}{h}$, where a is a real-valued constant

17) $\lim\limits_{\theta\to\pi}\dfrac{\sin\theta}{\tan\theta}$

Answer

When $\theta=\pi$, $\dfrac{\sin\theta}{\tan\theta}=\dfrac{\sin\pi}{\tan\pi}=\dfrac{0}{0}$;

then, $\lim\limits_{\theta\to\pi}\dfrac{\sin\theta}{\tan\theta}=\lim\limits_{\theta\to\pi}\dfrac{\sin\theta}{\frac{\sin\theta}{\cos\theta}}=\lim\limits_{\theta\to\pi}\cos\theta=\cos\pi=-1$

18) $\lim\limits_{x\to 1}\dfrac{x^3-1}{x^2-1}$

19) $\lim\limits_{x\to 1/2}\dfrac{2x^2+3x-2}{2x-1}$

Answer

When $x=1/2$, $\dfrac{2x^2+3x-2}{2x-1}=\dfrac{\frac{1}{2}+\frac{3}{2}-2}{1-1}=\dfrac{0}{0}$;

then, $\lim\limits_{x\to 1/2}\dfrac{2x^2+3x-2}{2x-1}=\lim\limits_{x\to 1/2}\dfrac{(2x-1)(x+2)}{2x-1}=\lim\limits_{x\to 1/2}(x+2)=\dfrac{1}{2}+2=\dfrac{5}{2}$

20) $\lim\limits_{x\to -3}\dfrac{\sqrt{x+4}-1}{x+3}$

In exercises 21 - 24, use direct substitution to obtain an undefined expression. Then, use the method used in Example 9 of this section to simplify the function and determine the limit.

21) $\lim\limits_{x\to -2^-}\dfrac{2x^2+7x-4}{x^2+x-2}$

Answer

$-\infty$

22) $\lim\limits_{x\to -2^+}\dfrac{2x^2+7x-4}{x^2+x-2}$

23) $\lim\limits_{x\to 1^-}\dfrac{2x^2+7x-4}{x^2+x-2}$

Answer

$-\infty$

24) $\lim\limits_{x\to 1^+}\dfrac{2x^2+7x-4}{x^2+x-2}$

In exercises 25 - 32, assume that $\lim\limits_{x\to 6}f(x)=4$, $\lim\limits_{x\to 6}g(x)=9$, and $\lim\limits_{x\to 6}h(x)=6$. Use these three facts and the limit laws to evaluate each limit.

25) $\lim_{x\to 6} 2f(x)g(x)$

Answer

$$\lim_{x\to 6} 2f(x)g(x) = 2\left(\lim_{x\to 6} f(x)\right)\left(\lim_{x\to 6} g(x)\right) = 2(4)(9) = 72$$

26) $\lim_{x\to 6} \dfrac{g(x)-1}{f(x)}$

27) $\lim_{x\to 6} \left(f(x) + \dfrac{1}{3}g(x)\right)$

Answer

$$\lim_{x\to 6} \left(f(x) + \frac{1}{3}g(x)\right) = \lim_{x\to 6} f(x) + \frac{1}{3}\lim_{x\to 6} g(x) = 4 + \frac{1}{3}(9) = 7$$

28) $\lim_{x\to 6} \dfrac{\left(h(x)\right)^3}{2}$

29) $\lim_{x\to 6} \sqrt{g(x) - f(x)}$

Answer

$$\lim_{x\to 6} \sqrt{g(x) - f(x)} = \sqrt{\lim_{x\to 6} g(x) - \lim_{x\to 6} f(x)} = \sqrt{9-4} = \sqrt{5}$$

30) $\lim_{x\to 6} x \cdot h(x)$

31) $\lim_{x\to 6}[(x+1)\cdot f(x)]$

Answer

$$\lim_{x\to 6}[(x+1)f(x)] = \left(\lim_{x\to 6}(x+1)\right)\left(\lim_{x\to 6} f(x)\right) = 7(4) = 28$$

32) $\lim_{x\to 6}(f(x)\cdot g(x) - h(x))$

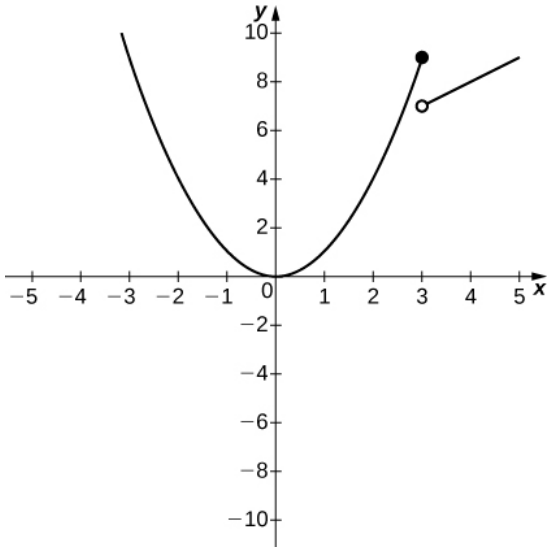
[T] In exercises 33 - 35, use a calculator to draw the graph of each piecewise-defined function and study the graph to evaluate the given limits.

33) $f(x) = \begin{cases} x^2, & x \le 3 \\ x+4, & x > 3 \end{cases}$

a. $\lim_{x\to 3^-} f(x)$

b. $\lim_{x\to 3^+} f(x)$

Answer



a. 9; b.7

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34) $g(x) = \begin{cases} x^3 - 1, & x \leq 0 \\ 1, & x > 0 \end{cases}$

a. $\lim_{x \to 0^-} g(x)$

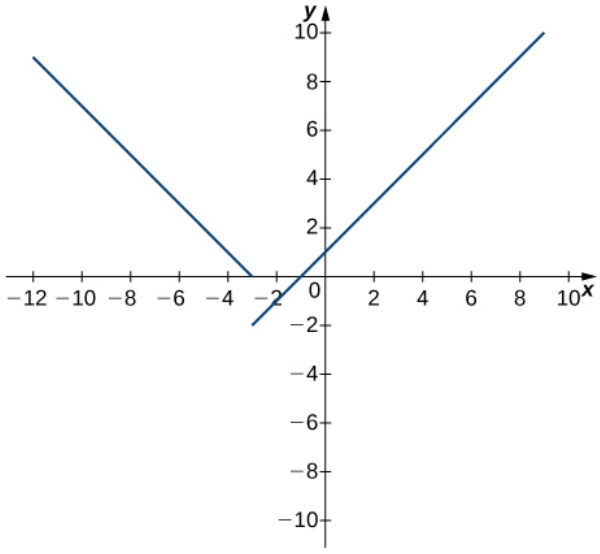
b. $\lim_{x \to 0^+} g(x)$

35) $h(x) = \begin{cases} x^2 - 2x + 1, & x < 2 \\ 3 - x, & x \geq 2 \end{cases}$

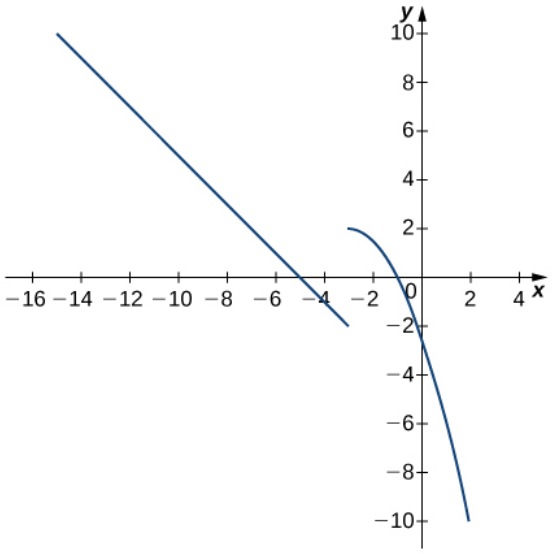
a. $\lim_{x \to 2^-} h(x)$

b. $\lim_{x \to 2^+} h(x)$

In exercises 36 - 43, use the following graphs and the limit laws to evaluate each limit.



$y = f(x)$



$y = g(x)$

36) $\lim_{x \to -3^+} (f(x) + g(x))$

37) $\lim_{x \to -3^-} (f(x) - 3g(x))$

Answer

$\lim_{x \to -3^-} (f(x) - 3g(x)) = \lim_{x \to -3^-} f(x) - 3 \lim_{x \to -3^-} g(x) = 0 + 6 = 6$

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38) $\lim\limits_{x\to 0}\dfrac{f(x)g(x)}{3}$

39) $\lim\limits_{x\to -5}\dfrac{2+g(x)}{f(x)}$

Answer

$$\lim_{x\to -5}\frac{2+g(x)}{f(x)}=\frac{2+\left(\lim\limits_{x\to -5}g(x)\right)}{\lim\limits_{x\to -5}f(x)}=\frac{2+0}{2}=1$$

40) $\lim\limits_{x\to 1}(f(x))^2$

41) $\lim\limits_{x\to 1}\sqrt[3]{f(x)-g(x)}$

Answer

$$\lim_{x\to 1}\sqrt[3]{f(x)-g(x)}=\sqrt[3]{\lim_{x\to 1}f(x)-\lim_{x\to 1}g(x)}=\sqrt[3]{2+5}=\sqrt[3]{7}$$

42) $\lim\limits_{x\to -7}(x\cdot g(x))$

43) $\lim\limits_{x\to -9}[x\cdot f(x)+2\cdot g(x)]$

Answer

$$\lim_{x\to -9}(xf(x)+2g(x))=\left(\lim_{x\to -9}x\right)\left(\lim_{x\to -9}f(x)\right)+2\lim_{x\to -9}g(x)=(-9)(6)+2(4)=-46$$

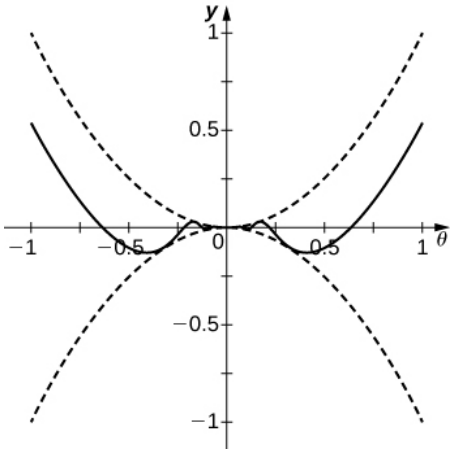
For exercises 44 - 46, evaluate the limit using the squeeze theorem. Use a calculator to graph the functions $f(x)$, $g(x)$, and $h(x)$ when possible.

44) [T] True or False? If $2x-1\le g(x)\le x^2-2x+3$, then $\lim\limits_{x\to 2}g(x)=0$.

45) [T] $\lim\limits_{\theta\to 0}\theta^2\cos\left(\dfrac{1}{\theta}\right)$

Answer

The limit is zero.



46) $\lim\limits_{x\to 0}f(x)$, where $f(x)=\begin{cases}0, & x \text{ rational}\\ x^2, & x \text{ irrational}\end{cases}$

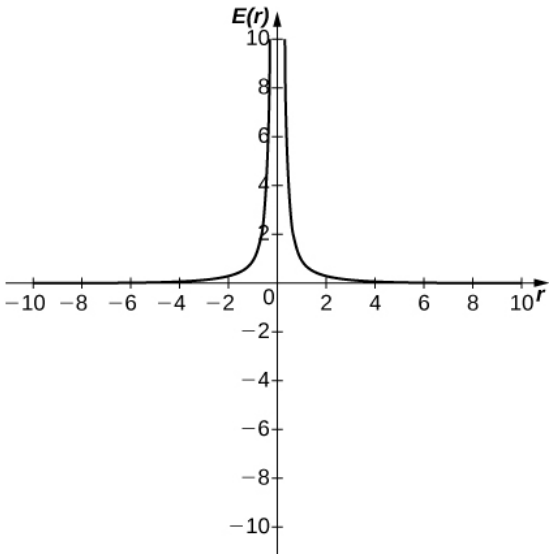
47) [T] In physics, the magnitude of an electric field generated by a point charge at a distance r in vacuum is governed by Coulomb's law: $E(r)=\dfrac{q}{4\pi\varepsilon_0 r^2}$, where E represents the magnitude of the electric field, q is the charge of the particle, r is the distance between the particle and where the strength of the field is measured, and $\dfrac{1}{4\pi\varepsilon_0}$ is Coulomb's constant: $8.988\times 109N\cdot m^2/C^2$.

a. Use a graphing calculator to graph $E(r)$ given that the charge of the particle is $q=10^{-10}$.

b. Evaluate $\lim\limits_{r\to 0^+}E(r)$. What is the physical meaning of this quantity? Is it physically relevant? Why are you evaluating from the right?

Answer

a.



b. ∞. The magnitude of the electric field as you approach the particle q becomes infinite. It does not make physical sense to evaluate negative distance.

48) [T] The density of an object is given by its mass divided by its volume: $\rho = m/V$.

a. Use a calculator to plot the volume as a function of density $(V = m/\rho)$, assuming you are examining something of mass 8 kg $(m = 8)$.

b. Evaluate $\lim_{x \to 0^+} V(\rho)$ and explain the physical meaning.

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2.4: Continuity

Learning Objectives

- Explain the three conditions for continuity at a point.
- Describe three kinds of discontinuities.
- Define continuity on an interval.
- State the theorem for limits of composite functions.
- Provide an example of the intermediate value theorem.

Many functions have the property that their graphs can be traced with a pencil without lifting the pencil from the page. Such functions are called *continuous*. Other functions have points at which a break in the graph occurs, but satisfy this property over intervals contained in their domains. They are continuous on these intervals and are said to have a *discontinuity* at a point where a break occurs.

We begin our investigation of continuity by exploring what it means for a function to have *continuity at a point*. Intuitively, a function is continuous at a particular point if there is no break in its graph at that point.

Continuity at a Point

Before we look at a formal definition of what it means for a function to be continuous at a point, let's consider various functions that fail to meet our intuitive notion of what it means to be continuous at a point. We then create a list of conditions that prevent such failures.

Our first function of interest is shown in Figure 2.4.1. We see that the graph of $f(x)$ has a hole at a. In fact, $f(a)$ is undefined. At the very least, for $f(x)$ to be continuous at a, we need the following condition:

i. $f(a)$ is defined

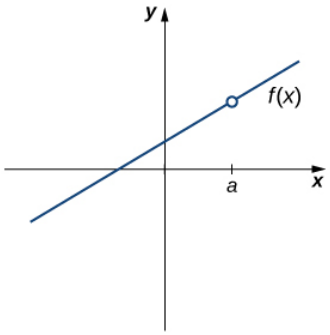

Figure 2.4.1: The function $f(x)$ is not continuous at a because $f(a)$ is undefined.

However, as we see in Figure 2.4.2, this condition alone is insufficient to guarantee continuity at the point a. Although $f(a)$ is defined, the function has a gap at a. In this example, the gap exists because $\lim_{x \to a} f(x)$ does not exist. We must add another condition for continuity at a—namely,

ii. $\lim_{x \to a} f(x)$ exists

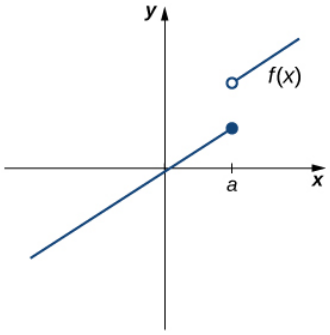

Figure 2.4.2: The function $f(x)$ is not continuous at a because $\lim_{x \to a} f(x)$ does not exist.

However, as we see in Figure 2.4.3, these two conditions by themselves do not guarantee continuity at a point. The function in this figure satisfies both of our first two conditions, but is still not continuous at a. We must add a third condition to our list:

iii. $\lim_{x \to a} f(x) = f(a)$

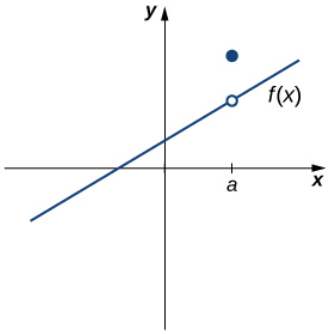

Figure 2.4.3: The function $f(x)$ is not continuous at a because $\lim_{x \to a} f(x) \neq f(a)$.

Now we put our list of conditions together and form a definition of continuity at a point.

Definition: Continuous at a Point

A function $f(x)$ is **continuous at a point** a if and only if the following three conditions are satisfied:

i. $f(a)$ is defined
ii. $\lim_{x \to a} f(x)$ exists
iii. $\lim_{x \to a} f(x) = f(a)$

A function is **discontinuous at a point** a if it fails to be continuous at a.

The following procedure can be used to analyze the continuity of a function at a point using this definition.

Problem-Solving Strategy: Determining Continuity at a Point

1. Check to see if $f(a)$ is defined. If $f(a)$ is undefined, we need go no further. The function is not continuous at a. If $f(a)$ is defined, continue to step 2.
2. Compute $\lim_{x \to a} f(x)$. In some cases, we may need to do this by first computing $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$. If $\lim_{x \to a} f(x)$ does not exist (that is, it is not a real number), then the function is not continuous at a and the problem is solved. If $\lim_{x \to a} f(x)$ exists, then continue to step 3.
3. Compare $f(a)$ and $\lim_{x \to a} f(x)$. If $\lim_{x \to a} f(x) \neq f(a)$, then the function is not continuous at a. If $\lim_{x \to a} f(x) = f(a)$, then the function is continuous at a.

The next three examples demonstrate how to apply this definition to determine whether a function is continuous at a given point. These examples illustrate situations in which each of the conditions for continuity in the definition succeed or fail.

Example 2.4.1A: Determining Continuity at a Point, Condition 1

Using the definition, determine whether the function $f(x) = \dfrac{x^2 - 4}{x - 2}$ is continuous at $x = 2$. Justify the conclusion.

Solution

Let's begin by trying to calculate $f(2)$. We can see that $f(2) = 0/0$, which is undefined. Therefore, $f(x) = \dfrac{x^2 - 4}{x - 2}$ is discontinuous at 2 because $f(2)$ is undefined. The graph of $f(x)$ is shown in Figure 2.4.4.

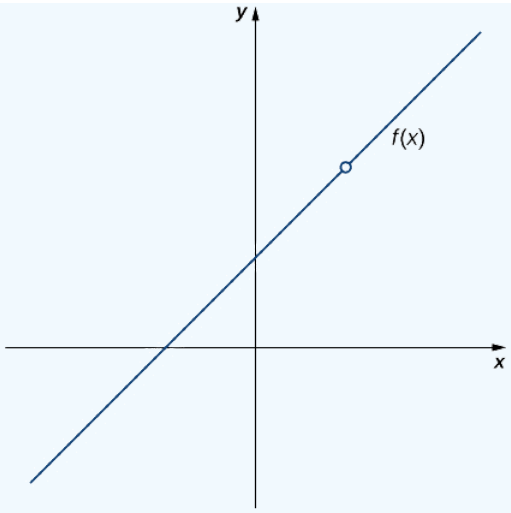

Figure 2.4.4: The function $f(x)$ is discontinuous at 2 because $f(2)$ is undefined.

Example 2.4.1B: Determining Continuity at a Point, Condition 2

Using the definition, determine whether the function $f(x) = \begin{cases} -x^2 + 4, & \text{if } x \leq 3 \\ 4x - 8, & \text{if } x > 3 \end{cases}$ is continuous at $x = 3$. Justify the conclusion.

Solution

Let's begin by trying to calculate $f(3)$.

$$f(3) = -(3^2) + 4 = -5.$$

Thus, $f(3)$ is defined. Next, we calculate $\lim\limits_{x \to 3} f(x)$. To do this, we must compute $\lim\limits_{x \to 3^-} f(x)$ and $\lim\limits_{x \to 3^+} f(x)$:

$$\lim_{x \to 3^-} f(x) = -(3^2) + 4 = -5$$

and

$$\lim_{x \to 3^+} f(x) = 4(3) - 8 = 4.$$

Therefore, $\lim\limits_{x \to 3} f(x)$ does not exist. Thus, $f(x)$ is not continuous at 3. The graph of $f(x)$ is shown in Figure 2.4.5.

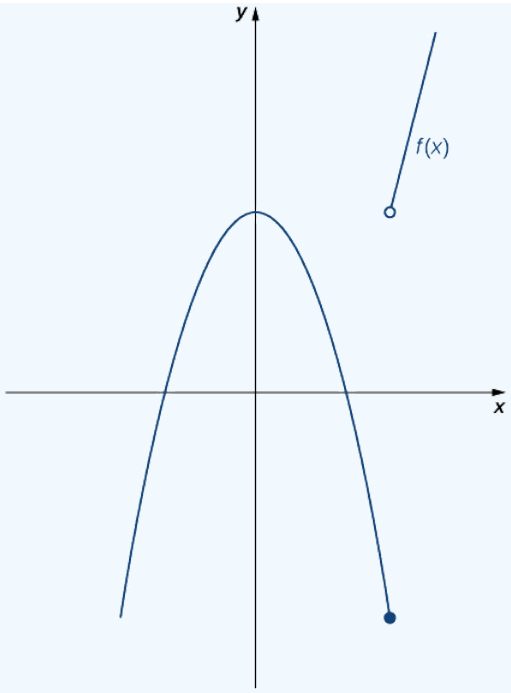

Figure 2.4.5: The function $f(x)$ is not continuous at 3 because $\lim_{x\to 3} f(x)$ does not exist.

Example 2.4.1C: Determining Continuity at a Point, Condition 3

Using the definition, determine whether the function $f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$ is continuous at $x = 0$.

Solution

First, observe that

$$f(0) = 1$$

Next,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Last, compare $f(0)$ and $\lim_{x\to 0} f(x)$. We see that

$$f(0) = 1 = \lim_{x\to 0} f(x).$$

Since all three of the conditions in the definition of continuity are satisfied, $f(x)$ is continuous at $x = 0$.

Exercise 2.4.1

Using the definition, determine whether the function $f(x) = \begin{cases} 2x + 1, & \text{if } x < 1 \\ 2, & \text{if } x = 1 \\ -x + 4, & \text{if } x > 1 \end{cases}$ is continuous at $x = 1$. If the function is not continuous at 1, indicate the condition for continuity at a point that fails to hold.

Hint

Check each condition of the definition.

Answer

f is not continuous at 1 because $f(1) = 2 \neq 3 = \lim_{x\to 1} f(x)$.

By applying the definition of continuity and previously established theorems concerning the evaluation of limits, we can state the following theorem.

Theorem 2.4.1: Continuity of Polynomials and Rational Functions

Polynomials and rational functions are continuous at every point in their domains.

Proof

Previously, we showed that if $p(x)$ and $q(x)$ are polynomials, $\lim_{x \to a} p(x) = p(a)$ for every polynomial $p(x)$ and $\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$ as long as $q(a) \neq 0$. Therefore, polynomials and rational functions are continuous on their domains.

□

We now apply Theorem 2.4.1 to determine the points at which a given rational function is continuous.

Example 2.4.2:Continuity of a Rational Function

For what values of x is $f(x) = \frac{x+1}{x-5}$ continuous?

Solution

The rational function $f(x) = \frac{x+1}{x-5}$ is continuous for every value of x except $x = 5$.

Exercise 2.4.2

For what values of x is $f(x) = 3x^4 - 4x^2$ continuous?

Hint

Use the Continuity of Polynomials and Rational Functions stated above.

Answer

$f(x)$ is continuous at every real number.

Types of Discontinuities

As we have seen in Example 2.4.1A and Example 2.4.1B, discontinuities take on several different appearances. We classify the types of discontinuities we have seen thus far as removable discontinuities, infinite discontinuities, or jump discontinuities. Intuitively, a **removable discontinuity** is a discontinuity for which there is a hole in the graph, a **jump discontinuity** is a noninfinite discontinuity for which the sections of the function do not meet up, and an **infinite discontinuity** is a discontinuity located at a vertical asymptote. Figure 2.4.6 illustrates the differences in these types of discontinuities. Although these terms provide a handy way of describing three common types of discontinuities, keep in mind that not all discontinuities fit neatly into these categories.

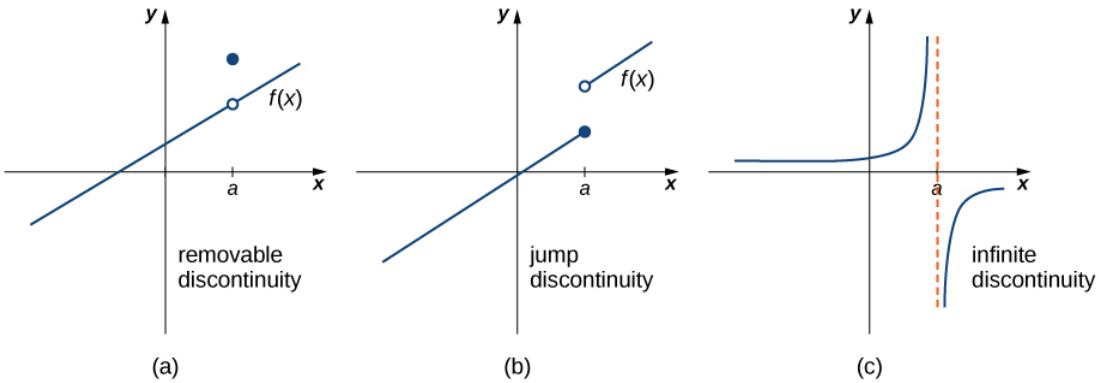

Figure 2.4.6: Discontinuities are classified as (a) removable, (b) jump, or (c) infinite.

These three discontinuities are formally defined as follows:

Definition

If $f(x)$ is discontinuous at a, then

1. f has a **removable discontinuity** at a if $\lim\limits_{x\to a} f(x)$ exists. (Note: When we state that $\lim\limits_{x\to a} f(x)$ exists, we mean that $\lim\limits_{x\to a} f(x) = L$, where L is a real number.)
2. f has a **jump discontinuity** at a if $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ both exist, but $\lim\limits_{x\to a^-} f(x) \neq \lim\limits_{x\to a^+} f(x)$. (Note: When we state that $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ both exist, we mean that both are real-valued and that neither take on the values $\pm\infty$.)
3. f has an **infinite discontinuity** at a if $\lim\limits_{x\to a^-} f(x) = \pm\infty$ or $\lim\limits_{x\to a^+} f(x) = \pm\infty$.

Example 2.4.3: Classifying a Discontinuity

In Example 2.4.1A, we showed that $f(x) = \dfrac{x^2-4}{x-2}$ is discontinuous at $x = 2$. Classify this discontinuity as removable, jump, or infinite.

Solution

To classify the discontinuity at 2 we must evaluate $\lim\limits_{x\to 2} f(x)$:

$$\begin{aligned}\lim_{x\to 2} f(x) &= \lim_{x\to 2} \frac{x^2-4}{x-2} \\ &= \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} \\ &= \lim_{x\to 2} (x+2) \\ &= 4.\end{aligned}$$

Since f is discontinuous at 2 and $\lim\limits_{x\to 2} f(x)$ exists, f has a removable discontinuity at $x = 2$.

Example 2.4.4: Classifying a Discontinuity

In Example 2.4.1B, we showed that $f(x) = \begin{cases} -x^2+4, & \text{if } x \leq 3 \\ 4x-8, & \text{if } x > 3 \end{cases}$ is discontinuous at $x = 3$. Classify this discontinuity as removable, jump, or infinite.

Solution

Earlier, we showed that f is discontinuous at 3 because $\lim\limits_{x\to 3} f(x)$ does not exist. However, since $\lim\limits_{x\to 3^-} f(x) = -5$ and $\lim\limits_{x\to 3^+} f(x) = 4$ both exist, we conclude that the function has a jump discontinuity at 3.

Example 2.4.5: Classifying a Discontinuity

Determine whether $f(x) = \dfrac{x+2}{x+1}$ is continuous at -1. If the function is discontinuous at -1, classify the discontinuity as removable, jump, or infinite.

Solution

The function value $f(-1)$ is undefined. Therefore, the function is not continuous at -1. To determine the type of discontinuity, we must determine the limit at -1. We see that $\lim\limits_{x\to -1^-} \dfrac{x+2}{x+1} = -\infty$ and $\lim\limits_{x\to -1^+} \dfrac{x+2}{x+1} = +\infty$. Therefore, the function has an infinite discontinuity at -1.

Exercise 2.4.3

For $f(x) = \begin{cases} x^2, & \text{if } x \neq 1 \\ 3, & \text{if } x = 1 \end{cases}$, decide whether f is continuous at 1. If f is not continuous at 1, classify the discontinuity as removable, jump, or infinite.

Hint

Consider the definitions of the various kinds of discontinuity stated above. If the function is discontinuous at 1, look at $\lim\limits_{x\to 1} f(x)$

Answer

Discontinuous at 1; removable

Continuity over an Interval

Now that we have explored the concept of continuity at a point, we extend that idea to **continuity over an interval**. As we develop this idea for different types of intervals, it may be useful to keep in mind the intuitive idea that a function is continuous over an interval if we can use a pencil to trace the function between any two points in the interval without lifting the pencil from the paper. In preparation for defining continuity on an interval, we begin by looking at the definition of what it means for a function to be continuous from the right at a point and continuous from the left at a point.

Definition: Continuity from the Right and from the Left

A function $f(x)$ is said to be **continuous from the right** at a if $\lim\limits_{x\to a^+} f(x) = f(a)$.

A function $f(x)$ is said to be **continuous from the left** at a if $\lim\limits_{x\to a^-} f(x) = f(a)$

A function is continuous over an open interval if it is continuous at every point in the interval. A function $f(x)$ is continuous over a closed interval of the form $[a,b]$ if it is continuous at every point in (a,b) and is continuous from the right at a and is continuous from the left at b. Analogously, a function $f(x)$ is continuous over an interval of the form $(a,b]$ if it is continuous over (a,b) and is continuous from the left at b. Continuity over other types of intervals are defined in a similar fashion.

Requiring that $\lim\limits_{x\to a^+} f(x) = f(a)$ and $\lim\limits_{x\to b^-} f(x) = f(b)$ ensures that we can trace the graph of the function from the point $(a, f(a))$ to the point $(b, f(b))$ without lifting the pencil. If, for example, $\lim\limits_{x\to a^+} f(x) \neq f(a)$, we would need to lift our pencil to jump from $f(a)$ to the graph of the rest of the function over $(a,b]$.

Example 2.4.6: Continuity on an Interval

State the interval(s) over which the function $f(x) = \dfrac{x-1}{x^2+2x}$ is continuous.

Solution

Since $f(x) = \dfrac{x-1}{x^2+2x}$ is a rational function, it is continuous at every point in its domain. The domain of $f(x)$ is the set $(-\infty,-2) \cup (-2,0) \cup (0,+\infty)$. Thus, $f(x)$ is continuous over each of the intervals $(-\infty,-2)$, $(-2,0)$, and $(0,+\infty)$.

Example 2.4.7: Continuity over an Interval

State the interval(s) over which the function $f(x) = \sqrt{4 - x^2}$ is continuous.

Solution

From the limit laws, we know that $\lim_{x \to a} \sqrt{4 - x^2} = \sqrt{4 - a^2}$ for all values of a in $(-2, 2)$. We also know that $\lim_{x \to -2^+} \sqrt{4 - x^2} = 0$ exists and $\lim_{x \to 2^-} \sqrt{4 - x^2} = 0$ exists. Therefore, $f(x)$ is continuous over the interval $[-2, 2]$.

Exercise 2.4.4

State the interval(s) over which the function $f(x) = \sqrt{x + 3}$ is continuous.

Hint

Use Example 2.4.7 as a guide.

Answer

$[-3, +\infty)$

Theorem 2.4.2 allows us to expand our ability to compute limits. In particular, this theorem ultimately allows us to demonstrate that trigonometric functions are continuous over their domains.

Theorem 2.4.2: Composite Function Theorem

If $f(x)$ is continuous at L and $\lim_{x \to a} g(x) = L$, then

$$\lim_{x \to a} f\big(g(x)\big) = f\Big(\lim_{x \to a} g(x)\Big) = f(L). \tag{2.4.1}$$

Before we move on to Example 2.4.8, recall that earlier, in the section on limit laws, we showed $\lim_{x \to 0} \cos x = 1 = \cos(0)$. Consequently, we know that $f(x) = \cos x$ is continuous at 0. In Example 2.4.8, we see how to combine this result with the composite function theorem.

Example 2.4.8: Limit of a Composite Cosine Function

Evaluate $\lim_{x \to \pi/2} \cos\left(x - \frac{\pi}{2}\right)$.

Solution

The given function is a composite of $\cos x$ and $x - \frac{\pi}{2}$. Since $\lim_{x \to \pi/2} \left(x - \frac{\pi}{2}\right) = 0$ and $\cos x$ is continuous at 0, we may apply the composite function theorem. Thus,

$$\lim_{x \to \pi/2} \cos\left(x - \frac{\pi}{2}\right) = \cos\left(\lim_{x \to \pi/2} \left(x - \frac{\pi}{2}\right)\right) = \cos(0) = 1.$$

Exercise 2.4.4:

Evaluate $\lim_{x \to \pi} \sin(x - \pi)$.

Hint

$f(x) = \sin x$ is continuous at 0. Use Example 2.4.8 as a guide.

Answer

0

The proof of the next theorem uses the composite function theorem as well as the continuity of $f(x) = \sin x$ and $g(x) = \cos x$ at the point 0 to show that trigonometric functions are continuous over their entire domains.

Theorem 2.4.3: Continuity of Trigonometric Functions

Trigonometric functions are continuous over their entire domains.

Proof

We begin by demonstrating that $\cos x$ is continuous at every real number. To do this, we must show that $\lim_{x \to a} \cos x = \cos a$ for all values of a.

$$\begin{aligned}
\lim_{x \to a} \cos x &= \lim_{x \to a} \cos((x - a) + a) && \text{Rewrite } x = x - a + a. \\
&= \lim_{x \to a} (\cos(x - a) \cos a - \sin(x - a) \sin a) && \text{Apply the identity for the cosine of the sum of two angles.} \\
&= \cos(\lim_{x \to a}(x - a)) \cos a - \sin(\lim_{x \to a}(x - a)) \sin a && \text{Since } \lim_{x \to a}(x - a) = 0, \text{ and } \sin x \text{ and } \cos x \text{ are continuous at } 0. \\
&= \cos(0) \cos a - \sin(0) \sin a && \text{Evaluate } \cos(0) \text{ and } \sin(0) \text{ and simplify.} \\
&= 1 \cdot \cos a - 0 \cdot \sin a = \cos a.
\end{aligned}$$

The proof that $\sin x$ is continuous at every real number is analogous. Because the remaining trigonometric functions may be expressed in terms of $\sin x$ and $\cos x$, their continuity follows from the quotient limit law.

□

As you can see, the composite function theorem is invaluable in demonstrating the continuity of trigonometric functions. As we continue our study of calculus, we revisit this theorem many times.

The Intermediate Value Theorem

Functions that are continuous over intervals of the form $[a, b]$, where a and b are real numbers, exhibit many useful properties. Throughout our study of calculus, we will encounter many powerful theorems concerning such functions. The first of these theorems is the **Intermediate Value Theorem.**

The Intermediate Value Theorem

Let f be continuous over a closed, bounded interval $[a, b]$. If z is any real number between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ satisfying $f(c) = z$ in Figure 2.4.7.

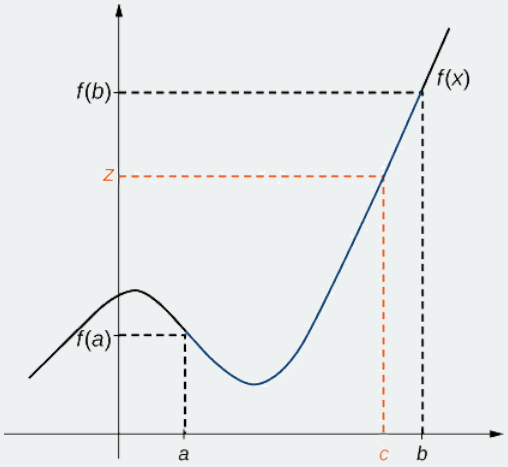

Figure 2.4.7: There is a number $c \in [a, b]$ that satisfies $f(c) = z$.

Example 2.4.9: Application of the Intermediate Value Theorem

Show that $f(x) = x - \cos x$ has at least one zero.

Solution

Since $f(x) = x - \cos x$ is continuous over $(-\infty, +\infty)$, it is continuous over any closed interval of the form $[a, b]$. If you can find an interval $[a, b]$ such that $f(a)$ and $f(b)$ have opposite signs, you can use the Intermediate Value Theorem to conclude there must be a real number c in (a, b) that satisfies $f(c) = 0$. Note that

$$f(0) = 0 - \cos(0) = -1 < 0$$

and

$$f(\tfrac{\pi}{2}) = \tfrac{\pi}{2} - \cos\tfrac{\pi}{2} = \tfrac{\pi}{2} > 0.$$

Using the Intermediate Value Theorem, we can see that there must be a real number c in $[0, \pi/2]$ that satisfies $f(c) = 0$. Therefore, $f(x) = x - \cos x$ has at least one zero.

Example 2.4.10: When Can You Apply the Intermediate Value Theorem?

If $f(x)$ is continuous over $[0, 2]$, $f(0) > 0$ and $f(2) > 0$, can we use the Intermediate Value Theorem to conclude that $f(x)$ has no zeros in the interval $[0, 2]$? Explain.

Solution

No. The Intermediate Value Theorem only allows us to conclude that we can find a value between $f(0)$ and $f(2)$; it doesn't allow us to conclude that we can't find other values. To see this more clearly, consider the function $f(x) = (x - 1)^2$. It satisfies $f(0) = 1 > 0, f(2) = 1 > 0$, and $f(1) = 0$.

Example 2.4.11: When Can You Apply the Intermediate Value Theorem?

For $f(x) = 1/x, f(-1) = -1 < 0$ and $f(1) = 1 > 0$. Can we conclude that $f(x)$ has a zero in the interval $[-1, 1]$?

Solution

No. The function is not continuous over $[-1, 1]$. The Intermediate Value Theorem does not apply here.

Exercise 2.4.5

Show that $f(x) = x^3 - x^2 - 3x + 1$ has a zero over the interval $[0, 1]$.

Hint

Find $f(0)$ and $f(1)$. Apply the Intermediate Value Theorem.

Answer

$f(0) = 1 > 0,\ f(1) = -2 < 0;\ f(x)$ is continuous over $[0, 1]$. It must have a zero on this interval.

Key Concepts

- For a function to be continuous at a point, it must be defined at that point, its limit must exist at the point, and the value of the function at that point must equal the value of the limit at that point.
- Discontinuities may be classified as removable, jump, or infinite.
- A function is continuous over an open interval if it is continuous at every point in the interval. It is continuous over a closed interval if it is continuous at every point in its interior and is continuous at its endpoints.
- The composite function theorem states: If $f(x)$ is continuous at L and $\lim_{x \to a} g(x) = L$, then $\lim_{x \to a} f\big(g(x)\big) = f\big(\lim_{x \to a} g(x)\big) = f(L)$.
- The Intermediate Value Theorem guarantees that if a function is continuous over a closed interval, then the function takes on every value between the values at its endpoints.

Glossary

continuity at a point

A function $f(x)$ is continuous at a point a if and only if the following three conditions are satisfied: (1) $f(a)$ is defined, (2) $\lim\limits_{x \to a} f(x)$ exists, and (3) $\lim x \to a f(x) = f(a)$

continuity from the left

A function is continuous from the left at b if $\lim\limits_{x \to b^-} f(x) = f(b)$

continuity from the right

A function is continuous from the right at a if $\lim\limits_{x \to a^+} f(x) = f(a)$

continuity over an interval

a function that can be traced with a pencil without lifting the pencil; a function is continuous over an open interval if it is continuous at every point in the interval; a function $f(x)$ is continuous over a closed interval of the form $[a, b]$ if it is continuous at every point in (a, b), and it is continuous from the right at a and from the left at b

discontinuity at a point

A function is discontinuous at a point or has a discontinuity at a point if it is not continuous at the point

infinite discontinuity

An infinite discontinuity occurs at a point a if $\lim\limits_{x \to a^-} f(x) = \pm\infty$ or $\lim\limits_{x \to a^+} f(x) = \pm\infty$

Intermediate Value Theorem

Let f be continuous over a closed bounded interval $[a, b]$ if z is any real number between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ satisfying $f(c) = z$

jump discontinuity

A jump discontinuity occurs at a point a if $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$ both exist, but $\lim\limits_{x \to a^-} f(x) \neq \lim\limits_{x \to a^+} f(x)$

removable discontinuity

A removable discontinuity occurs at a point a if $f(x)$ is discontinuous at a, but $\lim\limits_{x \to a} f(x)$ exists

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2.4E: EXERCISES FOR SECTION 2.4

For exercises 1 - 8, determine the point(s), if any, at which each function is discontinuous. Classify any discontinuity as *jump*, *removable*, *infinite*, or *other*.

1) $f(x) = \dfrac{1}{\sqrt{x}}$

Answer

The function is defined for all x in the interval $(0, \infty)$.

2) $f(x) = \dfrac{2}{x^2 + 1}$

3) $f(x) = \dfrac{x}{x^2 - x}$

Answer

Removable discontinuity at $x = 0$; infinite discontinuity at $x = 1$.

4) $g(t) = t^{-1} + 1$

5) $f(x) = \dfrac{5}{e^x - 2}$

Answer

Infinite discontinuity at $x = \ln 2$

6) $f(x) = \dfrac{|x - 2|}{x - 2}$

7) $H(x) = \tan 2x$

Answer

Infinite discontinuities at $x = \dfrac{(2k + 1)\pi}{4}$, for $k = 0, \pm 1, \pm 2, \pm 3, \ldots$

8) $f(t) = \dfrac{t + 3}{t^2 + 5t + 6}$

For exercises 9 - 14, decide if the function continuous at the given point. If it is discontinuous, what type of discontinuity is it?

9) $\dfrac{2x^2 - 5x + 3}{x - 1}$ at $x = 1$

Answer

No. It is a removable discontinuity.

10) $h(\theta) = \dfrac{\sin\theta - \cos\theta}{\tan\theta}$ at $\theta = \pi$

11) $g(u) = \begin{cases} \dfrac{6u^2 + u - 2}{2u - 1}, & \text{if } u \neq \frac{1}{2} \\ \frac{7}{2}, & \text{if } u = \frac{1}{2} \end{cases}$, at $u = \frac{1}{2}$

Answer

Yes. It is continuous.

12) $f(y) = \dfrac{\sin(\pi y)}{\tan(\pi y)}$, at $y = 1$

13) $f(x) = \begin{cases} x^2 - e^x, & \text{if } x < 0 \\ x - 1, & \text{if } x \geq 0 \end{cases}$, at $x = 0$

Answer

Yes. It is continuous.

14) $f(x) = \begin{cases} x\sin(x), & \text{if } x \leq \pi \\ x\tan(x), & \text{if } x > \pi \end{cases}$, at $x = \pi$

In exercises 15 - 19, find the value(s) of k that makes each function continuous over the given interval.

15) $f(x) = \begin{cases} 3x + 2, & \text{if } x < k \\ 2x - 3, & \text{if } k \leq x \leq 8 \end{cases}$

Answer

$k = -5$

16) $f(\theta) = \begin{cases} \sin\theta, & \text{if } 0 \le \theta < \frac{\pi}{2} \\ \cos(\theta + k), & \text{if } \frac{\pi}{2} \le \theta \le \pi \end{cases}$

17) $f(x) = \begin{cases} \dfrac{x^2 + 3x + 2}{x + 2}, & \text{if } x \neq -2 \\ k, & \text{if } x = -2 \end{cases}$

Answer

$k = -1$

18) $f(x) = \begin{cases} e^{kx}, & \text{if } 0 \le x < 4 \\ x + 3, & \text{if } 4 \le x \le 8 \end{cases}$

19) $f(x) = \begin{cases} \sqrt{kx}, & \text{if } 0 \le x \le 3 \\ x + 1, & \text{if } 3 < x \le 10 \end{cases}$

Answer

$k = \frac{16}{3}$

In exercises 20 - 21, use the Intermediate Value Theorem (IVT).

20) Let $h(x) = \begin{cases} 3x^2 - 4, & \text{if } x \le 2 \\ 5 + 4x, & \text{if } x > 2 \end{cases}$ Over the interval $[0, 4]$, there is no value of x such that $h(x) = 10$, although $h(0) < 10$ and $h(4) > 10$. Explain why this does not contradict the IVT.

21) A particle moving along a line for time t has a position function $s(t)$, which is continuous. Assume $s(2) = 5$ and $s(5) = 2$. Another particle moves such that its position is given by $h(t) = s(t) - t$. Explain why there must be a value c for $2 < c < 5$ such that $h(c) = 0$.

Answer

Since both s and $y = t$ are continuous everywhere, then $h(t) = s(t) - t$ is continuous everywhere and, in particular, it is continuous over the closed interval $[2, 5]$. Also, $h(2) = 3 > 0$ and $h(5) = -3 < 0$. Therefore, by the IVT, there is a value $x = c$ such that $h(c) = 0$.

22) [T] Use the statement "The cosine of t is equal to t cubed."

a. Write a mathematical equation of the statement.

b. Prove that the equation in part a. has at least one real solution.

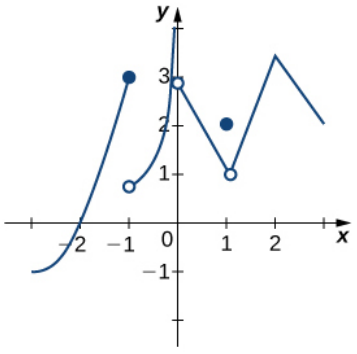
c. Use a calculator to find an interval of length 0.01 that contains a solution.

23) Apply the IVT to determine whether $2^x = x^3$ has a solution in one of the intervals $[1.25, 1.375]$ or $[1.375, 1.5]$. Briefly explain your response for each interval.

Answer

The function $f(x) = 2^x - x^3$ is continuous over the interval $[1.25, 1.375]$ and has opposite signs at the endpoints.

24) Consider the graph of the function $y = f(x)$ shown in the following graph.



a. Find all values for which the function is discontinuous.

b. For each value in part a., state why the formal definition of continuity does not apply.

c. Classify each discontinuity as either jump, removable, or infinite.

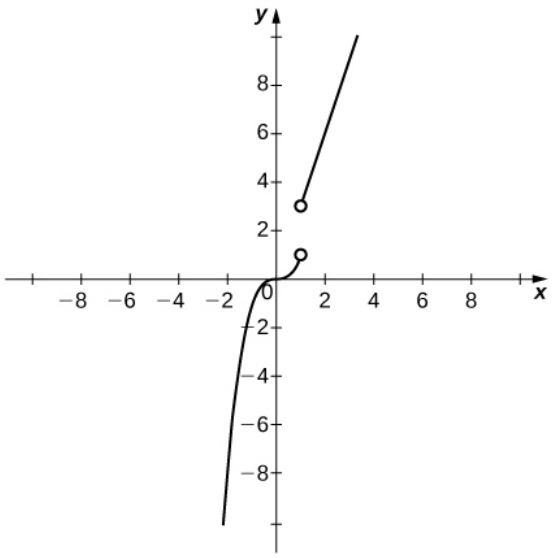
25) Let $f(x) = \begin{cases} 3x, & \text{if } x > 1 \\ x^3, & \text{if } x < 1 \end{cases}$.

a. Sketch the graph of f.

b. Is it possible to find a value k such that $f(1) = k$, which makes $f(x)$ continuous for all real numbers? Briefly explain.

Answer

a.



b. It is not possible to redefine $f(1)$ since the discontinuity is a jump discontinuity.

26) Let $f(x) = \dfrac{x^4 - 1}{x^2 - 1}$ for $x \neq -1, 1$.

a. Sketch the graph of f.

b. Is it possible to find values k_1 and k_2 such that $f(-1) = k$ and $f(1) = k_2$, and that makes $f(x)$ continuous for all real numbers? Briefly explain.

27) Sketch the graph of the function $y = f(x)$ with properties i. through vii.

i. The domain of f is $(-\infty, +\infty)$.

ii. f has an infinite discontinuity at $x = -6$.

iii. $f(-6) = 3$

iv. $\lim\limits_{x \to -3^-} f(x) = \lim\limits_{x \to -3^+} f(x) = 2$

v. $f(-3) = 3$

vi. f is left continuous but not right continuous at $x = 3$.

vii. $\lim\limits_{x \to -\infty} f(x) = -\infty$ and $\lim\limits_{x \to +\infty} f(x) = +\infty$

Answer

Answers may vary; see the following example:

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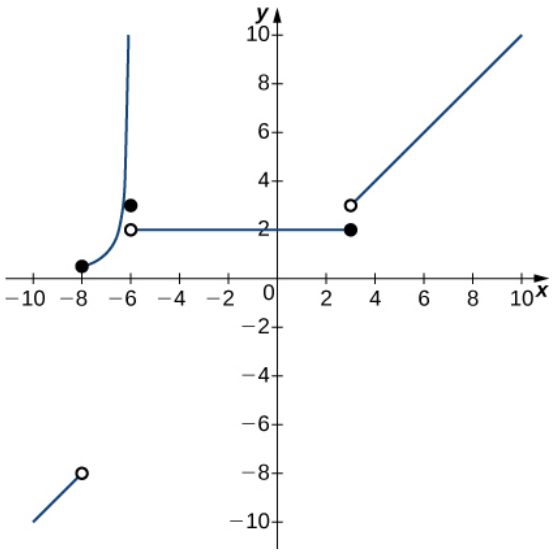


28) Sketch the graph of the function $y = f(x)$ with properties i. through iv.

i. The domain of f is $[0, 5]$.

ii. $\lim_{x \to 1^+} f(x)$ and $\lim_{x \to 1^-} f(x)$ exist and are equal.

iii. $f(x)$ is left continuous but not continuous at $x = 2$, and right continuous but not continuous at $x = 3$.

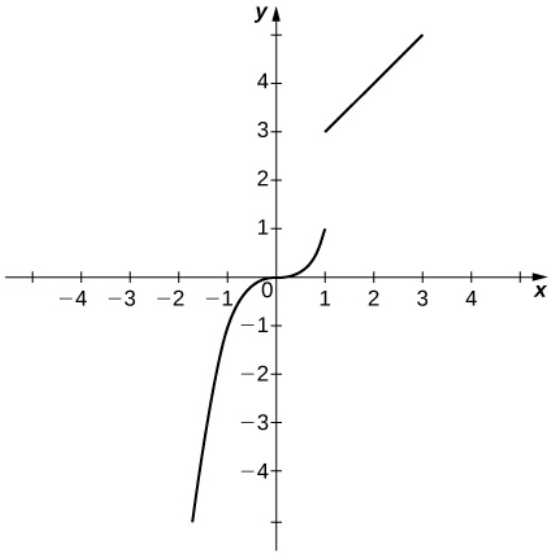
iv. $f(x)$ has a removable discontinuity at $x = 1$, a jump discontinuity at $x = 2$, and the following limits hold: $\lim_{x \to 3^-} f(x) = -\infty$ and $\lim_{x \to 3^+} f(x) = 2$.

In exercises 29 - 30, suppose $y = f(x)$ is defined for all x. For each description, sketch a graph with the indicated property.

29) Discontinuous at $x = 1$ with $\lim_{x \to -1} f(x) = -1$ and $\lim_{x \to 2} f(x) = 4$

Answer

Answers may vary; see the following example:



30) Discontinuous at $x = 2$ but continuous elsewhere with $\lim_{x \to 0} f(x) = \frac{1}{2}$

Determine whether each of the given statements is true. Justify your response with an explanation or counterexample.

31) $f(t) = \frac{2}{e^t - e^{-t}}$ is continuous everywhere.

Answer

False. It is continuous over $(-\infty, 0) \cup (0, \infty)$.

32) If the left- and right-hand limits of $f(x)$ as $x \to a$ exist and are equal, then f cannot be discontinuous at $x = a$.

33) If a function is not continuous at a point, then it is not defined at that point.

Answer

False. Consider $f(x) = \begin{cases} x, & \text{if } x \neq 0 \\ 4, & \text{if } x = 0 \end{cases}$.

34) According to the IVT, $\cos x - \sin x - x = 2$ has a solution over the interval $[-1, 1]$.

35) If $f(x)$ is continuous such that $f(a)$ and $f(b)$ have opposite signs, then $f(x) = 0$ has exactly one solution in $[a, b]$.

Answer

False. Consider $f(x) = \cos(x)$ on $[-\pi, 2\pi]$.

36) The function $f(x) = \dfrac{x^2 - 4x + 3}{x^2 - 1}$ is continuous over the interval $[0, 3]$.

37) If $f(x)$ is continuous everywhere and $f(a), f(b) > 0$, then there is no root of $f(x)$ in the interval $[a, b]$.

Answer

False. The IVT does not work in reverse! Consider $(x - 1)^2$ over the interval $[-2, 2]$.

[T] The following problems consider the scalar form of Coulomb's law, which describes the electrostatic force between two point charges, such as electrons. It is given by the equation $F(r) = k_e \dfrac{|q_1 q_2|}{r^2}$, where k_e is Coulomb's constant, q_i are the magnitudes of the charges of the two particles, and r is the distance between the two particles.

38) To simplify the calculation of a model with many interacting particles, after some threshold value $r = R$, we approximate F as zero.

a. Explain the physical reasoning behind this assumption.

b. What is the force equation?

c. Evaluate the force F using both Coulomb's law and our approximation, assuming two protons with a charge magnitude of 1.6022×10^{-19} **coulombs (C)**, and the Coulomb constant $k_e = 8.988 \times 10^9 Nm^2/C^2$ are 1 m apart. Also, assume $R < 1$ m. How much inaccuracy does our approximation generate? Is our approximation reasonable?

d. Is there any finite value of R for which this system remains continuous at R?

39) Instead of making the force 0 at R, we let the force be $10 - 20$ for $r \geq R$. Assume two protons, which have a magnitude of charge $1.6022 \times 10^{-19}\ C$, and the Coulomb constant $k_e = 8.988 \times 10^9\ Nm^2/C^2$. Is there a value R that can make this system continuous? If so, find it.

Answer

$R = 0.0001519$ m

Recall the discussion on spacecraft from the chapter opener. The following problems consider a rocket launch from Earth's surface. The force of gravity on the rocket is given by $F(d) = -mk/d^2$, where m is the mass of the rocket, d is the distance of the rocket from the center of Earth, and k is a constant.

40) [T] Determine the value and units of k given that the mass of the rocket on Earth is 3 million kg. (Hint: The distance from the center of Earth to its surface is 6378 km.)

41) [T] After a certain distance D has passed, the gravitational effect of Earth becomes quite negligible, so we can approximate the force function by $F(d) = \begin{cases} -\dfrac{mk}{d^2}, & \text{if } d < D \\ 10{,}000, & \text{if } d \geq D \end{cases}$. Find the necessary condition D such that the force function remains continuous.

Answer

$D = 63.78$ km

42) As the rocket travels away from Earth's surface, there is a distance D where the rocket sheds some of its mass, since it no longer needs the excess fuel storage. We can write this function as $F(d) = \begin{cases} -\dfrac{m_1 k}{d^2}, & \text{if } d < D \\ -\dfrac{m_2 k}{d^2}, & \text{if } d \geq D \end{cases}$. Is there a value of D such that this function is continuous, assuming $m_1 \neq m_2$?

In Exercises 43 - 44, prove each function is continuous everywhere.

43) $f(\theta) = \sin\theta$

Answer

For all values of a, $f(a)$ is defined, $\lim_{\theta \to a} f(\theta)$ exists, and $\lim_{\theta \to a} f(\theta) = f(a)$. Therefore, $f(\theta)$ is continuous everywhere.

44) $g(x) = |x|$

45) Where is $f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational} \\ 1, & \text{if } x \text{ is rational} \end{cases}$ continuous?

Answer

Nowhere

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2.5: The Precise Definition of a Limit

Learning Objectives

- Describe the epsilon-delta definition of a limit.
- Apply the epsilon-delta definition to find the limit of a function.
- Describe the epsilon-delta definitions of one-sided limits and infinite limits.
- Use the epsilon-delta definition to prove the limit laws.

By now you have progressed from the very informal definition of a limit in the introduction of this chapter to the intuitive understanding of a limit. At this point, you should have a very strong intuitive sense of what the limit of a function means and how you can find it. In this section, we convert this intuitive idea of a limit into a formal definition using precise mathematical language. The formal definition of a limit is quite possibly one of the most challenging definitions you will encounter early in your study of calculus; however, it is well worth any effort you make to reconcile it with your intuitive notion of a limit. Understanding this definition is the key that opens the door to a better understanding of calculus.

Quantifying Closeness

Before stating the formal definition of a limit, we must introduce a few preliminary ideas. Recall that the distance between two points a and b on a number line is given by $|a-b|$.

- The statement $|f(x)-L|<\varepsilon$ may be interpreted as: The distance between $f(x)$ and L is less than ε.
- The statement $0<|x-a|<\delta$ may be interpreted as: $x\neq a$ and the distance between x and a is less than δ.

It is also important to look at the following equivalences for absolute value:

- The statement $|f(x)-L|<\varepsilon$ is equivalent to the statement $L-\varepsilon<f(x)<L+\varepsilon$.
- The statement $0<|x-a|<\delta$ is equivalent to the statement $a-\delta<x<a+\delta$ and $x\neq a$.

With these clarifications, we can state the formal **epsilon-delta definition of the limit**.

Definition: Finite Limits (Formal)

Let $f(x)$ be defined for all $x\neq a$ over an open interval containing a. Let L be a real number. Then

$$\lim_{x\to a} f(x)=L$$

if, for every $\varepsilon>0$, there exists a $\delta>0$, such that if $0<|x-a|<\delta$, then $|f(x)-L|<\varepsilon$.

This definition may seem rather complex from a mathematical point of view, but it becomes easier to understand if we break it down phrase by phrase. The statement itself involves something called a **universal quantifier** (for every $\varepsilon>0$), an **existential quantifier** (there exists a $\delta>0$), and, last, a **conditional statement** (if $0<|x-a|<\delta$, then $|f(x)-L|<\varepsilon$). Let's take a look at Table 2.5.1, which breaks down the definition and translates each part.

Table 2.5.1

Definition	Translation
1. For every $\varepsilon>0$,	1. For every positive distance ε from L,
2. there exists a $\delta>0$,	2. There is a positive distance δ from a,
3. such that	3. such that
4. if $0<\|x-a\|<\delta$, then $\|f(x)-L\|<\varepsilon$.	4. if x is closer than δ to a and $x\neq a$, then $f(x)$ is closer than ε to L.

We can get a better handle on this definition by looking at the definition geometrically. Figure 2.5.1 shows possible values of δ for various choices of $\varepsilon>0$ for a given function $f(x)$, a number a, and a limit L at a. Notice that as we choose smaller values of ε (the distance between the function and the limit), we can always find a δ small enough so that if we have chosen an x value within δ of a, then the value of $f(x)$ is within ε of the limit L.

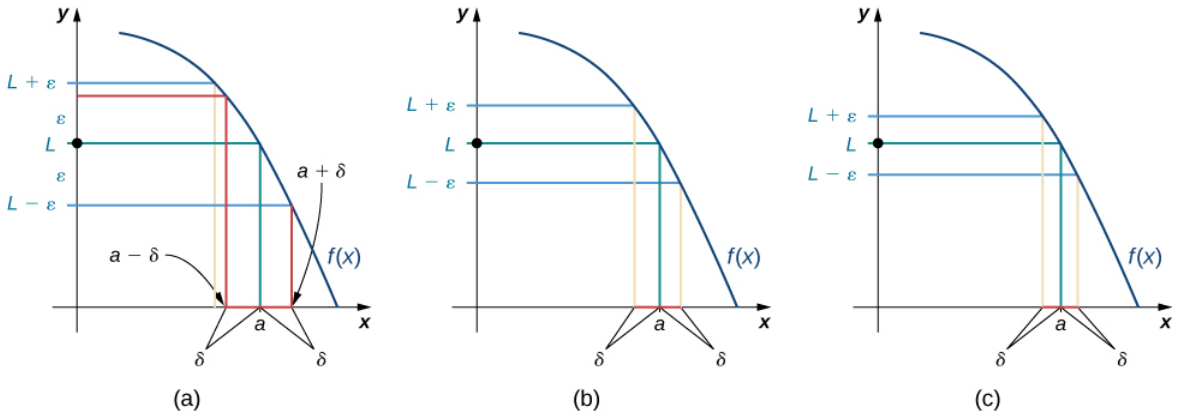


Figure 2.5.1: These graphs show possible values of δ, given successively smaller choices of ε.

Note

Visit the following applet to experiment with finding values of δ for selected values of ε:

- The epsilon-delta definition of limit

Example 2.5.1 shows how you can use this definition to prove a statement about the limit of a specific function at a specified value.

Example 2.5.1: Proving a Statement about the Limit of a Specific Function

Prove that $\lim\limits_{x\to 1}(2x+1)=3$.

Solution

Let $\varepsilon > 0$.

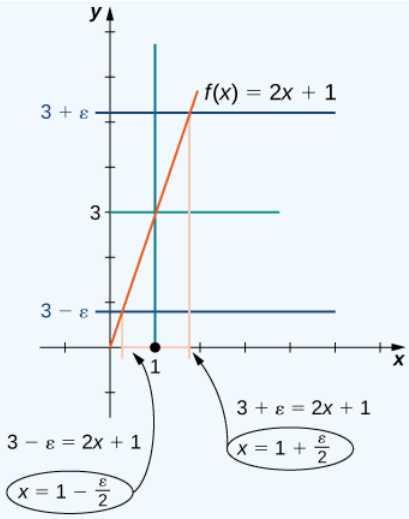
The first part of the definition begins "For every $\varepsilon > 0$."This means we must prove that whatever follows is true no matter what positive value of ε is chosen. By stating "Let $\varepsilon > 0$," we signal our intent to do so.

Choose $\delta = \frac{\varepsilon}{2}$.

The definition continues with "there exists a $\delta > 0$. " The phrase "there exists" in a mathematical statement is always a signal for a scavenger hunt. In other words, we must go and find δ. So, where exactly did $\delta = \varepsilon/2$ come from? There are two basic approaches to tracking down δ. One method is purely algebraic and the other is geometric.

We begin by tackling the problem from an algebraic point of view. Since ultimately we want $|(2x+1)-3| < \varepsilon$, we begin by manipulating this expression: $|(2x+1)-3| < \varepsilon$ is equivalent to $|2x-2| < \varepsilon$, which in turn is equivalent to $|2||x-1| < \varepsilon$. Last, this is equivalent to $|x-1| < \varepsilon/2$. Thus, it would seem that $\delta = \varepsilon/2$ is appropriate.

We may also find δ through geometric methods. Figure 2.5.2 demonstrates how this is done.



δ is the length of the smaller of the two distances marked in brown.

$\delta = \min\{1 + \frac{\varepsilon}{2} - 1, 1 - (1 - \frac{\varepsilon}{2})\}$

$= \min\{\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\}$

$= \frac{\varepsilon}{2}$

Figure 2.5.2: This graph shows how we find δ geometrically.

Assume $0 < |x-1| < \delta$. When δ has been chosen, our goal is to show that if $0 < |x-1| < \delta$, then $|(2x+1)-3| < \varepsilon$. To prove any statement of the form "If this, then that," we begin by assuming "this" and trying to get "that."

Thus,

$|(2x+1)-3| = |2x-2|$ **property of absolute value**

$= |2(x-1)|$

$= |2||x-1|$ $|2| = 2$

$= 2|x-1|$

$< 2 \cdot \delta$ **here's where we use the assumption that** $0 < |x-1| < \delta$

$= 2 \cdot \frac{\varepsilon}{2} = \varepsilon$ **here's where we use our choice of** $\delta = \varepsilon/2$

Analysis

In this part of the proof, we started with $|(2x+1)-3|$ and used our assumption $0 < |x-1| < \delta$ in a key part of the chain of inequalities to get $|(2x+1)-3|$ to be less than ε. We could just as easily have manipulated the assumed inequality $0 < |x-1| < \delta$ to arrive at $|(2x+1)-3| < \varepsilon$ as follows:

$0 < |x-1| < \delta \Rightarrow |x-1| < \delta$

$\Rightarrow -\delta < x-1 < \delta$

$\Rightarrow -\frac{\varepsilon}{2} < x-1 < \frac{\varepsilon}{2}$

$\Rightarrow -\varepsilon < 2x-2 < \varepsilon$

$\Rightarrow |2x-2| < \varepsilon$

$\Rightarrow |(2x+1)-3| < \varepsilon.$

Therefore, $\lim_{x \to 1}(2x+1) = 3$. (Having completed the proof, we state what we have accomplished.)

After removing all the remarks, here is a final version of the proof:

Let $\varepsilon > 0$.

Choose $\delta = \varepsilon/2$.

Assume $0 < |x-1| < \delta$.

Thus,

$$
\begin{aligned}
|(2x+1)-3| &= |2x-2| \\
&= |2(x-1)| \\
&= |2||x-1| \\
&= 2|x-1| \\
&< 2 \cdot \delta \\
&= 2 \cdot \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}
$$

Therefore, $\lim\limits_{x \to 1} (2x+1) = 3$.

The following Problem-Solving Strategy summarizes the type of proof we worked out in Example 2.5.1.

Problem-Solving Strategy: Proving That $\lim\limits_{x \to a} f(x) = L$ for a Specific Function $f(x)$

1. Let's begin the proof with the following statement: Let $\varepsilon > 0$.
2. Next, we need to obtain a value for δ. After we have obtained this value, we make the following statement, filling in the blank with our choice of δ: Choose $\delta =$_____.
3. The next statement in the proof should be (at this point, we fill in our given value for a): Assume $0 < |x-a| < \delta$.
4. Next, based on this assumption, we need to show that $|f(x) - L| < \varepsilon$, where $f(x)$ and L are our function $f(x)$ and our limit L. At some point, we need to use $0 < |x-a| < \delta$.
5. We conclude our proof with the statement: Therefore, $\lim\limits_{x \to a} f(x) = L$.

Example 2.5.2: Proving a Statement about a Limit

Complete the proof that $\lim\limits_{x \to -1} (4x+1) = -3$ by filling in the blanks.

Let _____.

Choose $\delta =$_____.

Assume $0 < |x-$_____$| < \delta$.

Thus, |_____−_____|=_____ε.

Solution

We begin by filling in the blanks where the choices are specified by the definition. Thus, we have

Let $\varepsilon > 0$.

Choose δ=_____.

Assume $0 < |x-(-1)| < \delta$. (or equivalently, $0 < |x+1| < \delta$.)

Thus, $|(4x+1)-(-3)| = |4x+4| = |4||x+1| < 4\delta$_____$\varepsilon$.

Focusing on the final line of the proof, we see that we should choose $\delta = \frac{\varepsilon}{4}$.

We now complete the final write-up of the proof:

Let $\varepsilon > 0$.

Choose $\delta = \frac{\varepsilon}{4}$.

Assume $0 < |x-(-1)| < \delta$ (or equivalently, $0 < |x+1| < \delta$.)

Thus, $|(4x+1)-(-3)| = |4x+4| = |4||x+1| < 4\delta = 4(\varepsilon/4) = \varepsilon$.

Exercise 2.5.1

Complete the proof that $\lim_{x\to 2}(3x-2)=4$ by filling in the blanks.

Let _____.

Choose δ =_____.

Assume $0<|x-$_____$|<$_____.

Thus,

|_____−_____|=_____ε.

Therefore, $\lim_{x\to 2}(3x-2)=4$.

Hint

Follow the outline in the Problem-Solving Strategy that we worked out in full in Example 2.5.2.

Answer

Let $\varepsilon>0$; choose $\delta=\frac{\varepsilon}{3}$; assume $0<|x-2|<\delta$.

Thus, $|(3x-2)-4|=|3x-6|=|3|\cdot|x-2|<3\cdot\delta=3\cdot(\varepsilon/3)=\varepsilon$.

Therefore, $\lim_{x\to 2}(3x-2)=4$.

In Examples 2.5.1 and 2.5.2, the proofs were fairly straightforward, since the functions with which we were working were linear. In Example 2.5.3, we see how to modify the proof to accommodate a nonlinear function.

Example 2.5.3: Proving a Statement about the Limit of a Specific Function (Geometric Approach)

Prove that $\lim_{x\to 2}x^2=4$.

Solution

1. Let $\varepsilon>0$. The first part of the definition begins "For every $\varepsilon>0$," so we must prove that whatever follows is true no matter what positive value of ε is chosen. By stating "Let $\varepsilon>0$," we signal our intent to do so.

2. Without loss of generality, assume $\varepsilon\le 4$. Two questions present themselves: Why do we want $\varepsilon\le 4$ and why is it okay to make this assumption? In answer to the first question: Later on, in the process of solving for δ, we will discover that δ involves the quantity $\sqrt{4-\varepsilon}$. Consequently, we need $\varepsilon\le 4$. In answer to the second question: If we can find $\delta>0$ that "works" for $\varepsilon\le 4$, then it will "work" for any $\varepsilon>4$ as well. Keep in mind that, although it is always okay to put an upper bound on ε, it is never okay to put a lower bound (other than zero) on ε.

3. Choose $\delta=\min\{2-\sqrt{4-\varepsilon},\sqrt{4+\varepsilon}-2\}$. Figure 2.5.3 shows how we made this choice of δ.

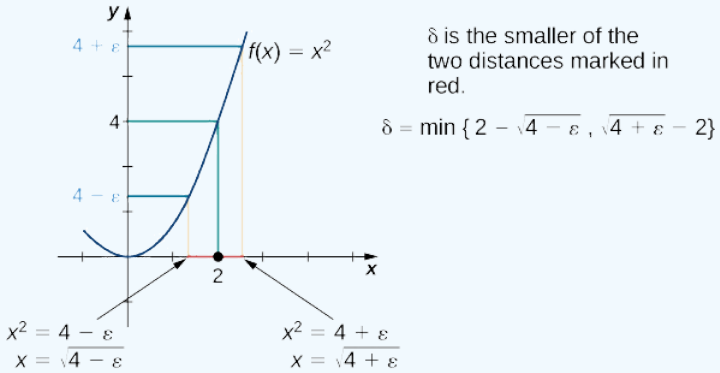


Figure 2.5.3: This graph shows how we find δ geometrically for a given ε for the proof in Example 2.5.3.

4. We must show: If $0<|x-2|<\delta$, then $|x^2-4|<\varepsilon$, so we must begin by assuming

$$0 < |x-2| < \delta.$$

We don't really need $0 < |x-2|$ (in other words, $x \neq 2$) for this proof. Since $0 < |x-2| < \delta \Rightarrow |x-2| < \delta$, it is okay to drop $0 < |x-2|$.

$$|x-2| < \delta.$$

Hence,

$$-\delta < x-2 < \delta.$$

Recall that $\delta = \min\{2-\sqrt{4-\varepsilon}, \sqrt{4+\varepsilon}-2\}$. Thus, $\delta \geq 2-\sqrt{4-\varepsilon}$ and consequently $-(2-\sqrt{4-\varepsilon}) \leq -\delta$. We also use $\delta \leq \sqrt{4+\varepsilon}-2$ here. We might ask at this point: Why did we substitute $2-\sqrt{4-\varepsilon}$ for δ on the left-hand side of the inequality and $\sqrt{4+\varepsilon}-2$ on the right-hand side of the inequality? If we look at Figure 2.5.3, we see that $2-\sqrt{4-\varepsilon}$ corresponds to the distance on the left of 2 on the x-axis and $\sqrt{4+\varepsilon}-2$ corresponds to the distance on the right. Thus,

$$-(2-\sqrt{4-\varepsilon}) \leq -\delta < x-2 < \delta \leq \sqrt{4+\varepsilon}-2.$$

We simplify the expression on the left:

$$-2+\sqrt{4-\varepsilon} < x-2 < \sqrt{4+\varepsilon}-2.$$

Then, we add 2 to all parts of the inequality:

$$\sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon}.$$

We square all parts of the inequality. It is okay to do so, since all parts of the inequality are positive:

$$4-\varepsilon < x^2 < 4+\varepsilon.$$

We subtract 4 from all parts of the inequality:

$$-\varepsilon < x^2-4 < \varepsilon.$$

Last,

$$|x^2-4| < \varepsilon.$$

5. Therefore,

$$\lim_{x\to 2} x^2 = 4.$$

Exercise 2.5.2

Find δ corresponding to $\varepsilon > 0$ for a proof that $\lim_{x\to 9} \sqrt{x} = 3$.

Hint

Draw a graph similar to the one in Example 2.5.3.

Answer

Choose $\delta = \min\{9-(3-\varepsilon)^2,\ (3+\varepsilon)^2-9\}$.

The geometric approach to proving that the limit of a function takes on a specific value works quite well for some functions. Also, the insight into the formal definition of the limit that this method provides is invaluable. However, we may also approach limit proofs from a purely algebraic point of view. In many cases, an algebraic approach may not only provide us with additional insight into the definition, it may prove to be simpler as well. Furthermore, an algebraic approach is the primary tool used in proofs of statements about limits. For Example 2.5.4, we take on a purely algebraic approach.

Example 2.5.4:Proving a Statement about the Limit of a Specific Function (Algebraic Approach)

Prove that $\lim_{x\to -1} (x^2-2x+3) = 6$.

Solution

Let's use our outline from the Problem-Solving Strategy:

1. Let $\varepsilon > 0$.

2. Choose $\delta = \min\{1, \varepsilon/5\}$. This choice of δ may appear odd at first glance, but it was obtained by taking a look at our ultimate desired inequality: $|\,(x^2 - 2x + 3) - 6\,| < \varepsilon$. This inequality is equivalent to $|x+1| \cdot |x-3| < \varepsilon$. At this point, the temptation simply to choose $\delta = \frac{\varepsilon}{x-3}$ is very strong. Unfortunately, our choice of δ must depend on ε only and no other variable. If we can replace $|x-3|$ by a numerical value, our problem can be resolved. This is the place where assuming $\delta \leq 1$ comes into play. The choice of $\delta \leq 1$ here is arbitrary. We could have just as easily used any other positive number. In some proofs, greater care in this choice may be necessary. Now, since $\delta \leq 1$ and $|x+1| < \delta \leq 1$, we are able to show that $|x-3| < 5$. Consequently, $|x+1| \cdot |x-3| < |x+1| \cdot 5$. At this point we realize that we also need $\delta \leq \varepsilon/5$. Thus, we choose $\delta = \min\{1, \varepsilon/5\}$.

3. Assume $0 < |x+1| < \delta$. Thus,

$$|x+1| < 1 \text{ and } |x+1| < \frac{\varepsilon}{5}.$$

Since $|x+1| < 1$, we may conclude that $-1 < x+1 < 1$. Thus, by subtracting 4 from all parts of the inequality, we obtain $-5 < x-3 < -1$. Consequently, $|x-3| < 5$. This gives us

$$|(x^2 - 2x + 3) - 6| = |x+1| \cdot |x-3| < \frac{\varepsilon}{5} \cdot 5 = \varepsilon.$$

Therefore,

$$\lim_{x \to -1} (x^2 - 2x + 3) = 6.$$

Exercise 2.5.3

Complete the proof that $\lim_{x \to 1} x^2 = 1$.

Let $\varepsilon > 0$; choose $\delta = \min\{1, \varepsilon/3\}$; assume $0 < |x-1| < \delta$.

Since $|x-1| < 1$, we may conclude that $-1 < x-1 < 1$. Thus, $1 < x+1 < 3$. Hence, $|x+1| < 3$.

Hint

Use Example 2.5.4 as a guide.

Answer

$|\,x^2 - 1\,| = |x-1| \cdot |x+1| < \varepsilon/3 \cdot 3 = \varepsilon$

You will find that, in general, the more complex a function, the more likely it is that the algebraic approach is the easiest to apply. The algebraic approach is also more useful in proving statements about limits.

Proving Limit Laws

We now demonstrate how to use the epsilon-delta definition of a limit to construct a rigorous proof of one of the limit laws. The **triangle inequality** is used at a key point of the proof, so we first review this key property of absolute value.

Definition: The Triangle Inequality

The **triangle inequality** states that if a and b are any real numbers, then $|a+b| \leq |a| + |b|$.

Proof

We prove the following limit law: If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then $\lim_{x \to a} (f(x) + g(x)) = L + M$.

Let $\varepsilon > 0$.

Choose $\delta_1 > 0$ so that if $0 < |x-a| < \delta_1$, then $|f(x) - L| < \varepsilon/2$.

Choose $\delta_2 > 0$ so that if $0 < |x-a| < \delta_2$, then $|g(x) - M| < \varepsilon/2$.

Choose $\delta = \min\{\delta_1, \delta_2\}$.

Assume $0 < |x-a| < \delta$.

Thus,

$$0 < |x-a| < \delta_1 \text{ and } 0 < |x-a| < \delta_2.$$

Hence,

$$\begin{aligned} |(f(x)+g(x)) - (L+M)| &= |(f(x)-L) + (g(x)-M)| \\ &\leq |f(x)-L| + |g(x)-M| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

□

We now explore what it means for a limit not to exist. The limit $\lim\limits_{x\to a} f(x)$ does not exist if there is no real number L for which $\lim\limits_{x\to a} f(x) = L$. Thus, for all real numbers L, $\lim\limits_{x\to a} f(x) \neq L$. To understand what this means, we look at each part of the definition of $\lim\limits_{x\to a} f(x) = L$ together with its opposite. A translation of the definition is given in Table 2.5.2.

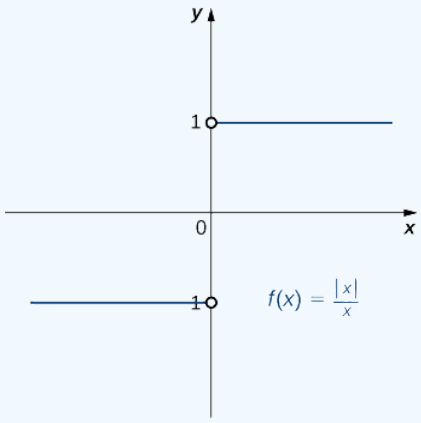
Table 2.5.2. Translation of the Definition of $\lim_{x\to a} f(x) = L$ and its Opposite

Definition	Opposite
1. For every $\varepsilon > 0$,	1. There exists $\varepsilon > 0$ so that
2. there exists a $\delta > 0$, so that	2. for every $\delta > 0$,
3. if $0 < \|x-a\| < \delta$, then $\|f(x) - L\| < \varepsilon$.	3. There is an x satisfying $0 < \|x-a\| < \delta$ so that $\|f(x) - L\| \geq \varepsilon$.

Finally, we may state what it means for a limit not to exist. The limit $\lim\limits_{x\to a} f(x)$ does not exist if for every real number L, there exists a real number $\varepsilon > 0$ so that for all $\delta > 0$, there is an x satisfying $0 < |x-a| < \delta$, so that $|f(x) - L| \geq \varepsilon$. Let's apply this in Example 2.5.5 to show that a limit does not exist.

Example 2.5.5: Showing That a Limit Does Not Exist

Show that $\lim\limits_{x\to 0} \dfrac{|x|}{x}$ does not exist. The graph of $f(x) = |x|/x$ is shown here:



Solution

Suppose that L is a candidate for a limit. Choose $\varepsilon = 1/2$.

Let $\delta > 0$. Either $L \geq 0$ or $L < 0$. If $L \geq 0$, then let $x = -\delta/2$.

Thus,

$$|x-0| = |-\tfrac{\delta}{2}-0| = \tfrac{\delta}{2} < \delta$$

and

$$\left|\frac{|-\frac{\delta}{2}|}{-\frac{\delta}{2}} - L\right| = |-1-L| = L+1 \geq 1 > \tfrac{1}{2} = \varepsilon.$$

On the other hand, if $L < 0$, then let $x = \delta/2$. Thus,

$$|x-0| = |\tfrac{\delta}{2}-0| = \tfrac{\delta}{2} < \delta$$

and

$$\left|\frac{|\frac{\delta}{2}|}{\frac{\delta}{2}} - L\right| = |1-L| = |L|+1 \geq 1 > \tfrac{1}{2} = \varepsilon.$$

Thus, for any value of L, $\lim\limits_{x\to 0}\dfrac{|x|}{x} \neq L$.

One-Sided Limits

Just as we first gained an intuitive understanding of limits and then moved on to a more rigorous definition of a limit, we now revisit one-sided limits. To do this, we modify the epsilon-delta definition of a limit to give formal epsilon-delta definitions for limits from the right and left at a point. These definitions only require slight modifications from the definition of the limit. In the definition of the limit from the right, the inequality $0 < x - a < \delta$ replaces $0 < |x-a| < \delta$, which ensures that we only consider values of x that are greater than (to the right of) a. Similarly, in the definition of the limit from the left, the inequality $-\delta < x - a < 0$ replaces $0 < |x-a| < \delta$, which ensures that we only consider values of x that are less than (to the left of) a.

Definition: One-Sided Limits (Formal)

Limit from the Right: Let $f(x)$ be defined over an open interval of the form (a,b) where $a < b$. Then

$$\lim_{x\to a^+} f(x) = L$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$, such that if $0 < x - a < \delta$, then $|f(x) - L| < \varepsilon$.

Limit from the Left: Let $f(x)$ be defined over an open interval of the form (b,c) where $b < c$. Then,

$$\lim_{x\to c^-} f(x) = L$$

if for every $\varepsilon > 0$,there exists a $\delta > 0$ such that if $-\delta < x - c < 0$, then $|f(x) - L| < \varepsilon$.

Example 2.5.6: Proving a Statement about a Limit From the Right

Prove that

$$\lim_{x\to 4^+} \sqrt{x-4} = 0.$$

Solution

Let $\varepsilon > 0$.

Choose $\delta = \varepsilon^2$. Since we ultimately want $|\sqrt{x-4} - 0| < \varepsilon$, we manipulate this inequality to get $\sqrt{x-4} < \varepsilon$ or, equivalently, $0 < x - 4 < \varepsilon^2$, making $\delta = \varepsilon^2$ a clear choice. We may also determine δ geometrically, as shown in Figure 2.5.4.

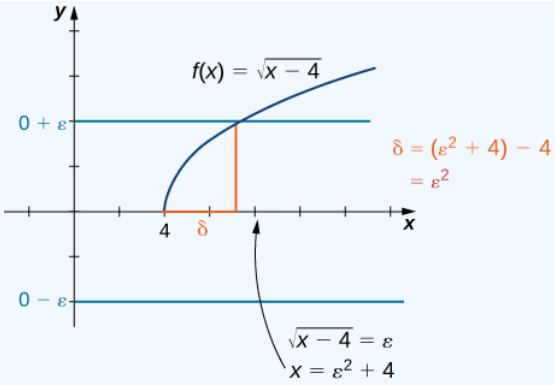


Figure 2.5.4: This graph shows how we find δ for the proof in Example 2.5.6.

Assume $0 < x - 4 < \delta$. Thus, $0 < x - 4 < \varepsilon^2$. Hence, $0 < \sqrt{x-4} < \varepsilon$. Finally, $\left|\sqrt{x-4} - 0\right| < \varepsilon$. Therefore, $\lim\limits_{x \to 4^+} \sqrt{x-4} = 0$.

Exercise 2.5.4

Find δ corresponding to ε for a proof that $\lim\limits_{x \to 1^-} \sqrt{1-x} = 0$.

Hint

Sketch the graph and use Example 2.5.6 as a solving guide.

Answer

$\delta = \varepsilon^2$

Infinite Limits

We conclude the process of converting our intuitive ideas of various types of limits to rigorous formal definitions by pursuing a formal definition of infinite limits. To have $\lim\limits_{x \to a} f(x) = +\infty$, we want the values of the function $f(x)$ to get larger and larger as x approaches a. Instead of the requirement that $|f(x) - L| < \varepsilon$ for arbitrarily small ε when $0 < |x - a| < \delta$ for small enough δ, we want $f(x) > M$ for arbitrarily large positive M when $0 < |x - a| < \delta$ for small enough δ. Figure 2.5.5 illustrates this idea by showing the value of δ for successively larger values of M.

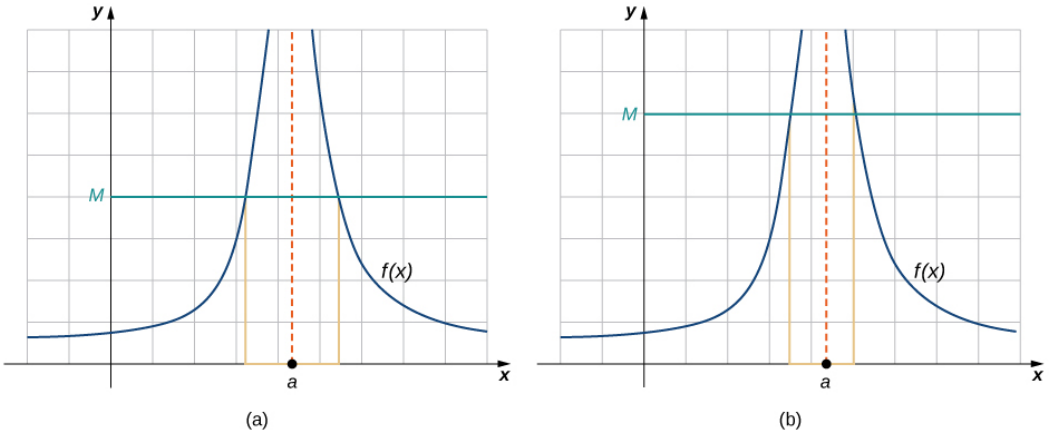


Figure 2.5.5: These graphs plot values of δ for M to show that $\lim\limits_{x \to a} f(x) = +\infty$.

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Definition: Infinite Limits (Formal)

Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a. Then, we have an infinite limit

$$\lim_{x \to a} f(x) = +\infty$$

if for every $M > 0$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $f(x) > M$.

Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a. Then, we have a negative infinite limit

$$\lim_{x \to a} f(x) = -\infty$$

if for every $M > 0$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $f(x) < -M$.

Example 2.5.7: Proving a Statement about an Infinite Limit

Prove that $\lim_{x \to 3} \dfrac{1}{(x-3)^2} = \infty$.

Solution

We use a very similar approach to our previous Problem-Solving Strategy. We first find an appropriate $\delta > 0$. Then we write our proof.

Step 1: First we find an appropriate $\delta > 0$.

1. Let M be any real number such that $M > 0$.

2. Let $f(x) = \dfrac{1}{(x-3)^2} > M$. Then we solve for the expression $x - 3$.

Multiplying both sides of the inequality by the positive quantity $(x-3)^2$ and dividing both sides by the positive quantity M gives us:

$$\frac{1}{M} > (x-3)^2$$

Taking the square root of both sides, we have,

$$\sqrt{\frac{1}{M}} > |x-3|. \qquad \left(\text{Remember that } \sqrt{x^2} = |x|.\right)$$

Rewriting this statement gives us, $0 < |x-3| < \sqrt{\dfrac{1}{M}}$. From this we choose $\delta = \sqrt{\dfrac{1}{M}}$.

Step 2: Now we write a proof.

3. Let $\delta = \sqrt{\dfrac{1}{M}}$ and assume $0 < |x-3| < \delta = \sqrt{\dfrac{1}{M}}$.

Thus,

$$|x-3| < \sqrt{\frac{1}{M}}.$$

Squaring both sides gives us,

$$(x-3)^2 < \frac{1}{M}.$$

Taking the reciprocal of both sides (and remembering that this will reverse the direction of the inequality),

$$\frac{1}{(x-3)^2} > M.$$

Therefore, we have proven that

$$\lim_{x \to 3} \frac{1}{(x-3)^2} = \infty.$$

A very similar proof will be needed for a limit that is equal to $-\infty$.

Note that a one-sided limit approach will often need to be taken with this type of limit. For example, to prove: $\lim\limits_{x\to 0^+} \dfrac{1}{x} = \infty$.

Key Concepts

- The intuitive notion of a limit may be converted into a rigorous mathematical definition known as the epsilon-delta definition of the limit.
- The epsilon-delta definition may be used to prove statements about limits.
- The epsilon-delta definition of a limit may be modified to define one-sided limits.
- A similar definition of an infinite limit can be used to prove statements about infinite limits.

Glossary

epsilon-delta definition of the limit

$\lim\limits_{x\to a} f(x) = L$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$

triangle inequality

If a and b are any real numbers, then $|a + b| \leq |a| + |b|$

formal definition of an infinite limit

$\lim\limits_{x\to a} f(x) = \infty$ if for every $M > 0$, there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $f(x) > M$
$\lim\limits_{x\to a} f(x) = -\infty$ if for every $M > 0$, there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $f(x) < -M$

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.
- Paul Seeburger (Monroe Community College), added Example 2.5.7 and entries for infinite limits under Key Concepts and the Glossary.

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2.5E: EXERCISES FOR SECTION 2.5

In exercises 1 - 4, write the appropriate $\varepsilon - \delta$ definition for each of the given statements.

1) $\lim\limits_{x \to a} f(x) = N$

2) $\lim\limits_{t \to b} g(t) = M$

Answer

For every $\varepsilon > 0$, there exists a $\delta > 0$, so that if $0 < |t - b| < \delta$, then $|g(t) - M| < \varepsilon$

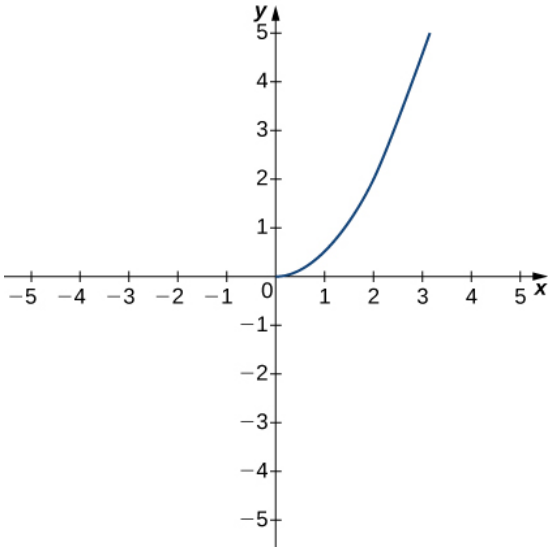
3) $\lim\limits_{x \to c} h(x) = L$

4) $\lim\limits_{x \to a} \varphi(x) = A$

Answer

For every $\varepsilon > 0$, there exists a $\delta > 0$, so that if $0 < |x - a| < \delta$, then $|\varphi(x) - A| < \varepsilon$

The following graph of the function f satisfies $\lim\limits_{x \to 2} f(x) = 2$. In the following exercises, determine a value of $\delta > 0$ that satisfies each statement.



5) If $0 < |x - 2| < \delta$, then $|f(x) - 2| < 1$.

6) If $0 < |x - 2| < \delta$, then $|f(x) - 2| < 0.5$.

Answer

$\delta \leq 0.25$

The following graph of the function f satisfies $\lim\limits_{x \to 3} f(x) = -1$. In the following exercises, determine a value of $\delta > 0$ that satisfies each statement.

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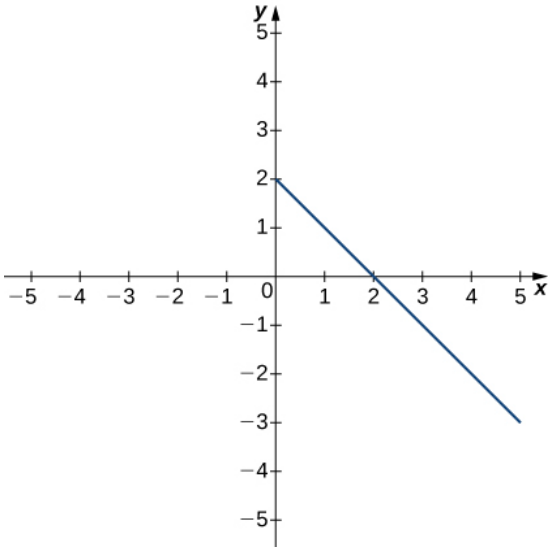


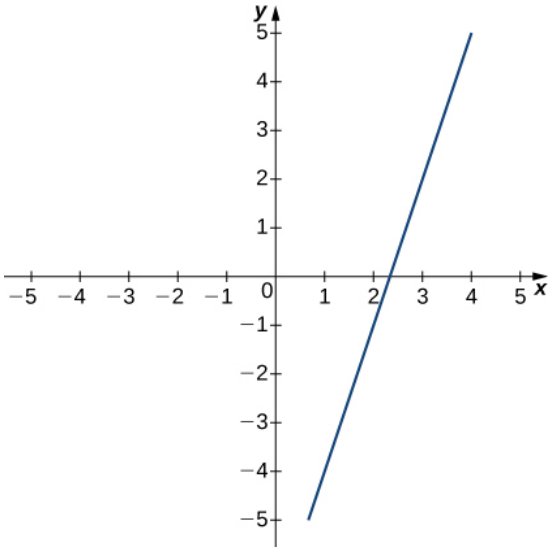
7) If $0 < |x - 3| < \delta$, then $|f(x) + 1| < 1$.

8) If $0 < |x - 3| < \delta$, then $|f(x) + 1| < 2$.

Answer

$\delta \leq 2$

The following graph of the function f satisfies $\lim_{x \to 3} f(x) = 2$. In the following exercises, for each value of ε, find a value of $\delta > 0$ such that the precise definition of limit holds true.



9) $\varepsilon = 1.5$

10) $\varepsilon = 3$

Answer

$\delta \leq 1$

[T] In exercises 11 - 12, use a graphing calculator to find a number δ such that the statements hold true.

11) $\left|\sin(2x) - \frac{1}{2}\right| < 0.1$, whenever $\left|x - \frac{\pi}{12}\right| < \delta$

12) $\left|\sqrt{x - 4} - 2\right| < 0.1$, whenever $|x - 8| < \delta$

Answer

$\delta < 0.3900$

In exercises 13 - 17, use the precise definition of limit to prove the given limits.

13) $\lim\limits_{x\to 2}(5x+8)=18$

14) $\lim\limits_{x\to 3}\dfrac{x^2-9}{x-3}=6$

Answer

Let $\delta=\varepsilon$. If $0<|x-3|<\varepsilon$, then $\left|\dfrac{x^2-9}{x-3}-6\right|=\left|\dfrac{(x+3)(x-3)}{x-3}-6\right|=|x+3-6|=|x-3|<\varepsilon$.

15) $\lim\limits_{x\to 2}\dfrac{2x^2-3x-2}{x-2}=5$

16) $\lim\limits_{x\to 0}x^4=0$

Answer

Let $\delta=\sqrt[4]{\varepsilon}$. If $0<|x|<\sqrt[4]{\varepsilon}$, then $|x^4-0|=x^4<\varepsilon$.

17) $\lim\limits_{x\to 2}(x^2+2x)=8$

In exercises 18 - 20, use the precise definition of limit to prove the given one-sided limits.

18) $\lim\limits_{x\to 5^-}\sqrt{5-x}=0$

Answer

Let $\delta=\varepsilon^2$. If $-\varepsilon^2<x-5<0$, we can multiply through by -1 to get $0<5-x<\varepsilon^2$.
Then $\left|\sqrt{5-x}-0\right|=\sqrt{5-x}<\sqrt{\varepsilon^2}=\varepsilon$.

19) $\lim\limits_{x\to 0^+}f(x)=-2$, where $f(x)=\begin{cases}8x-3, & \text{if } x<0\\ 4x-2, & \text{if } x\geq 0\end{cases}$.

20) $\lim\limits_{x\to 1^-}f(x)=3$, where $f(x)=\begin{cases}5x-2, & \text{if } x<1\\ 7x-1, & \text{if } x\geq 1\end{cases}$.

Answer

Let $\delta=\varepsilon/5$. If $-\varepsilon/5<x-1<0$, we can multiply through by -1 to get $0<1-x<\varepsilon/5$.
Then $|f(x)-3|=|5x-2-3|=|5x-5|=5(1-x)$, since $x<1$ here.
And $5(1-x)<5(\varepsilon/5)=\varepsilon$.

In exercises 21 - 23, use the precise definition of limit to prove the given infinite limits.

21) $\lim\limits_{x\to 0}\dfrac{1}{x^2}=\infty$

22) $\lim\limits_{x\to -1}\dfrac{3}{(x+1)^2}=\infty$

Answer

Let $\delta=\sqrt{\frac{3}{N}}$. If $0<|x+1|<\sqrt{\frac{3}{N}}$, then $f(x)=\frac{3}{(x+1)^2}>N$.

23) $\lim\limits_{x\to 2}-\dfrac{1}{(x-2)^2}=-\infty$

24) An engineer is using a machine to cut a flat square of Aerogel of area $144\,\text{cm}^2$. If there is a maximum error tolerance in the area of $8\,\text{cm}^2$, how accurately must the engineer cut on the side, assuming all sides have the same length? How do these numbers relate to δ, ε, a, and L?

Answer

0.033 cm, $\varepsilon=8$, $\delta=0.33$, $a=12$, $L=144$

25) Use the precise definition of limit to prove that the following limit does not exist: $\lim\limits_{x\to 1}\dfrac{|x-1|}{x-1}$.

26) Using precise definitions of limits, prove that $\lim\limits_{x\to 0}f(x)$ does not exist, given that $f(x)$ is the ceiling function. (Hint: Try any $\delta<1$.)

Answer

Answers may very.

27) Using precise definitions of limits, prove that $\lim_{x\to 0} f(x)$ does not exist: $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$. (Hint: Think about how you can always choose a rational number \(0 <d\), >

28) Using precise definitions of limits, determine $\lim_{x\to 0} f(x)$ for $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$. (Hint: Break into two cases, x rational and x irrational.)

Answer

0

29) Using the function from the previous exercise, use the precise definition of limits to show that $\lim_{x\to a} f(x)$ does not exist for $a \neq 0$

For exercises 30 - 32, suppose that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ both exist. Use the precise definition of limits to prove the following limit laws:

30) $\lim_{x\to a}(f(x) - g(x)) = L - M$

Answer

$f(x) - g(x) = f(x) + (-1)g(x)$

31) $\lim_{x\to a}[cf(x)] = cL$ for any real constant c (Hint: Consider two cases: $c = 0$ and $c \neq 0$.)

32) $\lim_{x\to a}[f(x)g(x)] = LM$. (Hint: $|f(x)g(x) - LM| = |f(x)g(x) - f(x)M + f(x)M - LM| \leq |f(x)||g(x) - M| + |M||f(x) - L|$.)

Answer

Answers may vary.

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2.R: CHAPTER 2 REVIEW EXERCISES

***True or False*. In exercises 1 - 4, justify your answer with a proof or a counterexample.**

1) A function has to be continuous at $x = a$ if the $\lim_{x \to a} f(x)$ exists.

2) You can use the quotient rule to evaluate $\lim_{x \to 0} \dfrac{\sin x}{x}$.

Answer

False, since we cannot have $\lim_{x \to 0} x = 0$ in the denominator.

3) If there is a vertical asymptote at $x = a$ for the function $f(x)$, then f is undefined at the point $x = a$.

4) If $\lim_{x \to a} f(x)$ does not exist, then f is undefined at the point $x = a$.

Answer

False. A jump discontinuity is possible.

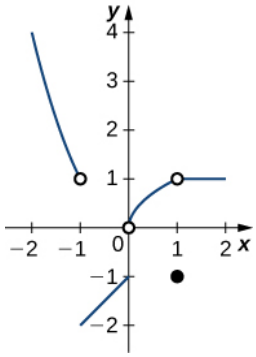
5) Using the graph, find each limit or explain why the limit does not exist.

a. $\lim_{x \to -1} f(x)$

b. $\lim_{x \to 1} f(x)$

c. $\lim_{x \to 0^+} f(x)$

d. $\lim_{x \to 2} f(x)$



In exercises 6 - 15, evaluate the limit algebraically or explain why the limit does not exist.

6) $\lim_{x \to 2} \dfrac{2x^2 - 3x - 2}{x - 2}$

Answer

5

7) $\lim_{x \to 0} 3x^2 - 2x + 4$

8) $\lim_{x \to 3} \dfrac{x^3 - 2x^2 - 1}{3x - 2}$

Answer

8/7

9) $\lim_{x \to \pi/2} \dfrac{\cot x}{\cos x}$

10) $\lim_{x \to -5} \dfrac{x^2 + 25}{x + 5}$

Answer

DNE

11) $\lim_{x \to 2} \dfrac{3x^2 - 2x - 8}{x^2 - 4}$

12) $\lim_{x\to 1}\dfrac{x^2-1}{x^3-1}$

Answer

$2/3$

13) $\lim_{x\to 1}\dfrac{x^2-1}{\sqrt{x}-1}$

14) $\lim_{x\to 4}\dfrac{4-x}{\sqrt{x}-2}$

Answer

-4

15) $\lim_{x\to 4}\dfrac{1}{\sqrt{x}-2}$

In exercises 16 - 17, use the squeeze theorem to prove the limit.

16) $\lim_{x\to 0} x^2\cos(2\pi x)=0$

Answer

Since $-1\le\cos(2\pi x)\le 1$, then $-x^2\le x^2\cos(2\pi x)\le x^2$. Since $\lim_{x\to 0}x^2=0=\lim_{x\to 0}-x^2$, it follows that $\lim_{x\to 0}x^2\cos(2\pi x)=0$.

17) $\lim_{x\to 0} x^3\sin\left(\dfrac{\pi}{x}\right)=0$

18) Determine the domain such that the function $f(x)=\sqrt{x-2}+xe^x$ is continuous over its domain.

Answer

$[2,\infty]$

In exercises 19 - 20, determine the value of c such that the function remains continuous. Draw your resulting function to ensure it is continuous.

19) $f(x)=\begin{cases}x^2+1, & \text{if } x>c\\ 2^x, & \text{if } x\le c\end{cases}$

20) $f(x)=\begin{cases}\sqrt{x+1}, & \text{if } x>-1\\ x^2+c, & \text{if } x\le -1\end{cases}$

In exercises 21 - 22, use the precise definition of limit to prove the limit.

21) $\lim_{x\to 1}(8x+16)=24$

22) $\lim_{x\to 0}x^3=0$

Answer

$\delta=\sqrt[3]{\varepsilon}$

23) A ball is thrown into the air and the vertical position is given by $x(t)=-4.9t^2+25t+5$. Use the Intermediate Value Theorem to show that the ball must land on the ground sometime between 5 sec and 6 sec after the throw.

24) A particle moving along a line has a displacement according to the function $x(t)=t^2-2t+4$, where x is measured in meters and t is measured in seconds. Find the average velocity over the time period $t=[0,2]$.

Answer

0 m/sec

25) From the previous exercises, estimate the instantaneous velocity at $t=2$ by checking the average velocity within $t=0.01$ sec.

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CHAPTER OVERVIEW

3: Derivatives

Calculating velocity and changes in velocity are important uses of calculus, but it is far more widespread than that. Calculus is important in all branches of mathematics, science, and engineering, and it is critical to analysis in business and health as well. In this chapter, we explore one of the main tools of calculus, the derivative, and show convenient ways to calculate derivatives. We apply these rules to a variety of functions in this chapter so that we can then explore applications of these techniques.

3.0: Prelude to Derivatives
3.1: Defining the Derivative
3.1E: Exercises for Section 3.1
3.2: The Derivative as a Function
3.2E: Exercises for Section 3.2
3.3: Differentiation Rules
3.3E: Exercises for Section 3.3
3.4: Derivatives as Rates of Change
3.4E: Exercises for Section 3.4
3.5: Derivatives of Trigonometric Functions
3.5E: Exercises for Section 3.5
3.6: The Chain Rule
3.6E: Exercises for Section 3.6
3.7: Derivatives of Inverse Functions
3.7E: Exercises for Section 3.7
3.8: Implicit Differentiation
3.8E: Exercises for Section 3.8
3.9: Derivatives of Exponential and Logarithmic Functions
3.9E: Exercises for Section 3.9
3.R: Chapter 3 Review Exercises

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3.0: Prelude to Derivatives

The Hennessey Venom GT is one of the fastest cars in the world. In 2014, it reached a record-setting speed of 270.49 mph. It can go from 0 to 200 mph in 14.51 seconds. The techniques in this chapter can be used to calculate the acceleration the Venom achieves in this feat.)

Figure 3.0.1: The Hennessey Venom GT can go from 0 to 200 mph in 14.51 seconds. (credit: modification of work by Codex41, Flickr)

Calculating velocity and changes in velocity are important uses of calculus, but it is far more widespread than that. Calculus is important in all branches of mathematics, science, and engineering, and it is critical to analysis in business and health as well. In this chapter, we explore one of the main tools of calculus, the derivative, and show convenient ways to calculate derivatives. We apply these rules to a variety of functions in this chapter so that we can then explore applications of these techniques.

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3.1: Defining the Derivative

Learning Objectives

- Recognize the meaning of the tangent to a curve at a point.
- Calculate the slope of a tangent line.
- Identify the derivative as the limit of a difference quotient.
- Calculate the derivative of a given function at a point.
- Describe the velocity as a rate of change.
- Explain the difference between average velocity and instantaneous velocity.
- Estimate the derivative from a table of values.

Now that we have both a conceptual understanding of a limit and the practical ability to compute limits, we have established the foundation for our study of calculus, the branch of mathematics in which we compute derivatives and integrals. Most mathematicians and historians agree that calculus was developed independently by the Englishman Isaac **Newton (1643–1727)** and the German Gottfried **Leibniz (1646–1716)**, whose images appear in Figure 3.1.1. When we credit Newton and Leibniz with developing calculus, we are really referring to the fact that Newton and Leibniz were the first to understand the relationship between the derivative and the integral. Both mathematicians benefited from the work of predecessors, such as Barrow, Fermat, and Cavalieri. The initial relationship between the two mathematicians appears to have been amicable; however, in later years a bitter controversy erupted over whose work took precedence. Although it seems likely that Newton did, indeed, arrive at the ideas behind calculus first, we are indebted to Leibniz for the notation that we commonly use today.

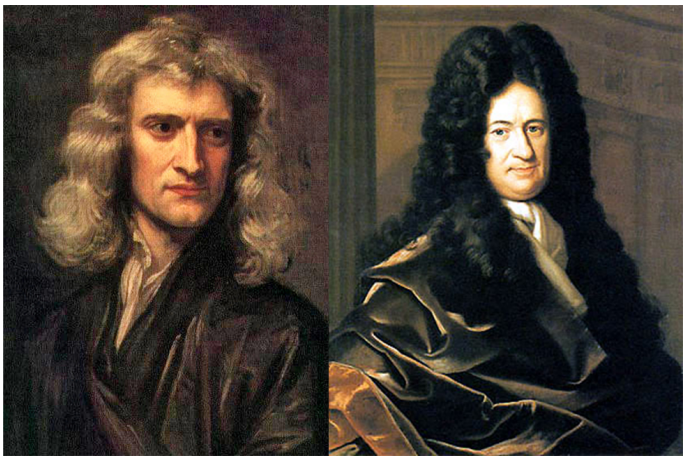

Figure 3.1.1: Newton and Leibniz are credited with developing calculus independently.

Tangent Lines

We begin our study of calculus by revisiting the notion of secant lines and tangent lines. Recall that we used the slope of a secant line to a function at a point $(a, f(a))$ to estimate the rate of change, or the rate at which one variable changes in relation to another variable. We can obtain the slope of the secant by choosing a value of x near a and drawing a line through the points $(a, f(a))$ and $(x, f(x))$, as shown in Figure 3.1.2. The slope of this line is given by an equation in the form of a difference quotient:

$$m_{sec} = \frac{f(x) - f(a)}{x - a}$$

We can also calculate the slope of a secant line to a function at a value a by using this equation and replacing x with $a + h$, where h is a value close to a. We can then calculate the slope of the line through the points $(a, f(a))$ and $(a + h, f(a + h))$. In this case, we find the secant line has a slope given by the following difference quotient with increment h:

$$m_{sec} = \frac{f(a+h) - f(a)}{a + h - a} = \frac{f(a+h) - f(a)}{h}$$

Definition: Difference Quotient

Let f be a function defined on an interval I containing a. If $x \neq a$ is in I, then

$$Q = \frac{f(x) - f(a)}{x - a}$$

is a *difference quotient.*

Also, if $h \neq 0$ is chosen so that $a + h$ is in I, then

$$Q = \frac{f(a + h) - f(a)}{h}$$

is a difference quotient with increment h.

These two expressions for calculating the slope of a secant line are illustrated in Figure 3.1.2. We will see that each of these two methods for finding the slope of a secant line is of value. Depending on the setting, we can choose one or the other. The primary consideration in our choice usually depends on ease of calculation.

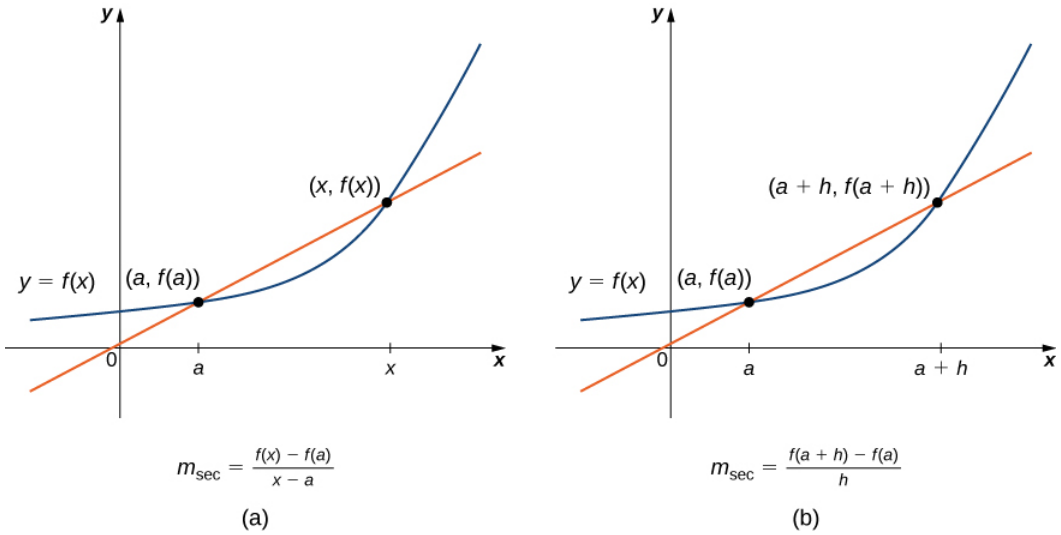

Figure 3.1.2: We can calculate the slope of a secant line in either of two ways.

In Figure 3.1.3a we see that, as the values of x approach a, the slopes of the secant lines provide better estimates of the rate of change of the function at a. Furthermore, the secant lines themselves approach the tangent line to the function at a, which represents the limit of the secant lines. Similarly, Figure 3.1.3b shows that as the values of h get closer to 0, the secant lines also approach the tangent line. The slope of the tangent line at a is the rate of change of the function at a, as shown in Figure 3.1.3c.

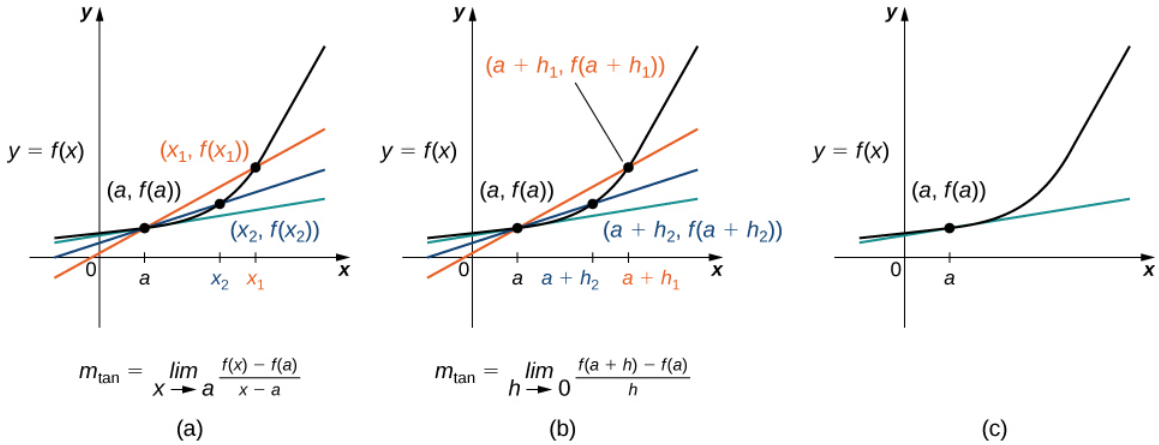

Figure 3.1.3: The secant lines approach the tangent line (shown in green) as the second point approaches the first.

In Figure 3.1.4 we show the graph of $f(x) = \sqrt{x}$ and its tangent line at $(1, 1)$ in a series of tighter intervals about $x = 1$. As the intervals become narrower, the graph of the function and its tangent line appear to coincide, making the values on the tangent line a good approximation to the values of the function for choices of x close to 1. In fact, the graph of $f(x)$ itself appears to be **locally linear** in the immediate vicinity of $x = 1$.

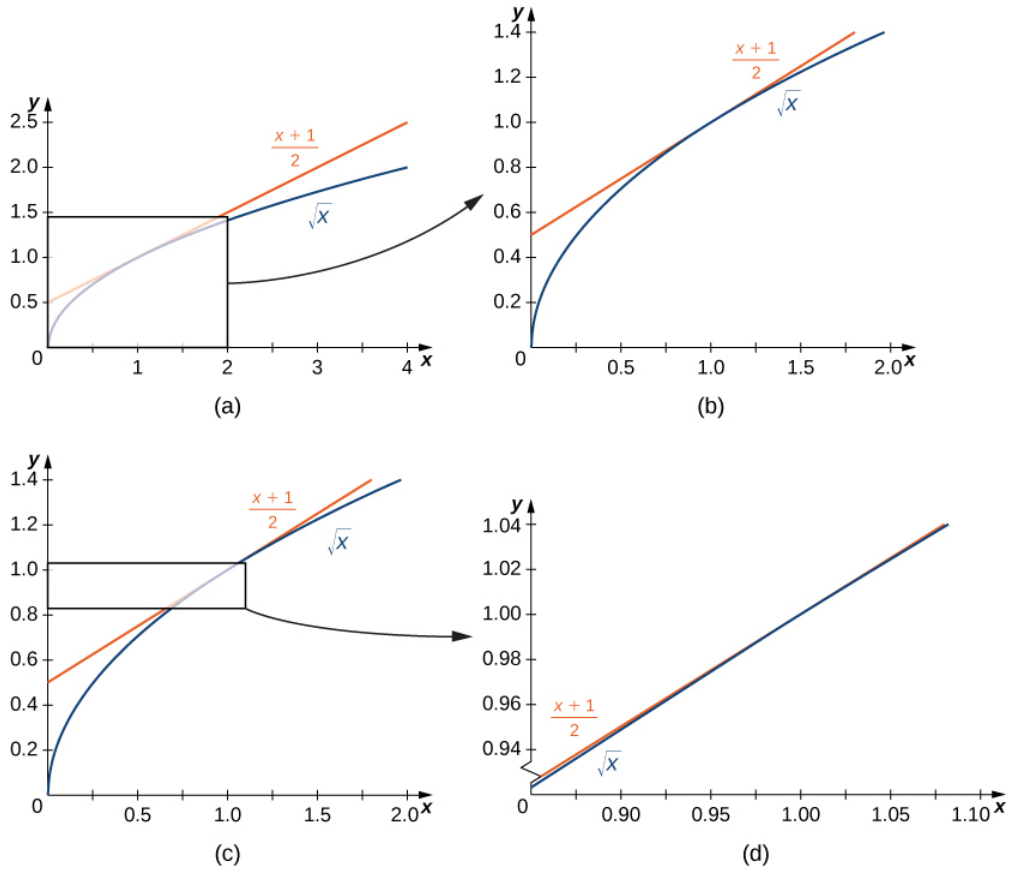

Figure 3.1.4: For values of x close to 1, the graph of $f(x) = \sqrt{x}$ and its tangent line appear to coincide.

Formally we may define the tangent line to the graph of a function as follows.

Definition: Tangent Line

Let $f(x)$ be a function defined in an open interval containing a. The tangent line to $f(x)$ at a is the line passing through the point $(a, f(a))$ having slope

$$m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{3.1.1}$$

provided this limit exists.

Equivalently, we may define the *tangent line* to $f(x)$ at a to be the line passing through the point $(a, f(a))$ having slope

$$m_{tan} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \tag{3.1.2}$$

provided this limit exists.

Just as we have used two different expressions to define the slope of a secant line, we use two different forms to define the slope of the tangent line. In this text we use both forms of the definition. As before, the choice of definition will depend on the setting. Now that we have formally defined a tangent line to a function at a point, we can use this definition to find equations of tangent lines.

Example 3.1.1: Finding a Tangent Line

Find the equation of the line tangent to the graph of $f(x) = x^2$ at $x = 3$.

Solution

First find the slope of the tangent line. In this example, use Equation 3.1.1.

$$m_{tan} = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} \qquad \text{Apply the definition.}$$

$$= \lim_{x \to 3} \frac{x^2 - 9}{x - 3} \qquad \text{Substitute } f(x) = x^2 \text{ and } f(3) = 9$$

$$= \lim_{x \to 3} \frac{(x - 3)(x + 3)}{x - 3} = \lim_{x \to 3}(x + 3) = 6 \qquad \text{Factor the numerator to evaluate the limit.}$$

Next, find a point on the tangent line. Since the line is tangent to the graph of $f(x)$ at $x = 3$, it passes through the point $(3, f(3))$. We have $f(3) = 9$, so the tangent line passes through the point $(3, 9)$.

Using the point-slope equation of the line with the slope $m = 6$ and the point $(3, 9)$, we obtain the line $y - 9 = 6(x - 3)$. Simplifying, we have $y = 6x - 9$. The graph of $f(x) = x^2$ and its tangent line at 3 are shown in Figure 3.1.5.

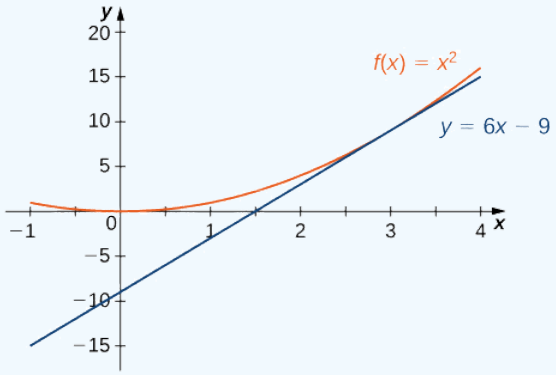

Figure 3.1.5: The tangent line to $f(x)$ at $x = 3$.

Example 3.1.2: The Slope of a Tangent Line Revisited

Use Equation 3.1.2 to find the slope of the line tangent to the graph of $f(x) = x^2$ at $x = 3$.

Solution

The steps are very similar to Example 3.1.1. See Equation 3.1.2 for the definition.

$$m_{tan} = \lim_{h \to 0} \frac{f(3 + h) - f(3)}{h} \qquad \text{Apply the definition.}$$

$$= \lim_{h \to 0} \frac{(3 + h)^2 - 9}{h} \qquad \text{Substitute } f(3 + h) = (3 + h)^2 \text{ and } f(3) = 9$$

$$= \lim_{h \to 0} \frac{9 + 6h + h^2 - 9}{h} \qquad \text{Expand and simplify to evaluate the limit.}$$

$$= \lim_{h \to 0} \frac{h(6 + h)}{h} = \lim_{h \to 0}(6 + h) = 6$$

We obtained the same value for the slope of the tangent line by using the other definition, demonstrating that the formulas can be interchanged.

Example 3.1.3: Finding the Equation of a Tangent Line

Find the equation of the line tangent to the graph of $f(x) = 1/x$ at $x = 2$.

Solution

We can use Equation 3.1.1, but as we have seen, the results are the same if we use Equation 3.1.2.

Misplaced &

We now know that the slope of the tangent line is $-\frac{1}{4}$. To find the equation of the tangent line, we also need a point on the line. We know that $f(2) = \frac{1}{2}$. Since the tangent line passes through the point $(2, \frac{1}{2})$ we can use the point-slope equation of a line to find the equation of the tangent line. Thus the tangent line has the equation $y = -\frac{1}{4}x + 1$. The graphs of $f(x) = \frac{1}{x}$ and $y = -\frac{1}{4}x + 1$ are shown in Figure 3.1.6.

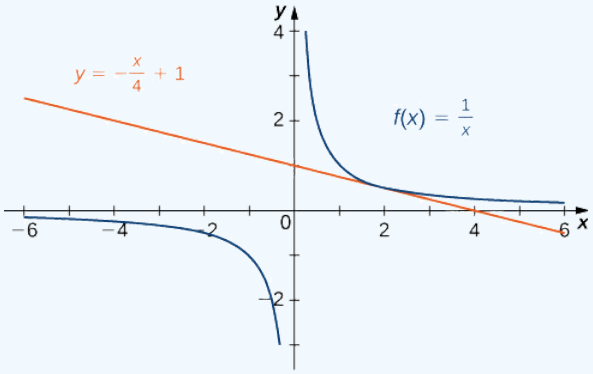

Figure 3.1.6:The line is tangent to $f(x)$ at $x = 2$.

Exercise 3.1.1

Find the slope of the line tangent to the graph of $f(x) = \sqrt{x}$ at $x = 4$.

Hint

Use either Equation 3.1.1 or Equation 3.1.2. Multiply the numerator and the denominator by a conjugate.

Answer

$\frac{1}{4}$

The Derivative of a Function at a Point

The type of limit we compute in order to find the slope of the line tangent to a function at a point occurs in many applications across many disciplines. These applications include velocity and acceleration in physics, marginal profit functions in business, and growth rates in biology. This limit occurs so frequently that we give this value a special name: the **derivative.** The process of finding a derivative is called **differentiation**.

Definition: Derivative

Let $f(x)$ be a function defined in an open interval containing a. The **derivative** of the function $f(x)$ at a, denoted by $f'(a)$, is defined by

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{3.1.3}$$

provided this limit exists.

Alternatively, we may also define the derivative of $f(x)$ at a as

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}. \tag{3.1.4}$$

Example 3.1.4: Estimating a Derivative

For $f(x) = x^2$, use a table to estimate $f'(3)$ using Equation 3.1.3.

Solution

Create a table using values of x just below 3 and just above 3.

x	$\frac{x^2-9}{x-3}$
2.9	5.9
2.99	5.99
2.999	5.999
3.001	6.001
3.01	6.01
3.1	6.1

After examining the table, we see that a good estimate is $f'(3) = 6$.

Exercise 3.1.2

For $f(x) = x^2$, use a table to estimate $f'(3)$ using Equation 3.1.4.

Hint

Evaluate $\frac{(x+h)^2 - x^2}{h}$ at $h = -0.1,\ -0.01,\ -0.001,\ 0.001,\ 0.01,\ 0.1$

Answer

6

Example 3.1.6: Finding a Derivative

For $f(x) = 3x^2 - 4x + 1$, find $f'(2)$ by using Equation 3.1.3.

Solution

Substitute the given function and value directly into the equation.

$$
\begin{aligned}
f'(x) &= \lim_{x\to 2} \frac{f(x) - f(2)}{x-2} && \text{Apply the definition.}\\
&= \lim_{x\to 2} \frac{(3x^2 - 4x + 1) - 5}{x-2} && \text{Substitute } f(x) = 3x^2 - 4x + 1 \text{ and } f(2) = 5.\\
&= \lim_{x\to 2} \frac{(x-2)(3x+2)}{x-2} && \text{Simplify and factor the numerator.}\\
&= \lim_{x\to 2} (3x+2) && \text{Cancel the common factor.}\\
&= 8 && \text{Evaluate the limit.}
\end{aligned}
$$

Example 3.1.7: Revisiting the Derivative

For $f(x) = 3x^2 - 4x + 1$, find $f'(2)$ by using Equation 3.1.4.

Solution

Using this equation, we can substitute two values of the function into the equation, and we should get the same value as in Example 3.1.6.

$$
\begin{aligned}
f'(2) &= \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} && \text{Apply the definition.}\\
&= \lim_{h\to 0}\frac{(3(2+h)^2-4(2+h)+1)-5}{h} && \text{Substitute } f(2)=5 \text{ and } f(2+h)=3(2+h)^2-4(2+h)+1.\\
&= \lim_{h\to 0}\frac{3(4+4h+h^2)-8-4h+1-5}{h} && \text{Expand the numerator.}\\
&= \lim_{h\to 0}\frac{12+12h+3h^2-12-4h}{h} && \text{Distribute and begin simplifying the numerator.}\\
&= \lim_{h\to 0}\frac{3h^2+8h}{h} && \text{Finish simplifying the numerator.}\\
&= \lim_{h\to 0}\frac{h(3h+8)}{h} && \text{Factor the numerator.}\\
&= \lim_{h\to 0}(3h+8) && \text{Cancel the common factor.}\\
&= 8 && \text{Evaluate the limit.}
\end{aligned}
$$

The results are the same whether we use Equation 3.1.3 or Equation 3.1.4.

? Exercise 3.1.4

For $f(x)=x^2+3x+2$, find $f'(1)$.

Hint

Use either Equation 3.1.3, Equation 3.1.4, or try both.

Answer

$f'(1)=5$

Velocities and Rates of Change

Now that we can evaluate a derivative, we can use it in velocity applications. Recall that if $s(t)$ is the position of an object moving along a coordinate axis, the **average velocity** of the object over a time interval $[a,t]$ if $t>a$ or $[t,a]$ if $t<a$ is given by the difference quotient

$$v_{ave}=\frac{s(t)-s(a)}{t-a}. \tag{3.1.5}$$

As the values of t approach a, the values of v_{ave} approach the value we call the **instantaneous velocity** at a. That is, instantaneous velocity at a, denoted $v(a)$, is given by

$$v(a)=s'(a)=\lim_{t\to a}\frac{s(t)-s(a)}{t-a}. \tag{3.1.6}$$

To better understand the relationship between average velocity and instantaneous velocity, see Figure 3.1.7. In this figure, the slope of the tangent line (shown in red) is the instantaneous velocity of the object at time $t=a$ whose position at time t is given by the function $s(t)$. The slope of the secant line (shown in green) is the average velocity of the object over the time interval $[a,t]$.

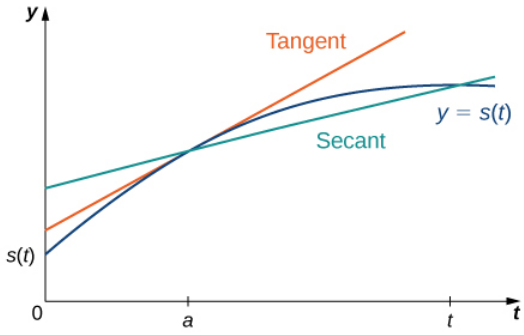

Figure 3.1.7: The slope of the secant line is the average velocity over the interval $[a, t]$. The slope of the tangent line is the instantaneous velocity.

We can use Equation 3.1.6 to calculate the instantaneous velocity, or we can estimate the velocity of a moving object by using a table of values. We can then confirm the estimate by using Equation 3.1.5.

Example 3.1.8: Estimating Velocity

A lead weight on a spring is oscillating up and down. Its position at time t with respect to a fixed horizontal line is given by $s(t) = \sin t$ (Figure 3.1.8). Use a table of values to estimate $v(0)$. Check the estimate by using Equation 3.1.6.

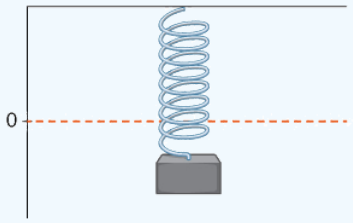

Figure 3.1.8: A lead weight suspended from a spring in vertical oscillatory motion.

Solution

We can estimate the instantaneous velocity at $t = 0$ by computing a table of average velocities using values of t approaching 0, as shown in Table 3.1.2.

Table 3.1.2: Average velocities using values of t approaching 0

t	$\frac{\sin t - \sin 0}{t-0} = \frac{\sin t}{t}$
−0.1	0.998334166
−0.01	0.9999833333
−0.001	0.999999833
0.001	0.999999833
0.01	0.9999833333
0.1	0.998334166

From the table we see that the average velocity over the time interval $[-0.1, 0]$ is 0.998334166, the average velocity over the time interval $[-0.01, 0]$ is 0.9999833333, and so forth. Using this table of values, it appears that a good estimate is $v(0) = 1$.

By using Equation 3.1.6, we can see that

$$v(0) = s'(0) = \lim_{t \to 0} \frac{\sin t - \sin 0}{t - 0} = \lim_{t \to 0} \frac{\sin t}{t} = 1.$$

Thus, in fact, $v(0) = 1$.

Exercise 3.1.5

A rock is dropped from a height of 64 feet. Its height above ground at time t seconds later is given by $s(t) = -16t^2 + 64,\ 0 \le t \le 2$. Find its instantaneous velocity 1 second after it is dropped, using Equation 3.1.6.

Hint

$v(t) = s'(t)$. Follow the earlier examples of the derivative using Equation 3.1.6.

Answer

-32 ft/s

As we have seen throughout this section, the slope of a tangent line to a function and instantaneous velocity are related concepts. Each is calculated by computing a derivative and each measures the instantaneous rate of change of a function, or the rate of change of a function at any point along the function.

Definition: Instantaneous Rate of Change

The **instantaneous rate of change** of a function $f(x)$ at a value a is its derivative $f'(a)$.

Example 3.1.9: Chapter Opener: Estimating Rate of Change of Velocity

Reaching a top speed of 270.49 mph, the Hennessey Venom GT is one of the fastest cars in the world. In tests it went from 0 to 60 mph in 3.05 seconds, from 0 to 100 mph in 5.88 seconds, from 0 to 200 mph in 14.51 seconds, and from 0 to 229.9 mph in 19.96 seconds. Use this data to draw a conclusion about the rate of change of velocity (that is, its **acceleration**) as it approaches 229.9 mph. Does the rate at which the car is accelerating appear to be increasing, decreasing, or constant?

Figure 3.1.9: (credit: modification of work by Codex41, Flickr)

Solution: First observe that 60 mph = 88 ft/s, 100 mph ≈146.67 ft/s, 200 mph ≈293.33 ft/s, and 229.9 mph ≈337.19 ft/s. We can summarize the information in a table.

Table 3.1.3: $v(t)$ at different values of t

t	$v(t)$
0	0
3.05	88
5.88	147.67
14.51	293.33
19.96	337.19

Now compute the average acceleration of the car in feet per second on intervals of the form $[t, 19.96]$ as t approaches 19.96, as shown in the following table.

Average acceleration

t	$\frac{v(t) - v(19.96)}{t - 19.96} = \frac{v(t) - 337.19}{t - 19.96}$

t	$\frac{v(t)-v(19.96)}{t-19.96}=\frac{v(t)-337.19}{t-19.96}$
0.0	16.89
3.05	14.74
5.88	13.46
14.51	8.05

The rate at which the car is accelerating is decreasing as its velocity approaches 229.9 mph (337.19 ft/s).

Example 3.1.10: Rate of Change of Temperature

A homeowner sets the thermostat so that the temperature in the house begins to drop from $70°F$ at 9 p.m., reaches a low of $60°$ during the night, and rises back to $70°$ by 7 a.m. the next morning. Suppose that the temperature in the house is given by $T(t) = 0.4t^2 - 4t + 70$ for $0 \le t \le 10$, where t is the number of hours past 9 p.m. Find the instantaneous rate of change of the temperature at midnight.

Solution

Since midnight is 3 hours past 9 p.m., we want to compute $T'(3)$. Refer to Equation 3.1.3.

$$\begin{aligned}
T'(3) &= \lim_{t\to 3}\frac{T(t)-T(3)}{t-3} && \text{Apply the definition.}\\
&= \lim_{t\to 3}\frac{0.4t^2-4t+70-61.6}{t-3} && \text{Substitute } T(t)=0.4t^2-4t+70 \text{ and } T(3)=61.6.\\
&= \lim_{t\to 3}\frac{0.4t^2-4t+8.4}{t-3} && \text{Simplify.}\\
&= \lim_{t\to 3}\frac{0.4(t-3)(t-7)}{t-3} &&\\
&= \lim_{t\to 3}0.4(t-7) && \text{Cancel.}\\
&= -1.6 && \text{Evaluate the limit.}
\end{aligned}$$

The instantaneous rate of change of the temperature at midnight is $-1.6°F$ per hour.

Example 3.1.11: Rate of Change of Profit

A toy company can sell x electronic gaming systems at a price of $p = -0.01x + 400$ dollars per gaming system. The cost of manufacturing x systems is given by $C(x) = 100x + 10,000$ dollars. Find the rate of change of profit when $10,000$ games are produced. Should the toy company increase or decrease production?

Solution

The profit $P(x)$ earned by producing x gaming systems is $R(x) - C(x)$, where $R(x)$ is the revenue obtained from the sale of x games. Since the company can sell x games at $p = -0.01x + 400$ per game,

$$R(x) = xp = x(-0.01x + 400) = -0.01x^2 + 400x.$$

Consequently,

$$P(x) = -0.01x^2 + 300x - 10,000.$$

Therefore, evaluating the rate of change of profit gives

$$
\begin{aligned}
P'(10000) &= \lim_{x\to 10000} \frac{P(x)-P(10000)}{x-10000} \\
&= \lim_{x\to 10000} \frac{-0.01x^2+300x-10000-1990000}{x-10000} \\
&= \lim_{x\to 10000} \frac{-0.01x^2+300x-2000000}{x-10000} \\
&= 100
\end{aligned}
$$

Since the rate of change of profit $P'(10,000) > 0$ and $P(10,000) > 0$, the company should increase production.

Exercise 3.1.6

A coffee shop determines that the daily profit on scones obtained by charging s dollars per scone is $P(s) = -20s^2 + 150s - 10$. The coffee shop currently charges $\$3.25$ per scone. Find $P'(3.25)$, the rate of change of profit when the price is $\$3.25$ and decide whether or not the coffee shop should consider raising or lowering its prices on scones.

Hint

Use Example 3.1.11 for a guide.

Answer

$P'(3.25) = 20 > 0$; raise prices

Key Concepts

- The slope of the tangent line to a curve measures the instantaneous rate of change of a curve. We can calculate it by finding the limit of the difference quotient or the difference quotient with increment h.
- The derivative of a function $f(x)$ at a value a is found using either of the definitions for the slope of the tangent line.
- Velocity is the rate of change of position. As such, the velocity $v(t)$ at time t is the derivative of the position $s(t)$ at time t. Average velocity is given by

$$v_{ave} = \frac{s(t)-s(a)}{t-a}.$$

  Instantaneous velocity is given by

$$v(a) = s'(a) = \lim_{t\to a} \frac{s(t)-s(a)}{t-a}.$$

- We may estimate a derivative by using a table of values.

Key Equations

- **Difference quotient**

$$Q = \frac{f(x)-f(a)}{x-a}$$

- **Difference quotient with increment h**

$$Q = \frac{f(a+h)-f(a)}{a+h-a} = \frac{f(a+h)-f(a)}{h}$$

- **Slope of tangent line**

$$m_{tan} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$$m_{tan} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

- **Derivative of f(x) at a**

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- **Average velocity**

$$v_{ave} = \frac{s(t) - s(a)}{t - a}$$

- **Instantaneous velocity**

$$v(a) = s'(a) = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}$$

Glossary

derivative

the slope of the tangent line to a function at a point, calculated by taking the limit of the difference quotient, is the derivative

difference quotient

of a function $f(x)$ at a is given by

$$\frac{f(a+h) - f(a)}{h} \text{ or } \frac{f(x) - f(a)}{x - a}$$

differentiation

the process of taking a derivative

instantaneous rate of change

the rate of change of a function at any point along the function a, also called $f'(a)$, or the derivative of the function at a

---

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3.1E: EXERCISES FOR SECTION 3.1

For exercises 1 - 10, use the equation $m_{\text{sec}} = \dfrac{f(x)-f(a)}{x-a}$ to find the slope of the secant line between the values x_1 and x_2 for each function $y = f(x)$.

1) $f(x) = 4x + 7; \quad x_1 = 2, \quad x_2 = 5$

Answer

$m_{\text{sec}} = 4$

2) $f(x) = 8x - 3; \quad x_1 = -1, \quad x_2 = 3$

3) $f(x) = x^2 + 2x + 1; \quad x_1 = 3, \quad x_2 = 3.5$

Answer

$m_{\text{sec}} = 8.5$

4) $f(x) = -x^2 + x + 2; \quad x_1 = 0.5, \quad x_2 = 1.5$

5) $f(x) = \dfrac{4}{3x-1}; \quad x_1 = 1, \quad x_2 = 3$

Answer

$m_{\text{sec}} = -\frac{3}{4}$

6) $f(x) = \dfrac{x-7}{2x+1}; \quad x_1 = -2, \quad x_2 = 0$

7) $f(x) = \sqrt{x}; \quad x_1 = 1, \quad x_2 = 16$

Answer

$m_{\text{sec}} = 0.2$

8) $f(x) = \sqrt{x-9}; \quad x_1 = 10, \quad x_2 = 13$

9) $f(x) = x^{1/3} + 1; \quad x_1 = 0, \quad x_2 = 8$

Answer

$m_{\text{sec}} = 0.25$

10) $f(x) = 6x^{2/3} + 2x^{1/3}; \quad x_1 = 1, \quad x_2 = 27$

For the functions in exercises 11 - 20,

a. use the equation $m_{\tan} = \lim\limits_{h \to 0} \dfrac{f(a+h)-f(a)}{h}$ to find the slope of the tangent line $m_{\tan} = f'(a)$, and

b. find the equation of the tangent line to f at $x = a$.

11) $f(x) = 3 - 4x, \quad a = 2$

Answer

a. $m_{\tan} = -4$

b. $y = -4x + 3$

12) $f(x) = \dfrac{x}{5} + 6, \quad a = -1$

13) $f(x) = x^2 + x, \quad a = 1$

Answer

a. $m_{\tan} = 3$

b. $y = 3x - 1$

14) $f(x) = 1 - x - x^2, \quad a = 0$

15) $f(x) = \dfrac{7}{x}, \quad a = 3$

Answer

a. $m_{\tan} = \frac{-7}{9}$

b. $y = \frac{-7}{9}x + \frac{14}{3}$

16) $f(x) = \sqrt{x+8}, \quad a = 1$

17) $f(x) = 2 - 3x^2, \quad a = -2$

Answer

a. $m_{\tan} = 12$

b. $y = 12x + 14$

18) $f(x)=\dfrac{-3}{x-1},\quad a=4$

19) $f(x)=\dfrac{2}{x+3},\quad a=-4$

Answer

a. $m_{\tan}=-2$

b. $y=-2x-10$

20) $f(x)=\dfrac{3}{x^2},\quad a=3$

For the functions $y=f(x)$ in exercises 21 - 30, find $f'(a)$ using the equation $f'(a)=\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}$.

21) $f(x)=5x+4,\quad a=-1$

Answer

$f'(-1)=5$

22) $f(x)=-7x+1,\quad a=3$

23) $f(x)=x^2+9x,\quad a=2$

Answer

$f'(2)=13$

24) $f(x)=3x^2-x+2,\quad a=1$

25) $f(x)=\sqrt{x},\quad a=4$

Answer

$f'(4)=\frac{1}{4}$

26) $f(x)=\sqrt{x-2},\quad a=6$

27) $f(x)=\dfrac{1}{x},\quad a=2$

Answer

$f'(2)=-\frac{1}{4}$

28) $f(x)=\dfrac{1}{x-3},\quad a=-1$

29) $f(x)=\dfrac{1}{x^3},\quad a=1$

Answer

$f'(1)=-3$

30) $f(x)=\dfrac{1}{\sqrt{x}},\quad a=4$

For the following exercises, given the function $y=f(x)$,

a. find the slope of the secant line PQ for each point $Q(x,f(x))$ with x value given in the table.

b. Use the answers from a. to estimate the value of the slope of the tangent line at P.

c. Use the answer from b. to find the equation of the tangent line to f at point P.

31) [T] $f(x)=x^2+3x+4,\quad P(1,8)$ (Round to 6 decimal places.)

x	Slope m_{PQ}	x	Slope m_{PQ}
1.1	(i)	0.9	(vii)
1.01	(ii)	0.99	(viii)
1.001	(iii)	0.999	(ix)
1.0001	(iv)	0.9999	(x)
1.00001	(v)	0.99999	(xi)
1.000001	(vi)	0.999999	(xii)

Answer

$a.(i)5.100000, (ii)5.010000, (iii)5.001000, (iv)5.000100, (v)5.000010, (vi)5.000001, (vii)4.900000, (viii)4.990000, (ix)4.999000, (x)4.999900, (xi)4.999990, (x)4.999999$

b. $m_{\tan}=5$

c. $y=5x+3$

32) [T] $f(x)=\dfrac{x+1}{x^2-1},\quad P(0,-1)$

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x	Slope m_{PQ}	x	Slope m_{PQ}
0.1	(i)	−0.1	(vii)
0.01	(ii)	−0.01	(viii)
0.001	(iii)	−0.001	(ix)
0.0001	(iv)	−0.0001	(x)
0.00001	(v)	−0.00001	(xi)
0.000001	(vi)	−0.000001	(xii)

33) [T] $f(x) = 10e^{0.5x}, \quad P(0, 10)$ (Round to 4 decimal places.)

x	Slope m_{PQ}
−0.1	(i)
−0.01	(ii)
−0.001	(iii)
−0.0001	(iv)
−0.00001	(v)
−0.000001	(vi)

Answer

a. $(i) 4.8771, (ii) 4.9875, (iii) 4.9988, (iv) 4.9999, (v) 4.9999, (vi) 4.9999$

b. $m_{\tan} = 5$

c. $y = 5x + 10$

34) [T] $f(x) = \tan(x), \quad P(\pi, 0)$

x	Slope m_{PQ}
3.1	(i)
3.14	(ii)
3.141	(iii)
3.1415	(iv)
3.14159	(v)
3.141592	(vi)

[T] For the following position functions $y = s(t)$, an object is moving along a straight line, where t is in seconds and s is in meters. Find

a. the simplified expression for the average velocity from $t = 2$ to $t = 2 + h$;

b. the average velocity between $t = 2$ and $t = 2 + h$, where $(i)\ h = 0.1, (ii)\ h = 0.01, (iii)\ h = 0.001$, and $(iv)\ h = 0.0001$; and

c. use the answer from a. to estimate the instantaneous velocity at $t = 2$ second.

35) $s(t) = \frac{1}{3}t + 5$

Answer

a. $\frac{1}{3}$;

b. $(i)\ \frac{1}{3}$ m/s, $(ii)\ \frac{1}{3}$ m/s, $(iii)\ \frac{1}{3}$ m/s, $(iv)\ \frac{1}{3}$ m/s;

c. $\frac{1}{3}$ m/s

36) $s(t) = t^2 - 2t$

37) $s(t) = 2t^3 + 3$

Answer

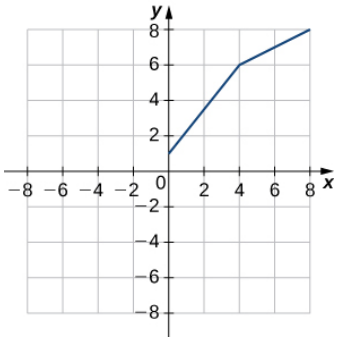
a. $2(h^2 + 6h + 12)$;

b. $(i)\ 25.22$ m/s, $(ii)\ 24.12$ m/s, $(iii)\ 24.01$ m/s, $(iv)\ 24$ m/s;

c. 24 m/s

38) $s(t) = \frac{16}{t^2} - \frac{4}{t}$

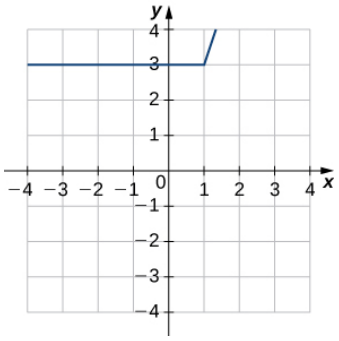
39) Use the following graph to evaluate a. $f'(1)$ and b. $f'(6)$.



Answer

a. 1.25; b. 0.5

40) Use the following graph to evaluate a. $f'(-3)$ and b. $f'(1.5)$.



For the following exercises, use the limit definition of derivative to show that the derivative does not exist at $x = a$ for each of the given functions.

41) $f(x) = x^{1/3}, \quad x = 0$

Answer

$$\lim_{x \to 0^-} \frac{x^{1/3} - 0}{x - 0} = \lim_{x \to 0^-} \frac{1}{x^{2/3}} = \infty$$

42) $f(x) = x^{2/3}, \quad x = 0$

43) $f(x) = \begin{cases} 1, & \text{if } x < 1 \\ x, & \text{if } x \geq 1 \end{cases}, \quad x = 1$

Answer

$$\lim_{x \to 1^-} \frac{1 - 1}{x - 1} = 0 \neq 1 = \lim_{x \to 1^+} \frac{x - 1}{x - 1}$$

44) $f(x) = \dfrac{|x|}{x}, \quad x = 0$

45) [T] The position in feet of a race car along a straight track after t seconds is modeled by the function $s(t) = 8t^2 - \frac{1}{16}t^3$.

a. Find the average velocity of the vehicle over the following time intervals to four decimal places:

i. $[4, 4.1]$

ii. $[4, 4.01]$

iii. $[4, 4.001]$

iv. $[4, 4.0001]$

b. Use a. to draw a conclusion about the instantaneous velocity of the vehicle at $t = 4$ seconds.

Answer

a. $(i) 61.7244 ft/s, \ (ii) 61.0725 ft/s, \ (iii) 61.0072 ft/s, \ (iv) 61.0007 ft/s$

b. At 4 seconds the race car is traveling at a rate/velocity of 61 ft/s.

46) [T] The distance in feet that a ball rolls down an incline is modeled by the function $s(t) = 14t^2$,

where t is seconds after the ball begins rolling.

a. Find the average velocity of the ball over the following time intervals:

i. [5, 5.1]

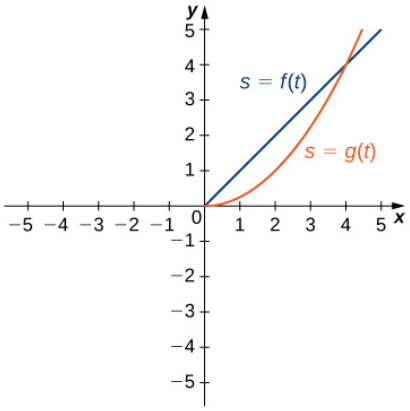
ii. [5, 5.01]

iii. [5, 5.001]

iv. [5, 5.0001]

b. Use the answers from a. to draw a conclusion about the instantaneous velocity of the ball at $t = 5$ seconds.

47) Two vehicles start out traveling side by side along a straight road. Their position functions, shown in the following graph, are given by $s = f(t)$ and $s = g(t)$, where s is measured in feet and t is measured in seconds.



a. Which vehicle has traveled farther at $t = 2$ seconds?

b. What is the approximate velocity of each vehicle at $t = 3$ seconds?

c. Which vehicle is traveling faster at $t = 4$ seconds?

d. What is true about the positions of the vehicles at $t = 4$ seconds?

Answer

a. The vehicle represented by $f(t)$, because it has traveled 2 feet, whereas $g(t)$ has traveled 1 foot.
b. The velocity of $f(t)$ is constant at 1 ft/s, while the velocity of $g(t)$ is approximately 2 ft/s.
c. The vehicle represented by $g(t)$, with a velocity of approximately 4 ft/s.
d. Both have traveled 4 feet in 4 seconds.

48) [T] The total cost $C(x)$, in hundreds of dollars, to produce x jars of mayonnaise is given by $C(x) = 0.000003x^3 + 4x + 300$.

a. Calculate the average cost per jar over the following intervals:

i. [100, 100.1]

ii. [100, 100.01]

iii. [100, 100.001]

iv. [100, 100.0001]

b. Use the answers from a. to estimate the average cost to produce 100 jars of mayonnaise.

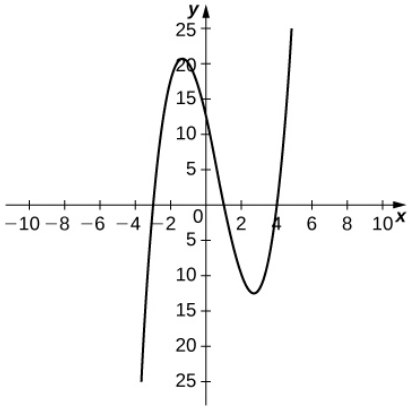
49) [T] For the function $f(x) = x^3 - 2x^2 - 11x + 12$, do the following.

a. Use a graphing calculator to graph f in an appropriate viewing window.

b. Use the ZOOM feature on the calculator to approximate the two values of $x = a$ for which $m_{tan} = f'(a) = 0$.

Answer

a.



b. $a \approx -1.361,\ 2.694$

50) [T] For the function $f(x) = \dfrac{x}{1+x^2}$, do the following.

a. Use a graphing calculator to graph f in an appropriate viewing window.

b. Use the ZOOM feature on the calculator to approximate the values of $x = a$ for which $m_{\tan} = f'(a) = 0$.

51) Suppose that $N(x)$ computes the number of gallons of gas used by a vehicle traveling x miles. Suppose the vehicle gets 30 mpg.

a. Find a mathematical expression for $N(x)$.

b. What is $N(100)$? Explain the physical meaning.

c. What is $N'(100)$? Explain the physical meaning.

Answer

a. $N(x) = \dfrac{x}{30}$

b. ~ 3.3 gallons. When the vehicle travels 100 miles, it has used 3.3 gallons of gas.

c. $\frac{1}{30}$. The rate of gas consumption in gallons per mile that the vehicle is achieving after having traveled 100 miles.

52) [T] For the function $f(x) = x^4 - 5x^2 + 4$, do the following.

a. Use a graphing calculator to graph f in an appropriate viewing window.

b. Use the *nDeriv* function, which numerically finds the derivative, on a graphing calculator to estimate $f'(-2)$, $f'(-0.5)$, $f'(1.7)$, and $f'(2.718)$.

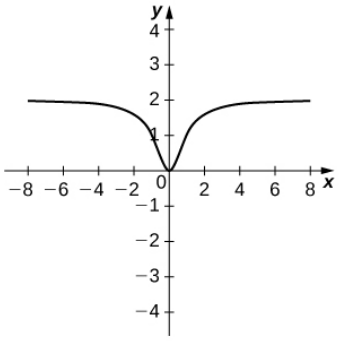
53) [T] For the function $f(x) = \dfrac{x^2}{x^2+1}$, do the following.

a. Use a graphing calculator to graph f in an appropriate viewing window.

b. Use the *nDeriv* function on a graphing calculator to find $f'(-4)$, $f'(-2)$, $f'(2)$, and $f'(4)$.

Answer

a.



b. $-0.028, -0.16, 0.16, 0.028$

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3.2: The Derivative as a Function

Learning Objectives

- Define the derivative function of a given function.
- Graph a derivative function from the graph of a given function.
- State the connection between derivatives and continuity.
- Describe three conditions for when a function does not have a derivative.
- Explain the meaning of a higher-order derivative.

As we have seen, the derivative of a function at a given point gives us the rate of change or slope of the tangent line to the function at that point. If we differentiate a position function at a given time, we obtain the velocity at that time. It seems reasonable to conclude that knowing the derivative of the function at every point would produce valuable information about the behavior of the function. However, the process of finding the derivative at even a handful of values using the techniques of the preceding section would quickly become quite tedious. In this section we define the derivative function and learn a process for finding it.

Derivative Functions

The derivative function gives the derivative of a function at each point in the domain of the original function for which the derivative is defined. We can formally define a derivative function as follows.

Definition: Derivative Function

Let f be a function. The **derivative function**, denoted by f', is the function whose domain consists of those values of x such that the following limit exists:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{3.2.1}$$

A function $f(x)$ is said to be **differentiable at** a if $f'(a)$ exists. More generally, a function is said to be **differentiable on** S if it is differentiable at every point in an open set S, and a differentiable function is one in which $f'(x)$ exists on its domain.

In the next few examples we use Equation 3.2.1 to find the derivative of a function.

Example 3.2.1: Finding the Derivative of a Square-Root Function

Find the derivative of $f(x) = \sqrt{x}$.

Solution

Start directly with the definition of the derivative function.

Substitute $f(x+h) = \sqrt{x+h}$ and $f(x) = \sqrt{x}$ into $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$	
$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$	Multiply numerator and denominator by $\sqrt{x+h} + \sqrt{x}$ without distributing in the denominator.
$= \lim_{h \to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$	Multiply the numerators and simplify.
$= \lim_{h \to 0} \frac{1}{\left(\sqrt{x+h} + \sqrt{x}\right)}$	Cancel the h.
$= \frac{1}{2\sqrt{x}}$	Evaluate the limit

Example 3.2.2: Finding the Derivative of a Quadratic Function

Find the derivative of the function $f(x) = x^2 - 2x$.

Solution

Follow the same procedure here, but without having to multiply by the conjugate.

Substitute $f(x+h) = (x+h)^2 - 2(x+h)$ and $f(x) = x^2 - 2x$ into $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$.

$f'(x) = \lim_{h\to 0} \frac{((x+h)^2 - 2(x+h)) - (x^2 - 2x)}{h}$	
$= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 2x - 2h - x^2 + 2x}{h}$	Expand $(x+h)^2 - 2(x+h)$.
$= \lim_{h\to 0} \frac{2xh - 2h + h^2}{h}$	Simplify
$= \lim_{h\to 0} \frac{h(2x - 2 + h)}{h}$	Factor out h from the numerator
$= \lim_{h\to 0} (2x - 2 + h)$	Cancel the common factor of h
$= 2x - 2$	Evaluate the limit

Exercise 3.2.1

Find the derivative of $f(x) = x^2$.

Hint

Use Equation 3.2.1 and follow the example.

Answer

$f'(x) = 2x$

We use a variety of different notations to express the derivative of a function. In Example 3.2.2 we showed that if $f(x) = x^2 - 2x$, then $f'(x) = 2x - 2$. If we had expressed this function in the form $y = x^2 - 2x$, we could have expressed the derivative as $y' = 2x - 2$ or $\frac{dy}{dx} = 2x - 2$. We could have conveyed the same information by writing $\frac{d}{dx}(x^2 - 2x) = 2x - 2$. Thus, for the function $y = f(x)$, each of the following notations represents the derivative of $f(x)$:

$$f'(x), \quad \frac{dy}{dx}, \quad y', \quad \frac{d}{dx}\big(f(x)\big).$$

In place of $f'(a)$ we may also use $\left.\frac{dy}{dx}\right|_{x=a}$. Use of the $\frac{dy}{dx}$ notation (called Leibniz notation) is quite common in engineering and physics. To understand this notation better, recall that the derivative of a function at a point is the limit of the slopes of secant lines as the secant lines approach the tangent line. The slopes of these secant lines are often expressed in the form $\frac{\Delta y}{\Delta x}$ where Δy is the difference in the y values corresponding to the difference in the x values, which are expressed as Δx (Figure 3.2.1). Thus the derivative, which can be thought of as the instantaneous rate of change of y with respect to x, is expressed as

$$\frac{dy}{dx} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x}.$$

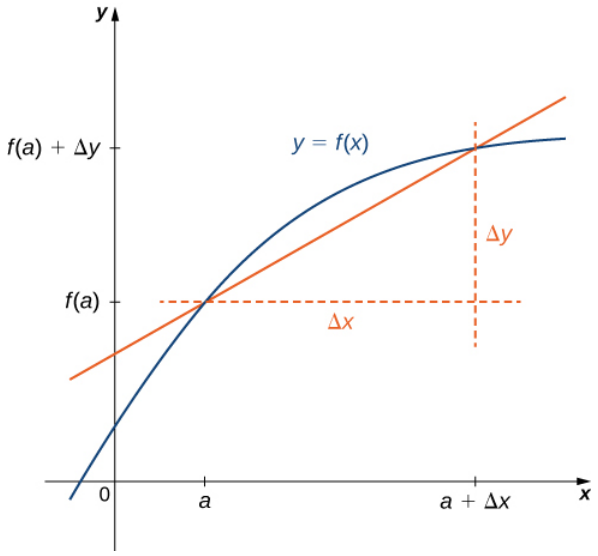


Figure 3.2.1: The derivative is expressed as $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$.

Graphing a Derivative

We have already discussed how to graph a function, so given the equation of a function or the equation of a derivative function, we could graph it. Given both, we would expect to see a correspondence between the graphs of these two functions, since $f'(x)$ gives the rate of change of a function $f(x)$ (or slope of the tangent line to $f(x)$).

In Example 3.2.1, we found that for $f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2\sqrt{x}}$. If we graph these functions on the same axes, as in Figure 3.2.2, we can use the graphs to understand the relationship between these two functions. First, we notice that $f(x)$ is increasing over its entire domain, which means that the slopes of its tangent lines at all points are positive. Consequently, we expect $f'(x) > 0$ for all values of x in its domain. Furthermore, as x increases, the slopes of the tangent lines to $f(x)$ are decreasing and we expect to see a corresponding decrease in $f'(x)$. We also observe that $f(0)$ is undefined and that $\lim_{x \to 0^+} f'(x) = +\infty$, corresponding to a vertical tangent to $f(x)$ at 0.

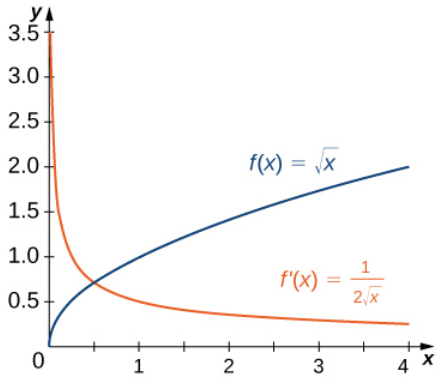


Figure 3.2.2: The derivative $f'(x)$ is positive everywhere because the function $f(x)$ is increasing.

In Example 3.2.2, we found that for $f(x) = x^2 - 2x$, $f'(x) = 2x - 2$. The graphs of these functions are shown in Figure 3.2.3. Observe that $f(x)$ is decreasing for $x < 1$. For these same values of x, $f'(x) < 0$. For values of $x > 1$, $f(x)$ is increasing and $f'(x) > 0$. Also, $f(x)$ has a horizontal tangent at $x = 1$ and $f'(1) = 0$.

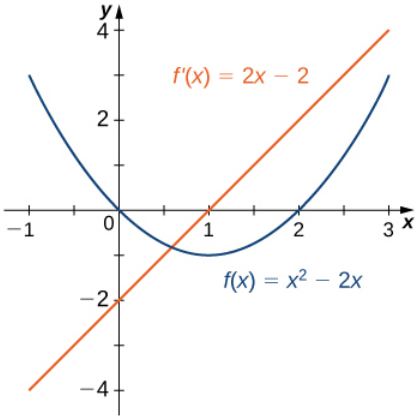
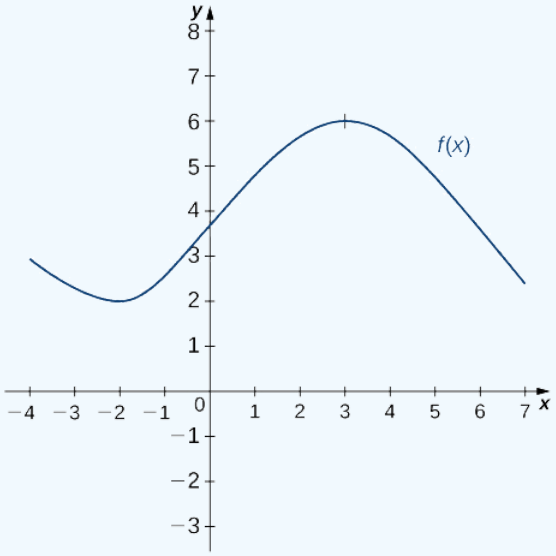


Figure 3.2.3: The derivative $f'(x) < 0$ where the function $f(x)$ is decreasing and $f'(x) > 0$ where $f(x)$ is increasing. The derivative is zero where the function has a horizontal tangent

Example 3.2.3: Sketching a Derivative Using a Function

Use the following graph of $f(x)$ to sketch a graph of $f'(x)$.



Solution

The solution is shown in the following graph. Observe that $f(x)$ is increasing and $f'(x) > 0$ on $(-2, 3)$. Also, $f(x)$ is decreasing and $f'(x) < 0$ on $(-\infty, -2)$ and on $(3, +\infty)$. Also note that $f(x)$ has horizontal tangents at -2 and 3, and $f'(-2) = 0$ and $f'(3) = 0$.

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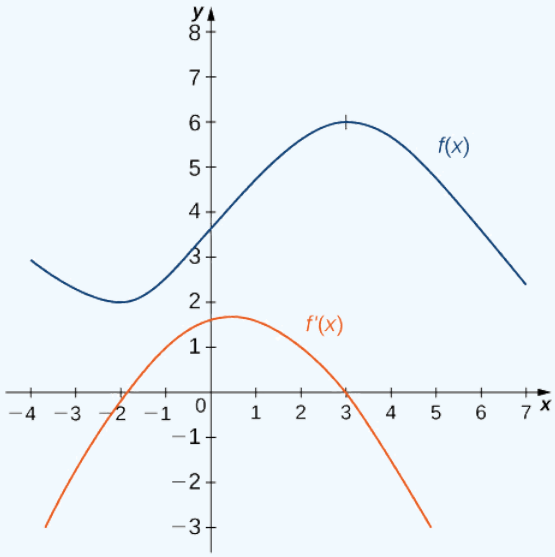

? Exercise 3.2.2

Sketch the graph of $f(x) = x^2 - 4$. On what interval is the graph of $f'(x)$ above the x-axis?

Hint

The graph of $f'(x)$ is positive where $f(x)$ is increasing.

Answer

$(0, +\infty)$

Derivatives and Continuity

Now that we can graph a derivative, let's examine the behavior of the graphs. First, we consider the relationship between differentiability and continuity. We will see that if a function is differentiable at a point, it must be continuous there; however, a function that is continuous at a point need not be differentiable at that point. In fact, a function may be continuous at a point and fail to be differentiable at the point for one of several reasons.

Differentiability Implies Continuity

Let $f(x)$ be a function and a be in its domain. If $f(x)$ is differentiable at a, then f is continuous at a.

Proof

If $f(x)$ is differentiable at a, then $f'(a)$ exists and, if we let $h = x - a$, we have $x = a + h$, and as $h = x - a \to 0$, we can see that $x \to a$.

Then

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

can be rewritten as

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

We want to show that $f(x)$ is continuous at a by showing that $\lim_{x \to a} f(x) = f(a)$. Thus,

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a}\ \big(f(x)-f(a)+f(a)\big)\\ &= \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\cdot(x-a)+f(a)\right) \qquad \text{Multiply and divide } (f(x)-f(a)) \text{ by } x-a.\\ &= \left(\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\right)\cdot\Big(\lim_{x\to a}(x-a)\Big)+\lim_{x\to a} f(a)\\ &= f'(a)\cdot 0+f(a)\\ &= f(a).\end{aligned}$$

Therefore, since $f(a)$ is defined and $\lim_{x\to a} f(x)=f(a)$, we conclude that f is continuous at a.

□

We have just proven that differentiability implies continuity, but now we consider whether continuity implies differentiability. To determine an answer to this question, we examine the function $f(x)=|x|$. This function is continuous everywhere; however, $f'(0)$ is undefined. This observation leads us to believe that continuity does not imply differentiability. Let's explore further. For $f(x)=|x|$,

$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{|x|-|0|}{x-0}=\lim_{x\to 0}\frac{|x|}{x}.$$

This limit does not exist because

$$\lim_{x\to 0^-}\frac{|x|}{x}=-1 \text{ and } \lim_{x\to 0^+}\frac{|x|}{x}=1.$$

See Figure 3.2.4.

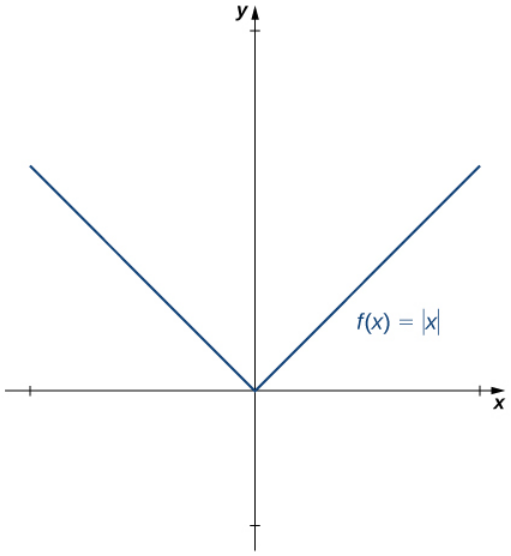

Figure 3.2.4: The function $f(x)=|x|$ is continuous at 0 but is not differentiable at 0.

Let's consider some additional situations in which a continuous function fails to be differentiable. Consider the function $f(x)=\sqrt[3]{x}$:

$$f'(0)=\lim_{x\to 0}\frac{\sqrt[3]{x}-0}{x-0}=\lim_{x\to 0}\frac{1}{\sqrt[3]{x^2}}=+\infty.$$

Thus $f'(0)$ does not exist. A quick look at the graph of $f(x)=\sqrt[3]{x}$ clarifies the situation. The function has a vertical tangent line at 0 (Figure 3.2.5).

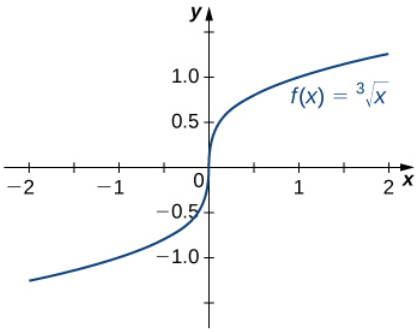

Figure 3.2.5: The function $f(x) = \sqrt[3]{x}$ has a vertical tangent at $x = 0$. It is continuous at 0 but is not differentiable at 0.

The function $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ also has a derivative that exhibits interesting behavior at 0.

We see that

$$f'(0) = \lim_{x \to 0} \frac{x \sin(1/x) - 0}{x - 0} = \lim_{x \to 0} \sin\left(\frac{1}{x}\right).$$

This limit does not exist, essentially because the slopes of the secant lines continuously change direction as they approach zero (Figure 3.2.6).

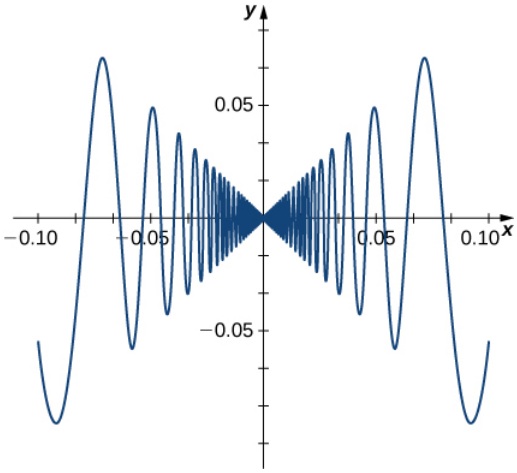

Figure 3.2.6: The function $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ is not differentiable at 0.

In summary:

1. We observe that if a function is not continuous, it cannot be differentiable, since every differentiable function must be continuous. However, if a function is continuous, it may still fail to be differentiable.
2. We saw that $f(x) = |x|$ failed to be differentiable at 0 because the limit of the slopes of the tangent lines on the left and right were not the same. Visually, this resulted in a sharp corner on the graph of the function at 0. From this we conclude that in order to be differentiable at a point, a function must be "smooth" at that point.
3. As we saw in the example of $f(x) = \sqrt[3]{x}$, a function fails to be differentiable at a point where there is a vertical tangent line.
4. As we saw with $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ a function may fail to be differentiable at a point in more complicated ways as well.

Example 3.2.4: A Piecewise Function that is Continuous and Differentiable

A toy company wants to design a track for a toy car that starts out along a parabolic curve and then converts to a straight line (Figure 3.2.7). The function that describes the track is to have the form $f(x) = \begin{cases} \frac{1}{10}x^2 + bx + c, & \text{if } x < -10 \\ -\frac{1}{4}x + \frac{5}{2}, & \text{if } x \geq -10 \end{cases}$ where x and $f(x)$ are in inches. For the car to move smoothly along the track, the function $f(x)$ must be both continuous and differentiable at -10. Find values of b and c that make $f(x)$ both continuous and differentiable.

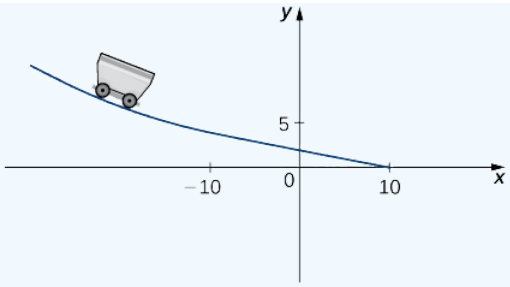

Figure 3.2.7: For the car to move smoothly along the track, the function must be both continuous and differentiable.

Solution

For the function to be continuous at $x = -10$, $\lim\limits_{x\to 10^-} f(x) = f(-10)$. Thus, since

$$\lim_{x\to -10^-} f(x) = \frac{1}{10}(-10)^2 - 10b + c = 10 - 10b + c$$

and $f(-10) = 5$, we must have $10 - 10b + c = 5$. Equivalently, we have $c = 10b - 5$.

For the function to be differentiable at -10,

$$f'(10) = \lim_{x\to -10} \frac{f(x) - f(-10)}{x + 10}$$

must exist. Since $f(x)$ is defined using different rules on the right and the left, we must evaluate this limit from the right and the left and then set them equal to each other:

$$\begin{aligned}
\lim_{x\to -10^-} \frac{f(x) - f(-10)}{x + 10} &= \lim_{x\to -10^-} \frac{\frac{1}{10}x^2 + bx + c - 5}{x + 10} \\
&= \lim_{x\to -10^-} \frac{\frac{1}{10}x^2 + bx + (10b - 5) - 5}{x + 10} && \text{Substitute } c = 10b - 5. \\
&= \lim_{x\to -10^-} \frac{x^2 - 100 + 10bx + 100b}{10(x + 10)} \\
&= \lim_{x\to -10^-} \frac{(x + 10)(x - 10 + 10b)}{10(x + 10)} && \text{Factor by grouping} \\
&= b - 2
\end{aligned}$$

We also have

$$\begin{aligned}
\lim_{x\to -10^+} \frac{f(x) - f(-10)}{x + 10} &= \lim_{x\to -10^+} \frac{-\frac{1}{4}x + \frac{5}{2} - 5}{x + 10} \\
&= \lim_{x\to -10^+} \frac{-(x + 10)}{4(x + 10)} \\
&= -\frac{1}{4}
\end{aligned}$$

This gives us $b - 2 = -\frac{1}{4}$. Thus $b = \frac{7}{4}$ and $c = 10(\frac{7}{4}) - 5 = \frac{25}{2}$.

Exercise 3.2.3

Find values of a and b that make $f(x) = \begin{cases} ax + b, & \text{if } x < 3 \\ x^2, & \text{if } x \geq 3 \end{cases}$ both continuous and differentiable at 3.

Hint

Use Example 3.2.4 as a guide.

Answer

$a = 6$ and $b = -9$

Higher-Order Derivatives

The derivative of a function is itself a function, so we can find the derivative of a derivative. For example, the derivative of a position function is the rate of change of position, or velocity. The derivative of velocity is the rate of change of velocity, which is acceleration. The new function obtained by differentiating the derivative is called the second derivative. Furthermore, we can continue to take derivatives to obtain the third derivative, fourth derivative, and so on. Collectively, these are referred to as **higher-order derivatives**. The notation for the higher-order derivatives of $y = f(x)$ can be expressed in any of the following forms:

$$f''(x),\ f'''(x),\ f^{(4)}(x),\ \ldots\ ,\ f^{(n)}(x)$$

$$y''(x),\ y'''(x),\ y^{(4)}(x),\ \ldots\ ,\ y^{(n)}(x)$$

$$\frac{d^2y}{dx^2},\ \frac{d^3y}{dy^3},\ \frac{d^4y}{dy^4},\ \ldots\ ,\ \frac{d^ny}{dy^n}.$$

It is interesting to note that the notation for $\frac{d^2y}{dx^2}$ may be viewed as an attempt to express $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ more compactly.

Analogously, $\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{dy}{dx}\right)\right) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$.

Example 3.2.5: Finding a Second Derivative

For $f(x) = 2x^2 - 3x + 1$, find $f''(x)$.

Solution

First find $f'(x)$.

Substitute $f(x) = 2x^2 - 3x + 1$ and $f(x+h) = 2(x+h)^2 - 3(x+h) + 1$ into $f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$.

$f'(x) = \lim_{h\to 0}\frac{(2(x+h)^2 - 3(x+h) + 1) - (2x^2 - 3x + 1)}{h}$	
$= \lim_{h\to 0}\frac{4xh + 2h^2 - 3h}{h}$	Simplify the numerator.
$= \lim_{h\to 0}(4x + 2h - 3)$	Factor out the h in the numerator and cancel with the h in the denominator.
$= 4x - 3$	Take the limit.

Next, find $f''(x)$ by taking the derivative of $f'(x) = 4x - 3$.

$f''(x) = \lim_{h\to 0}\frac{f'(x+h) - f'(x)}{h}$	Use $f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$ with $f'(x)$ in place of $f(x)$.
$= \lim_{h\to 0}\frac{(4(x+h) - 3) - (4x - 3)}{h}$	Substitute $f'(x+h) = 4(x+h) - 3$ and $f'(x) = 4x - 3$.
$= \lim_{h\to 0} 4$	Simplify.
$= 4$	Take the limit.

Exercise 3.2.4

Find $f''(x)$ for $f(x) = x^2$.

Hint

We found $f'(x) = 2x$ in a previous checkpoint. Use Equation 3.2.1 to find the derivative of $f'(x)$

Answer

$f''(x) = 2$

Example 3.2.6: Finding Acceleration

The position of a particle along a coordinate axis at time t (in seconds) is given by $s(t) = 3t^2 - 4t + 1$ (in meters). Find the function that describes its acceleration at time t.

Solution

Since $v(t) = s'(t)$ and $a(t) = v'(t) = s''(t)$, we begin by finding the derivative of $s(t)$:

$$\begin{aligned} s'(t) &= \lim_{h \to 0} \frac{s(t+h) - s(t)}{h} \\ &= \lim_{h \to 0} \frac{3(t+h)^2 - 4(t+h) + 1 - (3t^2 - 4t + 1)}{h} \\ &= 6t - 4. \end{aligned}$$

Next,

$$\begin{aligned} s''(t) &= \lim_{h \to 0} \frac{s'(t+h) - s'(t)}{h} \\ &= \lim_{h \to 0} \frac{6(t+h) - 4 - (6t - 4)}{h} \\ &= 6. \end{aligned}$$

Thus, $a = 6 \text{ m/s}^2$.

Exercise 3.2.5

For $s(t) = t^3$, find $a(t)$.

Hint

Use Example 3.2.6 as a guide.

Answer

$a(t) = 6t$

Key Concepts

- The derivative of a function $f(x)$ is the function whose value at x is $f'(x)$.
- The graph of a derivative of a function $f(x)$ is related to the graph of $f(x)$. Where $f(x)$ has a tangent line with positive slope, $f'(x) > 0$. Where $f(x)$ has a tangent line with negative slope, $f'(x) < 0$. Where $f(x)$ has a horizontal tangent line, $f'(x) = 0$.
- If a function is differentiable at a point, then it is continuous at that point. A function is not differentiable at a point if it is not continuous at the point, if it has a vertical tangent line at the point, or if the graph has a sharp corner or cusp.
- Higher-order derivatives are derivatives of derivatives, from the second derivative to the nth derivative.

Key Equations

- **The derivative function**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Glossary

derivative function

gives the derivative of a function at each point in the domain of the original function for which the derivative is defined

differentiable at a

a function for which $f'(a)$ exists is differentiable at a

differentiable on S

a function for which $f'(x)$ exists for each x in the open set S is differentiable on S

differentiable function

a function for which $f'(x)$ exists is a differentiable function

higher-order derivative

a derivative of a derivative, from the second derivative to the nth derivative, is called a higher-order derivative

- Gilbert Strang (MIT) and Edwin "Jed" Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.
- Paul Seeburger (Monroe Community College) added explanation of the alternative definition of the derivative used in the proof of that differentiability implies continuity.

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3.2E: EXERCISES FOR SECTION 3.2

For exercises 1 - 10, use the definition of a derivative to find $f'(x)$.

1) $f(x) = 6$

2) $f(x) = 2 - 3x$

Answer

$f'(x) = -3$

3) $f(x) = \frac{2x}{7} + 1$

4) $f(x) = 4x^2$

Answer

$f'(x) = 8x$

5) $f(x) = 5x - x^2$

6) $f(x) = \sqrt{2x}$

Answer

$f'(x) = \frac{1}{\sqrt{2x}}$

7) $f(x) = \sqrt{x - 6}$

8) $f(x) = \frac{9}{x}$

Answer

$f'(x) = \frac{-9}{x^2}$

9) $f(x) = x + \frac{1}{x}$

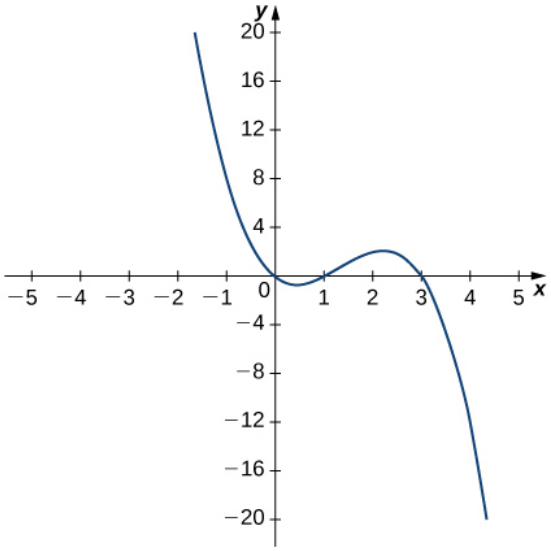
10) $f(x) = \frac{1}{\sqrt{x}}$

Answer

$f'(x) = \frac{-1}{2x^{3/2}}$

For the exercises 11 - 14, use the graph of $y = f(x)$ to sketch the graph of its derivative $f'(x)$.

11)



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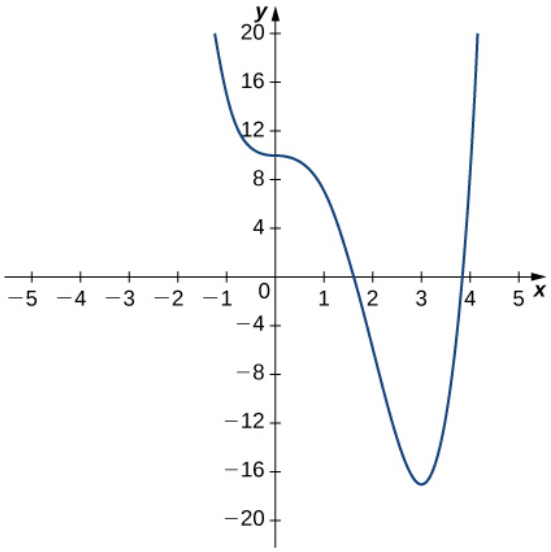
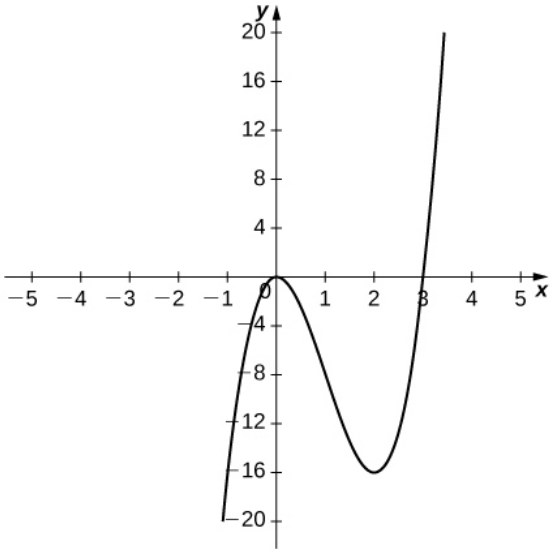


Answer



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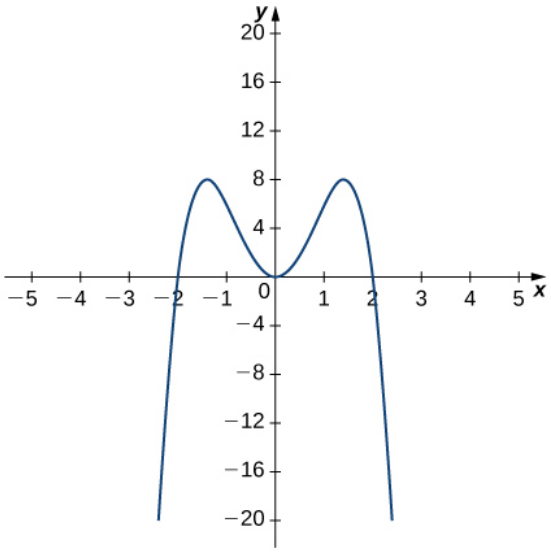
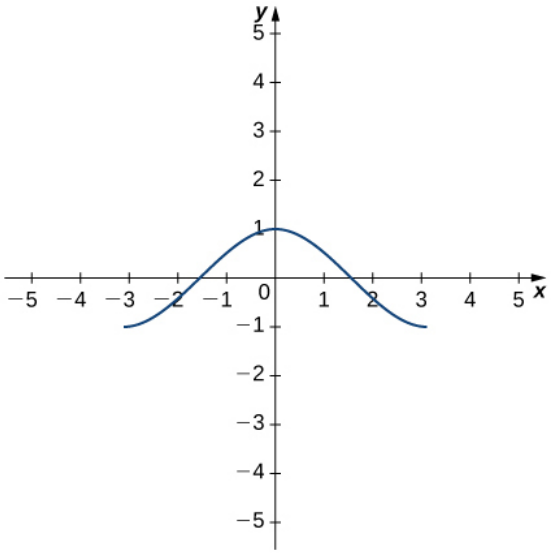
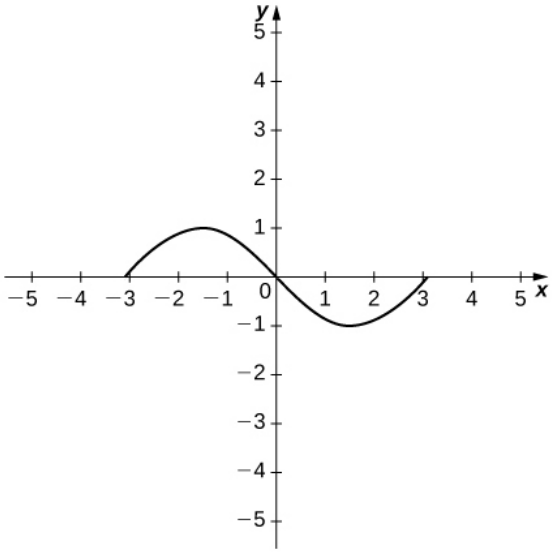

y
20
16
12
8
4
−5 −4 −3 −2 −1 0 1 2 3 4 5 x
−4
−8
−12
−16
−20

14)


y
5
4
3
2
1
−5 −4 −3 −2 −1 0 1 2 3 4 5 x
−1
−2
−3
−4
−5

Answer


y
5
4
3
2
1
−5 −4 −3 −2 −1 0 1 2 3 4 5 x
−1
−2
−3
−4
−5

For exercises 15 - 20, the given limit represents the derivative of a function $y = f(x)$ at $x = a$. Find $f(x)$ and a.

15) $\lim\limits_{h \to 0} \dfrac{(1+h)^{2/3} - 1}{h}$

16) $\lim\limits_{h \to 0} \dfrac{[3(2+h)^2 + 2] - 14}{h}$

Answer

$f(x) = 3x^2 + 2, \quad a = 2$

17) $\lim\limits_{h \to 0} \dfrac{\cos(\pi + h) + 1}{h}$

18) $\lim\limits_{h \to 0} \dfrac{(2+h)^4 - 16}{h}$

Answer

$f(x) = x^4, \quad a = 2$

19) $\lim\limits_{h \to 0} \dfrac{[2(3+h)^2 - (3+h)] - 15}{h}$

20) $\lim\limits_{h \to 0} \dfrac{e^h - 1}{h}$

Answer

$f(x) = e^x, \quad a = 0$

For the functions in exercises 21 - 24,

a. sketch the graph and

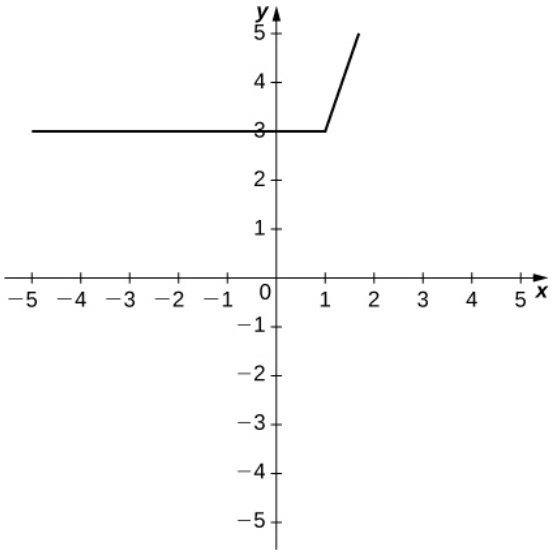
b. use the definition of a derivative to show that the function is not differentiable at $x = 1$.

21) $f(x) = \begin{cases} 2\sqrt{x}, & \text{if } 0 \le x \le 1 \\ 3x - 1, & \text{if } x > 1 \end{cases}$

22) $f(x) = \begin{cases} 3, & \text{if } x < 1 \\ 3x, & \text{if } x \ge 1 \end{cases}$

Answer

a.



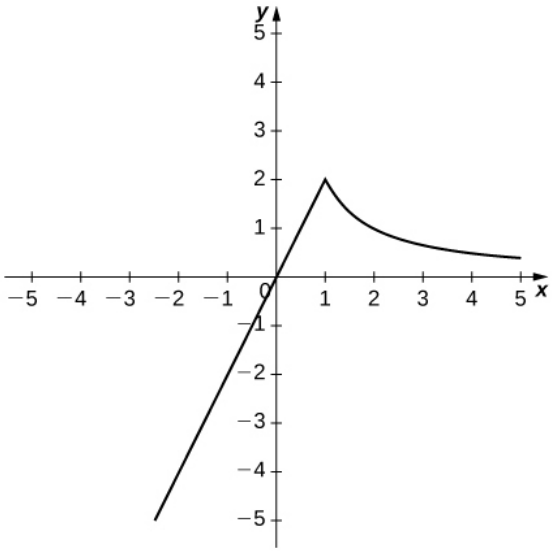
b. $\lim\limits_{h \to 1^-} \dfrac{3-3}{h} \ne \lim\limits_{h \to 1^+} \dfrac{3h}{h}$

23) $f(x) = \begin{cases} -x^2 + 2, & \text{if } x \le 1 \\ x, & \text{if } x > 1 \end{cases}$

24) $f(x) = \begin{cases} 2x, & \text{if } x \le 1 \\ \frac{2}{x}, & \text{if } x > 1 \end{cases}$

Answer

a.



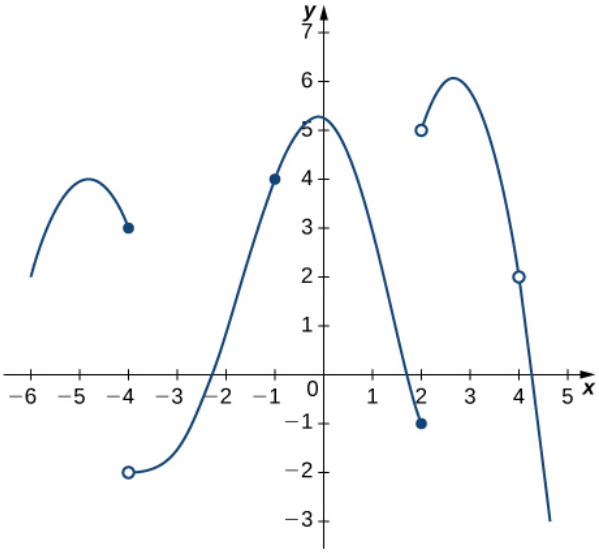
b. $\lim_{h\to 1^-} \frac{2h}{h} \neq \lim_{h\to 1^+} \frac{\frac{2}{x+h} - 2x}{h}$.

For the graphs in exercises 25 - 26,

a. determine for which values of $x = a$ the $\lim_{x\to a} f(x)$ exists but f is not continuous at $x = a$, and

b. determine for which values of $x = a$ the function is continuous but not differentiable at $x = a$.

25)



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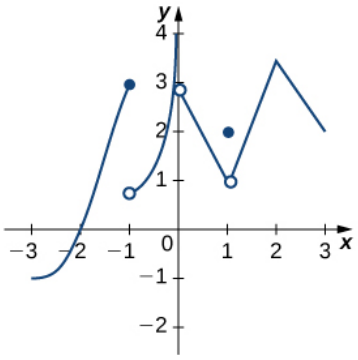
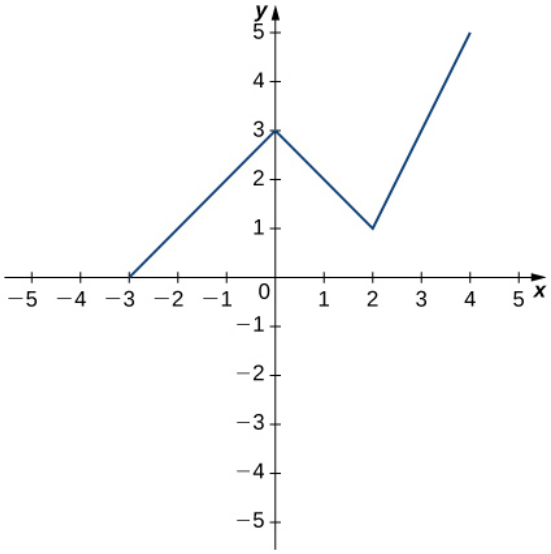


Answer

a. $x = 1$,

b. $x = 2$

27) Use the graph to evaluate a. $f'(-0.5)$, b. $f'(0)$, c. $f'(1)$, d. $f'(2)$, and e. $f'(3)$, if it exists.



For the functions in exercises 28 - 30, use $f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$ to find $f''(x)$.

28) $f(x) = 2 - 3x$

Answer

$f''(x) = 0$

29) $f(x) = 4x^2$

30) $f(x) = x + \frac{1}{x}$

Answer

$f''(x) = \frac{2}{x^3}$

For exercises 31 - 36, use a calculator to graph $f(x)$. Determine the function $f'(x)$, then use a calculator to graph $f'(x)$.

31) [T] $f(x) = -\frac{5}{x}$

32) [T] $f(x) = 3x^2 + 2x + 4$.

Answer

$f'(x) = 6x + 2$

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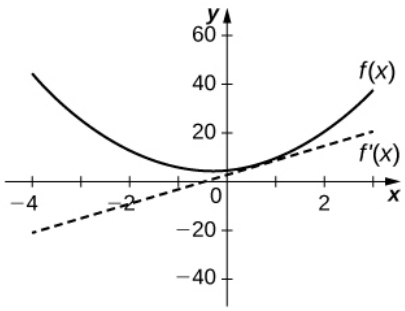
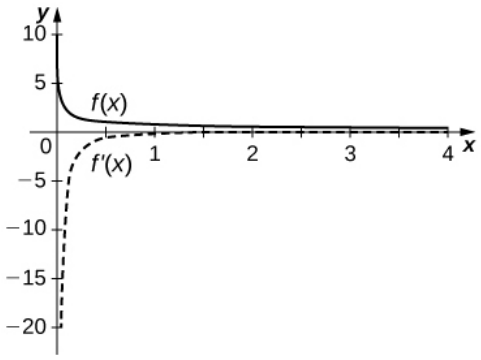


33) [T] $f(x) = \sqrt{x} + 3x$

34) [T] $f(x) = \dfrac{1}{\sqrt{2x}}$

Answer

$f'(x) = -\dfrac{1}{(2x)^{3/2}}$

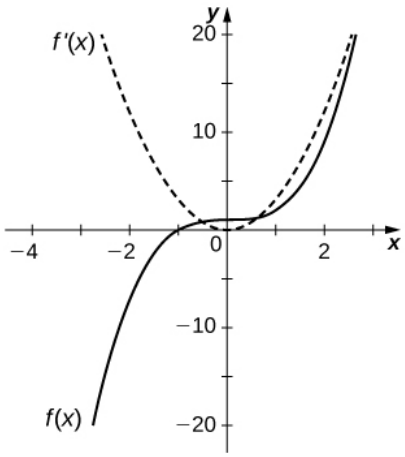


35) [T] $f(x) = 1 + x + \dfrac{1}{x}$

36) [T] $f(x) = x^3 + 1$

Answer

$f'(x) = 3x^2$



For exercises 37 - 42, describe what the two expressions represent in terms of each of the given situations. Be sure to include units.

a. $\dfrac{f(x+h) - f(x)}{h}$

b. $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

37) $P(x)$ denotes the population of a city at time x in years.

38) $C(x)$ denotes the total amount of money (in thousands of dollars) spent on concessions by x customers at an amusement park.

Answer

a. Average rate at which customers spent on concessions in thousands per customer.

b. Rate (in thousands per customer) at which x customers spent money on concessions in thousands per customer.

39) $R(x)$ denotes the total cost (in thousands of dollars) of manufacturing x clock radios

40) $g(x)$ denotes the grade (in percentage points) received on a test, given x hours of studying.

Answer

a. Average grade received on the test with an average study time between two values.

b. Rate (in percentage points per hour) at which the grade on the test increased or decreased for a given average study time of x hours.

41) $B(x)$denotes the cost (in dollars) of a sociology textbook at university bookstores in the United States in x years since 1990.

42) $p(x)$ denotes atmospheric pressure at an altitude of x feet.

Answer

a. Average change of atmospheric pressure between two different altitudes.

b. Rate (torr per foot) at which atmospheric pressure is increasing or decreasing at x feet.

43) Sketch the graph of a function $y = f(x)$ with all of the following properties:

a. $f'(x) > 0$ for $-2 \le x < 1$

b. $f'(2) = 0$

c. $f'(x) > 0$ for $x > 2$

d. $f(2) = 2$ and $f(0) = 1$

e. $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to \infty} f(x) = \infty$

f. $f'(1)$ does not exist.

44) Suppose temperature T in degrees Fahrenheit at a height x in feet above the ground is given by $y = T(x)$.

a. Give a physical interpretation, with units, of $T'(x)$.

b. If we know that $T'(1000) = -0.1$, explain the physical meaning.

Answer

a. The rate (in degrees per foot) at which temperature is increasing or decreasing for a given height x.

b. The rate of change of temperature as altitude changes at 1000 feet is -0.1 degrees per foot.

45) Suppose the total profit of a company is $y = P(x)$ thousand dollars when x units of an item are sold.

a. What does $\dfrac{P(b) - P(a)}{b - a}$ for $0 < a < b$ measure, and what are the units?

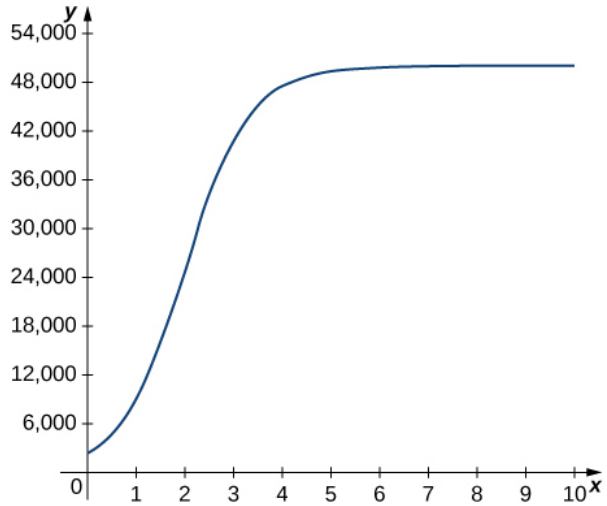
b. What does $P'(x)$ measure, and what are the units?

c. Suppose that $P'(30) = 5$, what is the approximate change in profit if the number of items sold increases from 30 to 31?

46) The graph in the following figure models the number of people $N(t)$ who have come down with the flu t weeks after its initial outbreak in a town with a population of 50,000 citizens.

a. Describe what $N'(t)$ represents and how it behaves as t increases.

b. What does the derivative tell us about how this town is affected by the flu outbreak?



Answer

a. The rate at which the number of people who have come down with the flu is changing t weeks after the initial outbreak.

b. The rate is increasing sharply up to the third week, at which point it slows down and then becomes constant.

For exercises 47 - 52, use the following table, which shows the height h of the Saturn V rocket for the Apollo 11 mission t seconds after launch.

Time (seconds)	Height (meters)
0	0
1	2
2	4
3	13
4	25
5	32

47) What is the physical meaning of $h'(t)$? What are the units?

48) [T] Construct a table of values for $h'(t)$ and graph both $h(t)$ and $h'(t)$ on the same graph. (Hint: for interior points, estimate both the left limit and right limit and average them.)

Answer

Time (seconds)	$h'(t)$ (m/s)
0	2
1	2
2	5.5
3	10.5
4	9.5
5	7

49) [T] The best linear fit to the data is given by $H(t) = 7.229t - 4.905$, where H is the height of the rocket (in meters) and t is the time elapsed since takeoff. From this equation, determine $H'(t)$. Graph $H(t$ with the given data and, on a separate coordinate plane, graph $H'(t)$.

50) [T] The best quadratic fit to the data is given by $G(t) = 1.429t^2 + 0.0857t - 0.1429$, where G is the height of the rocket (in meters) and t is the time elapsed since takeoff. From this equation, determine $G'(t)$. Graph $G(t)$ with the given data and, on a separate coordinate plane, graph $G'(t)$.

Answer

$G'(t) = 2.858t + 0.0857$

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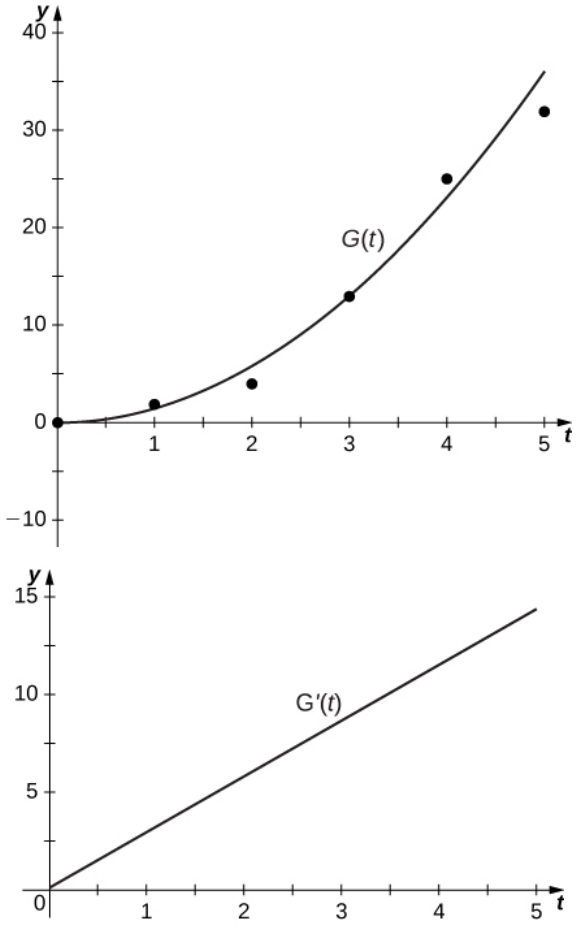

51) [T] The best cubic fit to the data is given by $F(t) = 0.2037t^3 + 2.956t^2 - 2.705t + 0.4683$, where F is the height of the rocket (in m) and t is the time elapsed since take off. From this equation, determine $F'(t)$. Graph $F(t)$ with the given data and, on a separate coordinate plane, graph $F'(t)$. Does the linear, quadratic, or cubic function fit the data best?

52) Using the best linear, quadratic, and cubic fits to the data, determine what $H''(t)$, $G''(t)$ and $F''(t)$ are. What are the physical meanings of $H''(t)$, $G''(t)$ and $F''(t)$, and what are their units?

Answer

$H''(t) = 0$, $G''(t) = 2.858$ and $f''(t) = 1.222t + 5.912$ represent the acceleration of the rocket, with units of meters per second squared (m/s^2).

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3.3: Differentiation Rules

Learning Objectives

- State the constant, constant multiple, and power rules.
- Apply the sum and difference rules to combine derivatives.
- Use the product rule for finding the derivative of a product of functions.
- Use the quotient rule for finding the derivative of a quotient of functions.
- Extend the power rule to functions with negative exponents.
- Combine the differentiation rules to find the derivative of a polynomial or rational function.

Finding derivatives of functions by using the definition of the derivative can be a lengthy and, for certain functions, a rather challenging process. For example, previously we found that

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$$

by using a process that involved multiplying an expression by a conjugate prior to evaluating a limit.

The process that we could use to evaluate $\frac{d}{dx}\left(\sqrt[3]{x}\right)$ using the definition, while similar, is more complicated.

In this section, we develop rules for finding derivatives that allow us to bypass this process. We begin with the basics.

The Basic Rules

The functions $f(x) = c$ and $g(x) = x^n$ where n is a positive integer are the building blocks from which all polynomials and rational functions are constructed. To find derivatives of polynomials and rational functions efficiently without resorting to the limit definition of the derivative, we must first develop formulas for differentiating these basic functions.

The Constant Rule

We first apply the limit definition of the derivative to find the derivative of the constant function, $f(x) = c$. For this function, both $f(x) = c$ and $f(x+h) = c$, so we obtain the following result:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{c-c}{h} \\ &= \lim_{h\to 0} \frac{0}{h} \\ &= \lim_{h\to 0} 0 = 0. \end{aligned}$$

The rule for differentiating constant functions is called the **constant rule**. It states that the derivative of a constant function is zero; that is, since a constant function is a horizontal line, the slope, or the rate of change, of a constant function is 0. We restate this rule in the following theorem.

The Constant Rule

Let c be a constant. If $f(x) = c$, then $f'(x) = 0$.

Alternatively, we may express this rule as

$$\frac{d}{dx}(c) = 0.$$

Example 3.3.1: Applying the Constant Rule

Find the derivative of $f(x) = 8$.

Solution

This is just a one-step application of the rule: $f'(8) = 0$.

Exercise 3.3.1

Find the derivative of $g(x) = -3$.

Hint

Use the preceding example as a guide

Answer

0

The Power Rule

We have shown that

$$\frac{d}{dx}\left(x^2\right) = 2x \quad \text{and} \quad \frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2}x^{-1/2}.$$

At this point, you might see a pattern beginning to develop for derivatives of the form $\frac{d}{dx}\left(x^n\right)$. We continue our examination of derivative formulas by differentiating power functions of the form $f(x) = x^n$ where n is a positive integer. We develop formulas for derivatives of this type of function in stages, beginning with positive integer powers. Before stating and proving the general rule for derivatives of functions of this form, we take a look at a specific case, $\frac{d}{dx}(x^3)$. As we go through this derivation, pay special attention to the portion of the expression in boldface, as the technique used in this case is essentially the same as the technique used to prove the general case.

Example 3.3.2: Differentiating x^3

Find $\frac{d}{dx}\left(x^3\right)$.

Solution:

$\frac{d}{dx}\left(x^3\right) = \lim_{h\to 0}\frac{(x+h)^3 - x^3}{h}$	
$= \lim_{h\to 0}\frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$	Notice that the first term in the expansion of $(x+h)^3$ is x^3 and the second term is $3x^2h$. All other terms contain powers of h that are two or greater
$= \lim_{h\to 0}\frac{3x^2h + 3xh^2 + h^3}{h}$	In this step the x^3 terms have been cancelled, leaving only terms containing h.
$= \lim_{h\to 0}\frac{h(3x^2 + 3xh + h^2)}{h}$	Factor out the common factor of h.
$= \lim_{h\to 0}(3x^2 + 3xh + h^2)$	After cancelling the common factor of h,the only term not containing h is $3x^2$.
$= 3x^2$	Let h go to 0.

Exercise 3.3.2

Find $\frac{d}{dx}\left(x^4\right)$.

Hint

Use $(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$ and follow the procedure outlined in the preceding example.

Answer

$\frac{d}{dx}\left(x^4\right) = 4x^3$

As we shall see, the procedure for finding the derivative of the general form $f(x) = x^n$ is very similar. Although it is often unwise to draw general conclusions from specific examples, we note that when we differentiate $f(x) = x^3$, the power on x becomes the coefficient of x^2 in the derivative and the power on x in the derivative decreases by 1. The following theorem states that the **power rule** holds for all positive integer powers of x. We will eventually extend this result to negative integer powers. Later, we will see that this rule may also be extended first to rational powers of x and then to arbitrary powers of x. Be aware, however, that this rule does not apply to functions in which a constant is raised to a variable power, such as $f(x) = 3^x$.

The Power Rule

Let n be a positive integer. If $f(x) = x^n$,then

$$f'(x) = nx^{n-1}.$$

Alternatively, we may express this rule as

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1.}$$

Proof

For $f(x) = x^n$ where n is a positive integer, we have

$$f'(x) = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}.$$

Since

$$(x+h)^n = x^n + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \ldots + nxh^{n-1} + h^n,$$

we see that

$$(x+h)^n - x^n = nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \ldots + nxh^{n-1} + h^n.$$

Next, divide both sides by h:

$$\frac{(x+h)^n - x^n}{h} = \frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \ldots + nxh^{n-1} + h^n}{h}.$$

Thus,

$$\frac{(x+h)^n - x^n}{h} = nx^{n-1} + \binom{n}{2}x^{n-2}h + \binom{n}{3}x^{n-3}h^2 + \ldots + nxh^{n-2} + h^{n-1}.$$

Finally,

$$f'(x) = \lim_{h\to 0}(nx^{n-1} + \binom{n}{2}x^{n-2}h + \binom{n}{3}x^{n-3}h^2 + \ldots + nxh^{n-2} + h^{n-1})$$

$$= nx^{n-1}.$$

□

Example 3.3.3: Applying the Power Rule

Find the derivative of the function $f(x) = x^{10}$ by applying the power rule.

Solution

Using the power rule with $n = 10$, we obtain

$$f'(x) = 10x^{10-1} = 10x^9.$$

Exercise 3.3.3

Find the derivative of $f(x) = x^7$.

Hint

Use the power rule with $n = 7$.

Answer

$f'(x) = 7x^6$

The Sum, Difference, and Constant Multiple Rules

We find our next differentiation rules by looking at derivatives of sums, differences, and constant multiples of functions. Just as when we work with functions, there are rules that make it easier to find derivatives of functions that we add, subtract, or multiply by a constant. These rules are summarized in the following theorem.

Sum, Difference, and Constant Multiple Rules

Let $f(x)$ and $g(x)$ be differentiable functions and k be a constant. Then each of the following equations holds.

Sum Rule. The derivative of the sum of a function f and a function g is the same as the sum of the derivative of f and the derivative of g.

$$\frac{d}{dx}\big(f(x) + g(x)\big) = \frac{d}{dx}\big(f(x)\big) + \frac{d}{dx}\big(g(x)\big);$$

that is,

$$\text{for } s(x) = f(x) + g(x), \quad s'(x) = f'(x) + g'(x).$$

Difference Rule. The derivative of the difference of a function f and a function g is the same as the difference of the derivative of f and the derivative of g :

$$\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x));$$

that is,

$$\text{for } d(x) = f(x) - g(x), \quad d'(x) = f'(x) - g'(x).$$

Constant Multiple Rule. The derivative of a constant k multiplied by a function f is the same as the constant multiplied by the derivative:

$$\frac{d}{dx}\big(kf(x)\big) = k\frac{d}{dx}\big(f(x)\big);$$

that is,

$$\text{for } m(x) = kf(x), \quad m'(x) = kf'(x).$$

Proof

We provide only the proof of the sum rule here. The rest follow in a similar manner.

For differentiable functions $f(x)$ and $g(x)$, we set $s(x) = f(x) + g(x)$. Using the limit definition of the derivative we have

$$s'(x) = \lim_{h \to 0} \frac{s(x+h) - s(x)}{h}.$$

By substituting $s(x+h) = f(x+h) + g(x+h)$ and $s(x) = f(x) + g(x)$, we obtain

$$s'(x) = \lim_{h \to 0} \frac{\big(f(x+h) + g(x+h)\big) - \big(f(x) + g(x)\big)}{h}.$$

Rearranging and regrouping the terms, we have

$$s'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right).$$

We now apply the sum law for limits and the definition of the derivative to obtain

$$s'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f'(x) + g'(x).$$

□

Example 3.3.4: Applying the Constant Multiple Rule

Find the derivative of $g(x) = 3x^2$ and compare it to the derivative of $f(x) = x^2$.

Solution

We use the power rule directly:

$$g'(x) = \frac{d}{dx}(3x^2) = 3\frac{d}{dx}(x^2) = 3(2x) = 6x.$$

Since $f(x) = x^2$ has derivative $f'(x) = 2x$, we see that the derivative of $g(x)$ is 3 times the derivative of $f(x)$. This relationship is illustrated in Figure 3.3.1.

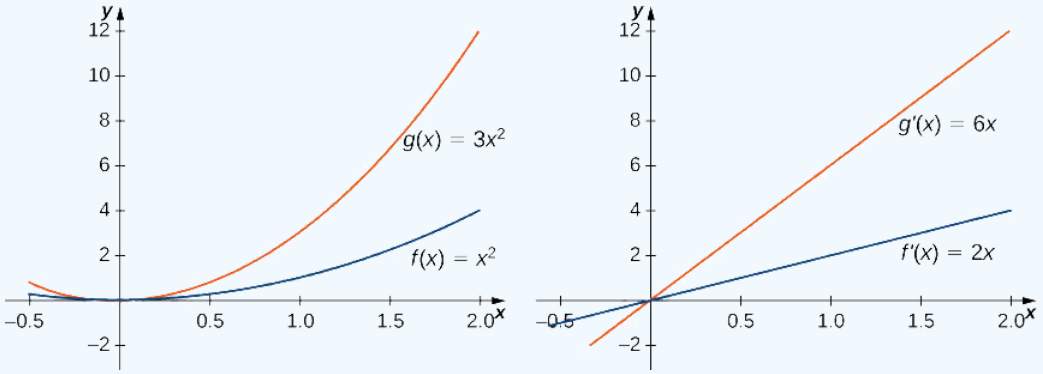

Figure 3.3.1: The derivative of $g(x)$ is 3 times the derivative of $f(x)$.

Example 3.3.5: Applying Basic Derivative Rules

Find the derivative of $f(x) = 2x^5 + 7$.

Solution

We begin by applying the rule for differentiating the sum of two functions, followed by the rules for differentiating constant multiples of functions and the rule for differentiating powers. To better understand the sequence in which the differentiation rules are applied, we use Leibniz notation throughout the solution:

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}\left(2x^5+7\right) \\
&= \frac{d}{dx}(2x^5)+\frac{d}{dx}(7) && \text{Apply the sum rule.} \\
&= 2\frac{d}{dx}(x^5)+\frac{d}{dx}(7) && \text{Apply the constant multiple rule.} \\
&= 2(5x^4)+0 && \text{Apply the power rule and the constant rule.} \\
&= 10x^4 && \text{Simplify.}
\end{aligned}
$$

Exercise 3.3.4

Find the derivative of $f(x)=2x^3-6x^2+3$.

Hint

Use the preceding example as a guide.

Answer

$f'(x)=6x^2-12x$.

Example 3.3.6: Finding the Equation of a Tangent Line

Find the equation of the line tangent to the graph of $f(x)=x^2-4x+6$ at $x=1$

Solution

To find the equation of the tangent line, we need a point and a slope. To find the point, compute

$$f(1)=1^2-4(1)+6=3.$$

This gives us the point $(1,3)$. Since the slope of the tangent line at 1 is $f'(1)$, we must first find $f'(x)$. Using the definition of a derivative, we have

$$f'(x)=2x-4$$

so the slope of the tangent line is $f'(1)=-2$. Using the point-slope formula, we see that the equation of the tangent line is

$$y-3=-2(x-1).$$

Putting the equation of the line in slope-intercept form, we obtain

$$y=-2x+5.$$

Exercise 3.3.5

Find the equation of the line tangent to the graph of $f(x)=3x^2-11$ at $x=2$. Use the point-slope form.

Hint

Use the preceding example as a guide.

Answer

$y=12x-23$

The Product Rule

Now that we have examined the basic rules, we can begin looking at some of the more advanced rules. The first one examines the derivative of the product of two functions. Although it might be tempting to assume that the derivative of the product is the product

of the derivatives, similar to the sum and difference rules, the **product rule** does not follow this pattern. To see why we cannot use this pattern, consider the function $f(x) = x^2$, whose derivative is $f'(x) = 2x$ and not $\frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1 \cdot 1 = 1$.

Product Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$\frac{d}{dx}(f(x)g(x)) = \frac{d}{dx}(f(x)) \cdot g(x) + \frac{d}{dx}(g(x)) \cdot f(x).$$

That is,

$$\text{if } p(x) = f(x)g(x), \quad \text{then } p'(x) = f'(x)g(x) + g'(x)f(x).$$

This means that the derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function.

Proof

We begin by assuming that $f(x)$ and $g(x)$ are differentiable functions. At a key point in this proof we need to use the fact that, since $g(x)$ is differentiable, it is also continuous. In particular, we use the fact that since $g(x)$ is continuous, $\lim_{h \to 0} g(x+h) = g(x)$.

By applying the limit definition of the derivative to $p(x) = f(x)g(x)$, we obtain

$$p'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}.$$

By adding and subtracting $f(x)g(x+h)$ in the numerator, we have

$$p'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}.$$

After breaking apart this quotient and applying the sum law for limits, the derivative becomes

$$p'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \lim_{h \to 0} \frac{f(x)g(x+h) - f(x)g(x)}{h}.$$

Rearranging, we obtain

$$\begin{aligned} p'(x) &= \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \cdot g(x+h) \right) + \lim_{h \to 0} \left(\frac{g(x+h) - g(x)}{h} \cdot f(x) \right) \\ &= \left(\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \right) \cdot \left(\lim_{h \to 0} g(x+h) \right) + \left(\lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \right) \cdot f(x) \end{aligned}$$

By using the continuity of $g(x)$, the definition of the derivatives of $f(x)$ and $g(x)$, and applying the limit laws, we arrive at the product rule,

$$p'(x) = f'(x)g(x) + g'(x)f(x).$$

□

Example 3.3.7: Applying the Product Rule to Constant Functions

For $p(x) = f(x)g(x)$, use the product rule to find $p'(2)$ if $f(2) = 3,\ f'(2) = -4,\ g(2) = 1$, and $g'(2) = 6$.

Solution

Since $p(x) = f(x)g(x)$, $p'(x) = f'(x)g(x) + g'(x)f(x)$, and hence

$p'(2) = f'(2)g(2) + g'(2)f(2) = (-4)(1) + (6)(3) = 14.$

Example 3.3.8: Applying the Product Rule to Binomials

For $p(x) = (x^2 + 2)(3x^3 - 5x)$, find $p'(x)$ by applying the product rule. Check the result by first finding the product and then differentiating.

Solution

If we set $f(x) = x^2 + 2$ and $g(x) = 3x^3 - 5x$, then $f'(x) = 2x$ and $g'(x) = 9x^2 - 5$. Thus,

$p'(x) = f'(x)g(x) + g'(x)f(x) = (2x)(3x^3 - 5x) + (9x^2 - 5)(x^2 + 2).$

Simplifying, we have

$$p'(x) = 15x^4 + 3x^2 - 10.$$

To check, we see that $p(x) = 3x^5 + x^3 - 10x$ and, consequently, $p'(x) = 15x^4 + 3x^2 - 10$.

Exercise 3.3.6

Use the product rule to obtain the derivative of $p(x) = 2x^5(4x^2 + x)$.

Hint

Set $f(x) = 2x^5$ and $g(x) = 4x^2 + x$ and use the preceding example as a guide.

Answer

$p'(x) = 10x^4(4x^2 + x) + (8x + 1)(2x^5) = 56x^6 + 12x^5$.

The Quotient Rule

Having developed and practiced the product rule, we now consider differentiating quotients of functions. As we see in the following theorem, the derivative of the quotient is not the quotient of the derivatives; rather, it is the derivative of the function in the numerator times the function in the denominator minus the derivative of the function in the denominator times the function in the numerator, all divided by the square of the function in the denominator. In order to better grasp why we cannot simply take the quotient of the derivatives, keep in mind that

$$\frac{d}{dx}(x^2) = 2x, \text{ not } \frac{\frac{d}{dx}(x^3)}{\frac{d}{dx}(x)} = \frac{3x^2}{1} = 3x^2.$$

The Quotient Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}(f(x)) \cdot g(x) - \frac{d}{dx}(g(x)) \cdot f(x)}{\big(g(x)\big)^2}.$$

That is, if

$$q(x) = \frac{f(x)}{g(x)}$$

then

$$q'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{\big(g(x)\big)^2}.$$

The proof of the quotient rule is very similar to the proof of the product rule, so it is omitted here. Instead, we apply this new rule for finding derivatives in the next example.

Example 3.3.9: Applying the Quotient Rule

Use the quotient rule to find the derivative of $q(x) = \dfrac{5x^2}{4x+3}$.

Solution

Let $f(x) = 5x^2$ and $g(x) = 4x + 3$. Thus, $f'(x) = 10x$ and $g'(x) = 4$.

Substituting into the quotient rule, we have

$$q'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2} = \frac{10x(4x+3) - 4(5x^2)}{(4x+3)^2}.$$

Simplifying, we obtain

$$q'(x) = \frac{20x^2 + 30x}{(4x+3)^2}$$

Exercise 3.3.7

Find the derivative of $h(x) = \dfrac{3x+1}{4x-3}$.

Hint

Apply the quotient rule with $f(x) = 3x + 1$ and $g(x) = 4x - 3$.

Answer

$h'(x) = -\dfrac{13}{(4x-3)^2}$.

It is now possible to use the quotient rule to extend the power rule to find derivatives of functions of the form x^k where k is a negative integer.

Extended Power Rule

If k is a negative integer, then

$$\frac{d}{dx}(x^k) = kx^{k-1}.$$

Proof

If k is a negative integer, we may set $n = -k$, so that n is a positive integer with $k = -n$. Since for each positive integer n,$x^{-n} = \dfrac{1}{x^n}$, we may now apply the quotient rule by setting $f(x) = 1$ and $g(x) = x^n$. In this case, $f'(x) = 0$ and $g'(x) = nx^{n-1}$. Thus,

$$\frac{d}{dx}(x^{-n}) = \frac{0(x^n) - 1(nx^{n-1})}{(x^n)^2}.$$

Simplifying, we see that

$$\begin{aligned}\frac{d}{dx}(x^{-n}) &= \frac{-nx^{n-1}}{x^{2n}} \\ &= -nx^{(n-1)-2n} \\ &= -nx^{-n-1}.\end{aligned}$$

Finally, observe that since $k = -n$, by substituting we have

$$\frac{d}{dx}(x^k) = kx^{k-1}.$$

□

Example 3.3.10: Using the Extended Power Rule

Find $\frac{d}{dx}(x^{-4})$.

Solution

By applying the extended power rule with $k = -4$, we obtain

$$\frac{d}{dx}(x^{-4}) = -4x^{-4-1} = -4x^{-5}.$$

Example 3.3.11: Using the Extended Power Rule and the Constant Multiple Rule

Use the extended power rule and the constant multiple rule to find $f(x) = \frac{6}{x^2}$.

Solution

It may seem tempting to use the quotient rule to find this derivative, and it would certainly not be incorrect to do so. However, it is far easier to differentiate this function by first rewriting it as $f(x) = 6x^{-2}$.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{6}{x^2}\right) = \frac{d}{dx}\left(6x^{-2}\right) && \text{Rewrite } \frac{6}{x^2} \text{ as } 6x^{-2}. \\ &= 6\frac{d}{dx}\left(x^{-2}\right) && \text{Apply the constant multiple rule.} \\ &= 6(-2x^{-3}) && \text{Use the extended power rule to differentiate } x^{-2}. \\ &= -12x^{-3} && \text{Simplify.} \end{aligned}$$

Exercise 3.3.8

Find the derivative of $g(x) = \frac{1}{x^7}$ using the extended power rule.

Hint

Rewrite $g(x) = \frac{1}{x^7} = x^{-7}$. Use the extended power rule with $k = -7$.

Answer

$g'(x) = -7x^{-8}$.

Combining Differentiation Rules

As we have seen throughout the examples in this section, it seldom happens that we are called on to apply just one differentiation rule to find the derivative of a given function. At this point, by combining the differentiation rules, we may find the derivatives of any polynomial or rational function. Later on we will encounter more complex combinations of differentiation rules. A good rule of thumb to use when applying several rules is to apply the rules in reverse of the order in which we would evaluate the function.

Example 3.3.12: Combining Differentiation Rules

For $k(x) = 3h(x) + x^2g(x)$, find $k'(x)$.

Solution: Finding this derivative requires the sum rule, the constant multiple rule, and the product rule.

$k'(x) = \frac{d}{dx}\left(3h(x) + x^2g(x)\right) = \frac{d}{dx}\left(3h(x)\right) + \frac{d}{dx}\left(x^2g(x)\right)$	Apply the sum rule.
$= 3\frac{d}{dx}\left(h(x)\right) + \left(\frac{d}{dx}(x^2)g(x) + \frac{d}{dx}(g(x))x^2\right)$	Apply the constant multiple rule to differentiate $3h(x)$ and the product rule to differentiate $x^2g(x)$.
$= 3h'(x) + 2xg(x) + g'(x)x^2$	

Example 3.3.13: Extending the Product Rule

For $k(x) = f(x)g(x)h(x)$, express $k'(x)$ in terms of $f(x), g(x), h(x)$, and their derivatives.

Solution

We can think of the function $k(x)$ as the product of the function $f(x)g(x)$ and the function $h(x)$. That is, $k(x) = (f(x)g(x)) \cdot h(x)$. Thus,

$$\begin{aligned} k'(x) &= \frac{d}{dx}\big(f(x)g(x)\big) \cdot h(x) + \frac{d}{dx}\big(h(x)\big) \cdot \big(f(x)g(x)\big) && \text{Apply the product rule to the product of } f(x)g(x) \text{ and } h(x). \\ &= \big(f'(x)g(x) + g'(x)f(x)\big)h(x) + h'(x)f(x)g(x) && \text{Apply the product rule to } f(x)g(x) \\ &= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x) && \text{Simplify.} \end{aligned}$$

Example 3.3.14: Combining the Quotient Rule and the Product Rule

For $h(x) = \dfrac{2x^3k(x)}{3x+2}$, find $h'(x)$.

Solution

$$\begin{aligned} h'(x) &= \frac{\frac{d}{dx}(2x^3k(x)) \cdot (3x+2) - \frac{d}{dx}(3x+2) \cdot (2x^3k(x))}{(3x+2)^2} && \text{Apply the quotient rule.} \\ &= \frac{(6x^2k(x) + k'(x) \cdot 2x^3)(3x+2) - 3(2x^3k(x))}{(3x+2)^2} && \text{Apply the product rule to find } \frac{d}{dx}(2x^3k(x)). \text{ Use } \frac{d}{dx}(3x+2) = 3. \\ &= \frac{-6x^3k(x) + 18x^3k(x) + 12x^2k(x) + 6x^4k'(x) + 4x^3k'(x)}{(3x+2)^2} && \text{Simplify} \end{aligned}$$

Exercise 3.3.9

Find $\dfrac{d}{dx}(3f(x) - 2g(x))$.

Hint

Apply the difference rule and the constant multiple rule.

Answer

$3f'(x) - 2g'(x)$.

Example 3.3.15: Determining Where a Function Has a Horizontal Tangent

Determine the values of x for which $f(x) = x^3 - 7x^2 + 8x + 1$ has a horizontal tangent line.

Solution

To find the values of x for which $f(x)$ has a horizontal tangent line, we must solve $f'(x) = 0$.

Since $f'(x) = 3x^2 - 14x + 8 = (3x-2)(x-4)$,

we must solve $(3x-2)(x-4) = 0$. Thus we see that the function has horizontal tangent lines at $x = \dfrac{2}{3}$ and $x = 4$ as shown in the following graph.

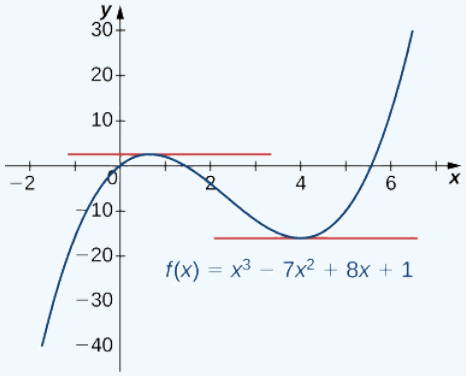

Figure 3.3.2: This function has horizontal tangent lines at $x = 2/3$ and $x = 4$.

Example 3.3.16: Finding a Velocity

The position of an object on a coordinate axis at time t is given by $s(t) = \dfrac{t}{t^2+1}$. What is the initial velocity of the object?

Solution

Since the initial velocity is $v(0) = s'(0)$, begin by finding $s'(t)$ by applying the quotient rule:

$$s'(t) = \frac{1(t^2+1) - 2t(t)}{(t^2+1)^2} = \frac{1-t^2}{(t^2+1)^2}.$$

After evaluating, we see that $v(0) = 1$.

Exercise 3.3.10

Find the values of x for which the line tangent to the graph of $f(x) = 4x^2 - 3x + 2$ has a tangent line parallel to the line $y = 2x + 3$.

Hint

Solve $f'(x) = 2$.

Answer

$\dfrac{5}{8}$

Formula One Grandstands

Formula One car races can be very exciting to watch and attract a lot of spectators. Formula One track designers have to ensure sufficient grandstand space is available around the track to accommodate these viewers. However, car racing can be dangerous, and safety considerations are paramount. The grandstands must be placed where spectators will not be in danger should a driver lose control of a car (Figure 3.3.3).

Figure 3.3.3: The grandstand next to a straightaway of the Circuit de Barcelona-Catalunya race track, located where the spectators are not in danger.

Safety is especially a concern on turns. If a driver does not slow down enough before entering the turn, the car may slide off the racetrack. Normally, this just results in a wider turn, which slows the driver down. But if the driver loses control completely, the car may fly off the track entirely, on a path tangent to the curve of the racetrack.

Suppose you are designing a new Formula One track. One section of the track can be modeled by the function $f(x) = x^3 + 3x^2 + x$ (Figure 3.3.4). The current plan calls for grandstands to be built along the first straightaway and around a portion of the first curve. The plans call for the front corner of the grandstand to be located at the point $(-1.9, 2.8)$. We want to determine whether this location puts the spectators in danger if a driver loses control of the car.

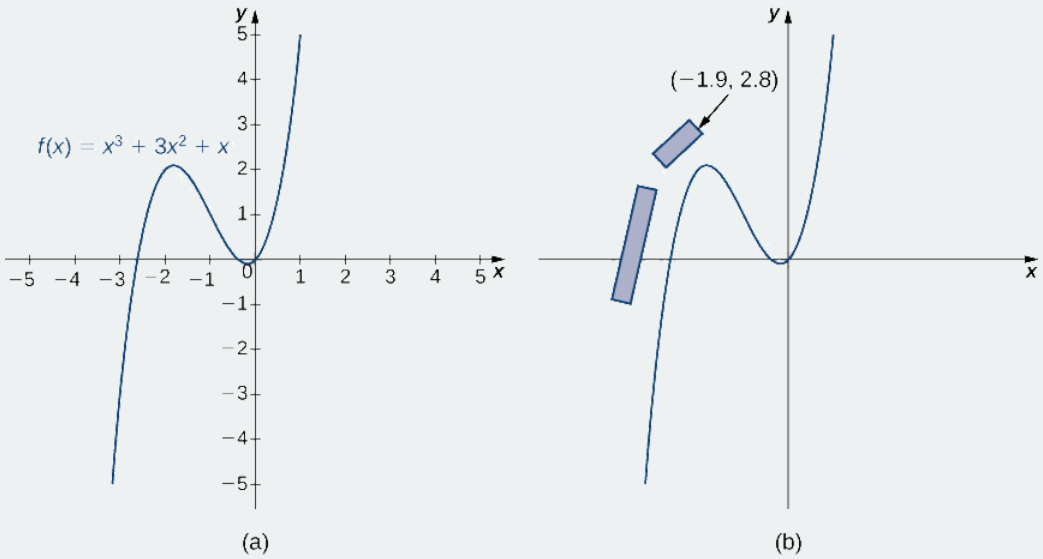

Figure 3.3.4: (a) One section of the racetrack can be modeled by the function $f(x) = x^3 + 3x^2 + x$. (b) The front corner of the grandstand is located at $(-1.9, 2.8)$.

1. Physicists have determined that drivers are most likely to lose control of their cars as they are coming into a turn, at the point where the slope of the tangent line is 1. Find the (x, y) coordinates of this point near the turn.
2. Find the equation of the tangent line to the curve at this point.
3. To determine whether the spectators are in danger in this scenario, find the x-coordinate of the point where the tangent line crosses the line $y = 2.8$. Is this point safely to the right of the grandstand? Or are the spectators in danger?
4. What if a driver loses control earlier than the physicists project? Suppose a driver loses control at the point $(-2.5, 0.625)$. What is the slope of the tangent line at this point?
5. If a driver loses control as described in part 4, are the spectators safe?
6. Should you proceed with the current design for the grandstand, or should the grandstands be moved?

Key Concepts

- The derivative of a constant function is zero.

- The derivative of a power function is a function in which the power on x becomes the coefficient of the term and the power on x in the derivative decreases by 1.
- The derivative of a constant c multiplied by a function f is the same as the constant multiplied by the derivative.
- The derivative of the sum of a function f and a function g is the same as the sum of the derivative of f and the derivative of g.
- The derivative of the difference of a function f and a function g is the same as the difference of the derivative of f and the derivative of g.
- The derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function.
- The derivative of the quotient of two functions is the derivative of the first function times the second function minus the derivative of the second function times the first function, all divided by the square of the second function.
- We used the limit definition of the derivative to develop formulas that allow us to find derivatives without resorting to the definition of the derivative. These formulas can be used singly or in combination with each other.

Glossary

constant multiple rule

the derivative of a constant c multiplied by a function f is the same as the constant multiplied by the derivative: $\frac{d}{dx}\big(cf(x)\big) = cf'(x)$

constant rule

the derivative of a constant function is zero: $\frac{d}{dx}(c) = 0$, where c is a constant

difference rule

the derivative of the difference of a function f and a function g is the same as the difference of the derivative of f and the derivative of g: $\frac{d}{dx}\big(f(x) - g(x)\big) = f'(x) - g'(x)$

power rule

the derivative of a power function is a function in which the power on x becomes the coefficient of the term and the power on x in the derivative decreases by 1: If n is an integer, then $\frac{d}{dx}\,(x^n) = nx^{n-1}$

product rule

the derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function: $\frac{d}{dx}\big(f(x)g(x)\big) = f'(x)g(x) + g'(x)f(x)$

quotient rule

the derivative of the quotient of two functions is the derivative of the first function times the second function minus the derivative of the second function times the first function, all divided by the square of the second function: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - g'(x)f(x)}{\big(g(x)\big)^2}$

sum rule

the derivative of the sum of a function f and a function g is the same as the sum of the derivative of f and the derivative of g: $\frac{d}{dx}\big(f(x) + g(x)\big) = f'(x) + g'(x)$

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3.3E: EXERCISES FOR SECTION 3.3

In exercises 1 - 12, find $f'(x)$ for each function.

1) $f(x) = x^7 + 10$

2) $f(x) = 5x^3 - x + 1$

Answer

$f'(x) = 15x^2 - 1$

3) $f(x) = 4x^2 - 7x$

4) $f(x) = 8x^4 + 9x^2 - 1$

Answer

$f'(x) = 32x^3 + 18x$

5) $f(x) = x^4 + 2x$

6) $f(x) = 3x\left(18x^4 + \dfrac{13}{x+1}\right)$

Answer

$f'(x) = 270x^4 + \dfrac{39}{(x+1)^2}$

7) $f(x) = (x+2)(2x^2 - 3)$

8) $f(x) = x^2\left(\dfrac{2}{x^2} + \dfrac{5}{x^3}\right)$

Answer

$f'(x) = \dfrac{-5}{x^2}$

9) $f(x) = \dfrac{x^3 + 2x^2 - 4}{3}$

10) $f(x) = \dfrac{4x^3 - 2x + 1}{x^2}$

Answer

$f'(x) = \dfrac{4x^4 + 2x^2 - 2x}{x^4}$

11) $f(x) = \dfrac{x^2 + 4}{x^2 - 4}$

12) $f(x) = \dfrac{x + 9}{x^2 - 7x + 1}$

Answer

$f'(x) = \dfrac{-x^2 - 18x + 64}{(x^2 - 7x + 1)^2}$

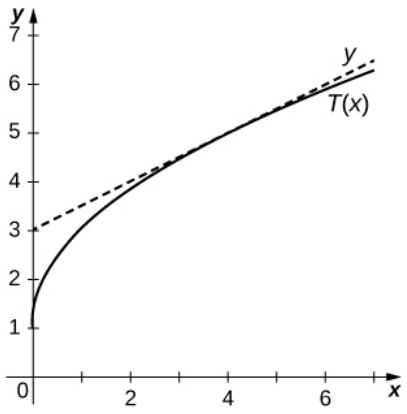
In exercises 13 - 16, find the equation of the tangent line $T(x)$ to the graph of the given function at the indicated point. Use a graphing calculator to graph the function and the tangent line.

13) [T] $y = 3x^2 + 4x + 1$ at $(0, 1)$

14) [T] $y = 2\sqrt{x} + 1$ at $(4, 5)$

Answer

$T(x) = \frac{1}{2}x + 3$

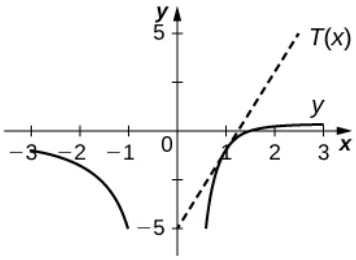


15) [T] $y = \dfrac{2x}{x-1}$ at $(-1, 1)$

16) [T] $y = \dfrac{2}{x} - \dfrac{3}{x^2}$ at $(1, -1)$

Answer

$T(x) = 4x - 5$



In exercises 17 - 20, assume that $f(x)$ and $g(x)$ are both differentiable functions for all x. Find the derivative of each of the functions $h(x)$.

17) $h(x) = 4f(x) + \dfrac{g(x)}{7}$

18) $h(x) = x^3 f(x)$

Answer

$h'(x) = 3x^2 f(x) + x^3 f'(x)$

19) $h(x) = \dfrac{f(x)g(x)}{2}$

20) $h(x) = \dfrac{3f(x)}{g(x)+2}$

Answer

$h'(x) = \dfrac{3f'(x)(g(x)+2) - 3f(x)g'(x)}{(g(x)+2)^2}$

For exercises 21 - 24, assume that $f(x)$ and $g(x)$ are both differentiable functions with values as given in the following table. Use the following table to calculate the following derivatives.

x	1	2	3	4
$f(x)$	3	5	−2	0
$g(x)$	2	3	−4	6
$f'(x)$	−1	7	8	−3
$g'(x)$	4	1	2	9

21) Find $h'(1)$ if $h(x) = xf(x) + 4g(x)$.

22) Find $h'(2)$ if $h(x) = \dfrac{f(x)}{g(x)}$.

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Answer

$h'(2) = \frac{16}{9}$

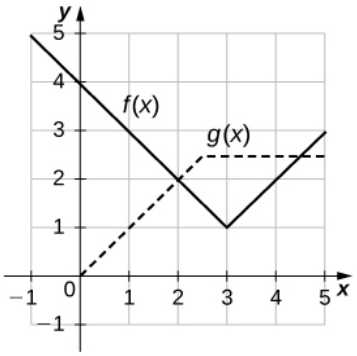
23) Find $h'(3)$ if $h(x) = 2x + f(x)g(x)$.

24) Find $h'(4)$ if $h(x) = \frac{1}{x} + \frac{g(x)}{f(x)}$.

Answer

$h'(4)$ is undefined.

In exercises 25 - 27, use the following figure to find the indicated derivatives, if they exist.



25) Let $h(x) = f(x) + g(x)$. Find

a) $h'(1)$,

b) $h'(3)$, and

c) $h'(4)$.

26) Let $h(x) = f(x)g(x)$. Find

a) $h'(1)$,

b) $h'(3)$, and

c) $h'(4)$.

Answer

a. $h'(1) = 2$,
b. $h'(3)$ does not exist,
c. $h'(4) = 2.5$

27) Let $h(x) = \frac{f(x)}{g(x)}$. Find

a) $h'(1)$,

b) $h'(3)$, and

c) $h'(4)$.

In exercises 28 - 31,

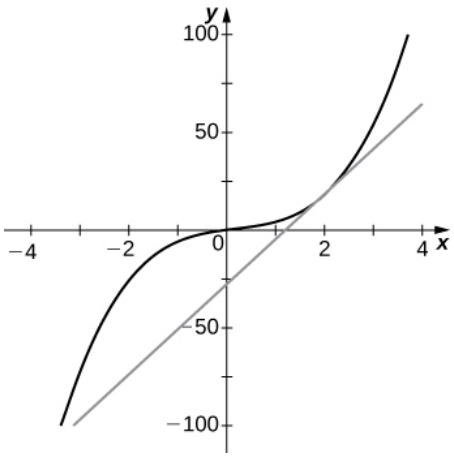
a) evaluate $f'(a)$, and

b) graph the function $f(x)$ and the tangent line at $x = a$.

28) [T] $f(x) = 2x^3 + 3x - x^2, \quad a = 2$

Answer

a. 23
b. $y = 23x - 28$



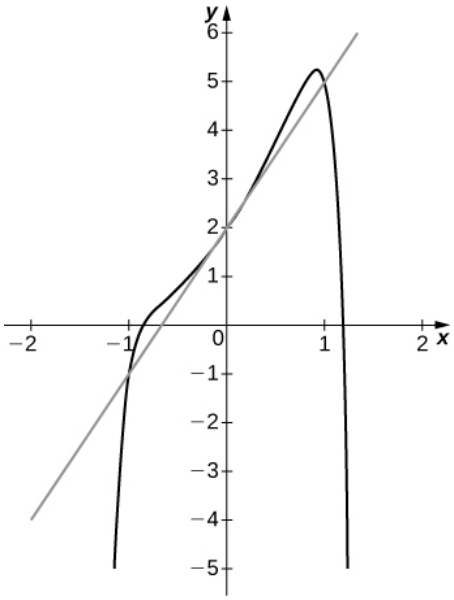
29) [T] $f(x)=\frac{1}{x}-x^2, \quad a=1$

30) [T] $f(x)=x^2-x^{12}+3x+2, \quad a=0$

Answer

a. 3

b. $y=3x+2$



31) [T] $f(x)=\frac{1}{x}-x^{2/3}, \quad a=-1$

32) Find the equation of the tangent line to the graph of $f(x)=2x^3+4x^2-5x-3$ at $x=-1$.

Answer

$y=-7x-3$

33) Find the equation of the tangent line to the graph of $f(x)=x^2+\frac{4}{x}-10$ at $x=8$.

34) Find the equation of the tangent line to the graph of $f(x)=(3x-x^2)(3-x-x^2)$ at $x=1$.

Answer

$y=-5x+7$

35) Find the point on the graph of $f(x)=x^3$ such that the tangent line at that point has an x-intercept of $(6,0)$.

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36) Find the equation of the line passing through the point $P(3,3)$ and tangent to the graph of $f(x)=\dfrac{6}{x-1}$.

Answer

$y=-\frac{3}{2}x+\frac{15}{2}$

37) Determine all points on the graph of $f(x)=x^3+x^2-x-1$ for which the slope of the tangent line is

a. horizontal

b. -1.

38) Find a quadratic polynomial such that $f(1)=5,\ f'(1)=3$ and $f''(1)=-6$.

Answer

$y=-3x^2+9x-1$

39) A car driving along a freeway with traffic has traveled $s(t)=t^3-6t^2+9t$ meters in t seconds.

a. Determine the time in seconds when the velocity of the car is 0.

b. Determine the acceleration of the car when the velocity is 0.

40) [T] A herring swimming along a straight line has traveled $s(t)=\dfrac{t^2}{t^2+2}$ feet in t seconds. Determine the velocity of the herring when it has traveled 3 seconds.

Answer

$\frac{12}{121}$ or 0.0992 ft/s

41) The population in millions of arctic flounder in the Atlantic Ocean is modeled by the function $P(t)=\dfrac{8t+3}{0.2t^2+1}$, where t is measured in years.

a. Determine the initial flounder population.

b. Determine $P'(10)$ and briefly interpret the result.

42) [T] The concentration of antibiotic in the bloodstream t hours after being injected is given by the function $C(t)=\dfrac{2t^2+t}{t^3+50}$, where C is measured in milligrams per liter of blood.

a. Find the rate of change of $C(t)$.

b. Determine the rate of change for $t=8,12,24$,and 36.

c. Briefly describe what seems to be occurring as the number of hours increases.

Answer

a. $\dfrac{-2t^4-2t^3+200t+50}{(t^3+50)^2}$

b. -0.02395 mg/L-hr, -0.01344 mg/L-hr, -0.003566 mg/L-hr, -0.001579 mg/L-hr

c. The rate at which the concentration of drug in the bloodstream decreases is slowing to 0 as time increases.

43) A book publisher has a cost function given by $C(x)=\dfrac{x^3+2x+3}{x^2}$, where x is the number of copies of a book in thousands and C is the cost, per book, measured in dollars. Evaluate $C'(2)$and explain its meaning.

44) [T] According to Newton's law of universal gravitation, the force F between two bodies of constant mass m_1 and m_2 is given by the formula $F=\dfrac{Gm_1m_2}{d^2}$, where G is the gravitational constant and d is the distance between the bodies.

a. Suppose that G, m_1, and m_2 are constants. Find the rate of change of force F with respect to distance d.

b. Find the rate of change of force F with gravitational constant $G=6.67\times10^{-11}\text{Nm}^2/\text{kg}^2$, on two bodies 10 meters apart, each with a mass of 1000 kilograms.

Answer

a. $F'(d)=\dfrac{-2Gm_1m_2}{d_3}$

b. -1.33×10^{-7} N/m

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3.4: Derivatives as Rates of Change

Learning Objectives

- Determine a new value of a quantity from the old value and the amount of change.
- Calculate the average rate of change and explain how it differs from the instantaneous rate of change.
- Apply rates of change to displacement, velocity, and acceleration of an object moving along a straight line.
- Predict the future population from the present value and the population growth rate.
- Use derivatives to calculate marginal cost and revenue in a business situation.

In this section we look at some applications of the derivative by focusing on the interpretation of the derivative as the rate of change of a function. These applications include **acceleration** and velocity in physics, **population growth rates** in biology, and marginal functions in economics.

Amount of Change Formula

One application for derivatives is to estimate an unknown value of a function at a point by using a known value of a function at some given point together with its rate of change at the given point. If $f(x)$ is a function defined on an interval $[a, a+h]$, then the **amount of change** of $f(x)$ over the interval is the change in the y values of the function over that interval and is given by

$$f(a+h) - f(a).$$

The **average rate of change** of the function f over that same interval is the ratio of the amount of change over that interval to the corresponding change in the x values. It is given by

$$\frac{f(a+h) - f(a)}{h}.$$

As we already know, the instantaneous rate of change of $f(x)$ at a is its derivative

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

For small enough values of h, $f'(a) \approx \frac{f(a+h)-f(a)}{h}$. We can then solve for $f(a+h)$ to get the amount of change formula:

$$f(a+h) \approx f(a) + f'(a)h. \tag{3.4.1}$$

We can use this formula if we know only $f(a)$and $f'(a)$ and wish to estimate the value of $f(a+h)$. For example, we may use the current population of a city and the rate at which it is growing to estimate its population in the near future. As we can see in Figure 3.4.1, we are approximating $f(a+h)$ by the y coordinate at a+h on the line tangent to $f(x)$ at $x = a$. Observe that the accuracy of this estimate depends on the value of h as well as the value of $f'(a)$.

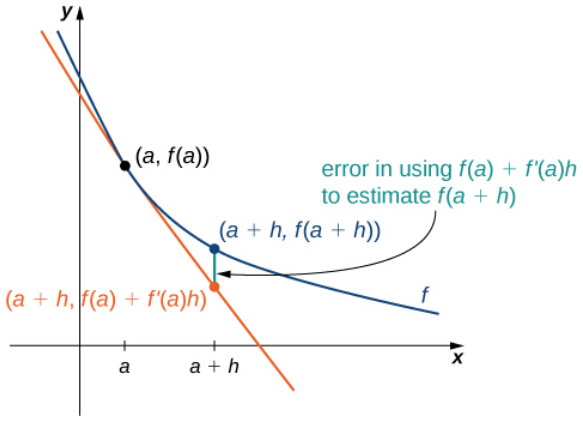


Figure 3.4.1: The new value of a changed quantity equals the original value plus the rate of change times the interval of change: $f(a+h) \approx f(a) + f'(a)h$.

Example 3.4.1: Estimating the Value of a Function

If $f(3) = 2$ and $f'(3) = 5$, estimate $f(3.2)$.

Solution

Begin by finding h. We have $h = 3.2 - 3 = 0.2$. Thus,

$$f(3.2) = f(3 + 0.2) \approx f(3) + (0.2)f'(3) = 2 + 0.2(5) = 3.$$

Exercise 3.4.1

Given $f(10) = -5$ and $f'(10) = 6$, estimate $f(10.1)$.

Hint

Use the same process as in the preceding example.

Answer

-4.4

Motion along a Line

Another use for the derivative is to analyze motion along a line. We have described velocity as the rate of change of position. If we take the derivative of the velocity, we can find the acceleration, or the rate of change of velocity. It is also important to introduce the idea of **speed**, which is the magnitude of velocity. Thus, we can state the following mathematical definitions.

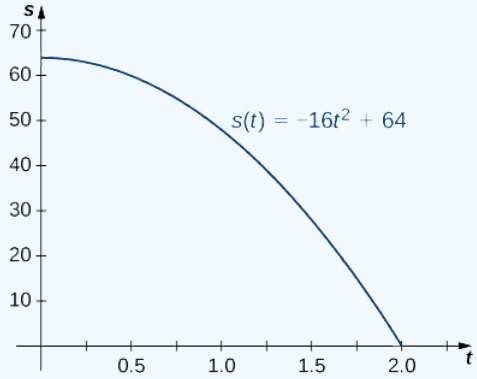
Definition

Let $s(t)$ be a function giving the position of an object at time t.

- The velocity of the object at time t is given by $v(t) = s'(t)$.
- The speed of the object at time t is given by $|v(t)|$.
- The acceleration of the object at t is given by $a(t) = v'(t) = s''(t)$.

Example 3.4.2: Comparing Instantaneous Velocity and Average Velocity

A ball is dropped from a height of 64 feet. Its height above ground (in feet) t seconds later is given by $s(t) = -16t^2 + 64$.



a. What is the instantaneous velocity of the ball when it hits the ground?
b. What is the average velocity during its fall?

Solution

The first thing to do is determine how long it takes the ball to reach the ground. To do this, set $s(t) = 0$. Solving $-16t^2 + 64 = 0$, we get $t = 2$, so it takes 2 seconds for the ball to reach the ground.

a. The instantaneous velocity of the ball as it strikes the ground is $v(2)$. Since $v(t) = s'(t) = -32t$, we obtain $v(t) = -64$ ft/s.

b. The average velocity of the ball during its fall is

$$v_{ave} = \frac{s(2)-s(0)}{2-0} = \frac{0-64}{2} = -32 \text{ ft/s}.$$

Example 3.4.3: Interpreting the Relationship between $v(t)$ and $a(t)$

A particle moves along a coordinate axis in the positive direction to the right. Its position at time t is given by $s(t) = t^3 - 4t + 2$. Find $v(1)$ and $a(1)$ and use these values to answer the following questions.

a. Is the particle moving from left to right or from right to left at time $t = 1$?
b. Is the particle speeding up or slowing down at time $t = 1$?

Solution

Begin by finding $v(t)$ and $a(t)$.

$v(t) = s'(t) = 3t^2 - 4$ and $a(t) = v'(t) = s''(t) = 6t$.

Evaluating these functions at $t = 1$, we obtain $v(1) = -1$ and $a(1) = 6$.

a. Because $v(1) < 0$, the particle is moving from right to left.
b. Because $v(1) < 0$ and $a(1) > 0$, velocity and acceleration are acting in opposite directions. In other words, the particle is being accelerated in the direction opposite the direction in which it is traveling, causing $|v(t)|$ to decrease. The particle is slowing down.

Example 3.4.4: Position and Velocity

The position of a particle moving along a coordinate axis is given by $s(t) = t^3 - 9t^2 + 24t + 4,\ t \geq 0$.

a. Find $v(t)$.
b. At what time(s) is the particle at rest?
c. On what time intervals is the particle moving from left to right? From right to left?
d. Use the information obtained to sketch the path of the particle along a coordinate axis.

Solution

a. The velocity is the derivative of the position function:

$$v(t) = s'(t) = 3t^2 - 18t + 24.$$

b. The particle is at rest when $v(t) = 0$, so set $3t^2 - 18t + 24 = 0$. Factoring the left-hand side of the equation produces $3(t - 2)(t - 4) = 0$. Solving, we find that the particle is at rest at $t = 2$ and $t = 4$.

c. The particle is moving from left to right when $v(t) > 0$ and from right to left when $v(t) < 0$. Figure 3.4.2 gives the analysis of the sign of $v(t)$ for $t \geq 0$, but it does not represent the axis along which the particle is moving.

Figure 3.4.2:The sign of $v(t)$ determines the direction of the particle.

- Since $3t^2 - 18t + 24 > 0$ on $[0, 2) \cup (4, +\infty)$, the particle is moving from left to right on these intervals.
- Since $3t^2 - 18t + 24 < 0$ on $(2, 4)$, the particle is moving from right to left on this interval.

d. Before we can sketch the graph of the particle, we need to know its position at the time it starts moving $(t = 0)$ and at the times that it changes direction $(t = 2, 4)$. We have $s(0) = 4$, $s(2) = 24$, and $s(4) = 20$. This means that the particle begins on the coordinate axis at 4 and changes direction at 24 and 20 on the coordinate axis. The path of the particle is shown on a coordinate axis in Figure 3.4.3.

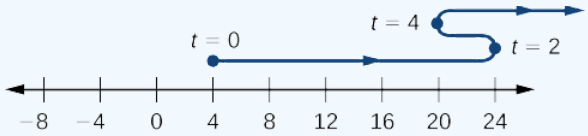


Figure 3.4.3: The path of the particle can be determined by analyzing $v(t)$.

Exercise 3.4.2

A particle moves along a coordinate axis. Its position at time t is given by $s(t) = t^2 - 5t + 1$. Is the particle moving from right to left or from left to right at time $t = 3$?

Hint

Find $v(3)$ and look at the sign.

Answer

left to right

Population Change

In addition to analyzing velocity, speed, acceleration, and position, we can use derivatives to analyze various types of populations, including those as diverse as bacteria colonies and cities. We can use a current population, together with a growth rate, to estimate the size of a population in the future. The population growth rate is the rate of change of a population and consequently can be represented by the derivative of the size of the population.

Definition

If $P(t)$ is the number of entities present in a population, then the population growth rate of $P(t)$ is defined to be $P'(t)$.

Example 3.4.5: Estimating a Population

The population of a city is tripling every 5 years. If its current population is 10,000, what will be its approximate population 2 years from now?

Solution

Let $P(t)$ be the population (in thousands) t years from now. Thus, we know that $P(0) = 10$ and based on the information, we anticipate $P(5) = 30$. Now estimate $P'(0)$, the current growth rate, using

$$P'(0) \approx \frac{P(5)-P(0)}{5-0} = \frac{30-10}{5} = 4.$$

By applying Equation 3.4.1 to $P(t)$, we can estimate the population 2 years from now by writing

$$P(2) \approx P(0) + (2)P'(0) \approx 10 + 2(4) = 18;$$

thus, in 2 years the population will be 18,000.

Exercise 3.4.3

The current population of a mosquito colony is known to be 3,000; that is, $P(0) = 3,000$. If $P'(0) = 100$, estimate the size of the population in 3 days, where t is measured in days.

Hint

Use $P(3) \approx P(0) + 3P'(0)$

Answer

3,300

Changes in Cost and Revenue

In addition to analyzing motion along a line and population growth, derivatives are useful in analyzing changes in cost, revenue, and profit. The concept of a marginal function is common in the fields of business and economics and implies the use of derivatives. The marginal cost is the derivative of the cost function. The marginal revenue is the derivative of the revenue function. The marginal profit is the derivative of the profit function, which is based on the cost function and the revenue function.

Definition

- If $C(x)$ is the cost of producing x items, then the **marginal cost** $MC(x)$ is $MC(x) = C'(x)$.
- If $R(x)$ is the revenue obtained from selling x items, then the marginal revenue $MR(x)$ is $MR(x) = R'(x)$.
- If $P(x) = R(x) - C(x)$ is the profit obtained from selling x items, then the **marginal profit** $MP(x)$ is defined to be $MP(x) = P'(x) = MR(x) - MC(x) = R'(x) - C'(x)$.

We can roughly approximate

$$MC(x) = C'(x) = \lim_{h \to 0} \frac{C(x+h) - C(x)}{h}$$

by choosing an appropriate value for h. Since x represents objects, a reasonable and small value for h is 1. Thus, by substituting $h = 1$, we get the approximation $MC(x) = C'(x) \approx C(x+1) - C(x)$. Consequently, $C'(x)$ for a given value of x can be thought of as the change in cost associated with producing one additional item. In a similar way, $MR(x) = R'(x)$ approximates the revenue obtained by selling one additional item, and $MP(x) = P'(x)$ approximates the profit obtained by producing and selling one additional item.

Example 3.4.6: Applying Marginal Revenue

Assume that the number of barbeque dinners that can be sold, x, can be related to the price charged, p, by the equation $p(x) = 9 - 0.03x, 0 \leq x \leq 300$.

In this case, the revenue in dollars obtained by selling x barbeque dinners is given by

$$R(x) = xp(x) = x(9 - 0.03x) = -0.03x^2 + 9x \quad \text{for } 0 \leq x \leq 300.$$

Use the marginal revenue function to estimate the revenue obtained from selling the 101st barbeque dinner. Compare this to the actual revenue obtained from the sale of this dinner.

Solution

First, find the marginal revenue function: $MR(x) = R'(x) = -0.06x + 9$.

Next, use $R'(100)$ to approximate $R(101) - R(100)$, the revenue obtained from the sale of the 101st dinner. Since $R'(100) = 3$, the revenue obtained from the sale of the 101st dinner is approximately \$3.

The actual revenue obtained from the sale of the 101st dinner is

$$R(101) - R(100) = 602.97 - 600 = 2.97, \text{ or } \$2.97.$$

The marginal revenue is a fairly good estimate in this case and has the advantage of being easy to compute.

Exercise 3.4.4

Suppose that the profit obtained from the sale of x fish-fry dinners is given by $P(x) = -0.03x^2 + 8x - 50$. Use the marginal profit function to estimate the profit from the sale of the 101st fish-fry dinner.

Hint

Use $P'(100)$ to approximate $P(101) - P(100)$.

Answer

\$2

Key Concepts

- Using $f(a+h) \approx f(a) + f'(a)h$, it is possible to estimate $f(a+h)$ given $f'(a)$ and $f(a)$.
- The rate of change of position is velocity, and the rate of change of velocity is acceleration. Speed is the absolute value, or magnitude, of velocity.
- The population growth rate and the present population can be used to predict the size of a future population.

- Marginal cost, marginal revenue, and marginal profit functions can be used to predict, respectively, the cost of producing one more item, the revenue obtained by selling one more item, and the profit obtained by producing and selling one more item.

Glossary

acceleration
is the rate of change of the velocity, that is, the derivative of velocity

amount of change
the amount of a function $f(x)$ over an interval $[x, x+h] is f(x+h) - f(x)$

average rate of change
is a function $f(x)$ over an interval $[x, x+h]$ is $\frac{f(x+h)-f(a)}{b-a}$

marginal cost
is the derivative of the cost function, or the approximate cost of producing one more item

marginal revenue
is the derivative of the revenue function, or the approximate revenue obtained by selling one more item

marginal profit
is the derivative of the profit function, or the approximate profit obtained by producing and selling one more item

population growth rate
is the derivative of the population with respect to time

speed
is the absolute value of velocity, that is, $|v(t)|$ is the speed of an object at time t whose velocity is given by $v(t)$

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3.4E: EXERCISES FOR SECTION 3.4

In exercises 1 - 3, the given functions represent the position of a particle traveling along a horizontal line.

a. Find the velocity and acceleration functions.

b. Determine the time intervals when the object is slowing down or speeding up.

1) $s(t) = 2t^3 - 3t^2 - 12t + 8$

2) $s(t) = 2t^3 - 15t^2 + 36t - 10$

Answer

a. $v(t) = 6t^2 - 30t + 36, \quad a(t) = 12t - 30$;
b. speeds up for $(2, 2.5) \cup (3, \infty)$, slows down for $(0, 2) \cup (2.5, 3)$

3) $s(t) = \dfrac{t}{1 + t^2}$

4) A rocket is fired vertically upward from the ground. The distance s in feet that the rocket travels from the ground after t seconds is given by $s(t) = -16t^2 + 560t$.

a. Find the velocity of the rocket 3 seconds after being fired.

b. Find the acceleration of the rocket 3 seconds after being fired.

Answer

a. 464 ft/s^2
b. -32 ft/s^2

5) A ball is thrown downward with a speed of 8 ft/s from the top of a 64-foot-tall building. After t seconds, its height above the ground is given by $s(t) = -16t^2 - 8t + 64$.

a. Determine how long it takes for the ball to hit the ground.

b. Determine the velocity of the ball when it hits the ground.

6) The position function $s(t) = t^2 - 3t - 4$ represents the position of the back of a car backing out of a driveway and then driving in a straight line, where s is in feet and t is in seconds. In this case, $s(t) = 0$ represents the time at which the back of the car is at the garage door, so $s(0) = -4$ is the starting position of the car, 4 feet inside the garage.

a. Determine the velocity of the car when $s(t) = 0$.

b. Determine the velocity of the car when $s(t) = 14$.

Answer

a. 5 ft/s
b. 9 ft/s

7) The position of a hummingbird flying along a straight line in t seconds is given by $s(t) = 3t^3 - 7t$ meters.

a. Determine the velocity of the bird at $t = 1$ sec.

b. Determine the acceleration of the bird at $t = 1$ sec.

c. Determine the acceleration of the bird when the velocity equals 0.

8) A potato is launched vertically upward with an initial velocity of 100 ft/s from a potato gun at the top of an 85-foot-tall building. The distance in feet that the potato travels from the ground after t seconds is given by $s(t) = -16t^2 + 100t + 85$.

a. Find the velocity of the potato after 0.5 s and 5.75 s.

b. Find the speed of the potato at 0.5 s and 5.75 s.

c. Determine when the potato reaches its maximum height.

d. Find the acceleration of the potato at 0.5 s and 1.5 s.

e. Determine how long the potato is in the air.

f. Determine the velocity of the potato upon hitting the ground.

Answer

a. 84 ft/s, −84 ft/s
b. 84 ft/s
c. $\frac{25}{8}$ s
d. $-32\ \text{ft/s}^2$ in both cases
e. $\frac{1}{8}(25+\sqrt{965})$ s
f. $-4\sqrt{965}$ ft/s

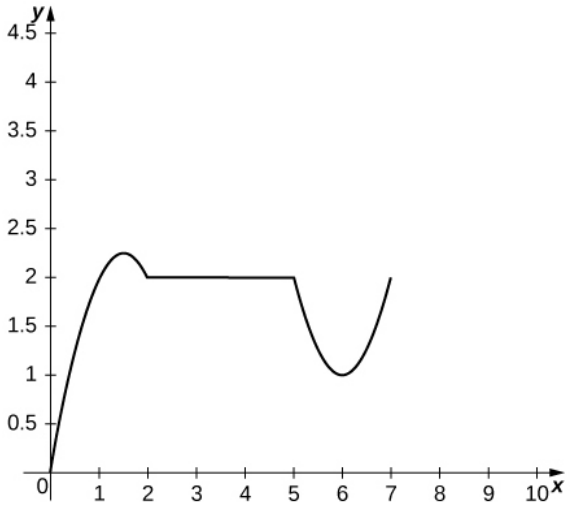
9) The position function $s(t)=t^3-8t$ gives the position in miles of a freight train where east is the positive direction and t is measured in hours.

a. Determine the direction the train is traveling when $s(t)=0$.

b. Determine the direction the train is traveling when $a(t)=0$.

c. Determine the time intervals when the train is slowing down or speeding up.

10) The following graph shows the position $y=s(t)$ of an object moving along a straight line.



a. Use the graph of the position function to determine the time intervals when the velocity is positive, negative, or zero.

b. Sketch the graph of the velocity function.

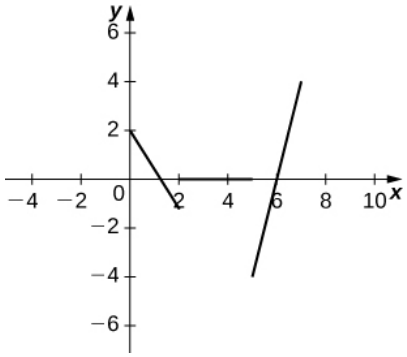
c. Use the graph of the velocity function to determine the time intervals when the acceleration is positive, negative, or zero.

d. Determine the time intervals when the object is speeding up or slowing down.

Answer

a. Velocity is positive on $(0,1.5)\cup(6,7)$, negative on $(1.5,2)\cup(5,6)$, and zero on $(2,5)$.

b.



c. Acceleration is positive on $(5,7)$, negative on $(0,2)$, and zero on $(2,5)$.

d. The object is speeding up on $(6,7)\cup(1.5,2)$ and slowing down on $(0,1.5)\cup(5,6)$.

11) The cost function, in dollars, of a company that manufactures food processors is given by $C(x)=200+\frac{7}{x}+\frac{x}{27}$, where x is the number of food processors manufactured.

https://math.libretexts.org/@go/page/51415

a. Find the marginal cost function.

b. Find the marginal cost of manufacturing 12 food processors.

c. Find the actual cost of manufacturing the thirteenth food processor.

12) The price p (in dollars) and the demand x for a certain digital clock radio is given by the price–demand function $p = 10 - 0.001x$.

a. Find the revenue function $R(x)$

b. Find the marginal revenue function.

c. Find the marginal revenue at $x = 2000$ and 5000.

Answer

a. $R(x) = 10x - 0.001x^2$
b.$R'(x) = 10 - 0.002x$
c. \$6 per item, \$0 per item

13) [T] A profit is earned when revenue exceeds cost. Suppose the profit function for a skateboard manufacturer is given by $P(x) = 30x - 0.3x^2 - 250$, where x is the number of skateboards sold.

a. Find the exact profit from the sale of the thirtieth skateboard.

b. Find the marginal profit function and use it to estimate the profit from the sale of the thirtieth skateboard.

14) [T] In general, the profit function is the difference between the revenue and cost functions: $P(x) = R(x) - C(x)$.

Suppose the price-demand and cost functions for the production of cordless drills is given respectively by $p = 143 - 0.03x$ and $C(x) = 75,000 + 65x$, where x is the number of cordless drills that are sold at a price of p dollars per drill and $C(x)$ is the cost of producing x cordless drills.

a. Find the marginal cost function.

b. Find the revenue and marginal revenue functions.

c. Find $R'(1000)$ and $R'(4000)$. Interpret the results.

d. Find the profit and marginal profit functions.

e. Find $P'(1000)$ and $P'(4000)$. Interpret the results.

Answer

a. $C'(x) = 65$
b. $R(x) = 143x - 0.03x^2, R'(x) = 143 - 0.06x$
c. $R'(1000) = 83, \quad R'(4000) = -97$. At a production level of 1000 cordless drills, revenue is increasing at a rate of \$83 per drill; at a production level of 4000 cordless drills, revenue is decreasing at a rate of \$97 per drill.
d. $P(x) = -0.03x^2 + 78x - 75000, \quad P'(x) = -0.06x + 78$
e. $P'(1000) = 18, \quad P'(4000) = -162$. At a production level of 1000 cordless drills, profit is increasing at a rate of \$18 per drill; at a production level of 4000 cordless drills, profit is decreasing at a rate of \$162 per drill.

15) A small town in Ohio commissioned an actuarial firm to conduct a study that modeled the rate of change of the town's population. The study found that the town's population (measured in thousands of people) can be modeled by the function $P(t) = -\frac{1}{3}t^3 + 64t + 3000$, where t is measured in years.

a. Find the rate of change function $P'(t)$ of the population function.

b. Find $P'(1), \ P'(2), \ P'(3)$, and $P'(4)$. Interpret what the results mean for the town.

c. Find $P''(1), \ P''(2), \ P''(3)$, and $P''(4)$. Interpret what the results mean for the town's population.

16) [T] A culture of bacteria grows in number according to the function $N(t) = 3000(1 + \frac{4t}{t^2 + 100})$, where t is measured in hours.

a. Find the rate of change of the number of bacteria.

b. Find $N'(0), \ N'(10), \ N'(20)$, and $N'(30)$.

c. Interpret the results in (b).

d. Find $N''(0), \ N''(10), \ N''(20)$, and $N''(30)$. Interpret what the answers imply about the bacteria population growth.

Answer

a. $N'(t) = 3000\left(\frac{-4t^2 + 400}{(t^2 + 100)^2}\right)$
b. $120, 0, -14.4, -9.6$

c. The bacteria population increases from time 0 to 10 hours; afterwards, the bacteria population decreases.

d. $0, -6, 0.384, 0.432$. The rate at which the bacteria is increasing is decreasing during the first 10 hours. Afterwards, the bacteria population is decreasing at a decreasing rate.

17) The centripetal force of an object of mass m is given by $F(r) = \frac{mv^2}{r}$, where v is the speed of rotation and r is the distance from the center of rotation.

a. Find the rate of change of centripetal force with respect to the distance from the center of rotation.

b. Find the rate of change of centripetal force of an object with mass 1000 kilograms, velocity of 13.89 m/s, and a distance from the center of rotation of 200 meters.

The following questions concern the population (in millions) of London by decade in the 19th century, which is listed in the following table.

Year Since 1800	Population (millions)
1	0.8975
11	1.040
21	1.264
31	1.516
41	1.661
51	2.000
61	2.634
71	3.272
81	3.911
91	4.422

Population of LondonSource: http://en.Wikipedia.org/wiki/Demographics_of_London

18) [T]

a. Using a calculator or a computer program, find the best-fit linear function to measure the population.

b. Find the derivative of the equation in a. and explain its physical meaning.

c. Find the second derivative of the equation and explain its physical meaning.

Answer

a. $P(t) = 0.03983 + 0.4280$

b. $P'(t) = 0.03983$. The population is increasing.

c. $P''(t) = 0$. The rate at which the population is increasing is constant.

19) [T]

a. Using a calculator or a computer program, find the best-fit quadratic curve through the data.

b. Find the derivative of the equation and explain its physical meaning.

c. Find the second derivative of the equation and explain its physical meaning.

For the following exercises, consider an astronaut on a large planet in another galaxy. To learn more about the composition of this planet, the astronaut drops an electronic sensor into a deep trench. The sensor transmits its vertical position every second in relation to the astronaut's position. The summary of the falling sensor data is displayed in the following table.

Time after dropping (s)	Position (m)
0	0
1	−1
2	−2
3	−5
4	−7
5	−14

20) [T]

a. Using a calculator or computer program, find the best-fit quadratic curve to the data.

b. Find the derivative of the position function and explain its physical meaning.

c. Find the second derivative of the position function and explain its physical meaning.

Answer

a. $p(t) = -0.6071x^2 + 0.4357x - 0.3571$

b. $p'(t) = -1.214x + 0.4357$. This is the velocity of the sensor.

c. $p''(t) = -1.214$. This is the acceleration of the sensor; it is a constant acceleration downward.

21) [T]

a. Using a calculator or computer program, find the best-fit cubic curve to the data.

b. Find the derivative of the position function and explain its physical meaning.

c. Find the second derivative of the position function and explain its physical meaning.

d. Using the result from c. explain why a cubic function is not a good choice for this problem.

The following problems deal with the Holling type I, II, and III equations. These equations describe the ecological event of growth of a predator population given the amount of prey available for consumption.

22) [T] The **Holling type I equation** is described by $f(x) = ax$, where x is the amount of prey available and $a > 0$ is the rate at which the predator meets the prey for consumption.

a. Graph the Holling type I equation, given $a = 0.5$.

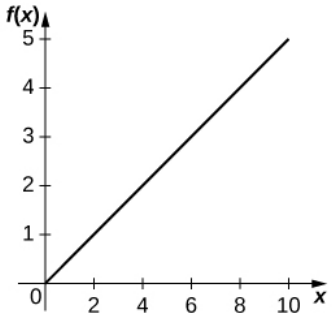
b. Determine the first derivative of the Holling type I equation and explain physically what the derivative implies.

c. Determine the second derivative of the Holling type I equation and explain physically what the derivative implies.

d. Using the interpretations from b. and c. explain why the Holling type I equation may not be realistic.

Answer

a.



b. $f'(x) = a$. The more increase in prey, the more growth for predators.

c. $f''(x) = 0$. As the amount of prey increases, the rate at which the predator population growth increases is constant.

d. This equation assumes that if there is more prey, the predator is able to increase consumption linearly. This assumption is unphysical because we would expect there to be some saturation point at which there is too much prey for the predator to consume adequately.

23) [T] The Holling type II equation is described by $f(x) = \frac{ax}{n+x}$, where x is the amount of prey available and $a > 0$ is the maximum consumption rate of the predator.

a. Graph the Holling type II equation given $a = 0.5$ and $n = 5$. What are the differences between the Holling type I and II equations?

b. Take the first derivative of the Holling type II equation and interpret the physical meaning of the derivative.

c. Show that $f(n) = \frac{1}{2}a$ and interpret the meaning of the parameter n.

d. Find and interpret the meaning of the second derivative. What makes the Holling type II function more realistic than the Holling type I function?

24) [T] The Holling type III equation is described by $f(x) = \frac{ax^2}{n^2+x^2}$, where x is the amount of prey available and $a > 0$ is the maximum consumption rate of the predator.

a. Graph the Holling type III equation given $a = 0.5$ and $n = 5$. What are the differences between the Holling type II and III equations?

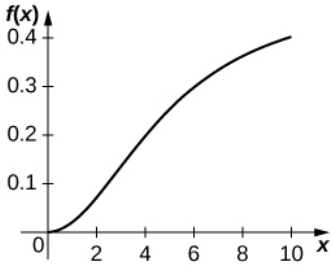
b. Take the first derivative of the Holling type III equation and interpret the physical meaning of the derivative.

c. Find and interpret the meaning of the second derivative (it may help to graph the second derivative).

d. What additional ecological phenomena does the Holling type III function describe compared with the Holling type II function?

Answer

a.



b. $f'(x) = \frac{2axn^2}{(n^2+x^2)^2}$. When the amount of prey increases, the predator growth increases.

c. $f''(x) = \frac{2an^2(n^2-3x^2)}{(n^2+x^2)^3}$. When the amount of prey is extremely small, the rate at which predator growth is increasing is increasing, but when the amount of prey reaches above a certain threshold, the rate at which predator growth is increasing begins to decrease.

d. At lower levels of prey, the prey is more easily able to avoid detection by the predator, so fewer prey individuals are consumed, resulting in less predator growth.

25) [T] The populations of the snowshoe hare (in thousands) and the lynx (in hundreds) collected over 7 years from 1937 to 1943 are shown in the following table. The snowshoe hare is the primary prey of the lynx.

Population of snowshoe hare (thousands)	Population of lynx (hundreds)
20	10
5	15
65	55
95	60

Snowshoe Hare and Lynx PopulationsSource: http://www.biotopics.co.uk/newgcse/predatorprey.html.

a. Graph the data points and determine which Holling-type function fits the data best.

b. Using the meanings of the parameters a and n, determine values for those parameters by examining a graph of the data. Recall that n measures what prey value results in the half-maximum of the predator value.

c. Plot the resulting Holling-type I, II, and III functions on top of the data. Was the result from part a. correct?

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3.5: Derivatives of Trigonometric Functions

Learning Objectives

- Find the derivatives of the sine and cosine function.
- Find the derivatives of the standard trigonometric functions.
- Calculate the higher-order derivatives of the sine and cosine.

One of the most important types of motion in physics is simple harmonic motion, which is associated with such systems as an object with mass oscillating on a spring. Simple harmonic motion can be described by using either sine or cosine functions. In this section we expand our knowledge of derivative formulas to include derivatives of these and other trigonometric functions. We begin with the derivatives of the sine and cosine functions and then use them to obtain formulas for the derivatives of the remaining four trigonometric functions. Being able to calculate the derivatives of the sine and cosine functions will enable us to find the velocity and acceleration of simple harmonic motion.

Derivatives of the Sine and Cosine Functions

We begin our exploration of the derivative for the sine function by using the formula to make a reasonable guess at its derivative. Recall that for a function $f(x)$,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Consequently, for values of h very close to 0,

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}.$$

We see that by using $h = 0.01$,

$$\frac{d}{dx}(\sin x) \approx \frac{\sin(x + 0.01) - \sin x}{0.01}$$

By setting

$$D(x) = \frac{\sin(x + 0.01) - \sin x}{0.01}$$

and using a graphing utility, we can get a graph of an approximation to the derivative of $\sin x$ (Figure 3.5.1).

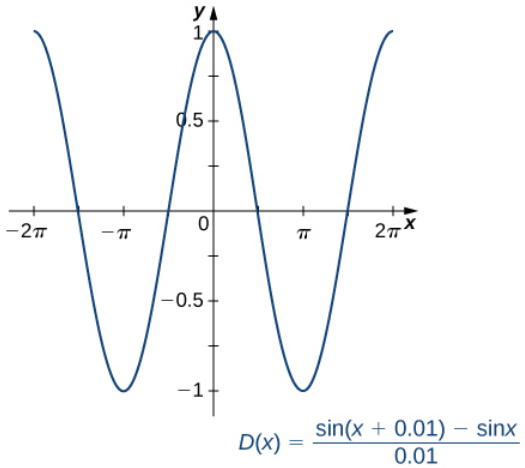

Figure 3.5.1: The graph of the function $D(x)$ looks a lot like a cosine curve.

Upon inspection, the graph of $D(x)$ appears to be very close to the graph of the cosine function. Indeed, we will show that

$$\frac{d}{dx}(\sin x) = \cos x.$$

If we were to follow the same steps to approximate the derivative of the cosine function, we would find that

$$\frac{d}{dx}(\cos x) = -\sin x.$$

The Derivatives of $\sin x$ and $\cos x$

The derivative of the sine function is the cosine and the derivative of the cosine function is the negative sine.

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

Proof

Because the proofs for $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$ use similar techniques, we provide only the proof for $\frac{d}{dx}(\sin x) = \cos x$. Before beginning, recall two important trigonometric limits:

$$\lim_{h\to 0}\frac{\sin h}{h} = 1 \text{ and } \lim_{h\to 0}\frac{\cos h - 1}{h} = 0.$$

The graphs of $y = \frac{\sin h}{h}$ and $y = \frac{\cos h - 1}{h}$ are shown in Figure 3.5.2.

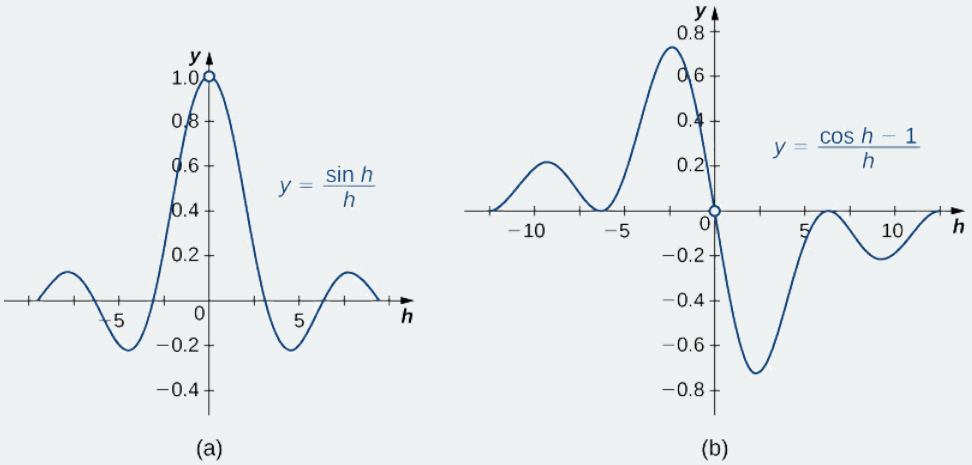

Figure 3.5.2: These graphs show two important limits needed to establish the derivative formulas for the sine and cosine functions.

We also recall the following trigonometric identity for the sine of the sum of two angles:

$$\sin(x + h) = \sin x \cos h + \cos x \sin h.$$

Now that we have gathered all the necessary equations and identities, we proceed with the proof.

$$
\begin{aligned}
\frac{d}{dx}(\sin x) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} && \text{Apply the definition of the derivative.}\\
&= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h} && \text{Use trig identity for the sine of the sum of two angles.}\\
&= \lim_{h\to 0}\left(\frac{\sin x\cos h-\sin x}{h}\right. && \text{Regroup.}\\
&\quad \left.+\frac{\cos x\sin h}{h}\right)\\
&= \lim_{h\to 0}\left(\sin x\left(\frac{\cos h-1}{h}\right)\right. && \text{Factor out } \sin x \text{ and } \cos x\\
&\quad \left.+(\cos x)\left(\frac{\sin h}{h}\right)\right)\\
&= (\sin x)\lim_{h\to 0}\left(\frac{\cos h-1}{h}\right) && \text{Factor } \sin x \text{ and } \cos x \text{ out of limits.}\\
&\quad +(\cos x)\lim_{h\to 0}\left(\frac{\sin h}{h}\right)\\
&= (\sin x)(0)+(\cos x)(1) && \text{Apply trig limit formulas.}\\
&= \cos x && \text{Simplify.}
\end{aligned}
$$

□

Figure 3.5.3 shows the relationship between the graph of $f(x)=\sin x$ and its derivative $f'(x)=\cos x$. Notice that at the points where $f(x)=\sin x$ has a horizontal tangent, its derivative $f'(x)=\cos x$ takes on the value zero. We also see that where f$(x)=\sin x$ is increasing, $f'(x)=\cos x>0$ and where $f(x)=\sin x$ is decreasing, $f'(x)=\cos x<0$.

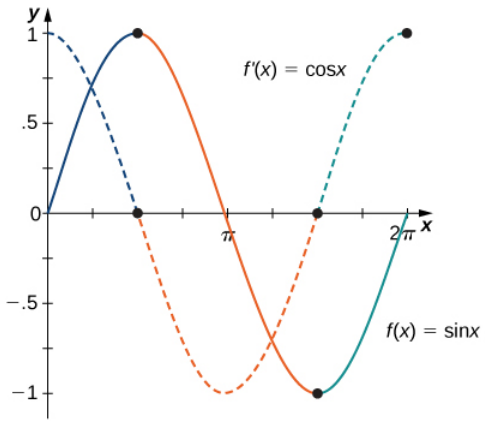

Figure 3.5.3: Where $f(x)$ has a maximum or a minimum, $f'(x)=0$ that is, $f'(x)=0$ where $f(x)$ has a horizontal tangent. These points are noted with dots on the graphs

Example 3.5.1: Differentiating a Function Containing $\sin x$

Find the derivative of $f(x)=5x^3\sin x$.

Solution

Using the product rule, we have

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}(5x^3)\cdot\sin x+\frac{d}{dx}(\sin x)\cdot 5x^3\\
&= 15x^2\cdot\sin x+\cos x\cdot 5x^3.
\end{aligned}
$$

After simplifying, we obtain

$$f'(x)=15x^2\sin x+5x^3\cos x.$$

Exercise 3.5.1

Find the derivative of $f(x) = \sin x \cos x$.

Hint

Don't forget to use the product rule.

Answer

$$f'(x) = \cos^2 x - \sin^2 x$$

Example 3.5.2: Finding the Derivative of a Function Containing cos x

Find the derivative of $g(x) = \dfrac{\cos x}{4x^2}$.

Solution

By applying the quotient rule, we have

$$g'(x) = \frac{(-\sin x)4x^2 - 8x(\cos x)}{(4x^2)^2}.$$

Simplifying, we obtain

$$g'(x) = \frac{-4x^2 \sin x - 8x \cos x}{16x^4} = \frac{-x \sin x - 2 \cos x}{4x^3}.$$

Exercise 3.5.2

Find the derivative of $f(x) = \dfrac{x}{\cos x}$.

Hint

Use the quotient rule.

Answer

$$f'(x) = \frac{\cos x + x \sin x}{\cos^2 x}$$

Example 3.5.3: An Application to Velocity

A particle moves along a coordinate axis in such a way that its position at time t is given by $s(t) = 2 \sin t - t$ for $0 \leq t \leq 2\pi$. At what times is the particle at rest?

Solution

To determine when the particle is at rest, set $s'(t) = v(t) = 0$. Begin by finding $s'(t)$. We obtain

$$s'(t) = 2 \cos t - 1,$$

so we must solve

$$2 \cos t - 1 = 0 \text{ for } 0 \leq t \leq 2\pi.$$

The solutions to this equation are $t = \dfrac{\pi}{3}$ and $t = \dfrac{5\pi}{3}$. Thus the particle is at rest at times $t = \dfrac{\pi}{3}$ and $t = \dfrac{5\pi}{3}$.

Exercise 3.5.3

A particle moves along a coordinate axis. Its position at time t is given by $s(t) = \sqrt{3}t + 2\cos t$ for $0 \le t \le 2\pi$. At what times is the particle at rest?

Hint

Use the previous example as a guide.

Answer

$t = \frac{\pi}{3}, \quad t = \frac{2\pi}{3}$

Derivatives of Other Trigonometric Functions

Since the remaining four trigonometric functions may be expressed as quotients involving sine, cosine, or both, we can use the quotient rule to find formulas for their derivatives.

Example 3.5.4: The Derivative of the Tangent Function

Find the derivative of $f(x) = \tan x$.

Solution

Start by expressing $\tan x$ as the quotient of $\sin x$ and $\cos x$:

$$f(x) = \tan x = \frac{\sin x}{\cos x}.$$

Now apply the quotient rule to obtain

$$f'(x) = \frac{\cos x \cos x - (-\sin x)\sin x}{(\cos x)^2}.$$

Simplifying, we obtain

$$f'(x) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}.$$

Recognizing that $\cos^2 x + \sin^2 x = 1$, by the Pythagorean theorem, we now have

$$f'(x) = \frac{1}{\cos^2 x}$$

Finally, use the identity $\sec x = \frac{1}{\cos x}$ to obtain

$$f'(x) = \sec^2 x.$$

Exercise 3.5.4

Find the derivative of $f(x) = \cot x$.

Hint

Rewrite $\cot x$ as $\frac{\cos x}{\sin x}$ and use the quotient rule.

Answer

$f'(x) = -\csc^2 x$

The derivatives of the remaining trigonometric functions may be obtained by using similar techniques. We provide these formulas in the following theorem.

Derivatives of $\tan x$, $\cot x$, $\sec x$, and $\csc x$

The derivatives of the remaining trigonometric functions are as follows:

$$\frac{d}{dx}(\tan x) = \sec^2 x \tag{3.5.1}$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x \tag{3.5.2}$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \tag{3.5.3}$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x. \tag{3.5.4}$$

Example 3.5.5: Finding the Equation of a Tangent Line

Find the equation of a line tangent to the graph of $f(x) = \cot x$ at $x = \frac{\pi}{4}$.

Solution

To find the equation of the tangent line, we need a point and a slope at that point. To find the point, compute

$$f\left(\tfrac{\pi}{4}\right) = \cot \tfrac{\pi}{4} = 1.$$

Thus the tangent line passes through the point $(\frac{\pi}{4}, 1)$. Next, find the slope by finding the derivative of $f(x) = \cot x$ and evaluating it at $\frac{\pi}{4}$:

$$f'(x) = -\csc^2 x \text{ and } f'\left(\tfrac{\pi}{4}\right) = -\csc^2\left(\tfrac{\pi}{4}\right) = -2.$$

Using the point-slope equation of the line, we obtain

$$y - 1 = -2\left(x - \tfrac{\pi}{4}\right)$$

or equivalently,

$$y = -2x + 1 + \tfrac{\pi}{2}.$$

Example 3.5.6: Finding the Derivative of Trigonometric Functions

Find the derivative of $f(x) = \csc x + x \tan x$.

Solution

To find this derivative, we must use both the sum rule and the product rule. Using the sum rule, we find

$$f'(x) = \frac{d}{dx}(\csc x) + \frac{d}{dx}(x \tan x).$$

In the first term, $\frac{d}{dx}(\csc x) = -\csc x \cot x$, and by applying the product rule to the second term we obtain

$$\frac{d}{dx}(x \tan x) = (1)(\tan x) + (\sec^2 x)(x).$$

Therefore, we have

$$f'(x) = -\csc x \cot x + \tan x + x \sec^2 x.$$

Exercise 3.5.5

Find the derivative of $f(x) = 2 \tan x - 3 \cot x$.

Hint

Use the rule for differentiating a constant multiple and the rule for differentiating a difference of two functions.

Answer

$f'(x) = 2 \sec^2 x + 3 \csc^2 x$

Exercise 3.5.6

Find the slope of the line tangent to the graph of $f(x) = \tan x$ at $x = \frac{\pi}{6}$.

Hint

Evaluate the derivative at $x = \frac{\pi}{6}$.

Answer

$\frac{4}{3}$

Higher-Order Derivatives

The higher-order derivatives of $\sin x$ and $\cos x$ follow a repeating pattern. By following the pattern, we can find any higher-order derivative of $\sin x$ and $\cos x$.

Example 3.5.7: Finding Higher-Order Derivatives of $y = \sin x$

Find the first four derivatives of $y = \sin x$.

Solution

Each step in the chain is straightforward:

$$\begin{aligned} y &= \sin x \\ \frac{dy}{dx} &= \cos x \\ \frac{d^2y}{dx^2} &= -\sin x \\ \frac{d^3y}{dx^3} &= -\cos x \\ \frac{d^4y}{dx^4} &= \sin x \end{aligned}$$

Analysis

Once we recognize the pattern of derivatives, we can find any higher-order derivative by determining the step in the pattern to which it corresponds. For example, every fourth derivative of $\sin x$ equals $\sin x$, so

$$\frac{d^4}{dx^4}(\sin x) = \frac{d^8}{dx^8}(\sin x) = \frac{d^{12}}{dx^{12}}(\sin x) = \ldots = \frac{d^{4n}}{dx^{4n}}(\sin x) = \sin x$$

$$\frac{d^5}{dx^5}(\sin x) = \frac{d^9}{dx^9}(\sin x) = \frac{d^{13}}{dx^{13}}(\sin x) = \ldots = \frac{d^{4n+1}}{dx^{4n+1}}(\sin x) = \cos x.$$

Exercise 3.5.7

For $y = \cos x$, find $\frac{d^4y}{dx^4}$.

Hint

See the previous example.

Answer

$\cos x$

Example 3.5.8: Using the Pattern for Higher-Order Derivatives of $y = \sin x$

Find $\frac{d^{74}}{dx^{74}}(\sin x)$.

Solution

We can see right away that for the 74th derivative of $\sin x$, $74 = 4(18) + 2$, so

$$\frac{d^{74}}{dx^{74}}(\sin x) = \frac{d^{72+2}}{dx^{72+2}}(\sin x) = \frac{d^2}{dx^2}(\sin x) = -\sin x.$$

Exercise 3.5.8

For $y = \sin x$, find $\frac{d^{59}}{dx^{59}}(\sin x)$.

Hint

$$\frac{d^{59}}{dx^{59}}(\sin x) = \frac{d^{4\cdot 14+3}}{dx^{4\cdot 14+3}}(\sin x)$$

Answer

$-\cos x$

Example 3.5.9: An Application to Acceleration

A particle moves along a coordinate axis in such a way that its position at time t is given by $s(t) = 2 - \sin t$. Find $v(\pi/4)$ and $a(\pi/4)$. Compare these values and decide whether the particle is speeding up or slowing down.

Solution

First find $v(t) = s'(t)$

$$v(t) = s'(t) = -\cos t.$$

Thus,

$$v\left(\tfrac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}.$$

Next, find $a(t) = v'(t)$. Thus, $a(t) = v'(t) = \sin t$ and we have

$$a\left(\tfrac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

Since $v\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} < 0$ and $a\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} > 0$, we see that velocity and acceleration are acting in opposite directions; that is, the object is being accelerated in the direction opposite to the direction in which it is traveling. Consequently, the particle is slowing down.

Exercise 3.5.9

A block attached to a spring is moving vertically. Its position at time t is given by $s(t) = 2\sin t$. Find $v\left(\frac{5\pi}{6}\right)$ and $a\left(\frac{5\pi}{6}\right)$. Compare these values and decide whether the block is speeding up or slowing down.

Hint

Use Example 3.5.9 as a guide.

Answer

$v\left(\frac{5\pi}{6}\right) = -\sqrt{3} < 0$ and $a\left(\frac{5\pi}{6}\right) = -1 < 0$. The block is speeding up.

Key Concepts

- We can find the derivatives of $\sin x$ and $\cos x$ by using the definition of derivative and the limit formulas found earlier. The results are

  $$\frac{d}{dx}\big(\sin x\big) = \cos x \quad \text{and} \quad \frac{d}{dx}\big(\cos x\big) = -\sin x.$$

- With these two formulas, we can determine the derivatives of all six basic trigonometric functions.

Key Equations

- **Derivative of sine function**

  $$\frac{d}{dx}(\sin x) = \cos x$$

- **Derivative of cosine function**

  $$\frac{d}{dx}(\cos x) = -\sin x$$

- **Derivative of tangent function**

  $$\frac{d}{dx}(\tan x) = \sec^2 x$$

- **Derivative of cotangent function**

  $$\frac{d}{dx}(\cot x) = -\csc^2 x$$

- **Derivative of secant function**

  $$\frac{d}{dx}(\sec x) = \sec x \tan x$$

- **Derivative of cosecant function**

  $$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

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3.5E: EXERCISES FOR SECTION 3.5

In exercises 1 - 10, find $\frac{dy}{dx}$ for the given functions.

1) $y = x^2 - \sec x + 1$

Answer

$$\frac{dy}{dx} = 2x - \sec x \tan x$$

2) $y = 3 \csc x + \frac{5}{x}$

3) $y = x^2 \cot x$

Answer

$$\frac{dy}{dx} = 2x \cot x - x^2 \csc^2 x$$

4) $y = x - x^3 \sin x$

5) $y = \frac{\sec x}{x}$

Answer

$$\frac{dy}{dx} = \frac{x \sec x \tan x - \sec x}{x^2}$$

6) $y = \sin x \tan x$

7) $y = (x + \cos x)(1 - \sin x)$

Answer

$$\frac{dy}{dx} = (1 - \sin x)(1 - \sin x) - \cos x(x + \cos x)$$

8) $y = \frac{\tan x}{1 - \sec x}$

9) $y = \frac{1 - \cot x}{1 + \cot x}$

Answer

$$\frac{dy}{dx} = \frac{2 \csc^2 x}{(1 + \cot x)^2}$$

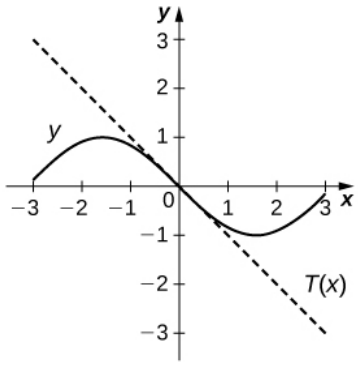
10) $y = (\cos x)(1 + \csc x)$

In exercises 11 - 16, find the equation of the tangent line to each of the given functions at the indicated values of x. Then use a calculator to graph both the function and the tangent line to ensure the equation for the tangent line is correct.

11) [T] $f(x) = -\sin x, \quad x = 0$

Answer

$y = -x$

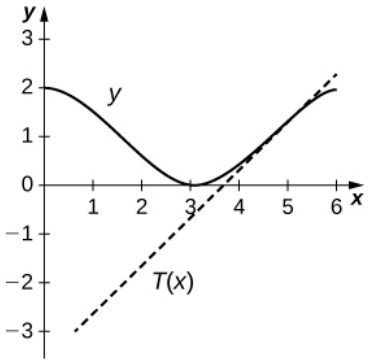


12) [T] $f(x) = \csc x, \quad x = \frac{\pi}{2}$

13) [T] $f(x) = 1 + \cos x, \quad x = \frac{3\pi}{2}$

Answer

$y = x + \frac{2-3\pi}{2}$

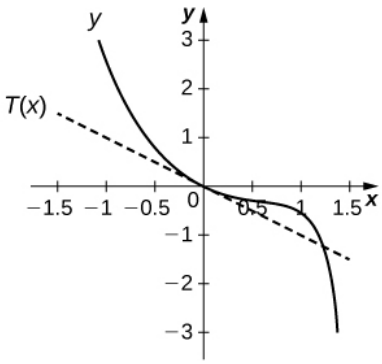


14) [T] $f(x) = \sec x, \quad x = \frac{\pi}{4}$

15) [T] $f(x) = x^2 - \tan x, \quad x = 0$

Answer

$y = -x$



16) [T] $f(x) = 5 \cot x, \quad x = \frac{\pi}{4}$

In exercises 17 - 22, find $\frac{d^2y}{dx^2}$ for the given functions.

17) $y = x \sin x - \cos x$

Answer

$\frac{d^2y}{dx^2} = 3 \cos x - x \sin x$

18) $y = \sin x \cos x$

19) $y = x - \frac{1}{2} \sin x$

Answer

$\frac{d^2y}{dx^2} = \frac{1}{2} \sin x$

20) $y = \frac{1}{x} + \tan x$

21) $y = 2 \csc x$

Answer

$\frac{d^2y}{dx^2} = 2 \csc(x) \left(\csc^2(x) + \cot^2(x)\right)$

22) $y = \sec^2 x$

23) Find all x values on the graph of $f(x) = -3 \sin x \cos x$ where the tangent line is horizontal.

Answer

$x = \dfrac{(2n+1)\pi}{4}$,where n is an integer

24) Find all x values on the graph of $f(x) = x - 2\cos x$ for $0 < x < 2\pi$ where the tangent line has slope 2.

25) Let $f(x) = \cot x$. Determine the points on the graph of f for $0 < x < 2\pi$ where the tangent line(s) is (are) parallel to the line $y = -2x$.

Answer

$\left(\frac{\pi}{4}, 1\right), \quad \left(\frac{3\pi}{4}, -1\right), \quad \left(\frac{5\pi}{4}, 1\right), \quad \left(\frac{7\pi}{4}, -1\right)$

26) [T] A mass on a spring bounces up and down in simple harmonic motion, modeled by the function $s(t) = -6 \cos t$ where s is measured in inches and t is measured in seconds. Find the rate at which the spring is oscillating at $t = 5$ s.

27) Let the position of a swinging pendulum in simple harmonic motion be given by $s(t) = a \cos t + b \sin t$. Find the constants a and b such that when the velocity is 3 cm/s, $s = 0$ and $t = 0$.

Answer

$a = 0, \quad b = 3$

28) After a diver jumps off a diving board, the edge of the board oscillates with position given by $s(t) = -5 \cos t$ cm at t seconds after the jump.

a. Sketch one period of the position function for $t \geq 0$.

b. Find the velocity function.

c. Sketch one period of the velocity function for $t \geq 0$.

d. Determine the times when the velocity is 0 over one period.

e. Find the acceleration function.

f. Sketch one period of the acceleration function for $t \geq 0$.

29) The number of hamburgers sold at a fast-food restaurant in Pasadena, California, is given by $y = 10 + 5 \sin x$ where y is the number of hamburgers sold and x represents the number of hours after the restaurant opened at 11 a.m. until 11 p.m., when the store closes. Find y' and determine the intervals where the number of burgers being sold is increasing.

Answer

$y' = 5\cos(x)$, increasing on $\left(0, \frac{\pi}{2}\right)$, $\left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$, and $\left(\frac{7\pi}{2}, 12\right)$

30) [T] The amount of rainfall per month in Phoenix, Arizona, can be approximated by $y(t) = 0.5 + 0.3 \cos t$, where t is months since January. Find $y'$and use a calculator to determine the intervals where the amount of rain falling is decreasing.

For exercises 31 - 33, use the quotient rule to derive the given equations.

31) $\dfrac{d}{dx}(\cot x) = -\csc^2 x$

32) $\dfrac{d}{dx}(\sec x) = \sec x \tan x$

33) $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$

34) Use the definition of derivative and the identity $\cos(x + h) = \cos x \cos h - \sin x \sin h$ to prove that $\dfrac{d}{dx}(\cos x) = -\sin x$.

For exercises 35 - 39, find the requested higher-order derivative for the given functions.

35) $\dfrac{d^3y}{dx^3}$ of $y = 3 \cos x$

Answer

$\dfrac{d^3y}{dx^3} = 3 \sin x$

36) $\dfrac{d^2y}{dx^2}$ of $y = 3 \sin x + x^2 \cos x$

37) $\dfrac{d^4y}{dx^4}$ of $y = 5 \cos x$

Answer

$$\frac{d^4y}{dx^4} = 5\cos x$$

38) $\frac{d^2y}{dx^2}$ of $y = \sec x + \cot x$

39) $\frac{d^3y}{dx^3}$ of $y = x^{10} - \sec x$

Answer

$$\frac{d^3y}{dx^3} = 720x^7 - 5\tan(x)\sec^3(x) - \tan^3(x)\sec(x)$$

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3.6: The Chain Rule

Learning Objectives

- State the chain rule for the composition of two functions.
- Apply the chain rule together with the power rule.
- Apply the chain rule and the product/quotient rules correctly in combination when both are necessary.
- Recognize the chain rule for a composition of three or more functions.
- Describe the proof of the chain rule.

We have seen the techniques for differentiating basic functions ($x^n, \sin x, \cos x,$ etc.) as well as sums, differences, products, quotients, and constant multiples of these functions. However, these techniques do not allow us to differentiate compositions of functions, such as $h(x) = \sin(x^3)$ or $k(x) = \sqrt{3x^2 + 1}$. In this section, we study the rule for finding the derivative of the composition of two or more functions.

Deriving the Chain Rule

When we have a function that is a composition of two or more functions, we could use all of the techniques we have already learned to differentiate it. However, using all of those techniques to break down a function into simpler parts that we are able to differentiate can get cumbersome. Instead, we use the *chain rule*, which states that the derivative of a composite function is the derivative of the outer function evaluated at the inner function times the derivative of the inner function.

To put this rule into context, let's take a look at an example: $h(x) = \sin(x^3)$. We can think of the derivative of this function with respect to x as the rate of change of $\sin(x^3)$ relative to the change in x. Consequently, we want to know how $\sin(x^3)$ changes as x changes. We can think of this event as a chain reaction: As x changes, x^3 changes, which leads to a change in $\sin(x^3)$. This chain reaction gives us hints as to what is involved in computing the derivative of $\sin(x^3)$. First of all, a change in x forcing a change in x^3 suggests that somehow the derivative of x^3 is involved. In addition, the change in x^3 forcing a change in $\sin(x^3)$ suggests that the derivative of $\sin(u)$ with respect to u, where $u = x^3$, is also part of the final derivative.

We can take a more formal look at the derivative of $h(x) = \sin(x^3)$ by setting up the limit that would give us the derivative at a specific value a in the domain of $h(x) = \sin(x^3)$.

$$h'(a) = \lim_{x \to a} \frac{\sin(x^3) - \sin(a^3)}{x - a}$$

This expression does not seem particularly helpful; however, we can modify it by multiplying and dividing by the expression $x^3 - a^3$ to obtain

$$h'(a) = \lim_{x \to a} \frac{\sin(x^3) - \sin(a^3)}{x^3 - a^3} \cdot \frac{x^3 - a^3}{x - a}.$$

From the definition of the derivative, we can see that the second factor is the derivative of x^3 at $x = a$. That is,

$$\lim_{x \to a} \frac{x^3 - a^3}{x - a} = \frac{d}{dx}(x^3)\Big|_{x=a} = 3a^2.$$

However, it might be a little more challenging to recognize that the first term is also a derivative. We can see this by letting $u = x^3$ and observing that as $x \to a, u \to a^3$:

$$\begin{aligned} \lim_{x \to a} \frac{\sin(x^3) - \sin(a^3)}{x^3 - a^3} &= \lim_{u \to a^3} \frac{\sin u - \sin(a^3)}{u - a^3} \\ &= \frac{d}{du}(\sin u)\Big|_{u=a^3} \\ &= \cos(a^3) \end{aligned}.$$

Thus, $h'(a) = \cos(a^3) \cdot 3a^2$.

In other words, if $h(x) = \sin(x^3)$, then $h'(x) = \cos(x^3) \cdot 3x^2$. Thus, if we think of $h(x) = \sin(x^3)$ as the composition $(f \circ g)(x) = f\big(g(x)\big)$ where $f(x) = \sin x$ and $g(x) = x^3$, then the derivative of $h(x) = \sin(x^3)$ is the product of the derivative of $g(x) = x^3$ and the

derivative of the function $f(x) = \sin x$ evaluated at the function $g(x) = x^3$. At this point, we anticipate that for $h(x) = \sin\big(g(x)\big)$, it is quite likely that $h'(x) = \cos\big(g(x)\big)g'(x)$. As we determined above, this is the case for $h(x) = \sin(x^3)$.

Now that we have derived a special case of the chain rule, we state the general case and then apply it in a general form to other composite functions. An informal proof is provided at the end of the section.

Rule: The Chain Rule

Let f and g be functions. For all x in the domain of g for which g is differentiable at x and f is differentiable at $g(x)$, the derivative of the composite function

$$h(x) = (f \circ g)(x) = f\big(g(x)\big)$$

is given by

$$h'(x) = f'\big(g(x)\big) \cdot g'(x).$$

Alternatively, if y is a function of u, and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Problem-Solving Strategy: Applying the Chain Rule

1. To differentiate $h(x) = f\big(g(x)\big)$, begin by identifying $f(x)$ and $g(x)$.
2. Find $f'(x)$ and evaluate it at $g(x)$ to obtain $f'\big(g(x)\big)$.
3. Find $g'(x)$.
4. Write $h'(x) = f'\big(g(x)\big) \cdot g'(x)$.

Note: When applying the chain rule to the composition of two or more functions, keep in mind that we work our way from the outside function in. It is also useful to remember that the derivative of the composition of two functions can be thought of as having two parts; the derivative of the composition of three functions has three parts; and so on. Also, remember that we never evaluate a derivative at a derivative.

The Chain and Power Rules Combined

We can now apply the chain rule to composite functions, but note that we often need to use it with other rules. For example, to find derivatives of functions of the form $h(x) = \big(g(x)\big)^n$, we need to use the chain rule combined with the power rule. To do so, we can think of $h(x) = \big(g(x)\big)^n$ as $f\big(g(x)\big)$ where $f(x) = x^n$. Then $f'(x) = nx^{n-1}$. Thus, $f'\big(g(x)\big) = n\big(g(x)\big)^{n-1}$. This leads us to the derivative of a power function using the chain rule,

$$h'(x) = n\big(g(x)\big)^{n-1} \cdot g'(x)$$

Rule: Power Rule for Composition of Functions (General Power Rule)

For all values of x for which the derivative is defined, if

$$h(x) = \big(g(x)\big)^n,$$

Then

$$h'(x) = n\big(g(x)\big)^{n-1} \cdot g'(x). \tag{3.6.1}$$

Example 3.6.1: Using the Chain and Power Rules

Find the derivative of $h(x) = \dfrac{1}{(3x^2+1)^2}$.

Solution

First, rewrite $h(x) = \dfrac{1}{(3x^2+1)^2} = (3x^2+1)^{-2}$.

Applying the power rule with $g(x) = 3x^2 + 1$, we have

$$h'(x) = -2(3x^2+1)^{-3} \cdot 6x.$$

Rewriting back to the original form gives us

$$h'(x) = \frac{-12x}{(3x^2+1)^3}$$

Exercise 3.6.1

Find the derivative of $h(x) = (2x^3 + 2x - 1)^4$.

Hint

Use the General Power Rule (Equation 3.6.1) with $g(x) = 2x^3 + 2x - 1$.

Answer

$h'(x) = 4(2x^3 + 2x - 1)^3(6x^2 + 2) = 8(3x^2 + 1)(2x^3 + 2x - 1)^3$

Example 3.6.2: Using the Chain and Power Rules with a Trigonometric Function

Find the derivative of $h(x) = \sin^3 x$.

Solution

First recall that $\sin^3 x = (\sin x)^3$, so we can rewrite $h(x) = \sin^3 x$ as $h(x) = (\sin x)^3$.

Applying the power rule with $g(x) = \sin x$, we obtain

$$h'(x) = 3(\sin x)^2 \cos x = 3\sin^2 x \cos x.$$

Example 3.6.3: Finding the Equation of a Tangent Line

Find the equation of a line tangent to the graph of $h(x) = \dfrac{1}{(3x-5)^2}$ at $x = 2$.

Solution

Because we are finding an equation of a line, we need a point. The x-coordinate of the point is 2. To find the y-coordinate, substitute 2 into $h(x)$. Since $h(2) = \dfrac{1}{(3(2)-5)^2} = 1$, the point is $(2, 1)$.

For the slope, we need $h'(2)$. To find $h'(x)$, first we rewrite $h(x) = (3x-5)^{-2}$ and apply the power rule to obtain

$$h'(x) = -2(3x-5)^{-3}(3) = -6(3x-5)^{-3}.$$

By substituting, we have $h'(2) = -6(3(2)-5)^{-3} = -6$.

Therefore, the line has equation $y - 1 = -6(x-2)$. Rewriting, the equation of the line is $y = -6x + 13$.

Exercise 3.6.2

Find the equation of the line tangent to the graph of $f(x) = (x^2-2)^3$ at $x = -2$.

Hint

Use the preceding example as a guide.

Answer

$y = -48x - 88$

Combining the Chain Rule with Other Rules

Now that we can combine the chain rule and the power rule, we examine how to combine the chain rule with the other rules we have learned. In particular, we can use it with the formulas for the derivatives of trigonometric functions or with the product rule.

Example 3.6.4: Using the Chain Rule on a General Cosine Function

Find the derivative of $h(x) = \cos\big(g(x)\big)$.

Solution

Think of $h(x) = \cos\big(g(x)\big)$ as $f\big(g(x)\big)$ where $f(x) = \cos x$. Since $f'(x) = -\sin x$, we have $f'\big(g(x)\big) = -\sin\big(g(x)\big)$. Then we do the following calculation.

$$\begin{aligned} h'(x) &= f'\big(g(x)\big) \cdot g'(x) && \text{Apply the chain rule.} \\ &= -\sin\big(g(x)\big) \cdot g'(x) && \text{Substitute } f'\big(g(x)\big) = -\sin\big(g(x)\big). \end{aligned}$$

Thus, the derivative of $h(x) = \cos\big(g(x)\big)$ is given by $h'(x) = -\sin\big(g(x)\big) \cdot g'(x)$.

In the following example we apply the rule that we have just derived.

Example 3.6.5: Using the Chain Rule on a Cosine Function

Find the derivative of $h(x) = \cos(5x^2)$.

Solution

Let $g(x) = 5x^2$. Then $g'(x) = 10x$. Using the result from the previous example,

$$h'(x) = -\sin(5x^2) \cdot 10x = -10x\sin(5x^2)$$

Example 3.6.6: Using the Chain Rule on Another Trigonometric Function

Find the derivative of $h(x) = \sec(4x^5 + 2x)$.

Solution

Apply the chain rule to $h(x) = \sec\big(g(x)\big)$ to obtain

$$h'(x) = \sec(g(x))\tan\big(g(x)\big) \cdot g'(x).$$

In this problem, $g(x) = 4x^5 + 2x$, so we have $g'(x) = 20x^4 + 2$. Therefore, we obtain

$$h'(x) = \sec(4x^5 + 2x)\tan(4x^5 + 2x)(20x^4 + 2) = (20x^4 + 2)\sec(4x^5 + 2x)\tan(4x^5 + 2x).$$

Exercise 3.6.3

Find the derivative of $h(x) = \sin(7x + 2)$.

Hint

Apply the chain rule to $h(x) = \sin\big(g(x)\big)$ first and then use $g(x) = 7x + 2$.

Answer

$h'(x) = 7\cos(7x + 2)$

At this point we provide a list of derivative formulas that may be obtained by applying the chain rule in conjunction with the formulas for derivatives of trigonometric functions. Their derivations are similar to those used in the examples above. For convenience, formulas are also given in Leibniz's notation, which some students find easier to remember. (We discuss the chain rule using Leibniz's notation at the end of this section.) It is not absolutely necessary to memorize these as separate formulas as they are all applications of the chain rule to previously learned formulas.

Using the Chain Rule with Trigonometric Functions

For all values of x for which the derivative is defined,

$\frac{d}{dx}\Big(\sin(g(x))\Big) = \cos(g(x)) \cdot g'(x)$	$\frac{d}{dx}\Big(\sin u\Big) = \cos u \cdot \frac{du}{dx}$
$\frac{d}{dx}\Big(\cos(g(x))\Big) = -\sin(g(x)) \cdot g'(x)$	$\frac{d}{dx}\Big(\cos u\Big) = -\sin u \cdot \frac{du}{dx}$
$\frac{d}{dx}\Big(\tan(g(x))\Big) = \sec^2(g(x)) \cdot g'(x)$	$\frac{d}{dx}\Big(\tan u\Big) = \sec^2 u \cdot \frac{du}{dx}$
$\frac{d}{dx}\Big(\cot(g(x))\Big) = -\csc^2(g(x)) \cdot g'(x)$	$\frac{d}{dx}\Big(\cot u\Big) = -\csc^2 u \cdot \frac{du}{dx}$
$\frac{d}{dx}\Big(\sec(g(x))\Big) = \sec(g(x))\tan(g(x)) \cdot g'(x)$	$\frac{d}{dx}\Big(\sec u\Big) = \sec u \tan u \cdot \frac{du}{dx}$
$\frac{d}{dx}\Big(\csc(g(x))\Big) = -\csc(g(x))\cot(g(x)) \cdot g'(x)$	$\frac{d}{dx}\Big(\csc u\Big) = -\csc u \cot u \cdot \frac{du}{dx}.$

Example 3.6.7: Combining the Chain Rule with the Product Rule

Find the derivative of $h(x) = (2x+1)^5(3x-2)^7$.

Solution

First apply the product rule, then apply the chain rule to each term of the product.

$$\begin{aligned} h'(x) &= \frac{d}{dx}\Big((2x+1)^5\Big) \cdot (3x-2)^7 + \frac{d}{dx}\Big((3x-2)^7\Big) \cdot (2x+1)^5 && \text{Apply the product rule.} \\ &= 5(2x+1)^4 \cdot 2 \cdot (3x-2)^7 + 7(3x-2)^6 \cdot 3 \cdot (2x+1)^5 && \text{Apply the chain rule.} \\ &= 10(2x+1)^4(3x-2)^7 + 21(3x-2)^6(2x+1)^5 && \text{Simplify.} \\ &= (2x+1)^4(3x-2)^6(10(3x-2) + 21(2x+1)) && \text{Factor out } (2x+1)^4(3x-2)^6 \\ &= (2x+1)^4(3x-2)^6(72x+1) && \text{Simplify.} \end{aligned}$$

Exercise 3.6.4

Find the derivative of $h(x) = \frac{x}{(2x+3)^3}$.

Hint

Start out by applying the quotient rule. Remember to use the chain rule to differentiate the denominator.

Answer

$$h'(x) = \frac{3-4x}{(2x+3)^4}$$

Composites of Three or More Functions

We can now combine the chain rule with other rules for differentiating functions, but when we are differentiating the composition of three or more functions, we need to apply the chain rule more than once. If we look at this situation in general terms, we can generate a formula, but we do not need to remember it, as we can simply apply the chain rule multiple times.

In general terms, first we let

$$k(x) = h\Big(f\big(g(x)\big)\Big).$$

Then, applying the chain rule once we obtain

$$k'(x) = \frac{d}{dx}\Big(h\big(f\big(g(x)\big)\big)\Big) = h'\Big(f\big(g(x)\big)\Big) \cdot \frac{d}{dx}\Big(f\big(g(x)\big)\Big).$$

Applying the chain rule again, we obtain

$$k'(x) = h'\Big(f\big(g(x)\big)\Big) \cdot f'\big(g(x)\big) \cdot g'(x).$$

Rule: Chain Rule for a Composition of Three Functions

For all values of x for which the function is differentiable, if

$$k(x) = h\Big(f\big(g(x)\big)\Big),$$

then

$$k'(x) = h'\Big(f\big(g(x)\big)\Big) \cdot f'\big(g(x)\big) \cdot g'(x).$$

In other words, we are applying the chain rule twice.

Notice that the derivative of the composition of three functions has three parts. (Similarly, the derivative of the composition of four functions has four parts, and so on.) Also, *remember, we can always work from the outside in, taking one derivative at a time.*

Example 3.6.8: Differentiating a Composite of Three Functions

Find the derivative of $k(x) = \cos^4(7x^2 + 1)$.

Solution

First, rewrite $k(x)$ as

$$k(x) = \big(\cos(7x^2 + 1)\big)^4.$$

Then apply the chain rule several times.

$$\begin{aligned} k'(x) &= 4(\cos(7x^2+1))^3 \cdot \frac{d}{dx}\big(\cos(7x^2+1)\big) && \text{Apply the chain rule.} \\ &= 4(\cos(7x^2+1))^3(-\sin(7x^2+1)) \cdot \frac{d}{dx}\big(7x^2+1\big) && \text{Apply the chain rule.} \\ &= 4(\cos(7x^2+1))^3(-\sin(7x^2+1))(14x) && \text{Apply the chain rule.} \\ &= -56x\sin(7x^2+1)\cos^3(7x^2+1) && \text{Simplify} \end{aligned}$$

Exercise 3.6.5

Find the derivative of $h(x) = \sin^6(x^3)$.

Hint

Rewrite $h(x) = \sin^6(x^3) = \big(\sin(x^3)\big)^6$ and use Example 3.6.8 as a guide.

Answer

$h'(x) = 18x^2\sin^5(x^3)\cos(x^3)$

Example 3.6.9: Using the Chain Rule in a Velocity Problem

A particle moves along a coordinate axis. Its position at time t is given by $s(t) = \sin(2t) + \cos(3t)$. What is the velocity of the particle at time $t = \frac{\pi}{6}$?

Solution

To find $v(t)$, the velocity of the particle at time t, we must differentiate $s(t)$. Thus,

$$v(t) = s'(t) = 2\cos(2t) - 3\sin(3t).$$

Proof of Chain Rule

At this point, we present a very informal proof of the chain rule. For simplicity's sake we ignore certain issues: For example, we assume that $g(x) \neq g(a)$ for $x \neq a$ in some open interval containing a. We begin by applying the limit definition of the derivative to the function $h(x)$ to obtain $h'(a)$:

$$h'(a) = \lim_{x \to a} \frac{f\big(g(x)\big) - f\big(g(a)\big)}{x - a}.$$

Rewriting, we obtain

$$h'(a) = \lim_{x \to a} \frac{f\big(g(x)\big) - f\big(g(a)\big)}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a}.$$

Although it is clear that

$$\lim_{x \to a} \frac{g(x) - g(a)}{x - a} = g'(a),$$

it is not obvious that

$$\lim_{x \to a} \frac{f\big(g(x)\big) - f\big(g(a)\big)}{g(x) - g(a)} = f'\big(g(a)\big).$$

To see that this is true, first recall that since g is differentiable at a, g is also continuous at a. Thus,

$$\lim_{x \to a} g(x) = g(a).$$

Next, make the substitution $y = g(x)$and $b = g(a)$ and use change of variables in the limit to obtain

$$\lim_{x \to a} \frac{f\big(g(x)\big) - f\big(g(a)\big)}{g(x) - g(a)} = \lim_{y \to b} \frac{f(y) - f(b)}{y - b} = f'(b) = f'\big(g(a)\big).$$

Finally,

$$h'(a) = \lim_{x \to a} \frac{f\big(g(x)\big) - f\big(g(a)\big)}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a} = f'\big(g(a)\big) \cdot g'(a).$$

□

Example 3.6.10: Using the Chain Rule with Functional Values

Let $h(x) = f\big(g(x)\big)$. If $g(1) = 4, g'(1) = 3$, and $f'(4) = 7$, find $h'(1)$.

Solution

Use the chain rule, then substitute.

$$\begin{aligned} h'(1) &= f'\big(g(1)\big) \cdot g'(1) && \text{Apply the chain rule.} \\ &= f'(4) \cdot 3 && \text{Substitute } g(1) = 4 \text{ and } g'(1) = 3. \\ &= 7 \cdot 3 && \text{Substitute } f'(4) = 7. \\ &= 21 && \text{Simplify.} \end{aligned}$$

Exercise 3.6.6

Given $h(x) = f(g(x))$. If $g(2) = -3, g'(2) = 4$, and $f'(-3) = 7$, find $h'(2)$.

Hint

Follow Example 3.6.10.

Answer

28

The Chain Rule Using Leibniz's Notation

As with other derivatives that we have seen, we can express the chain rule using Leibniz's notation. This notation for the chain rule is used heavily in physics applications.

For $h(x) = f(g(x))$, let $u = g(x)$ and $y = h(x) = f(u)$. Thus,

$$h'(x) = \frac{dy}{dx}$$

$$f'(g(x)) = f'(u) = \frac{dy}{du}$$

and

$$g'(x) = \frac{du}{dx}.$$

Consequently,

$$\frac{dy}{dx} = h'(x) = f'\big(g(x)\big) \cdot g'(x) = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Rule: Chain Rule Using Leibniz's Notation

If y is a function of u, and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Example 3.6.11: Taking a Derivative Using Leibniz's Notation I

Find the derivative of $y = \left(\frac{x}{3x+2}\right)^5$.

Solution

First, let $u = \frac{x}{3x+2}$. Thus, $y = u^5$. Next, find $\frac{du}{dx}$ and $\frac{dy}{du}$. Using the quotient rule,

$$\frac{du}{dx} = \frac{2}{(3x+2)^2}$$

and

$$\frac{dy}{du} = 5u^4.$$

Finally, we put it all together.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} && \text{Apply the chain rule.} \\
&= 5u^4 \cdot \frac{2}{(3x+2)^2} && \text{Substitute } \frac{dy}{du} = 5u^4 \text{ and } \frac{du}{dx} = \frac{2}{(3x+2)^2}. \\
&= 5\left(\frac{x}{3x+2}\right)^4 \cdot \frac{2}{(3x+2)^2} && \text{Substitute } u = \frac{x}{3x+2}. \\
&= \frac{10x^4}{(3x+2)^6} && \text{Simplify.}
\end{aligned}$$

It is important to remember that, when using the Leibniz form of the chain rule, the final answer must be expressed entirely in terms of the original variable given in the problem.

Example 3.6.12: Taking a Derivative Using Leibniz's Notation II

Find the derivative of $y = \tan(4x^2 - 3x + 1)$.

Solution

First, let $u = 4x^2 - 3x + 1$. Then $y = \tan u$. Next, find $\frac{du}{dx}$ and $\frac{dy}{du}$:

$$\frac{du}{dx} = 8x - 3 \text{ and } \frac{dy}{du} = \sec^2 u.$$

Finally, we put it all together.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} && \text{Apply the chain rule.} \\ &= \sec^2 u \cdot (8x - 3) && \text{Use } \frac{du}{dx} = 8x - 3 \text{ and } \frac{dy}{du} = \sec^2 u. \\ &= \sec^2(4x^2 - 3x + 1) \cdot (8x - 3) && \text{Substitute } u = 4x^2 - 3x + 1. \end{aligned}$$

Exercise 3.6.7

Use Leibniz's notation to find the derivative of $y = \cos(x^3)$. Make sure that the final answer is expressed entirely in terms of the variable x.

Hint

Let $u = x^3$.

Answer

$\frac{dy}{dx} = -3x^2 \sin(x^3)$.

Key Concepts

- The chain rule allows us to differentiate compositions of two or more functions. It states that for $h(x) = f\big(g(x)\big)$,

$$h'(x) = f'\big(g(x)\big) \cdot g'(x).$$

  In Leibniz's notation this rule takes the form

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

- We can use the chain rule with other rules that we have learned, and we can derive formulas for some of them.
- The chain rule combines with the power rule to form a new rule:

$$\text{If } h(x) = \big(g(x)\big)^n, \text{ then } h'(x) = n\big(g(x)\big)^{n-1} \cdot g'(x).$$

- When applied to the composition of three functions, the chain rule can be expressed as follows: If $h(x) = f\Big(g\big(k(x)\big)\Big)$, then $h'(x) = f'\Big(g\big(k(x)\big)\Big) \cdot g'\big(k(x)\big) \cdot k'(x)$.

Key Equations

- **The chain rule**

$$h'(x) = f'\big(g(x)\big) \cdot g'(x)$$

- **The power rule for functions**

$$h'(x) = n\big(g(x)\big)^{n-1} \cdot g'(x)$$

Glossary

chain rule

the chain rule defines the derivative of a composite function as the derivative of the outer function evaluated at the inner function times the derivative of the inner function

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3.6E: EXERCISES FOR SECTION 3.6

In exercises 1 - 6, given $y = f(u)$ and $u = g(x)$, find $\frac{dy}{dx}$ by using Leibniz's notation for the chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

1) $y = 3u - 6, \quad u = 2x^2$

2) $y = 6u^3, \quad u = 7x - 4$

Answer

$\frac{dy}{dx} = 18u^2 \cdot 7 = 18(7x-4)^2 \cdot 7 = 126(7x-4)^2$

3) $y = \sin u, \quad u = 5x - 1$

4) $y = \cos u, \quad u = -\frac{x}{8}$

Answer

$\frac{dy}{dx} = -\sin u \cdot \left(-\frac{1}{8}\right) = \frac{1}{8}\sin\left(-\frac{x}{8}\right)$

5) $y = \tan u, \quad u = 9x + 2$

6) $y = \sqrt{4u+3}, \quad u = x^2 - 6x$

Answer

$\frac{dy}{dx} = \frac{8x-24}{2\sqrt{4u+3}} = \frac{4x-12}{\sqrt{4x^2-24x+3}}$

For each of the following exercises,

a. decompose each function in the form $y = f(u)$ and $u = g(x)$, and

b. find $\frac{dy}{dx}$ as a function of x.

7) $y = (3x-2)^6$

8) $y = (3x^2+1)^3$

Answer

a. $f(u) = u^3, \quad u = 3x^2 + 1;$

b. $\frac{dy}{dx} = 18x(3x^2+1)^2$

9) $y = \sin^5(x)$

10) $y = \left(\frac{x}{7} + \frac{7}{x}\right)^7$

Answer

a. $f(u) = u^7, \quad u = \frac{x}{7} + \frac{7}{x};$

b. $\frac{dy}{dx} = 7\left(\frac{x}{7} + \frac{7}{x}\right)^6 \cdot \left(\frac{1}{7} - \frac{7}{x^2}\right)$

11) $y = \tan(\sec x)$

12) $y = \csc(\pi x + 1)$

Answer

a. $f(u) = \csc u, \quad u = \pi x + 1;$

b. $\frac{dy}{dx} = -\pi \csc(\pi x + 1) \cdot \cot(\pi x + 1)$

13) $y = \cot^2 x$

14) $y = -6\sin^{-3} x$

Answer

a. $f(u) = -6u^{-3}, \quad u = \sin x;$

b. $\dfrac{dy}{dx} = 18\sin^{-4} x \cdot \cos x$

In exercises 15 - 24, find $\dfrac{dy}{dx}$ for each function.

15) $y = (3x^2 + 3x - 1)^4$

16) $y = (5 - 2x)^{-2}$

Answer

$$\frac{dy}{dx} = \frac{4}{(5-2x)^3}$$

17) $y = \cos^3(\pi x)$

18) $y = (2x^3 - x^2 + 6x + 1)^3$

Answer

$$\frac{dy}{dx} = 6(2x^3 - x^2 + 6x + 1)^2 \cdot (3x^2 - x + 3)$$

19) $y = \dfrac{1}{\sin^2(x)}$

20) $y = \Big(\tan x + \sin x\Big)^{-3}$

Answer

$$\frac{dy}{dx} = -3\Big(\tan x + \sin x\Big)^{-4} \cdot (\sec^2 x + \cos x)$$

21) $y = x^2 \cos^4 x$

22) $y = \sin(\cos 7x)$

Answer

$$\frac{dy}{dx} = -7\cos(\cos 7x) \cdot \sin 7x$$

23) $y = \sqrt{6 + \sec \pi x^2}$

24) $y = \cot^3(4x + 1)$

Answer

$$\frac{dy}{dx} = -12\cot^2(4x+1) \cdot \csc^2(4x+1)$$

25) Let $y = \big[f(x)\big]^3$ and suppose that $f'(1) = 4$ and $\frac{dy}{dx} = 10$ for $x = 1$. Find $f(1)$.

26) Let $y = \Big(f(x) + 5x^2\Big)^4$ and suppose that $f(-1) = -4$ and $\frac{dy}{dx} = 3$ when $x = -1$. Find $f'(-1)$

Answer

$f'(-1) = 10\frac{3}{4}$

27) Let $y = (f(u) + 3x)^2$ and $u = x^3 - 2x$. If $f(4) = 6$ and $\frac{dy}{dx} = 18$ when $x = 2$, find $f'(4)$.

28) [T] Find the equation of the tangent line to $y = -\sin(\frac{x}{2})$ at the origin. Use a calculator to graph the function and the tangent line together.

Answer

$y = -\frac{1}{2}x$

29) [T] Find the equation of the tangent line to $y = \Big(3x + \frac{1}{x}\Big)^2$ at the point $(1, 16)$. Use a calculator to graph the function and the tangent line together.

30) Find the x -coordinates at which the tangent line to $y = \Big(x - \frac{6}{x}\Big)^8$ is horizontal.

Answer

$x = \pm\sqrt{6}$

31) [T] Find an equation of the line that is normal to $g(\theta) = \sin^2(\pi\theta)$ at the point $\left(\frac{1}{4}, \frac{1}{2}\right)$. Use a calculator to graph the function and the normal line together.

For exercises 32 - 39, use the information in the following table to find $h'(a)$ at the given value for a.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	2	5	0	2
1	1	−2	3	0
2	4	4	1	−1
3	3	−3	2	3

32) $h(x) = f\big(g(x)\big); \quad a = 0$

Answer

$h'(0) = 10$

33) $h(x) = g\big(f(x)\big); \quad a = 0$

34) $h(x) = \big(x^4 + g(x)\big)^{-2}; \quad a = 1$

Answer

$h'(1) = -\frac{1}{8}$

35) $h(x) = \left(\dfrac{f(x)}{g(x)}\right)^2; \quad a = 3$

36) $h(x) = f\big(x + f(x)\big); \quad a = 1$

Answer

$h'(1) = -4$

37) $h(x) = \big(1 + g(x)\big)^3; \quad a = 2$

38) $h(x) = g\big(2 + f(x^2)\big); \quad a = 1$

Answer

$h'(1) = -12$

39) $h(x) = f\big(g(\sin x)\big); \quad a = 0$

40) [T] The position function of a freight train is given by $s(t) = 100(t+1)^{-2}$, with s in meters and t in seconds. At time $t = 6$ s, find the train's

a. velocity and

b. acceleration.

c. Considering your results in parts a. and b., is the train speeding up or slowing down?

Answer

a. $v(6) = -\frac{200}{343}$ m/s,

b. $a(6) = \frac{600}{2401}$ m/s^2,

c. The train is slowing down since velocity and acceleration have opposite signs.

41) [T] A mass hanging from a vertical spring is in simple harmonic motion as given by the following position function, where t is measured in seconds and s is in inches:

$$s(t) = -3\cos\left(\pi t + \frac{\pi}{4}\right).$$

a. Determine the position of the spring at $t = 1.5$ s.

b. Find the velocity of the spring at $t = 1.5$ s.

42) [T] The total cost to produce x boxes of Thin Mint Girl Scout cookies is C dollars, where $C = 0.0001x^3 - 0.02x^2 + 3x + 300$. In t weeks production is estimated to be $x = 1600 + 100t$ boxes.

a. Find the marginal cost $C'(x)$.

b. Use Leibniz's notation for the chain rule, $\frac{dC}{dt} = \frac{dC}{dx} \cdot \frac{dx}{dt}$, to find the rate with respect to time t that the cost is changing.

c. Use your result in part b. to determine how fast costs are increasing when $t = 2$ weeks. Include units with the answer.

Answer

a. $C'(x) = 0.0003x^2 - 0.04x + 3$

b. $\frac{dC}{dt} = 100 \cdot (0.0003x^2 - 0.04x + 3) = 100 \cdot (0.0003(1600 + 100t)^2 - 0.04(1600 + 100t) + 3) = 300t^2 + 9200t + 70700$

c. Approximately $90,300 per week

43) [T] The formula for the area of a circle is $A = \pi r^2$, where r is the radius of the circle. Suppose a circle is expanding, meaning that both the area A and the radius r (in inches) are expanding.

a. Suppose $r = 2 - \frac{100}{(t+7)^2}$ where t is time in seconds. Use the chain rule $\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$ to find the rate at which the area is expanding.

b. Use your result in part a. to find the rate at which the area is expanding at $t = 4$ s.

44) [T] The formula for the volume of a sphere is $S = \frac{4}{3}\pi r^3$, where r (in feet) is the radius of the sphere. Suppose a spherical snowball is melting in the sun.

a. Suppose $r = \frac{1}{(t+1)^2} - \frac{1}{12}$ where t is time in minutes. Use the chain rule $\frac{dS}{dt} = \frac{dS}{dr} \cdot \frac{dr}{dt}$ to find the rate at which the snowball is melting.

b. Use your result in part a. to find the rate at which the volume is changing at $t = 1$ min.

Answer

a. $\frac{dS}{dt} = -\frac{8\pi r^2}{(t+1)^3} = -\frac{8\pi\left(\frac{1}{(t+1)^2} - \frac{1}{12}\right)^2}{(t+1)^3}$

b. The volume is decreasing at a rate of $-\frac{\pi}{36}$ ft^3/min

45) [T] The daily temperature in degrees Fahrenheit of Phoenix in the summer can be modeled by the function $T(x) = 94 - 10\cos\left[\frac{\pi}{12}(x-2)\right]$, where x is hours after midnight. Find the rate at which the temperature is changing at 4 p.m.

46) [T] The depth (in feet) of water at a dock changes with the rise and fall of tides. The depth is modeled by the function $D(t) = 5\sin\left(\frac{\pi}{6}t - \frac{7\pi}{6}\right) + 8$, where t is the number of hours after midnight. Find the rate at which the depth is changing at 6 a.m.

Answer

2.3 ft/hr

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3.7: Derivatives of Inverse Functions

Learning Objectives

- Calculate the derivative of an inverse function.
- Recognize the derivatives of the standard inverse trigonometric functions.

In this section we explore the relationship between the derivative of a function and the derivative of its inverse. For functions whose derivatives we already know, we can use this relationship to find derivatives of inverses without having to use the limit definition of the derivative. In particular, we will apply the formula for derivatives of inverse functions to trigonometric functions. This formula may also be used to extend the power rule to rational exponents.

The Derivative of an Inverse Function

We begin by considering a function and its inverse. If $f(x)$ is both invertible and differentiable, it seems reasonable that the inverse of $f(x)$ is also differentiable. Figure 3.7.1 shows the relationship between a function $f(x)$ and its inverse $f^{-1}(x)$. Look at the point $(a,\ f^{-1}(a))$ on the graph of $f^{-1}(x)$ having a tangent line with a slope of

$$\left(f^{-1}\right)'(a) = \frac{p}{q}.$$

This point corresponds to a point $(f^{-1}(a),\ a)$ on the graph of $f(x)$ having a tangent line with a slope of

$$f'\left(f^{-1}(a)\right) = \frac{q}{p}.$$

Thus, if $f^{-1}(x)$ is differentiable at a, then it must be the case that

$$\left(f^{-1}\right)'(a) = \frac{1}{f'\left(f^{-1}(a)\right)}.$$

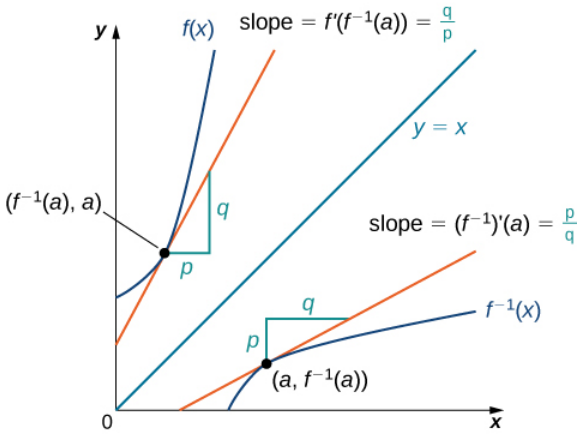

Figure 3.7.1:The tangent lines of a function and its inverse are related; so, too, are the derivatives of these functions.

We may also derive the formula for the derivative of the inverse by first recalling that $x = f\left(f^{-1}(x)\right)$. Then by differentiating both sides of this equation (using the chain rule on the right), we obtain

$$1 = f'\left(f^{-1}(x)\right)\left(f^{-1}\right)'(x)).$$

Solving for $\left(f^{-1}\right)'(x)$, we obtain

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}.$$

We summarize this result in the following theorem.

Inverse Function Theorem

Let $f(x)$ be a function that is both invertible and differentiable. Let $y = f^{-1}(x)$ be the inverse of $f(x)$. For all x satisfying $f'\left(f^{-1}(x)\right) \neq 0$,

$$\frac{dy}{dx} = \frac{d}{dx}\left(f^{-1}(x)\right) = \left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}. \tag{3.7.1}$$

Alternatively, if $y = g(x)$ is the inverse of $f(x)$, then

$$g'(x) = \frac{1}{f'\left(g(x)\right)}. \tag{3.7.2}$$

Example 3.7.1: Applying the Inverse Function Theorem

Use the inverse function theorem to find the derivative of $g(x) = \dfrac{x+2}{x}$. Compare the resulting derivative to that obtained by differentiating the function directly.

Solution

The inverse of $g(x) = \dfrac{x+2}{x}$ is $f(x) = \dfrac{2}{x-1}$.

We will use Equation 3.7.2 and begin by finding $f'(x)$. Thus,

$$f'(x) = \frac{-2}{(x-1)^2}$$

and

$$f'\left(g(x)\right) = \frac{-2}{(g(x)-1)^2} = \frac{-2}{\left(\dfrac{x+2}{x} - 1\right)^2} = -\frac{x^2}{2}.$$

Finally,

$$g'(x) = \frac{1}{f'\left(g(x)\right)} = -\frac{2}{x^2}.$$

We can verify that this is the correct derivative by applying the quotient rule to $g(x)$ to obtain

$$g'(x) = -\frac{2}{x^2}.$$

Exercise 3.7.1

Use the inverse function theorem to find the derivative of $g(x) = \dfrac{1}{x+2}$. Compare the result obtained by differentiating $g(x)$ directly.

Hint

Use the preceding example as a guide.

Answer

$g'(x) = -\dfrac{1}{(x+2)^2}$

Example 3.7.2: Applying the Inverse Function Theorem

Use the inverse function theorem to find the derivative of $g(x) = \sqrt[3]{x}$.

Solution

The function $g(x) = \sqrt[3]{x}$ is the inverse of the function $f(x) = x^3$. Since $g'(x) = \dfrac{1}{f'\big(g(x)\big)}$, begin by finding $f'(x)$. Thus,

$$f'(x) = 3x^2$$

and

$$f'\big(g(x)\big) = 3\big(\sqrt[3]{x}\big)^2 = 3x^{2/3}$$

Finally,

$$g'(x) = \frac{1}{3x^{2/3}}.$$

If we were to differentiate $g(x)$ directly, using the power rule, we would first rewrite $g(x) = \sqrt[3]{x}$ as a power of x to get,

$$g(x) = x^{1/3}$$

Then we would differentiate using the power rule to obtain

$$g'(x) = \tfrac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}.$$

Exercise 3.7.2

Find the derivative of $g(x) = \sqrt[5]{x}$ by applying the inverse function theorem.

Hint

$g(x)$ is the inverse of $f(x) = x^5$.

Answer

$g(x) = \tfrac{1}{5}x^{-4/5}$

From the previous example, we see that we can use the inverse function theorem to extend the power rule to exponents of the form $\dfrac{1}{n}$, where n is a positive integer. This extension will ultimately allow us to differentiate x^q, where q is any rational number.

Extending the Power Rule to Rational Exponents

The power rule may be extended to rational exponents. That is, if n is a positive integer, then

$$\frac{d}{dx}\big(x^{1/n}\big) = \frac{1}{n}x^{(1/n)-1}.$$

Also, if n is a positive integer and m is an arbitrary integer, then

$$\frac{d}{dx}\big(x^{m/n}\big) = \frac{m}{n}x^{(m/n)-1}.$$

Proof

The function $g(x) = x^{1/n}$ is the inverse of the function $f(x) = x^n$. Since $g'(x) = \dfrac{1}{f'\big(g(x)\big)}$, begin by finding $f'(x)$. Thus,

$$f'(x) = nx^{n-1} \text{ and } f'\big(g(x)\big) = n\big(x^{1/n}\big)^{n-1} = nx^{(n-1)/n}.$$

Finally,

$$g'(x) = \frac{1}{nx^{(n-1)/n}} = \frac{1}{n}x^{(1-n)/n} = \frac{1}{n}x^{(1/n)-1}.$$

To differentiate $x^{m/n}$ we must rewrite it as $(x^{1/n})^m$ and apply the chain rule. Thus,

$$\frac{d}{dx}\left(x^{m/n}\right) = \frac{d}{dx}\left(\left(x^{1/n}\right)^m\right) = m\left(x^{1/n}\right)^{m-1} \cdot \frac{1}{n}x^{(1/n)-1} = \frac{m}{n}x^{(m/n)-1}.$$

□

Example 3.7.3: Applying the Power Rule to a Rational Power

Find the equation of the line tangent to the graph of $y = x^{2/3}$ at $x = 8$.

Solution

First find $\frac{dy}{dx}$ and evaluate it at $x = 8$. Since

$$\frac{dy}{dx} = \frac{2}{3}x^{-1/3}$$

and

$$\left.\frac{dy}{dx}\right|_{x=8} = \frac{1}{3}$$

the slope of the tangent line to the graph at $x = 8$ is $\frac{1}{3}$.

Substituting $x = 8$ into the original function, we obtain $y = 4$. Thus, the tangent line passes through the point $(8, 4)$. Substituting into the point-slope formula for a line, we obtain the tangent line

$$y = \tfrac{1}{3}x + \tfrac{4}{3}.$$

Exercise 3.7.3

Find the derivative of $s(t) = \sqrt{2t+1}$.

Hint

Use the chain rule.

Answer

$s'(t) = (2t+1)^{-1/2}$

Derivatives of Inverse Trigonometric Functions

We now turn our attention to finding derivatives of inverse trigonometric functions. These derivatives will prove invaluable in the study of integration later in this text. The derivatives of inverse trigonometric functions are quite surprising in that their derivatives are actually algebraic functions. Previously, derivatives of algebraic functions have proven to be algebraic functions and derivatives of trigonometric functions have been shown to be trigonometric functions. Here, for the first time, we see that the derivative of a function need not be of the same type as the original function.

Example 3.7.4A: Derivative of the Inverse Sine Function

Use the inverse function theorem to find the derivative of $g(x) = \sin^{-1} x$.

Solution

Since for x in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $f(x) = \sin x$ is the inverse of $g(x) = \sin^{-1} x$, begin by finding $f'(x)$. Since

$$f'(x) = \cos x$$

and

$$f'\big(g(x)\big) = \cos\big(\sin^{-1} x\big) = \sqrt{1-x^2}$$

we see that

$$g'(x) = \frac{d}{dx}\big(\sin^{-1} x\big) = \frac{1}{f'\big(g(x)\big)} = \frac{1}{\sqrt{1-x^2}}$$

Analysis

To see that $\cos(\sin^{-1} x) = \sqrt{1-x^2}$, consider the following argument. Set $\sin^{-1} x = \theta$. In this case, $\sin\theta = x$ where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. We begin by considering the case where $0 < \theta < \frac{\pi}{2}$. Since θ is an acute angle, we may construct a right triangle having acute angle θ, a hypotenuse of length 1 and the side opposite angle θ having length x. From the Pythagorean theorem, the side adjacent to angle θ has length $\sqrt{1-x^2}$. This triangle is shown in Figure 3.7.2 Using the triangle, we see that $\cos(\sin^{-1} x) = \cos\theta = \sqrt{1-x^2}$.

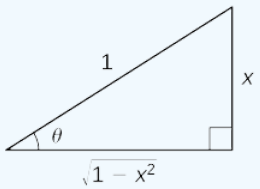

Figure 3.7.2: Using a right triangle having acute angle θ, a hypotenuse of length 1, and the side opposite angle θ having length x, we can see that $\cos(\sin^{-1} x) = \cos\theta = \sqrt{1-x^2}$.

In the case where $-\frac{\pi}{2} < \theta < 0$, we make the observation that $0 < -\theta < \frac{\pi}{2}$ and hence

$$\cos\big(\sin^{-1} x\big) = \cos\theta = \cos(-\theta) = \sqrt{1-x^2}.$$

Now if $\theta = \frac{\pi}{2}$ or $\theta = -\frac{\pi}{2}, x = 1$ or $x = -1$, and since in either case $\cos\theta = 0$ and $\sqrt{1-x^2} = 0$, we have

$$\cos\big(\sin^{-1} x\big) = \cos\theta = \sqrt{1-x^2}.$$

Consequently, in all cases,

$$\cos\big(\sin^{-1} x\big) = \sqrt{1-x^2}.$$

Example 3.7.4B: Applying the Chain Rule to the Inverse Sine Function

Apply the chain rule to the formula derived in Example 3.7.4A to find the derivative of $h(x) = \sin^{-1}\big(g(x)\big)$ and use this result to find the derivative of $h(x) = \sin^{-1}(2x^3)$.

Solution

Applying the chain rule to $h(x) = \sin^{-1}\big(g(x)\big)$, we have

$$h'(x) = \frac{1}{\sqrt{1-\big(g(x)\big)^2}} g'(x).$$

Now let $g(x) = 2x^3$, so $g'(x) = 6x^2$. Substituting into the previous result, we obtain

$$\begin{aligned} h'(x) &= \frac{1}{\sqrt{1-4x^6}} \cdot 6x^2 \\ &= \frac{6x^2}{\sqrt{1-4x^6}} \end{aligned}$$

Exercise 3.7.4

Use the inverse function theorem to find the derivative of $g(x) = \tan^{-1} x$.

Hint

The inverse of $g(x)$ is $f(x) = \tan x$. Use Example 3.7.4A as a guide.

Answer

$$g'(x) = \frac{1}{1+x^2}$$

The derivatives of the remaining inverse trigonometric functions may also be found by using the inverse function theorem. These formulas are provided in the following theorem.

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}} \tag{3.7.3}$$

$$\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}} \tag{3.7.4}$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2} \tag{3.7.5}$$

$$\frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2} \tag{3.7.6}$$

$$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}} \tag{3.7.7}$$

$$\frac{d}{dx}\left(\csc^{-1} x\right) = \frac{-1}{|x|\sqrt{x^2-1}} \tag{3.7.8}$$

Example 3.7.5A: Applying Differentiation Formulas to an Inverse Tangent Function

Find the derivative of $f(x) = \tan^{-1}(x^2)$.

Solution

Let $g(x) = x^2$, so $g'(x) = 2x$. Substituting into Equation 3.7.5, we obtain

$$f'(x) = \frac{1}{1+(x^2)^2} \cdot (2x).$$

Simplifying, we have

$$f'(x) = \frac{2x}{1+x^4}.$$

Example 3.7.5B: Applying Differentiation Formulas to an Inverse Sine Function

Find the derivative of $h(x) = x^2 \sin^{-1} x$.

Solution

By applying the product rule, we have

$$h'(x) = 2x \sin^{-1} x + \frac{1}{\sqrt{1-x^2}} \cdot x^2$$

Exercise 3.7.5

Find the derivative of $h(x) = \cos^{-1}(3x-1)$.

Hint

Use Equation 3.7.4. with $g(x) = 3x - 1$

Answer

$$h'(x) = \frac{-3}{\sqrt{6x-9x^2}}$$

Example 3.7.6: Applying the Inverse Tangent Function

The position of a particle at time t is given by $s(t) = \tan^{-1}\left(\frac{1}{t}\right)$ for $t \geq \frac{1}{2}$. Find the velocity of the particle at time $t = 1$.

Solution

Begin by differentiating $s(t)$ in order to find $v(t)$.Thus,

$$v(t) = s'(t) = \frac{1}{1 + \left(\frac{1}{t}\right)^2} \cdot \frac{-1}{t^2}.$$

Simplifying, we have

$$v(t) = -\frac{1}{t^2 + 1}.$$

Thus, $v(1) = -\frac{1}{2}$.

Exercise 3.7.6

Find the equation of the line tangent to the graph of $f(x) = \sin^{-1} x$ at $x = 0$.

Hint

$f'(0)$ is the slope of the tangent line.

Answer

$y = x$

Key Concepts

- The inverse function theorem allows us to compute derivatives of inverse functions without using the limit definition of the derivative.
- We can use the inverse function theorem to develop differentiation formulas for the inverse trigonometric functions.

Key Equations

- **Inverse function theorem**

  $(f^{-1})'(x) = \dfrac{1}{f'\left(f^{-1}(x)\right)}$ whenever $f'\left(f^{-1}(x)\right) \neq 0$ and $f(x)$ is differentiable.

- **Power rule with rational exponents**

  $\dfrac{d}{dx}\left(x^{m/n}\right) = \dfrac{m}{n}x^{(m/n)-1}$.

- **Derivative of inverse sine function**

  $\dfrac{d}{dx}\left(\sin^{-1} x\right) = \dfrac{1}{\sqrt{1-x^2}}$

- **Derivative of inverse cosine function**

  $\dfrac{d}{dx}\left(\cos^{-1} x\right) = \dfrac{-1}{\sqrt{1-x^2}}$

Derivative of inverse tangent function

$\dfrac{d}{dx}\left(\tan^{-1} x\right) = \dfrac{1}{1+x^2}$

Derivative of inverse cotangent function

$\dfrac{d}{dx}\left(\cot^{-1} x\right) = \dfrac{-1}{1+x^2}$

Derivative of inverse secant function

$$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}}$$

Derivative of inverse cosecant function

$$\frac{d}{dx}\left(\csc^{-1} x\right) = \frac{-1}{|x|\sqrt{x^2-1}}$$

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.
- Paul Seeburger (Monroe Community College) added the second half of Example 3.7.2.

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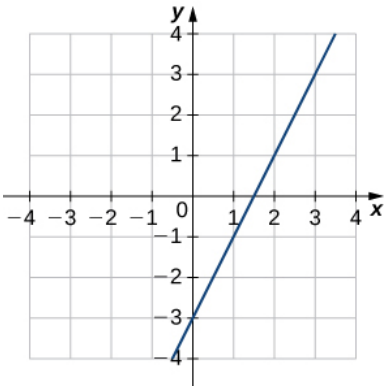
3.7E: EXERCISES FOR SECTION 3.7

In exercises 1 - 4, use the graph of $y = f(x)$ to

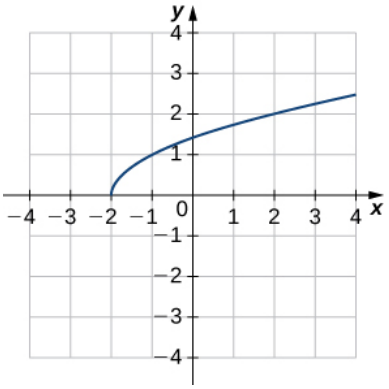
a. sketch the graph of $y = f^{-1}(x)$, and

b. use part a. to estimate $\left(f^{-1}\right)'(1)$.

1)

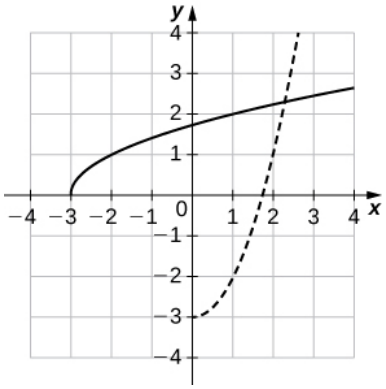


2)



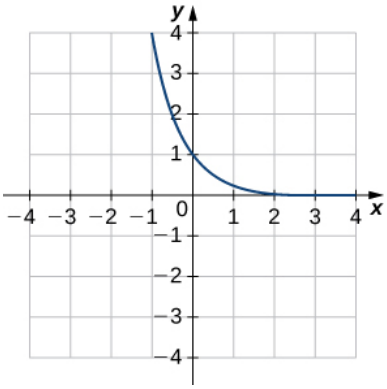
Answer

a.

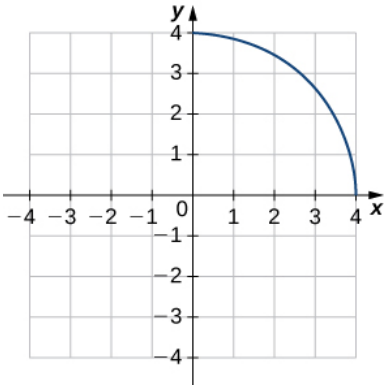


b. $(f^{-1})'(1) \approx 2$

3)

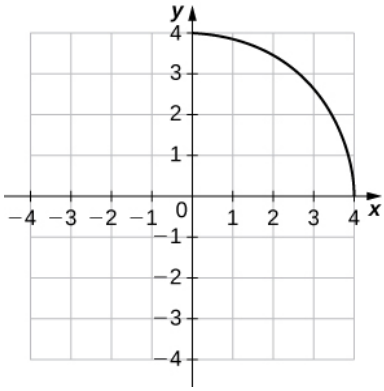


4)



Answer

a.



b. $(f^{-1})'(1) \approx -1/\sqrt{3}$

For exercises 5 - 8, use the given function $y = f(x)$ to find

a. $\frac{df}{dx}$ at $x = a$ and

b. $x = f^{-1}(y)$.

c. Then use part b. to find $\frac{df^{-1}}{dy}$ at $y = f(a)$.

5) $f(x) = 6x - 1,\ x = -2$

6) $f(x) = 2x^3 - 3,\ x = 1$

Answer

a. $\frac{df}{dx} = 6$
b. $x = f^{-1}(y) = \left(\frac{y+3}{2}\right)^{1/3}$
c. $\frac{df^{-1}}{dy} = \frac{1}{6}$

7) $f(x) = 9 - x^2,\ 0 \le x \le 3, x = 2$

8) $f(x) = \sin x,\ x = 0$

Answer

a. $\frac{df}{dx} = 1$
b. $x = f^{-1}(y) = \sin^{-1} y$
c. $\frac{df^{-1}}{dy} = 1$

For each function in exercises 9 - 14, find $\left(f^{-1}\right)'(a)$.

9) $f(x) = x^2 + 3x + 2,\ x \ge -1,\ a = 2$

10 $f(x) = x^3 + 2x + 3,\ a = 0$

Answer

$\left(f^{-1}\right)'(1) = \frac{1}{5}$

11) $f(x) = x + \sqrt{x},\ a = 2$

12) $f(x) = x - \frac{2}{x},\ x < 0,\ a = 1$

Answer

$\left(f^{-1}\right)'(1) = \frac{1}{3}$

13) $f(x) = x + \sin x,\ a = 0$

14) $f(x) = \tan x + 3x^2,\ a = 0$

Answer

$\left(f^{-1}\right)'(0) = 1$

For each function $y = f(x)$, given in exercises 15 - 19,

a. find the slope of the tangent line to its inverse function f^{-1} at the indicated point P, and

b. find the equation of the tangent line to the graph of f^{-1} at the indicated point.

15) $f(x) = \frac{4}{1+x^2},\quad P(2,1)$

16) $f(x) = \sqrt{x-4},\quad P(2,8)$

Answer

a. 4
b. $y = 4x$

17) $f(x) = (x^3+1)^4,\quad P(16,1)$

18) $f(x) = -x^3 - x + 2,\quad P(-8,2)$

Answer

a. $-\frac{1}{96}$
b. $y = -\frac{1}{13}x + \frac{18}{13}$

19) $f(x) = x^5 + 3x^3 - 4x - 8,\quad P(-8,1)$

In exercises 20 - 29, find $\frac{dy}{dx}$ for the given function.

20) $y = \sin^{-1}(x^2)$

Answer

$$\frac{dy}{dx} = \frac{2x}{\sqrt{1-x^4}}$$

21) $y = \cos^{-1}(\sqrt{x})$

22) $y = \sec^{-1}\left(\frac{1}{x}\right)$

Answer

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

23) $y = \sqrt{\csc^{-1} x}$

24) $y = (1+\tan^{-1} x)^3$

Answer

$$\frac{dy}{dx} = \frac{3(1+\tan^{-1} x)^2}{1+x^2}$$

25) $y = \cos^{-1}(2x) \cdot \sin^{-1}(2x)$

26) $y = \dfrac{1}{\tan^{-1}(x)}$

Answer

$$\frac{dy}{dx} = \frac{-1}{(1+x^2)(\tan^{-1} x)^2}$$

27) $y = \sec^{-1}(-x)$

28) $y = \cot^{-1}\sqrt{4-x^2}$

Answer

$$\frac{dy}{dx} = \frac{x}{(5-x^2)\sqrt{4-x^2}}$$

29) $y = x \cdot \csc^{-1} x$

In exercises 30 - 35, use the given values to find $\left(f^{-1}\right)'(a)$.

30) $f(\pi) = 0, f'(\pi) = -1, a = 0$

Answer

$$\left(f^{-1}\right)'(0) = -1$$

31) $f(6) = 2,\ f'(6) = \frac{1}{3},\ a = 2$

32) $f(\frac{1}{3}) = -8,\ f'(\frac{1}{3}) = 2,\ a = -8$

Answer

$$\left(f^{-1}\right)'(-8) = \tfrac{1}{2}$$

33) $f(\sqrt{3}) = \frac{1}{2}, f'(\sqrt{3}) = \frac{2}{3}, a = \frac{1}{2}$

34) $f(1) = -3,\ f'(1) = 10,\ a = -3$

Answer

$$\left(f^{-1}\right)'(-3) = \tfrac{1}{10}$$

35) $f(1) = 0,\ f'(1) = -2,\ a = 0$

36) [T] The position of a moving hockey puck after t seconds is $s(t) = tan^{-1}t$ where s is in meters.

a. Find the velocity of the hockey puck at any time t.

b. Find the acceleration of the puck at any time t.

c. Evaluate parts a. and b. for $t = 2$, 4,and 6 seconds.

d. What conclusion can be drawn from the results in c.?

Answer

a. $v(t) = \dfrac{1}{1+t^2}$

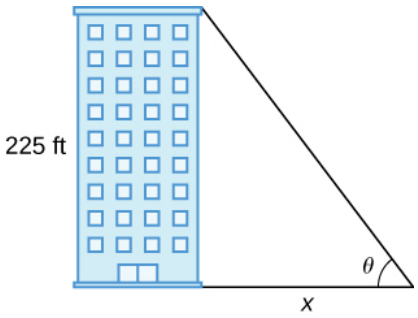
b. $a(t) = \dfrac{-2t}{(1+t^2)^2}$

c. (a) $0.2, 0.06, 0.03$; (b) $-0.16, -0.028, -0.0088$

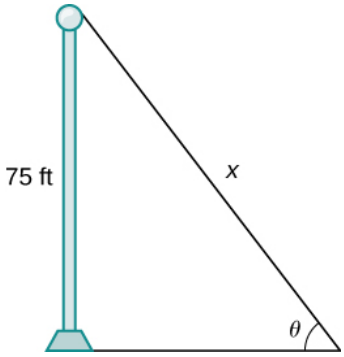
d. The hockey puck is decelerating/slowing down at 2, 4, and 6 seconds.

Solution:

37) [T] A building that is 225 feet tall casts a shadow of various lengths x as the day goes by. An angle of elevation θ is formed by lines from the top and bottom of the building to the tip of the shadow, as seen in the following figure. Find the rate of change of the angle of elevation $\frac{d\theta}{dx}$ when $x = 272$ feet.



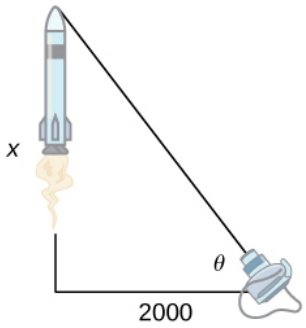
38) [T] A pole stands 75 feet tall. An angle θ is formed when wires of various lengths of x feet are attached from the ground to the top of the pole, as shown in the following figure. Find the rate of change of the angle $\frac{d\theta}{dx}$ when a wire of length 90 feet is attached.



Answer

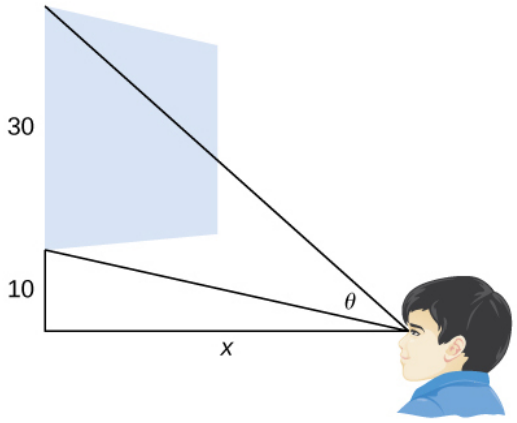
-0.0168 radians per foot

39) [T] A television camera at ground level is 2000 feet away from the launching pad of a space rocket that is set to take off vertically, as seen in the following figure. The angle of elevation of the camera can be found by $\theta = \tan^{-1}\left(\frac{x}{2000}\right)$, where x is the height of the rocket. Find the rate of change of the angle of elevation after launch when the camera and the rocket are 5000 feet apart.



40) [T] A local movie theater with a 30-foot-high screen that is 10 feet above a person's eye level when seated has a viewing angle θ (in radians) given by $\theta = \cot^{-1}\frac{x}{40} - \cot^{-1}\frac{x}{10}$,

where x is the distance in feet away from the movie screen that the person is sitting, as shown in the following figure.



a. Find $\frac{d\theta}{dx}$.

b. Evaluate $\frac{d\theta}{dx}$ for $x = 5, 10, 15,$ and 20.

c. Interpret the results in part b.

d. Evaluate $\frac{d\theta}{dx}$ for $x = 25, 30, 35$, and 40.

e. Interpret the results in part d. At what distance x should the person stand to maximize his or her viewing angle?

Answer

a. $\frac{d\theta}{dx} = \frac{10}{100 + x^2} - \frac{40}{1600 + x^2}$

b. $\frac{18}{325}, \frac{9}{340}, \frac{42}{4745}, 0$

c. As a person moves farther away from the screen, the viewing angle is increasing, which implies that as he or she moves farther away, his or her screen vision is widening. d. $-\frac{54}{12905}, -\frac{3}{500}, -\frac{198}{29945}, -\frac{9}{1360}$

e. As the person moves beyond 20 feet from the screen, the viewing angle is decreasing. The optimal distance the person should stand for maximizing the viewing angle is 20 feet.

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3.8: Implicit Differentiation

Learning Objectives

- Find the derivative of a complicated function by using implicit differentiation.
- Use implicit differentiation to determine the equation of a tangent line.

We have already studied how to find equations of tangent lines to functions and the rate of change of a function at a specific point. In all these cases we had the explicit equation for the function and differentiated these functions explicitly. Suppose instead that we want to determine the equation of a tangent line to an arbitrary curve or the rate of change of an arbitrary curve at a point. In this section, we solve these problems by finding the derivatives of functions that define y implicitly in terms of x.

Implicit Differentiation

In most discussions of math, if the dependent variable y is a function of the independent variable x, we express y in terms of x. If this is the case, we say that y is an explicit function of x. For example, when we write the equation $y = x^2 + 1$, we are defining y explicitly in terms of x. On the other hand, if the relationship between the function y and the variable x is expressed by an equation where y is not expressed entirely in terms of x, we say that the equation defines y implicitly in terms of x. For example, the equation $y - x^2 = 1$ defines the function $y = x^2 + 1$ implicitly.

Implicit differentiation allows us to find slopes of tangents to curves that are clearly not functions (they fail the vertical line test). We are using the idea that portions of y are functions that satisfy the given equation, but that y is not actually a function of x.

In general, an equation defines a function implicitly if the function satisfies that equation. An equation may define many different functions implicitly. For example, the functions

$$y = \sqrt{25 - x^2}$$

and

$$y = \begin{cases} \sqrt{25 - x^2}, & \text{if } -5 \leq x < 0 \\ -\sqrt{25 - x^2}, & \text{if } 0 \leq x \leq 5 \end{cases}$$

which are illustrated in Figure 3.8.1, are just two of the many functions defined implicitly by the equation $x^2 + y^2 = 25$.

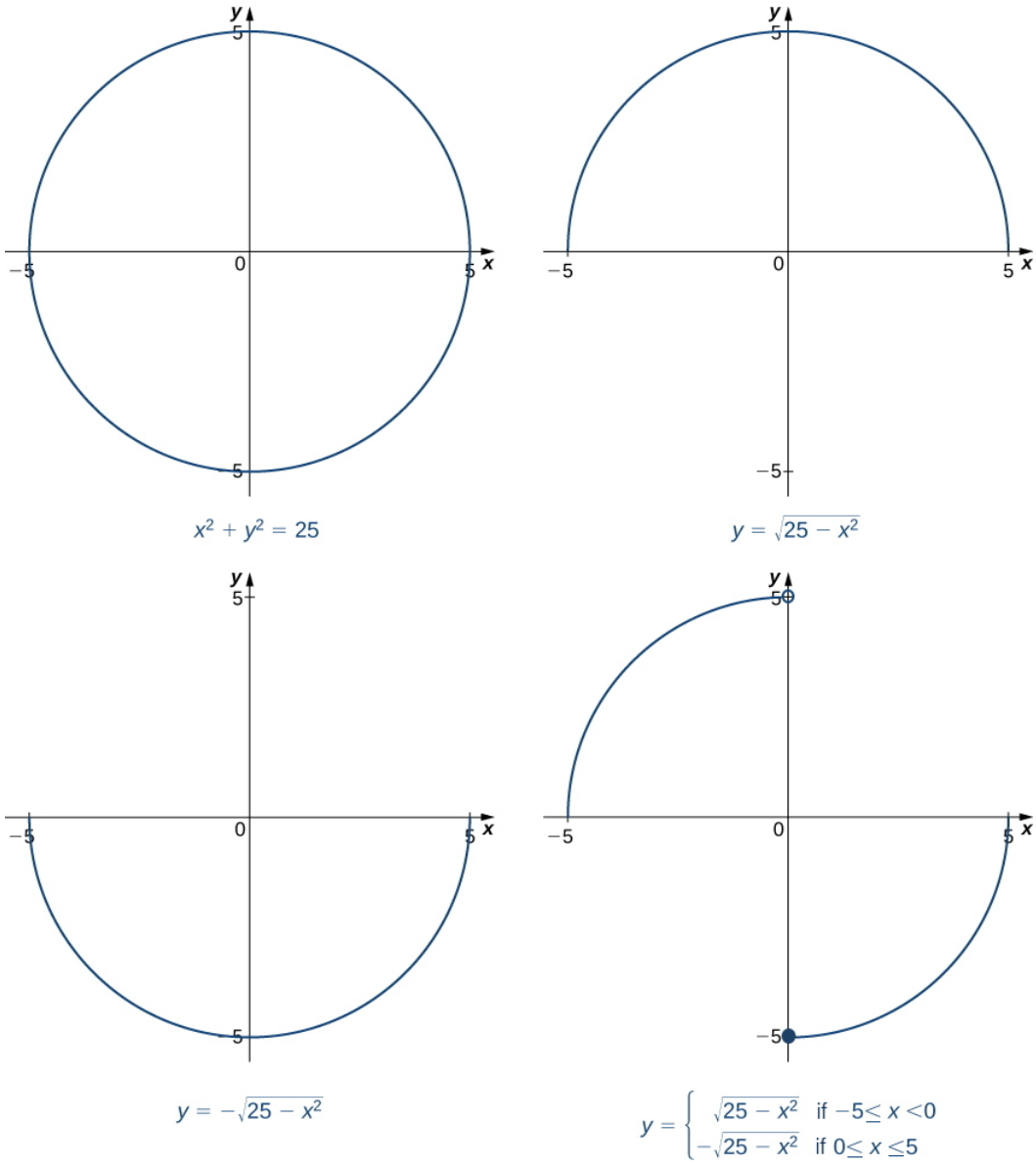

Figure 3.8.1:The equation $x^2 + y^2 = 25$ defines many functions implicitly.

If we want to find the slope of the line tangent to the graph of $x^2 + y^2 = 25$ at the point $(3, 4)$, we could evaluate the derivative of the function $y = \sqrt{25 - x^2}$ at $x = 3$. On the other hand, if we want the slope of the tangent line at the point $(3, -4)$, we could use the derivative of $y = -\sqrt{25 - x^2}$. However, it is not always easy to solve for a function defined implicitly by an equation. Fortunately, the technique of **implicit differentiation** allows us to find the derivative of an implicitly defined function without ever solving for the function explicitly. The process of finding $\frac{dy}{dx}$ using implicit differentiation is described in the following problem-solving strategy.

Problem-Solving Strategy: Implicit Differentiation

To perform *implicit differentiation* on an equation that defines a function y implicitly in terms of a variable x, use the following steps:

1. Take the derivative of both sides of the equation. Keep in mind that y is a function of x. Consequently, whereas

$$\frac{d}{dx}(\sin x) = \cos x$$

and

$$\frac{d}{dx}(\sin y) = \cos y \cdot \frac{dy}{dx}$$

because we must use the chain rule to differentiate $\sin y$ with respect to x.

2. Rewrite the equation so that all terms containing dy/dx are on the left and all terms that do not contain dy/dx are on the right.
3. Factor out dy/dx on the left.
4. Solve for dy/dx by dividing both sides of the equation by an appropriate algebraic expression.

Example 3.8.1: Using Implicit Differentiation

Assuming that y is defined implicitly by the equation $x^2 + y^2 = 25$, find $\frac{dy}{dx}$.

Solution

Follow the steps in the problem-solving strategy.

$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$	Step 1. Differentiate both sides of the equation.
$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$	Step 1.1. Use the sum rule on the left. On the right $\frac{d}{dx}(25) = 0$.
$2x + 2y\frac{dy}{dx} = 0$	Step 1.2. Take the derivatives, so $\frac{d}{dx}(x^2) = 2x$ and $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$.
$2y\frac{dy}{dx} = -2x$	Step 2. Keep the terms with $\frac{dy}{dx}$ on the left. Move the remaining terms to the right.
$\frac{dy}{dx} = -\frac{x}{y}$	Step 4. Divide both sides of the equation by $2y$. (Step 3 does not apply in this case.)

Analysis

Note that the resulting expression for $\frac{dy}{dx}$ is in terms of both the independent variable x and the dependent variable y. Although in some cases it may be possible to express $\frac{dy}{dx}$ in terms of x only, it is generally not possible to do so.

Example 3.8.2: Using Implicit Differentiation and the Product Rule

Assuming that y is defined implicitly by the equation $x^3 \sin y + y = 4x + 3$, find $\frac{dy}{dx}$.

Solution

$\frac{d}{dx}(x^3 \sin y + y) = \frac{d}{dx}(4x + 3)$	Step 1: Differentiate both sides of the equation.
$\frac{d}{dx}(x^3 \sin y) + \frac{d}{dx}(y) = 4$	Step 1.1: Apply the sum rule on the left. On the right, $\frac{d}{dx}(4x + 3) = 4$.
$\left(\frac{d}{dx}(x^3) \cdot \sin y + \frac{d}{dx}(\sin y) \cdot x^3\right) + \frac{dy}{dx} = 4$	Step 1.2: Use the product rule to find $\frac{d}{dx}(x^3 \sin y)$. Observe that $\frac{d}{dx}(y) = \frac{dy}{dx}$.
$3x^2 \sin y + (\cos y\frac{dy}{dx}) \cdot x^3 + \frac{dy}{dx} = 4$	Step 1.3: We know $\frac{d}{dx}(x^3) = 3x^2$. Use the chain rule to obtain $\frac{d}{dx}(\sin y) = \cos y\frac{dy}{dx}$.
$x^3 \cos y\frac{dy}{dx} + \frac{dy}{dx} = 4 - 3x^2 \sin y$	Step 2: Keep all terms containing $\frac{dy}{dx}$ on the left. Move all other terms to the right.
$\frac{dy}{dx}(x^3 \cos y + 1) = 4 - 3x^2 \sin y$	Step 3: Factor out $\frac{dy}{dx}$ on the left.

$\frac{dy}{dx} = \frac{4 - 3x^2 \sin y}{x^3 \cos y + 1}$	Step 4: Solve for $\frac{dy}{dx}$ by dividing both sides of the equation by $x^3 \cos y + 1$.

Example 3.8.3: Using Implicit Differentiation to Find a Second Derivative

Find $\frac{d^2y}{dx^2}$ if $x^2 + y^2 = 25$.

Solution

In Example 3.8.1, we showed that $\frac{dy}{dx} = -\frac{x}{y}$. We can take the derivative of both sides of this equation to find $\frac{d^2y}{dx^2}$.

$$\frac{d^2y}{dx^2} = \frac{d}{dy}\left(-\frac{x}{y}\right) \quad \text{Differentiate both sides of } \frac{dy}{dx} = -\frac{x}{y}.$$

$$= -\frac{\left(1 \cdot y - x\frac{dy}{dx}\right)}{y^2} \quad \text{Use the quotient rule to find } \frac{d}{dy}\left(-\frac{x}{y}\right).$$

$$= \frac{-y + x\frac{dy}{dx}}{y^2} \quad \text{Simplify.}$$

$$= \frac{-y + x\left(-\frac{x}{y}\right)}{y^2} \quad \text{Substitute } \frac{dy}{dx} = -\frac{x}{y}.$$

$$= \frac{-y^2 - x^2}{y^3} \quad \text{Simplify.}$$

At this point we have found an expression for $\frac{d^2y}{dx^2}$. If we choose, we can simplify the expression further by recalling that $x^2 + y^2 = 25$ and making this substitution in the numerator to obtain $\frac{d^2y}{dx^2} = -\frac{25}{y^3}$.

Exercise 3.8.1

Find $\frac{dy}{dx}$ for y defined implicitly by the equation $4x^5 + \tan y = y^2 + 5x$.

Hint

Follow the problem solving strategy, remembering to apply the chain rule to differentiate $\tan y$ and y^2.

Answer

$$\frac{dy}{dx} = \frac{5 - 20x^4}{\sec^2 y - 2y}$$

Finding Tangent Lines Implicitly

Now that we have seen the technique of implicit differentiation, we can apply it to the problem of finding equations of tangent lines to curves described by equations.

Example 3.8.4: Finding a Tangent Line to a Circle

Find the equation of the line tangent to the curve $x^2 + y^2 = 25$ at the point $(3, -4)$.

Solution

Although we could find this equation without using implicit differentiation, using that method makes it much easier. In Example 3.8.1, we found $\frac{dy}{dx} = -\frac{x}{y}$.

The slope of the tangent line is found by substituting $(3, -4)$ into this expression. Consequently, the slope of the tangent line is $\left.\frac{dy}{dx}\right|_{(3,-4)} = -\frac{3}{-4} = \frac{3}{4}$.

Using the point $(3, -4)$ and the slope $\frac{3}{4}$ in the point-slope equation of the line, we obtain the equation $y = \frac{3}{4}x - \frac{25}{4}$ (Figure 3.8.2).

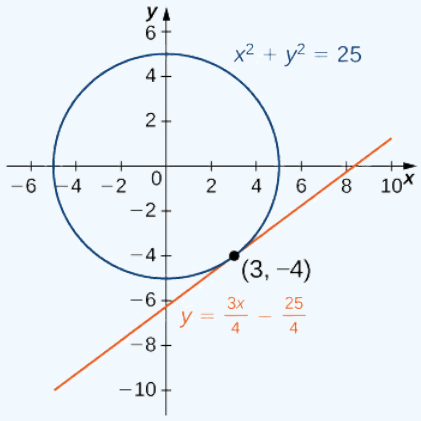

Figure 3.8.2: The line $y = \frac{3}{4}x - \frac{25}{4}$ is tangent to $x^2 + y^2 = 25$ at the point (3, −4).

Example 3.8.5: Finding the Equation of the Tangent Line to a Curve

Find the equation of the line tangent to the graph of $y^3 + x^3 - 3xy = 0$ at the point $\left(\frac{3}{2}, \frac{3}{2}\right)$ (Figure 3.8.3). This curve is known as the folium (or leaf) of Descartes.

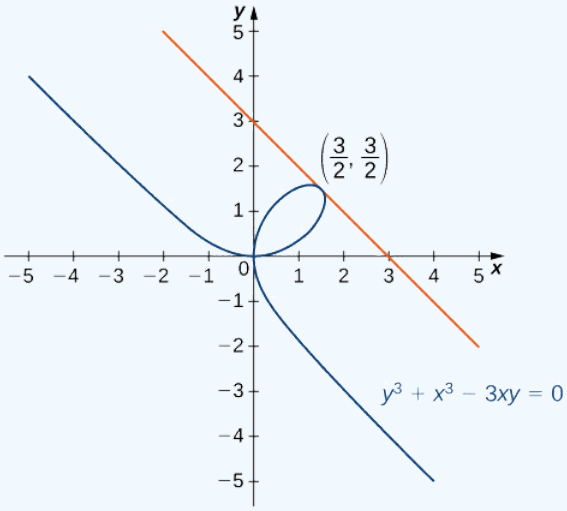

Figure 3.8.3: Finding the tangent line to the folium of Descartes at $\left(\frac{3}{2}, \frac{3}{2}\right)$.

Solution

Begin by finding $\frac{dy}{dx}$.

$$\frac{d}{dx}\left(y^3 + x^3 - 3xy\right) = \frac{d}{dx}\left(0\right)$$
$$3y^2\frac{dy}{dx} + 3x^2 - \left(3y + 3x\frac{dy}{dx}\right) = 0$$
$$3y^2\frac{dy}{dx} + 3x^2 - 3y - 3x\frac{dy}{dx} = 0$$
$$\left(3y^2 - 3x\right)\frac{dy}{dx} = 3y - 3x^2$$

$$\frac{dy}{dx} = \frac{3y - 3x^2}{3y^2 - 3x}.$$

Next, substitute $\left(\frac{3}{2}, \frac{3}{2}\right)$ into $\frac{dy}{dx} = \frac{3y - 3x^2}{3y^2 - 3x}$ to find the slope of the tangent line:

$$\left.\frac{dy}{dx}\right|_{\left(\frac{3}{2}, \frac{3}{2}\right)} = -1.$$

Finally, substitute into the point-slope equation of the line to obtain

$$y = -x + 3.$$

Example 3.8.6: Applying Implicit Differentiation

In a simple video game, a rocket travels in an elliptical orbit whose path is described by the equation $4x^2 + 25y^2 = 100$. The rocket can fire missiles along lines tangent to its path. The object of the game is to destroy an incoming asteroid traveling along the positive x-axis toward $(0, 0)$. If the rocket fires a missile when it is located at $\left(3, \frac{8}{5}\right)$, where will it intersect the x-axis?

Solution

To solve this problem, we must determine where the line tangent to the graph of

$4x^2 + 25y^2 = 100$ at $\left(3, \frac{8}{5}\right)$ intersects the x-axis. Begin by finding $\frac{dy}{dx}$ implicitly.

Differentiating, we have

$$8x + 50y\frac{dy}{dx} = 0.$$

Solving for $\frac{dy}{dx}$,

we have

$$\frac{dy}{dx} = -\frac{4x}{25y}.$$

The slope of the tangent line is $\left.\frac{dy}{dx}\right|_{\left(3, \frac{8}{5}\right)} = -\frac{3}{10}$. The equation of the tangent line is $y = -\frac{3}{10}x + \frac{5}{2}$. To determine where the line intersects the x-axis, solve $0 = -\frac{3}{10}x + \frac{5}{2}$. The solution is $x = \frac{25}{3}$. The missile intersects the x-axis at the point $\left(\frac{25}{3}, 0\right)$.

Exercise 3.8.2

Find the equation of the line tangent to the hyperbola $x^2 - y^2 = 16$ at the point $(5, 3)$.

Hint

$\frac{dy}{dx} = \frac{x}{y}$

Answer

$y = \frac{5}{3}x - \frac{16}{3}$

Key Concepts

- We use implicit differentiation to find derivatives of implicitly defined functions (functions defined by equations).
- By using implicit differentiation, we can find the equation of a tangent line to the graph of a curve.

Glossary

implicit differentiation

is a technique for computing $\frac{dy}{dx}$ for a function defined by an equation, accomplished by differentiating both sides of the equation (remembering to treat the variable y as a function) and solving for $\frac{dy}{dx}$

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3.8E: EXERCISES FOR SECTION 3.8

In exercises 1 - 10, use implicit differentiation to find $\frac{dy}{dx}$.

1) $x^2 - y^2 = 4$

2) $6x^2 + 3y^2 = 12$

Answer

$$\frac{dy}{dx} = \frac{-2x}{y}$$

3) $x^2y = y - 7$

4) $3x^3 + 9xy^2 = 5x^3$

Answer

$$\frac{dy}{dx} = \frac{x}{3y} - \frac{y}{2x}$$

5) $xy - \cos(xy) = 1$

6) $y\sqrt{x+4} = xy + 8$

Answer

$$\frac{dy}{dx} = \frac{y - \frac{y}{2\sqrt{x+4}}}{\sqrt{x+4} - x}$$

7) $-xy - 2 = \frac{x}{7}$

8) $y\sin(xy) = y^2 + 2$

Answer

$$\frac{dy}{dx} = \frac{y^2\cos(xy)}{2y - \sin(xy) - xy\cos(xy)}$$

9) $(xy)^2 + 3x = y^2$

10) $x^3y + xy^3 = -8$

Answer

$$\frac{dy}{dx} = \frac{-3x^2y - y^3}{x^3 + 3xy^2}$$

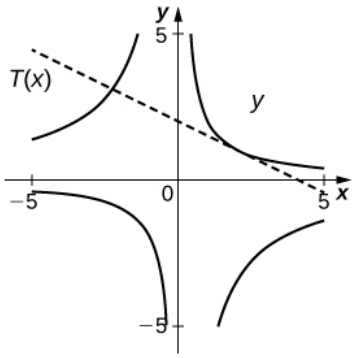
For exercises 11 - 16, find the equation of the tangent line to the graph of the given equation at the indicated point. Use a calculator or computer software to graph the function and the tangent line.

11) [T] $x^4y - xy^3 = -2, \quad (-1, -1)$

12) [T] $x^2y^2 + 5xy = 14, \quad (2, 1)$

Answer

$y = -\frac{1}{2}x + 2$

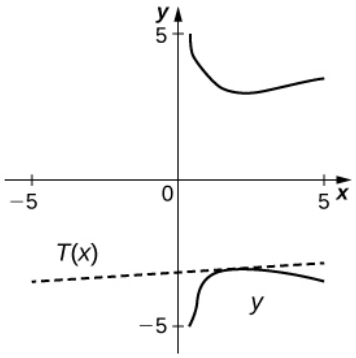


13) [T] $\tan(xy) = y, \quad (\frac{\pi}{4}, 1)$

14) [T] $xy^2 + \sin(\pi y) - 2x^2 = 10, \quad (2, -3)$

Answer

$y = \frac{1}{\pi+12}x - \frac{3\pi+38}{\pi+12}$

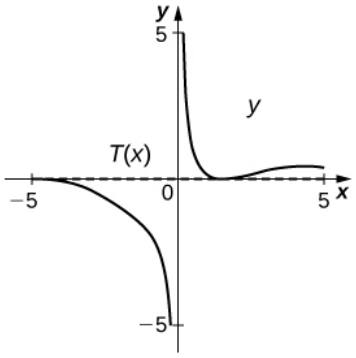


15) [T] $\frac{x}{y} + 5x - 7 = -\frac{3}{4}y, \quad (1, 2)$

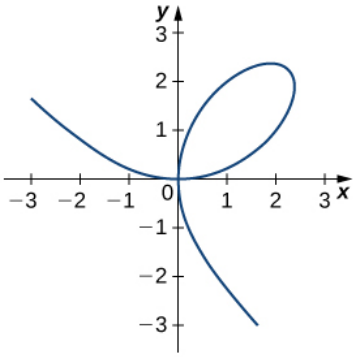
16) [T] $xy + \sin(x) = 1, \quad (\frac{\pi}{2}, 0)$

Answer

$y = 0$



17) [T] The graph of a folium of Descartes with equation $2x^3 + 2y^3 - 9xy = 0$ is given in the following graph.



a. Find the equation of the tangent line at the point $(2, 1)$. Graph the tangent line along with the folium.

b. Find the equation of the normal line to the tangent line in a. at the point $(2, 1)$.

18) For the equation $x^2 + 2xy - 3y^2 = 0$,

a. Find the equation of the normal to the tangent line at the point $(1, 1)$.

b. At what other point does the normal line in a. intersect the graph of the equation?

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Answer

a. $y = -x + 2$
b. $(3, -1)$

19) Find all points on the graph of $y^3 - 27y = x^2 - 90$ at which the tangent line is vertical.

20) For the equation $x^2 + xy + y^2 = 7$,

a. Find the x-intercept(s).

b.Find the slope of the tangent line(s) at the x-intercept(s).

c. What does the value(s) in part b. indicate about the tangent line(s)?

Answer

a. $\left(\pm\sqrt{7}, 0\right)$
b. -2
c. They are parallel since the slope is the same at both intercepts.

21) Find the equation of the tangent line to the graph of the equation $\sin^{-1} x + \sin^{-1} y = \frac{\pi}{6}$ at the point $\left(0, \frac{1}{2}\right)$.

22) Find the equation of the tangent line to the graph of the equation $\tan^{-1}(x + y) = x^2 + \frac{\pi}{4}$ at the point $(0, 1)$.

Answer

$y = -x + 1$

23) Find y' and y'' for $x^2 + 6xy - 2y^2 = 3$.

24) [T] The number of cell phones produced when x dollars is spent on labor and y dollars is spent on capital invested by a manufacturer can be modeled by the equation $60x^{3/4}y^{1/4} = 3240$.

a. Find $\frac{dy}{dx}$ and evaluate at the point $(81, 16)$.

b. Interpret the result of a.

Answer

a. $\frac{dy}{dx} = -0.5926$
b. When \$81 is spent on labor and \$16 is spent on capital, the amount spent on capital is decreasing by \$0.5926 per \$1 spent on labor.

25) [T] The number of cars produced when x dollars is spent on labor and y dollars is spent on capital invested by a manufacturer can be modeled by the equation $30x^{1/3}y^{2/3} = 360$.

(Both $x$and y are measured in thousands of dollars.)

a. Find $\frac{dy}{dx}$ and evaluate at the point $(27, 8)$.

b. Interpret the result of part a.

26) The volume of a right circular cone of radius x and height y is given by $V = \frac{1}{3}\pi x^2 y$. Suppose that the volume of the cone is $85\pi\ \text{cm}^3$. Find $\frac{dy}{dx}$ when $x = 4$ and $y = 16$.

Answer

$\frac{dy}{dx} = -8$

For exercises 27 - 28, consider a closed rectangular box with a square base with side x and height y.

27) Find an equation for the surface area of the rectangular box, $S(x, y)$.

28) If the surface area of the rectangular box is 78 square feet, find $\frac{dy}{dx}$ when $x = 3$ feet and $y = 5$ feet.

Answer

$\frac{dy}{dx} = -2.67$

In exercises 29 - 31, use implicit differentiation to determine y'. Does the answer agree with the formulas we have previously determined?

29) $x = \sin y$

30) $x = \cos y$

Answer

$$y' = -\frac{1}{\sqrt{1-x^2}}$$

31) $x = \tan y$

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3.9: Derivatives of Exponential and Logarithmic Functions

Learning Objectives

- Find the derivative of exponential functions.
- Find the derivative of logarithmic functions.
- Use logarithmic differentiation to determine the derivative of a function.

So far, we have learned how to differentiate a variety of functions, including trigonometric, inverse, and implicit functions. In this section, we explore derivatives of exponential and logarithmic functions. As we discussed in Introduction to Functions and Graphs, exponential functions play an important role in modeling population growth and the decay of radioactive materials. Logarithmic functions can help rescale large quantities and are particularly helpful for rewriting complicated expressions.

Derivative of the Exponential Function

Just as when we found the derivatives of other functions, we can find the derivatives of exponential and logarithmic functions using formulas. As we develop these formulas, we need to make certain basic assumptions. The proofs that these assumptions hold are beyond the scope of this course.

First of all, we begin with the assumption that the function $B(x) = b^x$, $b > 0$, is defined for every real number and is continuous. In previous courses, the values of exponential functions for all rational numbers were defined—beginning with the definition of b^n, where n is a positive integer—as the product of b multiplied by itself n times. Later, we defined $b^0 = 1, b^{-n} = \frac{1}{b^n}$, for a positive integer n, and $b^{s/t} = (\sqrt[t]{b})^s$ for positive integers s and t. These definitions leave open the question of the value of b^r where r is an arbitrary real number. By assuming the continuity of $B(x) = b^x, b > 0$, we may interpret b^r as $\lim_{x \to r} b^x$ where the values of x as we take the limit are rational. For example, we may view 4^π as the number satisfying

$$4^3 < 4^\pi < 4^4, \quad 4^{3.1} < 4^\pi < 4^{3.2}, \quad 4^{3.14} < 4^\pi < 4^{3.15},$$

$$4^{3.141} < 4^\pi < 4^{3.142}, \quad 4^{3.1415} < 4^\pi < 4^{3.1416}, \quad \ldots.$$

As we see in the following table, $4^\pi \approx 77.88$.

x	4^x	x	4^x
4^3	64	$4^{3.141593}$	77.8802710486
$4^{3.1}$	73.5166947198	$4^{3.1416}$	77.8810268071
$4^{3.14}$	77.7084726013	$4^{3.142}$	77.9242251944
$4^{3.141}$	77.8162741237	$4^{3.15}$	78.7932424541
$4^{3.1415}$	77.8702309526	$4^{3.2}$	84.4485062895
$4^{3.14159}$	77.8799471543	4^4	256

Approximating a Value of 4^π

We also assume that for $B(x) = b^x$, $b > 0$, the value $B'(0)$ of the derivative exists. In this section, we show that by making this one additional assumption, it is possible to prove that the function $B(x)$ is differentiable everywhere.

We make one final assumption: that there is a unique value of $b > 0$ for which $B'(0) = 1$. We define e to be this unique value, as we did in Introduction to Functions and Graphs. Figure 3.9.1 provides graphs of the functions $y = 2^x$, $y = 3^x$, $y = 2.7^x$, and $y = 2.8^x$. A visual estimate of the slopes of the tangent lines to these functions at 0 provides evidence that the value of e lies somewhere between 2.7 and 2.8. The function $E(x) = e^x$ is called the **natural exponential function**. Its inverse, $L(x) = \log_e x = \ln x$ is called the **natural logarithmic function.**

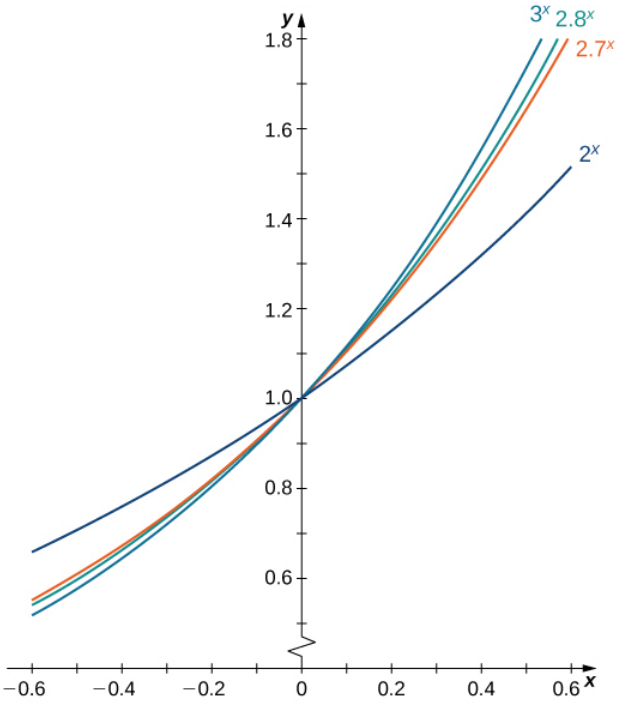

Figure 3.9.1: The graph of $E(x) = e^x$ is between $y = 2^x$ and $y = 3^x$.

For a better estimate of e, we may construct a table of estimates of $B'(0)$ for functions of the form $B(x) = b^x$. Before doing this, recall that

$$B'(0) = \lim_{x \to 0} \frac{b^x - b^0}{x - 0} = \lim_{x \to 0} \frac{b^x - 1}{x} \approx \frac{b^x - 1}{x}$$

for values of x very close to zero. For our estimates, we choose $x = 0.00001$ and $x = -0.00001$

to obtain the estimate

$$\frac{b^{-0.00001} - 1}{-0.00001} < B'(0) < \frac{b^{0.00001} - 1}{0.00001}.$$

See the following table.

Table : Estimating the value of e

b	$\frac{b^{-0.00001}-1}{-0.00001} < B'(0) < \frac{b^{0.00001}-1}{0.00001}$.	b	$\frac{b^{-0.00001}-1}{-0.00001} < B'(0) < \frac{b^{0.00001}-1}{0.00001}$.
2	$0.693145 < B'(0) < 0.69315$	2.7183	$1.000002 < B'(0) < 1.000012$
2.7	$0.993247 < B'(0) < 0.993257$	2.719	$1.000259 < B'(0) < 1.000269$
2.71	$0.996944 < B'(0) < 0.996954$	2.72	$1.000627 < B'(0) < 1.000637$
2.718	$0.999891 < B'(0) < 0.999901$	2.8	$1.029614 < B'(0) < 1.029625$
2.7182	$0.999965 < B'(0) < 0.999975$	3	$1.098606 < B'(0) < 1.098618$

The evidence from the table suggests that $2.7182 < e < 2.7183$.

The graph of $E(x) = e^x$ together with the line $y = x + 1$ are shown in Figure 3.9.2. This line is tangent to the graph of $E(x) = e^x$ at $x = 0$.

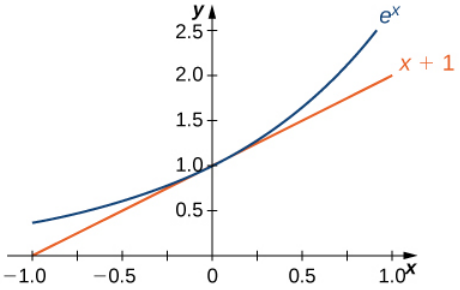

Figure 3.9.2: The tangent line to $E(x) = e^x$ at $x = 0$ has slope 1.

Now that we have laid out our basic assumptions, we begin our investigation by exploring the derivative of $B(x) = b^x$, $b > 0$. Recall that we have assumed that $B'(0)$ exists. By applying the limit definition to the derivative we conclude that

$$B'(0) = \lim_{h \to 0} \frac{b^{0+h} - b^0}{h} = \lim_{h \to 0} \frac{b^h - 1}{h}$$

Turning to $B'(x)$, we obtain the following.

$$\begin{aligned}
B'(x) &= \lim_{h \to 0} \frac{b^{x+h} - b^x}{h} && \text{Apply the limit definition of the derivative.} \\
&= \lim_{h \to 0} \frac{b^x b^h - b^x}{h} && \text{Note that } b^{x+h} = b^x b^h. \\
&= \lim_{h \to 0} \frac{b^x (b^h - 1)}{h} && \text{Factor out } b^x. \\
&= b^x \lim_{h \to 0} \frac{b^h - 1}{h} && \text{Apply a property of limits.} \\
&= b^x B'(0) && \text{Use } B'(0) = \lim_{h \to 0} \frac{b^{0+h} - b^0}{h} = \lim_{h \to 0} \frac{b^h - 1}{h}.
\end{aligned}$$

We see that on the basis of the assumption that $B(x) = b^x$ is differentiable at 0, $B(x)$ is not only differentiable everywhere, but its derivative is

$$B'(x) = b^x B'(0).$$

For $E(x) = e^x$, $E'(0) = 1$. Thus, we have $E'(x) = e^x$. (The value of $B'(0)$ for an arbitrary function of the form $B(x) = b^x$, $b > 0$, will be derived later.)

Derivative of the Natural Exponential Function

Let $E(x) = e^x$ be the natural exponential function. Then

$$E'(x) = e^x.$$

In general,

$$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)} g'(x)$$

Example 3.9.1: Derivative of an Exponential Function

Find the derivative of $f(x) = e^{\tan(2x)}$.

Solution:

Using the derivative formula and the chain rule,

$$f'(x) = e^{\tan(2x)} \frac{d}{dx}\left(\tan(2x)\right) = e^{\tan(2x)} \sec^2(2x) \cdot 2$$

Example 3.9.2: Combining Differentiation Rules

Find the derivative of $y = \dfrac{e^{x^2}}{x}$.

Solution

Use the derivative of the natural exponential function, the quotient rule, and the chain rule.

$$y' = \frac{(e^{x^2} \cdot 2)x \cdot x - 1 \cdot e^{x^2}}{x^2} \qquad \text{Apply the quotient rule.}$$

$$= \frac{e^{x^2}(2x^2 - 1)}{x^2} \qquad \text{Simplify.}$$

Exercise 3.9.1

Find the derivative of $h(x) = xe^{2x}$.

Hint

Don't forget to use the product rule.

Answer

$h'(x) = e^{2x} + 2xe^{2x}$

Example 3.9.3: Applying the Natural Exponential Function

A colony of mosquitoes has an initial population of 1000. After t days, the population is given by $A(t) = 1000e^{0.3t}$. Show that the ratio of the rate of change of the population, $A'(t)$, to the population, $A(t)$ is constant.

Solution

First find $A'(t)$. By using the chain rule, we have $A'(t) = 300e^{0.3t}$. Thus, the ratio of the rate of change of the population to the population is given by

$$\frac{A'(t)}{A(t)} = \frac{300e^{0.3t}}{1000e^{0.3t}} = 0.3.$$

The ratio of the rate of change of the population to the population is the constant 0.3.

Exercise 3.9.2

If $A(t) = 1000e^{0.3t}$ describes the mosquito population after t days, as in the preceding example, what is the rate of change of $A(t)$ after 4 days?

Hint

Find $A'(4)$.

Answer

996

Derivative of the Logarithmic Function

Now that we have the derivative of the natural exponential function, we can use implicit differentiation to find the derivative of its inverse, the natural logarithmic function.

Definition: The Derivative of the Natural Logarithmic Function

If $x > 0$ and $y = \ln x$, then

$$\frac{dy}{dx} = \frac{1}{x}.$$

More generally, let $g(x)$ be a differentiable function. For all values of x for which $g'(x) > 0$, the derivative of $h(x) = \ln(g(x))$ is given by

$$h'(x) = \frac{1}{g(x)}g'(x).$$

Proof

If $x > 0$ and $y = \ln x$, then $e^y = x$. Differentiating both sides of this equation results in the equation

$$e^y \frac{dy}{dx} = 1.$$

Solving for $\frac{dy}{dx}$ yields

$$\frac{dy}{dx} = \frac{1}{e^y}.$$

Finally, we substitute $x = e^y$ to obtain

$$\frac{dy}{dx} = \frac{1}{x}.$$

We may also derive this result by applying the inverse function theorem, as follows. Since $y = g(x) = \ln x$ is the inverse of $f(x) = e^x$, by applying the inverse function theorem we have

$$\frac{dy}{dx} = \frac{1}{f'(g(x))} = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

Using this result and applying the chain rule to $h(x) = \ln(g(x))$ yields

$$h'(x) = \frac{1}{g(x)}g'(x). \tag{3.9.1}$$

□

The graph of $y = \ln x$ and its derivative $\frac{dy}{dx} = \frac{1}{x}$ are shown in Figure 3.9.3.

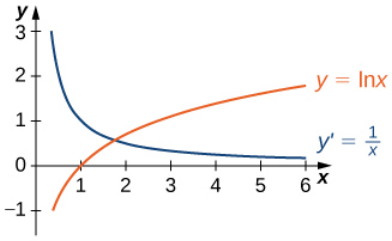

Figure 3.9.3: The function $y = \ln x$ is increasing on $(0, +\infty)$. Its derivative $y' = \frac{1}{x}$ is greater than zero on $(0, +\infty)$

Example 3.9.4: Taking a Derivative of a Natural Logarithm

Find the derivative of $f(x) = \ln(x^3 + 3x - 4)$.

Solution

Use Equation 3.9.1 directly.

$f'(x) = \dfrac{1}{x^3+3x-4} \cdot (3x^2+3)$ Use $g(x) = x^3+3x-4$ in $h'(x) = \dfrac{1}{g(x)}g'(x)$.

$= \dfrac{3x^2+3}{x^3+3x-4}$ Rewrite.

Example 3.9.5: Using Properties of Logarithms in a Derivative

Find the derivative of $f(x) = \ln\left(\dfrac{x^2 \sin x}{2x+1}\right)$.

Solution

At first glance, taking this derivative appears rather complicated. However, by using the properties of logarithms prior to finding the derivative, we can make the problem much simpler.

$f(x) = \ln\left(\dfrac{x^2 \sin x}{2x+1}\right) = 2\ln x + \ln(\sin x) - \ln(2x+1)$ Apply properties of logarithms.

$f'(x) = \dfrac{2}{x} + \cot x - \dfrac{2}{2x+1}$ Apply sum rule and $h'(x) = \dfrac{1}{g(x)}g'(x)$.

Exercise 3.9.3

Differentiate: $f(x) = \ln(3x+2)^5$.

Hint

Use a property of logarithms to simplify before taking the derivative.

Answer

$f'(x) = \dfrac{15}{3x+2}$

Now that we can differentiate the natural logarithmic function, we can use this result to find the derivatives of $y = \log_b x$ and $y = b^x$ for $b > 0,\ b \neq 1$.

Derivatives of General Exponential and Logarithmic Functions

Let $b > 0, b \neq 1$, and let $g(x)$ be a differentiable function.

i. If $y = \log_b x$, then

$$\frac{dy}{dx} = \frac{1}{x \ln b}.$$

More generally, if $h(x) = \log_b(g(x))$, then for all values of x for which $g(x) > 0$,

$$h'(x) = \frac{g'(x)}{g(x)\ln b}. \tag{3.9.2}$$

ii. If $y = b^x$, then

$$\frac{dy}{dx} = b^x \ln b.$$

More generally, if $h(x) = b^{g(x)}$, then

$$h'(x) = b^{g(x)} g'(x) \ln b \tag{3.9.3}$$

Proof

If $y = \log_b x$, then $b^y = x$. It follows that $\ln(b^y) = \ln x$. Thus $y \ln b = \ln x$. Solving for y, we have $y = \dfrac{\ln x}{\ln b}$. Differentiating and keeping in mind that $\ln b$ is a constant, we see that

$$\frac{dy}{dx} = \frac{1}{x \ln b}.$$

The derivative in Equation 3.9.2 now follows from the chain rule.

If $y = b^x$. then $\ln y = x \ln b$. Using implicit differentiation, again keeping in mind that $\ln b$ is constant, it follows that $\dfrac{1}{y}\dfrac{dy}{dx} = \ln b$. Solving for $\dfrac{dy}{dx}$ and substituting $y = b^x$, we see that

$$\frac{dy}{dx} = y \ln b = b^x \ln b.$$

The more general derivative (Equation 3.9.3) follows from the chain rule.

□

Example 3.9.6: Applying Derivative Formulas

Find the derivative of $h(x) = \dfrac{3^x}{3^x + 2}$.

Solution

Use the quotient rule and Note.

$$h'(x) = \frac{3^x \ln 3(3^x + 2) - 3^x \ln 3(3^x)}{(3^x + 2)^2} \qquad \text{Apply the quotient rule.}$$

$$= \frac{2 \cdot 3^x \ln 3}{(3x + 2)^2} \qquad \text{Simplify.}$$

Example 3.9.7: Finding the Slope of a Tangent Line

Find the slope of the line tangent to the graph of $y = \log_2(3x + 1)$ at $x = 1$.

Solution

To find the slope, we must evaluate $\dfrac{dy}{dx}$ at $x = 1$. Using Equation 3.9.2, we see that

$$\frac{dy}{dx} = \frac{3}{(3x + 1) \ln 2}.$$

By evaluating the derivative at $x = 1$, we see that the tangent line has slope

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{3}{4 \ln 2} = \frac{3}{\ln 16}.$$

Exercise 3.9.4

Find the slope for the line tangent to $y = 3^x$ at $x = 2$.

Hint

Evaluate the derivative at $x = 2$.

Answer

$9 \ln(3)$

Logarithmic Differentiation

At this point, we can take derivatives of functions of the form $y = (g(x))^n$ for certain values of n, as well as functions of the form $y = b^{g(x)}$, where $b > 0$ and $b \neq 1$. Unfortunately, we still do not know the derivatives of functions such as $y = x^x$ or $y = x^\pi$. These functions require a technique called **logarithmic differentiation**, which allows us to differentiate any function of the form $h(x) = g(x)^{f(x)}$. It can also be used to convert a very complex differentiation problem into a simpler one, such as finding the derivative of $y = \dfrac{x\sqrt{2x+1}}{e^x \sin^3 x}$. We outline this technique in the following problem-solving strategy.

Problem-Solving Strategy: Using Logarithmic Differentiation

1. To differentiate $y = h(x)$ using logarithmic differentiation, take the natural logarithm of both sides of the equation to obtain $\ln y = \ln(h(x))$.
2. Use properties of logarithms to expand $\ln(h(x))$ as much as possible.
3. Differentiate both sides of the equation. On the left we will have $\dfrac{1}{y}\dfrac{dy}{dx}$.
4. Multiply both sides of the equation by y to solve for $\dfrac{dy}{dx}$.
5. Replace y by $h(x)$.

Example 3.9.8: Using Logarithmic Differentiation

Find the derivative of $y = (2x^4 + 1)^{\tan x}$.

Solution

Use logarithmic differentiation to find this derivative.

$\ln y = \ln (2x^4 + 1)^{\tan x}$ — Step 1. Take the natural logarithm of both sides.

$\ln y = \tan x \ln(2x^4 + 1)$ — Step 2. Expand using properties of logarithms.

$\dfrac{1}{y}\dfrac{dy}{dx} = \sec^2 x \ln(2x^4 + 1) + \dfrac{8x^3}{2x^4 + 1} \cdot \tan x$ — Step 3. Differentiate both sides. Use the product rule on the right.

$\dfrac{dy}{dx} = y \cdot (\sec^2 x \ln(2x^4 + 1) + \dfrac{8x^3}{2x^4 + 1} \cdot \tan x)$ Step 4. Multiply by y on both sides.

$\dfrac{dy}{dx} = (2x^4 + 1)^{\tan x}(\sec^2 x \ln(2x^4 + 1) + \dfrac{8x^3}{2x^4 + 1} \cdot \tan x)$ Step 5. Substitute $y = (2x^4 + 1)^{\tan x}$.

Example 3.9.9: Extending the Power Rule

Find the derivative of $y = \dfrac{x\sqrt{2x+1}}{e^x \sin^3 x}$.

Solution

This problem really makes use of the properties of logarithms and the differentiation rules given in this chapter.

$\ln y = \ln \dfrac{x\sqrt{2x+1}}{e^x \sin^3 x}$	Step 1. Take the natural logarithm of both sides.
$\ln y = \ln x + \frac{1}{2}\ln(2x+1) - x \ln e - 3 \ln \sin x$	Step 2. Expand using properties of logarithms.
$\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{x} + \dfrac{1}{2x+1} - 1 - 3\dfrac{\cos x}{\sin x}$	Step 3. Differentiate both sides.
$\dfrac{dy}{dx} = y\left(\dfrac{1}{x} + \dfrac{1}{2x+1} - 1 - 3\cot x\right)$	Step 4. Multiply by y on both sides.
$\dfrac{dy}{dx} = \dfrac{x\sqrt{2x+1}}{e^x \sin^3 x}\left(\dfrac{1}{x} + \dfrac{1}{2x+1} - 1 - 3\cot x\right)$	Step 5. Substitute $y = \dfrac{x\sqrt{2x+1}}{e^x \sin^3 x}$.

Exercise 3.9.5

Use logarithmic differentiation to find the derivative of $y = x^x$.

Hint

Follow the problem solving strategy.

Answer

Solution: $\frac{dy}{dx} = x^x(1 + \ln x)$

Exercise 3.9.6

Find the derivative of $y = (\tan x)^\pi$.

Hint

Use the power rule (since the exponent π is a constant) and the chain rule.

Answer

$y' = \pi(\tan x)^{\pi-1} \sec^2 x$

Key Concepts

- On the basis of the assumption that the exponential function $y = b^x$, $b > 0$ is continuous everywhere and differentiable at 0, this function is differentiable everywhere and there is a formula for its derivative.
- We can use a formula to find the derivative of $y = \ln x$, and the relationship $\log_b x = \frac{\ln x}{\ln b}$ allows us to extend our differentiation formulas to include logarithms with arbitrary bases.
- Logarithmic differentiation allows us to differentiate functions of the form $y = g(x)^{f(x)}$ or very complex functions by taking the natural logarithm of both sides and exploiting the properties of logarithms before differentiating.

Key Equations

- **Derivative of the natural exponential function**

  $$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)}g'(x)$$

- **Derivative of the natural logarithmic function**

  $$\frac{d}{dx}\left(\ln g(x)\right) = \frac{1}{g(x)}g'(x)$$

- **Derivative of the general exponential function**

  $$\frac{d}{dx}\left(b^{g(x)}\right) = b^{g(x)}g'(x)\ln b$$

- **Derivative of the general logarithmic function**

  $$\frac{d}{dx}\left(\log_b g(x)\right) = \frac{g'(x)}{g(x)\ln b}$$

Glossary

logarithmic differentiation

is a technique that allows us to differentiate a function by first taking the natural logarithm of both sides of an equation, applying properties of logarithms to simplify the equation, and differentiating implicitly

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3.9E: EXERCISES FOR SECTION 3.9

In exercises 1 - 15, find $f'(x)$ for each function.

1) $f(x) = x^2 e^x$

Answer

$f'(x) = 2xe^x + x^2 e^x$

2) $f(x) = \dfrac{e^{-x}}{x}$

3) $f(x) = e^{x^3 \ln x}$

Answer

$f'(x) = e^{x^3 \ln x}\left(3x^2 \ln x + x^2\right)$

4) $f(x) = \sqrt{e^{2x} + 2x}$

5) $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Answer

$f'(x) = \dfrac{4}{(e^x + e^{-x})^2}$

6) $f(x) = \dfrac{10^x}{\ln 10}$

7) $f(x) = 2^{4x} + 4x^2$

Answer

$f'(x) = 2^{4x+2} \cdot \ln 2 + 8x$

8) $f(x) = 3^{\sin 3x}$

9) $f(x) = x^\pi \cdot \pi^x$

Answer

$f'(x) = \pi x^{\pi-1} \cdot \pi^x + x^\pi \cdot \pi^x \ln \pi$

10) $f(x) = \ln(4x^3 + x)$

11) $f(x) = \ln \sqrt{5x - 7}$

Answer

$f'(x) = \dfrac{5}{2(5x - 7)}$

12) $f(x) = x^2 \ln 9x$

13) $f(x) = \log(\sec x)$

Answer

$f'(x) = \dfrac{\tan x}{\ln 10}$

14) $f(x) = \log_7 (6x^4 + 3)^5$

15) $f(x) = 2^x \cdot \log_3 7^{x^2-4}$

Answer

$f'(x) = 2^x \cdot \ln 2 \cdot \log_3 7^{x^2-4} + 2^x \cdot \dfrac{2x \ln 7}{\ln 3}$

For exercises 16 - 23, use logarithmic differentiation to find $\dfrac{dy}{dx}$.

16) $y = x^{\sqrt{x}}$

17) $y = (\sin 2x)^{4x}$

Answer

$$\frac{dy}{dx} = (\sin 2x)^{4x}\left[4 \cdot \ln(\sin 2x) + 8x \cdot \cot 2x\right]$$

18) $y = (\ln x)^{\ln x}$

19) $y = x^{\log_2 x}$

Answer

$$\frac{dy}{dx} = x^{\log_2 x} \cdot \frac{2 \ln x}{x \ln 2}$$

20) $y = (x^2 - 1)^{\ln x}$

21) $y = x^{\cot x}$

Answer

$$\frac{dy}{dx} = x^{\cot x} \cdot \left[-\csc^2 x \cdot \ln x + \frac{\cot x}{x}\right]$$

22) $y = \dfrac{x + 11}{\sqrt[3]{x^2 - 4}}$

23) $y = x^{-1/2}(x^2 + 3)^{2/3}(3x - 4)^4$

Answer

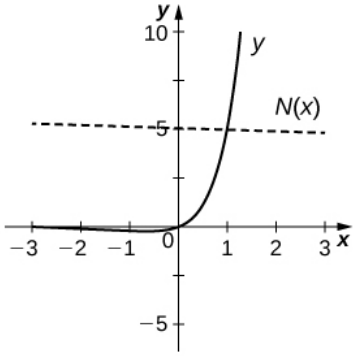
$$\frac{dy}{dx} = x^{-1/2}(x^2 + 3)^{2/3}(3x - 4)^4 \cdot \left[\frac{-1}{2x} + \frac{4x}{3(x^2 + 3)} + \frac{12}{3x - 4}\right]$$

24) [T] Find an equation of the tangent line to the graph of $f(x) = 4xe^{(x^2-1)}$ at the point where $x = -1$. Graph both the function and the tangent line.

25) [T] Find the equation of the line that is normal to the graph of $f(x) = x \cdot 5^x$ at the point where $x = 1$. Graph both the function and the normal line.

Answer

$y = \frac{-1}{5+5\ln 5}x + \left(5 + \frac{1}{5+5\ln 5}\right)$



26) [T] Find the equation of the tangent line to the graph of $x^3 - x\ln y + y^3 = 2x + 5$ at the point where $x = 2$. (Hint: Use implicit differentiation to find $\frac{dy}{dx}$.) Graph both the curve and the tangent line.

27) Consider the function $y = x^{1/x}$ for $x > 0$.

a. Determine the points on the graph where the tangent line is horizontal.

b. Determine the points on the graph where $y' > 0$ and those where $y' < 0$.

Answer

a. $x = e \approx 2.718$

b. $y' > 0$ for $(0, e)$ and $y' < 0$ for (e, ∞).

28) The formula $I(t) = \dfrac{\sin t}{e^t}$ is the formula for a decaying alternating current.

a. Complete the following table with the appropriate values.

t	$\frac{\sin t}{e^t}$
0	(i)
$\pi/2$	(ii)
π	(iii)
$3\pi/2$	(vi)
2π	(v)
2π	(vi)
3π	(vii)

b. Using only the values in the table, determine where the tangent line to the graph of $I(t)$ is horizontal.

29) [T] The population of Toledo, Ohio, in 2000 was approximately 500,000. Assume the population is increasing at a rate of 5% per year.

a. Write the exponential function that relates the total population as a function of t.

b. Use part a. to determine the rate at which the population is increasing in t years.

c. Use part b. to determine the rate at which the population is increasing in 10 years

Answer

a. $P = 500,000(1.05)^t$ individuals
b. $P'(t) = 24395 \cdot (1.05)^t$ individuals per year
c. $39,737$ individuals per year

30)[T] An isotope of the element erbium has a half-life of approximately 12 hours. Initially there are 9 grams of the isotope present.

a. Write the exponential function that relates the amount of substance remaining as a function of t, measured in hours.

b. Use a. to determine the rate at which the substance is decaying in t hours.

c. Use b. to determine the rate of decay at $t = 4$ hours.

31) [T] The number of cases of influenza in New York City from the beginning of 1960 to the beginning of 1964 is modeled by the function $N(t) = 5.3e^{0.093t^2 - 0.87t}, (0 \le t \le 4)$, where $N(t)$ gives the number of cases (in thousands) and t is measured in years, with $t = 0$ corresponding to the beginning of 1960.

a. Show work that evaluates $N(0)$ and $N(4)$. Briefly describe what these values indicate about the disease in New York City.

b. Show work that evaluates $N'(0)$ and $N'(3)$. Briefly describe what these values indicate about the disease in the United States.

Answer

a. At the beginning of 1960 there were 5.3 thousand cases of the disease in New York City. At the beginning of 1963 there were approximately 723 cases of the disease in the United States.
b. At the beginning of 1960 the number of cases of the disease was decreasing at rate of -4.611 thousand per year; at the beginning of 1963, the number of cases of the disease was decreasing at a rate of -0.2808 thousand per year.

32) [T] The relative rate of change of a differentiable function $y = f(x)$ is given by $\frac{100 \cdot f'(x)}{f(x)}$ One model for population growth is a Gompertz growth function, given by $P(x) = ae^{-b \cdot e^{-cx}}$ where a, b, and c are constants.

a. Find the relative rate of change formula for the generic Gompertz function.

b. Use part a. to find the relative rate of change of a population in $x = 20$ months when $a = 204,\ b = 0.0198$, and $c = 0.15$.

c. Briefly interpret what the result of part b. means.

For exercises 33 - 36, use the population of New York City from 1790 to 1860, given in the following table.

Year since 1790	Population
0	33,131
10	60,515
20	96,373
30	123,706
40	202,300
50	312,710
60	515,547
70	813,669

New York City Population Over TimeSource: http://en.Wikipedia.org/wiki/Largest..._United_States

_by_population_by_decade

33) [T] Using a computer program or a calculator, fit a growth curve to the data of the form $p = ab^t$.

Answer

$p = 35741(1.045)^t$

34) [T] Using the exponential best fit for the data, write a table containing the derivatives evaluated at each year.

35) [T] Using the exponential best fit for the data, write a table containing the second derivatives evaluated at each year.

Answer

Year since 1790	P''
0	69.25
10	107.5
20	167.0
30	259.4
40	402.8
50	625.5
60	971.4
70	1508.5

36) [T] Using the tables of first and second derivatives and the best fit, answer the following questions:

a. Will the model be accurate in predicting the future population of New York City? Why or why not?

b. Estimate the population in 2010. Was the prediction correct from part a.?

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3.R: Chapter 3 Review Exercises

Chapter Review Exercises

True or False? Justify the answer with a proof or a counterexample.

1) Every function has a derivative.

Answer

False

2) A continuous function has a continuous derivative.

3) A continuous function has a derivative.

Answer

False

4) If a function is differentiable, it is continuous.

In exercises 5 and 6, use the limit definition of the derivative to exactly evaluate the derivative.

5) $f(x) = \sqrt{x+4}$

Answer

$$f'(x) = \frac{1}{2\sqrt{x+4}}$$

6) $f(x) = \frac{3}{x}$

In exercises 7 - 15, find the derivatives of the given functions.

7) $f(x) = 3x^3 - \frac{4}{x^2}$

Answer

$f'(x) = 9x^2 + \frac{8}{x^3}$

9) $f(x) = (4 - x^2)^3$

10) $f(x) = e^{\sin x}$

Answer

$f'(x) = e^{\sin x} \cos x$

11) $f(x) = \ln(x+2)$

12) $f(x) = x^2 \cos x + x \tan x$

Answer

$f'(x) = x \sec^2 x + 2x \cos x + \tan x - x^2 \sin x$

13) $f(x) = \sqrt{3x^2 + 2}$

14) $f(x) = \frac{x}{4} \sin^{-1}(x)$

Answer

$f'(x) = \frac{1}{4}\left(\frac{x}{\sqrt{1-x^2}} + \sin^{-1} x\right)$

15) $x^2 y = (y+2) + xy \sin x$

In exercises 16 - 18, find the indicated derivatives of various orders.

16) First derivative of $y = x(\ln x)\cos x$

Answer

$$\frac{dy}{dx} = \cos x \cdot (\ln x + 1) - x(\ln x)\sin x$$

17) Third derivative of $y = (3x + 2)^2$

18) Second derivative of $y = 4^x + x^2 \sin x$

Answer

$$\frac{d^2y}{dx^2} = 4^x(\ln 4)^2 + 2\sin x + 4x\cos x - x^2 \sin x$$

In exercises 19 and 20, find the equation of the tangent line to the following equations at the specified point.

19) $y = \cos^{-1}(x) + x$ at $x = 0$

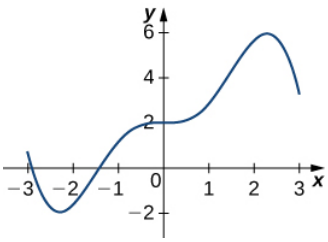
20) $y = x + e^x - \frac{1}{x}$ at $x = 1$

Answer

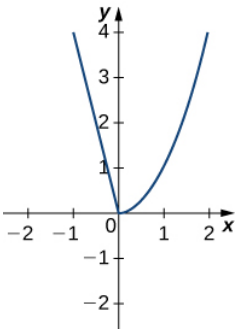
$y = (2 + e)x - 2$

In exercises 21 and 22, draw the derivative of the functions with the given graphs.

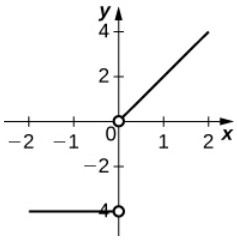
21)



22)



Answer



Questions 22 and 23 concern the water level in Ocean City, New Jersey, in January, which can be approximated by $w(t) = 1.9 + 2.9\cos(\frac{\pi}{6}t)$, where t is measured in hours after midnight, and the height is measured in feet.

22) Find and graph the derivative. What is the physical meaning?

23) Find $w'(3)$. What is the physical meaning of this value?

Answer

$w'(3) = -\frac{2.9\pi}{6}$. At 3 a.m. the tide is decreasing at a rate of 1.514 ft/hr.

Questions 24 and 25 consider the wind speeds of Hurricane Katrina, which affected New Orleans, Louisiana, in August 2005. The data are displayed in a table.

Hours after Midnight, August 26	Wind Speed (mph)
1	45
5	75
11	100
29	115
49	145
58	175
73	155
81	125
85	95
107	35

Wind Speeds of Hurricane KatrinaSource: news.nationalgeographic.com/n..._timeline.html.

24) Using the table, estimate the derivative of the wind speed at hour 39. What is the physical meaning?

25) Estimate the derivative of the wind speed at hour 83. What is the physical meaning?

Answer

-7.5. The wind speed is decreasing at a rate of 7.5 mph/hr

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CHAPTER OVERVIEW

4: Applications of Derivatives

A rocket launch involves two related quantities that change over time. Being able to solve this type of problem is just one application of derivatives introduced in this chapter. We also look at how derivatives are used to find maximum and minimum values of functions. As a result, we will be able to solve applied optimization problems, such as maximizing revenue and minimizing surface area. In addition, we examine how derivatives are used to evaluate complicated limits, to approximate roots of functions, and to provide accurate graphs of functions.

4.0: Prelude to Applications of Derivatives
4.1: Related Rates
4.1E: Exercises for Section 4.1
4.2: Linear Approximations and Differentials
4.2E: Exercises for Section 4.2
4.3: Maxima and Minima
4.3E: Exercises for Section 4.3
4.4: The Mean Value Theorem
4.4E: Exercises for Section 4.4
4.5: Derivatives and the Shape of a Graph
4.5E: Exercises for Section 4.5
4.6: Limits at Infinity and Asymptotes
4.6E: Exercises for Section 4.6
4.7: Applied Optimization Problems
4.7E: Exercises for Section 4.7
4.8: L'Hôpital's Rule
4.8E: Exercises for Section 4.8
4.9: Newton's Method
4.9E: Exercises for Section 4.9
4.10: Antiderivatives
4.10E: Exercises for Section 4.10
4.R: Chapter 4 Review Exercises

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4.0: Prelude to Applications of Derivatives

A rocket is being launched from the ground and cameras are recording the event. A video camera is located on the ground a certain distance from the launch pad. At what rate should the angle of inclination (the angle the camera makes with the ground) change to allow the camera to record the flight of the rocket as it heads upward?

Figure 4.0.1: As a rocket is being launched, at what rate should the angle of a video camera change to continue viewing the rocket? (credit: modification of work by Steve Jurvetson, Wikimedia Commons)

A rocket launch involves two related quantities that change over time. Being able to solve this type of problem is just one application of derivatives introduced in this chapter. We also look at how derivatives are used to find maximum and minimum values of functions. As a result, we will be able to solve applied optimization problems, such as maximizing revenue and minimizing surface area. In addition, we examine how derivatives are used to evaluate complicated limits, to approximate roots of functions, and to provide accurate graphs of functions.

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4.1: Related Rates

Learning Objectives

- Express changing quantities in terms of derivatives.
- Find relationships among the derivatives in a given problem.
- Use the chain rule to find the rate of change of one quantity that depends on the rate of change of other quantities.

We have seen that for quantities that are changing over time, the rates at which these quantities change are given by derivatives. If two related quantities are changing over time, the rates at which the quantities change are related. For example, if a balloon is being filled with air, both the radius of the balloon and the volume of the balloon are increasing. In this section, we consider several problems in which two or more related quantities are changing and we study how to determine the relationship between the rates of change of these quantities.

Setting up Related-Rates Problems

In many real-world applications, related quantities are changing with respect to time. For example, if we consider the balloon example again, we can say that the rate of change in the volume, V, is related to the rate of change in the radius, r. In this case, we say that $\frac{dV}{dt}$ and $\frac{dr}{dt}$ are **related rates** because V is related to r. Here we study several examples of related quantities that are changing with respect to time and we look at how to calculate one rate of change given another rate of change.

Example 4.1.1: Inflating a Balloon

A spherical balloon is being filled with air at the constant rate of $2\ \text{cm}^3/\text{sec}$ (Figure 4.1.1). How fast is the radius increasing when the radius is 3 cm?

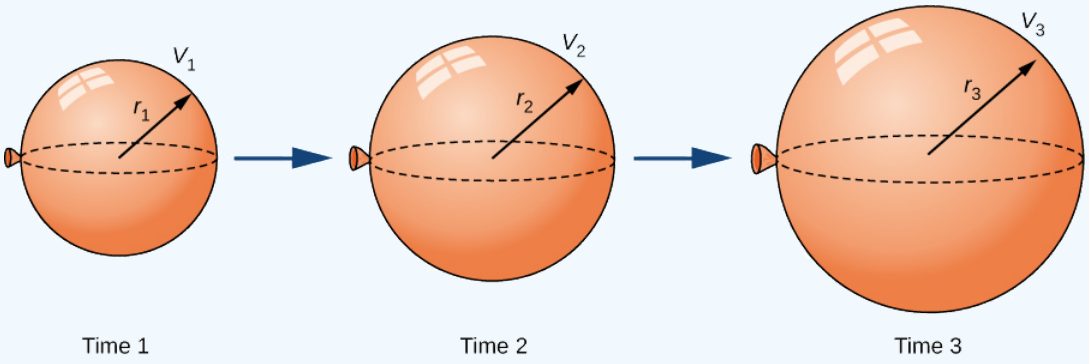


Figure 4.1.1: As the balloon is being filled with air, both the radius and the volume are increasing with respect to time.

Solution

The volume of a sphere of radius r centimeters is

$$V = \tfrac{4}{3}\pi r^3\ \text{cm}^3.$$

Since the balloon is being filled with air, both the volume and the radius are functions of time. Therefore, t seconds after beginning to fill the balloon with air, the volume of air in the balloon is

$$V(t) = \tfrac{4}{3}\pi\big[r(t)\big]^3\text{cm}^3.$$

Differentiating both sides of this equation with respect to time and applying the chain rule, we see that the rate of change in the volume is related to the rate of change in the radius by the equation

$$V'(t) = 4\pi\big[r(t)\big]^2 r'(t).$$

The balloon is being filled with air at the constant rate of $2\,\text{cm}^3/\text{sec}$, so $V'(t) = 2\,\text{cm}^3/\text{sec}$. Therefore,

$$2\,\text{cm}^3/\text{sec} = \Big(4\pi\big[r(t)\big]^2\ \text{cm}^2\Big)\cdot\Big(r'(t)\ \text{cm/s}\Big),$$

which implies

$$r'(t) = \frac{1}{2\pi\left[r(t)\right]^2} \text{ cm/sec.}$$

When the radius $r = 3$ cm,

$$r'(t) = \frac{1}{18\pi} \text{ cm/sec.}$$

? Exercise 4.1.1

What is the instantaneous rate of change of the radius when $r = 6$ cm?

Hint

$\frac{dr}{dt} = \frac{1}{2\pi r^2}$

Answer

$\frac{1}{72\pi}$ cm/sec, or approximately 0.0044 cm/sec

Before looking at other examples, let's outline the problem-solving strategy we will be using to solve related-rates problems.

Problem-Solving Strategy: Solving a Related-Rates Problem

1. Assign symbols to all variables involved in the problem. Draw a figure if applicable.
2. State, in terms of the variables, the information that is given and the rate to be determined.
3. Find an equation relating the variables introduced in step 1.
4. Using the chain rule, differentiate both sides of the equation found in step 3 with respect to the independent variable. This new equation will relate the derivatives.
5. Substitute all known values into the equation from step 4, then solve for the unknown rate of change

Note that when solving a related-rates problem, it is crucial not to substitute known values too soon. For example, if the value for a changing quantity is substituted into an equation before both sides of the equation are differentiated, then that quantity will behave as a constant and its derivative will not appear in the new equation found in step 4. We examine this potential error in the following example.

Examples of the Process

Let's now implement the strategy just described to solve several related-rates problems. The first example involves a plane flying overhead. The relationship we are studying is between the speed of the plane and the rate at which the distance between the plane and a person on the ground is changing.

✓ Example 4.1.2: An Airplane Flying at a Constant Elevation

An airplane is flying overhead at a constant elevation of 4000 ft. A man is viewing the plane from a position 3000 ft from the base of a radio tower. The airplane is flying horizontally away from the man. If the plane is flying at the rate of 600 ft/sec, at what rate is the distance between the man and the plane increasing when the plane passes over the radio tower?

Solution

Step 1. Draw a picture, introducing variables to represent the different quantities involved.

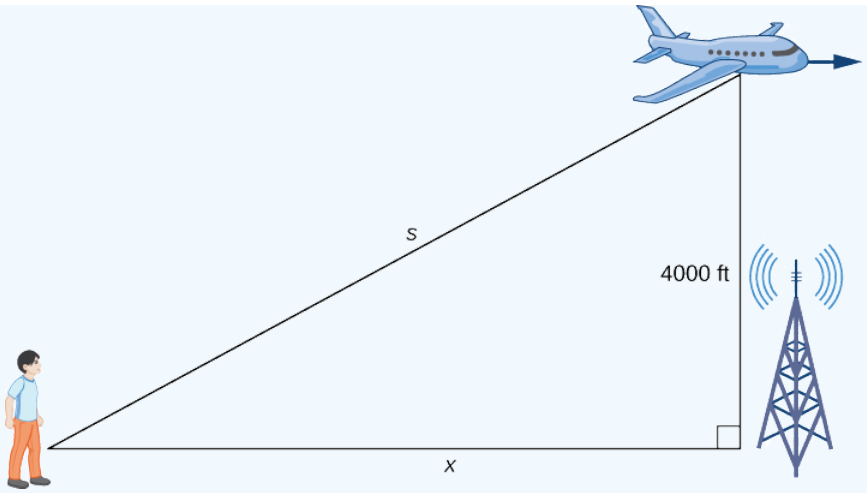


Figure 4.1.2: An airplane is flying at a constant height of 4000 ft. The distance between the person and the airplane and the person and the place on the ground directly below the airplane are changing. We denote those quantities with the variables s and x, respectively.

As shown, x denotes the distance between the man and the position on the ground directly below the airplane. The variable s denotes the distance between the man and the plane. Note that both x and s are functions of time. We do not introduce a variable for the height of the plane because it remains at a constant elevation of 4000 ft. Since an object's height above the ground is measured as the shortest distance between the object and the ground, the line segment of length 4000 ft is perpendicular to the line segment of length x feet, creating a right triangle.

Step 2. Since x denotes the horizontal distance between the man and the point on the ground below the plane, dx/dt represents the speed of the plane. We are told the speed of the plane is 600 ft/sec. Therefore, $\frac{dx}{dt} = 600$ ft/sec. Since we are asked to find the rate of change in the distance between the man and the plane when the plane is directly above the radio tower, we need to find ds/dt when $x = 3000$ ft.

Step 3. From the figure, we can use the **Pythagorean theorem** to write an equation relating x and s:

$$[x(t)]^2 + 4000^2 = [s(t)]^2.$$

Step 4. Differentiating this equation with respect to time and using the fact that the derivative of a constant is zero, we arrive at the equation

$$x\frac{dx}{dt} = s\frac{ds}{dt}.$$

Step 5. Find the rate at which the distance between the man and the plane is increasing when the plane is directly over the radio tower. That is, find $\frac{ds}{dt}$ when $x = 3000$ ft. Since the speed of the plane is 600 ft/sec, we know that $\frac{dx}{dt} = 600$ ft/sec. We are not given an explicit value for s; however, since we are trying to find $\frac{ds}{dt}$ when $x = 3000$ ft, we can use the Pythagorean theorem to determine the distance s when $x = 3000$ ft and the height is 4000 ft. Solving the equation

$$3000^2 + 4000^2 = s^2$$

for s, we have $s = 5000$ ft at the time of interest. Using these values, we conclude that ds/dt

is a solution of the equation

$$(3000)(600) = (5000) \cdot \frac{ds}{dt}.$$

Therefore,

$$\frac{ds}{dt} = \frac{3000 \cdot 600}{5000} = 360 \text{ ft/sec}.$$

Note: When solving related-rates problems, it is important not to substitute values for the variables too soon. For example, in step 3, we related the variable quantities $x(t)$ and $s(t)$ by the equation

$$[x(t)]^2 + 4000^2 = [s(t)]^2.$$

Since the plane remains at a constant height, it is not necessary to introduce a variable for the height, and we are allowed to use the constant 4000 to denote that quantity. However, the other two quantities are changing. If we mistakenly substituted $x(t) = 3000$ into the equation before differentiating, our equation would have been

$$3000^2 + 4000^2 = [s(t)]^2.$$

After differentiating, our equation would become

$$0 = s(t)\frac{ds}{dt}.$$

As a result, we would incorrectly conclude that $\frac{ds}{dt} = 0$.

? Exercise 4.1.2

What is the speed of the plane if the distance between the person and the plane is increasing at the rate of 300 ft/sec?

Hint

$\frac{ds}{dt} = 300$ ft/sec

Answer

500 ft/sec

We now return to the problem involving the rocket launch from the beginning of the chapter.

✓ Example 4.1.3: Chapter Opener - A Rocket Launch

A rocket is launched so that it rises vertically. A camera is positioned 5000 ft from the launch pad. When the rocket is 1000 ft above the launch pad, its velocity is 600 ft/sec.

Figure 4.1.3: (credit: modification of work by Steve Jurvetson, Wikimedia Commons)

Find the necessary rate of change of the camera's angle as a function of time so that it stays focused on the rocket.

Solution

Step 1. Draw a picture introducing the variables.

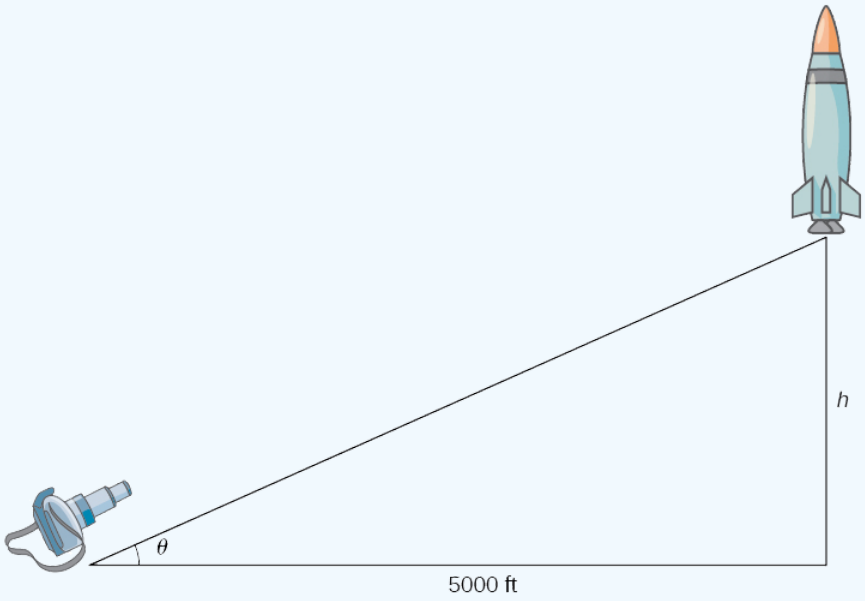


Figure 4.1.4: A camera is positioned 5000 ft from the launch pad of the rocket. The height of the rocket and the angle of the camera are changing with respect to time. We denote those quantities with the variables h and θ,respectively.

Let h denote the height of the rocket above the launch pad and θ be the angle between the camera lens and the ground.

Step 2. We are trying to find the rate of change in the angle of the camera with respect to time when the rocket is 1000 ft off the ground. That is, we need to find $\frac{d\theta}{dt}$ when $h = 1000$ ft. At that time, we know the velocity of the rocket is $\frac{dh}{dt} = 600$ ft/sec.

Step 3. Now we need to find an equation relating the two quantities that are changing with respect to time: h and θ. How can we create such an equation? Using the fact that we have drawn a right triangle, it is natural to think about trigonometric functions. Recall that $\tan\theta$ is the ratio of the length of the opposite side of the triangle to the length of the adjacent side. Thus, we have

$$\tan\theta = \frac{h}{5000}.$$

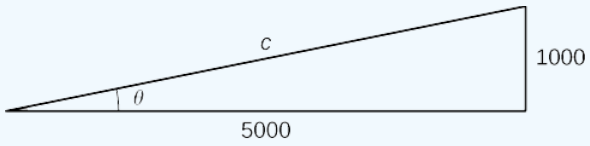
This gives us the equation

$$h = 5000\tan\theta.$$

Step 4. Differentiating this equation with respect to time t, we obtain

$$\frac{dh}{dt} = 5000\sec^2\theta\frac{d\theta}{dt}.$$

Step 5. We want to find $\frac{d\theta}{dt}$ when $h = 1000$ ft. At this time, we know that $\frac{dh}{dt} = 600$ ft/sec. We need to determine $\sec^2\theta$. Recall that $\sec\theta$ is the ratio of the length of the hypotenuse to the length of the adjacent side. We know the length of the adjacent side is 5000 ft. To determine the length of the hypotenuse, we use the Pythagorean theorem, where the length of one leg is 5000 ft, the length of the other leg is $h = 1000$ ft, and the length of the hypotenuse is c feet as shown in the following figure.



We see that

$$1000^2 + 5000^2 = c^2$$

and we conclude that the hypotenuse is

$$c = 1000\sqrt{26}\,\text{ft}.$$

Therefore, when $h = 1000$, we have

$$\sec^2\theta = \left(\frac{1000\sqrt{26}}{5000}\right)^2 = \frac{26}{25}.$$

Recall from step 4 that the equation relating $\frac{d\theta}{dt}$ to our known values is

$$\frac{dh}{dt} = 5000\sec^2\theta\frac{d\theta}{dt}.$$

When $h = 1000$ ft, we know that $\frac{dh}{dt} = 600$ ft/sec and $\sec^2\theta = \frac{26}{25}$. Substituting these values into the previous equation, we arrive at the equation

$$600 = 5000\left(\tfrac{26}{25}\right)\frac{d\theta}{dt}.$$

Therefore, $\frac{d\theta}{dt} = \frac{3}{26}$ rad/sec.

Exercise 4.1.3

What rate of change is necessary for the elevation angle of the camera if the camera is placed on the ground at a distance of 4000 ft from the launch pad and the velocity of the rocket is 500 ft/sec when the rocket is 2000 ft off the ground?

Hint

Find $\frac{d\theta}{dt}$ when $h = 2000$ ft. At that time, $\frac{dh}{dt} = 500$ ft/sec.

Answer

$\frac{1}{10}$ rad/sec

In the next example, we consider water draining from a cone-shaped funnel. We compare the rate at which the level of water in the cone is decreasing with the rate at which the volume of water is decreasing.

Example 4.1.4: Water Draining from a Funnel

Water is draining from the bottom of a cone-shaped funnel at the rate of 0.03 ft^3/sec. The height of the funnel is 2 ft and the radius at the top of the funnel is 1 ft. At what rate is the height of the water in the funnel changing when the height of the water is $\frac{1}{2}$ ft?

Solution

Step 1: Draw a picture introducing the variables.

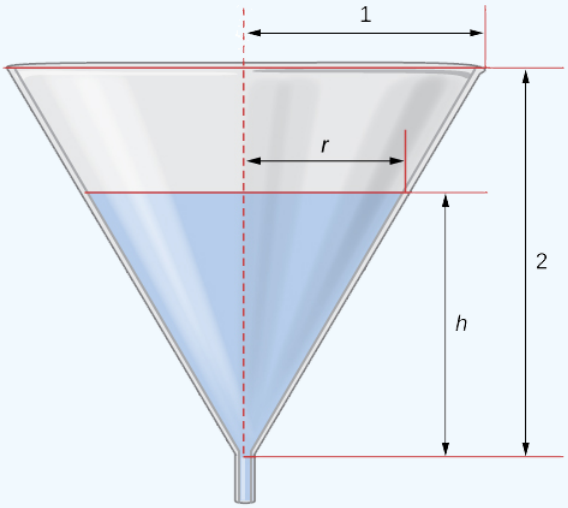


Figure 4.1.5: Water is draining from a funnel of height 2 ft and radius 1 ft. The height of the water and the radius of water are changing over time. We denote these quantities with the variables h and r, respectively.

Let h denote the height of the water in the funnel, r denote the radius of the water at its surface, and V denote the volume of the water.

Step 2: We need to determine $\frac{dh}{dt}$ when $h = \frac{1}{2}$ ft. We know that $\frac{dV}{dt} = -0.03$ ft/sec.

Step 3: The volume of water in the cone is

$$V = \tfrac{1}{3}\pi r^2 h.$$

From the figure, we see that we have similar triangles. Therefore, the ratio of the sides in the two triangles is the same. Therefore, $\frac{r}{h} = \frac{1}{2}$ or $r = \frac{h}{2}$. Using this fact, the equation for volume can be simplified to

$$V = \tfrac{1}{3}\pi\left(\tfrac{h}{2}\right)^2 h = \tfrac{\pi}{12}h^3.$$

Step 4: Applying the chain rule while differentiating both sides of this equation with respect to time t, we obtain

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}.$$

Step 5: We want to find $\frac{dh}{dt}$ when $h = \frac{1}{2}$ ft. Since water is leaving at the rate of $0.03\,\text{ft}^3/\text{sec}$, we know that $\frac{dV}{dt} = -0.03\,\text{ft}^3/\text{sec}$. Therefore,

$$-0.03 = \frac{\pi}{4}\left(\frac{1}{2}\right)^2\frac{dh}{dt},$$

which implies

$$-0.03 = \frac{\pi}{16}\frac{dh}{dt}.$$

It follows that

$$\frac{dh}{dt} = -\frac{0.48}{\pi} = -0.153\,\text{ft/sec}.$$

? Exercise 4.1.4

At what rate is the height of the water changing when the height of the water is $\frac{1}{4}$ ft?

Hint

We need to find $\frac{dh}{dt}$ when $h = \frac{1}{4}$.

Answer

-0.61 ft/sec

Key Concepts

- To solve a related rates problem, first draw a picture that illustrates the relationship between the two or more related quantities that are changing with respect to time.
- In terms of the quantities, state the information given and the rate to be found.
- Find an equation relating the quantities.
- Use differentiation, applying the chain rule as necessary, to find an equation that relates the rates.
- Be sure not to substitute a variable quantity for one of the variables until after finding an equation relating the rates.

Glossary

related rates

are rates of change associated with two or more related quantities that are changing over time

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4.1E: EXERCISES FOR SECTION 4.1

In exercises 1 - 3, find the quantities for the given equation.

1) Find $\frac{dy}{dt}$ at $x = 1$ and $y = x^2 + 3$ if $\frac{dx}{dt} = 4$.

Answer

$$\frac{dy}{dt} = 8$$

2) Find $\frac{dx}{dt}$ at $x = -2$ and $y = 2x^2 + 1$ if $\frac{dy}{dt} = -1$.

3) Find $\frac{dz}{dt}$ at $(x, y) = (1, 3)$ and $z^2 = x^2 + y^2$ if $\frac{dx}{dt} = 4$ and $\frac{dy}{dt} = 3$.

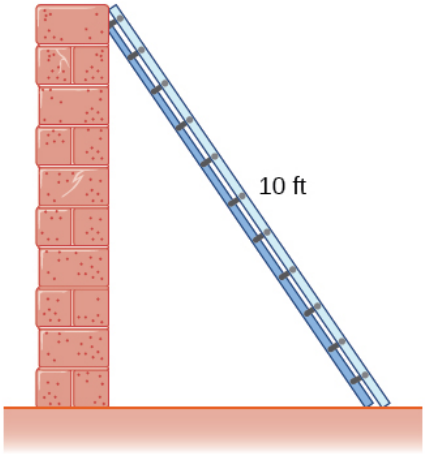
Answer

$$\frac{dz}{dt} = \frac{13}{\sqrt{10}}$$

In exercises 4 - 15, sketch the situation if necessary and used related rates to solve for the quantities.

4) [T] If two electrical resistors are connected in parallel, the total resistance (measured in ohms, denoted by the Greek capital letter omega, Ω) is given by the equation $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. If R_1 is increasing at a rate of $0.5\Omega/\text{min}$ and R_2 decreases at a rate of $1.1\Omega/\text{min}$, at what rate does the total resistance change when $R_1 = 20\Omega$ and $R_2 = 50\Omega/\text{min}$?

5) A 10-ft ladder is leaning against a wall. If the top of the ladder slides down the wall at a rate of 2 ft/sec, how fast is the bottom moving along the ground when the bottom of the ladder is 5 ft from the wall?



Answer

$2\sqrt{3}$ ft/sec

6) A 25-ft ladder is leaning against a wall. If we push the ladder toward the wall at a rate of 1 ft/sec, and the bottom of the ladder is initially 20 ft away from the wall, how fast does the ladder move up the wall 5 sec after we start pushing?

7) Two airplanes are flying in the air at the same height: airplane A is flying east at 250 mi/h and airplane B is flying north at 300 mi/h. If they are both heading to the same airport, located 30 miles east of airplane A and 40 miles north of airplane B, at what rate is the distance between the airplanes changing?

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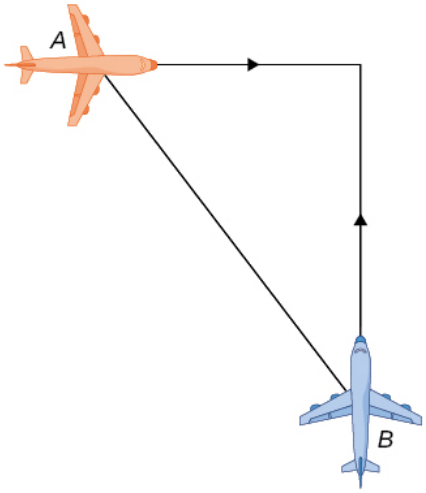


Answer

The distance is decreasing at 390 mi/h.

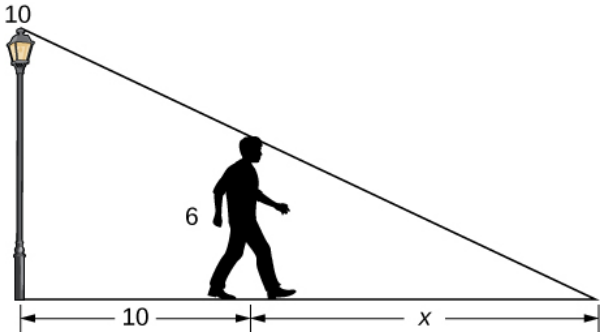
8) You and a friend are riding your bikes to a restaurant that you think is east; your friend thinks the restaurant is north. You both leave from the same point, with you riding at 16 mph east and your friend riding 12 mph north. After you traveled 4 mi, at what rate is the distance between you changing?

9) Two buses are driving along parallel freeways that are 5 mi apart, one heading east and the other heading west. Assuming that each bus drives a constant 55 mph, find the rate at which the distance between the buses is changing when they are 13 mi apart (as the crow flies), heading toward each other.

Answer

The distance between them shrinks at a rate of $\frac{1320}{13} \approx 101.5$ mph.

10) A 6-ft-tall person walks away from a 10-ft lamppost at a constant rate of 3 ft/sec. What is the rate that the tip of the shadow moves away from the pole when the person is 10 ft away from the pole?



11) Using the previous problem, what is the rate at which the tip of the shadow moves away from the person when the person is 10 ft from the pole?

Answer

$\frac{9}{2}$ ft/sec

12) A 5-ft-tall person walks toward a wall at a rate of 2 ft/sec. A spotlight is located on the ground 40 ft from the wall. How fast does the height of the person's shadow on the wall change when the person is 10 ft from the wall?

13) Using the previous problem, what is the rate at which the shadow changes when the person is 10 ft from the wall, if the person is walking away from the wall at a rate of 2 ft/sec?

Answer

It grows at a rate $\frac{4}{9}$ ft/sec

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14) A helicopter starting on the ground is rising directly into the air at a rate of 25 ft/sec. You are running on the ground starting directly under the helicopter at a rate of 10 ft/sec. Find the rate of change of the distance between the helicopter and yourself after 5 sec.

15) Using the previous problem, assuming the helicopter is again rising at a rate of 25 ft/sec and you are running on the ground starting directly under the helicopter at a rate of 10 ft/sec, what is the rate at which the distance between you and the helicopter is changing when the helicopter has risen to a height of 60 ft in the air, assuming that, initially, it was 30 ft above you?

Answer

The distance is increasing at $\frac{135\sqrt{26}}{26}$ ft/sec

In exercises 16 - 24, draw and label diagrams to help solve the related-rates problems.

16) The side of a cube increases at a rate of $\frac{1}{2}$ m/sec. Find the rate at which the volume of the cube increases when the side of the cube is 4 m.

17) The volume of a cube decreases at a rate of $10\ \text{m}^3$/sec. Find the rate at which the side of the cube changes when the side of the cube is 2 m.

Answer

$-\frac{5}{6}$ m/sec

18) The radius of a circle increases at a rate of 2 m/sec. Find the rate at which the area of the circle increases when the radius is 5 m.

19) The radius of a sphere decreases at a rate of 3 m/sec. Find the rate at which the surface area decreases when the radius is 10 m.

Answer

$240\pi\ \text{m}^2/\text{sec}$

20) The radius of a sphere increases at a rate of 1 m/sec. Find the rate at which the volume increases when the radius is 20 m.

21) The radius of a sphere is increasing at a rate of 9 cm/sec. Find the radius of the sphere when the volume and the radius of the sphere are increasing at the same numerical rate.

Answer

$\frac{1}{2\sqrt{\pi}}$ cm

22) The base of a triangle is shrinking at a rate of 1 cm/min and the height of the triangle is increasing at a rate of 5 cm/min. Find the rate at which the area of the triangle changes when the height is 22 cm and the base is 10 cm.

23) A triangle has two constant sides of length 3 ft and 5 ft. The angle between these two sides is increasing at a rate of 0.1 rad/sec. Find the rate at which the area of the triangle is changing when the angle between the two sides is $\pi/6$.

Answer

The area is increasing at a rate $\frac{3\sqrt{3}}{8}\ \text{ft}^2/\text{sec}$.

24) A triangle has a height that is increasing at a rate of 2 cm/sec and its area is increasing at a rate of $4\ \text{cm}^2/\text{sec}$. Find the rate at which the base of the triangle is changing when the height of the triangle is 4 cm and the area is $20\ \text{cm}^2$.

In exercises 25 - 27, consider an inverted right cone that is leaking water. (Inverted means the cone's point is facing down, like a funnel.) The dimensions of the conical tank are a height of 16 ft and a radius of 5 ft.

25) How fast does the depth of the water change when the water is 10 ft high if the cone leaks water at a rate of $10\ \text{ft}^3/\text{min}$?

Answer

The depth of the water decreases at $\frac{128}{125\pi}$ ft/min.

26) Find the rate at which the surface area of the water changes when the water is 10 ft high if the cone leaks water at a rate of $10\ \text{ft}^3/\text{min}$.

27) If the water level is decreasing at a rate of 3 in/min when the depth of the water is 8 ft, determine the rate at which water is leaking out of the cone.

Answer

The volume is decreasing at a rate of $\frac{25\pi}{16}\ \text{ft}^3/\text{min}$.

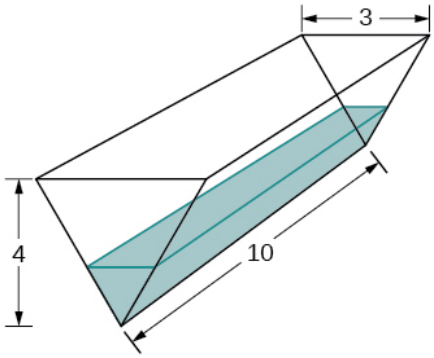
28) A vertical cylinder is leaking water at a rate of $1\ \text{ft}^3/\text{sec}$. If the cylinder has a height of 10 ft and a radius of 1 ft, at what rate is the height of the water changing when the height is 6 ft?

29) A cylinder is leaking water but you are unable to determine at what rate. The cylinder has a height of 2 m and a radius of 2 m. Find the rate at which the water is leaking out of the cylinder if the rate at which the height is decreasing is 10 cm/min when the height is 1 m.

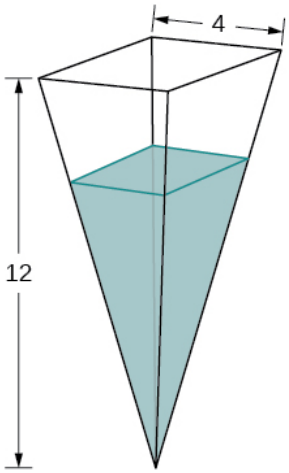
Answer

The water flows out at rate $\frac{2\pi}{5}\ \mathrm{m}^3/\mathrm{min}$.

30) A trough has ends shaped like isosceles triangles, with width 3 m and height 4 m, and the trough is 10 m long. Water is being pumped into the trough at a rate of $5\ \mathrm{m}^3/\mathrm{min}$. At what rate does the height of the water change when the water is 1 m deep?



31) A tank is shaped like an upside-down square pyramid, with base of 4 m by 4 m and a height of 12 m (see the following figure). How fast does the height increase when the water is 2 m deep if water is being pumped in at a rate of $\frac{2}{3}\ \mathrm{m}^3$/sec?



Answer

$\frac{3}{2}$ m/sec

For exercises 32 - 34, consider a pool shaped like the bottom half of a sphere, that is being filled at a rate of $25\ \mathrm{ft}^3/\mathrm{min}$. The radius of the pool is 10 ft.

32) Find the rate at which the depth of the water is changing when the water has a depth of 5 ft.

33) Find the rate at which the depth of the water is changing when the water has a depth of 1 ft.

Answer

$\frac{25}{19\pi}$ ft/min

34) If the height is increasing at a rate of 1 in/sec when the depth of the water is 2 ft, find the rate at which water is being pumped in.

35) Gravel is being unloaded from a truck and falls into a pile shaped like a cone at a rate of $10\ \mathrm{ft}^3/\mathrm{min}$. The radius of the cone base is three times the height of the cone. Find the rate at which the height of the gravel changes when the pile has a height of 5 ft.

Answer

$\frac{2}{45\pi}$ ft/min

36) Using a similar setup from the preceding problem, find the rate at which the gravel is being unloaded if the pile is 5 ft high and the height is increasing at a rate of 4 in/min.

In exercises 37 - 41, draw the situations and solve the related-rate problems.

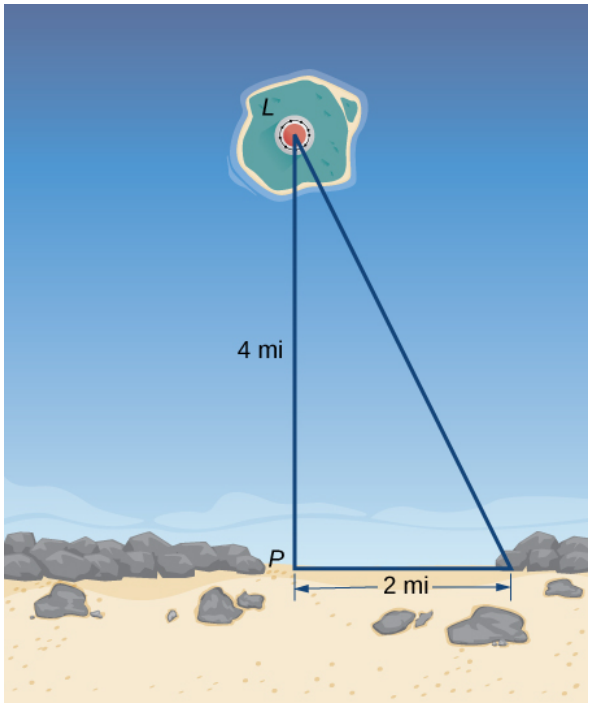
37) You are stationary on the ground and are watching a bird fly horizontally at a rate of 10 m/sec. The bird is located 40 m above your head. How fast does the angle of elevation change when the horizontal distance between you and the bird is 9 m?

Answer

The angle decreases at $\frac{400}{1681}$ rad/sec.

38) You stand 40 ft from a bottle rocket on the ground and watch as it takes off vertically into the air at a rate of 20 ft/sec. Find the rate at which the angle of elevation changes when the rocket is 30 ft in the air.

39) A lighthouse, L, is on an island 4 mi away from the closest point, P, on the beach (see the following image). If the lighthouse light rotates clockwise at a constant rate of 10 revolutions/min, how fast does the beam of light move across the beach 2 mi away from the closest point on the beach?



Answer

100π mi/min

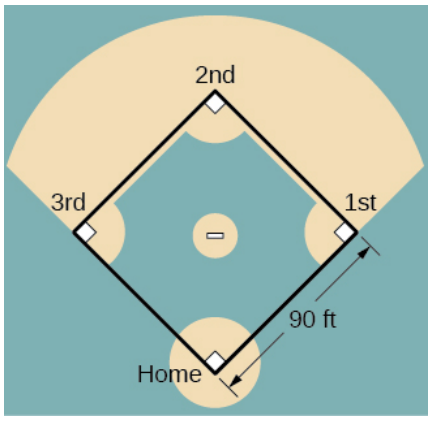
40) Using the same setup as the previous problem, determine at what rate the beam of light moves across the beach 1 mi away from the closest point on the beach.

41) You are walking to a bus stop at a right-angle corner. You move north at a rate of 2 m/sec and are 20 m south of the intersection. The bus travels west at a rate of 10 m/sec away from the intersection – you have missed the bus! What is the rate at which the angle between you and the bus is changing when you are 20 m south of the intersection and the bus is 10 m west of the intersection?

Answer

The angle is changing at a rate of $\frac{11}{25}$ rad/sec.

In exercises 42 - 45, refer to the figure of baseball diamond, which has sides of 90 ft.



42) [T] A batter hits a ball toward third base at **75** ft/sec and runs toward first base at a rate of **24** ft/sec. At what rate does the distance between the ball and the batter change when **2** sec have passed?

43) [T] A batter hits a ball toward second base at **80** ft/sec and runs toward first base at a rate of **30** ft/sec. At what rate does the distance between the ball and the batter change when the runner has covered one-third of the distance to first base? (Hint: Recall the law of cosines.)

Answer

The distance is increasing at a rate of **62.50** ft/sec.

44) [T] A batter hits the ball and runs toward first base at a speed of **22** ft/sec. At what rate does the distance between the runner and second base change when the runner has run **30** ft?

45) [T] Runners start at first and second base. When the baseball is hit, the runner at first base runs at a speed of **18** ft/sec toward second base and the runner at second base runs at a speed of **20** ft/sec toward third base. How fast is the distance between runners changing 1 sec after the ball is hit?

Answer

The distance is decreasing at a rate of **11.99** ft/sec.

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4.2: Linear Approximations and Differentials

Learning Objectives

- Describe the linear approximation to a function at a point.
- Write the linearization of a given function.
- Draw a graph that illustrates the use of differentials to approximate the change in a quantity.
- Calculate the relative error and percentage error in using a differential approximation.

We have just seen how derivatives allow us to compare related quantities that are changing over time. In this section, we examine another application of derivatives: the ability to approximate functions locally by linear functions. Linear functions are the easiest functions with which to work, so they provide a useful tool for approximating function values. In addition, the ideas presented in this section are generalized later in the text when we study how to approximate functions by higher-degree polynomials Introduction to Power Series and Functions.

Linear Approximation of a Function at a Point

Consider a function f that is differentiable at a point $x = a$. Recall that the tangent line to the graph of f at a is given by the equation

$$y = f(a) + f'(a)(x - a).$$

For example, consider the function $f(x) = \frac{1}{x}$ at $a = 2$. Since f is differentiable at $x = 2$ and $f'(x) = -\frac{1}{x^2}$, we see that $f'(2) = -\frac{1}{4}$. Therefore, the tangent line to the graph of f at $a = 2$ is given by the equation

$$y = \frac{1}{2} - \frac{1}{4}(x - 2).$$

Figure $4.2.1a$ shows a graph of $f(x) = \frac{1}{x}$ along with the tangent line to f at $x = 2$. Note that for x near 2, the graph of the tangent line is close to the graph of f. As a result, we can use the equation of the tangent line to approximate $f(x)$ for x near 2. For example, if $x = 2.1$, the y value of the corresponding point on the tangent line is

$$y = \frac{1}{2} - \frac{1}{4}(2.1 - 2) = 0.475.$$

The actual value of $f(2.1)$ is given by

$$f(2.1) = \frac{1}{2.1} \approx 0.47619.$$

Therefore, the tangent line gives us a fairly good approximation of $f(2.1)$ (Figure $4.2.1b$). However, note that for values of x far from 2, the equation of the tangent line does not give us a good approximation. For example, if $x = 10$, the y-value of the corresponding point on the tangent line is

$$y = \frac{1}{2} - \frac{1}{4}(10 - 2) = \frac{1}{2} - 2 = -1.5,$$

whereas the value of the function at $x = 10$ is $f(10) = 0.1$.

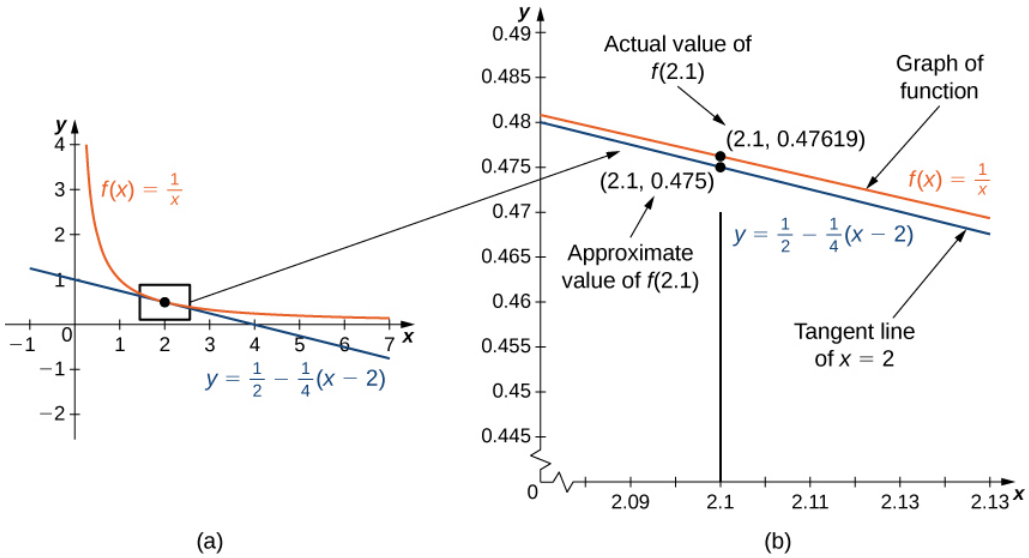

Figure 4.2.1: (a) The tangent line to $f(x) = 1/x$ at $x = 2$ provides a good approximation to f for x near 2. (b) At $x = 2.1$, the value of y on the tangent line to $f(x) = 1/x$ is 0.475. The actual value of $f(2.1)$ is $1/2.1$, which is approximately 0.47619.

In general, for a differentiable function f, the equation of the tangent line to f at $x = a$ can be used to approximate $f(x)$ for x near a. Therefore, we can write

$$f(x) \approx f(a) + f'(a)(x - a) \text{ for } x \text{ near } a.$$

We call the linear function

$$L(x) = f(a) + f'(a)(x - a) \tag{4.2.1}$$

the **linear approximation,** or **tangent line approximation**, of f at $x = a$. This function L is also known as the **linearization** of f at $x = a$.

To show how useful the linear approximation can be, we look at how to find the linear approximation for $f(x) = \sqrt{x}$ at $x = 9$.

Example 4.2.1: Linear Approximation of $\sqrt{x}$

Find the linear approximation of $f(x) = \sqrt{x}$ at $x = 9$ and use the approximation to estimate $\sqrt{9.1}$.

Solution

Since we are looking for the linear approximation at $x = 9$, using Equation 4.2.1 we know the linear approximation is given by

$$L(x) = f(9) + f'(9)(x - 9).$$

We need to find $f(9)$ and $f'(9)$.

$$f(x) = \sqrt{x} \Rightarrow f(9) = \sqrt{9} = 3$$

$$f'(x) = \tfrac{1}{2\sqrt{x}} \Rightarrow f'(9) = \tfrac{1}{2\sqrt{9}} = \tfrac{1}{6}$$

Therefore, the linear approximation is given by Figure 4.2.2.

$$L(x) = 3 + \frac{1}{6}(x - 9)$$

Using the linear approximation, we can estimate $\sqrt{9.1}$ by writing

$$\sqrt{9.1} = f(9.1) \approx L(9.1) = 3 + \frac{1}{6}(9.1 - 9) \approx 3.0167.$$

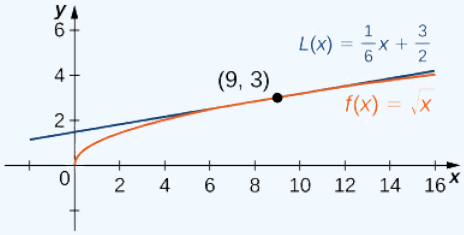

Figure 4.2.2: The local linear approximation to $f(x) = \sqrt{x}$ at $x = 9$ provides an approximation to f for x near 9.

Analysis

Using a calculator, the value of $\sqrt{9.1}$ to four decimal places is 3.0166. The value given by the linear approximation, 3.0167, is very close to the value obtained with a calculator, so it appears that using this linear approximation is a good way to estimate $\sqrt{x}$, at least for x near 9. At the same time, it may seem odd to use a linear approximation when we can just push a few buttons on a calculator to evaluate $\sqrt{9.1}$. However, how does the calculator evaluate $\sqrt{9.1}$? The calculator uses an approximation! In fact, calculators and computers use approximations all the time to evaluate mathematical expressions; they just use higher-degree approximations.

? Exercise 4.2.1

Find the local linear approximation to $f(x) = \sqrt[3]{x}$ at $x = 8$. Use it to approximate $\sqrt[3]{8.1}$ to five decimal places.

Hint

$L(x) = f(a) + f'(a)(x - a)$

Answer

$L(x) = 2 + \frac{1}{12}(x - 8)$; 2.00833

✓ Example 4.2.2: Linear Approximation of $\sin x$

Find the linear approximation of $f(x) = \sin x$ at $x = \frac{\pi}{3}$ and use it to approximate $\sin(62°)$.

Solution

First we note that since $\frac{\pi}{3}$ rad is equivalent to 60°, using the linear approximation at $x = \pi/3$ seems reasonable. The linear approximation is given by

$$L(x) = f(\tfrac{\pi}{3}) + f'(\tfrac{\pi}{3})(x - \tfrac{\pi}{3}).$$

We see that

$$f(x) = \sin x \Rightarrow f(\tfrac{\pi}{3}) = \sin(\tfrac{\pi}{3}) = \tfrac{\sqrt{3}}{2}$$

$$f'(x) = \cos x \Rightarrow f'(\tfrac{\pi}{3}) = \cos(\tfrac{\pi}{3}) = \tfrac{1}{2}$$

Therefore, the linear approximation of f at $x = \pi/3$ is given by Figure 4.2.3.

$$L(x) = \tfrac{\sqrt{3}}{2} + \tfrac{1}{2}(x - \tfrac{\pi}{3})$$

To estimate $\sin(62°)$ using L, we must first convert 62° to radians. We have $62° = \frac{62\pi}{180}$ radians, so the estimate for $\sin(62°)$ is given by

$$\sin(62°) = f(\tfrac{62\pi}{180}) \approx L(\tfrac{62\pi}{180}) = \tfrac{\sqrt{3}}{2} + \tfrac{1}{2}(\tfrac{62\pi}{180} - \tfrac{\pi}{3}) = \tfrac{\sqrt{3}}{2} + \tfrac{1}{2}(\tfrac{2\pi}{180}) = \tfrac{\sqrt{3}}{2} + \tfrac{\pi}{180} \approx 0.88348.$$

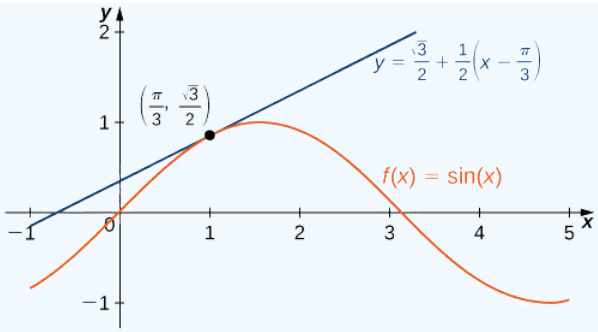

Figure 4.2.3: The linear approximation to $f(x) = \sin x$ at $x = \pi/3$ provides an approximation to $\sin x$ for x near $\pi/3$.

Exercise 4.2.2

Find the linear approximation for $f(x) = \cos x$ at $x = \frac{\pi}{2}$.

Hint

$L(x) = f(a) + f'(a)(x - a)$

Answer

$L(x) = -x + \frac{\pi}{2}$

Linear approximations may be used in estimating roots and powers. In the next example, we find the linear approximation for $f(x) = (1 + x)^n$ at $x = 0$, which can be used to estimate roots and powers for real numbers near 1. The same idea can be extended to a function of the form $f(x) = (m + x)^n$ to estimate roots and powers near a different number m.

Example 4.2.3: Approximating Roots and Powers

Find the linear approximation of $f(x) = (1 + x)^n$ at $x = 0$. Use this approximation to estimate $(1.01)^3$.

Solution

The linear approximation at $x = 0$ is given by

$$L(x) = f(0) + f'(0)(x - 0).$$

Because

$$f(x) = (1 + x)^n \Rightarrow f(0) = 1$$

$$f'(x) = n(1 + x)^{n-1} \Rightarrow f'(0) = n,$$

the linear approximation is given by Figure 4.2.4*a*.

$$L(x) = 1 + n(x - 0) = 1 + nx$$

We can approximate $(1.01)^3$ by evaluating $L(0.01)$ when $n = 3$. We conclude that

$$(1.01)^3 = f(1.01) \approx L(1.01) = 1 + 3(0.01) = 1.03.$$

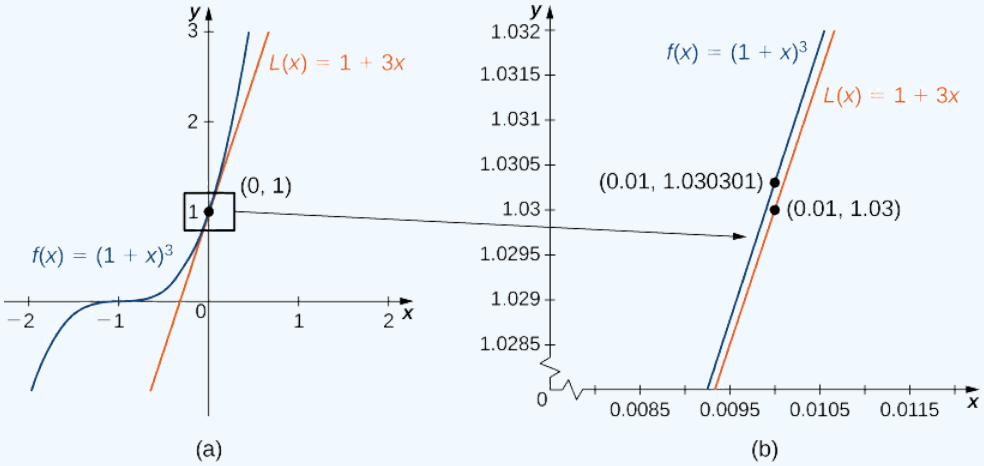

Figure 4.2.4: (a) The linear approximation of $f(x)$ at $x = 0$ is $L(x)$. (b) The actual value of 1.01^3 is 1.030301. The linear approximation of $f(x)$ at $x = 0$ estimates 1.01^3 to be 1.03.

? Exercise 4.2.3

Find the linear approximation of $f(x) = (1 + x)^4$ at $x = 0$ without using the result from the preceding example.

Hint

$f'(x) = 4(1 + x)^3$

Answer

$L(x) = 1 + 4x$

Differentials

We have seen that linear approximations can be used to estimate function values. They can also be used to estimate the amount a function value changes as a result of a small change in the input. To discuss this more formally, we define a related concept: **differentials**. Differentials provide us with a way of estimating the amount a function changes as a result of a small change in input values.

When we first looked at derivatives, we used the Leibniz notation dy/dx to represent the derivative of y with respect to x. Although we used the expressions dy and dx in this notation, they did not have meaning on their own. Here we see a meaning to the expressions dy and dx. Suppose $y = f(x)$ is a differentiable function. Let dx be an independent variable that can be assigned any nonzero real number, and define the dependent variable dy by

$$dy = f'(x)\,dx. \tag{4.2.2}$$

It is important to notice that dy is a function of both x and dx. The expressions dy and dx are called differentials. We can divide both sides of Equation 4.2.2 by dx, which yields

$$\frac{dy}{dx} = f'(x). \tag{4.2.3}$$

This is the familiar expression we have used to denote a derivative. Equation 4.2.3 is known as the **differential form** of Equation 4.2.2.

✓ Example 4.2.4: Computing Differentials

For each of the following functions, find dy and evaluate when $x = 3$ and $dx = 0.1$.

a. $y = x^2 + 2x$
b. $y = \cos x$

Solution

The key step is calculating the derivative. When we have that, we can obtain dy directly.

a. Since $f(x) = x^2 + 2x$, we know $f'(x) = 2x + 2$, and therefore

$$dy = (2x + 2)\,dx.$$

When $x = 3$ and $dx = 0.1$,

$$dy = (2 \cdot 3 + 2)(0.1) = 0.8.$$

b. Since $f(x) = \cos x$, $f'(x) = -\sin(x)$. This gives us

$$dy = -\sin x\,dx.$$

When $x = 3$ and $dx = 0.1$,

$$dy = -\sin(3)(0.1) = -0.1\sin(3).$$

? Exercise 4.2.4

For $y = e^{x^2}$, find dy.

Hint

$dy = f'(x)\,dx$

Answer

$dy = 2xe^{x^2}dx$

We now connect differentials to linear approximations. Differentials can be used to estimate the change in the value of a function resulting from a small change in input values. Consider a function f that is differentiable at point a. Suppose the input x changes by a small amount. We are interested in how much the output y changes. If x changes from a to $a + dx$, then the change in x is dx (also denoted Δx), and the change in y is given by

$$\Delta y = f(a + dx) - f(a).$$

Instead of calculating the exact change in y, however, it is often easier to approximate the change in y by using a linear approximation. For x near a, $f(x)$ can be approximated by the linear approximation (Equation 4.2.1)

$$L(x) = f(a) + f'(a)(x - a).$$

Therefore, if dx is small,

$$f(a + dx) \approx L(a + dx) = f(a) + f'(a)(a + dx - a).$$

That is,

$$f(a + dx) - f(a) \approx L(a + dx) - f(a) = f'(a)\,dx.$$

In other words, the actual change in the function f if x increases from a to $a + dx$ is approximately the difference between $L(a + dx)$ and $f(a)$, where $L(x)$ is the linear approximation of f at a. By definition of $L(x)$, this difference is equal to $f'(a)\,dx$. In summary,

$$\Delta y = f(a + dx) - f(a) \approx L(a + dx) - f(a) = f'(a)\,dx = dy.$$

Therefore, we can use the differential $dy = f'(a)\,dx$ to approximate the change in y if x increases from $x = a$ to $x = a + dx$. We can see this in the following graph.

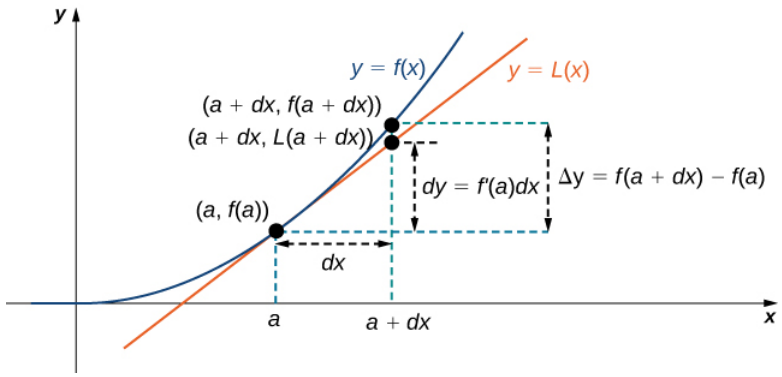

Figure 4.2.5: The differential $dy = f'(a)\,dx$ is used to approximate the actual change in y if x increases from a to $a + dx$.

We now take a look at how to use differentials to approximate the change in the value of the function that results from a small change in the value of the input. Note the calculation with differentials is much simpler than calculating actual values of functions and the result is very close to what we would obtain with the more exact calculation.

Example 4.2.5: Approximating Change with Differentials

Let $y = x^2 + 2x$. Compute Δy and dy at $x = 3$ if $dx = 0.1$.

Solution

The actual change in y if x changes from $x = 3$ to $x = 3.1$ is given by

$$\Delta y = f(3.1) - f(3) = [(3.1)^2 + 2(3.1)] - [3^2 + 2(3)] = 0.81.$$

The approximate change in y is given by $dy = f'(3)\,dx$. Since $f'(x) = 2x + 2$, we have

$$dy = f'(3)\,dx = (2(3) + 2)(0.1) = 0.8.$$

Exercise 4.2.5

For $y = x^2 + 2x$, find Δy and dy at $x = 3$ if $dx = 0.2$.

Hint

$dy = f'(3)\,dx,\ \Delta y = f(3.2) - f(3)$

Answer

$dy = 1.6,\ \Delta y = 1.64$

Calculating the Amount of Error

Any type of measurement is prone to a certain amount of error. In many applications, certain quantities are calculated based on measurements. For example, the area of a circle is calculated by measuring the radius of the circle. An error in the measurement of the radius leads to an error in the computed value of the area. Here we examine this type of error and study how differentials can be used to estimate the error.

Consider a function f with an input that is a measured quantity. Suppose the exact value of the measured quantity is a, but the measured value is $a + dx$. We say the measurement error is dx (or Δx). As a result, an error occurs in the calculated quantity $f(x)$. This type of error is known as a **propagated error** and is given by

$$\Delta y = f(a + dx) - f(a).$$

Since all measurements are prone to some degree of error, we do not know the exact value of a measured quantity, so we cannot calculate the propagated error exactly. However, given an estimate of the accuracy of a measurement, we can use differentials to approximate the propagated error Δy. Specifically, if f is a differentiable function at a,the propagated error is

$$\Delta y \approx dy = f'(a)\,dx.$$

Unfortunately, we do not know the exact value a. However, we can use the measured value $a + dx$, and estimate

$$\Delta y \approx dy \approx f'(a + dx)\, dx.$$

In the next example, we look at how differentials can be used to estimate the error in calculating the volume of a box if we assume the measurement of the side length is made with a certain amount of accuracy.

Example 4.2.6: Volume of a Cube

Suppose the side length of a cube is measured to be 5 cm with an accuracy of 0.1 cm.

a. Use differentials to estimate the error in the computed volume of the cube.
b. Compute the volume of the cube if the side length is (i) 4.9 cm and (ii) 5.1 cm to compare the estimated error with the actual potential error.

Solution

a. The measurement of the side length is accurate to within ± 0.1 cm. Therefore,

$$-0.1 \leq dx \leq 0.1.$$

The volume of a cube is given by $V = x^3$, which leads to

$$dV = 3x^2 dx.$$

Using the measured side length of 5 cm, we can estimate that

$$-3(5)^2(0.1) \leq dV \leq 3(5)^2(0.1).$$

Therefore,

$$-7.5 \leq dV \leq 7.5.$$

b. If the side length is actually 4.9 cm, then the volume of the cube is

$$V(4.9) = (4.9)^3 = 117.649 \text{cm}^3.$$

If the side length is actually 5.1 cm, then the volume of the cube is

$$V(5.1) = (5.1)^3 = 132.651 \text{cm}^3.$$

Therefore, the actual volume of the cube is between 117.649 and 132.651. Since the side length is measured to be 5 cm, the computed volume is $V(5) = 5^3 = 125$. Therefore, the error in the computed volume is

$$117.649 - 125 \leq \Delta V \leq 132.651 - 125.$$

That is,

$$-7.351 \leq \Delta V \leq 7.651.$$

We see the estimated error dV is relatively close to the actual potential error in the computed volume.

Exercise 4.2.6

Estimate the error in the computed volume of a cube if the side length is measured to be 6 cm with an accuracy of 0.2 cm.

Hint

$dV = 3x^2 dx$

Answer

The volume measurement is accurate to within $21.6\,\text{cm}^3$.

The measurement error dx $(= \Delta x)$ and the propagated error Δy are absolute errors. We are typically interested in the size of an error relative to the size of the quantity being measured or calculated. Given an absolute error Δq for a particular quantity, we define the **relative error** as $\frac{\Delta q}{q}$, where q is the actual value of the quantity. The **percentage error** is the relative error expressed as a

percentage. For example, if we measure the height of a ladder to be 63 in. when the actual height is 62 in., the absolute error is 1 in. but the relative error is $\frac{1}{62} = 0.016$, or 1.6%. By comparison, if we measure the width of a piece of cardboard to be 8.25 in. when the actual width is 8 in., our absolute error is $\frac{1}{4}$ in., whereas the relative error is $\frac{0.25}{8} = \frac{1}{32}$, or 3.1%. Therefore, the percentage error in the measurement of the cardboard is larger, even though 0.25 in. is less than 1 in.

Example 4.2.7: Relative and Percentage Error

An astronaut using a camera measures the radius of Earth as 4000 mi with an error of ± 80 mi. Let's use differentials to estimate the relative and percentage error of using this radius measurement to calculate the volume of Earth, assuming the planet is a perfect sphere.

Solution: If the measurement of the radius is accurate to within ± 80, we have

$$-80 \leq dr \leq 80.$$

Since the volume of a sphere is given by $V = (\frac{4}{3})\pi r^3$, we have

$$dV = 4\pi r^2 dr.$$

Using the measured radius of 4000 mi, we can estimate

$$-4\pi(4000)^2(80) \leq dV \leq 4\pi(4000)^2(80).$$

To estimate the relative error, consider $\frac{dV}{V}$. Since we do not know the exact value of the volume V, use the measured radius $r = 4000$ mi to estimate V. We obtain $V \approx (\frac{4}{3})\pi(4000)^3$. Therefore the relative error satisfies

$$\frac{-4\pi(4000)^2(80)}{4\pi(4000)^3/3} \leq \frac{dV}{V} \leq \frac{4\pi(4000)^2(80)}{4\pi(4000)^3/3},$$

which simplifies to

$$-0.06 \leq \frac{dV}{V} \leq 0.06.$$

The relative error is 0.06 and the percentage error is 6%.

Exercise 4.2.7

Determine the percentage error if the radius of Earth is measured to be 3950 mi with an error of ± 100 mi.

Hint

Use the fact that $dV = 4\pi r^2 dr$ to find dV/V.

Answer

7.6%

Key Concepts

- A differentiable function $y = f(x)$ can be approximated at a by the linear function
$$L(x) = f(a) + f'(a)(x - a).$$
- For a function $y = f(x)$, if x changes from a to $a + dx$, then
$$dy = f'(x)\,dx$$
is an approximation for the change in y. The actual change in y is
$$\Delta y = f(a + dx) - f(a).$$
- A measurement error dx can lead to an error in a calculated quantity $f(x)$. The error in the calculated quantity is known as the propagated error. The propagated error can be estimated by
$$dy \approx f'(x)\,dx.$$
- To estimate the relative error of a particular quantity q, we estimate $\frac{\Delta q}{q}$.

Key Equations

- **Linear approximation**

$L(x) = f(a) + f'(a)(x - a)$

- **A differential**

$dy = f'(x)\,dx$

Glossary

differential

the differential dx is an independent variable that can be assigned any nonzero real number; the differential dy is defined to be $dy = f'(x)\,dx$

differential form

given a differentiable function $y = f'(x)$, the equation $dy = f'(x)\,dx$ is the differential form of the derivative of y with respect to x

linear approximation

the linear function $L(x) = f(a) + f'(a)(x - a)$ is the linear approximation of f at $x = a$

percentage error

the relative error expressed as a percentage

propagated error

the error that results in a calculated quantity $f(x)$ resulting from a measurement error dx

relative error

given an absolute error Δq for a particular quantity, $\frac{\Delta q}{q}$ is the relative error.

tangent line approximation (linearization)

since the linear approximation of f at $x = a$ is defined using the equation of the tangent line, the linear approximation of f at $x = a$ is also known as the tangent line approximation to f at $x = a$

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4.2E: EXERCISES FOR SECTION 4.2

1) What is the linear approximation for any generic linear function $y = mx + b$?

2) Determine the necessary conditions such that the linear approximation function is constant. Use a graph to prove your result.

Answer

$f'(a) = 0$

3) Explain why the linear approximation becomes less accurate as you increase the distance between x and a. Use a graph to prove your argument.

4) When is the linear approximation exact?

Answer

The linear approximation exact when $y = f(x)$ is linear or constant.

In exercises 5 - 10, find the linear approximation $L(x)$ to $y = f(x)$ near $x = a$ for the function.

5) [T] $f(x) = x + x^4, \quad a = 0$

6) [T] $f(x) = \dfrac{1}{x}, \quad a = 2$

Answer

$L(x) = \frac{1}{2} - \frac{1}{4}(x - 2)$

7) [T] $f(x) = \tan x, \quad a = \frac{\pi}{4}$

8) [T] $f(x) = \sin x, \quad a = \frac{\pi}{2}$

Answer

$L(x) = 1$

9) [T] $f(x) = x \sin x, \quad a = 2\pi$

10) [T] $f(x) = \sin^2 x, \quad a = 0$

Answer

$L(x) = 0$

In exercises 11 - 16, compute the values given within 0.01 by deciding on the appropriate $f(x)$ and a, and evaluating $L(x) = f(a) + f'(a)(x - a)$. Check your answer using a calculator.

11) [T] $(2.001)^6$

12) [T] $\sin(0.02)$

Answer

$\sin(0.02) \approx 0.02$

13) [T] $\cos(0.03)$

14) [T] $(15.99)^{1/4}$

Answer

$(15.99)^{1/4} \approx 1.9996875$

15) [T] $\dfrac{1}{0.98}$

16) [T] $\sin(3.14)$

Answer

$\sin(3.14) \approx 0.001593$

In exercises 17 - 22, determine the appropriate $f(x)$ and a, and evaluate $L(x) = f(a) + f'(a)(x - a)$. Calculate the numerical error in the linear approximations that follow.

17) $(1.01)^3$

18) $\cos(0.01)$

Answer

$\cos(0.01) \approx L(0.01) = f(0) + f'(0)(0 - 0.01) = 1$; error, 0.00005

19) $(\sin(0.01))^2$

20) $(1.01)^{-3}$

Answer

$(1.01)^{-3} \approx L(1.01) = f(1) + f'(1)(1.01 - 1) = 0.97$; error, 0.0006

21) $\left(1 + \frac{1}{10}\right)^{10}$

22) $\sqrt{8.99}$

Answer

$\sqrt{8.99} \approx L(8.99) = f(9) + f'(9)(8.99 - 9) = 3 - \frac{1}{600}$; error, 4.632×10^{-7}

In exercises 23 - 26, find the differential of the function.

23) $y = 3x^4 + x^2 - 2x + 1$

24) $y = x \cos x$

Answer

$dy = (\cos x - x \sin x)\, dx$

25) $y = \sqrt{1 + x}$

26) $y = \dfrac{x^2 + 2}{x - 1}$

Answer

$dy = \left(\dfrac{x^2 - 2x - 2}{(x-1)^2}\right)dx$

In exercises 27 - 32, find the differential and evaluate for the given x and dx.

27) $y = 3x^2 - x + 6,\ x = 2,\ dx = 0.1$

28) $y = \dfrac{1}{x+1},\ x = 1,\ dx = 0.25$

Answer

$dy = -\dfrac{1}{(x+1)^2}dx, \quad dy = -\frac{1}{16}$

29) $y = \tan x,\ x = 0,\ dx = \frac{\pi}{10}$

30) $y = \dfrac{3x^2 + 2}{\sqrt{x+1}},\ x = 0,\ dx = 0.1$

Answer

$dy = \dfrac{9x^2 + 12x - 2}{2(x+1)^{3/2}}dx, \quad dy = -0.1$

31) $y = \dfrac{\sin(2x)}{x},\ x = \pi,\ dx = 0.25$

32) $y = x^3 + 2x + \dfrac{1}{x},\ x = 1,\ dx = 0.05$

Answer

$dy = \left(3x^2 + 2 - \dfrac{1}{x^2}\right)dx, \quad dy = 0.2$

In exercises 33 - 38, find the change in volume dV or in surface area dA.

33) dV if the sides of a cube change from 10 to 10.1.

34) dA if the sides of a cube change from x to $x + dx$.

Answer

$dA = 12x\,dx$

35) dA if the radius of a sphere changes from r by dr.

36) dV if the radius of a sphere changes from r by dr.

Answer

$dV = 4\pi r^2 dr$

37) dV if a circular cylinder with $r = 2$ changes height from 3 cm to $3.05cm$.

38) dV if a circular cylinder of height 3 changes from $r = 2$ to $r = 1.9$ cm.

Answer

$dV = -1.2\pi\ \text{cm}^3$

In exercises 39 - 41, use differentials to estimate the maximum and relative error when computing the surface area or volume.

39) A spherical golf ball is measured to have a radius of 5 mm, with a possible measurement error of 0.1 mm. What is the possible change in volume?

40) A pool has a rectangular base of 10 ft by 20 ft and a depth of 6 ft. What is the change in volume if you only fill it up to 5.5 ft?

Answer

$-100\ \text{ft}^3$

41) An ice cream cone has height 4 in. and radius 1 in. If the cone is 0.1 in. thick, what is the difference between the volume of the cone, including the shell, and the volume of the ice cream you can fit inside the shell?

In exercises 42 - 44, confirm the approximations by using the linear approximation at $x = 0$.

42) $\sqrt{1-x} \approx 1 - \frac{1}{2}x$

43) $\dfrac{1}{\sqrt{1-x^2}} \approx 1$

44) $\sqrt{c^2 + x^2} \approx c$

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4.3: Maxima and Minima

Learning Objectives

- Define absolute extrema.
- Define local extrema.
- Explain how to find the critical points of a function over a closed interval.
- Describe how to use critical points to locate absolute extrema over a closed interval.

Given a particular function, we are often interested in determining the largest and smallest values of the function. This information is important in creating accurate graphs. Finding the maximum and minimum values of a function also has practical significance, because we can use this method to solve optimization problems, such as maximizing profit, minimizing the amount of material used in manufacturing an aluminum can, or finding the maximum height a rocket can reach. In this section, we look at how to use derivatives to find the largest and smallest values for a function.

Absolute Extrema

Consider the function $f(x) = x^2 + 1$ over the interval $(-\infty, \infty)$. As $x \to \pm\infty, f(x) \to \infty$. Therefore, the function does not have a largest value. However, since $x^2 + 1 \geq 1$ for all real numbers x and $x^2 + 1 = 1$ when $x = 0$, the function has a smallest value, 1, when $x = 0$. We say that 1 is the absolute minimum of $f(x) = x^2 + 1$ and it occurs at $x = 0$. We say that $f(x) = x^2 + 1$ does not have an absolute maximum (Figure 4.3.1).

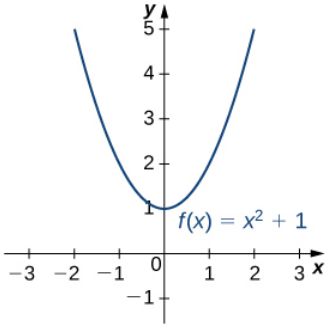


Figure 4.3.1: The given function has an absolute minimum of 1 at $x = 0$. The function does not have an absolute maximum.

Definition: Absolute Extrema

Let f be a function defined over an interval I and let $c \in I$. We say f has an *absolute maximum* on I at c if $f(c) \geq f(x)$ for all $x \in I$. We say f has an *absolute minimum* on I at c if $f(c) \leq f(x)$ for all $x \in I$. If f has an absolute maximum on I at c or an absolute minimum on I at c, we say f has *an absolute extremum* on I at c.

Before proceeding, let's note two important issues regarding this definition. First, the term absolute here does not refer to absolute value. An absolute extremum may be positive, negative, or zero. Second, if a function f has an absolute extremum over an interval I at c, the absolute extremum is $f(c)$. The real number c is a point in the domain at which the absolute extremum occurs. For example, consider the function $f(x) = 1/(x^2 + 1)$ over the interval $(-\infty, \infty)$. Since

$$f(0) = 1 \geq \frac{1}{x^2 + 1} = f(x)$$

for all real numbers x, we say f has an absolute maximum over $(-\infty, \infty)$ at $x = 0$. The absolute maximum is $f(0) = 1$. It occurs at $x = 0$, as shown in Figure 4.3.2(b).

A function may have both an absolute maximum and an absolute minimum, just one extremum, or neither. Figure 4.3.2 shows several functions and some of the different possibilities regarding absolute extrema. However, the following theorem, called the **Extreme Value Theorem**, guarantees that a continuous function f over a closed, bounded interval $[a, b]$ has both an absolute maximum and an absolute minimum.

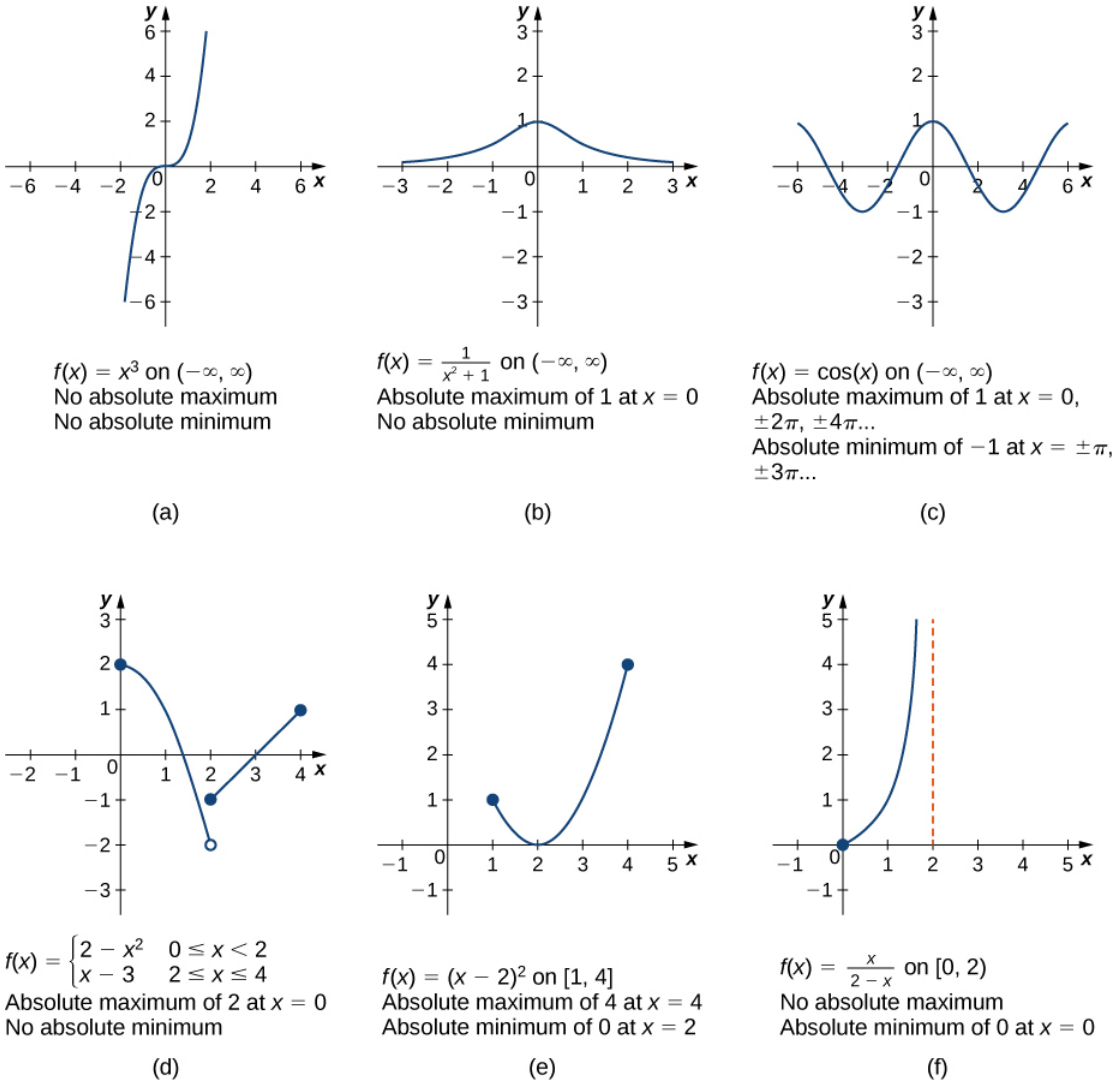


Figure 4.3.2: Graphs (a), (b), and (c) show several possibilities for absolute extrema for functions with a domain of $(-\infty, \infty)$. Graphs (d), (e), and (f) show several possibilities for absolute extrema for functions with a domain that is a bounded interval.

Theorem 4.3.1: Extreme Value Theorem

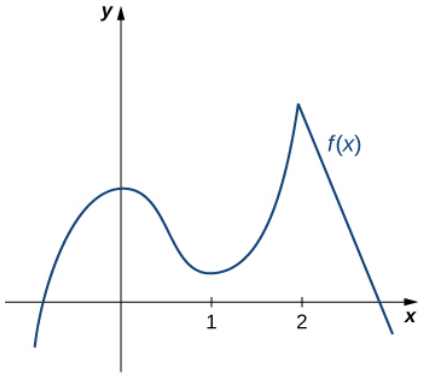
If f is a continuous function over the closed, bounded interval $[a, b]$, then there is a point in $[a, b]$ at which f has an absolute maximum over $[a, b]$ and there is a point in $[a, b]$ at which f has an absolute minimum over $[a, b]$.

The proof of the extreme value theorem is beyond the scope of this text. Typically, it is proved in a course on real analysis. There are a couple of key points to note about the statement of this theorem. For the extreme value theorem to apply, the function must be continuous over a closed, bounded interval. If the interval I is open or the function has even one point of discontinuity, the function may not have an absolute maximum or absolute minimum over I. For example, consider the functions shown in Figure 4.3.2 (d), (e), and (f). All three of these functions are defined over bounded intervals. However, the function in graph (e) is the only one that has both an absolute maximum and an absolute minimum over its domain. The extreme value theorem cannot be applied to the functions in graphs (d) and (f) because neither of these functions is continuous over a closed, bounded interval. Although the function in graph (d) is defined over the closed interval $[0, 4]$, the function is discontinuous at $x = 2$. The function has an absolute maximum over $[0, 4]$ but does not have an absolute minimum. The function in graph (f) is continuous over the half-open interval $[0, 2)$, but is not defined at $x = 2$, and therefore is not continuous over a closed, bounded interval. The function has an absolute minimum over $[0, 2)$, but does not have an absolute maximum over $[0, 2)$. These two graphs illustrate why a function over a bounded interval may fail to have an absolute maximum and/or absolute minimum.

Before looking at how to find absolute extrema, let's examine the related concept of local extrema. This idea is useful in determining where absolute extrema occur.

Local Extrema and Critical Points

Consider the function f shown in Figure 4.3.3. The graph can be described as two mountains with a valley in the middle. The absolute maximum value of the function occurs at the higher peak, at $x = 2$. However, $x = 0$ is also a point of interest. Although $f(0)$ is not the largest value of f, the value $f(0)$ is larger than $f(x)$ for all x near 0. We say f has a local maximum at $x = 0$. Similarly, the function f does not have an absolute minimum, but it does have a local minimum at $x = 1$ because $f(1)$ is less than $f(x)$ for x near 1.



f(*x*) defined on $(-\infty, \infty)$
Local maxima at $x = 0$ and $x = 2$
Local minimum at $x = 1$

Figure 4.3.3: This function f has two local maxima and one local minimum. The local maximum at $x = 2$ is also the absolute maximum.

Definition: Local Extrema

A function f has a *local maximum* at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \geq f(x)$ for all $x \in I$. A function f has a *local minimum* at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \leq f(x)$ for all $x \in I$. A function f has a *local extremum* at c if f has a local maximum at c or f has a local minimum at c.

Note that if f has an absolute extremum at c and f is defined over an interval containing c, then $f(c)$ is also considered a local extremum. If an absolute extremum for a function f occurs at an endpoint, we do not consider that to be a local extremum, but instead refer to that as an endpoint extremum.

Given the graph of a function f, it is sometimes easy to see where a local maximum or local minimum occurs. However, it is not always easy to see, since the interesting features on the graph of a function may not be visible because they occur at a very small scale. Also, we may not have a graph of the function. In these cases, how can we use a formula for a function to determine where these extrema occur?

To answer this question, let's look at Figure 4.3.3 again. The local extrema occur at $x = 0, x = 1$, and $x = 2$. Notice that at $x = 0$ and $x = 1$, the derivative $f'(x) = 0$. At $x = 2$, the derivative $f'(x)$ does not exist, since the function f has a corner there. In fact, if f has a local extremum at a point $x = c$, the derivative $f'(c)$ must satisfy one of the following conditions: either $f'(c) = 0$ or $f'(c)$ is undefined. Such a value c is known as a critical point and it is important in finding extreme values for functions.

Definition: Critical Points

Let c be an interior point in the domain of f. We say that c is a *critical point* of f if $f'(c) = 0$ or $f'(c)$ is undefined.

As mentioned earlier, if f has a local extremum at a point $x = c$, then c must be a critical point of f. This fact is known as **Fermat's theorem.**

Theorem 4.3.2: Fermat's Theorem

If f has a local extremum at c and f is differentiable at c, then $f'(c) = 0$.

Proof

Suppose f has a local extremum at c and f is differentiable at c. We need to show that $f'(c) = 0$. To do this, we will show that $f'(c) \geq 0$ and $f'(c) \leq 0$, and therefore $f'(c) = 0$. Since f has a local extremum at c, f has a local maximum or local minimum at c. Suppose f has a local maximum at c. The case in which f has a local minimum at c can be handled similarly. There then exists an open interval I such that $f(c) \geq f(x)$ for all $x \in I$. Since f is differentiable at c, from the definition of the derivative, we know that

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

Since this limit exists, both one-sided limits also exist and equal $f'(c)$. Therefore,

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c,} \tag{4.3.1}$$

and

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}.$$

Since $f(c)$ is a local maximum, we see that $f(x) - f(c) \leq 0$ for x near c. Therefore, for x near c, but $x > c$, we have $\frac{f(x)-f(c)}{x-c} \leq 0$. From Equation 4.3.1 we conclude that $f'(c) \leq 0$. Similarly, it can be shown that $f'(c) \geq 0$. Therefore, $f'(c) = 0$.

□

From Fermat's theorem, we conclude that if f has a local extremum at c, then either $f'(c) = 0$ or $f'(c)$ is undefined. In other words, local extrema can only occur at critical points.

Note this theorem does not claim that a function f must have a local extremum at a critical point. Rather, it states that critical points are candidates for local extrema. For example, consider the function $f(x) = x^3$. We have $f'(x) = 3x^2 = 0$ when $x = 0$. Therefore, $x = 0$ is a critical point. However, $f(x) = x^3$ is increasing over $(-\infty, \infty)$, and thus f does not have a local extremum at $x = 0$. In Figure 4.3.4, we see several different possibilities for critical points. In some of these cases, the functions have local extrema at critical points, whereas in other cases the functions do not. Note that these graphs do not show all possibilities for the behavior of a function at a critical point.

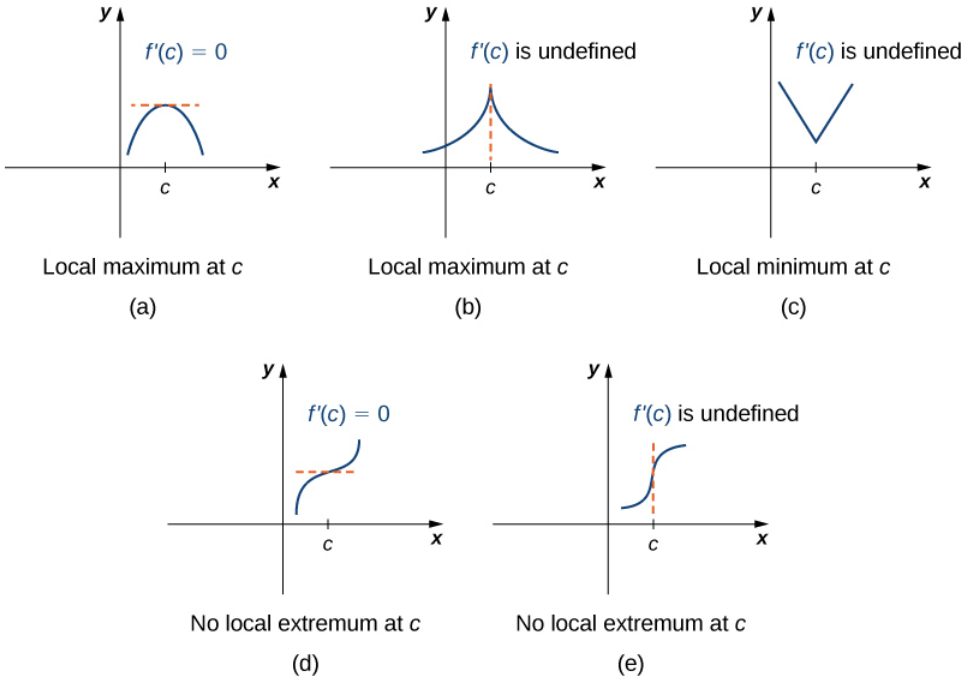


Figure 4.3.4: (a–e) A function f has a critical point at c if $f'(c) = 0$ or $f'(c)$ is undefined. A function may or may not have a local extremum at a critical point.

Later in this chapter we look at analytical methods for determining whether a function actually has a local extremum at a critical point. For now, let's turn our attention to finding critical points. We will use graphical observations to determine whether a critical point is associated with a local extremum.

Example 4.3.1: Locating Critical Points

For each of the following functions, find all critical points. Use a graphing utility to determine whether the function has a local extremum at each of the critical points.

a. $f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x$
b. $f(x) = (x^2 - 1)^3$
c. $f(x) = \frac{4x}{1+x^2}$

Solution

a. The derivative $f'(x) = x^2 - 5x + 4$ is defined for all real numbers x. Therefore, we only need to find the values for x where $f'(x) = 0$. Since $f'(x) = x^2 - 5x + 4 = (x - 4)(x - 1)$, the critical points are $x = 1$ and $x = 4$. From the graph of f in Figure 4.3.5, we see that f has a local maximum at $x = 1$ and a local minimum at $x = 4$.

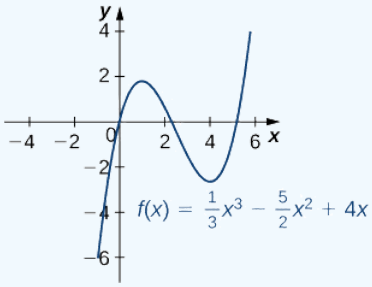


Figure 4.3.5: This function has a local maximum and a local minimum.

b. Using the chain rule, we see the derivative is

$$f'(x) = 3(x^2 - 1)^2(2x) = 6x(x^2 - 1)^2.$$

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Therefore, f has critical points when $x = 0$ and when $x^2 - 1 = 0$. We conclude that the critical points are $x = 0, \pm 1$. From the graph of f in Figure 4.3.6, we see that f has a local (and absolute) minimum at $x = 0$, but does not have a local extremum at $x = 1$ or $x = -1$.

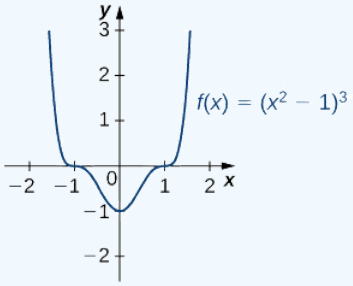

Figure 4.3.6: This function has three critical points: $x = 0$, $x = 1$, and $x = -1$. The function has a local (and absolute) minimum at $x = 0$, but does not have extrema at the other two critical points.

c. By the quotient rule, we see that the derivative is

$$f'(x) = \frac{4(1+x^2) - 4x(2x)}{(1+x^2)^2} = \frac{4-4x^2}{(1+x^2)^2}.$$

The derivative is defined everywhere. Therefore, we only need to find values for x where $f'(x) = 0$. Solving $f'(x) = 0$, we see that $4 - 4x^2 = 0$, which implies $x = \pm 1$. Therefore, the critical points are $x = \pm 1$. From the graph of f in Figure 4.3.7, we see that f has an absolute maximum at $x = 1$ and an absolute minimum at $x = -1$. Hence, f has a local maximum at $x = 1$ and a local minimum at $x = -1$. (Note that if f has an absolute extremum over an interval I at a point c that is not an endpoint of I, then f has a local extremum at c.)

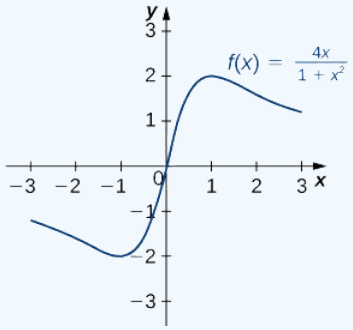

Figure 4.3.7: This function has an absolute maximum and an absolute minimum.

? Exercise 4.3.1

Find all critical points for $f(x) = x^3 - \frac{1}{2}x^2 - 2x + 1$.

Hint

Calculate $f'(x)$.

Answer

$x = \frac{-2}{3}, x = 1$

Locating Absolute Extrema

The extreme value theorem states that a continuous function over a closed, bounded interval has an absolute maximum and an absolute minimum. As shown in Figure 4.3.2, one or both of these absolute extrema could occur at an endpoint. If an absolute extremum does not occur at an endpoint, however, it must occur at an interior point, in which case the absolute extremum is a local extremum. Therefore, by Fermat's Theorem, the point c at which the local extremum occurs must be a critical point. We summarize this result in the following theorem.

Theorem 4.3.3: Location of Absolute Extrema

Let f be a continuous function over a closed, bounded interval I. The absolute maximum of f over I and the absolute minimum of f over I must occur at endpoints of I or at critical points of f in I.

With this idea in mind, let's examine a procedure for locating absolute extrema.

Problem-Solving Strategy: Locating Absolute Extrema over a Closed Interval

Consider a continuous function f defined over the closed interval $[a, b]$.

1. Evaluate f at the endpoints $x = a$ and $x = b$.
2. Find all critical points of f that lie over the interval (a, b) and evaluate f at those critical points.
3. Compare all values found in (1) and (2). From "Location of Absolute Extrema," the absolute extrema must occur at endpoints or critical points. Therefore, the largest of these values is the absolute maximum of f. The smallest of these values is the absolute minimum of f.

Now let's look at how to use this strategy to find the absolute maximum and absolute minimum values for continuous functions.

Example 4.3.2: Locating Absolute Extrema

For each of the following functions, find the absolute maximum and absolute minimum over the specified interval and state where those values occur.

a. $f(x) = -x^2 + 3x - 2$ over $[1, 3]$.
b. $f(x) = x^2 - 3x^{2/3}$ over $[0, 2]$.

Solution

a. Step 1. Evaluate f at the endpoints $x = 1$ and $x = 3$.

$$f(1) = 0 \text{ and } f(3) = -2$$

Step 2. Since $f'(x) = -2x + 3$, f' is defined for all real numbers x. Therefore, there are no critical points where the derivative is undefined. It remains to check where $f'(x) = 0$. Since $f'(x) = -2x + 3 = 0$ at $x = \frac{3}{2}$ and $\frac{3}{2}$ is in the interval $[1, 3]$, $f(\frac{3}{2})$ is a candidate for an absolute extremum of f over $[1, 3]$. We evaluate $f(\frac{3}{2})$ and find

$$f\left(\tfrac{3}{2}\right) = \tfrac{1}{4}.$$

Step 3. We set up the following table to compare the values found in steps 1 and 2.

x	$f(x)$	Conclusion
1	0	
$\frac{3}{2}$	$\frac{1}{4}$	Absolute maximum
3	-2	Absolute minimum

From the table, we find that the absolute maximum of f over the interval [1, 3] is $\frac{1}{4}$, and it occurs at $x = \frac{3}{2}$. The absolute minimum of f over the interval $[1, 3]$ is -2, and it occurs at $x = 3$ as shown in Figure 4.3.8.

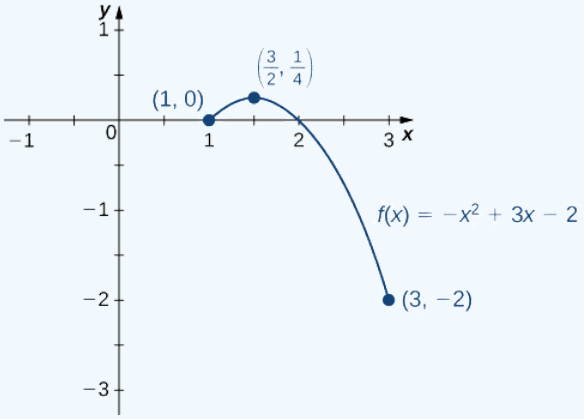

Figure 4.3.8: This function has both an absolute maximum and an absolute minimum.

b. Step 1. Evaluate f at the endpoints $x = 0$ and $x = 2$.

$$f(0) = 0 \text{ and } f(2) = 4 - 3\,(2)^{2/3} \approx -0.762$$

Step 2. The derivative of f is given by

$$f'(x) = 2x - \tfrac{2}{x^{1/3}} = \frac{2x^{4/3} - 2}{x^{1/3}}$$

for $x \neq 0$. The derivative is zero when $2x^{4/3} - 2 = 0$, which implies $x = \pm 1$. The derivative is undefined at $x = 0$. Therefore, the critical points of f are $x = 0, 1, -1$. The point $x = 0$ is an endpoint, so we already evaluated $f(0)$ in step 1. The point $x = -1$ is not in the interval of interest, so we need only evaluate $f(1)$. We find that

$$f(1) = -2.$$

Step 3. We compare the values found in steps 1 and 2, in the following table.

x	$f(x)$	Conclusion
0	0	Absolute maximum
1	-2	Absolute minimum
2	-0.762	

We conclude that the absolute maximum of f over the interval $[0, 2]$ is zero, and it occurs at $x = 0$. The absolute minimum is -2, and it occurs at $x = 1$ as shown in Figure 4.3.9.

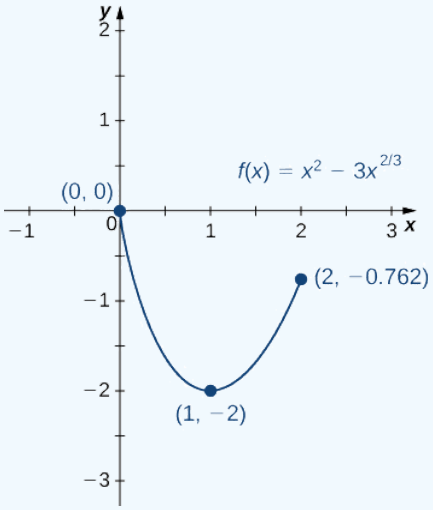

Figure 4.3.9: This function has an absolute maximum at an endpoint of the interval.

? Exercise 4.3.2

Find the absolute maximum and absolute minimum of $f(x) = x^2 - 4x + 3$ over the interval $[1, 4]$.

Hint

Look for critical points. Evaluate f at all critical points and at the endpoints.

Answer

The absolute maximum is 3 and it occurs at $x = 4$. The absolute minimum is -1 and it occurs at $x = 2$.

At this point, we know how to locate absolute extrema for continuous functions over closed intervals. We have also defined local extrema and determined that if a function f has a local extremum at a point c, then c must be a critical point of f. However, c being a critical point is not a sufficient condition for f to have a local extremum at c. Later in this chapter, we show how to determine whether a function actually has a local extremum at a critical point. First, however, we need to introduce the Mean Value Theorem, which will help as we analyze the behavior of the graph of a function.

Key Concepts

- A function may have both an absolute maximum and an absolute minimum, have just one absolute extremum, or have no absolute maximum or absolute minimum.
- If a function has a local extremum, the point at which it occurs must be a critical point. However, a function need not have a local extremum at a critical point.
- A continuous function over a closed, bounded interval has an absolute maximum and an absolute minimum. Each extremum occurs at a critical point or an endpoint.

Glossary

absolute extremum

if f has an absolute maximum or absolute minimum at c, we say f has an absolute extremum at c

absolute maximum

if $f(c) \geq f(x)$ for all x in the domain of f, we say f has an absolute maximum at c

absolute minimum

if $f(c) \leq f(x)$ for all x in the domain of f, we say f has an absolute minimum at c

critical point

if $f'(c) = 0$ or $f'(c)$ is undefined, we say that c is a critical point of f

extreme value theorem

if f is a continuous function over a finite, closed interval, then f has an absolute maximum and an absolute minimum

Fermat's theorem

if f has a local extremum at c, then c is a critical point of f

local extremum

if f has a local maximum or local minimum at c, we say f has a local extremum at c

local maximum

if there exists an interval I such that $f(c) \geq f(x)$ for all $x \in I$, we say f has a local maximum at c

local minimum

if there exists an interval I such that $f(c) \leq f(x)$ for all $x \in I$, we say f has a local minimum at c

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4.3E: EXERCISES FOR SECTION 4.3

1) In precalculus, you learned a formula for the position of the maximum or minimum of a quadratic equation $y = ax^2 + bx + c$, which was $m = -\frac{b}{2a}$. Prove this formula using calculus.

2) If you are finding an absolute minimum over an interval $[a, b]$, why do you need to check the endpoints? Draw a graph that supports your hypothesis.

Answer

On a closed interval, the endpoints often lie above or below any local (relative) extrema. Answers may vary for the graph.

3) If you are examining a function over an interval (a, b), for a and b finite, is it possible not to have an absolute maximum or absolute minimum?

4) When you are checking for critical points to locate the extrema of a function f, explain why you also need to determine points where $f'(x)$ is undefined. Draw a graph to support your explanation.

Answer

Points on the graph of f where there is a corner, a cusp, or a jump discontinuity or removable discontinuity can easily be absolute (or local) extrema of the function. Answers may vary for the graph.

5) Can you have a finite absolute maximum for $y = ax^2 + bx + c$ over $(-\infty, \infty)$? Explain why or why not using graphical arguments.

6) Can you have a finite absolute maximum for $y = ax^3 + bx^2 + cx + d$ over $(-\infty, \infty)$ assuming a is non-zero? Explain why or why not using graphical arguments.

Answer

No; answers will vary

7) Let m be the number of local minima and M be the number of local maxima. Can you create a function where $M > m + 2$? Draw a graph to support your explanation.

8) Is it possible to have more than one absolute maximum? Use a graphical argument to prove your hypothesis.

Answer

Since the absolute maximum is the function (output) value rather than the x value, the answer is no; answers will vary

9) Is it possible to have no absolute minimum or maximum for a function? If so, construct such a function. If not, explain why this is not possible.

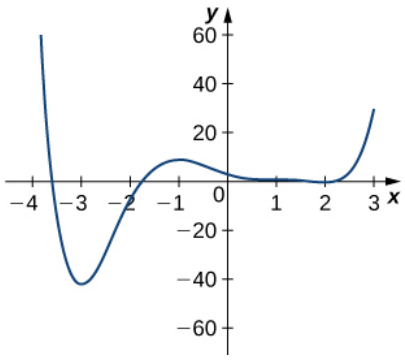
10) [T] Graph the function $y = e^{ax}$. For which values of a, on any infinite domain, will you have an absolute minimum and absolute maximum?

Answer

When $a = 0$

In exercises 11 - 14, determine where the local and absolute maxima and minima occur on the graph given. Assume domains are closed intervals unless otherwise specified.

11)



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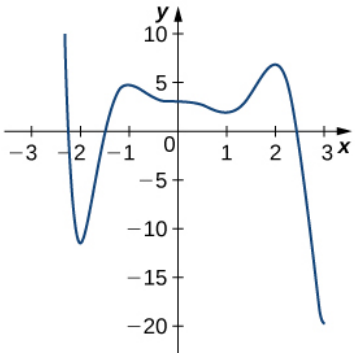
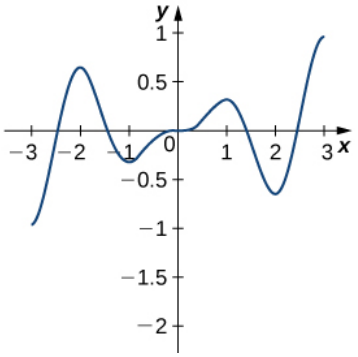


Answer

Absolute minimum at 3; Absolute maximum at −2.2; local minima at −2, 1; local maxima at −1, 2

13)



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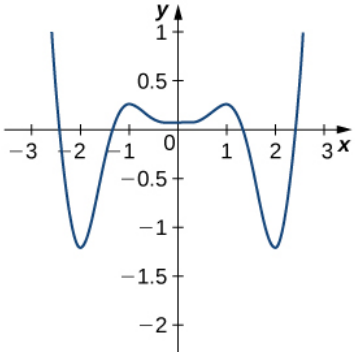


Answer

Absolute minima at −2, 2; absolute maxima at −2.5, 2.5; local minimum at 0; local maxima at −1, 1

For exercises 15 - 18, draw graphs of $f(x)$, which is continuous, over the interval $[-4, 4]$ with the following properties:

15) Absolute maximum at $x = 2$ and absolute minima at $x = \pm 3$

16) Absolute minimum at $x = 1$ and absolute maximum at $x = 2$

Answer

Answers may vary.

17) Absolute maximum at $x = 4$, absolute minimum at $x = -1$, local maximum at $x = -2$, and a critical point that is not a maximum or minimum at $x = 2$

18) Absolute maxima at $x = 2$ and $x = -3$, local minimum at $x = 1$, and absolute minimum at $x = 4$

Answer

Answers may vary.

In exercises 19 - 28, find the critical points in the domains of the given functions.

19) $y = 4x^3 - 3x$

20) $y = 4\sqrt{x} - x^2$

Answer

$x = 1$

21) $y = \dfrac{1}{x-1}$

22) $y = \ln(x-2)$

Answer

None

23) $y = \tan(x)$

24) $y = \sqrt{4-x^2}$

Answer

$x = 0$

25) $y = x^{3/2} - 3x^{5/2}$

26) $y = \dfrac{x^2-1}{x^2+2x-3}$

Answer

None

27) $y = \sin^2(x)$

28) $y = x + \dfrac{1}{x}$

Answer

$x = -1$ and $x = 1$

In exercises 29 - 39, find the local and/or absolute maxima for the functions over the specified domain.

29) $f(x) = x^2 + 3$ over $[-1, 4]$

30) $y = x^2 + \dfrac{2}{x}$ over $[1, 4]$

Answer

Absolute maximum: $x = 4, y = \frac{33}{2}$; absolute minimum: $x = 1, y = 3$

31) $y = (x - x^2)^2$ over $[-1, 1]$

32) $y = \dfrac{1}{x - x^2}$ over $[0, 1]$

Answer

Absolute minimum: $x = \frac{1}{2}, y = 4$

33) $y = \sqrt{9-x}$ over $[1, 9]$

34) $y = x + \sin(x)$ over $[0, 2\pi]$

Answer

Absolute maximum: $x = 2\pi, y = 2\pi$; absolute minimum: $x = 0, y = 0$

35) $y = \dfrac{x}{1+x}$ over $[0, 100]$

36) $y = |x+1| + |x-1|$ over $[-3, 2]$

Answer

Absolute maximum: $x=-3, y=6$; absolute minimum: $-1 \le x \le 1, y=2$

37) $y=\sqrt{x}-\sqrt{x^3}$ over $[0,4]$

38) $y=\sin x+\cos x$ over $[0,2\pi]$

Answer

Absolute maximum: $x=\frac{\pi}{4}, y=\sqrt{2}$; absolute minimum: $x=\frac{5\pi}{4}, y=-\sqrt{2}$

39) $y=4\sin\theta-3\cos\theta$ over $[0,2\pi]$

In exercises 40 - 45, find the local and absolute minima and maxima for the functions over $(-\infty,\infty)$.

40) $y=x^2+4x+5$

Answer

Absolute minimum: $x=-2, y=1$

41) $y=x^3-12x$

42) $y=3x^4+8x^3-18x^2$

Answer

Absolute minimum: $x=-3, y=-135$; local maximum: $x=0, y=0$; local minimum: $x=1, y=-7$

43) $y=x^3(1-x)^6$

44) $y=\dfrac{x^2+x+6}{x-1}$

Answer

Local maximum: $x=1-2\sqrt{2}, y=3-4\sqrt{2}$; local minimum: $x=1+2\sqrt{2}, y=3+4\sqrt{2}$

45) $y=\dfrac{x^2-1}{x-1}$

In exercises 46 - 50, use a calculator to graph the function and to estimate the absolute and local maxima and minima. Then, solve for them explicitly.

46) [T] $y=3x\sqrt{1-x^2}$

Answer

Absolute maximum: $x=\frac{\sqrt{2}}{2}, y=\frac{3}{2}$; absolute minimum: $x=-\frac{\sqrt{2}}{2}, y=-\frac{3}{2}$

47) [T] $y=x+\sin(x)$

48) [T] $y=12x^5+45x^4+20x^3-90x^2-120x+3$

Answer

Local maximum: $x=-2, y=59$; local minimum: $x=1, y=-130$

49) [T] $y=\dfrac{x^3+6x^2-x-30}{x-2}$

50) [T] $y=\dfrac{\sqrt{4-x^2}}{\sqrt{4+x^2}}$

Answer

Absolute maximum: $x=0, y=1$; absolute minimum: $x=-2, 2, y=0$

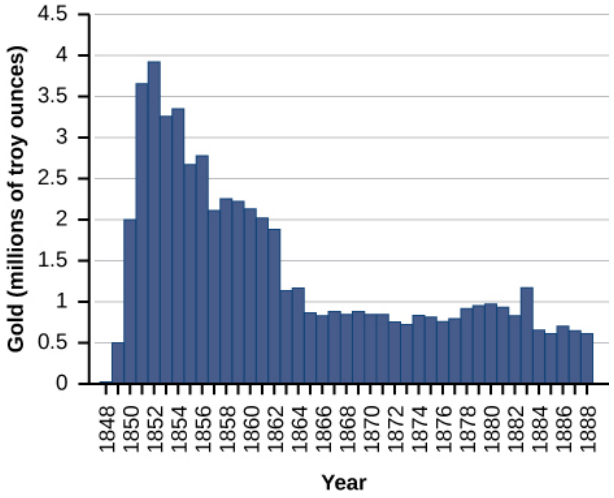
51) A company that produces cell phones has a cost function of $C=x^2-1200x+36,400$, where C is cost in dollars and x is number of cell phones produced (in thousands). How many units of cell phone (in thousands) minimizes this cost function?

52) A ball is thrown into the air and its position is given by $h(t)=-4.9t^2+60t+5m$. Find the height at which the ball stops ascending. How long after it is thrown does this happen?

Answer

$h=\frac{9245}{49}$ m, $t=\frac{300}{49}$ s

For exercises 53-54, consider the production of gold during the California gold rush (1848–1888). The production of gold can be modeled by $G(t) = \frac{(25t)}{(t^2+16)}$, where t is the number of years since the rush began $(0 \le t \le 40)$ and G is ounces of gold produced (in millions). A summary of the data is shown in the following figure.



53) Find when the maximum (local and global) gold production occurred, and the amount of gold produced during that maximum.

54) Find when the minimum (local and global) gold production occurred. What was the amount of gold produced during this minimum?

Answer

The global minimum was in 1848, when no gold was produced.

In exercises 55 & 56, find the critical points, maxima, and minima for the given piecewise functions.

55) $y = \begin{cases} x^2 - 4x, & \text{if } 0 \le x \le 1 \\ x^2 - 4, & \text{if } 1 < x \le 2 \end{cases}$

56) $y = \begin{cases} x^2 + 1, & \text{if } x \le 1 \\ x^2 - 4x + 5, & \text{if } x > 1 \end{cases}$

Answer

Absolute minima: $x = 0, x = 2, y = 1$; local maximum at $x = 1, y = 2$

In exercises 57 - 58, find the critical points of the following generic functions. Are they maxima, minima, or neither? State the necessary conditions.

57) $y = ax^2 + bx + c$, given that $a > 0$

58) $y = (x-1)^a$, given that $a > 1$

Answer

No maxima/minima if a is odd, minimum at $x = 1$ if a is even

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.
- Paul Seeburger (Monroe Community College) added answers for exercises 2 and 4.

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4.4: The Mean Value Theorem

Learning Objectives

- Explain the meaning of Rolle's theorem.
- Describe the significance of the Mean Value Theorem.
- State three important consequences of the Mean Value Theorem.

The *Mean Value Theorem* is one of the most important theorems in calculus. We look at some of its implications at the end of this section. First, let's start with a special case of the Mean Value Theorem, called Rolle's theorem.

Rolle's Theorem

Informally, **Rolle's theorem** states that if the outputs of a differentiable function f are equal at the endpoints of an interval, then there must be an interior point c where $f'(c) = 0$. Figure 4.4.1 illustrates this theorem.

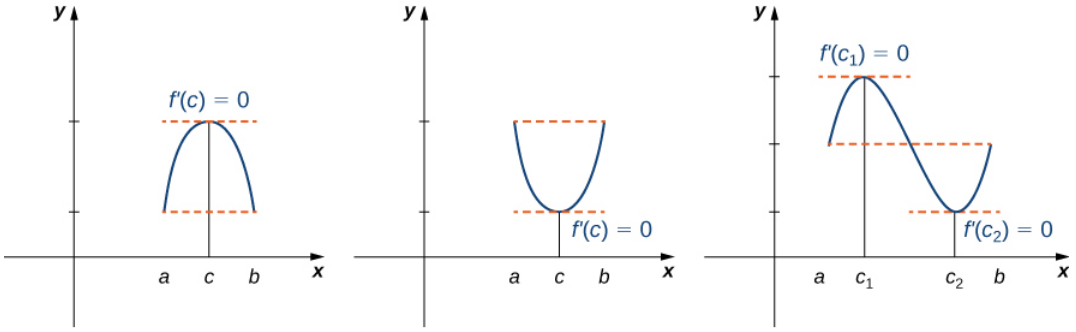

Figure 4.4.1: If a differentiable function f satisfies $f(a) = f(b)$, then its derivative must be zero at some point(s) between a and b.

Rolle's Theorem

Let f be a continuous function over the closed interval $[a, b]$ and differentiable over the open interval (a, b) such that $f(a) = f(b)$. There then exists at least one $c \in (a, b)$ such that $f'(c) = 0$.

Proof

Let $k = f(a) = f(b)$. We consider three cases:

1. $f(x) = k$ for all $x \in (a, b)$.
2. There exists $x \in (a, b)$ such that $f(x) > k$.
3. There exists $x \in (a, b)$ such that $f(x) < k$.

Case 1: If $f(x) = k$ for all $x \in (a, b)$, then $f'(x) = 0$ for all $x \in (a, b)$.

Case 2: Since f is a continuous function over the closed, bounded interval $[a, b]$, by the extreme value theorem, it has an absolute maximum. Also, since there is a point $x \in (a, b)$ such that $f(x) > k$, the absolute maximum is greater than k. Therefore, the absolute maximum does not occur at either endpoint. As a result, the absolute maximum must occur at an interior point $c \in (a, b)$. Because f has a maximum at an interior point c, and f is differentiable at c, by Fermat's theorem, $f'(c) = 0$.

Case 3: The case when there exists a point $x \in (a, b)$ such that $f(x) < k$ is analogous to case 2, with maximum replaced by minimum.

□

An important point about Rolle's theorem is that the differentiability of the function f is critical. If f is not differentiable, even at a single point, the result may not hold. For example, the function $f(x) = |x| - 1$ is continuous over $[-1, 1]$ and $f(-1) = 0 = f(1)$, but $f'(c) \neq 0$ for any $c \in (-1, 1)$ as shown in the following figure.

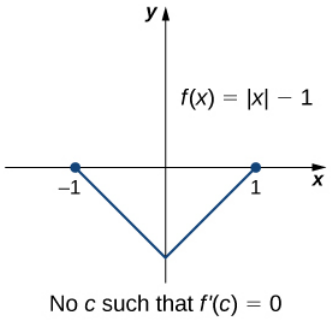

Figure 4.4.2: Since $f(x) = |x| - 1$ is not differentiable at $x = 0$, the conditions of Rolle's theorem are not satisfied. In fact, the conclusion does not hold here; there is no $c \in (-1, 1)$ such that $f'(c) = 0$.

Let's now consider functions that satisfy the conditions of Rolle's theorem and calculate explicitly the points c where $f'(c) = 0$.

Example 4.4.1: Using Rolle's Theorem

For each of the following functions, verify that the function satisfies the criteria stated in Rolle's theorem and find all values c in the given interval where $f'(c) = 0$.

a. $f(x) = x^2 + 2x$ over $[-2, 0]$
b. $f(x) = x^3 - 4x$ over $[-2, 2]$

Solution

a. Since f is a polynomial, it is continuous and differentiable everywhere. In addition, $f(-2) = 0 = f(0)$. Therefore, f satisfies the criteria of Rolle's theorem. We conclude that there exists at least one value $c \in (-2, 0)$ such that $f'(c) = 0$. Since $f'(x) = 2x + 2 = 2(x + 1)$, we see that $f'(c) = 2(c + 1) = 0$ implies $c = -1$ as shown in the following graph.

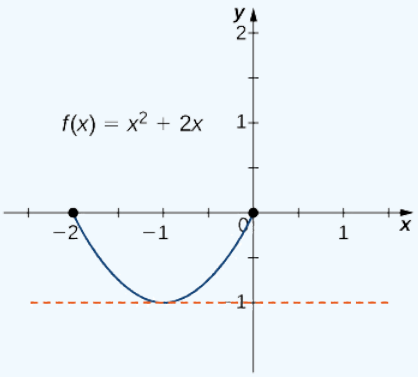

Figure 4.4.3: This function is continuous and differentiable over [−2,0], $f'(c) = 0$ when $c = -1$.

b. As in part a. f is a polynomial and therefore is continuous and differentiable everywhere. Also, $f(-2) = 0 = f(2)$. That said, f satisfies the criteria of Rolle's theorem. Differentiating, we find that $f'(x) = 3x^2 - 4$. Therefore, $f'(c) = 0$ when $x = \pm\frac{2}{\sqrt{3}}$. Both points are in the interval $[-2, 2]$, and, therefore, both points satisfy the conclusion of Rolle's theorem as shown in the following graph.

https://math.libretexts.org/@go/page/2503

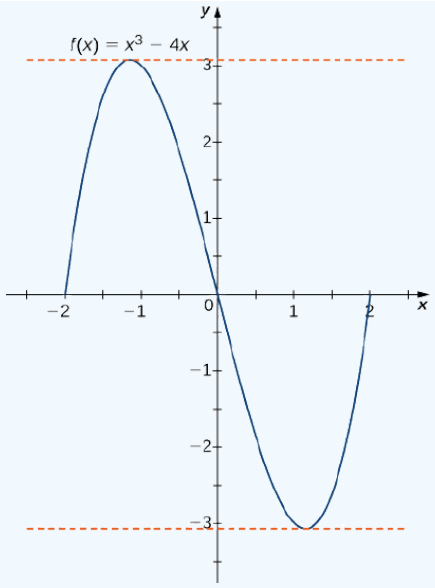

Figure 4.4.4: For this polynomial over $[-2, 2]$, $f'(c) = 0$ at $x = \pm 2/\sqrt{3}$.

Exercise 4.4.1

Verify that the function $f(x) = 2x^2 - 8x + 6$ defined over the interval $[1, 3]$ satisfies the conditions of Rolle's theorem. Find all points c guaranteed by Rolle's theorem.

Hint

Find all values c, where $f'(c) = 0$.

Answer

$c = 2$

The Mean Value Theorem and Its Meaning

Rolle's theorem is a special case of the Mean Value Theorem. In Rolle's theorem, we consider differentiable functions f that are zero at the endpoints. The Mean Value Theorem generalizes Rolle's theorem by considering functions that are not necessarily zero at the endpoints. Consequently, we can view the Mean Value Theorem as a slanted version of Rolle's theorem (Figure 4.4.5). The Mean Value Theorem states that if f is continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b), then there exists a point $c \in (a, b)$ such that the tangent line to the graph of f at c is parallel to the secant line connecting $(a, f(a))$ and $(b, f(b))$.

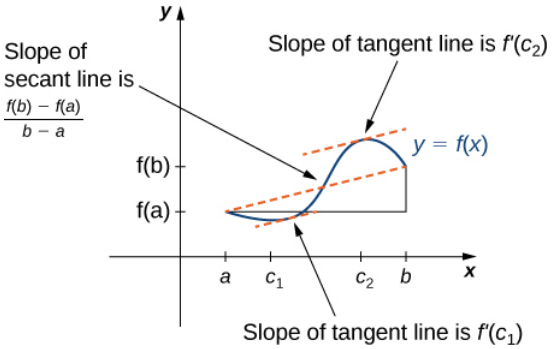

Figure 4.4.5: The Mean Value Theorem says that for a function that meets its conditions, at some point the tangent line has the same slope as the secant line between the ends. For this function, there are two values c_1 and c_2 such that the tangent line to f at c_1 and c_2 has the same slope as the secant line.

Mean Value Theorem

Let f be continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b). Then, there exists at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof

The proof follows from Rolle's theorem by introducing an appropriate function that satisfies the criteria of Rolle's theorem. Consider the line connecting $(a, f(a))$ and $(b, f(b))$. Since the slope of that line is

$$\frac{f(b) - f(a)}{b - a}$$

and the line passes through the point $(a, f(a))$, the equation of that line can be written as

$$y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a).$$

Let $g(x)$ denote the vertical difference between the point $(x, f(x))$ and the point (x, y) on that line. Therefore,

$$g(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right].$$

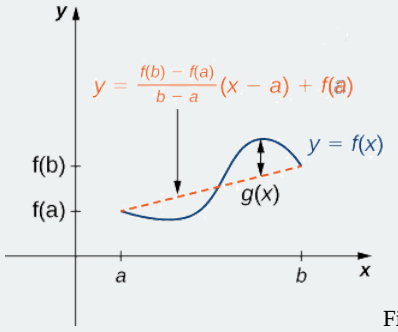

Figure 4.4.6: The value $g(x)$ is the vertical difference between the point $(x, f(x))$ and the point (x, y) on the secant line connecting $(a, f(a))$ and $(b, f(b))$.

Since the graph of f intersects the secant line when $x = a$ and $x = b$, we see that $g(a) = 0 = g(b)$. Since f is a differentiable function over (a, b), g is also a differentiable function over (a, b). Furthermore, since f is continuous over $[a, b]$, g is also continuous over $[a, b]$. Therefore, g satisfies the criteria of Rolle's theorem. Consequently, there exists a point $c \in (a, b)$ such that $g'(c) = 0$. Since

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a},$$

we see that

$$g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

Since $g'(c) = 0$, we conclude that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

□

In the next example, we show how the Mean Value Theorem can be applied to the function $f(x) = \sqrt{x}$ over the interval $[0, 9]$. The method is the same for other functions, although sometimes with more interesting consequences.

Example 4.4.2: Verifying that the Mean Value Theorem Applies

For $f(x) = \sqrt{x}$ over the interval $[0, 9]$, show that f satisfies the hypothesis of the Mean Value Theorem, and therefore there exists at least one value $c \in (0, 9)$ such that $f'(c)$ is equal to the slope of the line connecting $(0, f(0))$ and $(9, f(9))$. Find these values c guaranteed by the Mean Value Theorem.

Solution

We know that $f(x) = \sqrt{x}$ is continuous over $[0, 9]$ and differentiable over $(0, 9)$. Therefore, f satisfies the hypotheses of the Mean Value Theorem, and there must exist at least one value $c \in (0, 9)$ such that $f'(c)$ is equal to the slope of the line connecting $(0, f(0))$ and $(9, f(9))$ (Figure 4.4.7). To determine which value(s) of c are guaranteed, first calculate the derivative of f. The derivative $f'(x) = \frac{1}{(2\sqrt{x})}$. The slope of the line connecting $(0, f(0))$ and $(9, f(9))$ is given by

$$\frac{f(9) - f(0)}{9 - 0} = \frac{\sqrt{9} - \sqrt{0}}{9 - 0} = \frac{3}{9} = \frac{1}{3}.$$

We want to find c such that $f'(c) = \frac{1}{3}$. That is, we want to find c such that

$$\frac{1}{2\sqrt{c}} = \frac{1}{3}.$$

Solving this equation for c, we obtain $c = \frac{9}{4}$. At this point, the slope of the tangent line equals the slope of the line joining the endpoints.

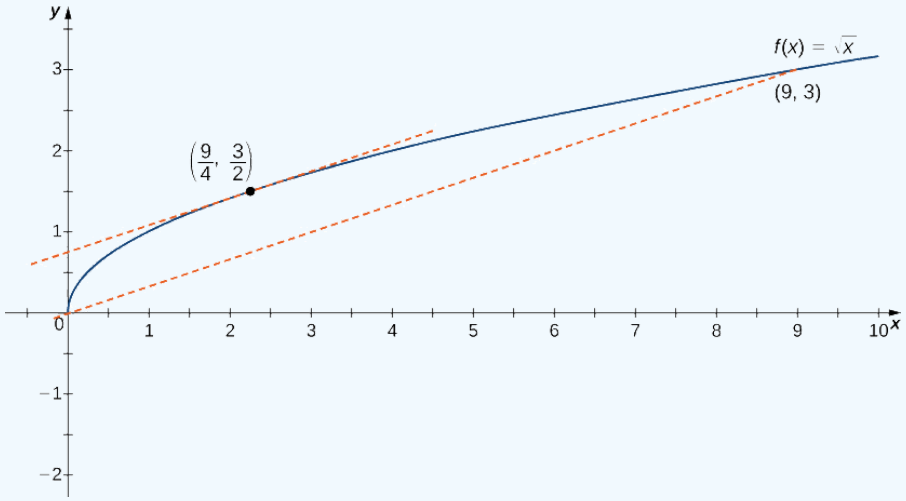

Figure 4.4.7: The slope of the tangent line at $c = 9/4$ is the same as the slope of the line segment connecting (0,0) and (9,3).

One application that helps illustrate the Mean Value Theorem involves velocity. For example, suppose we drive a car for 1 h down a straight road with an average velocity of 45 mph. Let $s(t)$ and $v(t)$ denote the position and velocity of the car, respectively, for $0 \le t \le 1$ h. Assuming that the position function $s(t)$ is differentiable, we can apply the Mean Value Theorem to conclude that, at some time $c \in (0, 1)$, the speed of the car was exactly

$$v(c) = s'(c) = \frac{s(1) - s(0)}{1 - 0} = 45 \text{ mph}.$$

Example 4.4.3: Mean Value Theorem and Velocity

If a rock is dropped from a height of 100 ft, its position t seconds after it is dropped until it hits the ground is given by the function $s(t) = -16t^2 + 100$.

a. Determine how long it takes before the rock hits the ground.
b. Find the average velocity v_{avg} of the rock for when the rock is released and the rock hits the ground.
c. Find the time t guaranteed by the Mean Value Theorem when the instantaneous velocity of the rock is v_{avg}.

Solution

a. When the rock hits the ground, its position is $s(t) = 0$. Solving the equation $-16t^2 + 100 = 0$ for t, we find that $t = \pm\frac{5}{2} sec$. Since we are only considering $t \geq 0$, the ball will hit the ground $\frac{5}{2}$ sec after it is dropped.

b. The average velocity is given by

$$v_{avg} = \frac{s(5/2) - s(0)}{5/2 - 0} = \frac{0 - 100}{5/2} = -40 \text{ ft/sec}.$$

c. The instantaneous velocity is given by the derivative of the position function. Therefore, we need to find a time t such that $v(t) = s'(t) = v_{avg} = -40$ ft/sec. Since $s(t)$ is continuous over the interval $[0, 5/2]$ and differentiable over the interval $(0, 5/2)$, by the Mean Value Theorem, there is guaranteed to be a point $c \in (0, 5/2)$ such that

$$s'(c) = \frac{s(5/2) - s(0)}{5/2 - 0} = -40.$$

Taking the derivative of the position function $s(t)$, we find that $s'(t) = -32t$. Therefore, the equation reduces to $s'(c) = -32c = -40$. Solving this equation for c, we have $c = \frac{5}{4}$. Therefore, $\frac{5}{4}$ sec after the rock is dropped, the instantaneous velocity equals the average velocity of the rock during its free fall: -40 ft/sec.

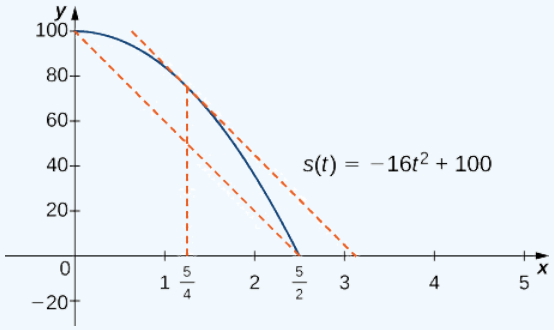

Figure 4.4.8: At time $t = 5/4$ sec, the velocity of the rock is equal to its average velocity from the time it is dropped until it hits the ground.

Exercise 4.4.2

Suppose a ball is dropped from a height of 200 ft. Its position at time t is $s(t) = -16t^2 + 200$. Find the time t when the instantaneous velocity of the ball equals its average velocity.

Hint

First, determine how long it takes for the ball to hit the ground. Then, find the average velocity of the ball from the time it is dropped until it hits the ground.

Answer

$\frac{5}{2\sqrt{2}}$ sec

Corollaries of the Mean Value Theorem

Let's now look at three corollaries of the Mean Value Theorem. These results have important consequences, which we use in upcoming sections.

At this point, we know the derivative of any constant function is zero. The Mean Value Theorem allows us to conclude that the converse is also true. In particular, if $f'(x) = 0$ for all x in some interval I, then $f(x)$ is constant over that interval. This result may seem intuitively obvious, but it has important implications that are not obvious, and we discuss them shortly.

Corollary 1: Functions with a Derivative of Zero

Let f be differentiable over an interval I. If $f'(x) = 0$ for all $x \in I$, then $f(x) =$ constant for all $x \in I$.

Proof

Since f is differentiable over I, f must be continuous over I. Suppose $f(x)$ is not constant for all x in I. Then there exist $a, b \in I$, where $a \neq b$ and $f(a) \neq f(b)$. Choose the notation so that $a < b$. Therefore,

$$\frac{f(b) - f(a)}{b - a} \neq 0.$$

Since f is a differentiable function, by the Mean Value Theorem, there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Therefore, there exists $c \in I$ such that $f'(c) \neq 0$, which contradicts the assumption that $f'(x) = 0$ for all $x \in I$.

□

From "Corollary 1: Functions with a Derivative of Zero," it follows that if two functions have the same derivative, they differ by, at most, a constant.

Corollary 2: Constant Difference Theorem

If f and g are differentiable over an interval I and $f'(x) = g'(x)$ for all $x \in I$, then $f(x) = g(x) + C$ for some constant C.

Proof

Let $h(x) = f(x) - g(x)$. Then, $h'(x) = f'(x) - g'(x) = 0$ for all $x \in I$. By Corollary 1, there is a constant C such that $h(x) = C$ for all $x \in I$. Therefore, $f(x) = g(x) + C$ for all $x \in I$.

□

The third corollary of the Mean Value Theorem discusses when a function is increasing and when it is decreasing. Recall that a function f is increasing over I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, whereas f is decreasing over I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$. Using the Mean Value Theorem, we can show that if the derivative of a function is positive, then the function is increasing; if the derivative is negative, then the function is decreasing (Figure 4.4.9). We make use of this fact in the next section, where we show how to use the derivative of a function to locate local maximum and minimum values of the function, and how to determine the shape of the graph.

This fact is important because it means that for a given function f, if there exists a function F such that $F'(x) = f(x)$; then, the only other functions that have a derivative equal to f are $F(x) + C$ for some constant C. We discuss this result in more detail later in the chapter.

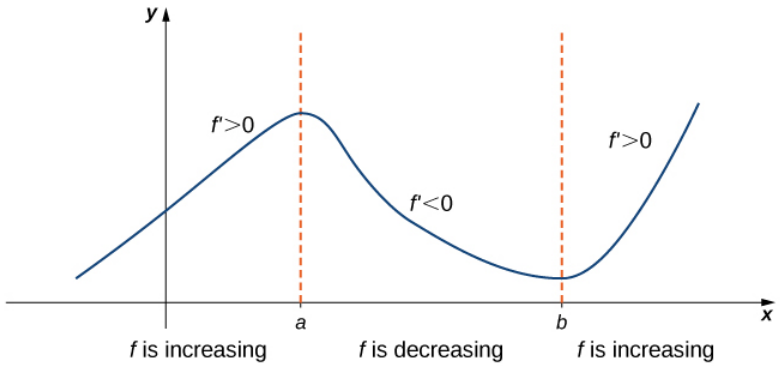

Figure 4.4.9: If a function has a positive derivative over some interval I, then the function increases over that interval I; if the derivative is negative over some interval I, then the function decreases over that interval I.

Corollary 3: Increasing and Decreasing Functions

Let f be continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b).

i. If $f'(x) > 0$ for all $x \in (a, b)$, then f is an increasing function over $[a, b]$.
ii. If $f'(x) < 0$ for all $x \in (a, b)$, then f is a decreasing function over $[a, b]$.

Proof

We will prove i.; the proof of ii. is similar. Suppose f is not an increasing function on I. Then there exist a and b in I such that $a < b$, but $f(a) \geq f(b)$. Since f is a differentiable function over I, by the Mean Value Theorem there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Since $f(a) \geq f(b)$, we know that $f(b) - f(a) \leq 0$. Also, $a < b$ tells us that $b - a > 0$. We conclude that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \leq 0.$$

However, $f'(x) > 0$ for all $x \in I$. This is a contradiction, and therefore f must be an increasing function over I.

□

Key Concepts

- If f is continuous over $[a, b]$ and differentiable over (a, b) and $f(a) = f(b)$, then there exists a point $c \in (a, b)$ such that $f'(c) = 0$. This is Rolle's theorem.
- If f is continuous over $[a, b]$ and differentiable over (a, b), then there exists a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

  This is the Mean Value Theorem.
- If $f'(x) = 0$ over an interval I, then f is constant over I.
- If two differentiable functions f and g satisfy $f'(x) = g'(x)$ over I, then $f(x) = g(x) + C$ for some constant C.
- If $f'(x) > 0$ over an interval I, then f is increasing over I. If $f'(x) < 0$ over I, then f is decreasing over I.

Glossary

mean value theorem

if f is continuous over $[a, b]$ and differentiable over (a, b), then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

rolle's theorem

if f is continuous over $[a, b]$ and differentiable over (a, b), and if $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$

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4.4E: EXERCISES FOR SECTION 4.4

1)Why do you need continuity to apply the Mean Value Theorem? Construct a counterexample.

2) Why do you need differentiability to apply the Mean Value Theorem? Find a counterexample.

Answer

One example is $f(x) = |x| + 3, -2 \le x \le 2$

3) When are Rolle's theorem and the Mean Value Theorem equivalent?

4) If you have a function with a discontinuity, is it still possible to have $f'(c)(b-a) = f(b) - f(a)$? Draw such an example or prove why not.

Answer

Yes, but the Mean Value Theorem still does not apply

In exercises 5 - 9, determine over what intervals (if any) the Mean Value Theorem applies. Justify your answer.

5) $y = \sin(\pi x)$

6) $y = \dfrac{1}{x^3}$

Answer

$(-\infty, 0), (0, \infty)$

7) $y = \sqrt{4 - x^2}$

8) $y = \sqrt{x^2 - 4}$

Answer

$(-\infty, -2), (2, \infty)$

9) $y = \ln(3x - 5)$

In exercises 10 - 13, graph the functions on a calculator and draw the secant line that connects the endpoints. Estimate the number of points c such that $f'(c)(b-a) = f(b) - f(a)$.

10) [T] $y = 3x^3 + 2x + 1$ over $[-1, 1]$

Answer

2 points

11) [T] $y = \tan\left(\frac{\pi}{4}x\right)$ over $\left[-\frac{3}{2}, \frac{3}{2}\right]$

12) [T] $y = x^2 \cos(\pi x)$ over $[-2, 2]$

Answer

5 points

13) [T] $y = x^6 - \frac{3}{4}x^5 - \frac{9}{8}x^4 + \frac{15}{16}x^3 + \frac{3}{32}x^2 + \frac{3}{16}x + \frac{1}{32}$ over $[-1, 1]$

In exercises 14 - 19, use the Mean Value Theorem and find all points $0 < c < 2$ such that $f(2) - f(0) = f'(c)(2-0)$.

14) $f(x) = x^3$

Answer

$c = \frac{2\sqrt{3}}{3}$

15) $f(x) = \sin(\pi x)$

16) $f(x) = \cos(2\pi x)$

Answer

$c = \frac{1}{2}, 1, \frac{3}{2}$

17) $f(x) = 1 + x + x^2$

18) $f(x) = (x-1)^{10}$

Answer

$c = 1$

19) $f(x) = (x-1)^9$

In exercises 20 - 23, show there is no c such that $f(1) - f(-1) = f'(c)(2)$. Explain why the Mean Value Theorem does not apply over the interval $[-1, 1]$.

20) $f(x) = \left|x - \frac{1}{2}\right|$

Answer

Not differentiable

21) $f(x) = \dfrac{1}{x^2}$

22) $f(x) = \sqrt{|x|}$

Answer

Not differentiable

23) $f(x) = \lfloor x \rfloor$ (Hint: This is called the floor function and it is defined so that $f(x)$ is the largest integer less than or equal to x.)

In exercises 24 - 34, determine whether the Mean Value Theorem applies for the functions over the given interval $[a, b]$. Justify your answer.

24) $y = e^x$ over $[0, 1]$

Answer

Yes

25) $y = \ln(2x + 3)$ over $[-\frac{3}{2}, 0]$

26) $f(x) = \tan(2\pi x)$ over $[0, 2]$

Answer

The Mean Value Theorem does not apply since the function is discontinuous at $x = \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}$.

27) $y = \sqrt{9 - x^2}$ over $[-3, 3]$

28) $y = \dfrac{1}{|x+1|}$ over $[0, 3]$

Answer

Yes

29) $y = x^3 + 2x + 1$ over $[0, 6]$

30) $y = \dfrac{x^2 + 3x + 2}{x}$ over $[-1, 1]$

Answer

The Mean Value Theorem does not apply; discontinuous at $x = 0$.

31) $y = \dfrac{x}{\sin(\pi x) + 1}$ over $[0, 1]$

32) $y = \ln(x + 1)$ over $[0, e - 1]$

Answer

Yes

33) $y = x \sin(\pi x)$ over $[0, 2]$

34) $y = 5 + |x|$ over $[-1, 1]$

Answer

The Mean Value Theorem does not apply; not differentiable at $x = 0$.

For exercises 35 - 37, consider the roots of each equation.

35) Show that the equation $y = x^3 + 3x^2 + 16$ has exactly one real root. What is it?

36) Find the conditions for exactly one root (double root) for the equation $y = x^2 + bx + c$

Answer

$b = \pm 2\sqrt{c}$

37) Find the conditions for $y = e^x - b$ to have one root. Is it possible to have more than one root?

In exercises 38 - 42, use a calculator to graph the function over the interval $[a, b]$ and graph the secant line from a to b. Use the calculator to estimate all values of c as guaranteed by the Mean Value Theorem. Then, find the exact value of c, if possible, or write the final equation and use a calculator to estimate to four digits.

38) [T] $y = \tan(\pi x)$ over $\left[-\frac{1}{4}, \frac{1}{4}\right]$

Answer

$c \approx \pm 0.1533$

$c = \pm \frac{1}{\pi} \cos^{-1}\left(\frac{\sqrt{\pi}}{2}\right)$

39) [T] $y = \dfrac{1}{\sqrt{x+1}}$ over $[0, 3]$

40) [T] $y = |x^2 + 2x - 4|$ over $[-4, 0]$

Answer

The Mean Value Theorem does not apply.

41) [T] $y = x + \dfrac{1}{x}$ over $\left[\frac{1}{2}, 4\right]$

42) [T] $y = \sqrt{x+1} + \dfrac{1}{x^2}$ over $[3, 8]$

Answer

$$\frac{1}{2\sqrt{c+1}} - \frac{2}{c^3} = \frac{521}{2880}$$

$c \approx 3.133, 5.867$

43) At 10:17 a.m., you pass a police car at 55 mph that is stopped on the freeway. You pass a second police car at 55 mph at 10:53 a.m., which is located 39 mi from the first police car. If the speed limit is 60 mph, can the police cite you for speeding?

44) Two cars drive from one stoplight to the next, leaving at the same time and arriving at the same time. Is there ever a time when they are going the same speed? Prove or disprove.

Answer

Yes

45) Show that $y = \sec^2 x$ and $y = \tan^2 x$ have the same derivative. What can you say about $y = \sec^2 x - \tan^2 x$?

46) Show that $y = \csc^2 x$ and $y = \cot^2 x$ have the same derivative. What can you say about $y = \csc^2 x - \cot^2 x$?

Answer

It is constant.

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4.5: Derivatives and the Shape of a Graph

Learning Objectives

- Explain how the sign of the first derivative affects the shape of a function's graph.
- State the first derivative test for critical points.
- Use concavity and inflection points to explain how the sign of the second derivative affects the shape of a function's graph.
- Explain the concavity test for a function over an open interval.
- Explain the relationship between a function and its first and second derivatives.
- State the second derivative test for local extrema.

Earlier in this chapter we stated that if a function f has a local extremum at a point c, then c must be a critical point of f. However, a function is not guaranteed to have a local extremum at a critical point. For example, $f(x) = x^3$ has a critical point at $x = 0$ since $f'(x) = 3x^2$ is zero at $x = 0$, but f does not have a local extremum at $x = 0$. Using the results from the previous section, we are now able to determine whether a critical point of a function actually corresponds to a local extreme value. In this section, we also see how the second derivative provides information about the shape of a graph by describing whether the graph of a function curves upward or curves downward.

The First Derivative Test

Corollary 3 of the Mean Value Theorem showed that if the derivative of a function is positive over an interval I then the function is increasing over I. On the other hand, if the derivative of the function is negative over an interval I, then the function is decreasing over I as shown in the following figure.

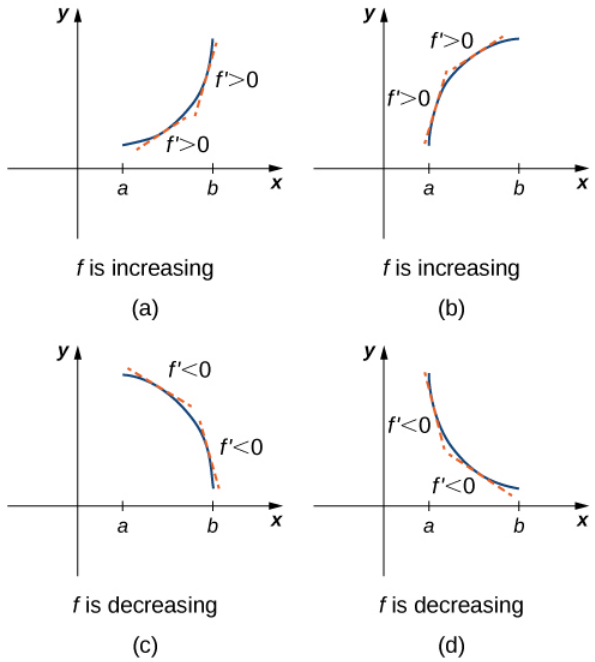

Figure 4.5.1: Both functions are increasing over the interval (a, b). At each point x, the derivative $f'(x) > 0$. Both functions are decreasing over the interval (a, b). At each point x, the derivative $f'(x) < 0$.

A continuous function f has a local maximum at point c if and only if f switches from increasing to decreasing at point c. Similarly, f has a local minimum at c if and only if f switches from decreasing to increasing at c. If f is a continuous function over an interval I containing c and differentiable over I, except possibly at c, the only way f can switch from increasing to decreasing (or vice versa) at point c is if f' changes sign as x increases through c. If f is differentiable at c, the only way that f' can change sign as x increases through c is if $f'(c) = 0$. Therefore, for a function f that is continuous over an interval I containing c and differentiable over I, except possibly at c, the only way f can switch from increasing to decreasing (or vice versa) is if $f'(c) = 0$ or $f'(c)$ is undefined. Consequently, to locate local extrema for a function f, we look for points c in the domain of f such that $f'(c) = 0$ or $f'(c)$ is undefined. Recall that such points are called critical points of f.

Note that f need not have a local extrema at a critical point. The critical points are candidates for local extrema only. In Figure 4.5.2, we show that if a continuous function f has a local extremum, it must occur at a critical point, but a function may not have a local extremum at a critical point. We show that if f has a local extremum at a critical point, then the sign of f' switches as x increases through that point.

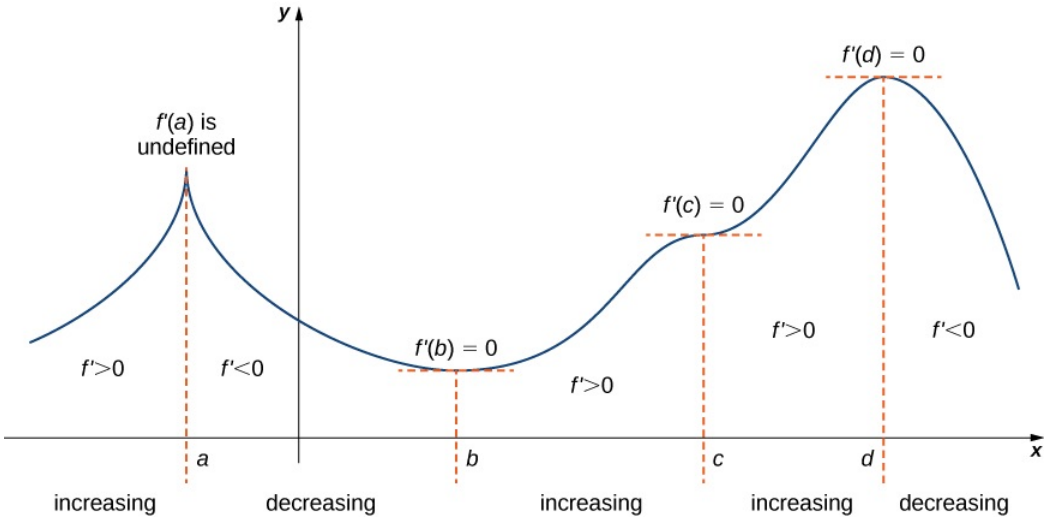

Figure 4.5.2: The function f has four critical points: a, b, c,and d. The function f has local maxima at a and d, and a local minimum at b. The function f does not have a local extremum at c. The sign of f' changes at all local extrema.

Using Figure 4.5.2, we summarize the main results regarding local extrema.

- If a continuous function f has a local extremum, it must occur at a critical point c.
- The function has a local extremum at the critical point c if and only if the derivative f' switches sign as x increases through c.
- Therefore, to test whether a function has a local extremum at a critical point c, we must determine the sign of $f'(x)$ to the left and right of c.

This result is known as the **first derivative test.**

First Derivative Test

Suppose that f is a continuous function over an interval I containing a critical point c. If f is differentiable over I, except possibly at point c, then $f(c)$ satisfies one of the following descriptions:

i. If f' changes sign from positive when $x < c$ to negative when $x > c$, then $f(c)$ is a local maximum of f.
ii. If f' changes sign from negative when $x < c$ to positive when $x > c$, then $f(c)$ is a local minimum of f.
iii. If f' has the same sign for $x < c$ and $x > c$, then $f(c)$ is neither a local maximum nor a local minimum of f

Now let's look at how to use this strategy to locate all local extrema for particular functions.

Example 4.5.1: Using the First Derivative Test to Find Local Extrema

Use the first derivative test to find the location of all local extrema for $f(x) = x^3 - 3x^2 - 9x - 1$. Use a graphing utility to confirm your results.

Solution

Step 1. The derivative is $f'(x) = 3x^2 - 6x - 9$. To find the critical points, we need to find where $f'(x) = 0$. Factoring the polynomial, we conclude that the critical points must satisfy

$$3(x^2 - 2x - 3) = 3(x - 3)(x + 1) = 0.$$

Therefore, the critical points are $x = 3, -1$. Now divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, -1), (-1, 3)$ and $(3, \infty)$.

Step 2. Since f' is a continuous function, to determine the sign of $f'(x)$ over each subinterval, it suffices to choose a point over each of the intervals $(-\infty, -1), (-1, 3)$ and $(3, \infty)$ and determine the sign of f' at each of these points. For example, let's choose

$x = -2$, $x = 0$, and $x = 4$ as test points.

Table: 4.5.1: First Derivative Test for $f(x) = x^3 - 3x^2 - 9x - 1$.

Interval	Test Point	Sign of $f'(x) = 3(x-3)(x+1)$ at Test Point	Conclusion
$(-\infty, -1)$	$x = -2$	(+)(−)(−)=+	f is increasing.
$(-1, 3)$	$x = 0$	(+)(−)(+)=-	f is decreasing.
$(3, \infty)$	$x = 4$	(+)(+)(+)=+	f is increasing.

Step 3. Since f' switches sign from positive to negative as x increases through -1, f has a local maximum at $x = -1$. Since f' switches sign from negative to positive as x increases through 3, f has a local minimum at $x = 3$. These analytical results agree with the following graph.

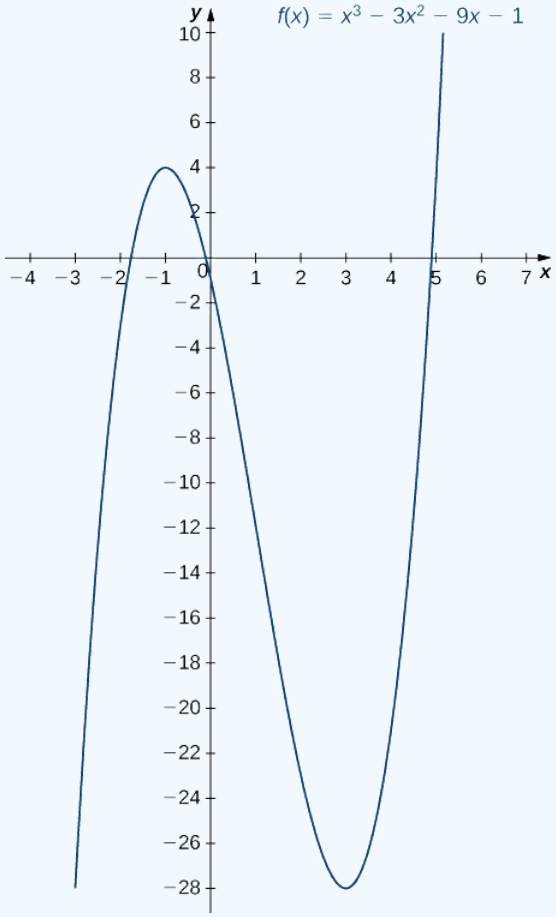

Figure 4.5.3: The function f has a maximum at $x = -1$ and a minimum at $x = 3$

Exercise 4.5.1

Use the first derivative test to locate all local extrema for $f(x) = -x^3 + \frac{3}{2}x^2 + 18x$.

Hint

Find all critical points of f and determine the signs of $f'(x)$ over particular intervals determined by the critical points.

Answer

f has a local minimum at -2 and a local maximum at 3.

Example 4.5.2: Using the First Derivative Test

Use the first derivative test to find the location of all local extrema for $f(x) = 5x^{1/3} - x^{5/3}$. Use a graphing utility to confirm your results.

Solution

Step 1. The derivative is

$$f'(x) = \frac{5}{3}x^{-2/3} - \frac{5}{3}x^{2/3} = \frac{5}{3x^{2/3}} - \frac{5x^{2/3}}{3} = \frac{5 - 5x^{4/3}}{3x^{2/3}} = \frac{5(1 - x^{4/3})}{3x^{2/3}}.$$

The derivative $f'(x) = 0$ when $1 - x^{4/3} = 0$. Therefore, $f'(x) = 0$ at $x = \pm 1$. The derivative $f'(x)$ is undefined at $x = 0$. Therefore, we have three critical points: $x = 0$, $x = 1$, and $x = -1$. Consequently, divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, \infty)$.

Step 2: Since f' is continuous over each subinterval, it suffices to choose a test point x in each of the intervals from step 1 and determine the sign of f' at each of these points. The points $x = -2$, $x = -\frac{1}{2}$, $x = \frac{1}{2}$, and $x = 2$ are test points for these intervals.

Table: 4.5.2: First Derivative Test for $f(x) = 5x^{1/3} - x^{5/3}$.

Interval	Test Point	Sign of $f'(x) = \frac{5(1-x^{4/3})}{3x^{2/3}}$ at Test Point	Conclusion
$(-\infty, -1)$	$x = -2$	$\frac{(+)(-)}{+} = -$	f is decreasing.
$(-1, 0)$	$x = -\frac{1}{2}$	$\frac{(+)(+)}{+} = +$	f is increasing.
$(0, 1)$	$x = \frac{1}{2}$	$\frac{(+)(+)}{+} = +$	f is increasing.
$(1, \infty)$	$x = 2$	$\frac{(+)(-)}{+} = -$	f is decreasing.

Step 3: Since f is decreasing over the interval $(-\infty, -1)$ and increasing over the interval $(-1, 0)$, f has a local minimum at $x = -1$. Since f is increasing over the interval $(-1, 0)$ and the interval $(0, 1)$, f does not have a local extremum at $x = 0$. Since f is increasing over the interval $(0, 1)$ and decreasing over the interval $(1, \infty)$, f has a local maximum at $x = 1$. The analytical results agree with the following graph.

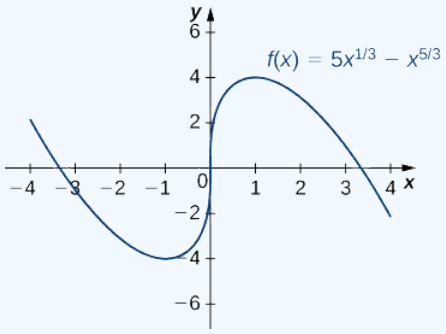

Figure 4.5.4: The function f has a local minimum at $x = -1$ and a local maximum at $x = 1$

Exercise 4.5.2

Use the first derivative test to find all local extrema for $f(x) = \frac{3}{x - 1}$.

Hint

The only critical point of f is $x = 1$.

Answer

f has no local extrema because f' does not change sign at $x = 1$.

Concavity and Points of Inflection

We now know how to determine where a function is increasing or decreasing. However, there is another issue to consider regarding the shape of the graph of a function. If the graph curves, does it curve upward or curve downward? This notion is called the **concavity** of the function.

Figure $4.5.5a$ shows a function f with a graph that curves upward. As x increases, the slope of the tangent line increases. Thus, since the derivative increases as x increases, f' is an increasing function. We say this function f is concave up. Figure $4.5.5b$ shows a function f that curves downward. As x increases, the slope of the tangent line decreases. Since the derivative decreases as x increases, f' is a decreasing function. We say this function f is concave down.

Definition: concavity test

Let f be a function that is differentiable over an open interval I. If f' is increasing over I, we say f is **concave up** over I. If f' is decreasing over I, we say f is **concave down** over I.

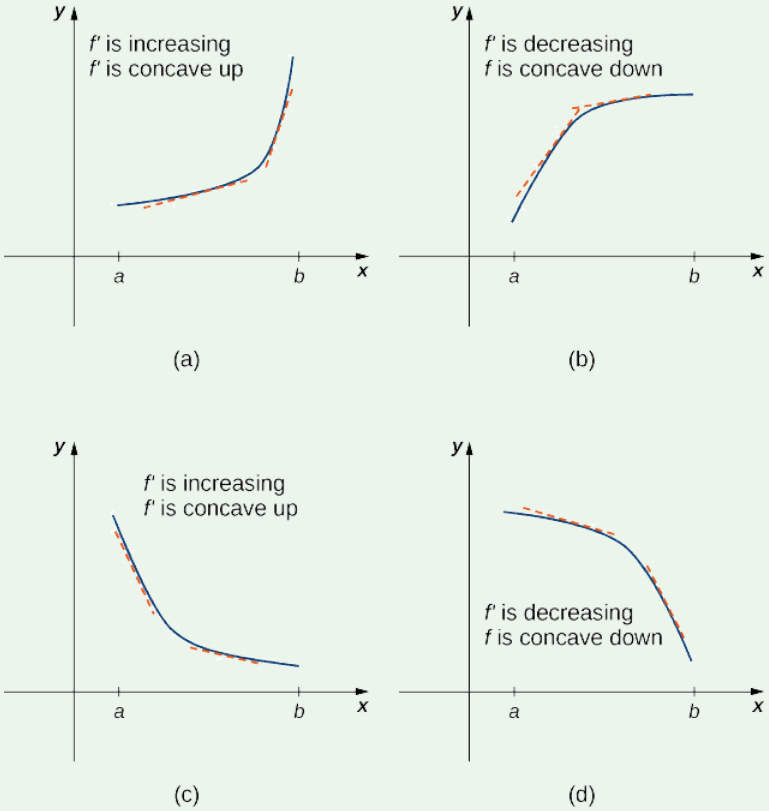

Figure $4.5.5$: (a), (c) Since f' is increasing over the interval (a,b), we say f is concave up over $(a,b).(b),(d)$ Since f' is decreasing over the interval (a,b), we say f is concave down over (a,b).

In general, without having the graph of a function f, how can we determine its concavity? By definition, a function f is concave up if f' is increasing. From Corollary 3, we know that if f' is a differentiable function, then f' is increasing if its derivative $f''(x) > 0$. Therefore, a function f that is twice differentiable is concave up when $f''(x) > 0$. Similarly, a function f is concave down if f' is decreasing. We know that a differentiable function f' is decreasing if its derivative $f''(x) < 0$. Therefore, a twice-differentiable function f is concave down when $f''(x) < 0$. Applying this logic is known as the *concavity test*.

Test for Concavity

Let f be a function that is twice differentiable over an interval I.

i. If $f''(x) > 0$ for all $x \in I$, then f is concave up over I
ii. If $f''(x) < 0$ for all $x \in I$, then f is concave down over I.

We conclude that we can determine the concavity of a function f by looking at the second derivative of f. In addition, we observe that a function f can switch concavity (Figure 4.5.6). However, a continuous function can switch concavity only at a point x if $f''(x) = 0$ or $f''(x)$ is undefined. Consequently, to determine the intervals where a function f is concave up and concave down, we look for those values of x where $f''(x) = 0$ or $f''(x)$ is undefined. When we have determined these points, we divide the domain of f into smaller intervals and determine the sign of f'' over each of these smaller intervals. If f'' changes sign as we pass through a point x, then f changes concavity. It is important to remember that a function f may not change concavity at a point x even if $f''(x) = 0$ or $f''(x)$ is undefined. If, however, f does change concavity at a point a and f is continuous at a, we say the point $(a, f(a))$ is an **inflection point** of f.

Definition: inflection point

If f is continuous at a and f changes concavity at a, the point $(a,\ f(a))$ is an *inflection point* of f.

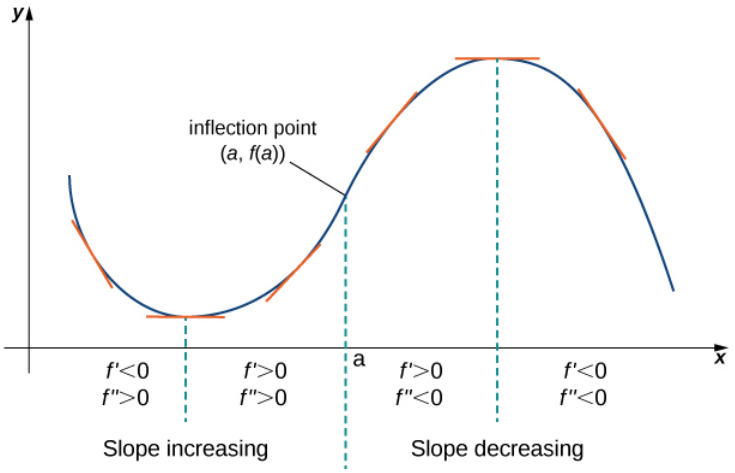

Figure 4.5.6: Since $f''(x) > 0$ for $x < a$, the function f is concave up over the interval $(-\infty, a)$. Since $f''(x) < 0$ for $x > a$, the function f is concave down over the interval (a, ∞). The point $(a, f(a))$ is an inflection point of f.

Example 4.5.3: Testing for Concavity

For the function $f(x) = x^3 - 6x^2 + 9x + 30$, determine all intervals where f is concave up and all intervals where f is concave down. List all inflection points for f. Use a graphing utility to confirm your results.

Solution

To determine concavity, we need to find the second derivative $f''(x)$. The first derivative is $f'(x) = 3x^2 - 12x + 9$, so the second derivative is $f''(x) = 6x - 12$. If the function changes concavity, it occurs either when $f''(x) = 0$ or $f''(x)$ is undefined. Since f'' is defined for all real numbers x, we need only find where $f''(x) = 0$. Solving the equation $6x - 12 = 0$, we see that $x = 2$ is the only place where f could change concavity. We now test points over the intervals $(-\infty, 2)$ and $(2, \infty)$ to determine the concavity of f. The points $x = 0$ and $x = 3$ are test points for these intervals.

Table: 4.5.3: Test for Concavity for $f(x) = x^3 - 6x^2 + 9x + 30$.

Interval	Test Point	Sign of $f''(x) = 6x - 12$ at Test Point	Conclusion
$(-\infty, 2)$	$x = 0$	$-$	f is concave down
$(2, \infty)$	$x = 3$	$+$	f is concave up

We conclude that f is concave down over the interval $(-\infty, 2)$ and concave up over the interval $(2, \infty)$. Since f changes concavity at $x = 2$, the point $(2, f(2)) = (2, 32)$ is an inflection point. Figure 4.5.7 confirms the analytical results.

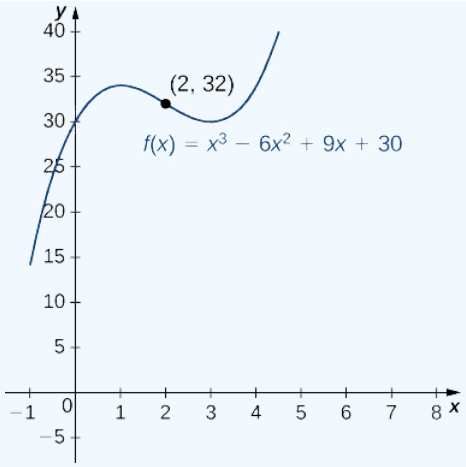

Figure 4.5.7: The given function has a point of inflection at $(2, 32)$ where the graph changes concavity.

? Exercise 4.5.3

For $f(x) = -x^3 + \frac{3}{2}x^2 + 18x$, find all intervals where f is concave up and all intervals where f is concave down.

Hint

Find where $f''(x) = 0$

Answer

f is concave up over the interval $(-\infty, \frac{1}{2})$ and concave down over the interval $(\frac{1}{2}, \infty)$

We now summarize, in Table 4.5.4, the information that the first and second derivatives of a function f provide about the graph of f, and illustrate this information in Figure 4.5.8.

Table: 4.5.4: What Derivatives Tell Us about Graphs

Sign of f'	Sign of f''	Is f increasing or decreasing?	Concavity
Positive	Positive	Increasing	Concave up
Positive	Negative	Increasing	Concave down
Negative	Positive	Decreasing	Concave up
Negative	Negative	Decreasing	Concave down

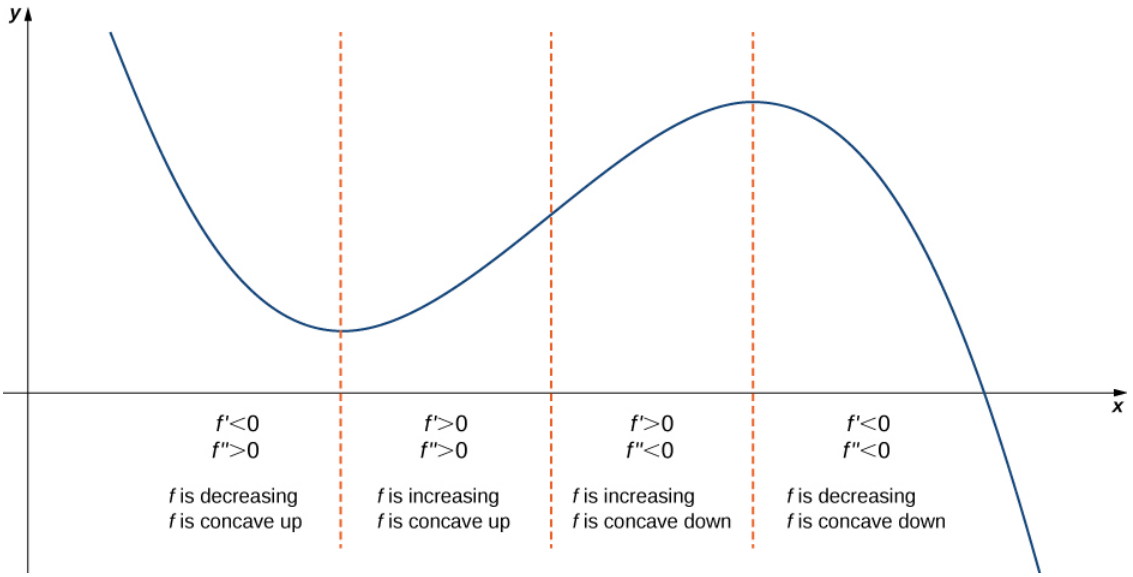

Figure 4.5.8:Consider a twice-differentiable function f over an open interval I. If $f'(x) > 0$ for all $x \in I$, the function is increasing over I. If $f'(x) < 0$ for all $x \in I$, the function is decreasing over I. If $f''(x) > 0$ for all $x \in I$, the function is concave up. If $f''(x) < 0$ for all $x \in I$, the function is concave down on I.

The Second Derivative Test

The first derivative test provides an analytical tool for finding local extrema, but the second derivative can also be used to locate extreme values. Using the second derivative can sometimes be a simpler method than using the first derivative.

We know that if a continuous function has a local extremum, it must occur at a critical point. However, a function need not have a local extremum at a critical point. Here we examine how the **second derivative test** can be used to determine whether a function has a local extremum at a critical point. Let f be a twice-differentiable function such that $f'(a) = 0$ and f'' is continuous over an open interval I containing a. Suppose $f''(a) < 0$. Since f'' is continuous over I, $f''(x) < 0$ for all $x \in I$ (Figure 4.5.9). Then, by Corollary 3, f' is a decreasing function over I. Since $f'(a) = 0$, we conclude that for all $x \in I$, $f'(x) > 0$ if $x < a$ and $f'(x) < 0$ if $x > a$. Therefore, by the first derivative test, f has a local maximum at $x = a$.

On the other hand, suppose there exists a point b such that $f'(b) = 0$ but $f''(b) > 0$. Since f'' is continuous over an open interval I containing b, then $f''(x) > 0$ for all $x \in I$ (Figure 4.5.9). Then, by Corollary 3, f' is an increasing function over I. Since $f'(b) = 0$, we conclude that for all $x \in I$, $f'(x) < 0$ if $x < b$ and $f'(x) > 0$ if $x > b$. Therefore, by the first derivative test, f has a local minimum at $x = b$.

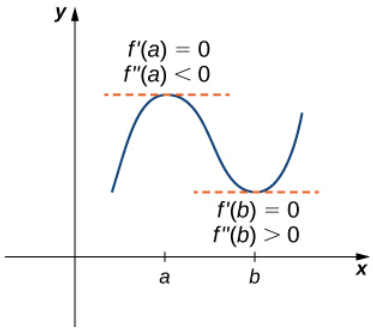

Figure 4.5.9: Consider a twice-differentiable function f such that f'' is continuous. Since $f'(a) = 0$ and $f''(a) < 0$, there is an interval I containing a such that for all x in I, f is increasing if $x < a$ and f is decreasing if $x > a$. As a result, f has a local maximum at $x = a$. Since $f'(b) = 0$ and $f''(b) > 0$, there is an interval I containing b such that for all x in I, f is decreasing if $x < b$ and f is increasing if $x > b$. As a result, f has a local minimum at $x = b$.

Second Derivative Test

Suppose $f'(c) = 0$ and f'' is continuous over an interval containing c.

i. If $f''(c) > 0$, then f has a local minimum at c.

ii. If $f''(c) < 0$, then f has a local maximum at c.
iii. If $f''(c) = 0$, then the test is inconclusive.

Note that for case iii. when $f''(c) = 0$, then f may have a local maximum, local minimum, or neither at c. For example, the functions $f(x) = x^3$, $f(x) = x^4$, and $f(x) = -x^4$ all have critical points at $x = 0$. In each case, the second derivative is zero at $x = 0$. However, the function $f(x) = x^4$ has a local minimum at $x = 0$ whereas the function $f(x) = -x^4$ has a local maximum at $x = 0$, and the function $f(x) = x^3$ does not have a local extremum at $x = 0$.

Let's now look at how to use the second derivative test to determine whether f has a local maximum or local minimum at a critical point c where $f'(c) = 0$.

Example 4.5.4: Using the Second Derivative Test

Use the second derivative to find the location of all local extrema for $f(x) = x^5 - 5x^3$.

Solution

To apply the second derivative test, we first need to find critical points c where $f'(c) = 0$. The derivative is $f'(x) = 5x^4 - 15x^2$. Therefore, $f'(x) = 5x^4 - 15x^2 = 5x^2(x^2 - 3) = 0$ when $x = 0, \pm\sqrt{3}$.

To determine whether f has a local extremum at any of these points, we need to evaluate the sign of f'' at these points. The second derivative is

$$f''(x) = 20x^3 - 30x = 10x(2x^2 - 3).$$

In the following table, we evaluate the second derivative at each of the critical points and use the second derivative test to determine whether f has a local maximum or local minimum at any of these points.

Table: 4.5.5: Second Derivative Test for $f(x) = x^5 - 5x^3$.

x	$f''(x)$	**Conclusion**
$-\sqrt{3}$	$-30\sqrt{3}$	Local maximum
0	0	Second derivative test is inconclusive
$\sqrt{3}$	$30\sqrt{3}$	Local minimum

By the second derivative test, we conclude that f has a local maximum at $x = -\sqrt{3}$ and f has a local minimum at $x = \sqrt{3}$. The second derivative test is inconclusive at $x = 0$. To determine whether f has a local extrema at $x = 0$, we apply the first derivative test. To evaluate the sign of $f'(x) = 5x^2(x^2 - 3)$ for $x \in (-\sqrt{3}, 0)$ and $x \in (0, \sqrt{3})$, let $x = -1$ and $x = 1$ be the two test points. Since $f'(-1) < 0$ and $f'(1) < 0$, we conclude that f is decreasing on both intervals and, therefore, f does not have a local extrema at $x = 0$ as shown in the following graph.

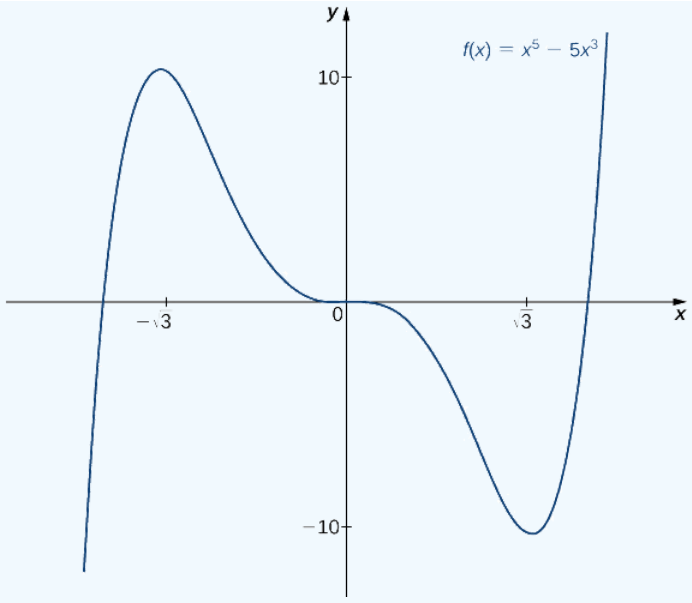

Figure 4.5.10:The function f has a local maximum at $x=-\sqrt{3}$ and a local minimum at $x=\sqrt{3}$

? Exercise 4.5.4

Consider the function $f(x)=x^3-(\frac{3}{2})x^2-18x$. The points $c=3,\ -2$ satisfy $f'(c)=0$. Use the second derivative test to determine whether f has a local maximum or local minimum at those points.

Hint

$f''(x)=6x-3$

Answer

f has a local maximum at -2 and a local minimum at 3.

We have now developed the tools we need to determine where a function is increasing and decreasing, as well as acquired an understanding of the basic shape of the graph. In the next section we discuss what happens to a function as $x\to\pm\infty$. At that point, we have enough tools to provide accurate graphs of a large variety of functions.

Key Concepts

- If c is a critical point of f and $f'(x)>0$ for $x<c$ and $f'(x)<0$ for $x>c$, then f has a local maximum at c.
- If c is a critical point of f and $f'(x)<0$ for $x<c$ and $f'(x)>0$ for $x>c$, then f has a local minimum at c.
- If $f''(x)>0$ over an interval I, then f is concave up over I.
- If $f''(x)<0$ over an interval I, then f is concave down over I.
- If $f'(c)=0$ and $f''(c)>0$, then f has a local minimum at c.
- If $f'(c)=0$ and $f''(c)<0$, then f has a local maximum at c.
- If $f'(c)=0$ and $f''(c)=0$, then evaluate $f'(x)$ at a test point x to the left of c and a test point x to the right of c, to determine whether f has a local extremum at c.

Glossary

concave down

if f is differentiable over an interval I and f' is decreasing over I, then f is concave down over I

concave up

if f is differentiable over an interval I and f' is increasing over I, then f is concave up over I

concavity

the upward or downward curve of the graph of a function

concavity test

suppose f is twice differentiable over an interval I; if $f'' > 0$ over I, then f is concave up over I; if $f'' <$ over I, then f is concave down over I

first derivative test

let f be a continuous function over an interval I containing a critical point c such that f is differentiable over I except possibly at c; if f' changes sign from positive to negative as x increases through c, then f has a local maximum at c; if f' changes sign from negative to positive as x increases through c, then f has a local minimum at c; if f' does not change sign as x increases through c, then f does not have a local extremum at c

inflection point

if f is continuous at c and f changes concavity at c, the point $(c, f(c))$ is an inflection point of f

second derivative test

suppose $f'(c) = 0$ and f'' is continuous over an interval containing c; if $f''(c) > 0$, then f has a local minimum at c; if $f''(c) < 0$, then f has a local maximum at c; if $f''(c) = 0$, then the test is inconclusive

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4.5E: EXERCISES FOR SECTION 4.5

1) If c is a critical point of $f(x)$, when is there no local maximum or minimum at c? Explain.

2) For the function $y = x^3$, is $x = 0$ both an inflection point and a local maximum/minimum?

Answer

It is not a local maximum/minimum because f' does not change sign

3) For the function $y = x^3$, is $x = 0$ an inflection point?

4) Is it possible for a point c to be both an inflection point and a local extremum of a twice differentiable function?

Answer

No

5) Why do you need continuity for the first derivative test? Come up with an example.

6) Explain whether a concave-down function has to cross $y = 0$ for some value of x.

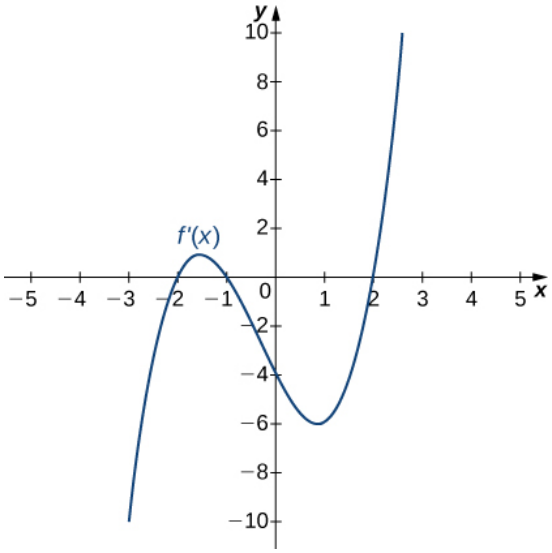
Answer

False; for example, $y = \sqrt{x}$.

7) Explain whether a polynomial of degree 2 can have an inflection point.

In exercises 8 - 12, analyze the graphs of f', then list all intervals where f is increasing or decreasing.

8)



Answer

Increasing for $-2 < x < -1$ and $x > 2$;
Decreasing for $x < -2$ and $-1 < x < 2$

9)

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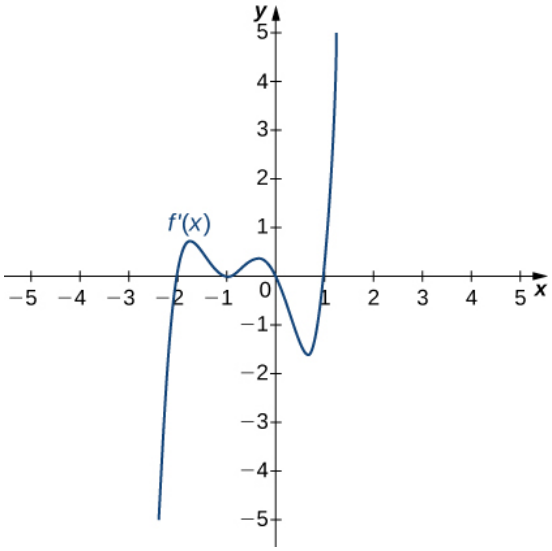
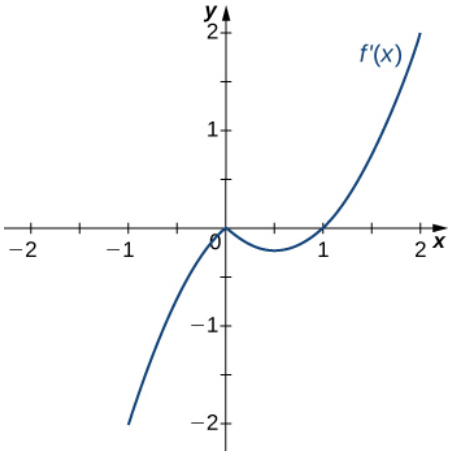


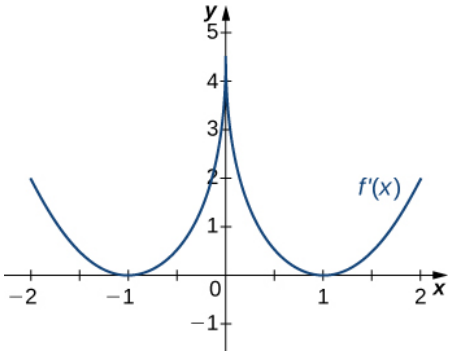
10)



Answer

Decreasing for $x < 1$,
Increasing for $x > 1$

11)



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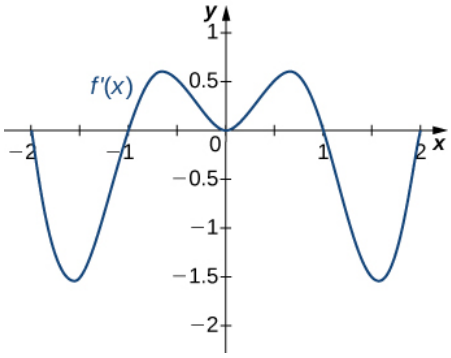


Answer

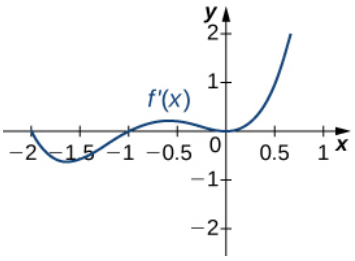
Decreasing for $-2 < x < -1$ and $1 < x < 2$;
Increasing for $-1 < x < 1$ and $x < -2$ and $x > 2$

In exercises 13 - 17, analyze the graphs of f', then list all intervals where

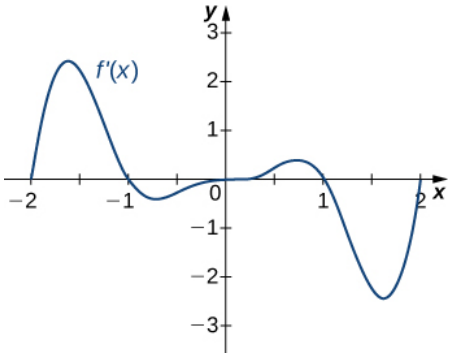
a. f is increasing and decreasing and

b. the minima and maxima are located.

13)



14)



Answer

a. Increasing over $-2 < x < -1,\ 0 < x < 1, x > 2$, Decreasing over $x < -2,\ -1 < x < 0,\ 1 < x < 2$;
b. Maxima at $x = -1$ and $x = 1$, Minima at $x = -2$ and $x = 0$ and $x = 2$

15)

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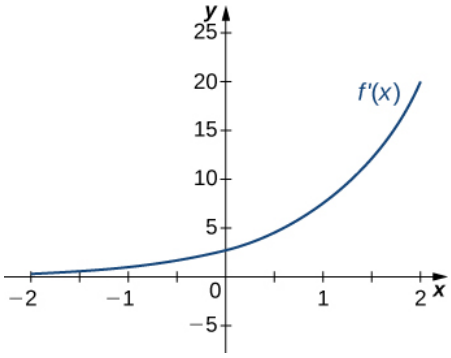
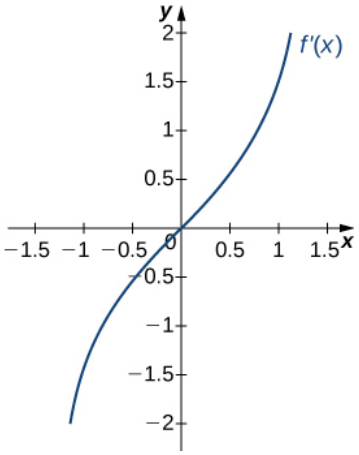


16)

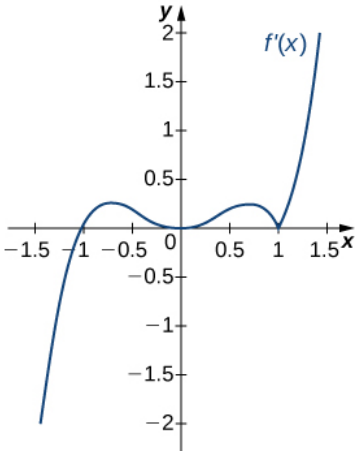


Answer

a. Increasing over $x > 0$, Decreasing over $x < 0$;

b. Minimum at $x = 0$

17)



In exercises 18 - 22, analyze the graphs of f', then list all inflection points and intervals f that are concave up and concave down.

18)

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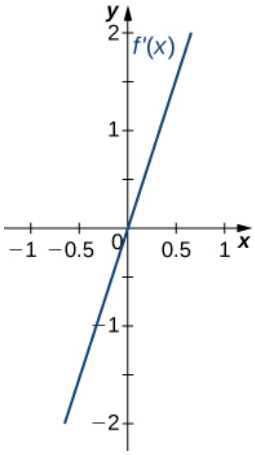
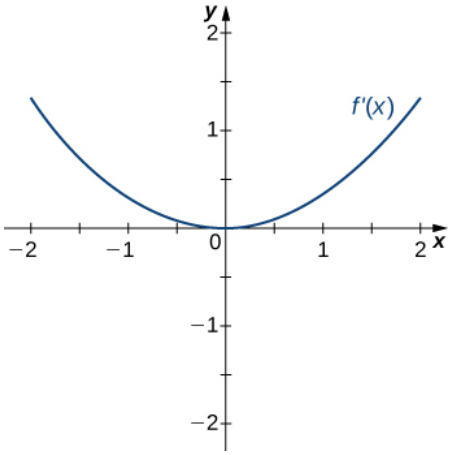


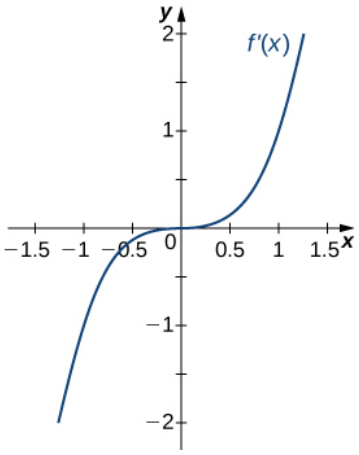
Answer

Concave up for all x,
No inflection points

19)



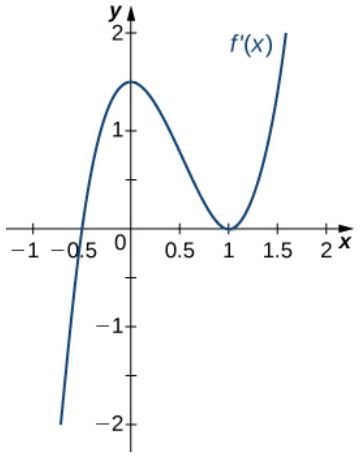
20)



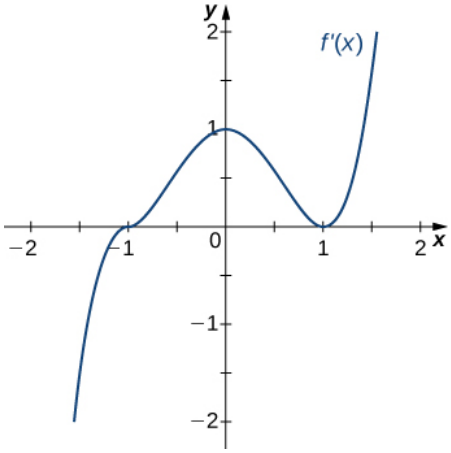
Answer

Concave up for all x,
No inflection points

21)



22)



Answer

Concave up for $x < 0$ and $x > 1$,
Concave down for $0 < x < 1$,
Inflection points at $x = 0$ and $x = 1$

For exercises 23 - 27, draw a graph that satisfies the given specifications for the domain $x = [-3, 3]$. The function does not have to be continuous or differentiable.

23) $f(x) > 0,\ f'(x) > 0$ over $x > 1,\ -3 < x < 0,\ f'(x) = 0$ over $0 < x < 1$

24) $f'(x) > 0$ over $x > 2,\ -3 < x < -1,\ f'(x) < 0$ over $-1 < x < 2,\ f''(x) < 0$ for all x

Answer

Answers will vary

25) $f''(x) < 0$ over $-1 < x < 1,\ f''(x) > 0,\ -3 < x < -1,\ 1 < x < 3$, local maximum at $x = 0$, local minima at $x = \pm 2$

26) There is a local maximum at $x = 2$, local minimum at $x = 1$, and the graph is neither concave up nor concave down.

Answer

Answers will vary

27) There are local maxima at $x = \pm 1$, the function is concave up for all x, and the function remains positive for all x.

For the following exercises, determine

a. intervals where f is increasing or decreasing and

b. local minima and maxima of f.

28) $f(x) = \sin x + \sin^3 x$ over $-\pi < x < \pi$

Answer

a. Increasing over $-\frac{\pi}{2} < x < \frac{\pi}{2}$, decreasing over $x < -\frac{\pi}{2}$, $x > \frac{\pi}{2}$

b. Local maximum at $x = \frac{\pi}{2}$; local minimum at $x = -\frac{\pi}{2}$

29) $f(x) = x^2 + \cos x$

For exercise 30, determine

a. intervals where f is concave up or concave down, and

b. the inflection points of f.

30) $f(x) = x^3 - 4x^2 + x + 2$

Answer

a. Concave up for $x > \frac{4}{3}$, concave down for $x < \frac{4}{3}$

b. Inflection point at $x = \frac{4}{3}$

For exercises 31 - 37, determine

a. intervals where f is increasing or decreasing,

b. local minima and maxima of f,

c. intervals where f is concave up and concave down, and

d. the inflection points of f.

31) $f(x) = x^2 - 6x$

32) $f(x) = x^3 - 6x^2$

Answer

a. Increasing over $x < 0$ and $x > 4$, decreasing over $0 < x < 4$

b. Maximum at $x = 0$, minimum at $x = 4$

c. Concave up for $x > 2$, concave down for $x < 2$

d. Inflection point at $x = 2$

33) $f(x) = x^4 - 6x^3$

34) $f(x) = x^{11} - 6x^{10}$

Answer

a. Increasing over $x < 0$ and $x > \frac{60}{11}$, decreasing over $0 < x < \frac{60}{11}$

b. Maximum at $x = 0$, minimum at $x = \frac{60}{11}$

c. Concave down for $x < \frac{54}{11}$, concave up for $x > \frac{54}{11}$

d. Inflection point at $x = \frac{54}{11}$

35) $f(x) = x + x^2 - x^3$

36) $f(x) = x^2 + x + 1$

Answer

a. Increasing over $x > -\frac{1}{2}$, decreasing over $x < -\frac{1}{2}$

b. Minimum at $x = -\frac{1}{2}$

c. Concave up for all x

d. No inflection points

37) $f(x) = x^3 + x^4$

For exercises 38 - 47, determine

a. intervals where f is increasing or decreasing,

b. local minima and maxima of f,

c. intervals where f is concave up and concave down, and

d. the inflection points of f. Sketch the curve, then use a calculator to compare your answer. If you cannot determine the exact answer analytically, use a calculator.

38) [T] $f(x) = \sin(\pi x) - \cos(\pi x)$ over $x = [-1, 1]$

Answer

a. Increases over $-\frac{1}{4} < x < \frac{3}{4}$, decreases over $x > \frac{3}{4}$ and $x < -\frac{1}{4}$
b. Minimum at $x = -\frac{1}{4}$, maximum at $x = \frac{3}{4}$
c. Concave up for $-\frac{3}{4} < x < \frac{1}{4}$, concave down for $x < -\frac{3}{4}$ and $x > \frac{1}{4}$
d. Inflection points at $x = -\frac{3}{4},\ x = \frac{1}{4}$

39) [T] $f(x) = x + \sin(2x)$ over $x = [-\frac{\pi}{2}, \frac{\pi}{2}]$

40) [T] $f(x) = \sin x + \tan x$ over $(-\frac{\pi}{2}, \frac{\pi}{2})$

Answer

a. Increasing for all x
b. No local minimum or maximum
c. Concave up for $x > 0$, concave down for $x < 0$
d. Inflection point at $x = 0$

41) [T] $f(x) = (x-2)^2(x-4)^2$

42) [T] $f(x) = \dfrac{1}{1-x}, \quad x \neq 1$

Answer

a. Increasing for all x where defined
b. No local minima or maxima
c. Concave up for $x < 1$; concave down for $x > 1$
d. No inflection points in domain

43) [T] $f(x) = \dfrac{\sin x}{x}$ over $x = [-2\pi, 0) \cup (0, 2\pi]$

44) $f(x) = \sin(x)e^x$ over $x = [-\pi, \pi]$

Answer

a. Increasing over $-\frac{\pi}{4} < x < \frac{3\pi}{4}$, decreasing over $x > \frac{3\pi}{4},\ x < -\frac{\pi}{4}$
b. Minimum at $x = -\frac{\pi}{4}$, maximum at $x = \frac{3\pi}{4}$
c. Concave up for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, concave down for $x < -\frac{\pi}{2},\ x > \frac{\pi}{2}$
d. Inflection points at $x = \pm\frac{\pi}{2}$

45) $f(x) = \ln x\sqrt{x}, \quad x > 0$

46) $f(x) = \frac{1}{4}\sqrt{x} + \frac{1}{x}, \quad x > 0$

Answer

a. Increasing over $x > 4$, decreasing over $0 < x < 4$
b. Minimum at $x = 4$
c. Concave up for $0 < x < 8\sqrt[3]{2}$, concave down for $x > 8\sqrt[3]{2}$
d. Inflection point at $x = 8\sqrt[3]{2}$

47) $f(x) = \dfrac{e^x}{x}, \quad x \neq 0$

In exercises 48 - 52, interpret the sentences in terms of f, f', and f''.

48) The population is growing more slowly. Here f is the population.

Answer

$f > 0,\ f' > 0,\ f'' < 0$

49) A bike accelerates faster, but a car goes faster. Here $f =$ Bike's position minus Car's position.

50) The airplane lands smoothly. Here f is the plane's altitude.

Answer

$f > 0,\ f' < 0,\ f'' > 0$

51) Stock prices are at their peak. Here f is the stock price.

52) The economy is picking up speed. Here f is a measure of the economy, such as GDP.

Answer

$f > 0,\ f' > 0,\ f'' > 0$

For exercises 53 - 57, consider a third-degree polynomial $f(x)$, which has the properties $f'(1) = 0$ and $f'(3) = 0$. Determine whether the following statements are true or false. Justify your answer.

53) $f(x) = 0$ for some $1 \le x \le 3$.

54) $f''(x) = 0$ for some $1 \le x \le 3$.

Answer

True, by the Mean Value Theorem

55) There is no absolute maximum at $x = 3$.

56) If $f(x)$ has three roots, then it has 1 inflection point.

Answer

True, examine derivative

57) If $f(x)$ has one inflection point, then it has three real roots.

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4.6: Limits at Infinity and Asymptotes

Learning Objectives

- Calculate the limit of a function as x increases or decreases without bound.
- Recognize a horizontal asymptote on the graph of a function.
- Estimate the end behavior of a function as x increases or decreases without bound.
- Recognize an oblique asymptote on the graph of a function.
- Analyze a function and its derivatives to draw its graph.

We have shown how to use the first and second derivatives of a function to describe the shape of a graph. To graph a function f defined on an unbounded domain, we also need to know the behavior of f as $x \to \pm\infty$. In this section, we define limits at infinity and show how these limits affect the graph of a function. At the end of this section, we outline a strategy for graphing an arbitrary function f.

We begin by examining what it means for a function to have a finite limit at infinity. Then we study the idea of a function with an infinite limit at infinity. Back in Introduction to Functions and Graphs, we looked at vertical asymptotes; in this section we deal with horizontal and oblique asymptotes.

Limits at Infinity and Horizontal Asymptotes

Recall that $\lim\limits_{x\to a} f(x) = L$ means $f(x)$ becomes arbitrarily close to L as long as x is sufficiently close to a. We can extend this idea to limits at infinity. For example, consider the function $f(x) = 2 + \frac{1}{x}$. As can be seen graphically in Figure 4.6.1 and numerically in Table 4.6.1, as the values of x get larger, the values of $f(x)$ approach 2. We say the limit as x approaches ∞ of $f(x)$ is 2 and write $\lim\limits_{x\to\infty} f(x) = 2$. Similarly, for $x < 0$, as the values $|x|$ get larger, the values of $f(x)$ approach 2. We say the limit as x approaches $-\infty$ of $f(x)$ is 2 and write $\lim\limits_{x\to-\infty} f(x) = 2$.

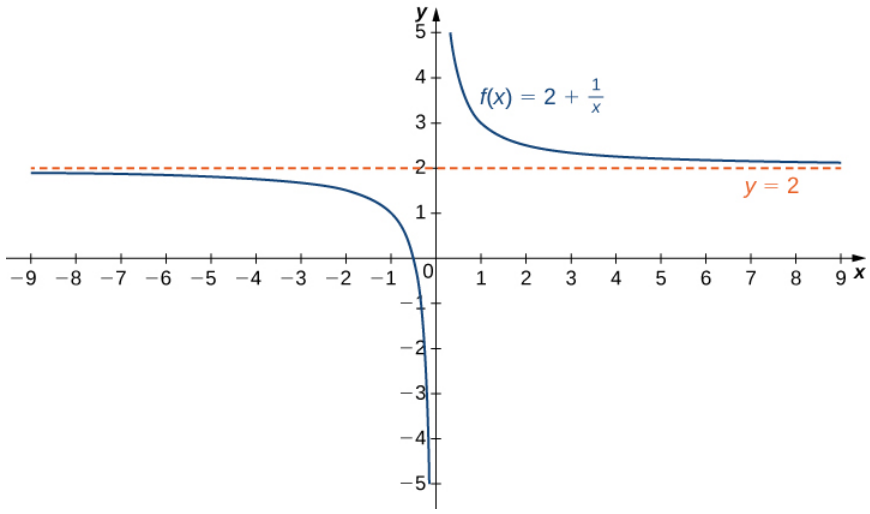


Figure 4.6.1:The function approaches the asymptote $y = 2$ as x approaches $\pm\infty$.

Table 4.6.1: Values of a function f as $x \to \pm\infty$

x	10	100	1,000	10,000
$2 + \frac{1}{x}$	2.1	2.01	2.001	2.0001
x	−10	−100	−1000	−10,000
$2 + \frac{1}{x}$	1.9	1.99	1.999	1.9999

More generally, for any function f, we say the limit as $x \to \infty$ of $f(x)$ is L if $f(x)$ becomes arbitrarily close to L as long as x is sufficiently large. In that case, we write $\lim\limits_{x\to\infty} f(x) = L$. Similarly, we say the limit as $x \to -\infty$ of $f(x)$ is L if $f(x)$ becomes arbitrarily

close to L as long as $x < 0$ and $|x|$ is sufficiently large. In that case, we write $\lim_{x \to -\infty} f(x) = L$. We now look at the definition of a function having a limit at infinity.

Definition: Limit at Infinity (Informal)

If the values of $f(x)$ become arbitrarily close to L as x becomes sufficiently large, we say the function f has a **limit at infinity** and write

$$\lim_{x \to \infty} f(x) = L.$$

If the values of $f(x)$ becomes arbitrarily close to L for $x < 0$ as $|x|$ becomes sufficiently large, we say that the function f has a limit at negative infinity and write

$$\lim_{x \to -\infty} f(x) = L.$$

If the values $f(x)$ are getting arbitrarily close to some finite value L as $x \to \infty$ or $x \to -\infty$, the graph of f approaches the line $y = L$. In that case, the line $y = L$ is a horizontal asymptote of f (Figure 4.6.2). For example, for the function $f(x) = \frac{1}{x}$, since $\lim_{x \to \infty} f(x) = 0$, the line $y = 0$ is a horizontal asymptote of $f(x) = \frac{1}{x}$.

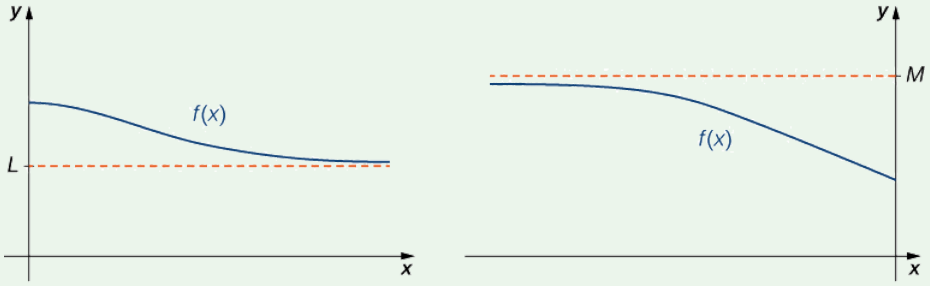


Figure 4.6.2: (a) As $x \to \infty$, the values of f are getting arbitrarily close to L. The line $y = L$ is a horizontal asymptote of f. (b) As $x \to -\infty$, the values of f are getting arbitrarily close to M. The line $y = M$ is a horizontal asymptote of f.

Definition: Horizontal Asymptote

If $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$, we say the line $y = L$ is a *horizontal asymptote* of f.

A function cannot cross a vertical asymptote because the graph must approach infinity (or $-\infty$) from at least one direction as x approaches the vertical asymptote. However, a function may cross a horizontal asymptote. In fact, a function may cross a horizontal asymptote an unlimited number of times. For example, the function $f(x) = \frac{\cos x}{x} + 1$ shown in Figure 4.6.3 intersects the horizontal asymptote $y = 1$ an infinite number of times as it oscillates around the asymptote with ever-decreasing amplitude.

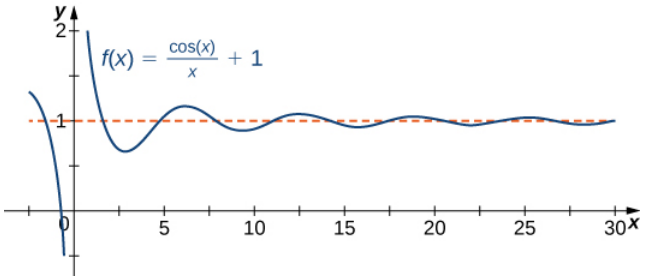


Figure 4.6.3: The graph of $f(x) = (\cos x)/x + 1$ crosses its horizontal asymptote $y = 1$ an infinite number of times.

The algebraic limit laws and squeeze theorem we introduced in Introduction to Limits also apply to limits at infinity. We illustrate how to use these laws to compute several limits at infinity.

Example 4.6.1: Computing Limits at Infinity

For each of the following functions f, evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. Determine the horizontal asymptote(s) for f.

a. $f(x) = 5 - \dfrac{2}{x^2}$
b. $f(x) = \dfrac{\sin x}{x}$
c. $f(x) = \tan^{-1}(x)$

Solution

a. Using the algebraic limit laws, we have

$$\lim_{x\to\infty}\left(5-\frac{2}{x^2}\right) = \lim_{x\to\infty} 5 - 2\left(\lim_{x\to\infty}\frac{1}{x}\right)\cdot\left(\lim_{x\to\infty}\frac{1}{x}\right) = 5 - 2\cdot 0 = 5.$$

Similarly, $\lim_{x\to-\infty} f(x) = 5$. Therefore, $f(x) = 5 - \dfrac{2}{x^2}$ has a horizontal asymptote of $y = 5$ and f approaches this horizontal asymptote as $x \to \pm\infty$ as shown in the following graph.

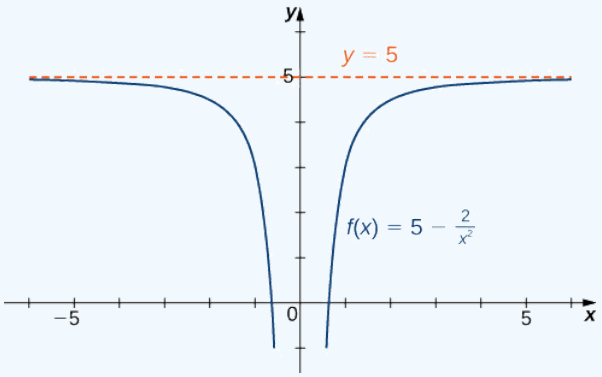


Figure 4.6.4: This function approaches a horizontal asymptote as $x \to \pm\infty$.

b. Since $-1 \le \sin x \le 1$ for all x, we have

$$\frac{-1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$$

for all $x \neq 0$. Also, since

$$\lim_{x\to\infty}\frac{-1}{x} = 0 = \lim_{x\to\infty}\frac{1}{x},$$

we can apply the squeeze theorem to conclude that

$$\lim_{x\to\infty}\frac{\sin x}{x} = 0.$$

Similarly,

$$\lim_{x\to-\infty}\frac{\sin x}{x} = 0.$$

Thus, $f(x) = \dfrac{\sin x}{x}$ has a horizontal asymptote of $y = 0$ and $f(x)$ approaches this horizontal asymptote as $x \to \pm\infty$ as shown in the following graph.

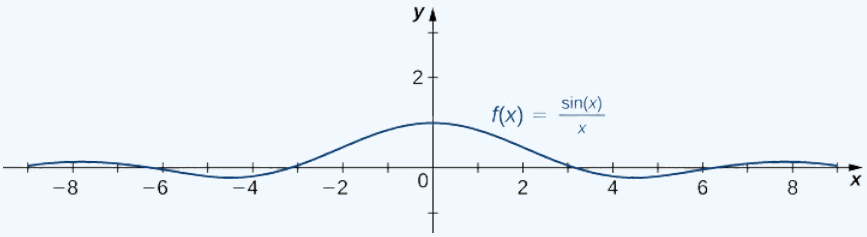


Figure 4.6.5: This function crosses its horizontal asymptote multiple times.

c. To evaluate $\lim_{x\to\infty} \tan^{-1}(x)$ and $\lim_{x\to-\infty} \tan^{-1}(x)$, we first consider the graph of $y = \tan(x)$ over the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ as shown in the following graph.

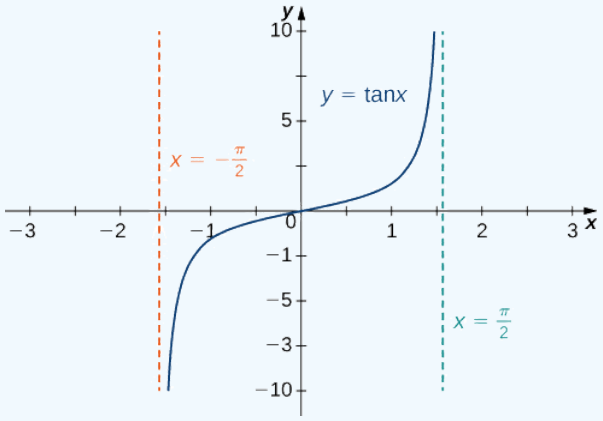


Figure 4.6.6: The graph of $y = \tan x$ has vertical asymptotes at $x = \pm\frac{\pi}{2}$

Since

$$\lim_{x\to\frac{\pi}{2}^-} \tan x = \infty,$$

it follows that

$$\lim_{x\to\infty} \tan^{-1}(x) = \frac{\pi}{2}.$$

Similarly, since

$$\lim_{x\to-\frac{\pi}{2}^+} \tan x = -\infty,$$

it follows that

$$\lim_{x\to-\infty} \tan^{-1}(x) = -\frac{\pi}{2}.$$

As a result, $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$ are horizontal asymptotes of $f(x) = \tan^{-1}(x)$ as shown in the following graph.

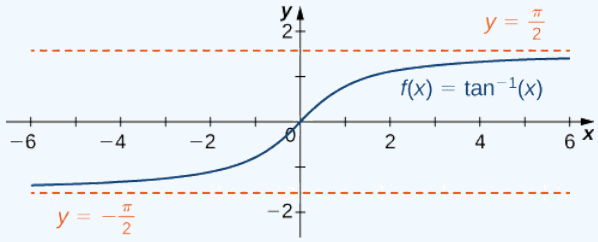


Figure 4.6.7: This function has two horizontal asymptotes.

? Exercise 4.6.1

Evaluate $\lim_{x\to-\infty}\left(3+\frac{4}{x}\right)$ and $\lim_{x\to\infty}\left(3+\frac{4}{x}\right)$. Determine the horizontal asymptotes of $f(x) = 3 + \frac{4}{x}$, if any.

Hint

$\lim_{x\to\pm\infty} \frac{1}{x} = 0$

Answer

Both limits are 3. The line $y = 3$ is a horizontal asymptote.

Infinite Limits at Infinity

Sometimes the values of a function f become arbitrarily large as $x \to \infty$(or as $x \to -\infty$). In this case, we write $\lim\limits_{x\to\infty} f(x) = \infty$ (or $\lim\limits_{x\to-\infty} f(x) = \infty$). On the other hand, if the values of f are negative but become arbitrarily large in magnitude as $x \to \infty$ (or as $x \to -\infty$), we write $\lim\limits_{x\to\infty} f(x) = -\infty$ (or $\lim\limits_{x\to-\infty} f(x) = -\infty$).

For example, consider the function $f(x) = x^3$. As seen in Table 4.6.2 and Figure 4.6.8, as $x \to \infty$ the values $f(x)$ become arbitrarily large. Therefore, $\lim\limits_{x\to\infty} x^3 = \infty$. On the other hand, as $x \to -\infty$, the values of $f(x) = x^3$ are negative but become arbitrarily large in magnitude. Consequently, $\lim\limits_{x\to-\infty} x^3 = -\infty$.

Table 4.6.2: Values of a power function as $x \to \pm\infty$

x	10	20	50	100	1000
x^3	1000	8000	125,000	1,000,000	1,000,000,000
x	−10	−20	−50	−100	−1000
x^3	−1000	−8000	−125,000	−1,000,000	−1,000,000,000

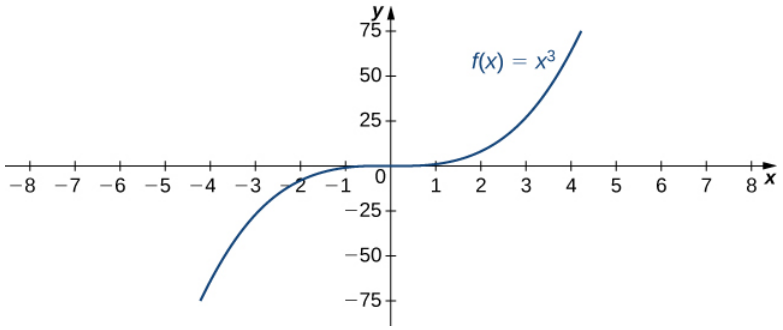


Figure 4.6.8: For this function, the functional values approach ±infinity as $x \to \pm\infty$.

Definition: Infinite Limit at Infinity (Informal)

We say a function f has an infinite limit at infinity and write

$$\lim_{x\to\infty} f(x) = \infty.$$

if $f(x)$ becomes arbitrarily large for x sufficiently large. We say a function has a negative infinite limit at infinity and write

$$\lim_{x\to\infty} f(x) = -\infty.$$

if $f(x) < 0$ and $|f(x)|$ becomes arbitrarily large for x sufficiently large. Similarly, we can define infinite limits as $x \to -\infty$.

Formal Definitions

Earlier, we used the terms arbitrarily close, arbitrarily large, and sufficiently large to define limits at infinity informally. Although these terms provide accurate descriptions of limits at infinity, they are not precise mathematically. Here are more formal definitions of limits at infinity. We then look at how to use these definitions to prove results involving limits at infinity.

Definition: Limit at Infinity (Formal)

We say a function f has a *limit at infinity*, if there exists a real number L such that for all $\varepsilon > 0$, there exists $N > 0$ such that

$$|f(x) - L| < \varepsilon$$

for all $x > N$. in that case, we write

$$\lim_{x\to\infty} f(x) = L$$

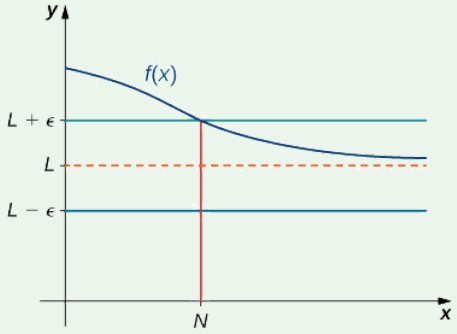


Figure 4.6.9: For a function with a limit at infinity, for all $x > N, |f(x) - L| < \varepsilon$.

Earlier in this section, we used graphical evidence in Figure 4.6.1 and numerical evidence in Table 4.6.1 to conclude that $\lim_{x\to\infty}\left(2+\frac{1}{x}\right)=2$. Here we use the formal definition of limit at infinity to prove this result rigorously.

Example 4.6.2:

Use the formal definition of limit at infinity to prove that $\lim_{x\to\infty}\left(2+\frac{1}{x}\right)=2$.

Solution

Let $\varepsilon > 0$. Let $N = \frac{1}{\varepsilon}$. Therefore, for all $x > N$, we have

$$\left|2+\frac{1}{x}-2\right| = \left|\frac{1}{x}\right| = \frac{1}{x} < \frac{1}{N} = \varepsilon$$

Exercise 4.6.2

Use the formal definition of limit at infinity to prove that $\lim_{x\to\infty}\left(3-\frac{1}{x^2}\right)=3$.

Hint

Let $N = \frac{1}{\sqrt{\varepsilon}}$.

Answer

Let $\varepsilon > 0$. Let $N = \frac{1}{\sqrt{\varepsilon}}$. Therefore, for all $x > N$, we have

$$\left|3-\frac{1}{x^2}-3\right| = \frac{1}{x^2} < \frac{1}{N^2} = \varepsilon$$

Therefore, $\lim_{x\to\infty}(3-1/x^2)=3$.

We now turn our attention to a more precise definition for an infinite limit at infinity.

Definition: Infinite Limit at Infinity (Formal)

We say a function f has an *infinite limit at infinity* and write

$$\lim_{x\to\infty} f(x) = \infty$$

if for all $M > 0$, there exists an $N > 0$ such that

$$f(x) > M$$

for all $x > N$ (see Figure 4.6.10).

We say a function has a negative infinite limit at infinity and write

$$\lim_{x\to\infty} f(x) = -\infty$$

if for all $M < 0$, there exists an $N > 0$ such that

$$f(x) < M$$

for all $x > N$.

Similarly we can define limits as $x \to -\infty$.

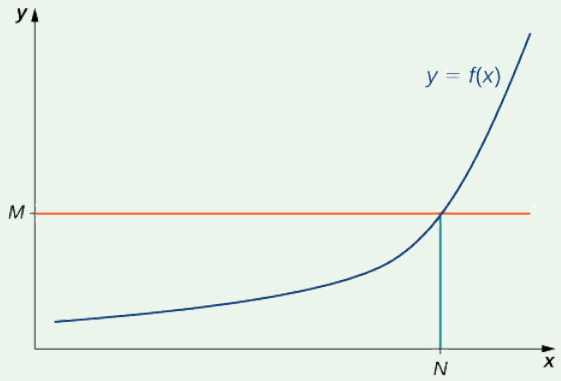


Figure 4.6.10: For a function with an infinite limit at infinity, for all $x > N,\ f(x) > M$.

Earlier, we used graphical evidence (Figure 4.6.8) and numerical evidence (Table 4.6.2) to conclude that $\lim_{x\to\infty} x^3 = \infty$. Here we use the formal definition of infinite limit at infinity to prove that result.

Example 4.6.3

Use the formal definition of infinite limit at infinity to prove that $\lim_{x\to\infty} x^3 = \infty$.

Solution

Let $M > 0$. Let $N = \sqrt[3]{M}$. Then, for all $x > N$, we have

$$x^3 > N^3 = (\sqrt[3]{M})^3 = M.$$

Therefore, $\lim_{x\to\infty} x^3 = \infty$.

Exercise 4.6.3

Use the formal definition of infinite limit at infinity to prove that $\lim_{x\to\infty} 3x^2 = \infty$.

Hint

Let $N = \sqrt{\frac{M}{3}}$.

Answer

Let $M > 0$. Let $N = \sqrt{\frac{M}{3}}$. Then, for all $x > N$, we have

$$3x^2 > 3N^2 = 3\left(\sqrt{\frac{M}{3}}\right)^2 = \frac{3M}{3} = M$$

End Behavior

The behavior of a function as $x \to \pm\infty$ is called the function's **end behavior**. At each of the function's ends, the function could exhibit one of the following types of behavior:

1. The function $f(x)$ approaches a horizontal asymptote $y = L$.
2. The function $f(x) \to \infty$ or $f(x) \to -\infty$.
3. The function does not approach a finite limit, nor does it approach ∞ or $-\infty$. In this case, the function may have some oscillatory behavior.

Let's consider several classes of functions here and look at the different types of end behaviors for these functions.

End Behavior for Polynomial Functions

Consider the power function $f(x) = x^n$ where n is a positive integer. From Figure 4.6.11 and Figure 4.6.12, we see that

$$\lim_{x\to\infty} x^n = \infty;\ n = 1, 2, 3, \ldots$$

and

$$\lim_{x\to-\infty} x^n = \begin{cases} \infty, & n = 2, 4, 6, \ldots \\ -\infty, & n = 1, 3, 5, \ldots . \end{cases}$$

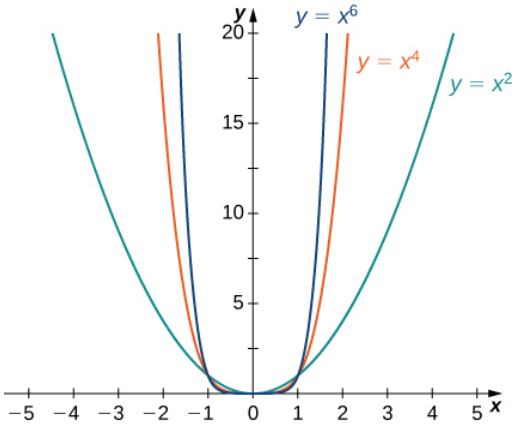


Figure 4.6.11: For power functions with an even power n, $\lim_{x\to\infty} x^n = \infty = \lim_{x\to-\infty} x^n$.

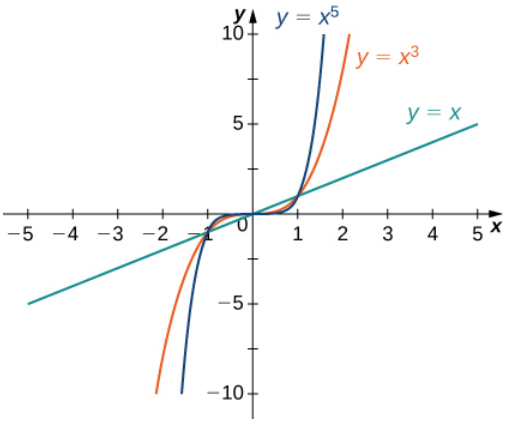


Figure 4.6.12: For power functions with an odd power n, $\lim_{x\to\infty} x^n = \infty$ and $\lim_{x\to-\infty} x^n = -\infty$.

Using these facts, it is not difficult to evaluate $\lim_{x\to\infty} cx^n$ and $\lim_{x\to-\infty} cx^n$, where c is any constant and n is a positive integer. If $c > 0$, the graph of $y = cx^n$ is a vertical stretch or compression of $y = x^n$, and therefore

$$\lim_{x\to\infty} cx^n = \lim_{x\to\infty} x^n \text{ and } \lim_{x\to-\infty} cx^n = \lim_{x\to-\infty} x^n \text{ if } c > 0.$$

If $c < 0$, the graph of $y = cx^n$ is a vertical stretch or compression combined with a reflection about the x-axis, and therefore

$$\lim_{x\to\infty} cx^n = -\lim_{x\to\infty} x^n \text{ and } \lim_{x\to-\infty} cx^n = -\lim_{x\to-\infty} x^n \text{ if } c < 0.$$

If $c = 0, y = cx^n = 0$, in which case $\lim_{x\to\infty} cx^n = 0 = \lim_{x\to-\infty} cx^n$.

Example 4.6.4: Limits at Infinity for Power Functions

For each function f, evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

a. $f(x) = -5x^3$
b. $f(x) = 2x^4$

Solution

a. Since the coefficient of x^3 is -5, the graph of $f(x) = -5x^3$ involves a vertical stretch and reflection of the graph of $y = x^3$ about the x-axis. Therefore, $\lim_{x\to\infty}(-5x^3) = -\infty$ and $\lim_{x\to-\infty}(-5x^3) = \infty$.

b. Since the coefficient of x^4 is 2, the graph of $f(x) = 2x^4$ is a vertical stretch of the graph of $y = x^4$. Therefore, $\lim_{x\to\infty} 2x^4 = \infty$ and $\lim_{x\to-\infty} 2x^4 = \infty$.

? Exercise 4.6.4

Let $f(x) = -3x^4$. Find $\lim_{x\to\infty} f(x)$.

Hint

The coefficient -3 is negative.

Answer

$-\infty$

We now look at how the limits at infinity for power functions can be used to determine $\lim_{x\to\pm\infty} f(x)$ for any polynomial function f. Consider a polynomial function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a^1 x + a^0$$

of degree $n \geq 1$ so that $a_n \neq 0$.

Factoring, we see that

$$f(x) = a_n x^n \left(1 + \frac{a_{n-1}}{a_n}\frac{1}{x} + ... + \frac{a_1}{a_n}\frac{1}{x^{n-1}} + \frac{a_0}{a_n}\frac{1}{x^n}\right).$$

As $x \to \pm\infty$, all the terms inside the parentheses approach zero except the first term. We conclude that

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} a_n x^n.$$

For example, the function $f(x) = 5x^3 - 3x^2 + 4$ behaves like $g(x) = 5x^3$ as $x \to \pm\infty$ as shown in Figure 4.6.13 and Table 4.6.3.

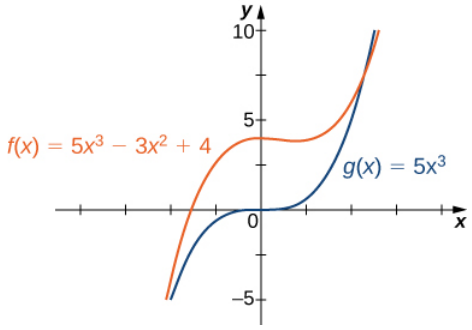


Figure 4.6.13: The end behavior of a polynomial is determined by the behavior of the term with the largest exponent.

Table 4.6.3: A polynomial's end behavior is determined by the term with the largest exponent

x	10	100	1000
$f(x) = 5x^3 - 3x^2 + 4$	4704	4,970,004	4,997,000,004
$g(x) = 5x^3$	5000	5,000,000	5,000,000,000
x	−10	−100	−000
$f(x) = 5x^3 - 3x^2 + 4$	−5296	−5,029,996	−5,002,999,996
$g(x) = 5x^3$	−5000	−5,000,000	−5,000,000,000

End Behavior for Algebraic Functions

The end behavior for rational functions and functions involving radicals is a little more complicated than for polynomials. In Example 4.6.5, we show that the limits at infinity of a rational function $f(x) = \dfrac{p(x)}{q(x)}$ depend on the relationship between the degree of the numerator and the degree of the denominator. To evaluate the limits at infinity for a rational function, we divide the numerator and denominator by the highest power of x appearing in the denominator. This determines which term in the overall expression dominates the behavior of the function at large values of x.

Example 4.6.5: Determining End Behavior for Rational Functions

For each of the following functions, determine the limits as $x \to \infty$ and $x \to -\infty$. Then, use this information to describe the end behavior of the function.

a. $f(x) = \dfrac{3x-1}{2x+5}$ (Note: The degree of the numerator and the denominator are the same.)

b. $f(x) = \dfrac{3x^2+2x}{4x^3-5x+7}$ (Note: The degree of numerator is less than the degree of the denominator.)

c. $f(x) = \dfrac{3x^2+4x}{x+2}$ (Note: The degree of the numerator is greater than the degree of the denominator.)

Solution

a. The highest power of x in the denominator is x. Therefore, dividing the numerator and denominator by x and applying the algebraic limit laws, we see that

$$\begin{aligned}\lim_{x\to\pm\infty}\frac{3x-1}{2x+5} &= \lim_{x\to\pm\infty}\frac{3-1/x}{2+5/x}\\ &= \frac{\lim_{x\to\pm\infty}(3-1/x)}{\lim_{x\to\pm\infty}(2+5/x)}\\ &= \frac{\lim_{x\to\pm\infty}3-\lim_{x\to\pm\infty}1/x}{\lim_{x\to\pm\infty}2+\lim_{x\to\pm\infty}5/x}\\ &= \frac{3-0}{2+0}=\frac{3}{2}.\end{aligned}$$

Since $\lim_{x\to\pm\infty} f(x) = \dfrac{3}{2}$, we know that $y = \frac{3}{2}$ is a horizontal asymptote for this function as shown in the following graph.

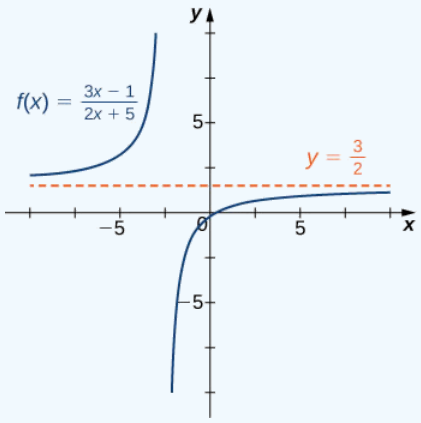


Figure 4.6.14: The graph of this rational function approaches a horizontal asymptote as $x \to \pm\infty$.

b. Since the largest power of x appearing in the denominator is x^3, divide the numerator and denominator by x^3. After doing so and applying algebraic limit laws, we obtain

$$\lim_{x\to\pm\infty}\frac{3x^2+2x}{4x^3-5x+7} = \lim_{x\to\pm\infty}\frac{3/x+2/x^2}{4-5/x^2+7/x^3} = \frac{3\cdot 0+2\cdot 0}{4-5\cdot 0+7\cdot 0} = \frac{0}{4} = 0.$$

Therefore f has a horizontal asymptote of $y = 0$ as shown in the following graph.

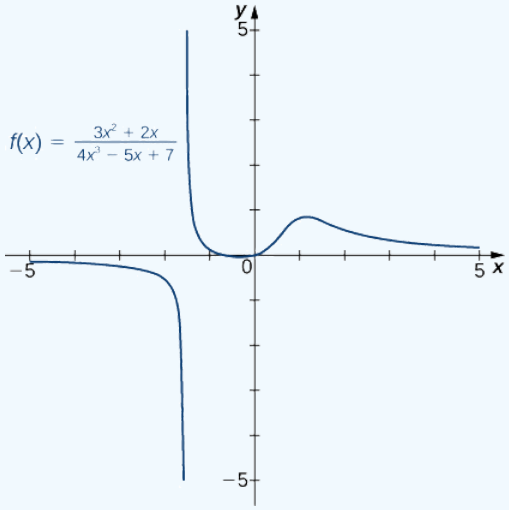


Figure 4.6.15: The graph of this rational function approaches the horizontal asymptote $y = 0$ as $x \to \pm\infty$.

c. Dividing the numerator and denominator by x, we have

$$\lim_{x\to\pm\infty} \frac{3x^2+4x}{x+2} = \lim_{x\to\pm\infty} \frac{3x+4}{1+2/x}.$$

As $x \to \pm\infty$, the denominator approaches 1. As $x \to \infty$, the numerator approaches $+\infty$. As $x \to -\infty$, the numerator approaches $-\infty$. Therefore $\lim_{x\to\infty} f(x) = \infty$, whereas $\lim_{x\to-\infty} f(x) = -\infty$ as shown in the following figure.

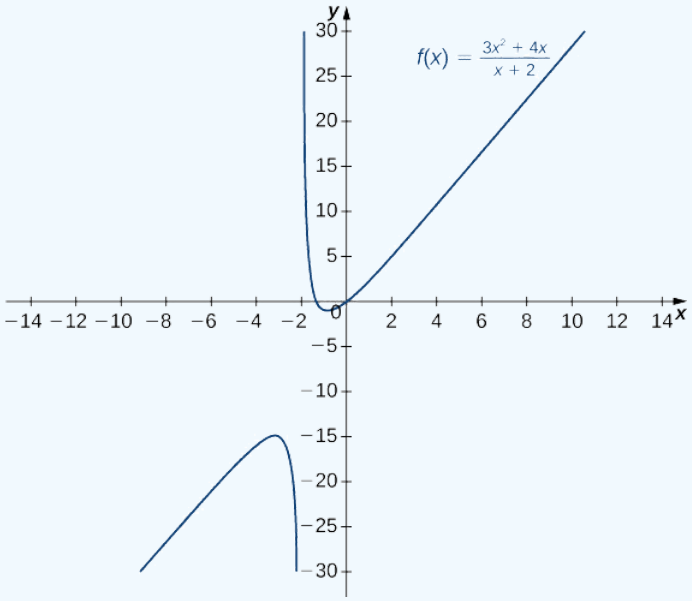


Figure 4.6.16: As $x \to \infty$, the values $f(x) \to \infty$. As $x \to -\infty$, the values $f(x) \to -\infty$.

Exercise 4.6.5

Evaluate $\lim_{x\to\pm\infty} \frac{3x^2+2x-1}{5x^2-4x+7}$ and use these limits to determine the end behavior of $f(x) = \frac{3x^2+2x-1}{5x^2-4x+7}$.

Hint

Divide the numerator and denominator by x^2.

Answer

$\frac{3}{5}$

Before proceeding, consider the graph of $f(x) = \dfrac{3x^2 + 4x}{x + 2}$ shown in Figure 4.6.16. As $x \to \infty$ and $x \to -\infty$, the graph of f appears almost linear. Although f is certainly not a linear function, we now investigate why the graph of f seems to be approaching a linear function. First, using long division of polynomials, we can write

$$f(x) = \frac{3x^2 + 4x}{x + 2} = 3x - 2 + \frac{4}{x + 2}.$$

Since $\dfrac{4}{x + 2} \to 0$ as $x \to \pm\infty$, we conclude that

$$\lim_{x \to \pm\infty} (f(x) - (3x - 2)) = \lim_{x \to \pm\infty} \frac{4}{x + 2} = 0.$$

Therefore, the graph of f approaches the line $y = 3x - 2$ as $x \to \pm\infty$. This line is known as an **oblique asymptote** for f (Figure 4.6.17).

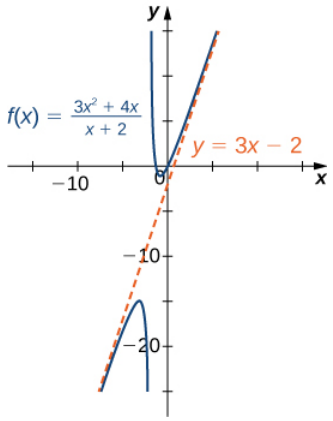


Figure 4.6.17: The graph of the rational function $f(x) = (3x^2 + 4x)/(x + 2)$ approaches the oblique asymptote $y = 3x - 2$ as $x \to \pm\infty$.

We can summarize the results of Example 4.6.5 to make the following conclusion regarding end behavior for rational functions. Consider a rational function

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \ldots + b_1 x + b_0},$$

where $a_n \neq 0$ and $b_m \neq 0$.

1. If the degree of the numerator is the same as the degree of the denominator $(n = m)$, then f has a horizontal asymptote of $y = a_n/b_m$ as $x \to \pm\infty$.
2. If the degree of the numerator is less than the degree of the denominator $(n < m)$, then f has a horizontal asymptote of $y = 0$ as $x \to \pm\infty$.
3. If the degree of the numerator is greater than the degree of the denominator $(n > m)$, then f does not have a horizontal asymptote. The limits at infinity are either positive or negative infinity, depending on the signs of the leading terms. In addition, using long division, the function can be rewritten as

$$f(x) = \frac{p(x)}{q(x)} = g(x) + \frac{r(x)}{q(x)},$$

where the degree of $r(x)$ is less than the degree of $q(x)$. As a result, $\lim_{x \to \pm\infty} r(x)/q(x) = 0$. Therefore, the values of $[f(x) - g(x)]$ approach zero as $x \to \pm\infty$. If the degree of $p(x)$ is exactly one more than the degree of $q(x)$ (i.e., $n = m + 1$), the function $g(x)$ is a linear function. In this case, we call $g(x)$ an oblique asymptote.

Now let's consider the end behavior for functions involving a radical.

Example 4.6.6: Determining End Behavior for a Function Involving a Radical

Find the limits as $x \to \infty$ and $x \to -\infty$ for $f(x) = \dfrac{3x - 2}{\sqrt{4x^2 + 5}}$ and describe the end behavior of f.

Solution

Let's use the same strategy as we did for rational functions: divide the numerator and denominator by a power of x. To determine the appropriate power of x, consider the expression $\sqrt{4x^2+5}$ in the denominator. Since

$$\sqrt{4x^2+5} \approx \sqrt{4x^2} = 2|x|$$

for large values of x in effect x appears just to the first power in the denominator. Therefore, we divide the numerator and denominator by $|x|$. Then, using the fact that $|x| = x$ for $x > 0$, $|x| = -x$ for $x < 0$, and $|x| = \sqrt{x^2}$ for all x, we calculate the limits as follows:

$$\begin{aligned}\lim_{x\to\infty} \frac{3x-2}{\sqrt{4x^2+5}} &= \lim_{x\to\infty} \frac{(1/|x|)(3x-2)}{(1/|x|)\sqrt{4x^2+5}} \\ &= \lim_{x\to\infty} \frac{(1/x)(3x-2)}{\sqrt{(1/x^2)(4x^2+5)}} \\ &= \lim_{x\to\infty} \frac{3-2/x}{\sqrt{4+5/x^2}} = \frac{3}{\sqrt{4}} = \frac{3}{2}\end{aligned}$$

$$\begin{aligned}\lim_{x\to-\infty} \frac{3x-2}{\sqrt{4x^2+5}} &= \lim_{x\to-\infty} \frac{(1/|x|)(3x-2)}{(1/|x|)\sqrt{4x^2+5}} \\ &= \lim_{x\to-\infty} \frac{(-1/x)(3x-2)}{\sqrt{(1/x^2)(4x^2+5)}} \\ &= \lim_{x\to-\infty} \frac{-3+2/x}{\sqrt{4+5/x^2}} = \frac{-3}{\sqrt{4}} = \frac{-3}{2}.\end{aligned}$$

Therefore, $f(x)$ approaches the horizontal asymptote $y = \frac{3}{2}$ as $x \to \infty$ and the horizontal asymptote $y = -\frac{3}{2}$ as $x \to -\infty$ as shown in the following graph.

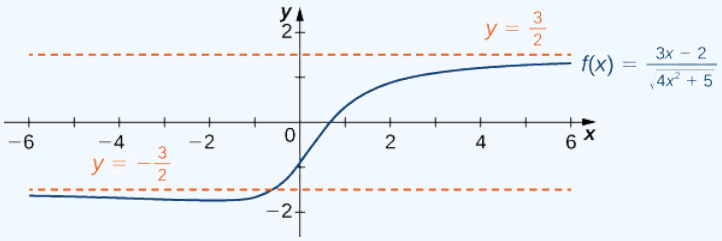


Figure 4.6.18: This function has two horizontal asymptotes and it crosses one of the asymptotes.

? Exercise 4.6.6

Evaluate $\lim_{x\to\infty} \frac{\sqrt{3x^2+4}}{x+6}$.

Hint

Divide the numerator and denominator by x.

Answer

$\sqrt{3}$

Determining End Behavior for Transcendental Functions

The six basic trigonometric functions are periodic and do not approach a finite limit as $x \to \pm\infty$. For example, $\sin x$ oscillates between 1 and −1 (Figure 4.6.19). The tangent function x has an infinite number of vertical asymptotes as $x \to \pm\infty$; therefore, it does not approach a finite limit nor does it approach $\pm\infty$ as $x \to \pm\infty$ as shown in Figure 4.6.20.

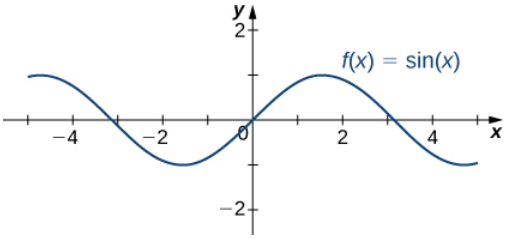


Figure 4.6.19: The function $f(x) = \sin x$ oscillates between 1 and -1 as $x \to \pm\infty$

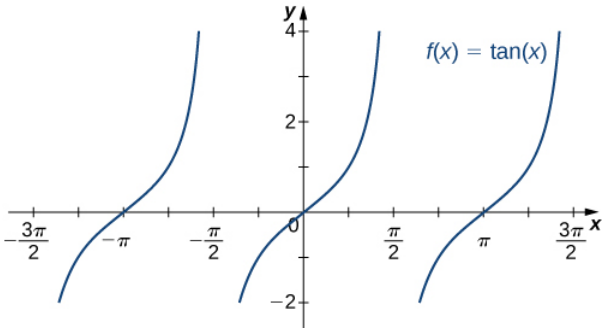


Figure 4.6.20: The function $f(x) = \tan x$ does not approach a limit and does not approach $\pm\infty$ as $x \to \pm\infty$

Recall that for any base $b > 0,\ b \neq 1$, the function $y = b^x$ is an exponential function with domain $(-\infty, \infty)$ and range $(0, \infty)$. If $b > 1,\ y = b^x$ is increasing over $(-\infty, \infty)$. If $0 < b < 1,\ y = b^x$ is decreasing over $(-\infty, \infty)$. For the natural exponential function $f(x) = e^x$, $e \approx 2.718\ > 1$. Therefore, $f(x) = e^x$ is increasing on $(-\infty, \infty)$ and the range is $(0, \infty)$. The exponential function $f(x) = e^x$ approaches ∞ as $x \to \infty$ and approaches 0 as $x \to -\infty$ as shown in Table 4.6.4 and Figure 4.6.21.

Table 4.6.4: End behavior of the natural exponential function

x	−5	−2	0	2	5
e^x	0.00674	0.135	1	7.389	148.413

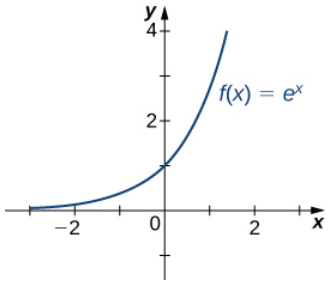


Figure 4.6.21: The exponential function approaches zero as $x \to -\infty$ and approaches ∞ as $x \to \infty$.

Recall that the natural logarithm function $f(x) = \ln(x)$ is the inverse of the natural exponential function $y = e^x$. Therefore, the domain of $f(x) = \ln(x)$ is $(0, \infty)$ and the range is $(-\infty, \infty)$. The graph of $f(x) = \ln(x)$ is the reflection of the graph of $y = e^x$ about the line $y = x$. Therefore, $\ln(x) \to -\infty$ as $x \to 0^+$ and $\ln(x) \to \infty$ as $x \to \infty$ as shown in Figure 4.6.22 and Table 4.6.5.

Table 4.6.5: End behavior of the natural logarithm function

x	0.01	0.1	1	10	100
$\ln(x)$	−4.605	−2.303	0	2.303	4.605

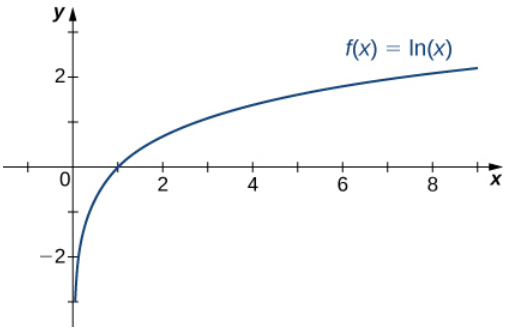


Figure 4.6.22: The natural logarithm function approaches ∞ as $x \to \infty$.

Example 4.6.7: Determining End Behavior for a Transcendental Function

Find the limits as $x \to \infty$ and $x \to -\infty$ for $f(x) = \dfrac{2+3e^x}{7-5e^x}$ and describe the end behavior of f.

Solution

To find the limit as $x \to \infty$, divide the numerator and denominator by e^x:

$$\begin{aligned} \lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{2+3e^x}{7-5e^x} \\ &= \lim_{x\to\infty} \frac{(2/e^x)+3}{(7/e^x)-5.} \end{aligned}$$

As shown in Figure 4.6.21, $e^x \to \infty$ as $x \to \infty$. Therefore,

$$\lim_{x\to\infty} \frac{2}{e^x} = 0 = \lim_{x\to\infty} \frac{7}{e^x}.$$

We conclude that $\lim_{x\to\infty} f(x) = -\dfrac{3}{5}$, and the graph of f approaches the horizontal asymptote $y = -\frac{3}{5}$ as $x \to \infty$. To find the limit as $x \to -\infty$, use the fact that $e^x \to 0$ as $x \to -\infty$ to conclude that $\lim_{x\to-\infty} f(x) = \dfrac{2}{7}$, and therefore the graph of $f(x)$ approaches the horizontal asymptote $y = \frac{2}{7}$ as $x \to -\infty$.

Exercise 4.6.7

Find the limits as $x \to \infty$ and $x \to -\infty$ for $f(x) = \dfrac{3e^x - 4}{5e^x + 2}$.

Hint

$\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to-\infty} e^x = 0$.

Answer

$\lim_{x\to\infty} f(x) = \dfrac{3}{5}, \quad \lim_{x\to-\infty} f(x) = -2$

Guidelines for Drawing the Graph of a Function

We now have enough analytical tools to draw graphs of a wide variety of algebraic and transcendental functions. Before showing how to graph specific functions, let's look at a general strategy to use when graphing any function.

Problem-Solving Strategy: Drawing the Graph of a Function

Given a function f, use the following steps to sketch a graph of f:

1. Determine the domain of the function.
2. Locate the x- and y-intercepts.

3. Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ to determine the end behavior. If either of these limits is a finite number L, then $y = L$ is a horizontal asymptote. If either of these limits is ∞ or $-\infty$, determine whether f has an oblique asymptote. If f is a rational function such that $f(x) = \dfrac{p(x)}{q(x)}$, where the degree of the numerator is greater than the degree of the denominator, then f can be written as

$$f(x) = \frac{p(x)}{q(x)} = g(x) + \frac{r(x)}{q(x),}$$

where the degree of $r(x)$ is less than the degree of $q(x)$. The values of $f(x)$ approach the values of $g(x)$ as $x \to \pm\infty$. If $g(x)$ is a linear function, it is known as an oblique asymptote.
4. Determine whether f has any vertical asymptotes.
5. Calculate f'. Find all critical points and determine the intervals where f is increasing and where f is decreasing. Determine whether f has any local extrema.
6. Calculate f''. Determine the intervals where f is concave up and where f is concave down. Use this information to determine whether f has any inflection points. The second derivative can also be used as an alternate means to determine or verify that f has a local extremum at a critical point.

Now let's use this strategy to graph several different functions. We start by graphing a polynomial function.

Example 4.6.8: Sketching a Graph of a Polynomial

Sketch a graph of $f(x) = (x-1)^2(x+2)$.

Solution

Step 1: Since f is a polynomial, the domain is the set of all real numbers.

Step 2: When $x = 0,\ f(x) = 2$. Therefore, the y-intercept is $(0, 2)$. To find the x-intercepts, we need to solve the equation $(x-1)^2(x+2) = 0$, which gives us the x-intercepts $(1, 0)$ and $(-2, 0)$

Step 3: We need to evaluate the end behavior of f. As $x \to \infty,\ (x-1)^2 \to \infty$ and $(x+2) \to \infty$. Therefore, $\lim_{x\to\infty} f(x) = \infty$.

As $x \to -\infty,\ (x-1)^2 \to \infty$ and $(x+2) \to -\infty$. Therefore, $\lim_{x\to-\infty} f(x) = -\infty$.

To get even more information about the end behavior of f, we can multiply the factors of f. When doing so, we see that

$$f(x) = (x-1)^2(x+2) = x^3 - 3x + 2.$$

Since the leading term of f is x^3, we conclude that f behaves like $y = x^3$ as $x \to \pm\infty$.

Step 4: Since f is a polynomial function, it does not have any vertical asymptotes.

Step 5: The first derivative of f is

$$f'(x) = 3x^2 - 3.$$

Therefore, f has two critical points: $x = 1, -1$. Divide the interval $(-\infty, \infty)$ into the three smaller intervals: $(-\infty, -1),\ (-1, 1)$, and $(1, \infty)$. Then, choose test points $x = -2, x = 0$, and $x = 2$ from these intervals and evaluate the sign of $f'(x)$ at each of these test points, as shown in the following table.

Interval	Test point	Sign of Derivative $f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$	Conclusion
$(-\infty, -1)$	$x = -2$	$(+)(-)(-) = +$	f is increasing
$(-1, 1)$	$x = 0$	$(+)(-)(+) = -$	f decreasing
$(1, \infty)$	$x = 2$	$(+)(+)(+) = +$	f is increasing

From the table, we see that f has a local maximum at $x = -1$ and a local minimum at $x = 1$. Evaluating $f(x)$ at those two points, we find that the local maximum value is $f(-1) = 4$ and the local minimum value is $f(1) = 0$.

Step 6: The second derivative of f is

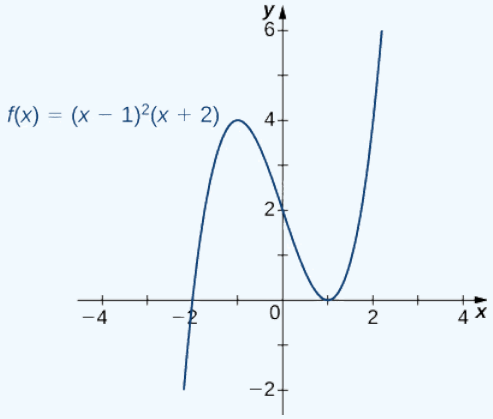
$$f''(x) = 6x.$$

The second derivative is zero at $x = 0$. Therefore, to determine the concavity of f, divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, 0)$ and $(0, \infty)$, and choose test points $x = -1$ and $x = 1$ to determine the concavity of f on each of these smaller intervals as shown in the following table.

Interval	Test Point	Sign of $f''(x) = 6x$	Conclusion
$(-\infty, 0)$	$x = -1$	$-$	f is concave down.
$(0, \infty)$	$x = 1$	$+$	f is concave up.

We note that the information in the preceding table confirms the fact, found in step 5, that f has a local maximum at $x = -1$ and a local minimum at $x = 1$. In addition, the information found in step 5—namely, f has a local maximum at $x = -1$ and a local minimum at $x = 1$, and $f'(x) = 0$ at those points—combined with the fact that f'' changes sign only at $x = 0$ confirms the results found in step 6 on the concavity of f.

Combining this information, we arrive at the graph of $f(x) = (x-1)^2(x+2)$ shown in the following graph.



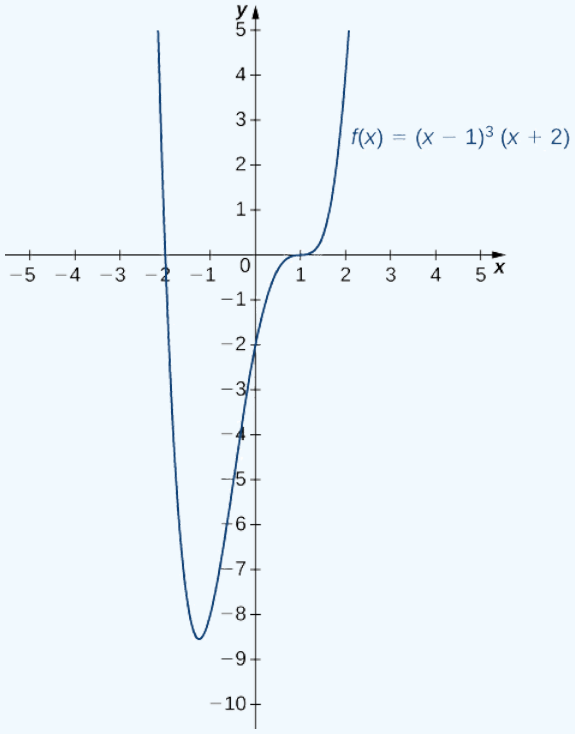
Exercise 4.6.8

Sketch a graph of $f(x) = (x-1)^3(x+2)$.

Hint

f is a fourth-degree polynomial.

Answer



Example 4.6.9: Sketching a Rational Function

Sketch the graph of $f(x) = \dfrac{x^2}{1-x^2}$.

Solution

Step 1: The function f is defined as long as the denominator is not zero. Therefore, the domain is the set of all real numbers x except $x = \pm 1$.

Step 2: Find the intercepts. If $x = 0$, then $f(x) = 0$, so 0 is an intercept. If $y = 0$, then $\dfrac{x^2}{1-x^2} = 0$, which implies $x = 0$. Therefore, $(0, 0)$ is the only intercept.

Step 3: Evaluate the limits at infinity. Since $f$is a rational function, divide the numerator and denominator by the highest power in the denominator: x^2.We obtain

$$\lim_{x\to\pm\infty} \frac{x^2}{1-x^2} = \lim_{x\to\pm\infty} \frac{1}{\frac{1}{x^2} - 1} = -1.$$

Therefore, f has a horizontal asymptote of $y = -1$ as $x \to \infty$ and $x \to -\infty$.

Step 4: To determine whether f has any vertical asymptotes, first check to see whether the denominator has any zeroes. We find the denominator is zero when $x = \pm 1$. To determine whether the lines $x = 1$ or $x = -1$ are vertical asymptotes of f, evaluate $\lim\limits_{x\to 1} f(x)$ and $\lim\limits_{x\to -1} f(x)$. By looking at each one-sided limit as $x \to 1$, we see that

$$\lim_{x\to 1^+} \frac{x^2}{1-x^2} = -\infty \text{ and } \lim_{x\to 1^-} \frac{x^2}{1-x^2} = \infty.$$

In addition, by looking at each one-sided limit as $x \to -1$, we find that

$$\lim_{x\to -1^+} \frac{x^2}{1-x^2} = \infty \text{ and } \lim_{x\to -1^-} \frac{x^2}{1-x^2} = -\infty.$$

Step 5: Calculate the first derivative:

$$f'(x) = \frac{(1-x^2)(2x) - x^2(-2x)}{\left(1-x^2\right)^2} = \frac{2x}{\left(1-x^2\right)^2}.$$

Critical points occur at points x where $f'(x) = 0$ or $f'(x)$ is undefined. We see that $f'(x) = 0$ when $x = 0$. The derivative f' is not undefined at any point in the domain of f. However, $x = \pm 1$ are not in the domain of f. Therefore, to determine where f is increasing and where f is decreasing, divide the interval $(-\infty, \infty)$ into four smaller intervals: $(-\infty, -1), (-1, 0), (0, 1)$, and $(1, \infty)$, and choose a test point in each interval to determine the sign of $f'(x)$ in each of these intervals. The values $x = -2$, $x = -\frac{1}{2}$, $x = \frac{1}{2}$, and $x = 2$ are good choices for test points as shown in the following table.

Interval	Test point	Sign of $f'(x) = \frac{2x}{(1-x^2)^2}$	Conclusion
$(-\infty, -1)$	$x = -2$	$-/+ = -$	f is decreasing.
$(-1, 0)$	$x = -1/2$	$-/+ = -$	f is decreasing.
$(0, 1)$	$x = 1/2$	$+/+ = +$	f is increasing.
$(1, \infty)$	$x = 2$	$+/+ = +$	f is increasing.

From this analysis, we conclude that f has a local minimum at $x = 0$ but no local maximum.

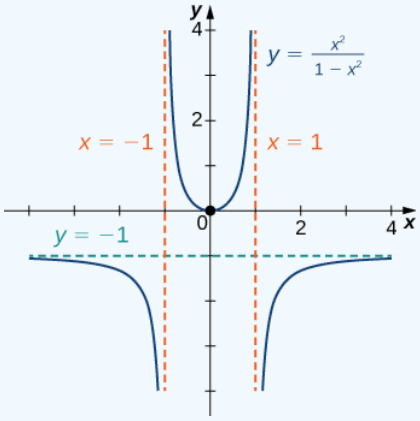
Step 6: Calculate the second derivative:

$$\begin{aligned} f''(x) &= \frac{(1-x^2)^2(2) - 2x(2(1-x^2)(-2x))}{(1-x^2)^4} \\ &= \frac{(1-x^2)[2(1-x^2) + 8x^2]}{\left(1-x^2\right)^4} \\ &= \frac{2(1-x^2) + 8x^2}{\left(1-x^2\right)^3} \\ &= \frac{6x^2 + 2}{\left(1-x^2\right)^3}. \end{aligned}$$

To determine the intervals where f is concave up and where f is concave down, we first need to find all points x where $f''(x) = 0$ or $f''(x)$ is undefined. Since the numerator $6x^2 + 2 \neq 0$ for any x, $f''(x)$ is never zero. Furthermore, f'' is not undefined for any x in the domain of f. However, as discussed earlier, $x = \pm 1$ are not in the domain of f. Therefore, to determine the concavity of f, we divide the interval $(-\infty, \infty)$ into the three smaller intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$, and choose a test point in each of these intervals to evaluate the sign of $f''(x)$. The values $x = -2$, $x = 0$, and $x = 2$ are possible test points as shown in the following table.

Interval	Test Point	Sign of $f''(x) = \frac{6x^2+2}{(1-x^2)^3}$	Conclusion
$(-\infty, -1)$	$x = -2$	$+/- = -$	f is concave down.
$(-1, 1)$	$x = 0$	$+/+ = +$	f is concave up
$(1, \infty)$	$x = 2$	$+/- = -$	f is concave down.

Combining all this information, we arrive at the graph of f shown below. Note that, although f changes concavity at $x = -1$ and $x = 1$, there are no inflection points at either of these places because f is not continuous at $x = -1$ or $x = 1$.



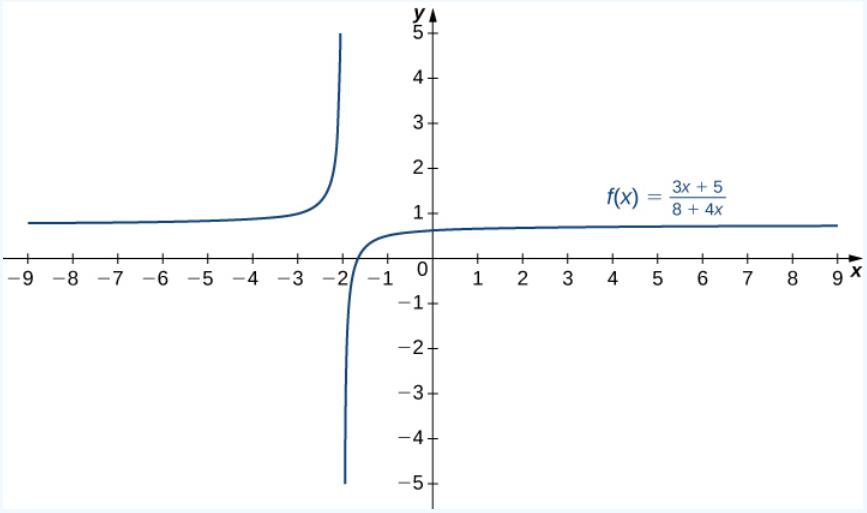
Exercise 4.6.9

Sketch a graph of $f(x) = \dfrac{3x+5}{8+4x}$.

Hint

A line $y = L$ is a horizontal asymptote of f if the limit as $x \to \infty$ or the limit as $x \to -\infty$ of $f(x)$ is L. A line $x = a$ is a vertical asymptote if at least one of the one-sided limits of f as $x \to a$ is ∞ or $-\infty$.

Answer



Example 4.6.10: Sketching a Rational Function with an Oblique Asymptote

Sketch the graph of $f(x) = \dfrac{x^2}{x-1}$

Solution

Step 1: The domain of f is the set of all real numbers x except $x = 1$.

Step 2: Find the intercepts. We can see that when $x = 0,\ f(x) = 0$, so $(0, 0)$ is the only intercept.

Step 3: Evaluate the limits at infinity. Since the degree of the numerator is one more than the degree of the denominator, f must have an oblique asymptote. To find the oblique asymptote, use long division of polynomials to write

$$f(x) = \frac{x^2}{x-1} = x + 1 + \frac{1}{x-1}.$$

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Since $\frac{1}{x-1} \to 0$ as $x \to \pm\infty$, $f(x)$ approaches the line $y = x + 1$ as $x \to \pm\infty$. The line $y = x + 1$ is an oblique asymptote for f.

Step 4: To check for vertical asymptotes, look at where the denominator is zero. Here the denominator is zero at $x = 1$. Looking at both one-sided limits as $x \to 1$, we find

$$\lim_{x\to 1^+} \frac{x^2}{x-1} = \infty \text{ and } \lim_{x\to 1^-} \frac{x^2}{x-1} = -\infty.$$

Therefore, $x = 1$ is a vertical asymptote, and we have determined the behavior of f as x approaches 1 from the right and the left.

Step 5: Calculate the first derivative:

$$f'(x) = \frac{(x-1)(2x) - x^2(1)}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2}.$$

We have $f'(x) = 0$ when $x^2 - 2x = x(x-2) = 0$. Therefore, $x = 0$ and $x = 2$ are critical points. Since f is undefined at $x = 1$, we need to divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, 0), (0, 1), (1, 2)$, and $(2, \infty)$, and choose a test point from each interval to evaluate the sign of $f'(x)$ in each of these smaller intervals. For example, let $x = -1, x = \frac{1}{2}, x = \frac{3}{2}$, and $x = 3$ be the test points as shown in the following table.

Interval	**Test Point**	**Sign of** $f'(x) = \frac{x^2-2x}{(x-1)^2}$	**Conclusion**
$(-\infty, 0)$	$x = -1$	$(-)(-)/+=+$	f is increasing.
$(0, 1)$	$x = 1/2$	$(+)(-)/+=-$	f is decreasing.
$(1, 2)$	$x = 3/2$	$(+)(-)/+=-$	f is decreasing.
$(2, \infty)$	$x = 3$	$(+)(+)/+=+$	f is increasing.

From this table, we see that f has a local maximum at $x = 0$ and a local minimum at $x = 2$. The value of f at the local maximum is $f(0) = 0$ and the value of f at the local minimum is $f(2) = 4$. Therefore, $(0, 0)$ and $(2, 4)$ are important points on the graph.

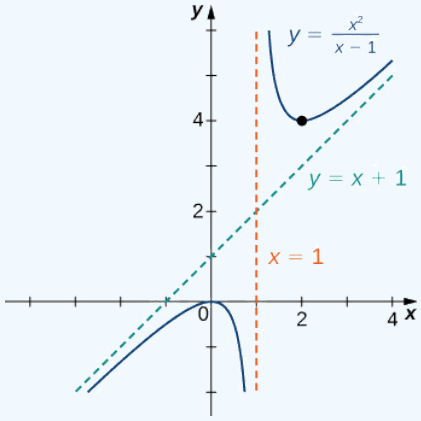
Step 6. Calculate the second derivative:

$$\begin{aligned} f''(x) &= \frac{(x-1)^2(2x-2) - 2(x-1)(x^2-2x)}{(x-1)^4} \\ &= \frac{2(x-1)[(x-1)^2 - (x^2-2x)]}{(x-1)^4} \\ &= \frac{2[x^2-2x+1-x^2+2x]}{(x-1)^3} \\ &= \frac{2}{(x-1)^3}. \end{aligned}$$

We see that $f''(x)$ is never zero or undefined for x in the domain of f. Since f is undefined at $x = 1$, to check concavity we just divide the interval $(-\infty, \infty)$ into the two smaller intervals $(-\infty, 1)$ and $(1, \infty)$, and choose a test point from each interval to evaluate the sign of $f''(x)$ in each of these intervals. The values $x = 0$ and $x = 2$ are possible test points as shown in the following table.

Interval	**Test Point**	**Sign of** $f''(x) = \frac{2}{(x-1)^3}$	**Conclusion**
$(-\infty, 1)$	$x = 0$	$+/- = -$	f is concave down.
$(1, \infty)$	$x = 2$	$+/+ = +$	f is concave up

From the information gathered, we arrive at the following graph for f.



Exercise 4.6.10

Find the oblique asymptote for $f(x) = \dfrac{3x^3 - 2x + 1}{2x^2 - 4}$.

Hint

Use long division of polynomials.

Answer

$y = \frac{3}{2}x$

Example 4.6.11: Sketching the Graph of a Function with a Cusp

Sketch a graph of $f(x) = (x-1)^{2/3}$

Solution

Step 1: Since the cube-root function is defined for all real numbers x and $(x-1)^{2/3} = (\sqrt[3]{x-1})^2$, the domain of f is all real numbers.

Step 2: To find the y-intercept, evaluate $f(0)$. Since $f(0) = 1$, the y-intercept is $(0, 1)$. To find the x-intercept, solve $(x-1)^{2/3} = 0$. The solution of this equation is $x = 1$, so the x-intercept is $(1, 0)$.

Step 3: Since $\lim\limits_{x \to \pm\infty} (x-1)^{2/3} = \infty$, the function continues to grow without bound as $x \to \infty$ and $x \to -\infty$.

Step 4: The function has no vertical asymptotes.

Step 5: To determine where f is increasing or decreasing, calculate f'. We find

$$f'(x) = \frac{2}{3}(x-1)^{-1/3} = \frac{2}{3(x-1)^{1/3}}$$

This function is not zero anywhere, but it is undefined when $x = 1$. Therefore, the only critical point is $x = 1$. Divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, 1)$ and $(1, \infty)$, and choose test points in each of these intervals to determine the sign of $f'(x)$ in each of these smaller intervals. Let $x = 0$ and $x = 2$ be the test points as shown in the following table.

Interval	Test Point	Sign of $f'(x) = \frac{2}{3(x-1)^{1/3}}$	Conclusion
$(-\infty, 1)$	$x = 0$	$+/- = -$	f is decreasing
$(1, \infty)$	$x = 2$	$+/+ = +$	f is increasing

We conclude that f has a local minimum at $x = 1$. Evaluating f at $x = 1$, we find that the value of f at the local minimum is zero. Note that $f'(1)$ is undefined, so to determine the behavior of the function at this critical point, we need to examine $\lim\limits_{x \to 1} f'(x)$. Looking at the one-sided limits, we have

$$\lim_{x\to 1^+}\frac{2}{3(x-1)^{1/3}}=\infty \text{ and } \lim_{x\to 1^-}\frac{2}{3(x-1)^{1/3}}=-\infty.$$

Therefore, f has a cusp at $x=1$.

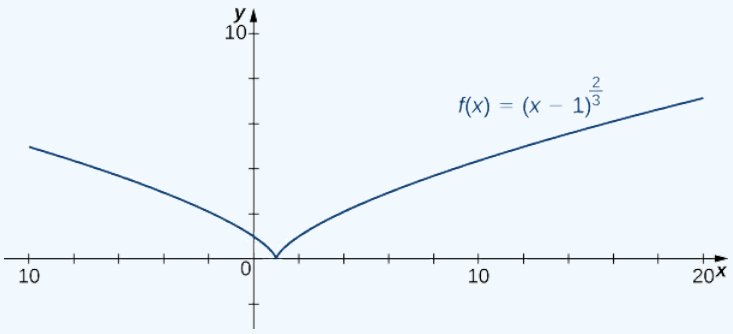
Step 6: To determine concavity, we calculate the second derivative of f :

$$f''(x)=-\frac{2}{9}(x-1)^{-4/3}=\frac{-2}{9(x-1)^{4/3}}.$$

We find that $f''(x)$ is defined for all x, but is undefined when $x=1$. Therefore, divide the interval $(-\infty,\infty)$ into the smaller intervals $(-\infty,1)$ and $(1,\infty)$, and choose test points to evaluate the sign of $f''(x)$ in each of these intervals. As we did earlier, let $x=0$ and $x=2$ be test points as shown in the following table.

Interval	Test Point	Sign of $f''(x)=\frac{-2}{9(x-1)^{4/3}}$	Conclusion
$(-\infty,1)$	$x=0$	$-/+=-$	f is concave down
$(1,\infty)$	$x=2$	$-/+=-$	f is concave down

From this table, we conclude that f is concave down everywhere. Combining all of this information, we arrive at the following graph for f.



? Exercise 4.6.11

Consider the function $f(x)=5-x^{2/3}$. Determine the point on the graph where a cusp is located. Determine the end behavior of f.

Hint

A function f has a cusp at a point a if $f(a)$ exists, $f'(a)$ is undefined, one of the one-sided limits as $x\to a$ of $f'(x)$ is $+\infty$, and the other one-sided limit is $-\infty$.

Answer

The function f has a cusp at $(0,5)$, since $\lim_{x\to 0^-}f'(x)=\infty$ and $\lim_{x\to 0^+}f'(x)=-\infty$. For end behavior, $\lim_{x\to\pm\infty}f(x)=-\infty$.

Key Concepts

- The limit of $f(x)$ is L as $x\to\infty$ (or as $x\to-\infty$) if the values $f(x)$ become arbitrarily close to L as x becomes sufficiently large.
- The limit of $f(x)$ is ∞ as $x\to\infty$ if $f(x)$ becomes arbitrarily large as x becomes sufficiently large. The limit of $f(x)$ is $-\infty$ as $x\to\infty$ if $f(x)<0$ and $|f(x)|$ becomes arbitrarily large as x becomes sufficiently large. We can define the limit of $f(x)$ as x approaches $-\infty$ similarly.
- For a polynomial function $p(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$, where $a_n\neq 0$, the end behavior is determined by the leading term a_nx^n. If $n\neq 0, p(x)$ approaches ∞ or $-\infty$at each end.
- For a rational function $f(x)=\frac{p(x)}{q(x),}$ the end behavior is determined by the relationship between the degree of p and the degree of q. If the degree of p is less than the degree of q, the line $y=0$ is a horizontal asymptote for f. If the degree of p is equal to the

degree of q, then the line $y = \frac{a_n}{b_n}$ is a horizontal asymptote, where a_n and b_n are the leading coefficients of p and q, respectively. If the degree of p is greater than the degree of q, then f approaches ∞ or $-\infty$ at each end.

Glossary

end behavior

the behavior of a function as $x \to \infty$ and $x \to -\infty$

horizontal asymptote

if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then $y = L$ is a horizontal asymptote of f

infinite limit at infinity

a function that becomes arbitrarily large as x becomes large

limit at infinity

a function that approaches a limit value L as x becomes large

oblique asymptote

the line $y = mx + b$ if $f(x)$ approaches it as $x \to \infty$ or$x \to -\infty$

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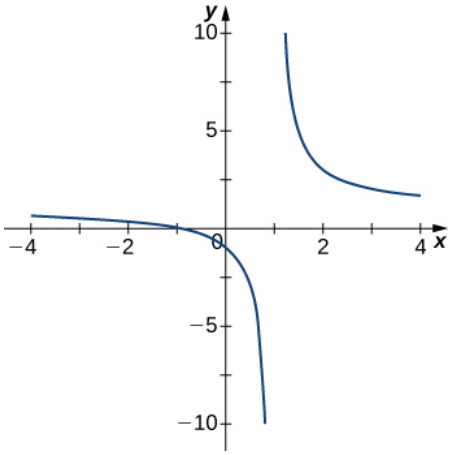
4.6E: EXERCISES FOR SECTION 4.6

For exercises 1 - 5, examine the graphs. Identify where the vertical asymptotes are located.

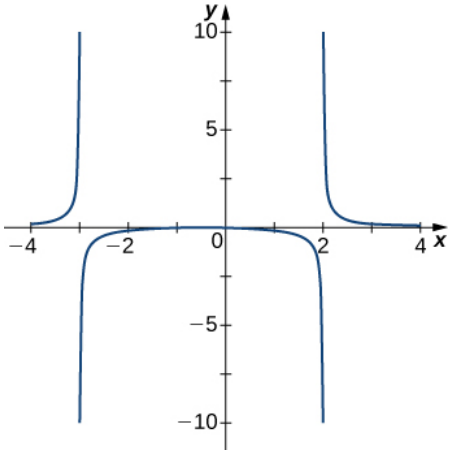
1)



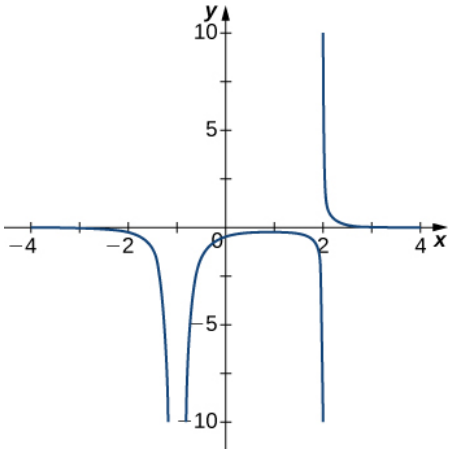
Answer

$x = 1$

2)



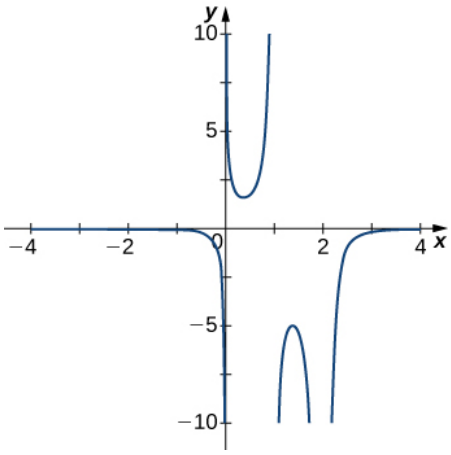
3)



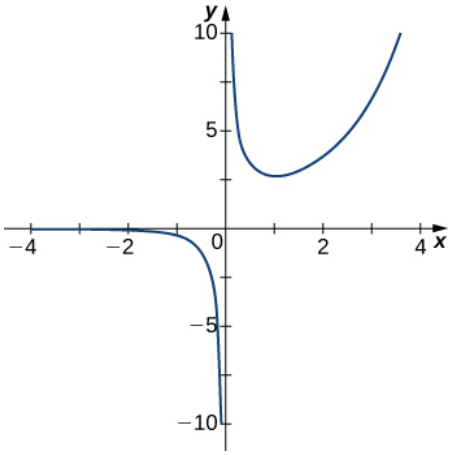
Answer

$x = -1,\ x = 2$

4)



5)



Answer

$x = 0$

For the functions $f(x)$ in exercises 6 - 10, determine whether there is an asymptote at $x = a$. Justify your answer without graphing on a calculator.

6) $f(x) = \dfrac{x+1}{x^2+5x+4},\quad a = -1$

7) $f(x) = \dfrac{x}{x-2},\quad a = 2$

Answer

Yes, there is a vertical asymptote at $x = 2$.

8) $f(x) = (x+2)^{3/2},\quad a = -2$

9) $f(x) = (x-1)^{-1/3},\quad a = 1$

Answer

Yes, there is vertical asymptote at $x = 1$.

10) $f(x) = 1 + x^{-2/5},\quad a = 1$

In exercises 11 - 20, evaluate the limit.

11) $\lim\limits_{x\to\infty} \dfrac{1}{3x+6}$

Answer

$$\lim_{x\to\infty}\frac{1}{3x+6}=0$$

12) $\lim_{x\to\infty}\frac{2x-5}{4x}$

13) $\lim_{x\to\infty}\frac{x^2-2x+5}{x+2}$

Answer

$$\lim_{x\to\infty}\frac{x^2-2x+5}{x+2}=\infty$$

14) $\lim_{x\to-\infty}\frac{3x^3-2x}{x^2+2x+8}$

15) $\lim_{x\to-\infty}\frac{x^4-4x^3+1}{2-2x^2-7x^4}$

Answer

$$\lim_{x\to-\infty}\frac{x^4-4x^3+1}{2-2x^2-7x^4}=-\frac{1}{7}$$

16) $\lim_{x\to\infty}\frac{3x}{\sqrt{x^2+1}}$

17) $\lim_{x\to-\infty}\frac{\sqrt{4x^2-1}}{x+2}$

Answer

$$\lim_{x\to-\infty}\frac{\sqrt{4x^2-1}}{x+2}=-2$$

18) $\lim_{x\to\infty}\frac{4x}{\sqrt{x^2-1}}$

19) $\lim_{x\to-\infty}\frac{4x}{\sqrt{x^2-1}}$

Answer

$$\lim_{x\to-\infty}\frac{4x}{\sqrt{x^2-1}}=-4$$

20) $\lim_{x\to\infty}\frac{2\sqrt{x}}{x-\sqrt{x}+1}$

For exercises 21 - 25, find the horizontal and vertical asymptotes.

21) $f(x)=x-\frac{9}{x}$

Answer

Horizontal: none,
Vertical: $x=0$

22) $f(x)=\frac{1}{1-x^2}$

23) $f(x)=\frac{x^3}{4-x^2}$

Answer

Horizontal: none,
Vertical: $x=\pm 2$

24) $f(x)=\frac{x^2+3}{x^2+1}$

25) $f(x)=\sin(x)\sin(2x)$

Answer

Horizontal: none,
Vertical: none

26) $f(x) = \cos x + \cos(3x) + \cos(5x)$

27) $f(x) = \dfrac{x \sin(x)}{x^2 - 1}$

Answer

Horizontal: $y = 0$,
Vertical: $x = \pm 1$

28) $f(x) = \dfrac{x}{\sin(x)}$

29) $f(x) = \dfrac{1}{x^3 + x^2}$

Answer

Horizontal: $y = 0$,
Vertical: $x = 0$ and $x = -1$

30) $f(x) = \dfrac{1}{x - 1} - 2x$

31) $f(x) = \dfrac{x^3 + 1}{x^3 - 1}$

Answer

Horizontal: $y = 1$,
Vertical: $x = 1$

32) $f(x) = \dfrac{\sin x + \cos x}{\sin x - \cos x}$

33) $f(x) = x - \sin x$

Answer

Horizontal: none,
Vertical: none

34) $f(x) = \dfrac{1}{x} - \sqrt{x}$

For exercises 35 - 38, construct a function $f(x)$ that has the given asymptotes.

35) $x = 1$ and $y = 2$

Answer

Answers will vary, for example: $y = \dfrac{2x}{x - 1}$

36) $x = 1$ and $y = 0$

37) $y = 4$, $x = -1$

Answer

Answers will vary, for example: $y = \dfrac{4x}{x + 1}$

38) $x = 0$

In exercises 39 - 43, graph the function on a graphing calculator on the window $x = [-5, 5]$ and estimate the horizontal asymptote or limit. Then, calculate the actual horizontal asymptote or limit.

39) [T] $f(x) = \dfrac{1}{x + 10}$

Answer

$\lim\limits_{x \to \infty} \dfrac{1}{x + 10} = 0$ so f has a horizontal asymptote of $y = 0$.

40) [T] $f(x) = \dfrac{x+1}{x^2+7x+6}$

41) [T] $\lim\limits_{x \to -\infty} x^2 + 10x + 25$

Answer

$\lim\limits_{x \to -\infty} x^2 + 10x + 25 = \infty$

42) [T] $\lim\limits_{x \to -\infty} \dfrac{x+2}{x^2+7x+6}$

43) [T] $\lim\limits_{x \to \infty} \dfrac{3x+2}{x+5}$

Answer

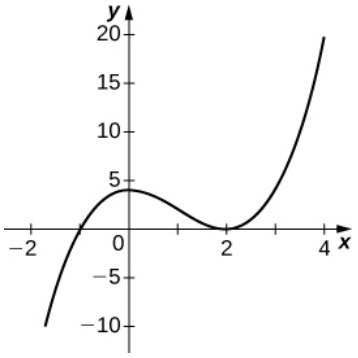
$\lim\limits_{x \to \infty} \dfrac{3x+2}{x+5} = 3$ so this function has a horizontal asymptote of $y = 3$.

In exercises 44 - 55, draw a graph of the functions without using a calculator. Be sure to notice all important features of the graph: local maxima and minima, inflection points, and asymptotic behavior.

44) $y = 3x^2 + 2x + 4$

45) $y = x^3 - 3x^2 + 4$

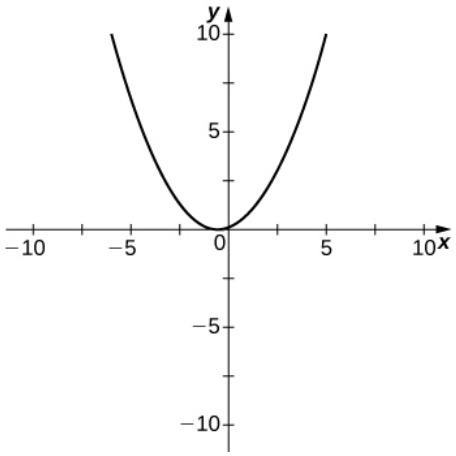
Answer



46) $y = \dfrac{2x+1}{x^2+6x+5}$

47) $y = \dfrac{x^3+4x^2+3x}{3x+9}$

Answer



48) $y = \dfrac{x^2+x-2}{x^2-3x-4}$

49) $y = \sqrt{x^2-5x+4}$

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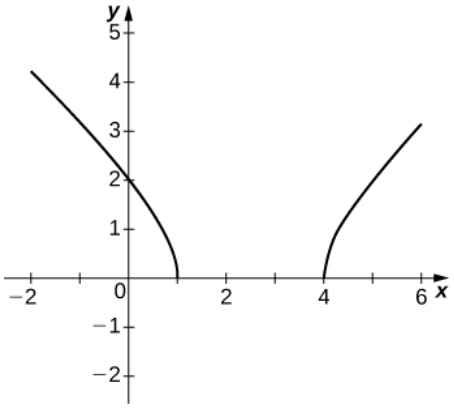
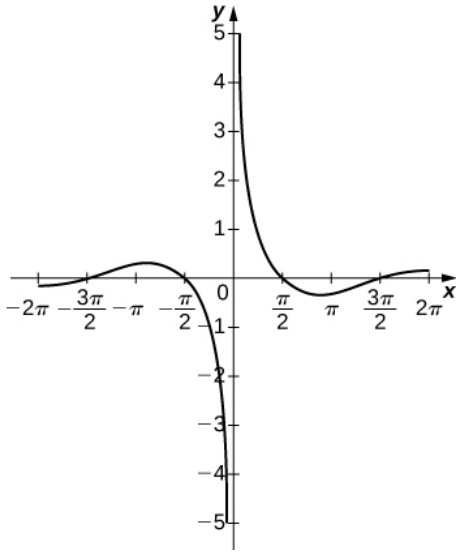
Answer



50) $y = 2x\sqrt{16 - x^2}$

51) $y = \dfrac{\cos x}{x}$, on $x = [-2\pi, 2\pi]$

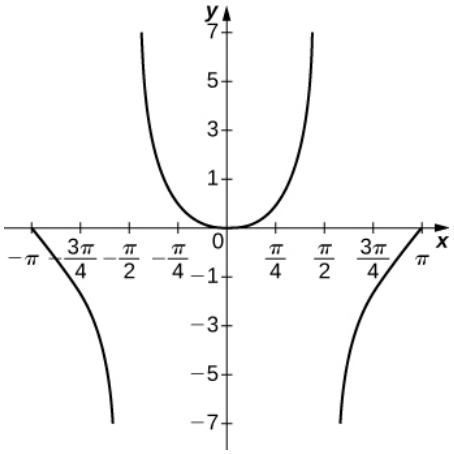
Answer



52) $y = e^x - x^3$

53) $y = x \tan x, \quad x = [-\pi, \pi]$

Answer



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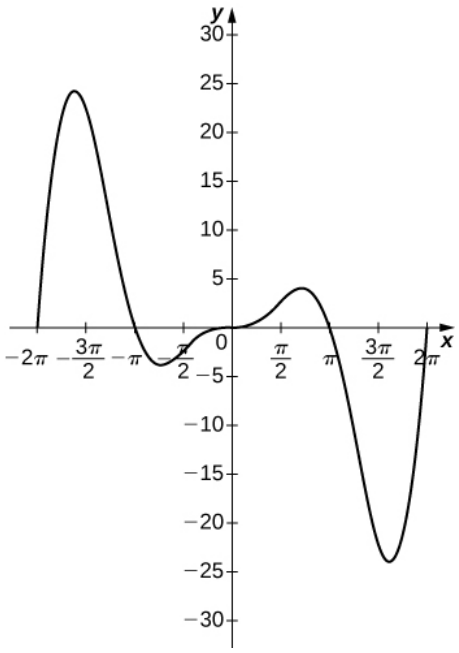
54) $y = x\ln(x), \quad x > 0$

55) $y = x^2\sin(x), \quad x = [-2\pi, 2\pi]$

Answer



56) For $f(x) = \dfrac{P(x)}{Q(x)}$ to have an asymptote at $y = 2$ then the polynomials $P(x)$ and $Q(x)$ must have what relation?

57) For $f(x) = \dfrac{P(x)}{Q(x)}$ to have an asymptote at $x = 0$, then the polynomials $P(x)$ and $Q(x)$. must have what relation?

Answer

$Q(x)$. must have have x^{k+1} as a factor, where $P(x)$ has x^k as a factor.

58) If $f'(x)$ has asymptotes at $y = 3$ and $x = 1$, then $f(x)$ has what asymptotes?

59) Both $f(x) = \dfrac{1}{x-1}$ and $g(x) = \dfrac{1}{(x-1)^2}$ have asymptotes at $x = 1$ and $y = 0$. What is the most obvious difference between these two functions?

Answer

$\lim\limits_{x\to 1^-} f(x) = -\infty$ and $\lim\limits_{x\to 1^-} g(x) = \infty$

60) True or false: Every ratio of polynomials has vertical asymptotes.

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4.7: Applied Optimization Problems

Learning Objectives

- Set up and solve optimization problems in several applied fields.

One common application of calculus is calculating the minimum or maximum value of a function. For example, companies often want to minimize production costs or maximize revenue. In manufacturing, it is often desirable to minimize the amount of material used to package a product with a certain volume. In this section, we show how to set up these types of minimization and maximization problems and solve them by using the tools developed in this chapter.

Solving Optimization Problems over a Closed, Bounded Interval

The basic idea of the optimization problems that follow is the same. We have a particular quantity that we are interested in maximizing or minimizing. However, we also have some auxiliary condition that needs to be satisfied. For example, in Example 4.7.1, we are interested in maximizing the area of a rectangular garden. Certainly, if we keep making the side lengths of the garden larger, the area will continue to become larger. However, what if we have some restriction on how much fencing we can use for the perimeter? In this case, we cannot make the garden as large as we like. Let's look at how we can maximize the area of a rectangle subject to some constraint on the perimeter.

Example 4.7.1: Maximizing the Area of a Garden

A rectangular garden is to be constructed using a rock wall as one side of the garden and wire fencing for the other three sides (Figure 4.7.1). Given $100\,\text{ft}$ of wire fencing, determine the dimensions that would create a garden of maximum area. What is the maximum area?

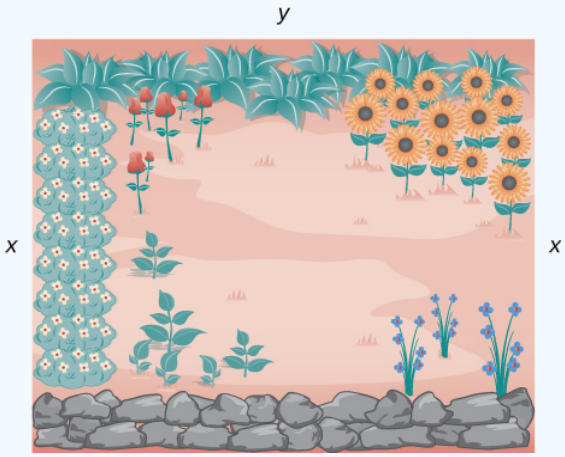

Figure 4.7.1: We want to determine the measurements x and y that will create a garden with a maximum area using $100\,\text{ft}$ of fencing.

Solution

Let x denote the length of the side of the garden perpendicular to the rock wall and y denote the length of the side parallel to the rock wall. Then the area of the garden is

$$A = x \cdot y.$$

We want to find the maximum possible area subject to the constraint that the total fencing is $100\,\text{ft}$. From Figure 4.7.1, the total amount of fencing used will be $2x + y$. Therefore, the constraint equation is

$$2x + y = 100.$$

Solving this equation for y, we have $y = 100 - 2x$. Thus, we can write the area as

$$A(x) = x \cdot (100 - 2x) = 100x - 2x^2.$$

Before trying to maximize the area function $A(x) = 100x - 2x^2$, we need to determine the domain under consideration. To construct a rectangular garden, we certainly need the lengths of both sides to be positive. Therefore, we need $x > 0$ and $y > 0$. Since $y = 100 - 2x$, if $y > 0$, then $x < 50$. Therefore, we are trying to determine the maximum value of $A(x)$ for x over the open interval $(0, 50)$. We do not know that a function necessarily has a maximum value over an open interval. However, we do know that a continuous function has an absolute maximum (and absolute minimum) over a closed interval. Therefore, let's consider the function $A(x) = 100x - 2x^2$ over the closed interval $[0, 50]$. If the maximum value occurs at an interior point, then we have found the value x in the open interval $(0, 50)$ that maximizes the area of the garden.

Therefore, we consider the following problem:

Maximize $A(x) = 100x - 2x^2$ over the interval $[0, 50]$.

As mentioned earlier, since A is a continuous function on a closed, bounded interval, by the extreme value theorem, it has a maximum and a minimum. These extreme values occur either at endpoints or critical points. At the endpoints, $A(x) = 0$. Since the area is positive for all x in the open interval $(0, 50)$, the maximum must occur at a critical point. Differentiating the function $A(x)$, we obtain

$$A'(x) = 100 - 4x.$$

Therefore, the only critical point is $x = 25$ (Figure 4.7.2). We conclude that the maximum area must occur when $x = 25$.

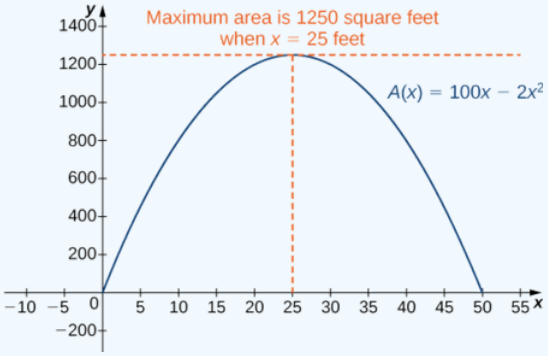

Figure 4.7.2: To maximize the area of the garden, we need to find the maximum value of the function $A(x) = 100x - 2x^2$.

Then we have $y = 100 - 2x = 100 - 2(25) = 50$. To maximize the area of the garden, let $x = 25\,\text{ft}$ and $y = 50\,\text{ft}$. The area of this garden is $1250\,\text{ft}^2$.

? Exercise 4.7.1

Determine the maximum area if we want to make the same rectangular garden as in Figure 4.7.2, but we have $200\,\text{ft}$ of fencing.

Hint

We need to maximize the function $A(x) = 200x - 2x^2$ over the interval $[0, 100]$.

Answer

The maximum area is $5000\,\text{ft}^2$.

Now let's look at a general strategy for solving optimization problems similar to Example 4.7.1.

Problem-Solving Strategy: Solving Optimization Problems

1. Introduce all variables. If applicable, draw a figure and label all variables.
2. Determine which quantity is to be maximized or minimized, and for what range of values of the other variables (if this can be determined at this time).
3. Write a formula for the quantity to be maximized or minimized in terms of the variables. This formula may involve more than one variable.
4. Write any equations relating the independent variables in the formula from step 3. Use these equations to write the quantity to be maximized or minimized as a function of one variable.

5. Identify the domain of consideration for the function in step 4 based on the physical problem to be solved.
6. Locate the maximum or minimum value of the function from step 4. This step typically involves looking for critical points and evaluating a function at endpoints.

Now let's apply this strategy to maximize the volume of an open-top box given a constraint on the amount of material to be used.

Example 4.7.2: Maximizing the Volume of a Box

An open-top box is to be made from a $24\,\text{in.}$ by $36\,\text{in.}$ piece of cardboard by removing a square from each corner of the box and folding up the flaps on each side. What size square should be cut out of each corner to get a box with the maximum volume?

Solution

Step 1: Let x be the side length of the square to be removed from each corner (Figure 4.7.3). Then, the remaining four flaps can be folded up to form an open-top box. Let V be the volume of the resulting box.

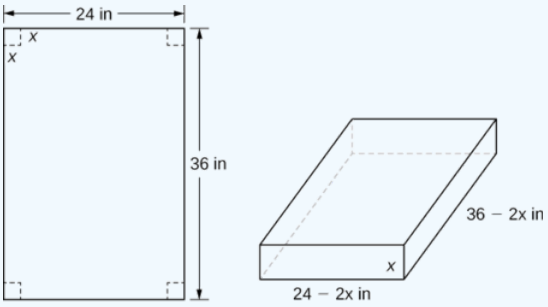

Figure 4.7.3: A square with side length x inches is removed from each corner of the piece of cardboard. The remaining flaps are folded to form an open-top box.

Step 2: We are trying to maximize the volume of a box. Therefore, the problem is to maximize V.

Step 3: As mentioned in step 2, are trying to maximize the volume of a box. The volume of a box is

$$V = L \cdot W \cdot H,$$

where L, W,and H are the length, width, and height, respectively.

Step 4: From Figure 4.7.3, we see that the height of the box is x inches, the length is $36 - 2x$ inches, and the width is $24 - 2x$ inches. Therefore, the volume of the box is

$$\begin{aligned} V(x) &= (36 - 2x)(24 - 2x)x \\ &= 4x^3 - 120x^2 + 864x \end{aligned}.$$

Step 5: To determine the domain of consideration, let's examine Figure 4.7.3. Certainly, we need $x > 0$. Furthermore, the side length of the square cannot be greater than or equal to half the length of the shorter side, $24\,\text{in.}$; otherwise, one of the flaps would be completely cut off. Therefore, we are trying to determine whether there is a maximum volume of the box for x over the open interval $(0, 12)$. Since V is a continuous function over the closed interval $[0, 12]$, we know V will have an absolute maximum over the closed interval. Therefore, we consider V over the closed interval $[0, 12]$ and check whether the absolute maximum occurs at an interior point.

Step 6: Since $V(x)$ is a continuous function over the closed, bounded interval $[0, 12]$, V must have an absolute maximum (and an absolute minimum). Since $V(x) = 0$ at the endpoints and $V(x) > 0$ for $0 < x < 12$, the maximum must occur at a critical point. The derivative is

$$V'(x) = 12x^2 - 240x + 864.$$

To find the critical points, we need to solve the equation

$$12x^2 - 240x + 864 = 0.$$

Dividing both sides of this equation by 12, the problem simplifies to solving the equation

$$x^2 - 20x + 72 = 0.$$

Using the quadratic formula, we find that the critical points are

$$\begin{aligned} x &= \frac{20 \pm \sqrt{(-20)^2 - 4(1)(72)}}{2} \\ &= \frac{20 \pm \sqrt{112}}{2} \\ &= \frac{20 \pm 4\sqrt{7}}{2} \\ &= 10 \pm 2\sqrt{7} \end{aligned}.$$

Since $10 + 2\sqrt{7}$ is not in the domain of consideration, the only critical point we need to consider is $10 - 2\sqrt{7}$. Therefore, the volume is maximized if we let $x = 10 - 2\sqrt{7}$ in. The maximum volume is

$$V(10 - 2\sqrt{7}) = 640 + 448\sqrt{7} \approx 1825 \text{ in}^3.$$

as shown in the following graph.

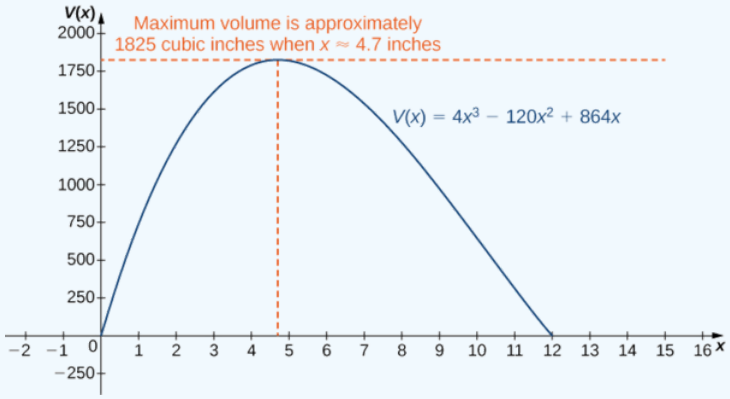

Figure 4.7.4: Maximizing the volume of the box leads to finding the maximum value of a cubic polynomial.

Exercise 4.7.2

Suppose the dimensions of the cardboard in Example 4.7.2 are 20 in. by 30 in. Let x be the side length of each square and write the volume of the open-top box as a function of x. Determine the domain of consideration for x.

Hint

The volume of the box is $L \cdot W \cdot H$.

Answer

$V(x) = x(20 - 2x)(30 - 2x)$. The domain is $[0, 10]$.

Example 4.7.3: Minimizing Travel Time

An island is 2 mi due north of its closest point along a straight shoreline. A visitor is staying at a cabin on the shore that is 6 mi west of that point. The visitor is planning to go from the cabin to the island. Suppose the visitor runs at a rate of 8 mph and swims at a rate of 3 mph. How far should the visitor run before swimming to minimize the time it takes to reach the island?

Solution

Step 1: Let x be the distance running and let y be the distance swimming (Figure 4.7.5). Let T be the time it takes to get from the cabin to the island.

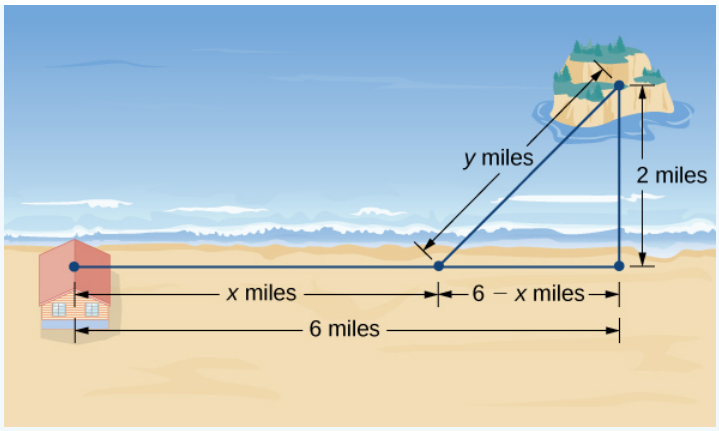

Figure 4.7.5: How can we choose x and y to minimize the travel time from the cabin to the island?

Step 2: The problem is to minimize T.

Step 3: To find the time spent traveling from the cabin to the island, add the time spent running and the time spent swimming. Since Distance = Rate × Time $(D = R \times T)$, the time spent running is

$$T_{running} = \frac{D_{running}}{R_{running}} = \frac{x}{8},$$

and the time spent swimming is

$$T_{swimming} = \frac{D_{swimming}}{R_{swimming}} = \frac{y}{3}.$$

Therefore, the total time spent traveling is

$$T = \frac{x}{8} + \frac{y}{3}.$$

Step 4: From Figure 4.7.5, the line segment of y miles forms the hypotenuse of a right triangle with legs of length 2 mi and $6 - x$ mi. Therefore, by the Pythagorean theorem, $2^2 + (6-x)^2 = y^2$, and we obtain $y = \sqrt{(6-x)^2 + 4}$. Thus, the total time spent traveling is given by the function

$$T(x) = \frac{x}{8} + \frac{\sqrt{(6-x)^2 + 4}}{3}.$$

Step 5: From Figure 4.7.5, we see that $0 \le x \le 6$. Therefore, $[0, 6]$ is the domain of consideration.

Step 6: Since $T(x)$ is a continuous function over a closed, bounded interval, it has a maximum and a minimum. Let's begin by looking for any critical points of T over the interval $[0, 6]$. The derivative is

$$\begin{aligned} T'(x) &= \frac{1}{8} - \frac{1}{2}\frac{[(6-x)^2 + 4]^{-1/2}}{3} \cdot 2(6-x) \\ &= \frac{1}{8} - \frac{(6-x)}{3\sqrt{(6-x)^2 + 4}} \end{aligned}$$

If $T'(x) = 0$,, then

$$\frac{1}{8} = \frac{6-x}{3\sqrt{(6-x)^2 + 4}} \tag{4.7.1}$$

Therefore,

$$3\sqrt{(6-x)^2 + 4} = 8(6-x). \tag{4.7.2}$$

Squaring both sides of this equation, we see that if x satisfies this equation, then x must satisfy

$$9[(6-x)^2 + 4] = 64(6-x)^2,$$

which implies

$$55(6-x)^2 = 36.$$

We conclude that if x is a critical point, then x satisfies

$$(x-6)^2 = \frac{36}{55}.$$

[Note that since we are squaring, $(x-6)^2 = (6-x)^2$.]

Therefore, the possibilities for critical points are

$$x = 6 \pm \frac{6}{\sqrt{55}}.$$

Since $x = 6 + 6/\sqrt{55}$ is not in the domain, it is not a possibility for a critical point. On the other hand, $x = 6 - 6/\sqrt{55}$ is in the domain. Since we squared both sides of Equation 4.7.2 to arrive at the possible critical points, it remains to verify that $x = 6 - 6/\sqrt{55}$ satisfies Equation 4.7.1. Since $x = 6 - 6/\sqrt{55}$ does satisfy that equation, we conclude that $x = 6 - 6/\sqrt{55}$ is a critical point, and it is the only one. To justify that the time is minimized for this value of x, we just need to check the values of $T(x)$ at the endpoints $x = 0$ and $x = 6$, and compare them with the value of $T(x)$ at the critical point $x = 6 - 6/\sqrt{55}$. We find that $T(0) \approx 2.108\,\text{h}$ and $T(6) \approx 1.417\,\text{h}$, whereas

$$T(6 - 6/\sqrt{55}) \approx 1.368\,\text{h}.$$

Therefore, we conclude that T has a local minimum at $x \approx 5.19$ mi.

? Exercise 4.7.3

Suppose the island is 1 mi from shore, and the distance from the cabin to the point on the shore closest to the island is 15 mi. Suppose a visitor swims at the rate of 2.5 mph and runs at a rate of 6 mph. Let x denote the distance the visitor will run before swimming, and find a function for the time it takes the visitor to get from the cabin to the island.

Hint

The time $T = T_{running} + T_{swimming}$.

Answer

$$T(x) = \frac{x}{6} + \frac{\sqrt{(15-x)^2+1}}{2.5}$$

In business, companies are interested in **maximizing revenue.** In the following example, we consider a scenario in which a company has collected data on how many cars it is able to lease, depending on the price it charges its customers to rent a car. Let's use these data to determine the price the company should charge to maximize the amount of money it brings in.

✓ Example 4.7.4: Maximizing Revenue

Owners of a car rental company have determined that if they charge customers p dollars per day to rent a car, where $50 \le p \le 200$, the number of cars n they rent per day can be modeled by the linear function $n(p) = 1000 - 5p$. If they charge $\$50$ per day or less, they will rent all their cars. If they charge $\$200$ per day or more, they will not rent any cars. Assuming the owners plan to charge customers between $\$50$ per day and $\$200$ per day to rent a car, how much should they charge to maximize their revenue?

Solution

Step 1: Let p be the price charged per car per day and let n be the number of cars rented per day. Let R be the revenue per day.

Step 2: The problem is to maximize R.

Step 3: The revenue (per day) is equal to the number of cars rented per day times the price charged per car per day—that is, $R = n \times p$.

Step 4: Since the number of cars rented per day is modeled by the linear function $n(p) = 1000 - 5p$, the revenue R can be represented by the function

$$\begin{aligned} R(p) &= n \times p \\ &= (1000 - 5p)p \\ &= -5p^2 + 1000p. \end{aligned}$$

Step 5: Since the owners plan to charge between $\$50$ per car per day and $\$200$ per car per day, the problem is to find the maximum revenue $R(p)$ for p in the closed interval $[50, 200]$.

Step 6: Since R is a continuous function over the closed, bounded interval $[50, 200]$, it has an absolute maximum (and an absolute minimum) in that interval. To find the maximum value, look for critical points. The derivative is $R'(p) = -10p + 1000$. Therefore, the critical point is $p = 100$. When $p = 100, R(100) = \$50,000$. When $p = 50, R(p) = \$37,500$. When $p = 200, R(p) = \$0$.

Therefore, the absolute maximum occurs at $p = \$100$. The car rental company should charge $\$100$ per day per car to maximize revenue as shown in the following figure.

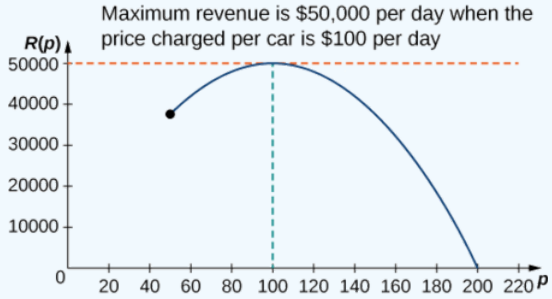

Figure 4.7.6: To maximize revenue, a car rental company has to balance the price of a rental against the number of cars people will rent at that price.

Exercise 4.7.4

A car rental company charges its customers p dollars per day, where $60 \leq p \leq 150$. It has found that the number of cars rented per day can be modeled by the linear function $n(p) = 750 - 5p$. How much should the company charge each customer to maximize revenue?

Hint

$R(p) = n \times p$, where n is the number of cars rented and p is the price charged per car.

Answer

The company should charge $\$75$ per car per day.

Example 4.7.5: Maximizing the Area of an Inscribed Rectangle

A rectangle is to be inscribed in the ellipse

$$\frac{x^2}{4} + y^2 = 1.$$

What should the dimensions of the rectangle be to maximize its area? What is the maximum area?

Solution

Step 1: For a rectangle to be inscribed in the ellipse, the sides of the rectangle must be parallel to the axes. Let L be the length of the rectangle and W be its width. Let A be the area of the rectangle.

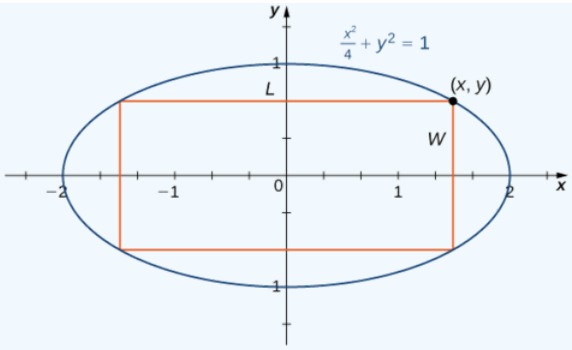

Figure 4.7.7: We want to maximize the area of a rectangle inscribed in an ellipse.

Step 2: The problem is to maximize A.

Step 3: The area of the rectangle is $A = LW$.

Step 4: Let (x, y) be the corner of the rectangle that lies in the first quadrant, as shown in Figure 4.7.7. We can write length $L = 2x$ and width $W = 2y$. Since $\frac{x^2}{4} + y^2 = 1$ and $y > 0$, we have $y = \sqrt{1 - \frac{x^2}{4}}$. Therefore, the area is

$$A = LW = (2x)(2y) = 4x\sqrt{1 - \frac{x^2}{4}} = 2x\sqrt{4 - x^2}$$

Step 5: From Figure 4.7.7, we see that to inscribe a rectangle in the ellipse, the x-coordinate of the corner in the first quadrant must satisfy $0 < x < 2$. Therefore, the problem reduces to looking for the maximum value of $A(x)$ over the open interval $(0, 2)$. Since $A(x)$ will have an absolute maximum (and absolute minimum) over the closed interval $[0, 2]$, we consider $A(x) = 2x\sqrt{4 - x^2}$ over the interval $[0, 2]$. If the absolute maximum occurs at an interior point, then we have found an absolute maximum in the open interval.

Step 6: As mentioned earlier, $A(x)$ is a continuous function over the closed, bounded interval $[0, 2]$. Therefore, it has an absolute maximum (and absolute minimum). At the endpoints $x = 0$ and $x = 2$, $A(x) = 0$. For $0 < x < 2$, $A(x) > 0$.

Therefore, the maximum must occur at a critical point. Taking the derivative of $A(x)$, we obtain

$$\begin{aligned} A'(x) &= 2\sqrt{4 - x^2} + 2x \cdot \frac{1}{2\sqrt{4 - x^2}}(-2x) \\ &= 2\sqrt{4 - x^2} - \frac{2x^2}{\sqrt{4 - x^2}} \\ &= \frac{8 - 4x^2}{\sqrt{4 - x^2}}. \end{aligned}$$

To find critical points, we need to find where $A'(x) = 0$. We can see that if x is a solution of

$$\frac{8 - 4x^2}{\sqrt{4 - x^2}} = 0, \tag{4.7.3}$$

then x must satisfy

$$8 - 4x^2 = 0.$$

Therefore, $x^2 = 2$. Thus, $x = \pm\sqrt{2}$ are the possible solutions of Equation 4.7.3. Since we are considering x over the interval $[0, 2]$, $x = \sqrt{2}$ is a possibility for a critical point, but $x = -\sqrt{2}$ is not. Therefore, we check whether $\sqrt{2}$ is a solution of Equation 4.7.3. Since $x = \sqrt{2}$ is a solution of Equation 4.7.3, we conclude that $\sqrt{2}$ is the only critical point of $A(x)$ in the interval $[0, 2]$.

Therefore, $A(x)$ must have an absolute maximum at the critical point $x = \sqrt{2}$. To determine the dimensions of the rectangle, we need to find the length L and the width W. If $x = \sqrt{2}$ then

$$y = \sqrt{1 - \frac{(\sqrt{2})^2}{4}} = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}}.$$

Therefore, the dimensions of the rectangle are $L = 2x = 2\sqrt{2}$ and $W = 2y = \dfrac{2}{\sqrt{2}} = \sqrt{2}$. The area of this rectangle is $A = LW = (2\sqrt{2})(\sqrt{2}) = 4$.

Exercise 4.7.5

Modify the area function A if the rectangle is to be inscribed in the unit circle $x^2 + y^2 = 1$. What is the domain of consideration?

Hint

If (x, y) is the vertex of the square that lies in the first quadrant, then the area of the square is $A = (2x)(2y) = 4xy$.

Answer

$A(x) = 4x\sqrt{1 - x^2}$. The domain of consideration is $[0, 1]$.

Solving Optimization Problems when the Interval Is Not Closed or Is Unbounded

In the previous examples, we considered functions on closed, bounded domains. Consequently, by the extreme value theorem, we were guaranteed that the functions had absolute extrema. Let's now consider functions for which the domain is neither closed nor bounded.

Many functions still have at least one absolute extrema, even if the domain is not closed or the domain is unbounded. For example, the function $f(x) = x^2 + 4$ over $(-\infty, \infty)$ has an absolute minimum of 4 at $x = 0$. Therefore, we can still consider functions over unbounded domains or open intervals and determine whether they have any absolute extrema. In the next example, we try to minimize a function over an unbounded domain. We will see that, although the domain of consideration is $(0, \infty)$, the function has an absolute minimum.

In the following example, we look at constructing a box of least surface area with a prescribed volume. It is not difficult to show that for a closed-top box, by symmetry, among all boxes with a specified volume, a cube will have the smallest surface area. Consequently, we consider the modified problem of determining which open-topped box with a specified volume has the smallest surface area.

Example 4.7.6: Minimizing Surface Area

A rectangular box with a square base, an open top, and a volume of $216\,\text{in}^3$ is to be constructed. What should the dimensions of the box be to minimize the surface area of the box? What is the minimum surface area?

Solution

Step 1: Draw a rectangular box and introduce the variable x to represent the length of each side of the square base; let y represent the height of the box. Let S denote the surface area of the open-top box.

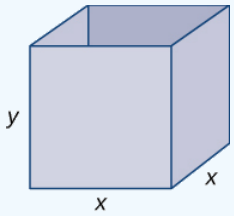

Figure 4.7.8: We want to minimize the surface area of a square-based box with a given volume.

Step 2: We need to minimize the surface area. Therefore, we need to minimize S.

Step 3: Since the box has an open top, we need only determine the area of the four vertical sides and the base. The area of each of the four vertical sides is $x \cdot y$. The area of the base is x^2. Therefore, the surface area of the box is

$$S = 4xy + x^2.$$

Step 4: Since the volume of this box is x^2y and the volume is given as $216\,\text{in}^3$, the constraint equation is

$$x^2y = 216.$$

Solving the constraint equation for y, we have $y = \frac{216}{x^2}$. Therefore, we can write the surface area as a function of x only:

$$S(x) = 4x\left(\frac{216}{x^2}\right) + x^2.$$

Therefore, $S(x) = \frac{864}{x} + x^2$.

Step 5: Since we are requiring that $x^2y = 216$, we cannot have $x = 0$. Therefore, we need $x > 0$. On the other hand, x is allowed to have any positive value. Note that as x becomes large, the height of the box y becomes correspondingly small so that $x^2y = 216$. Similarly, as x becomes small, the height of the box becomes correspondingly large. We conclude that the domain is the open, unbounded interval $(0, \infty)$. Note that, unlike the previous examples, we cannot reduce our problem to looking for an absolute maximum or absolute minimum over a closed, bounded interval. However, in the next step, we discover why this function must have an absolute minimum over the interval $(0, \infty)$.

Step 6: Note that as $x \to 0^+$, $S(x) \to \infty$. Also, as $x \to \infty$, $S(x) \to \infty$. Since S is a continuous function that approaches infinity at the ends, it must have an absolute minimum at some $x \in (0, \infty)$. This minimum must occur at a critical point of S. The derivative is

$$S'(x) = -\frac{864}{x^2} + 2x.$$

Therefore, $S'(x) = 0$ when $2x = \frac{864}{x^2}$. Solving this equation for x, we obtain $x^3 = 432$, so $x = \sqrt[3]{432} = 6\sqrt[3]{2}$. Since this is the only critical point of S, the absolute minimum must occur at $x = 6\sqrt[3]{2}$ (see Figure 4.7.9).

When $x = 6\sqrt[3]{2}$, $y = \frac{216}{(6\sqrt[3]{2})^2} = 3\sqrt[3]{2}$ in. Therefore, the dimensions of the box should be $x = 6\sqrt[3]{2}$ in. and $y = 3\sqrt[3]{2}$ in. With these dimensions, the surface area is

$$S(6\sqrt[3]{2}) = \frac{864}{6\sqrt[3]{2}} + (6\sqrt[3]{2})^2 = 108\sqrt[3]{4}\,\text{in}^2$$

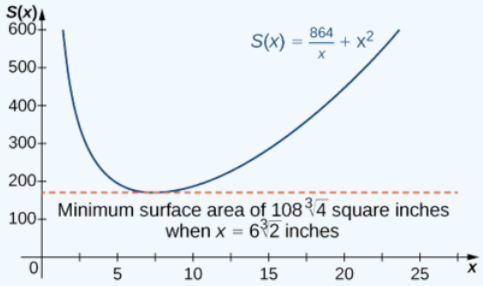

Figure 4.7.9: We can use a graph to determine the dimensions of a box of given the volume and the minimum surface area.

? Exercise 4.7.6

Consider the same open-top box, which is to have volume $216\,\text{in}^3$. Suppose the cost of the material for the base is $20¢/\text{in}^2$ and the cost of the material for the sides is $30¢/\text{in}^2$ and we are trying to minimize the cost of this box. Write the cost as a function of the side lengths of the base. (Let x be the side length of the base and y be the height of the box.)

Hint

If the cost of one of the sides is $30¢/\text{in}^2$, the cost of that side is $0.30xy$ dollars.

Answer

$c(x) = \frac{259.2}{x} + 0.2x^2$ dollars

Key Concepts

- To solve an optimization problem, begin by drawing a picture and introducing variables.
- Find an equation relating the variables.

- Find a function of one variable to describe the quantity that is to be minimized or maximized.
- Look for critical points to locate local extrema.

Glossary

optimization problems

problems that are solved by finding the maximum or minimum value of a function

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4.7E: EXERCISES FOR SECTION 4.7

For exercises 1 - 4, answer by proof, counterexample, or explanation.

1) When you find the maximum for an optimization problem, why do you need to check the sign of the derivative around the critical points?

Answer

The critical points can be the minima, maxima, or neither.

2) Why do you need to check the endpoints for optimization problems?

3) True or False. For every continuous nonlinear function, you can find the value x that maximizes the function.

Answer

False; $y = -x^2$ has a minimum only

4) True or False. For every continuous non-constant function on a closed, finite domain, there exists at least one x that minimizes or maximizes the function.

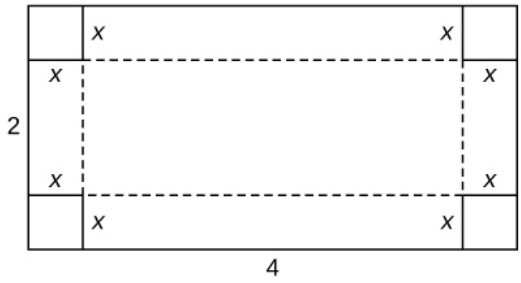
In exercises 5 - 8, set up and evaluate each optimization problem.

5) To carry a suitcase on an airplane, the $\text{length} + \text{width} + \text{height}$ of the box must be less than or equal to 62 in. Assuming the height is fixed, show that the maximum volume is $V = h\left(31 - \frac{1}{2}h\right)^2$. What height allows you to have the largest volume?

Answer

$h = \frac{62}{3}$ in.

6) You are constructing a cardboard box with the dimensions 2 m by 4 m. You then cut equal-size squares from each corner so you may fold the edges. What are the dimensions of the box with the largest volume?



7) Find the positive integer that minimizes the sum of the number and its reciprocal.

Answer

1

8) Find two positive integers such that their sum is 10, and minimize and maximize the sum of their squares.

In exercises 9 - 11, consider the construction of a pen to enclose an area.

9) You have $400\,\text{ft}$ of fencing to construct a rectangular pen for cattle. What are the dimensions of the pen that maximize the area?

Answer

$100\,\text{ft}$ by $100\,\text{ft}$

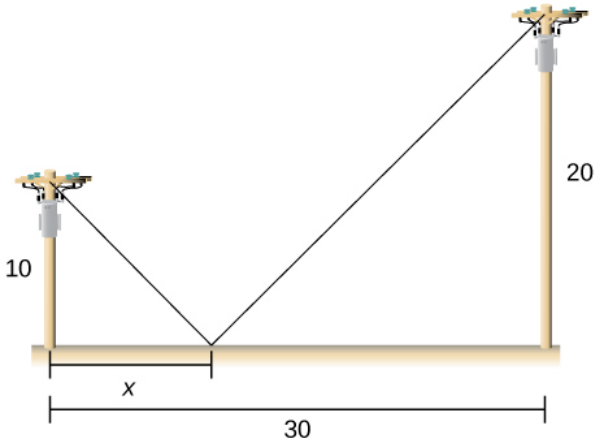
10) You have $800\,\text{ft}$ of fencing to make a pen for hogs. If you have a river on one side of your property, what is the dimension of the rectangular pen that maximizes the area?

11) You need to construct a fence around an area of $1600\,\text{ft}^2$. What are the dimensions of the rectangular pen to minimize the amount of material needed?

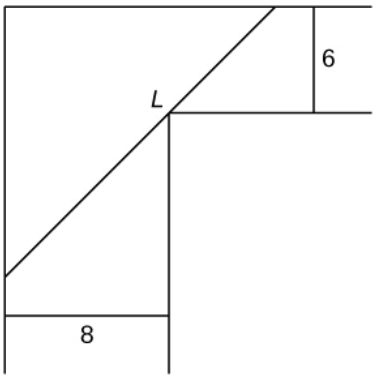
Answer

$40\,\text{ft}$ by $40\,\text{ft}$

12) Two poles are connected by a wire that is also connected to the ground. The first pole is $20\,\text{ft}$ tall and the second pole is $10\,\text{ft}$ tall. There is a distance of $30\,\text{ft}$ between the two poles. Where should the wire be anchored to the ground to minimize the amount of wire needed?



13) [T] You are moving into a new apartment and notice there is a corner where the hallway narrows from $8\,\text{ft}$ to $6\,\text{ft}$. What is the length of the longest item that can be carried horizontally around the corner?



Answer

$19.73\,\text{ft}$

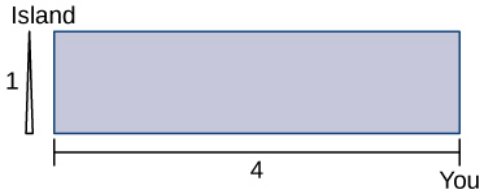
14) A patient's pulse measures $70\,\text{bpm}$, $80\,\text{bpm}$, then $120\,\text{bpm}$. To determine an accurate measurement of pulse, the doctor wants to know what value minimizes the expression $(x-70)^2+(x-80)^2+(x-120)^2$? What value minimizes it?

15) In the previous problem, assume the patient was nervous during the third measurement, so we only weight that value half as much as the others. What is the value that minimizes $(x-70)^2+(x-80)^2+\frac{1}{2}(x-120)^2$?

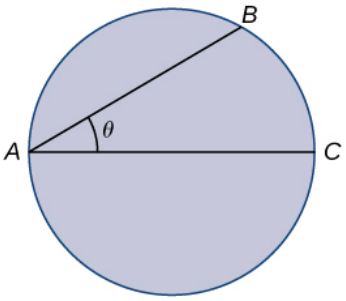
Answer

$84\,\text{bpm}$

16) You can run at a speed of 6 mph and swim at a speed of 3 mph and are located on the shore, 4 miles east of an island that is 1 mile north of the shoreline. How far should you run west to minimize the time needed to reach the island?



For exercises 17 - 19, consider a lifeguard at a circular pool with diameter 40 m. He must reach someone who is drowning on the exact opposite side of the pool, at position C. The lifeguard swims with a speed v and runs around the pool at speed $w=3v$.



17) Find a function that measures the total amount of time it takes to reach the drowning person as a function of the swim angle, θ.

Answer

$$T(\theta) = \frac{40\theta}{3v} + \frac{40\cos\theta}{v}$$

18) Find at what angle θ the lifeguard should swim to reach the drowning person in the least amount of time.

19) A truck uses gas as $g(v) = av + \frac{b}{v}$, where v represents the speed of the truck and g represents the gallons of fuel per mile. At what speed is fuel consumption minimized?

Answer

$$v = \sqrt{\frac{b}{a}}$$

For exercises 20 - 21, consider a limousine that gets $m(v) = \frac{120-2v}{5}$ mi/gal at speed v, the chauffeur costs \$15/h, and gas is \$3.50/gal.

20) Find the cost per mile at speed v.

21) Find the cheapest driving speed.

Answer

approximately 34.02 mph

For exercises 22 - 24, consider a pizzeria that sell pizzas for a revenue of $R(x) = ax$ and costs $C(x) = b + cx + dx^2$, where x represents the number of pizzas.

22) Find the profit function for the number of pizzas. How many pizzas gives the largest profit per pizza?

23) Assume that $R(x) = 10x$ and $C(x) = 2x + x^2$. How many pizzas sold maximizes the profit?

Answer

Selling 4 pizzas will maximize the profit.

24) Assume that $R(x) = 15x$, and $C(x) = 60 + 3x + \frac{1}{2}x^2$. How many pizzas sold maximizes the profit?

For exercises 25 - 26, consider a wire 4 ft long cut into two pieces. One piece forms a circle with radius r and the other forms a square of side x.

25) Choose x to maximize the sum of their areas.

Answer

$x = 0$

26) Choose x to minimize the sum of their areas.

For exercises 27 - 30, consider two nonnegative numbers x and y such that $x + y = 10$. Maximize and minimize the quantities.

27) xy

Answer

Maximal: $x = 5, y = 5$;
Minimal: $x = 0, y = 10$ and $y = 0, x = 10$

28 x^2y^2

29) $y - \frac{1}{x}$

Answer

Maximal: $x = 1, y = 9$;
Minimal: none

30) $x^2 - y$

In exercises 31 - 36, draw the given optimization problem and solve.

31) Find the volume of the largest right circular cylinder that fits in a sphere of radius 1.

Answer

$V = \frac{4\pi}{3\sqrt{3}}$

32) Find the volume of the largest right cone that fits in a sphere of radius 1.

33) Find the area of the largest rectangle that fits into the triangle with sides $x = 0$, $y = 0$ and $\frac{x}{4} + \frac{y}{6} = 1$.

Answer

$A = 6$

34) Find the largest volume of a cylinder that fits into a cone that has base radius R and height h.

35) Find the dimensions of the closed cylinder volume $V = 16\pi$ that has the least amount of surface area.

Answer

$r = 2, h = 4$

36) Find the dimensions of a right cone with surface area $S = 4\pi$ that has the largest volume.

For exercises 37 - 40, consider the points on the graphs of the given equations. Use a calculator to graph the functions.

37) [T] Where is the line $y = 5 - 2x$ closest to the origin?

Answer

$(2, 1)$

38) [T] Where is the line $y = 5 - 2x$ closest to point $(1, 1)$?

39) [T] Where is the parabola $y = x^2$ closest to point $(2, 0)$?

Answer

$(0.8351, 0.6974)$

40) [T] Where is the parabola $y = x^2$ closest to point $(0, 3)$?

In exercises 41 - 45, set up, but do not evaluate, each optimization problem.

41) A window is composed of a semicircle placed on top of a rectangle. If you have $20\,\text{ft}$ of window-framing materials for the outer frame, what is the maximum size of the window you can create? Use r to represent the radius of the semicircle.

Answer

$A = 20r - 2r^2 - \frac{1}{2}\pi r^2$

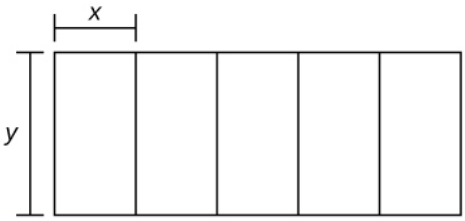
42) You have a garden row of 20 watermelon plants that produce an average of 30 watermelons apiece. For any additional watermelon plants planted, the output per watermelon plant drops by one watermelon. How many extra watermelon plants should you plant?

43) You are constructing a box for your cat to sleep in. The plush material for the square bottom of the box costs $\$5/\text{ft}^2$ and the material for the sides costs $\$2/\text{ft}^2$. You need a box with volume $4\,\text{ft}^3$. Find the dimensions of the box that minimize cost. Use x to represent the length of the side of the box.

Answer

$C(x) = 5x^2 + \frac{32}{x}$

44) You are building five identical pens adjacent to each other with a total area of $1000\ \text{m}^2$, as shown in the following figure. What dimensions should you use to minimize the amount of fencing?



45) You are the manager of an apartment complex with 50 units. When you set rent at $\$800/\text{month}$, all apartments are rented. As you increase rent by $\$25/\text{month}$, one fewer apartment is rented. Maintenance costs run $\$50/\text{month}$ for each occupied unit. What is the rent that maximizes the total amount of profit?

Answer

$P(x) = (50 - x)(800 + 25x - 50)$

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4.8: L'Hôpital's Rule

Learning Objectives

- Recognize when to apply L'Hôpital's rule.
- Identify indeterminate forms produced by quotients, products, subtractions, and powers, and apply L'Hôpital's rule in each case.
- Describe the relative growth rates of functions.

In this section, we examine a powerful tool for evaluating limits. This tool, known as **L'Hôpital's rule,** uses derivatives to calculate limits. With this rule, we will be able to evaluate many limits we have not yet been able to determine. Instead of relying on numerical evidence to conjecture that a limit exists, we will be able to show definitively that a limit exists and to determine its exact value.

Applying L'Hôpital's Rule

L'Hôpital's rule can be used to evaluate limits involving the quotient of two functions. Consider

$$\lim_{x\to a}\frac{f(x)}{g(x)}.$$

If $\lim_{x\to a} f(x)=L_1$ and $\lim_{x\to a} g(x)=L_2\neq 0$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{L_1}{L_2}.$$

However, what happens if $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$? We call this one of the **indeterminate forms,** of type $\frac{0}{0}$. This is considered an indeterminate form because we cannot determine the exact behavior of $\frac{f(x)}{g(x)}$ as $x\to a$ without further analysis. We have seen examples of this earlier in the text. For example, consider

$$\lim_{x\to 2}\frac{x^2-4}{x-2}$$

and

$$\lim_{x\to 0}\frac{\sin x}{x}.$$

For the first of these examples, we can evaluate the limit by factoring the numerator and writing

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x+2)(x-2)}{x-2}=\lim_{x\to 2}(x+2)=2+2=4.$$

For $\lim_{x\to 0}\frac{\sin x}{x}$ we were able to show, using a geometric argument, that

$$\lim_{x\to 0}\frac{\sin x}{x}=1.$$

Here we use a different technique for evaluating limits such as these. Not only does this technique provide an easier way to evaluate these limits, but also, and more importantly, it provides us with a way to evaluate many other limits that we could not calculate previously.

The idea behind L'Hôpital's rule can be explained using local linear approximations. Consider two differentiable functions f and g such that $\lim_{x\to a} f(x)=0=\lim_{x\to a} g(x)$ and such that $g'(a)\neq 0$ For x near a,we can write

$$f(x)\approx f(a)+f'(a)(x-a)$$

and

$$g(x)\approx g(a)+g'(a)(x-a).$$

Therefore,

$$\frac{f(x)}{g(x)} \approx \frac{f(a)+f'(a)(x-a)}{g(a)+g'(a)(x-a)}.$$

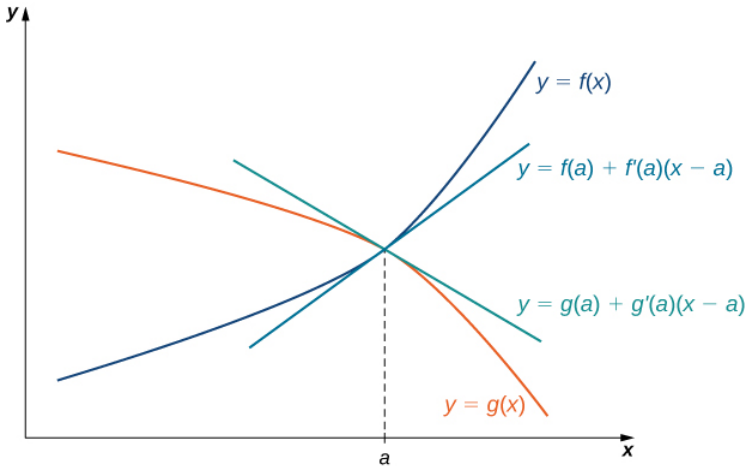

Figure 4.8.1: If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, then the ratio $f(x)/g(x)$ is approximately equal to the ratio of their linear approximations near a.

Since f is differentiable at a, then f is continuous at a, and therefore $f(a) = \lim_{x\to a} f(x) = 0$. Similarly, $g(a) = \lim_{x\to a} g(x) = 0$. If we also assume that f' and g' are continuous at $x = a$, then $f'(a) = \lim_{x\to a} f'(x)$ and $g'(a) = \lim_{x\to a} g'(x)$. Using these ideas, we conclude that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)(x-a)}{g'(x)(x-a)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Note that the assumption that f' and g' are continuous at a and $g'(a) \neq 0$ can be loosened. We state L'Hôpital's rule formally for the indeterminate form $\frac{0}{0}$. Also note that the notation $\frac{0}{0}$ does not mean we are actually dividing zero by zero. Rather, we are using the notation $\frac{0}{0}$ to represent a quotient of limits, each of which is zero.

L'Hôpital's Rule (0/0 Case)

Suppose f and g are differentiable functions over an open interval containing a, except possibly at a. If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

assuming the limit on the right exists or is ∞ or $-\infty$. This result also holds if we are considering one-sided limits, or if $a = \infty$ or $a = -\infty$.

Proof

We provide a proof of this theorem in the special case when f, g, f', and g' are all continuous over an open interval containing a. In that case, since $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$ and f and g are continuous at a, it follows that $f(a) = 0 = g(a)$. Therefore,

$$\begin{aligned}
\lim_{x\to a}\frac{f(x)}{g(x)} &= \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)} && \text{Since } f(a)=0=g(a)\\
&= \lim_{x\to a}\frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}} && \text{Multiply numerator and denominator by } \frac{1}{x-a}\\
&= \frac{\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}}{\lim\limits_{x\to a}\dfrac{g(x)-g(a)}{x-a}} && \text{The limit of a quotient is the quotient of the limits.}\\
&= \frac{f'(a)}{g'(a)} && \text{By the definition of the derivative}\\
&= \frac{\lim\limits_{x\to a}f'(x)}{\lim\limits_{x\to a}g'(x)} && \text{By the continuity of } f' \text{ and } g'\\
&= \lim_{x\to a}\frac{f'(x)}{g'(x)}. && \text{The limit of a quotient}
\end{aligned}$$

Note that L'Hôpital's rule states we can calculate the limit of a quotient $\frac{f}{g}$ by considering the limit of the quotient of the derivatives $\frac{f'}{g'}$. It is important to realize that we are not calculating the derivative of the quotient $\frac{f}{g}$.

□

Example 4.8.1: Applying L'Hôpital's Rule (0/0 Case)

Evaluate each of the following limits by applying L'Hôpital's rule.

a. $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x}$
b. $\lim\limits_{x\to 1}\dfrac{\sin(\pi x)}{\ln x}$
c. $\lim\limits_{x\to\infty}\dfrac{e^{1/x}-1}{1/x}$
d. $\lim\limits_{x\to 0}\dfrac{\sin x - x}{x^2}$

Solution

a. Since the numerator $1-\cos x\to 0$ and the denominator $x\to 0$, we can apply L'Hôpital's rule to evaluate this limit. We have

$$\lim_{x\to 0}\frac{1-\cos x}{x}=\lim_{x\to 0}\frac{\frac{d}{dx}\left(1-\cos x\right)}{\frac{d}{dx}\left(x\right)}=\lim_{x\to 0}\frac{\sin x}{1}=\frac{\lim\limits_{x\to 0}\sin x}{\lim\limits_{x\to 0}1}=\frac{0}{1}=0.$$

b. As $x\to 1$, the numerator $\sin(\pi x)\to 0$ and the denominator $\ln(x)\to 0$. Therefore, we can apply L'Hôpital's rule. We obtain

$$\begin{aligned}
\lim_{x\to 1}\frac{\sin(\pi x)}{\ln x} &= \lim_{x\to 1}\frac{\pi\cos(\pi x)}{1/x}\\
&= \lim_{x\to 1}(\pi x)\cos(\pi x)\\
&= (\pi\cdot 1)(-1)=-\pi.
\end{aligned}$$

c. As $x\to\infty$, the numerator $e^{1/x}-1\to 0$ and the denominator $\frac{1}{x}\to 0$. Therefore, we can apply L'Hôpital's rule. We obtain

$$\lim_{x\to\infty}\frac{e^{1/x}-1}{\frac{1}{x}}=\lim_{x\to\infty}\frac{e^{1/x}\left(\frac{-1}{x^2}\right)}{\left(\frac{-1}{x^2}\right)}=\lim_{x\to\infty}e^{1/x}=e^0=1.$$

d. As $x\to 0$, both the numerator and denominator approach zero. Therefore, we can apply L'Hôpital's rule. We obtain

$$\lim_{x\to 0}\frac{\sin x - x}{x^2}=\lim_{x\to 0}\frac{\cos x-1}{2x}.$$

Since the numerator and denominator of this new quotient both approach zero as $x \to 0$, we apply L'Hôpital's rule again. In doing so, we see that

$$\lim_{x\to 0} \frac{\cos x - 1}{2x} = \lim_{x\to 0} \frac{-\sin x}{2} = 0.$$

Therefore, we conclude that

$$\lim_{x\to 0} \frac{\sin x - x}{x^2} = 0.$$

? Exercise 4.8.1

Evaluate $\lim_{x\to 0} \frac{x}{\tan x}$.

Hint

$\frac{d}{dx}\left(\tan x\right) = \sec^2 x$

Answer

1

We can also use L'Hôpital's rule to evaluate limits of quotients $\frac{f(x)}{g(x)}$ in which $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$. Limits of this form are classified as indeterminate forms of type ∞/∞. Again, note that we are not actually dividing ∞ by ∞. Since ∞ is not a real number, that is impossible; rather, ∞/∞ is used to represent a quotient of limits, each of which is ∞ or $-\infty$.

L'Hôpital's Rule (∞/∞ Case)

Suppose f and g are differentiable functions over an open interval containing a, except possibly at a. Suppose $\lim_{x\to a} f(x) = \infty$ (or $-\infty$) and $\lim_{x\to a} g(x) = \infty$ (or $-\infty$). Then,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

assuming the limit on the right exists or is ∞ or $-\infty$. This result also holds if the limit is infinite, if $a = \infty$ or $-\infty$, or the limit is one-sided.

✓ Example 4.8.2: Applying L'Hôpital's Rule (∞/∞) Case

Evaluate each of the following limits by applying L'Hôpital's rule.

a. $\lim_{x\to\infty} \frac{3x+5}{2x+1}$
b. $\lim_{x\to 0^+} \frac{\ln x}{\cot x}$

Solution

a. Since $3x + 5$ and $2x + 1$ are first-degree polynomials with positive leading coefficients, $\lim_{x\to\infty}(3x + 5) = \infty$ and $\lim_{x\to\infty}(2x + 1) = \infty$. Therefore, we apply L'Hôpital's rule and obtain

$$\lim_{x\to\infty} \frac{3x+5}{2x+1} = \lim_{x\to\infty} \frac{3}{2} = \frac{3}{2}.$$

Note that this limit can also be calculated without invoking L'Hôpital's rule. Earlier in the chapter we showed how to evaluate such a limit by dividing the numerator and denominator by the highest power of x in the denominator. In doing so, we saw that

$$\lim_{x\to\infty} \frac{3x+5}{2x+1} = \lim_{x\to\infty} \frac{3+5/x}{2+1/x} = \frac{3}{2}.$$

L'Hôpital's rule provides us with an alternative means of evaluating this type of limit.

b. Here, $\lim_{x\to 0^+} \ln x = -\infty$ and $\lim_{x\to 0^+} \cot x = \infty$. Therefore, we can apply L'Hôpital's rule and obtain

$$\lim_{x\to 0^+} \frac{\ln x}{\cot x} = \lim_{x\to 0^+} \frac{1/x}{-\csc^2 x} = \lim_{x\to 0^+} \frac{1}{-x\csc^2 x}.$$

Now as $x \to 0^+, \csc^2 x \to \infty$. Therefore, the first term in the denominator is approaching zero and the second term is getting really large. In such a case, anything can happen with the product. Therefore, we cannot make any conclusion yet. To evaluate the limit, we use the definition of $\csc x$ to write

$$\lim_{x\to 0^+} \frac{1}{-x\csc^2 x} = \lim_{x\to 0^+} \frac{\sin^2 x}{-x}.$$

Now $\lim_{x\to 0^+} \sin^2 x = 0$ and $\lim_{x\to 0^+} -x = 0$, so we apply L'Hôpital's rule again. We find

$$\lim_{x\to 0^+} \frac{\sin^2 x}{-x} = \lim_{x\to 0^+} \frac{2\sin x\cos x}{-1} = \frac{0}{-1} = 0.$$

We conclude that

$$\lim_{x\to 0^+} \frac{\ln x}{\cot x} = 0.$$

? Exercise 4.8.2

Evaluate $\lim_{x\to\infty} \frac{\ln x}{5x}$.

Hint

$\frac{d}{dx}\big(\ln x\big) = \frac{1}{x}$

Answer

0

As mentioned, L'Hôpital's rule is an extremely useful tool for evaluating limits. It is important to remember, however, that to apply L'Hôpital's rule to a quotient $\frac{f(x)}{g(x)}$, it is essential that the limit of $\frac{f(x)}{g(x)}$ be of the form $\frac{0}{0}$ or ∞/∞. Consider the following example.

✓ Example 4.8.3: When L'Hôpital's Rule Does Not Apply

Consider $\lim_{x\to 1} \frac{x^2+5}{3x+4}$.

Show that the limit cannot be evaluated by applying L'Hôpital's rule.

Solution

Because the limits of the numerator and denominator are not both zero and are not both infinite, we cannot apply L'Hôpital's rule. If we try to do so, we get

$$\frac{d}{dx}(x^2+5) = 2x$$

and

$$\frac{d}{dx}(3x+4) = 3.$$

At which point we would conclude erroneously that

$$\lim_{x\to 1} \frac{x^2+5}{3x+4} = \lim_{x\to 1} \frac{2x}{3} = \frac{2}{3}.$$

However, since $\lim_{x\to 1}(x^2+5) = 6$ and $\lim_{x\to 1}(3x+4) = 7$, we actually have

$$\lim_{x\to 1}\frac{x^2+5}{3x+4}=\frac{6}{7}.$$

We can conclude that

$$\lim_{x\to 1}\frac{x^2+5}{3x+4}\neq\lim_{x\to 1}\frac{\frac{d}{dx}(x^2+5)}{\frac{d}{dx}(3x+4)}.$$

Exercise 4.8.3

Explain why we cannot apply L'Hôpital's rule to evaluate $\lim_{x\to 0^+}\frac{\cos x}{x}$. Evaluate $\lim_{x\to 0^+}\frac{\cos x}{x}$ by other means.

Hint

Determine the limits of the numerator and denominator separately.

Answer

$\lim_{x\to 0^+}\cos x=1$. Therefore, we cannot apply L'Hôpital's rule. The limit of the quotient is ∞.

Other Indeterminate Forms

L'Hôpital's rule is very useful for evaluating limits involving the indeterminate forms $\frac{0}{0}$ and ∞/∞. However, we can also use L'Hôpital's rule to help evaluate limits involving other indeterminate forms that arise when evaluating limits. The expressions $0\cdot\infty, \infty-\infty, 1^\infty, \infty^0$, and 0^0 are all considered indeterminate forms. These expressions are not real numbers. Rather, they represent forms that arise when trying to evaluate certain limits. Next we realize why these are indeterminate forms and then understand how to use L'Hôpital's rule in these cases. The key idea is that we must rewrite the indeterminate forms in such a way that we arrive at the indeterminate form $\frac{0}{0}$ or ∞/∞.

Indeterminate Form of Type $0\cdot\infty$

Suppose we want to evaluate $\lim_{x\to a}(f(x)\cdot g(x))$, where $f(x)\to 0$ and $g(x)\to\infty$ (or $-\infty$) as $x\to a$. Since one term in the product is approaching zero but the other term is becoming arbitrarily large (in magnitude), anything can happen to the product. We use the notation $0\cdot\infty$ to denote the form that arises in this situation. The expression $0\cdot\infty$ is considered indeterminate because we cannot determine without further analysis the exact behavior of the product $f(x)g(x)$ as $x\to\infty$. For example, let n be a positive integer and consider

$f(x)=\frac{1}{(x^n+1)}$ and $g(x)=3x^2$.

As $x\to\infty, f(x)\to 0$ and $g(x)\to\infty$. However, the limit as $x\to\infty$ of $f(x)g(x)=\frac{3x^2}{(x^n+1)}$ varies, depending on n. If $n=2$, then $\lim_{x\to\infty}f(x)g(x)=3$. If $n=1$, then $\lim_{x\to\infty}f(x)g(x)=\infty$. If $n=3$, then $\lim_{x\to\infty}f(x)g(x)=0$. Here we consider another limit involving the indeterminate form $0\cdot\infty$ and show how to rewrite the function as a quotient to use L'Hôpital's rule.

Example 4.8.4: Indeterminate Form of Type $0\cdot\infty$

Evaluate $\lim_{x\to 0^+}x\ln x$.

Solution

First, rewrite the function $x\ln x$ as a quotient to apply L'Hôpital's rule. If we write

$$x\ln x=\frac{\ln x}{1/x}$$

we see that $\ln x\to-\infty$ as $x\to 0^+$ and $\frac{1}{x}\to\infty$ as $x\to 0^+$. Therefore, we can apply L'Hôpital's rule and obtain

$$\ln x \qquad \frac{d}{}(\ln x) \qquad 1/x$$

$$\lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{dx}{\frac{d}{dx}(1/x)} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0.$$

We conclude that

$$\lim_{x\to 0^+} x\ln x = 0.$$

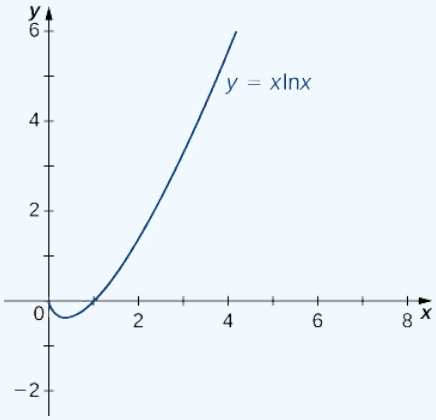

Figure 4.8.2: Finding the limit at $x = 0$ of the function $f(x) = x\ln x$.

? Exercise 4.8.4

Evaluate $\lim_{x\to 0} x\cot x$.

Hint

Write $x\cot x = \dfrac{x\cos x}{\sin x}$

Answer

1

Indeterminate Form of Type ∞-∞

Another type of indeterminate form is $\infty - \infty$. Consider the following example. Let n be a positive integer and let $f(x) = 3x^n$ and $g(x) = 3x^2 + 5$. As $x \to \infty$, $f(x) \to \infty$ and $g(x) \to \infty$. We are interested in $\lim_{x\to\infty}(f(x) - g(x))$. Depending on whether $f(x)$ grows faster, $g(x)$ grows faster, or they grow at the same rate, as we see next, anything can happen in this limit. Since $f(x) \to \infty$ and $g(x) \to \infty$, we write $\infty - \infty$ to denote the form of this limit. As with our other indeterminate forms, $\infty - \infty$ has no meaning on its own and we must do more analysis to determine the value of the limit. For example, suppose the exponent n in the function $f(x) = 3x^n$ is $n = 3$, then

$$\lim_{x\to\infty}(f(x) - g(x)) = \lim_{x\to\infty}(3x^3 - 3x^2 - 5) = \infty.$$

On the other hand, if $n = 2$, then

$$\lim_{x\to\infty}(f(x) - g(x)) = \lim_{x\to\infty}(3x^2 - 3x^2 - 5) = -5.$$

However, if $n = 1$, then

$$\lim_{x\to\infty}(f(x) - g(x)) = \lim_{x\to\infty}(3x - 3x^2 - 5) = -\infty.$$

Therefore, the limit cannot be determined by considering only $\infty - \infty$. Next we see how to rewrite an expression involving the indeterminate form $\infty - \infty$ as a fraction to apply L'Hôpital's rule.

Example 4.8.5: Indeterminate Form of Type $\infty - \infty$

Evaluate $\lim_{x \to 0^+} \left(\frac{1}{x^2} - \frac{1}{\tan x} \right)$.

Solution

By combining the fractions, we can write the function as a quotient. Since the least common denominator is $x^2 \tan x$, we have

$$\frac{1}{x^2} - \frac{1}{\tan x} = \frac{(\tan x) - x^2}{x^2 \tan x}.$$

As $x \to 0^+$, the numerator $\tan x - x^2 \to 0$ and the denominator $x^2 \tan x \to 0$. Therefore, we can apply L'Hôpital's rule. Taking the derivatives of the numerator and the denominator, we have

$$\lim_{x \to 0^+} \frac{(\tan x) - x^2}{x^2 \tan x} = \lim_{x \to 0^+} \frac{(\sec^2 x) - 2x}{x^2 \sec^2 x + 2x \tan x}.$$

As $x \to 0^+$, $(\sec^2 x) - 2x \to 1$ and $x^2 \sec^2 x + 2x \tan x \to 0$. Since the denominator is positive as x approaches zero from the right, we conclude that

$$\lim_{x \to 0^+} \frac{(\sec^2 x) - 2x}{x^2 \sec^2 x + 2x \tan x} = \infty.$$

Therefore,

$$\lim_{x \to 0^+} \left(\frac{1}{x^2} - \frac{1}{\tan x} \right) = \infty.$$

Exercise 4.8.5

Evaluate $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x} \right)$.

Hint

Rewrite the difference of fractions as a single fraction.

Answer

0

Indeterminate Forms Involving Exponents

Another type of indeterminate form that arises when evaluating limits involves exponents. The expressions $0^0, \infty^0$, and 1^∞ are all indeterminate forms. On their own, these expressions are meaningless because we cannot actually evaluate these expressions as we would evaluate an expression involving real numbers. Rather, these expressions represent forms that arise when finding limits. Now we examine how L'Hôpital's rule can be used to evaluate limits involving these indeterminate forms.

Since L'Hôpital's rule applies to quotients, we use the natural logarithm function and its properties to reduce a problem evaluating a limit involving exponents to a related problem involving a limit of a quotient. For example, suppose we want to evaluate $\lim_{x \to a} f(x)^{g(x)}$ and we arrive at the indeterminate form ∞^0. (The indeterminate forms 0^0 and 1^∞ can be handled similarly.) We proceed as follows. Let

$$y = f(x)^{g(x)}.$$

Then,

$$\ln y = \ln(f(x)^{g(x)}) = g(x) \ln(f(x)).$$

Therefore,

$$\lim_{x \to a} [\ln(y)] = \lim_{x \to a} [g(x) \ln(f(x))].$$

Since $\lim_{x\to a} f(x) = \infty$, we know that $\lim_{x\to a} \ln(f(x)) = \infty$. Therefore, $\lim_{x\to a} g(x)\ln(f(x))$ is of the indeterminate form $0\cdot\infty$, and we can use the techniques discussed earlier to rewrite the expression $g(x)\ln(f(x))$ in a form so that we can apply L'Hôpital's rule. Suppose $\lim_{x\to a} g(x)\ln(f(x)) = L$, where L may be ∞ or $-\infty$. Then

$$\lim_{x\to a}[\ln(y)] = L.$$

Since the natural logarithm function is continuous, we conclude that

$$\ln\left(\lim_{x\to a} y\right) = L,$$

which gives us

$$\lim_{x\to a} y = \lim_{x\to a} f(x)^{g(x)} = e^L.$$

Example 4.8.6: Indeterminate Form of Type ∞^0

Evaluate $\lim_{x\to\infty} x^{1/x}$.

Solution

Let $y = x^{1/x}$.Then,

$$\ln(x^{1/x}) = \frac{1}{x}\ln x = \frac{\ln x}{x}.$$

We need to evaluate $\lim_{x\to\infty} \dfrac{\ln x}{x}$. Applying L'Hôpital's rule, we obtain

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

Therefore, $\lim_{x\to\infty} \ln y = 0$. Since the natural logarithm function is continuous, we conclude that

$$\ln\left(\lim_{x\to\infty} y\right) = 0,$$

which leads to

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} y = e^{\ln\left(\lim_{x\to\infty} y\right)} = e^0 = 1.$$

Hence,

$$\lim_{x\to\infty} x^{1/x} = 1.$$

Exercise 4.8.6

Evaluate $\lim_{x\to\infty} x^{1/\ln(x)}$.

Hint

Let $y = x^{1/\ln(x)}$ and apply the natural logarithm to both sides of the equation.

Answer

e

Example 4.8.7: Indeterminate Form of Type 0^0

Evaluate $\lim_{x\to 0^+} x^{\sin x}$.

Solution

Let

$$y = x^{\sin x}.$$

Therefore,

$$\ln y = \ln(x^{\sin x}) = \sin x \ln x.$$

We now evaluate $\lim\limits_{x\to 0^+} \sin x \ln x$. Since $\lim\limits_{x\to 0^+} \sin x = 0$ and $\lim\limits_{x\to 0^+} \ln x = -\infty$, we have the indeterminate form $0 \cdot \infty$. To apply L'Hôpital's rule, we need to rewrite $\sin x \ln x$ as a fraction. We could write

$$\sin x \ln x = \frac{\sin x}{1/\ln x}$$

or

$$\sin x \ln x = \frac{\ln x}{1/\sin x} = \frac{\ln x}{\csc x}.$$

Let's consider the first option. In this case, applying L'Hôpital's rule, we would obtain

$$\lim_{x\to 0^+} \sin x \ln x = \lim_{x\to 0^+} \frac{\sin x}{1/\ln x} = \lim_{x\to 0^+} \frac{\cos x}{-1/(x(\ln x)^2)} = \lim_{x\to 0^+} (-x(\ln x)^2 \cos x).$$

Unfortunately, we not only have another expression involving the indeterminate form $0 \cdot \infty$, but the new limit is even more complicated to evaluate than the one with which we started. Instead, we try the second option. By writing

$$\sin x \ln x = \frac{\ln x}{1/\sin x} = \frac{\ln x}{\csc x,}$$

and applying L'Hôpital's rule, we obtain

$$\lim_{x\to 0^+} \sin x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\csc x} = \lim_{x\to 0^+} \frac{1/x}{-\csc x \cot x} = \lim_{x\to 0^+} \frac{-1}{x \csc x \cot x}.$$

Using the fact that $\csc x = \dfrac{1}{\sin x}$ and $\cot x = \dfrac{\cos x}{\sin x}$, we can rewrite the expression on the right-hand side as

$$\lim_{x\to 0^+} \frac{-\sin^2 x}{x \cos x} = \lim_{x\to 0^+} \left[\frac{\sin x}{x} \cdot (-\tan x)\right] = \left(\lim_{x\to 0^+} \frac{\sin x}{x}\right) \cdot \left(\lim_{x\to 0^+} (-\tan x)\right) = 1 \cdot 0 = 0.$$

We conclude that $\lim\limits_{x\to 0^+} \ln y = 0$. Therefore, $\ln\left(\lim\limits_{x\to 0^+} y\right) = 0$ and we have

$$\lim_{x\to 0^+} y = \lim_{x\to 0^+} x^{\sin x} = e^0 = 1.$$

Hence,

$$\lim_{x\to 0^+} x^{\sin x} = 1.$$

? Exercise 4.8.7

Evaluate $\lim\limits_{x\to 0^+} x^x$.

Hint

Let $y = x^x$ and take the natural logarithm of both sides of the equation.

Answer

1

Growth Rates of Functions

Suppose the functions f and g both approach infinity as $x \to \infty$. Although the values of both functions become arbitrarily large as the values of x become sufficiently large, sometimes one function is growing more quickly than the other. For example, $f(x) = x^2$ and $g(x) = x^3$ both approach infinity as $x \to \infty$. However, as Table 4.8.1 shows, the values of x^3 are growing much faster than the values of x^2.

Table 4.8.1: Comparing the Growth Rates of x^2 and x^3

x	10	100	1000	10,000
$f(x) = x^2$	100	10,000	1,000,000	100,000,000
$g(x) = x^3$	1000	1,000,000	1,000,000,000	1,000,000,000,000

In fact,

$$\lim_{x\to\infty} \frac{x^3}{x^2} = \lim_{x\to\infty} x = \infty.$$

or, equivalently

$$\lim_{x\to\infty} \frac{x^2}{x^3} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

As a result, we say x^3 is growing more rapidly than x^2 as $x \to \infty$. On the other hand, for $f(x) = x^2$ and $g(x) = 3x^2 + 4x + 1$, although the values of $g(x)$ are always greater than the values of $f(x)$ for $x > 0$, each value of $g(x)$ is roughly three times the corresponding value of $f(x)$ as $x \to \infty$, as shown in Table 4.8.2. In fact,

$$\lim_{x\to\infty} \frac{x^2}{3x^2 + 4x + 1} = \frac{1}{3}.$$

Table 4.8.2: Comparing the Growth Rates of x^2 and $3x^2 + 4x + 1$

x	10	100	1000	10,000
$f(x) = x^2$	100	10,000	1,000,000	100,000,000
$g(x) = 3x^2 + 4x + 1$	341	30,401	3,004,001	300,040,001

In this case, we say that x^2 and $3x^2 + 4x + 1$ are growing at the same rate as $x \to \infty$.

More generally, suppose f and g are two functions that approach infinity as $x \to \infty$. We say g grows more rapidly than f as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = \infty \quad \text{or, equivalently,} \quad \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

On the other hand, if there exists a constant $M \neq 0$ such that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = M,$$

we say f and g grow at the same rate as $x \to \infty$.

Next we see how to use L'Hôpital's rule to compare the growth rates of power, exponential, and logarithmic functions.

Example 4.8.8: Comparing the Growth Rates of $\ln(x)$, x^2, and e^x

For each of the following pairs of functions, use L'Hôpital's rule to evaluate

$$\lim_{x\to\infty} \frac{f(x)}{g(x)}.$$

a. $f(x) = x^2$ and $g(x) = e^x$

b. $f(x) = \ln(x)$ and $g(x) = x^2$

Solution

a. Since $\lim_{x\to\infty} x^2 = \infty$ and $\lim_{x\to\infty} e^x = \infty$, we can use L'Hôpital's rule to evaluate $\lim_{x\to\infty}\left[\frac{x^2}{e^x}\right]$. We obtain

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x}.$$

Since $\lim_{x\to\infty} 2x = \infty$ and $\lim_{x\to\infty} e^x = \infty$, we can apply L'Hôpital's rule again. Since

$$\lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0,$$

we conclude that

$$\lim_{x\to\infty} \frac{x^2}{e^x} = 0.$$

Therefore, e^x grows more rapidly than x^2 as $x \to \infty$ (See Figure 4.8.3 and Table 4.8.3)

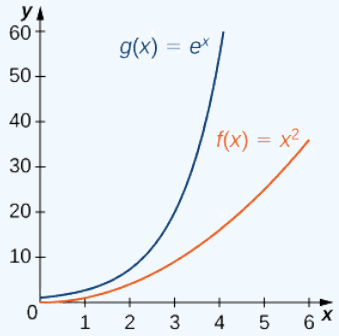

Figure 4.8.3: An exponential function grows at a faster rate than a power function.

Table 4.8.3: Growth rates of a power function and an exponential function.

x	5	10	15	20
x^2	25	100	225	400
e^x	148	22,026	3,269,017	485,165,195

b. Since $\lim_{x\to\infty} \ln x = \infty$ and $\lim_{x\to\infty} x^2 = \infty$, we can use L'Hôpital's rule to evaluate $\lim_{x\to\infty} \frac{\ln x}{x^2}$. We obtain

$$\lim_{x\to\infty} \frac{\ln x}{x^2} = \lim_{x\to\infty} \frac{1/x}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0.$$

Thus, x^2 grows more rapidly than $\ln x$ as $x \to \infty$ (see Figure 4.8.4 and Table 4.8.4).

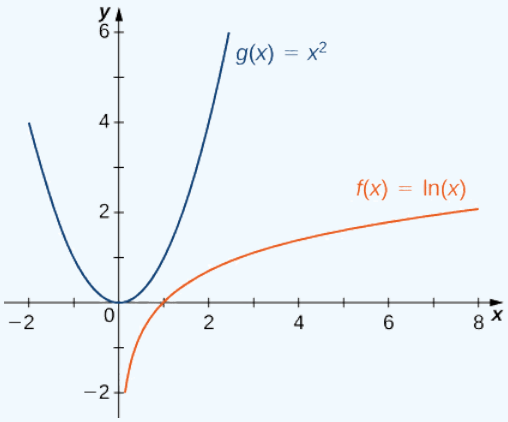

Figure 4.8.4: A power function grows at a faster rate than a logarithmic function.

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Table 4.8.4: Growth rates of a power function and a logarithmic function

x	10	100	1000	10,000
$\ln(x)$	2.303	4.605	6.908	9.210
x^2	100	10,000	1,000,000	100,000,000

? Exercise 4.8.8

Compare the growth rates of x^{100} and 2^x.

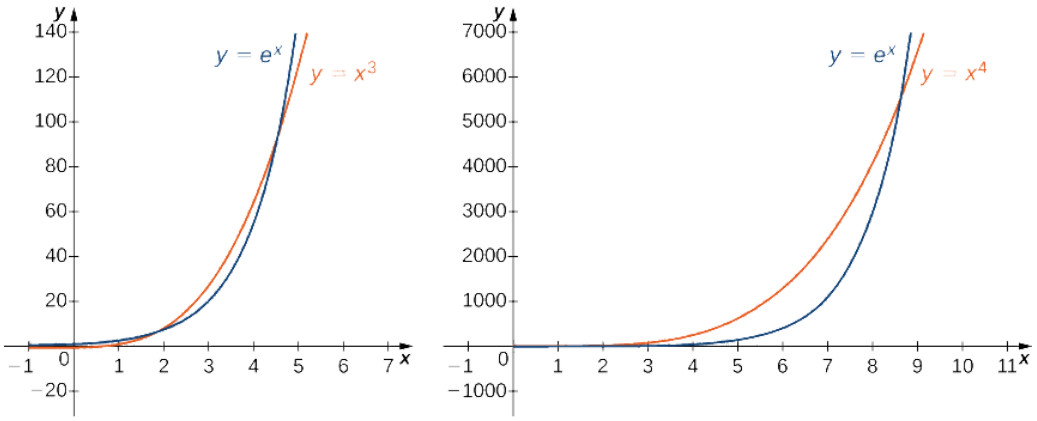
Hint

Apply L'Hôpital's rule to $x^{100}/2^x$.

Answer

The function 2^x grows faster than x^{100}.

Using the same ideas as in Example 4.8.8a. it is not difficult to show that e^x grows more rapidly than x^p for any $p > 0$. In Figure 4.8.5 and Table 4.8.5, we compare e^x with x^3 and x^4 as $x \to \infty$.



(a) (b)

Figure 4.8.5: The exponential function e^x grows faster than x^p for any $p > 0$. (a) A comparison of e^x with x^3. (b) A comparison of e^x with x^4.

Table 4.8.5: An exponential function grows at a faster rate than any power function

x	5	10	15	20
x^3	125	1000	3375	8000
x^4	625	10,000	50,625	160,000
e^x	148	22,026	3,269,017	485,165,195

Similarly, it is not difficult to show that x^p grows more rapidly than $\ln x$ for any $p > 0$. In Figure 4.8.6 and Table 4.8.6, we compare $\ln x$ with $\sqrt[3]{x}$ and $\sqrt{x}$.

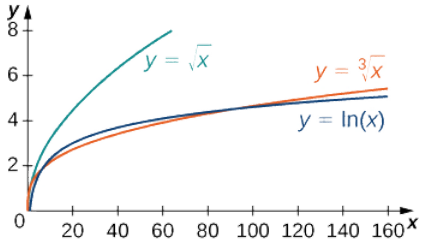


Figure 4.8.6: The function $y = \ln(x)$ grows more slowly than x^p for any $p > 0$ as $x \to \infty$.

Table 4.8.6: A logarithmic function grows at a slower rate than any root function

x	10	100	1000	10,000
$\ln(x)$	2.303	4.605	6.908	9.210
$\sqrt[3]{x}$	2.154	4.642	10	21.544
$\sqrt{x}$	3.162	10	31.623	100

Key Concepts

- L'Hôpital's rule can be used to evaluate the limit of a quotient when the indeterminate form $\frac{0}{0}$ or ∞/∞ arises.
- L'Hôpital's rule can also be applied to other indeterminate forms if they can be rewritten in terms of a limit involving a quotient that has the indeterminate form $\frac{0}{0}$ or ∞/∞.
- The exponential function e^x grows faster than any power function $x^p, p > 0$.
- The logarithmic function $\ln x$ grows more slowly than any power function $x^p, p > 0$.

Glossary

indeterminate forms

When evaluating a limit, the forms $\frac{0}{0}, \infty/\infty, 0 \cdot \infty, \infty - \infty, 0^0, \infty^0$, and 1^∞ are considered indeterminate because further analysis is required to determine whether the limit exists and, if so, what its value is.

L'Hôpital's rule

If f and g are differentiable functions over an interval a, except possibly at a, and $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$ or $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ are infinite, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$, assuming the limit on the right exists or is ∞ or $-\infty$.

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4.8E: EXERCISES FOR SECTION 4.8

In exercises 1 - 6, evaluate the limit.

1) Evaluate the limit $\lim\limits_{x\to\infty}\dfrac{e^x}{x}$.

2) Evaluate the limit $\lim\limits_{x\to\infty}\dfrac{e^x}{x^k}$.

Answer

$$\lim_{x\to\infty}\frac{e^x}{x^k} \quad = \quad \infty$$

3) Evaluate the limit $\lim\limits_{x\to\infty}\dfrac{\ln x}{x^k}$.

4) Evaluate the limit $\lim\limits_{x\to a}\dfrac{x-a}{x^2-a^2}$.

Answer

$$\lim_{x\to a}\frac{x-a}{x^2-a^2} \quad = \quad \frac{1}{2a}$$

5. Evaluate the limit $\lim\limits_{x\to a}\dfrac{x-a}{x^3-a^3}$.

6. Evaluate the limit $\lim\limits_{x\to a}\dfrac{x-a}{x^n-a^n}$.

Answer

$$\lim_{x\to a}\frac{x-a}{x^n-a^n} \quad = \quad \frac{1}{na^{n-1}}$$

In exercises 7 - 11, determine whether you can apply L'Hôpital's rule directly. Explain why or why not. Then, indicate if there is some way you can alter the limit so you can apply L'Hôpital's rule.

7) $\lim\limits_{x\to 0^+} x^2\ln x$

8) $\lim\limits_{x\to\infty} x^{1/x}$

Answer

Cannot apply directly; use logarithms

9) $\lim\limits_{x\to 0} x^{2/x}$

10) $\lim\limits_{x\to 0}\dfrac{x^2}{1/x}$

Answer

Cannot apply directly; rewrite as $\lim\limits_{x\to 0} x^3$

11) $\lim\limits_{x\to\infty}\dfrac{e^x}{x}$

In exercises 12 - 40, evaluate the limits with either L'Hôpital's rule or previously learned methods.

12) $\lim\limits_{x\to 3}\dfrac{x^2-9}{x-3}$

Answer

$$\lim_{x\to 3}\frac{x^2-9}{x-3} \quad = \quad 6$$

13) $\lim\limits_{x\to 3}\dfrac{x^2-9}{x+3}$

14) $\lim\limits_{x\to 0}\dfrac{(1+x)^{-2}-1}{x}$

Answer

$$\lim_{x\to 0}\frac{(1+x)^{-2}-1}{x} \quad = \quad -2$$

15) $\lim_{x\to\pi/2}\dfrac{\cos x}{\frac{\pi}{2}-x}$

16) $\lim_{x\to\pi}\dfrac{x-\pi}{\sin x}$

Answer

$$\lim_{x\to\pi}\frac{x-\pi}{\sin x} = -1$$

17) $\lim_{x\to1}\dfrac{x-1}{\sin x}$

18) $\lim_{x\to0}\dfrac{(1+x)^n-1}{x}$

Answer

$$\lim_{x\to0}\frac{(1+x)^n-1}{x} = n$$

19) $\lim_{x\to0}\dfrac{(1+x)^n-1-nx}{x^2}$

20) $\lim_{x\to0}\dfrac{\sin x-\tan x}{x^3}$

Answer

$$\lim_{x\to0}\frac{\sin x-\tan x}{x^3} = -\frac{1}{2}$$

21) $\lim_{x\to0}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$

22) $\lim_{x\to0}\dfrac{e^x-x-1}{x^2}$

Answer

$$\lim_{x\to0}\frac{e^x-x-1}{x^2} = \frac{1}{2}$$

23) $\lim_{x\to0}\dfrac{\tan x}{\sqrt{x}}$

24) $\lim_{x\to1}\dfrac{x-1}{\ln x}$

Answer

$$\lim_{x\to1}\frac{x-1}{\ln x} = 1$$

25) $\lim_{x\to0}(x+1)^{1/x}$

26) $\lim_{x\to1}\dfrac{\sqrt{x}-\sqrt[3]{x}}{x-1}$

Answer

$$\lim_{x\to1}\frac{\sqrt{x}-\sqrt[3]{x}}{x-1} = \frac{1}{6}$$

27) $\lim_{x\to0^+}x^{2x}$

28) $\lim_{x\to\infty}x\sin\left(\frac{1}{x}\right)$

Answer

$$\lim_{x\to\infty}x\sin\left(\frac{1}{x}\right) = 1$$

29) $\lim_{x\to0}\dfrac{\sin x-x}{x^2}$

30) $\lim_{x\to0^+}x\ln\left(x^4\right)$

Answer

$$\lim_{x\to0^+}x\ln\left(x^4\right) = 0$$

31) $\lim_{x\to\infty}(x-e^x)$

32) $\lim_{x\to\infty} x^2e^{-x}$

Answer

$$\lim_{x\to\infty} x^2e^{-x} = 0$$

33) $\lim_{x\to 0}\dfrac{3^x-2^x}{x}$

34) $\lim_{x\to 0}\dfrac{1+1/x}{1-1/x}$

Answer

$$\lim_{x\to 0}\dfrac{1+1/x}{1-1/x} = -1$$

35) $\lim_{x\to\pi/4}(1-\tan x)\cot x$

36) $\lim_{x\to\infty} xe^{1/x}$

Answer

$$\lim_{x\to\infty} xe^{1/x} = \infty$$

37) $\lim_{x\to 0} x^{1/\cos x}$

38) $\lim_{x\to 0^+} x^{1/x}$

Answer

$$\lim_{x\to 0^+} x^{1/x} = 0$$

39) $\lim_{x\to 0}\left(1-\dfrac{1}{x}\right)^x$

40) $\lim_{x\to\infty}\left(1-\dfrac{1}{x}\right)^x$

Answer

$$\lim_{x\to\infty}\left(1-\dfrac{1}{x}\right)^x = \dfrac{1}{e}$$

For exercises 41 - 50, use a calculator to graph the function and estimate the value of the limit, then use L'Hôpital's rule to find the limit directly.

41) [T] $\lim_{x\to 0}\dfrac{e^x-1}{x}$

42) [T] $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$

Answer

$$\lim_{x\to 0} x\sin\left(\tfrac{1}{x}\right) = 0$$

43) [T] $\lim_{x\to 1}\dfrac{x-1}{1-\cos(\pi x)}$

44) [T] $\lim_{x\to 1}\dfrac{e^{x-1}-1}{x-1}$

Answer

$$\lim_{x\to 1}\dfrac{e^{x-1}-1}{x-1} = 1$$

45) [T] $\lim_{x\to 1}\dfrac{(x-1)^2}{\ln x}$

46) [T] $\lim_{x\to\pi}\dfrac{1+\cos x}{\sin x}$

Answer

$$\lim_{x \to \pi} \frac{1 + \cos x}{\sin x} = 0$$

47) [T] $\lim_{x \to 0} \left(\csc x - \frac{1}{x} \right)$

48) [T] $\lim_{x \to 0^+} \tan \left(x^x \right)$

Answer

$$\lim_{x \to 0^+} \tan \left(x^x \right) = \tan 1$$

49) [T] $\lim_{x \to 0^+} \frac{\ln x}{\sin x}$

50) [T] $\lim_{x \to 0} \frac{e^x - e^{-x}}{x}$

Answer

$$\lim_{x \to 0} \frac{e^x - e^{-x}}{x} = 2$$

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4.9: Newton's Method

Learning Objectives

- Describe the steps of Newton's method.
- Explain what an iterative process means.
- Recognize when Newton's method does not work.
- Apply iterative processes to various situations.

In many areas of pure and applied mathematics, we are interested in finding solutions to an equation of the form $f(x)=0.$ For most functions, however, it is difficult—if not impossible—to calculate their zeroes explicitly. In this section, we take a look at a technique that provides a very efficient way of approximating the **zeroes of functions**. This technique makes use of tangent line approximations and is behind the method used often by calculators and computers to find zeroes.

Describing Newton's Method

Consider the task of finding the solutions of $f(x)=0.$ If f is the first-degree polynomial $f(x)=ax+b$, then the solution of $f(x)=0$ is given by the formula $x=-\frac{b}{a}$. If f is the second-degree polynomial $f(x)=ax^2+bx+c$, the solutions of $f(x)=0$ can be found by using the quadratic formula. However, for polynomials of degree **3** or more, finding roots of f becomes more complicated. Although formulas exist for third- and fourth-degree polynomials, they are quite complicated. Also, if f is a polynomial of degree **5** or greater, it is known that no such formulas exist. For example, consider the function

$$f(x)=x^5+8x^4+4x^3−2x−7.\nonumber$$

No formula exists that allows us to find the solutions of $f(x)=0.$ Similar difficulties exist for nonpolynomial functions. For example, consider the task of finding solutions of $\tan(x)−x=0.$ No simple formula exists for the solutions of this equation. In cases such as these, we can use Newton's method to approximate the roots.

Newton's method makes use of the following idea to approximate the solutions of $f(x)=0.$ By sketching a graph of f, we can estimate a root of $f(x)=0$. Let's call this estimate x_0. We then draw the tangent line to f at x_0. If $f'(x_0)≠0$, this tangent line intersects the x-axis at some point $(x_1,0)$. Now let x_1 be the next approximation to the actual root. Typically, x_1 is closer than x_0 to an actual root. Next we draw the tangent line to f at x_1. If $f'(x_1)≠0$, this tangent line also intersects the x-axis, producing another approximation, x_2. We continue in this way, deriving a list of approximations: $x_0,\, x_1,\, x_2,\, ….$ Typically, the numbers $x_0,\, x_1,\, x_2,\, …$ quickly approach an actual root x^*, as shown in the following figure.

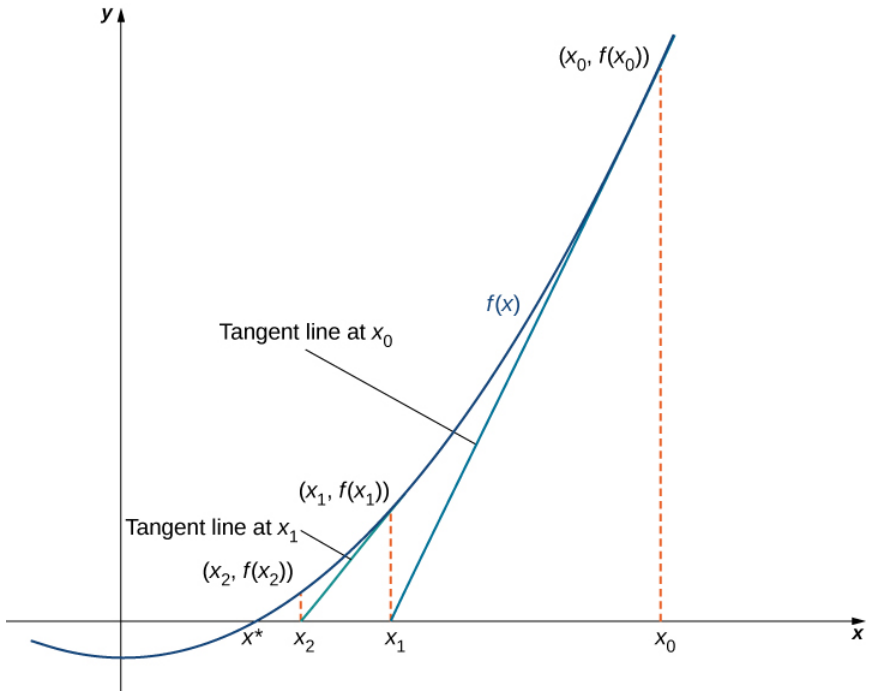

Figure \(\PageIndex{1}\):The approximations \(x_0,\, x_1,\, x_2,\, … \) approach the actual root \(x^*\). The approximations are derived by looking at tangent lines to the graph of \(f\).

Now let's look at how to calculate the approximations \(x_0,\, x_1,\, x_2,\, ….\) If \(x_0\) is our first approximation, the approximation \(x_1\) is defined by letting \((x_1,0)\) be the \(x\)-intercept of the tangent line to \(f\) at \(x_0\). The equation of this tangent line is given by

$$y=f(x_0)+f'(x_0)(x−x_0). \nonumber$$

Therefore, \(x_1\) must satisfy

$$f(x_0)+f'(x_0)(x_1−x_0)=0.\nonumber$$

Solving this equation for \(x_1\), we conclude that

$$x_1=x_0−\frac{f(x_0)}{f'(x_0)}.\nonumber$$

Similarly, the point \((x_2,0)\) is the \(x\)-intercept of the tangent line to \(f\) at \(x_1\). Therefore, \(x_2\) satisfies the equation

$$x_2=x_1−\frac{f(x_1)}{f'(x_1)}.\nonumber$$

In general, for \(n>0,x_n\) satisfies

$$x_n=x_{n−1}−\frac{f(x_{n−1})}{f'(x_{n−1})}.\label{Newton}$$

Next we see how to make use of this technique to approximate the root of the polynomial \(f(x)=x^3−3x+1.\)

Example \(\PageIndex{1}\): Finding a Root of a Polynomial

Use Newton's method to approximate a root of \(f(x)=x^3−3x+1\) in the interval \([1,2]\). Let \(x_0=2\) and find \(x_1,\, x_2, \,x_3, \,x_4,\) and \(x_5\).

Solution

From Figure \(\PageIndex{2}\), we see that \(f\) has one root over the interval \([1,2]\). Therefore \(x_0=2\) seems like a reasonable first approximation. To find the next approximation, we use Equation \ref{Newton}. Since \(f(x)=x^3−3x+1\), the derivative is \(f'(x)=3x^2−3\). Using Equation \ref{Newton} with \(n=1\) (and a calculator that displays \(10\) digits), we obtain

$$x_1=x_0−\frac{f(x_0)}{f'(x_0)}=2−\frac{f(2)}{f'(2)}=2−\frac{3}{9}≈1.666666667.\nonumber$$

To find the next approximation, x_2, we use Equation \ref{Newton} with $n=2$ and the value of x_1 stored on the calculator. We find that

$$x_2=x_1-\frac{f(x_1)}{f'(x_1)}\approx 1.548611111.\nonumber$$

Continuing in this way, we obtain the following results:

- $x_1\approx 1.666666667$
- $x_2\approx 1.548611111$
- $x_3\approx 1.532390162$
- $x_4\approx 1.532088989$
- $x_5\approx 1.532088886$
- $x_6\approx 1.532088886.$

We note that we obtained the same value for x_5 and x_6. Therefore, any subsequent application of Newton's method will most likely give the same value for x_n.

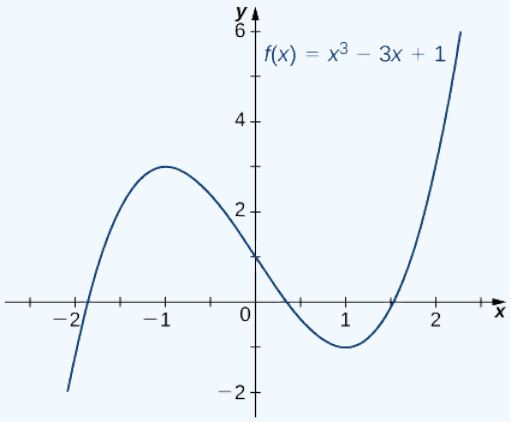

Figure $\PageIndex{2}$: The function $f(x)=x^3-3x+1$ has one root over the interval $[1,2].$

Exercise $\PageIndex{1}$

Letting $x_0=0$, let's use Newton's method to approximate the root of $f(x)=x^3-3x+1$ over the interval $[0,1]$ by calculating x_1 and x_2.

Hint

Use Equation \ref{Newton}.

Answer

$x_1\approx 0.33333333$
$x_2\approx 0.347222222$

Newton's method can also be used to approximate square roots. Here we show how to approximate $\sqrt{2}$. This method can be modified to approximate the square root of any positive number.

Example $\PageIndex{2}$: Finding a Square Root

Use Newton's method to approximate $\sqrt{2}$ (Figure $\PageIndex{3}$). Let $f(x)=x^2-2$, let $x_0=2$, and calculate $x_1,\, x_2,\, x_3,\, x_4,\, x_5$. (We note that since $f(x)=x^2-2$ has a zero at $\sqrt{2}$, the initial value $x_0=2$ is a reasonable choice to approximate $\sqrt{2}$).

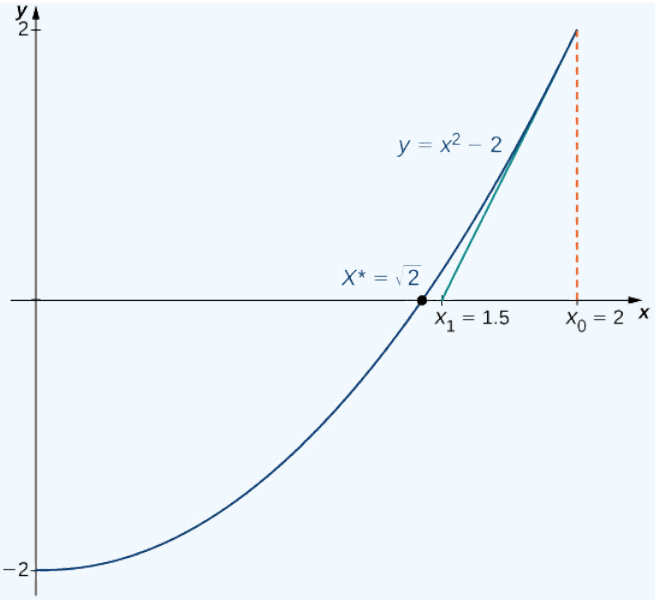

Figure \(\PageIndex{3}\): We can use Newton's method to find \(\sqrt{2}\).

Solution

For \(f(x)=x^2−2,\; f'(x)=2x.\) From Equation \ref{Newton}, we know that

$$\begin{aligned} x_n&=x_{n−1}−\frac{f(x_{n−1})}{f'(x_{n−1})}\\[4pt] &=x_{n−1}−\frac{x^2_{n−1}−2}{2x_{n−1}}\\[4pt] &=\frac{1}{2}x_{n−1}+\frac{1}{x_{n−1}}\\[4pt] &=\frac{1}{2}\left(x_{n−1}+\frac{2}{x_{n−1}}\right).\end{aligned} \nonumber$$

Therefore,

$$x_1=\frac{1}{2}\left(x_0+\frac{2}{x_0}\right)=\frac{1}{2}\left(2+\frac{2}{2}\right)=1.5$$

$$x_2=\frac{1}{2}\left(x_1+\frac{2}{x_1}\right)=\frac{1}{2}\left(1.5+\frac{2}{1.5}\right)≈1.416666667.$$

Continuing in this way, we find that

$$x_1=1.5$$

$$x_2≈1.416666667$$

$$x_3≈1.414215686$$

$$x_4≈1.414213562$$

$$x_5≈1.414213562.$$

Since we obtained the same value for \(x_4\) and \(x_5\), it is unlikely that the value \(x_n\) will change on any subsequent application of Newton's method. We conclude that \(\sqrt{2}≈1.414213562.\)

? Exercise \(\PageIndex{2}\)

Use Newton's method to approximate \(\sqrt{3}\) by letting \(f(x)=x^2−3\) and \(x_0=3\). Find \(x_1\) and \(x_2\).

Hint

For \(f(x)=x^2−3\), Equation \ref{Newton} reduces to \(x_n=\frac{x_{n−1}}{2}+\frac{3}{2x_{n−1}}\).

Answer

\(x_1=2\)
\(x_2=1.75\)

When using Newton's method, each approximation after the initial guess is defined in terms of the previous approximation by using the same formula. In particular, by defining the function $F(x)=x-\left[\frac{f(x)}{f'(x)}\right]$, we can rewrite Equation \ref{Newton} as $x_n=F(x_{n-1})$. This type of process, where each x_n is defined in terms of x_{n-1} by repeating the same function, is an example of an iterative process. Shortly, we examine other iterative processes. First, let's look at the reasons why Newton's method could fail to find a root.

Failures of Newton's Method

Typically, Newton's method is used to find roots fairly quickly. However, things can go wrong. Some reasons why Newton's method might fail include the following:

1. At one of the approximations x_n, the derivative f' is zero at x_n, but $f(x_n)\neq 0$. As a result, the tangent line of f at x_n does not intersect the x-axis. Therefore, we cannot continue the iterative process.
2. The approximations $x_0,\, x_1,\, x_2,\, \ldots$ may approach a different root. If the function f has more than one root, it is possible that our approximations do not approach the one for which we are looking, but approach a different root (see Figure \PageIndex{4}). This event most often occurs when we do not choose the approximation x_0 close enough to the desired root.
3. The approximations may fail to approach a root entirely. In Example \PageIndex{3}, we provide an example of a function and an initial guess x_0 such that the successive approximations never approach a root because the successive approximations continue to alternate back and forth between two values.

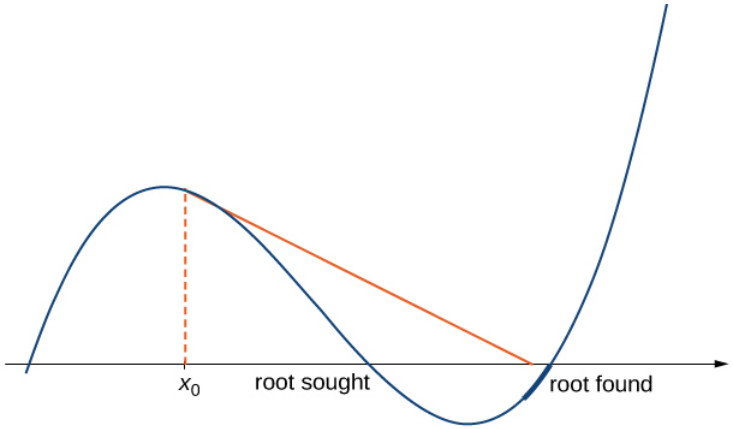

Figure \PageIndex{4}: If the initial guess x_0 is too far from the root sought, it may lead to approximations that approach a different root.

Example \PageIndex{3}: When Newton's Method Fails

Consider the function $f(x)=x^3-2x+2$. Let $x_0=0$. Show that the sequence $x_1,\, x_2,\, \ldots$ fails to approach a root of f.

Solution

For $f(x)=x^3-2x+2,$ the derivative is $f'(x)=3x^2-2$.Therefore,

$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}=0-\frac{f(0)}{f'(0)}=-\frac{2}{-2}=1. \nonumber$$

In the next step,

$$x_2=x_1-\frac{f(x_1)}{f'(x_1)}=1-\frac{f(1)}{f'(1)}=1-\frac{1}{1}=0. \nonumber$$

Consequently, the numbers $x_0,\, x_1,\, x_2,\, \ldots$ continue to bounce back and forth between 0 and 1 and never get closer to the root of f which is over the interval $[-2,-1]$ (Figure \PageIndex{5}). Fortunately, if we choose an initial approximation x_0 closer to the actual root, we can avoid this situation.

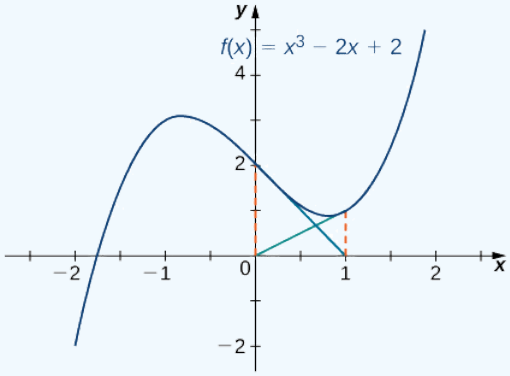

Figure \(\PageIndex{5}\): The approximations continue to alternate between \(0\) and \(1\) and never approach the root of \(f\).

? Exercise \(\PageIndex{3}\)

For \(f(x)=x^3−2x+2,\) let \(x_0=−1.5\) and find \(x_1\) and \(x_2\).

Hint

Use Equation \ref{Newton}.

Answer

\(x_1≈−1.842105263\)
\(x_2≈−1.772826920\)

From Example \(\PageIndex{3}\), we see that Newton's method does not always work. However, when it does work, the sequence of approximations approaches the root very quickly. Discussions of how quickly the sequence of approximations approach a root found using Newton's method are included in texts on numerical analysis.

Other Iterative Processes

As mentioned earlier, Newton's method is a type of iterative process. We now look at an example of a different type of iterative process.

Consider a function \(F\) and an initial number \(x_0\). Define the subsequent numbers \(x_n\) by the formula \(x_n=F(x_{n−1})\). This process is an iterative process that creates a list of numbers \(x_0,\, x_1,\, x_2,\, …,\, x_n,\, ….\) This list of numbers may approach a finite number \(x^*\) as \(n\) gets larger, or it may not. In Example \(\PageIndex{4}\), we see an example of a function \(F\) and an initial guess \(x_0\) such that the resulting list of numbers approaches a finite value.

✓ Example \(\PageIndex{4}\): Finding a Limit for an Iterative Process

Let \(F(x)=\frac{1}{2}x+4\) and let \(x_0=0\). For all \(n≥1\), let \(x_n=F(x_{n−1})\). Find the values \(x_1,\, x_2,\, x_3,\, x_4,\, x_5\). Make a conjecture about what happens to this list of numbers \(x_1,\, x_2,\, x_3,\, …,\, x_n,\, …\) as \(n→∞\). If the list of numbers \(x_1,\, x_2,\, x_3,\, …\) approaches a finite number \(x^*\), then \(x^*\) satisfies \(x^*=F(x^*)\), and \(x^*\) is called a fixed point of \(F\).

Solution

If \(x_0=0\), then

- \(x_1=\frac{1}{2}(0)+4=4\)
- \(x_2=\frac{1}{2}(4)+4=6\)
- \(x_3=\frac{1}{2}(6)+4=7\)
- \(x_4=\frac{1}{2}(7)+4=7.5\)
- \(x_5=\frac{1}{2}(7.5)+4=7.75\)
- \(x_6=\frac{1}{2}(7.75)+4=7.875\)
- \(x_7=\frac{1}{2}(7.875)+4=7.9375\)

- \(x_8=\frac{1}{2}(7.9375)+4=7.96875\)
- \(x _9=\frac{1}{2}(7.96875)+4=7.984375.\)

From this list, we conjecture that the values \(x_n\) approach \(8\).

Figure \(\PageIndex{6}\) provides a graphical argument that the values approach \(8\) as \(n→∞\). Starting at the point \((x_0,x_0)\), we draw a vertical line to the point \((x_0,F(x_0))\). The next number in our list is \(x_1=F(x_0)\). We use \(x_1\) to calculate \(x_2\). Therefore, we draw a horizontal line connecting \((x_0,x_1)\) to the point \((x_1,x_1)\) on the line \(y=x\), and then draw a vertical line connecting \((x_1,x_1)\) to the point \((x_1,F(x_1))\). The output \(F(x_1)\) becomes \(x_2\). Continuing in this way, we could create an infinite number of line segments. These line segments are trapped between the lines \(F(x)=\frac{x}{2}+4\) and \(y=x\). The line segments get closer to the intersection point of these two lines, which occurs when \(x=F(x)\). Solving the equation \(x=\frac{x}{2}+4,\) we conclude they intersect at \(x=8\). Therefore, our graphical evidence agrees with our numerical evidence that the list of numbers \(x_0,\, x_1,\, x_2,\, …\) approaches \(x^*=8\) as \(n→∞\).

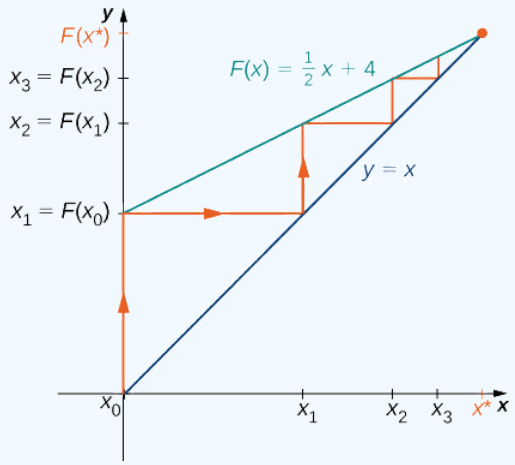

Figure \(\PageIndex{6}\): This iterative process approaches the value \(x^*=8.\)

? Exercise \(\PageIndex{4}\)

Consider the function \(F(x)=\frac{1}{3}x+6\). Let \(x_0=0\) and let \(x_n=F(x_{n−1})\) for \(n≥2\). Find \(x_1,\, x_2,\, x_3,\, x_4,\, x_5\). Make a conjecture about what happens to the list of numbers \(x_1,\, x_2,\, x_3,\, …\, x_n,\, … \) as \(n→∞.\)

Hint

Consider the point where the lines \(y=x\) and \(y=F(x)\) intersect.

Answer

\(x_1=6,\;x_2=8,\;x_3=\frac{26}{3},\;x_4=\frac{80}{9},\;x_5=\frac{242}{27};\;x^*=9\)

Iterative Processes and Chaos

Iterative processes can yield some very interesting behavior. In this section, we have seen several examples of iterative processes that converge to a fixed point. We also saw in Example \(\PageIndex{3}\) that the iterative process bounced back and forth between two values. We call this kind of behavior a 2-cycle. Iterative processes can converge to cycles with various periodicities, such as 2−cycles, 4−cycles (where the iterative process repeats a sequence of four values), 8-cycles, and so on.

Some iterative processes yield what mathematicians call chaos. In this case, the iterative process jumps from value to value in a seemingly random fashion and never converges or settles into a cycle. Although a complete exploration of **chaos** is beyond the scope of this text, in this project we look at one of the key properties of a chaotic iterative process: sensitive dependence on initial conditions. This property refers to the concept that small changes in initial conditions can generate drastically different behavior in the iterative process.

Probably the best-known example of chaos is the **Mandelbrot set** (see Figure \(\PageIndex{7}\)), named after Benoit Mandelbrot (1924–2010), who investigated its properties and helped popularize the field of chaos theory. The Mandelbrot set is usually generated by computer and shows fascinating details on enlargement, including self-replication of the set. Several colorized versions of the set have been shown in museums and can be found online and in popular books on the subject.

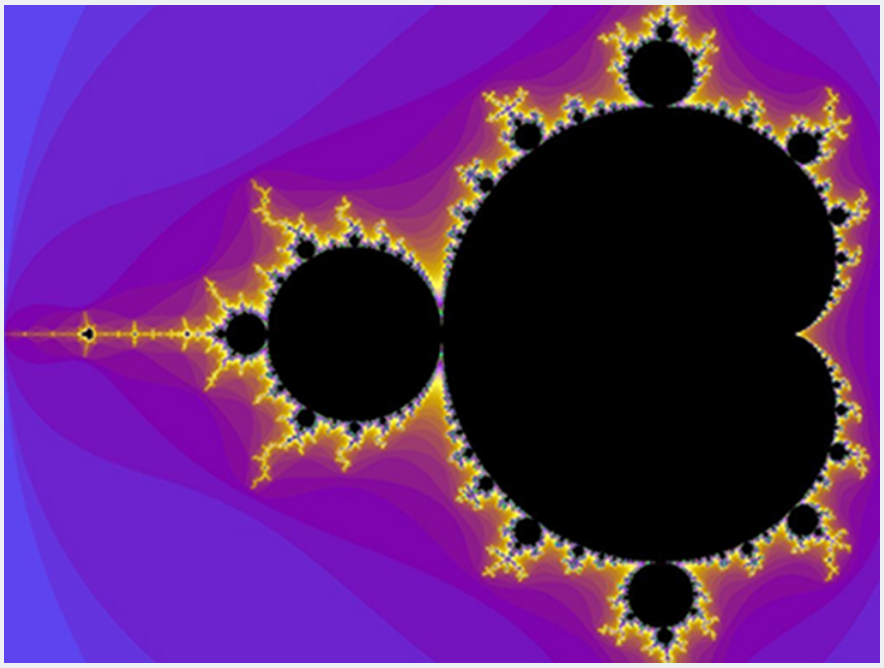

Figure \(\PageIndex{7}\): The Mandelbrot set is a well-known example of a set of points generated by the iterative chaotic behavior of a relatively simple function.

In this project we use the logistic map

$$f(x)=rx(1−x) \nonumber$$

where $x∈[0,1]$ and $r>0$

as the function in our iterative process. The logistic map is a deceptively simple function; but, depending on the value of r, the resulting iterative process displays some very interesting behavior. It can lead to fixed points, cycles, and even chaos.

To visualize the long-term behavior of the iterative process associated with the logistic map, we will use a tool called a cobweb diagram. As we did with the iterative process we examined earlier in this section, we first draw a vertical line from the point $(x_0,0)$ to the point $(x_0,f(x_0))=(x_0,x_1)$. We then draw a horizontal line from that point to the point $(x_1,x_1),$ then draw a vertical line to $(x_1,f(x_1))=(x_1,x_2)$, and continue the process until the long-term behavior of the system becomes apparent. Figure \(\PageIndex{8}\) shows the long-term behavior of the logistic map when $r=3.55$ and $x_0=0.2$. (The first 100 iterations are not plotted.) The long-term behavior of this iterative process is an 8-cycle.

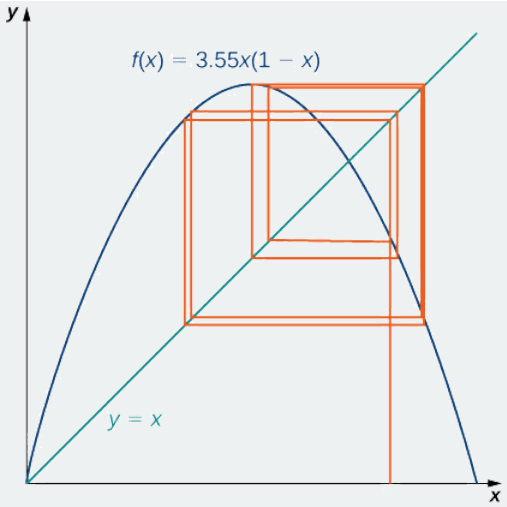

Figure \(\PageIndex{8}\): A cobweb diagram for \(f(x)=3.55x(1−x)\) is presented here. The sequence of values results in an 8-cycle.

1. Let \(r=0.5\) and choose \(x_0=0.2\). Either by hand or by using a computer, calculate the first \(10\) values in the sequence. Does the sequence appear to converge? If so, to what value? Does it result in a cycle? If so, what kind of cycle (for example, \(2\)−cycle, \(4\)−cycle.)?
2. What happens when \(r=2\)?
3. For \(r=3.2\) and \(r=3.5\), calculate the first \(100\) sequence values. Generate a cobweb diagram for each iterative process. (Several free applets are available online that generate cobweb diagrams for the logistic map.) What is the long-term behavior in each of these cases?
4. Now let \(r=4.\) Calculate the first \(100\) sequence values and generate a cobweb diagram. What is the long-term behavior in this case?
5. Repeat the process for \(r=4,\) but let \(x_0=0.201.\) How does this behavior compare with the behavior for \(x_0=0.2\)?

Key Concepts

- Newton's method approximates roots of \(f(x)=0\) by starting with an initial approximation \(x_0\), then uses tangent lines to the graph of \(f\) to create a sequence of approximations \(x_1,\, x_2,\, x_3,\, ….\)
- Typically, Newton's method is an efficient method for finding a particular root. In certain cases, Newton's method fails to work because the list of numbers \(x_0,\, x_1,\, x_2,\, …\) does not approach a finite value or it approaches a value other than the root sought.
- Any process in which a list of numbers \(x_0,\, x_1,\, x_2,\, …\) is generated by defining an initial number \(x_0\) and defining the subsequent numbers by the equation \(x_n=F(x_{n−1})\) for some function \(F\) is an iterative process. Newton's method is an example of an iterative process, where the function \(F(x)=x−\left[\frac{f(x)}{f′(x)}\right]\) for a given function \(f\).

Glossary

iterative process

process in which a list of numbers \(x_0,x_1,x_2,x_3…\) is generated by starting with a number \(x_0\) and defining \(x_n=F(x_{n−1})\) for \(n≥1\)

Newton's method

method for approximating roots of \(f(x)=0;\) using an initial guess \(x_0\); each subsequent approximation is defined by the equation \(x_n=x_{n−1}−\frac{f(x_{n−1})}{f′(x_{n−1})}\)

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4.9E: EXERCISES FOR SECTION 4.9

In exercises 1 - 5, write Newton's formula as $x_{n+1} = F(x_n)$ for solving $f(x) = 0$.

1) $f(x) = x^2 + 1$

2) $f(x) = x^3 + 2x + 1$

Answer

$$F(x_n) = x_n - \frac{x_n^3 + 2x_n + 1}{3x_n^2 + 2}$$

3) $f(x) = \sin x$

4) $f(x) = e^x$

Answer

$$F(x_n) = x_n - \frac{e^{x_n}}{e^{x_n}}$$

5) $f(x) = x^3 + 3xe^x$

In exercises 6 - 8, solve $f(x) = 0$ using the iteration $x_{n+1} = x_{n-c}f(x_n)$, which differs slightly from Newton's method. Find a c that works and a c that fails to converge, with the exception of $c = 0$.

6) $f(x) = x^2 - 4$, with $x_0 = 0$

Answer

$|c| > 0.5$ fails, $|c| \leq 0.5$ works

7) $f(x) = x^2 - 4x + 3$, with $x_0 = 2$

8) What is the value of "c" for Newton's method?

Answer

$$c = \frac{1}{f'(x_n)}$$

In exercises 9 - 16, compute x_1 and x_2 using the specified iterative method.

Start at

a. $x_0 = 0.6$ and

b. $x_0 = 2$.

9) $x_{n+1} = x_n^2 - \frac{1}{2}$

10) $x_{n+1} = 2x_n(1 - x_n)$

Answer

a. $x_1 = \frac{12}{25},\ x_2 = \frac{312}{625}$;
b. $x_1 = -4,\ x_2 = -40$

11) $x_{n+1} = \sqrt{x_n}$

12) $x_{n+1} = \frac{1}{\sqrt{x_n}}$

Answer

a. $x_1 = 1.291,\ x_2 = 0.8801$;
b. $x_1 = 0.7071,\ x_2 = 1.189$

13) $x_{n+1} = 3x_n(1 - x_n)$

14) $x_{n+1} = x_n^2 + x_{n-2}$

Answer

a. $x_1 = -\frac{26}{25},\ x_2 = -\frac{1224}{625}$;
b. $x_1 = 4,\ x_2 = 18$

15) $x_{n+1} = \frac{1}{2}x_n - 1$

16) $x_{n+1} = |x_n|$

Answer

a. $x_1 = \frac{6}{10},\ x_2 = \frac{6}{10}$;

b. $x_1 = 2,\ x_2 = 2$

In exercises 17 - 26, solve to four decimal places using Newton's method and a computer or calculator. Choose any initial guess x_0 that is not the exact root.

17) $x^2 - 10 = 0$

18) $x^4 - 100 = 0$

Answer

3.1623 or -3.1623

19) $x^2 - x = 0$

20) $x^3 - x = 0$

Answer

$0, -1$ or 1

21) $x + 5\cos x = 0$

22) $x + \tan x = 0$, choose $x_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$

Answer

0

23) $\frac{1}{1-x} = 2$

24) $1 + x + x^2 + x^3 + x^4 = 2$

Answer

0.5188 or -1.2906

25) $x^3 + (x+1)^3 = 10^3$

26) $x = \sin^2(x)$

Answer

0

In exercises 27 - 30, use Newton's method to find the fixed points of the function where $f(x) = x$; round to three decimals.

27) $\sin x$

28) $\tan x$ on $x = (\frac{\pi}{2}, \frac{3\pi}{2})$

Answer

4.493

29) $e^x - 2$

30) $\ln(x) + 2$

Answer

$0.159,\ 3.146$

Newton's method can be used to find maxima and minima of functions in addition to the roots. In this case apply Newton's method to the derivative function $f'(x)$ to find its roots, instead of the original function. In exercises 31 - 32, consider the formulation of the method.

31) To find candidates for maxima and minima, we need to find the critical points $f'(x) = 0$. Show that to solve for the critical points of a function $f(x)$, Newton's method is given by $x_{n+1} = x_n - \dfrac{f'(x_n)}{f''(x_n)}$.

32) What additional restrictions are necessary on the function f?

Answer

We need f to be twice continuously differentiable.

In exercises 33 - 40, use Newton's method to find the location of the local minima and/or maxima of the following functions; round to three decimals.

33) Minimum of $f(x) = x^2 + 2x + 4$

34) Minimum of $f(x) = 3x^3 + 2x^2 - 16$

Answer

$x = 0$

35) Minimum of $f(x) = x^2 e^x$

36) Maximum of $f(x) = x + \dfrac{1}{x}$

Answer

$x = -1$

37) Maximum of $f(x) = x^3 + 10x^2 + 15x - 2$

38) Maximum of $f(x) = \dfrac{\sqrt{x} - \sqrt[3]{x}}{x}$

Answer

$x = 5.619$

39) Minimum of $f(x) = x^2 \sin x$, closest non-zero minimum to $x = 0$

40) Minimum of $f(x) = x^4 + x^3 + 3x^2 + 12x + 6$

Answer

$x = -1.326$

In exercises 41 - 44, use the specified method to solve the equation. If it does not work, explain why it does not work.

41) Newton's method, $x^2 + 2 = 0$

42) Newton's method, $0 = e^x$

Answer

There is no solution to the equation.

43) Newton's method, $0 = 1 + x^2$ starting at $x_0 = 0$

44) Solving $x_{n+1} = -x_n^3$ starting at $x_0 = -1$

Answer

It enters a cycle.

In exercises 45 - 48, use the *secant method*, an alternative iterative method to Newton's method. The formula is given by

$$x_n = x_{n-1} - f(x_{n-1}) \frac{x_{n-1} - x_{n-2}}{f(x_{n-1}) - f(x_{n-2})}.$$

45) a root to $0 = x^2 - x - 3$ accurate to three decimal places.

46) Find a root to $0 = \sin x + 3x$ accurate to four decimal places.

Answer

0

47) Find a root to $0 = e^x - 2$ accurate to four decimal places.

48) Find a root to $\ln(x+2)=\frac{1}{2}$ accurate to four decimal places.

Answer

-0.3513

49) Why would you use the secant method over Newton's method? What are the necessary restrictions on f?

In exercises 50 - 54, use both Newton's method and the secant method to calculate a root for the following equations. Use a calculator or computer to calculate how many iterations of each are needed to reach within three decimal places of the exact answer. For the secant method, use the first guess from Newton's method.

50) $f(x)=x^2+2x+1, \quad x_0=1$

Answer

Newton: 11 iterations, secant: 16 iterations

51) $f(x)=x^2, \quad x_0=1$

52) $f(x)=\sin x, \quad x_0=1$

Answer

Newton: three iterations, secant: six iterations

53) $f(x)=e^x-1, \quad x_0=2$

54) $f(x)=x^3+2x+4, \quad x_0=0$

Answer

Newton: five iterations, secant: eight iterations

In exercises 55 - 56, consider Kepler's equation regarding planetary orbits, $M=E-\varepsilon\sin(E)$, where M is the mean anomaly, E is eccentric anomaly, and ε measures eccentricity.

55) Use Newton's method to solve for the eccentric anomaly E when the mean anomaly $M=\frac{\pi}{3}$ and the eccentricity of the orbit $\varepsilon=0.25$; round to three decimals.

56) Use Newton's method to solve for the eccentric anomaly E when the mean anomaly $M=\frac{3\pi}{2}$ and the eccentricity of the orbit $\varepsilon=0.8$; round to three decimals.

Answer

$E=4.071$

In exercises 57 - 58, consider a bank investment. The initial investment is $\$10,000$. After 25 years, the investment has tripled to $\$30,000$.

57) Use Newton's method to determine the interest rate if the interest was compounded annually.

58) Use Newton's method to determine the interest rate if the interest was compounded continuously.

Answer

4.394

59) The cost for printing a book can be given by the equation $C(x)=1000+12x+\frac{1}{2}x^{2/3}$. Use Newton's method to find the break-even point if the printer sells each book for $\$20$.

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4.10: Antiderivatives

Learning Objectives

- Find the general antiderivative of a given function.
- Explain the terms and notation used for an indefinite integral.
- State the power rule for integrals.
- Use antidifferentiation to solve simple initial-value problems.

At this point, we have seen how to calculate derivatives of many functions and have been introduced to a variety of their applications. We now ask a question that turns this process around: Given a function f, how do we find a function with the derivative f and why would we be interested in such a function?

We answer the first part of this question by defining antiderivatives. The antiderivative of a function f is a function with a derivative f. Why are we interested in antiderivatives? The need for antiderivatives arises in many situations, and we look at various examples throughout the remainder of the text. Here we examine one specific example that involves rectilinear motion. In our examination in Derivatives of rectilinear motion, we showed that given a position function $s(t)$ of an object, then its velocity function $v(t)$ is the derivative of $s(t)$—that is, $v(t) = s'(t)$. Furthermore, the acceleration $a(t)$ is the derivative of the velocity $v(t)$—that is, $a(t) = v'(t) = s''(t)$. Now suppose we are given an acceleration function a, but not the velocity function v or the position function s. Since $a(t) = v'(t)$, determining the velocity function requires us to find an antiderivative of the acceleration function. Then, since $v(t) = s'(t)$, determining the position function requires us to find an antiderivative of the velocity function. Rectilinear motion is just one case in which the need for antiderivatives arises. We will see many more examples throughout the remainder of the text. For now, let's look at the terminology and notation for antiderivatives, and determine the antiderivatives for several types of functions. We examine various techniques for finding antiderivatives of more complicated functions later in the text (Introduction to Techniques of Integration).

The Reverse of Differentiation

At this point, we know how to find derivatives of various functions. We now ask the opposite question. Given a function f, how can we find a function with derivative f? If we can find a function F with derivative f, we call F an antiderivative of f.

Definition: Antiderivative

A function F is an **antiderivative** of the function f if

$$F'(x) = f(x)$$

for all x in the domain of f.

Consider the function $f(x) = 2x$. Knowing the power rule of differentiation, we conclude that $F(x) = x^2$ is an antiderivative of f since $F'(x) = 2x$.

Are there any other antiderivatives of f?

Yes; since the derivative of any constant C is zero, $x^2 + C$ is also an antiderivative of $2x$. Therefore, $x^2 + 5$ and $x^2 - \sqrt{2}$ are also antiderivatives.

Are there any others that are not of the form $x^2 + C$ for some constant C?

The answer is no. From Corollary 2 of the Mean Value Theorem, we know that if F and G are differentiable functions such that $F'(x) = G'(x)$, then $F(x) - G(x) = C$ for some constant C. This fact leads to the following important theorem.

Theorem 4.10.1: General Form of an Antiderivative

Let F be an antiderivative of f over an interval I. Then,

I. for each constant C, the function $F(x) + C$ is also an antiderivative of f over I;

II. if G is an antiderivative of f over I, there is a constant C for which $G(x) = F(x) + C$ over I.

In other words, the most general form of the antiderivative of f over I is $F(x) + C$.

We use this fact and our knowledge of derivatives to find all the antiderivatives for several functions.

Example 4.10.1: Finding Antiderivatives

For each of the following functions, find all antiderivatives.

a. $f(x) = 3x^2$
b. $f(x) = \dfrac{1}{x}$
c. $f(x) = \cos x$
d. $f(x) = e^x$

Solution:

a. Because

$$\frac{d}{dx}\left(x^3\right) = 3x^2$$

then $F(x) = x^3$ is an antiderivative of $3x^2$. Therefore, every antiderivative of $3x^2$ is of the form $x^3 + C$ for some constant C, and every function of the form $x^3 + C$ is an antiderivative of $3x^2$.

b. Let $f(x) = \ln|x|$.

For $x > 0,\ f(x) = \ln|x| = \ln(x)$ and

$$\frac{d}{dx}\Big(\ln x\Big) = \frac{1}{x}.$$

For $x < 0,\ f(x) = \ln|x| = \ln(-x)$ and

$$\frac{d}{dx}\Big(\ln(-x)\Big) = -\frac{1}{-x} = \frac{1}{x}.$$

Therefore,

$$\frac{d}{dx}\Big(\ln|x|\Big) = \frac{1}{x}.$$

Thus, $F(x) = \ln|x|$ is an antiderivative of $\dfrac{1}{x}$. Therefore, every antiderivative of $\dfrac{1}{x}$ is of the form $\ln|x| + C$ for some constant C and every function of the form $\ln|x| + C$ is an antiderivative of $\dfrac{1}{x}$.

c. We have

$$\frac{d}{dx}\Big(\sin x\Big) = \cos x,$$

so $F(x) = \sin x$ is an antiderivative of $\cos x$. Therefore, every antiderivative of $\cos x$ is of the form $\sin x + C$ for some constant C and every function of the form $\sin x + C$ is an antiderivative of $\cos x$.

d. Since

$$\frac{d}{dx}\left(e^x\right) = e^x,$$

then $F(x) = e^x$ is an antiderivative of e^x. Therefore, every antiderivative of e^x is of the form $e^x + C$ for some constant C and every function of the form $e^x + C$ is an antiderivative of e^x.

Exercise 4.10.1

Find all antiderivatives of $f(x) = \sin x$.

Hint

What function has a derivative of $\sin x$?

Answer

$F(x) = -\cos x + C$

Indefinite Integrals

We now look at the formal notation used to represent antiderivatives and examine some of their properties. These properties allow us to find antiderivatives of more complicated functions. Given a function f, we use the notation $f'(x)$ or $\frac{df}{dx}$ to denote the derivative of f. Here we introduce notation for antiderivatives. If F is an antiderivative of f, we say that $F(x) + C$ is the most general antiderivative of f and write

$$\int f(x)\,dx = F(x) + C.$$

The symbol $\int$ is called an integral sign, and $\int f(x)\,dx$ is called the indefinite integral of f.

Definition: Indefinite Integrals

Given a function f, the indefinite integral of f, denoted

$$\int f(x)\,dx,$$

is the most general antiderivative of f. If F is an antiderivative of f, then

$$\int f(x)\,dx = F(x) + C.$$

The expression $f(x)$ is called the integrand and the variable x is the variable of integration.

Given the terminology introduced in this definition, the act of finding the antiderivatives of a function f is usually referred to as integrating f.

For a function f and an antiderivative F, the functions $F(x) + C$, where C is any real number, is often referred to as the family of antiderivatives of f. For example, since x^2 is an antiderivative of $2x$ and any antiderivative of $2x$ is of the form $x^2 + C$, we write

$$\int 2x\,dx = x^2 + C.$$

The collection of all functions of the form $x^2 + C$, where C is any real number, is known as the family of antiderivatives of $2x$. Figure 4.10.1 shows a graph of this family of antiderivatives.

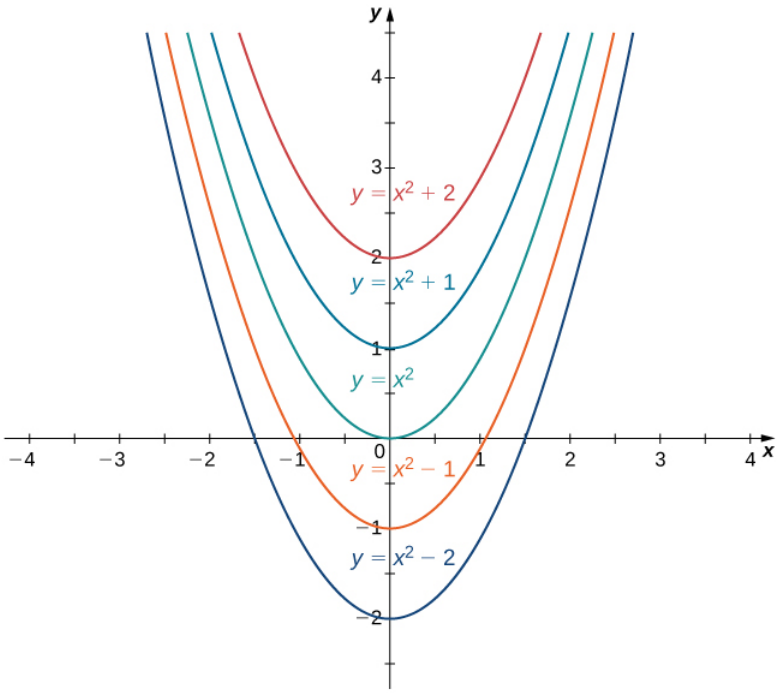

Figure 4.10.1: The family of antiderivatives of $2x$ consists of all functions of the form $x^2 + C$, where C is any real number.

For some functions, evaluating indefinite integrals follows directly from properties of derivatives. For example, for $n \neq -1$,

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C,$$

which comes directly from

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = (n+1)\frac{x^n}{n+1} = x^n.$$

This fact is known as the power rule for integrals.

Power Rule for Integrals

For $n \neq -1$,

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C.$$

Evaluating indefinite integrals for some other functions is also a straightforward calculation. The following table lists the indefinite integrals for several common functions. A more complete list appears in Appendix B.

Table 4.10.1: Integration Formulas

Differentiation Formula	Indefinite Integral
$\frac{d}{dx}\left(k\right) = 0$	$\int k\,dx = \int kx^0\,dx = kx + C$
$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$
$\frac{d}{dx}\left(\ln\lvert x\rvert\right) = \frac{1}{x}$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$
$\frac{d}{dx}\left(e^x\right) = e^x$	$\int e^x\,dx = e^x + C$
$\frac{d}{dx}\left(\sin x\right) = \cos x$	$\int \cos x\,dx = \sin x + C$
$\frac{d}{dx}\left(\cos x\right) = -\sin x$	$\int \sin x\,dx = -\cos x + C$
$\frac{d}{dx}\left(\tan x\right) = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
$\frac{d}{dx}\left(\csc x\right) = -\csc x\cot x$	$\int \csc x\cot x\,dx = -\csc x + C$
$\frac{d}{dx}\left(\sec x\right) = \sec x\tan x$	$\int \sec x\tan x\,dx = \sec x + C$
$\frac{d}{dx}\left(\cot x\right) = -\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} = \sin^{-1} x + C$
$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$
$\frac{d}{dx}\left(\sec^{-1}\lvert x\rvert\right) = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}\lvert x\rvert + C$

From the definition of indefinite integral of f, we know

$$\int f(x)\,dx = F(x) + C$$

if and only if F is an antiderivative of f.

Therefore, when claiming that

$$\int f(x)\,dx = F(x) + C$$

it is important to check whether this statement is correct by verifying that $F'(x) = f(x)$.

Example 4.10.2: Verifying an Indefinite Integral

Each of the following statements is of the form $\int f(x)\,dx = F(x) + C$. Verify that each statement is correct by showing that $F'(x) = f(x)$.

a. $\int \left(x + e^x\right)dx = \frac{x^2}{2} + e^x + C$

b. $\int xe^x\,dx = xe^x - e^x + C$

Solution:

a. Since

$$\frac{d}{dx}\left(\frac{x^2}{2} + e^x + C\right) = x + e^x,$$

the statement

$$\int \left(x + e^x\right) dx = \frac{x^2}{2} + e^x + C$$

is correct.

Note that we are verifying an indefinite integral for a sum. Furthermore, $\frac{x^2}{2}$ and e^x are antiderivatives of x and e^x, respectively, and the sum of the antiderivatives is an antiderivative of the sum. We discuss this fact again later in this section.

b. Using the product rule, we see that

$$\frac{d}{dx}\left(xe^x - e^x + C\right) = e^x + xe^x - e^x = xe^x.$$

Therefore, the statement

$$\int xe^x\, dx = xe^x - e^x + C$$

is correct.

Note that we are verifying an indefinite integral for a product. The antiderivative $xe^x - e^x$ is not a product of the antiderivatives. Furthermore, the product of antiderivatives, $x^2e^x/2$ is not an antiderivative of xe^x since

$$\frac{d}{dx}\left(\frac{x^2e^x}{2}\right) = xe^x + \frac{x^2e^x}{2} \neq xe^x.$$

In general, the product of antiderivatives is not an antiderivative of a product.

? Exercise 4.10.2

Verify that $\int x\cos x\ dx = x\sin x + \cos x + C$.

Hint

Calculate $\frac{d}{dx}\left(x\sin x + \cos x + C\right)$.

Answer

$\frac{d}{dx}\left(x\sin x + \cos x + C\right) = \sin x + x\cos x - \sin x = x\cos x$

In Table 4.10.1, we listed the indefinite integrals for many elementary functions. Let's now turn our attention to evaluating indefinite integrals for more complicated functions. For example, consider finding an antiderivative of a sum $f + g$. In Example 4.10.2a we showed that an antiderivative of the sum $x + e^x$ is given by the sum $\frac{x^2}{2} + e^x$—that is, an antiderivative of a sum is given by a sum of antiderivatives. This result was not specific to this example. In general, if F and G are antiderivatives of any functions f and g, respectively, then

$$\frac{d}{dx}\left(F(x) + G(x)\right) = F'(x) + G'(x) = f(x) + g(x).$$

Therefore, $F(x) + G(x)$ is an antiderivative of $f(x) + g(x)$ and we have

$$\int \left(f(x) + g(x)\right) dx = F(x) + G(x) + C.$$

Similarly,

$$\int \left(f(x) - g(x)\right) dx = F(x) - G(x) + C.$$

In addition, consider the task of finding an antiderivative of $kf(x)$, where k is any real number. Since

$$\frac{d}{dx}\left(kF(x)\right) = k\frac{d}{dx}\left(F(x)\right) = kF'(x)$$

for any real number k, we conclude that

$$\int kf(x)\,dx = kF(x) + C.$$

These properties are summarized next.

Properties of Indefinite Integrals

Let F and G be antiderivatives of f and g, respectively, and let k be any real number.

Sums and Differences

$$\int \big(f(x) \pm g(x)\big)\,dx = F(x) \pm G(x) + C$$

Constant Multiples

$$\int kf(x)\,dx = kF(x) + C$$

From this theorem, we can evaluate any integral involving a sum, difference, or constant multiple of functions with antiderivatives that are known. Evaluating integrals involving products, quotients, or compositions is more complicated. (See Example $4.10.2b$ for an example involving an antiderivative of a product.) We look at and address integrals involving these more complicated functions in Introduction to Integration. In the next example, we examine how to use this theorem to calculate the indefinite integrals of several functions.

Example $4.10.3$: Evaluating Indefinite Integrals

Evaluate each of the following indefinite integrals:

a. $\int \big(5x^3 - 7x^2 + 3x + 4\big)\,dx$

b. $\int \dfrac{x^2 + 4\sqrt[3]{x}}{x}\,dx$

c. $\int \dfrac{4}{1 + x^2}\,dx$

d. $\int \tan x \cos x\,dx$

Solution:

a. Using Properties of Indefinite Integrals, we can integrate each of the four terms in the integrand separately. We obtain

$$\int \big(5x^3 - 7x^2 + 3x + 4\big)\,dx = \int 5x^3\,dx - \int 7x^2\,dx + \int 3x\,dx + \int 4\,dx.$$

From the second part of Properties of Indefinite Integrals, each coefficient can be written in front of the integral sign, which gives

$$\int 5x^3\,dx - \int 7x^2\,dx + \int 3x\,dx + \int 4\,dx = 5\int x^3\,dx - 7\int x^2\,dx + 3\int x\,dx + 4\int 1\,dx.$$

Using the power rule for integrals, we conclude that

$$\int \big(5x^3 - 7x^2 + 3x + 4\big)\,dx = \frac{5}{4}x^4 - \frac{7}{3}x^3 + \frac{3}{2}x^2 + 4x + C.$$

b. Rewrite the integrand as

$$\frac{x^2 + 4\sqrt[3]{x}}{x} = \frac{x^2}{x} + \frac{4\sqrt[3]{x}}{x}.$$

Then, to evaluate the integral, integrate each of these terms separately. Using the power rule, we have

$$\int \left(x + \frac{4}{x^{2/3}}\right) dx = \int x\, dx + 4\int x^{-2/3}\, dx$$
$$= \frac{1}{2}x^2 + 4\frac{1}{\left(\frac{-2}{3}\right)+1}x^{(-2/3)+1} + C$$
$$= \frac{1}{2}x^2 + 12x^{1/3} + C.$$

c. Using Properties of Indefinite Integrals, write the integral as

$$4\int \frac{1}{1+x^2}\, dx.$$

Then, use the fact that $\tan^{-1}(x)$ is an antiderivative of $\frac{1}{1+x^2}$ to conclude that

$$\int \frac{4}{1+x^2}\, dx = 4\tan^{-1}(x) + C.$$

d. Rewrite the integrand as

$$\tan x \cos x = \frac{\sin x}{\cos x} \cdot \cos x = \sin x.$$

Therefore,

$$\int \tan x \cos x\, dx = \int \sin x\, dx = -\cos x + C.$$

? Exercise 4.10.3

Evaluate $\int \left(4x^3 - 5x^2 + x - 7\right) dx$.

Hint

Integrate each term in the integrand separately, making use of the power rule.

Answer

$$\int \left(4x^3 - 5x^2 + x - 7\right) dx = \quad x^4 - \frac{5}{3}x^3 + \frac{1}{2}x^2 - 7x + C$$

Initial-Value Problems

We look at techniques for integrating a large variety of functions involving products, quotients, and compositions later in the text. Here we turn to one common use for antiderivatives that arises often in many applications: solving differential equations.

A differential equation is an equation that relates an unknown function and one or more of its derivatives. The equation

$$\frac{dy}{dx} = f(x) \tag{4.10.1}$$

is a simple example of a differential equation. Solving this equation means finding a function y with a derivative f. Therefore, the solutions of Equation 4.10.1 are the antiderivatives of f. If F is one antiderivative of f, every function of the form $y = F(x) + C$ is a solution of that differential equation. For example, the solutions of

$$\frac{dy}{dx} = 6x^2$$

are given by

$$y = \int 6x^2\, dx = 2x^3 + C.$$

Sometimes we are interested in determining whether a particular solution curve passes through a certain point (x_0, y_0) —that is, $y(x_0) = y_0$. The problem of finding a function y that satisfies a differential equation

$$\frac{dy}{dx} = f(x)$$

with the additional condition

$$y(x_0) = y_0$$

is an example of an initial-value problem. The condition $y(x_0) = y_0$ is known as an initial condition. For example, looking for a function y that satisfies the differential equation

$$\frac{dy}{dx} = 6x^2$$

and the initial condition

$$y(1) = 5$$

is an example of an initial-value problem. Since the solutions of the differential equation are $y = 2x^3 + C$, to find a function y that also satisfies the initial condition, we need to find C such that $y(1) = 2(1)^3 + C = 5$. From this equation, we see that $C = 3$, and we conclude that $y = 2x^3 + 3$ is the solution of this initial-value problem as shown in the following graph.

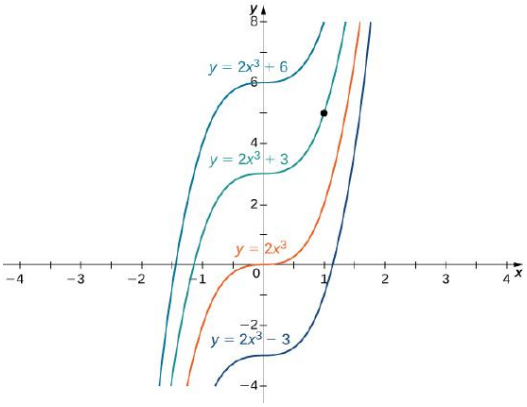

Figure 4.10.2: Some of the solution curves of the differential equation $\frac{dy}{dx} = 6x^2$ are displayed. The function $y = 2x^3 + 3$ satisfies the differential equation and the initial condition $y(1) = 5$.

Example 4.10.4: Solving an Initial-Value Problem

Solve the initial-value problem

$$\frac{dy}{dx} = \sin x, \quad y(0) = 5.$$

Solution

First we need to solve the differential equation. If $\frac{dy}{dx} = \sin x$, then

$$y = \int \sin(x)\, dx = -\cos x + C.$$

Next we need to look for a solution y that satisfies the initial condition. The initial condition $y(0) = 5$ means we need a constant C such that $-\cos x + C = 5$. Therefore,

$$C = 5 + \cos(0) = 6.$$

The solution of the initial-value problem is $y = -\cos x + 6$.

Exercise 4.10.4

Solve the initial value problem $\frac{dy}{dx} = 3x^{-2}, \quad y(1) = 2.$

Hint

Find all antiderivatives of $f(x) = 3x^{-2}$.

Answer

$$y = -\frac{3}{x} + 5$$

Initial-value problems arise in many applications. Next we consider a problem in which a driver applies the brakes in a car. We are interested in how long it takes for the car to stop. Recall that the velocity function $v(t)$ is the derivative of a position function $s(t)$, and the acceleration $a(t)$ is the derivative of the velocity function. In earlier examples in the text, we could calculate the velocity from the position and then compute the acceleration from the velocity. In the next example we work the other way around. Given an acceleration function, we calculate the velocity function. We then use the velocity function to determine the position function.

Example 4.10.5:

A car is traveling at the rate of 88 ft/sec (60 mph) when the brakes are applied. The car begins decelerating at a constant rate of 15 ft/sec^2.

a. How many seconds elapse before the car stops?
b. How far does the car travel during that time?

Solution

a. First we introduce variables for this problem. Let t be the time (in seconds) after the brakes are first applied. Let $a(t)$ be the acceleration of the car (in feet per seconds squared) at time t. Let $v(t)$ be the velocity of the car (in feet per second) at time t. Let $s(t)$ be the car's position (in feet) beyond the point where the brakes are applied at time t.

The car is traveling at a rate of 88 ft/sec. Therefore, the initial velocity is $v(0) = 88$ ft/sec. Since the car is decelerating, the acceleration is

$$a(t) = -15\,\text{ft/sec}^2.$$

The acceleration is the derivative of the velocity,

$$v'(t) = -15.$$

Therefore, we have an initial-value problem to solve:

$$v'(t) = -15, \quad v(0) = 88.$$

Integrating, we find that

$$v(t) = -15t + C.$$

Since $v(0) = 88, C = 88$. Thus, the velocity function is

$$v(t) = -15t + 88.$$

To find how long it takes for the car to stop, we need to find the time t such that the velocity is zero. Solving $-15t + 88 = 0$, we obtain $t = \dfrac{88}{15}$ sec.

b. To find how far the car travels during this time, we need to find the position of the car after $\dfrac{88}{15}$ sec. We know the velocity $v(t)$ is the derivative of the position $s(t)$. Consider the initial position to be $s(0) = 0$. Therefore, we need to solve the initial-value problem

$$s'(t) = -15t + 88, \quad s(0) = 0.$$

Integrating, we have

$$s(t) = -\frac{15}{2}t^2 + 88t + C.$$

Since $s(0) = 0$, the constant is $C = 0$. Therefore, the position function is

$$s(t) = -\frac{15}{2}t^2 + 88t.$$

After $t = \frac{88}{15}$ sec, the position is $s\left(\frac{88}{15}\right) \approx 258.133$ ft.

Exercise 4.10.5

Suppose the car is traveling at the rate of 44 ft/sec. How long does it take for the car to stop? How far will the car travel?

Hint

$v(t) = -15t + 44$.

Answer

2.93 sec, 64.5 ft

Key Concepts

- If F is an antiderivative of f, then every antiderivative of f is of the form $F(x) + C$ for some constant C.
- Solving the initial-value problem

$$\frac{dy}{dx} = f(x), \quad y(x_0) = y_0$$

requires us first to find the set of antiderivatives of f and then to look for the particular antiderivative that also satisfies the initial condition.

Glossary

antiderivative

a function F such that $F'(x) = f(x)$ for all x in the domain of f is an antiderivative of f

indefinite integral

the most general antiderivative of $f(x)$ is the indefinite integral of f; we use the notation $\int f(x)\,dx$ to denote the indefinite integral of f

initial value problem

a problem that requires finding a function y that satisfies the differential equation $\frac{dy}{dx} = f(x)$ together with the initial condition $y(x_0) = y_0$

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4.10E: EXERCISES FOR SECTION 4.10

In exercises 1 - 20, find the antiderivative $F(x)$ of each function $f(x)$.

1) $f(x) = \dfrac{1}{x^2} + x$

2) $f(x) = e^x - 3x^2 + \sin x$

Answer

$F(x) = e^x - x^3 - \cos x + C$

3) $f(x) = e^x + 3x - x^2$

4) $f(x) = x - 1 + 4\sin(2x)$

Answer

$F(x) = \dfrac{x^2}{2} - x - 2\cos(2x) + C$

5) $f(x) = 5x^4 + 4x^5$

6) $f(x) = x + 12x^2$

Answer

$F(x) = \frac{1}{2}x^2 + 4x^3 + C$

7) $f(x) = \dfrac{1}{\sqrt{x}}$

8) $f(x) = (\sqrt{x})^3$

Answer

$F(x) = \frac{2}{5}(\sqrt{x})^5 + C$

9) $f(x) = x^{1/3} + \left(2x\right)^{1/3}$

10) $f(x) = \dfrac{x^{1/3}}{x^{2/3}}$

Answer

$F(x) = \frac{3}{2}x^{2/3} + C$

11) $f(x) = 2\sin(x) + \sin(2x)$

12) $f(x) = \sec^2 x + 1$

Answer

$F(x) = x + \tan x + C$

13) $f(x) = \sin x \cos x$

14) $f(x) = \sin^2(x)\cos(x)$

Answer

$F(x) = \frac{1}{3}\sin^3(x) + C$

15) $f(x) = 0$

16) $f(x) = \frac{1}{2}\csc^2 x + \dfrac{1}{x^2}$

Answer

$F(x) = -\frac{1}{2}\cot x - \dfrac{1}{x} + C$

17) $f(x) = \csc x \cot x + 3x$

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18) $f(x) = 4\csc x \cot x - \sec x \tan x$

Answer

$$F(x) = -\sec x - 4\csc x + C$$

19) $f(x) = 8(\sec x)\left(\sec x - 4\tan x\right)$

20) $f(x) = \frac{1}{2}e^{-4x} + \sin x$

Answer

$$F(x) = -\frac{1}{8}e^{-4x} - \cos x + C$$

For exercises 21 - 29, evaluate the integral.

21) $\int (-1)\, dx$

22) $\int \sin x\, dx$

Answer

$$\int \sin x\, dx = -\cos x + C$$

23) $\int \left(4x + \sqrt{x}\right) dx$

24) $\int \frac{3x^2 + 2}{x^2}\, dx$

Answer

$$\int \frac{3x^2 + 2}{x^2}\, dx = 3x - \frac{2}{x} + C$$

25) $\int \left(\sec x \tan x + 4x\right) dx$

26) $\int \left(4\sqrt{x} + \sqrt[4]{x}\right) dx$

Answer

$$\int \left(4\sqrt{x} + \sqrt[4]{x}\right) dx = \frac{8}{3}x^{3/2} + \frac{4}{5}x^{5/4} + C$$

27) $\int \left(x^{-1/3} - x^{2/3}\right)\, dx$

28) $\int \frac{14x^3 + 2x + 1}{x^3}\, dx$

Answer

$$\int \frac{14x^3 + 2x + 1}{x^3}\, dx = 14x - \frac{2}{x} - \frac{1}{2x^2} + C$$

29) $\int \left(e^x + e^{-x}\right) dx$

In exercises 30 - 34, solve the initial value problem.

30) $f'(x) = x^{-3}, \quad f(1) = 1$

Answer

$$f(x) = -\frac{1}{2x^2} + \frac{3}{2}$$

31) $f'(x) = \sqrt{x} + x^2, \quad f(0) = 2$

32) $f'(x) = \cos x + \sec^2(x), \quad f(\frac{\pi}{4}) = 2 + \frac{\sqrt{2}}{2}$

Answer

$$f(x) = \sin x + \tan x + 1$$

33) $f'(x) = x^3 - 8x^2 + 16x + 1, \quad f(0) = 0$

34) $f'(x) = \frac{2}{x^2} - \frac{x^2}{2}, \quad f(1) = 0$

Answer

$f(x) = -\frac{1}{6}x^3 - \frac{2}{x} + \frac{13}{6}$

In exercises 35 - 39, find two possible functions f given the second- or third-order derivatives

35) $f''(x) = x^2 + 2$

36) $f''(x) = e^{-x}$

Answer

Answers may vary; one possible answer is $f(x) = e^{-x}$

37) $f''(x) = 1 + x$

38) $f'''(x) = \cos x$

Answer

Answers may vary; one possible answer is $f(x) = -\sin x$

39) $f'''(x) = 8e^{-2x} - \sin x$

40) A car is being driven at a rate of 40 mph when the brakes are applied. The car decelerates at a constant rate of $10\,\text{ft/sec}^2$. How long before the car stops?

Answer

5.867 sec

41) In the preceding problem, calculate how far the car travels in the time it takes to stop.

42) You are merging onto the freeway, accelerating at a constant rate of $12\,\text{ft/sec}^2$. How long does it take you to reach merging speed at 60 mph?

Answer

7.333 sec

43) Based on the previous problem, how far does the car travel to reach merging speed?

44) A car company wants to ensure its newest model can stop in 8 sec when traveling at 75 mph. If we assume constant deceleration, find the value of deceleration that accomplishes this.

Answer

$13.75\,\text{ft/sec}^2$

45) A car company wants to ensure its newest model can stop in less than 450 ft when traveling at 60 mph. If we assume constant deceleration, find the value of deceleration that accomplishes this.

In exercises 46 - 51, find the antiderivative of the function, assuming $F(0) = 0$.

46) [T] $f(x) = x^2 + 2$

Answer

$F(x) = \frac{1}{3}x^3 + 2x$

47) [T] $f(x) = 4x - \sqrt{x}$

48) [T] $f(x) = \sin x + 2x$

Answer

$F(x) = x^2 - \cos x + 1$

49) [T] $f(x) = e^x$

50) [T] $f(x) = \dfrac{1}{(x+1)^2}$

Answer

$F(x) = -\dfrac{1}{x+1} + 1$

51) [T] $f(x) = e^{-2x} + 3x^2$

In exercises 52 - 55, determine whether the statement is true or false. Either prove it is true or find a counterexample if it is false.

52) If $f(x)$ is the antiderivative of $v(x)$, then $2f(x)$ is the antiderivative of $2v(x)$.

Answer

True

53) If $f(x)$ is the antiderivative of $v(x)$, then $f(2x)$ is the antiderivative of $v(2x)$.

54) If $f(x)$ is the antiderivative of $v(x)$, then $f(x) + 1$ is the antiderivative of $v(x) + 1$.

Answer

False

55) If $f(x)$ is the antiderivative of $v(x)$, then $(f(x))^2$ is the antiderivative of $(v(x))^2$.

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4.R: CHAPTER 4 REVIEW EXERCISES

True or False? Justify your answer with a proof or a counterexample. Assume that $f(x)$ is continuous and differentiable unless stated otherwise.

1) If $f(-1) = -6$ and $f(1) = 2$, then there exists at least one point $x \in [-1, 1]$ such that $f'(x) = 4$.

Answer

True, by Mean Value Theorem

2) If $f'(c) = 0$, there is a maximum or minimum at $x = c$.

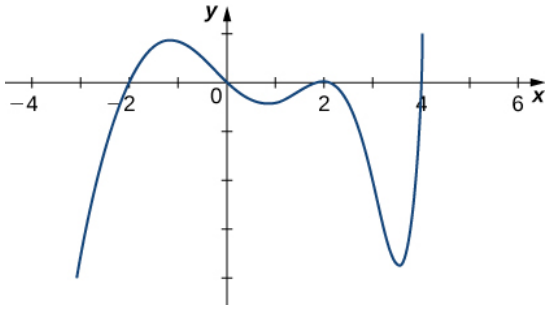
3) There is a function such that $f(x) < 0, f'(x) > 0$, and $f''(x) < 0$. (A graphical "proof" is acceptable for this answer.)

Answer

True

4) There is a function such that there is both an inflection point and a critical point for some value $x = a$.

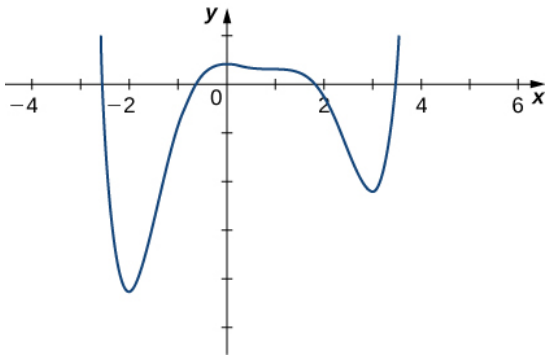
5) Given the graph of f', determine where f is increasing or decreasing.



Answer

Increasing: $(-2, 0) \cup (4, \infty)$, decreasing: $(-\infty, -2) \cup (0, 4)$

6) The graph of f is given below. Draw f'.



7) Find the linear approximation $L(x)$ to $y = x^2 + \tan(\pi x)$ near $x = \frac{1}{4}$.

Answer

$L(x) = \frac{17}{16} + \frac{1}{2}(1 + 4\pi)\left(x - \frac{1}{4}\right)$

8) Find the differential of $y = x^2 - 5x - 6$ and evaluate for $x = 2$ with $dx = 0.1$.

Find the critical points and the local and absolute extrema of the following functions on the given interval.

9) $f(x) = x + \sin^2(x)$ over $[0, \pi]$

Answer

Critical point: $x = \frac{3\pi}{4}$,
Absolute minimum: 0 when $x = 0$,

Absolute maximum: π when $x = \pi$

Solution:

10) $f(x) = 3x^4 - 4x^3 - 12x^2 + 6$ over $[-3, 3]$

Determine over which intervals the following functions are increasing, decreasing, concave up, and concave down.

11) $x(t) = 3t^4 - 8t^3 - 18t^2$

Answer

Increasing: $(-1, 0) \cup (3, \infty)$,
Decreasing: $(-\infty, -1) \cup (0, 3)$,
Concave up: $\left(-\infty, \frac{1}{3}\left(2 - \sqrt{13}\right)\right) \cup \left(\frac{1}{3}\left(2 + \sqrt{13}\right), \infty\right)$,
Concave down: $\left(\frac{1}{3}\left(2 - \sqrt{13}\right), \frac{1}{3}\left(2 + \sqrt{13}\right)\right)$

12) $y = x + \sin(\pi x)$

13) $g(x) = x - \sqrt{x}$

Answer

Increasing: $\left(\frac{1}{4}, \infty\right)$,
Decreasing: $\left(0, \frac{1}{4}\right)$,
Concave up: $(0, \infty)$,
Concave down: nowhere

14) $f(\theta) = \sin(3\theta)$

Evaluate the following limits.

15) $\lim\limits_{x \to \infty} \dfrac{3x\sqrt{x^2 + 1}}{\sqrt{x^4 - 1}}$

Answer

3

16) $\lim\limits_{x \to \infty} \cos\left(\dfrac{1}{x}\right)$

17) $\lim\limits_{x \to 1} \dfrac{x - 1}{\sin(\pi x)}$

Answer

$-\frac{1}{\pi}$

18) $\lim\limits_{x \to \infty} (3x)^{1/x}$

Use Newton's method to find the first two iterations, given the starting point.

19) $y = x^3 + 1, \quad x_0 = 0.5$

Answer

$x_1 = -1, \ x_2 = -1$

20) $\dfrac{1}{x + 1} = \dfrac{1}{2}, \quad x_0 = 0$

Find the antiderivatives $F(x)$ of the following functions.

21) $g(x) = \sqrt{x} - \dfrac{1}{x^2}$

Answer

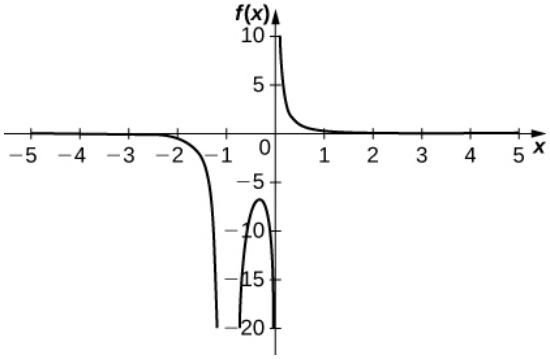
$F(x) = \dfrac{2x^{3/2}}{3} + \dfrac{1}{x} + C$

22) $f(x) = 2x + 6\cos x, \quad F(\pi) = \pi^2 + 2$

Graph the following functions by hand. Make sure to label the inflection points, critical points, zeros, and asymptotes.

23) $y = \dfrac{1}{x(x + 1)^2}$

Answer



Inflection points: none;
Critical points: $x = -\frac{1}{3}$;
Zeros: none;
Vertical asymptotes: $x = -1,\ x = 0$;
Horizontal asymptote: $y = 0$

24) $y = x - \sqrt{4 - x^2}$

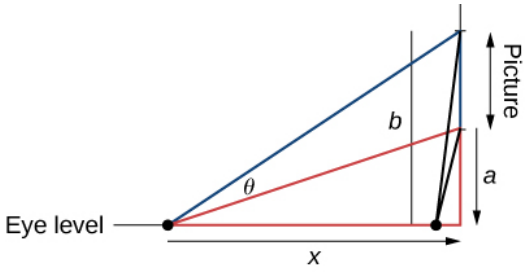
25) A car is being compacted into a rectangular solid. The volume is decreasing at a rate of $2\,\text{m}^3/\text{sec}$. The length and width of the compactor are square, but the height is not the same length as the length and width. If the length and width walls move toward each other at a rate of 0.25 m/sec, find the rate at which the height is changing when the length and width are 2 m and the height is 1.5 m.

Answer

The height is decreasing at a rate of 0.125 m/sec

26) A rocket is launched into space; its kinetic energy is given by $K(t) = \frac{1}{2}m(t)v(t)^2$, where K is the kinetic energy in joules, m is the mass of the rocket in kilograms, and v is the velocity of the rocket in meters/second. Assume the velocity is increasing at a rate of $15\,\text{m/sec}^2$ and the mass is decreasing at a rate of 10 kg/sec because the fuel is being burned. At what rate is the rocket's kinetic energy changing when the mass is 2000 kg and the velocity is 5000 m/sec? Give your answer in mega-Joules (MJ), which is equivalent to 10^6 J.

27) The famous **Regiomontanus' problem** for angle maximization was proposed during the 15th century. A painting hangs on a wall with the bottom of the painting a distance a feet above eye level, and the top b feet above eye level. What distance x (in feet) from the wall should the viewer stand to maximize the angle subtended by the painting, θ?



Answer

$x = \sqrt{ab}$ feet

28) An airline sells tickets from Tokyo to Detroit for $\$1200$. There are 500 seats available and a typical flight books 350 seats. For every $\$10$ decrease in price, the airline observes an additional five seats sold. What should the fare be to maximize profit? How many passengers would be onboard?

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CHAPTER OVERVIEW

5: Integration

5.0: Prelude to Integration
5.1: Approximating Areas
5.1E: Exercises for Section 5.1
5.2: The Definite Integral
5.2E: Exercises for Section 5.2
5.3: The Fundamental Theorem of Calculus
5.3E: Exercises for Section 5.3
5.4: Integration Formulas and the Net Change Theorem
5.4E: Exercises for Section 5.4
5.5: Substitution
5.5E: Exercises for Section 5.5
5.6: Integrals Involving Exponential and Logarithmic Functions
5.6E: Exercises for Section 5.6
5.7: Integrals Resulting in Inverse Trigonometric Functions
5.7E: Exercises for Section 5.7
5.7: Integrals Resulting in Inverse Trigonometric Functions
5.R: Chapter 5 Review Exercises

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5.0: Prelude to Integration

Iceboats are a common sight on the lakes of Wisconsin and Minnesota on winter weekends. Iceboats are similar to sailboats, but they are fitted with runners, or “skates,” and are designed to run over the ice, rather than on water. Iceboats can move very quickly, and many ice boating enthusiasts are drawn to the sport because of the speed. Top iceboat racers can attain speeds up to five times the wind speed. If we know how fast an iceboat is moving, we can use integration to determine how far it travels. We revisit this question later in the chapter.

Iceboating is a popular winter sport in parts of the northern United States and Europe. (credit: modification of work by Carter Brown, Flickr)

Determining distance from velocity is just one of many applications of integration. In fact, integrals are used in a wide variety of mechanical and physical applications. In this chapter, we first introduce the theory behind integration and use integrals to calculate areas. From there, we develop the Fundamental Theorem of Calculus, which relates differentiation and integration. We then study some basic integration techniques and briefly examine some applications.

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5.1: Approximating Areas

Learning Objectives

- Use sigma (summation) notation to calculate sums and powers of integers.
- Use the sum of rectangular areas to approximate the area under a curve.
- Use Riemann sums to approximate area.

Archimedes was fascinated with calculating the areas of various shapes—in other words, the amount of space enclosed by the shape. He used a process that has come to be known as the **method of exhaustion**, which used smaller and smaller shapes, the areas of which could be calculated exactly, to fill an irregular region and thereby obtain closer and closer approximations to the total area. In this process, an area bounded by curves is filled with rectangles, triangles, and shapes with exact area formulas. These areas are then summed to approximate the area of the curved region.

In this section, we develop techniques to approximate the area between a curve, defined by a function $f(x)$, and the x-axis on a closed interval $[a, b]$. Like Archimedes, we first approximate the area under the curve using shapes of known area (namely, rectangles). By using smaller and smaller rectangles, we get closer and closer approximations to the area. Taking a limit allows us to calculate the exact area under the curve.

Let's start by introducing some notation to make the calculations easier. We then consider the case when $f(x)$ is continuous and nonnegative. Later in the chapter, we relax some of these restrictions and develop techniques that apply in more general cases.

Sigma (Summation) Notation

As mentioned, we will use shapes of known area to approximate the area of an irregular region bounded by curves. This process often requires adding up long strings of numbers. To make it easier to write down these lengthy sums, we look at some new notation here, called **sigma notation** (also known as **summation notation**). The Greek capital letter Σ, sigma, is used to express long sums of values in a compact form. For example, if we want to add all the integers from 1 to 20 without sigma notation, we have to write

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 + 20.$$

We could probably skip writing a couple of terms and write

$$1 + 2 + 3 + 4 + \cdots + 19 + 20,$$

which is better, but still cumbersome. With sigma notation, we write this sum as

$$\sum_{i=1}^{20} i$$

which is much more compact. Typically, sigma notation is presented in the form

$$\sum_{i=1}^{n} a_i$$

where a_i describes the terms to be added, and the i is called the $index$. Each term is evaluated, then we sum all the values, beginning with the value when $i = 1$ and ending with the value when $i = n$. For example, an expression like $\sum_{i=2}^{7} s_i$ is interpreted as $s_2 + s_3 + s_4 + s_5 + s_6 + s_7$. Note that the index is used only to keep track of the terms to be added; it does not factor into the calculation of the sum itself. The index is therefore called a **dummy variable**. We can use any letter we like for the index. Typically, mathematicians use i, j, k, m, and n for indices.

Let's try a couple of examples of using sigma notation.

Example 5.1.1: Using Sigma Notation

a. Write in sigma notation and evaluate the sum of terms 3^i for $i = 1, 2, 3, 4, 5$.
b. Write the sum in sigma notation:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}.$$

Solution

a. Write

$$\sum_{i=1}^{5} 3^i = 3 + 3^2 + 3^3 + 3^4 + 3^5 = 363.$$

b. The denominator of each term is a perfect square. Using sigma notation, this sum can be written as $\sum_{i=1}^{5} \frac{1}{i^2}$.

Exercise 5.1.1

Write in sigma notation and evaluate the sum of terms 2^i for $i = 3, 4, 5, 6$.

Hint

Use the solving steps in Example 5.1.1 as a guide.

Answer

$$\sum_{i=3}^{6} 2^i = 2^3 + 2^4 + 2^5 + 2^6 = 120$$

The properties associated with the summation process are given in the following rule.

Rule: Properties of Sigma Notation

Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ represent two sequences of terms and let c be a constant. The following properties hold for all positive integers n and for integers m, with $1 \leq m \leq n$.

i. $\sum_{i=1}^{n} c = nc$
ii. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$
iii. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$
iv. $\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$
v. $\sum_{i=1}^{n} a_i = \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i$

Proof

We prove properties (ii.) and (iii.) here, and leave proof of the other properties to the Exercises.

(ii.) We have

$$\sum_{i=1}^{n} ca_i = ca_1 + ca_2 + ca_3 + \cdots + ca_n = c(a_1 + a_2 + a_3 + \cdots + a_n) = c\sum_{i=1}^{n} a_i.$$

(iii.) We have

$$\sum_{i=1}^{n} (a_i + b_i) = (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \cdots + (a_n + b_n) \tag{5.1.1}$$

$$= (a_1 + a_2 + a_3 + \cdots + a_n) + (b_1 + b_2 + b_3 + \cdots + b_n) \tag{5.1.2}$$

$$= \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i. \tag{5.1.3}$$

□

A few more formulas for frequently found functions simplify the summation process further. These are shown in the next rule, for **sums and powers of integers**, and we use them in the next set of examples.

Rule: Sums and Powers of Integers

1. The sum of n integers is given by

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}. \tag{5.1.4}$$

2. The sum of consecutive integers squared is given by

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}. \tag{5.1.5}$$

3. The sum of consecutive integers cubed is given by

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}. \tag{5.1.6}$$

Example 5.1.2: Evaluation Using Sigma Notation

Write using sigma notation and evaluate:

a. The sum of the terms $(i-3)^2$ for $i = 1, 2, \ldots, 200$.
b. The sum of the terms $(i^3 - i^2)$ for $i = 1, 2, 3, 4, 5, 6$

Solution

a. Multiplying out $(i-3)^2$, we can break the expression into three terms.

$$\begin{aligned}
\sum_{i=1}^{200} (i-3)^2 &= \sum_{i=1}^{200} (i^2 - 6i + 9) \\
&= \sum_{i=1}^{200} i^2 - \sum_{i=1}^{200} 6i + \sum_{i=1}^{200} 9 \\
&= \sum_{i=1}^{200} i^2 - 6\sum_{i=1}^{200} i + \sum_{i=1}^{200} 9 \\
&= \frac{200(200+1)(400+1)}{6} - 6\left[\frac{200(200+1)}{2}\right] + 9(200) \\
&= 2,686,700 - 120,600 + 1800 \\
&= 2,567,900
\end{aligned}$$

b. Use sigma notation property iv. and the rules for the sum of squared terms and the sum of cubed terms.

$$\begin{aligned}
\sum_{i=1}^{6} (i^3 - i^2) &= \sum_{i=1}^{6} i^3 - \sum_{i=1}^{6} i^2 \\
&= \frac{6^2(6+1)^2}{4} - \frac{6(6+1)(2(6)+1)}{6} \\
&= \frac{1764}{4} - \frac{546}{6} \\
&= 350
\end{aligned}$$

Exercise 5.1.2

Find the sum of the values of $4+3i$ for $i=1,2,\ldots,100$.

Hint

Use the properties of sigma notation to solve the problem.

Answer

$15,550$

Example 5.1.3: Finding the Sum of the Function Values

Find the sum of the values of $f(x)=x^3$ over the integers $1,2,3,\ldots,10$.

Solution

Using Equation 5.1.6, we have

$$\sum_{i=0}^{10} i^3 = \frac{(10)^2(10+1)^2}{4} = \frac{100(121)}{4} = 3025$$

Exercise 5.1.3

Evaluate the sum indicated by the notation $\sum_{k=1}^{20}(2k+1)$.

Hint

Use the rule on sum and powers of integers (Equations 5.1.4-5.1.6).

Answer

440

Approximating Area

Now that we have the necessary notation, we return to the problem at hand: approximating the area under a curve. Let $f(x)$ be a continuous, nonnegative function defined on the closed interval $[a,b]$. We want to approximate the area A bounded by $f(x)$ above, the x-axis below, the line $x=a$ on the left, and the line $x=b$ on the right (Figure 5.1.1).

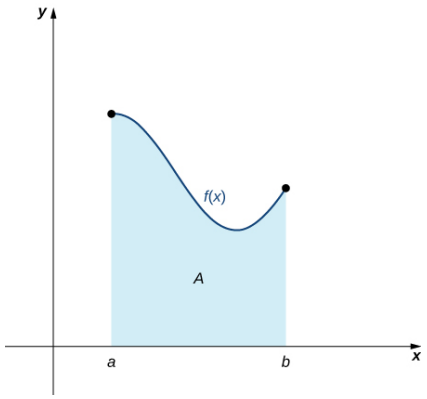


Figure 5.1.1: An area (shaded region) bounded by the curve $f(x)$ at top, the x-axis at bottom, the line $x=a$ to the left, and the line $x=b$ at right.

How do we approximate the area under this curve? The approach is a geometric one. By dividing a region into many small shapes that have known area formulas, we can sum these areas and obtain a reasonable estimate of the true area. We begin by dividing the

interval $[a, b]$ into n subintervals of equal width, $\frac{b-a}{n}$. We do this by selecting equally spaced points $x_0, x_1, x_2, \ldots, x_n$ with $x_0 = a, x_n = b$, and

$$x_i - x_{i-1} = \frac{b-a}{n}$$

for $i = 1, 2, 3, \ldots, n$.

We denote the width of each subinterval with the notation Δx, so $\Delta x = \frac{b-a}{n}$ and

$$x_i = x_0 + i\Delta x$$

for $i = 1, 2, 3, \ldots, n$. This notion of dividing an interval $[a, b]$ into subintervals by selecting points from within the interval is used quite often in approximating the area under a curve, so let's define some relevant terminology.

Definition: Partitions

A set of points $P = x_i$ for $i = 0, 1, 2, \ldots, n$ with $a = x_0 < x_1 < x_2 < \ldots < x_n = b$, which divides the interval $[a, b]$ into subintervals of the form $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ is called a **partition** of $[a, b]$. If the subintervals all have the same width, the set of points forms a **regular partition** (or uniform partition) of the interval $[a, b]$.

We can use this regular partition as the basis of a method for estimating the area under the curve. We next examine two methods: the left-endpoint approximation and the right-endpoint approximation.

Rule: Left-Endpoint Approximation

On each subinterval $[x_{i-1}, x_i]$ (for $i = 1, 2, 3, \ldots, n$), construct a rectangle with width Δx and height equal to $f(x_{i-1})$, which is the function value at the left endpoint of the subinterval. Then the area of this rectangle is $f(x_{i-1})\Delta x$. Adding the areas of all these rectangles, we get an approximate value for A (Figure 5.1.2). We use the notation L_n to denote that this is a **left-endpoint approximation** of A using n subintervals.

$$A \approx L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x = \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

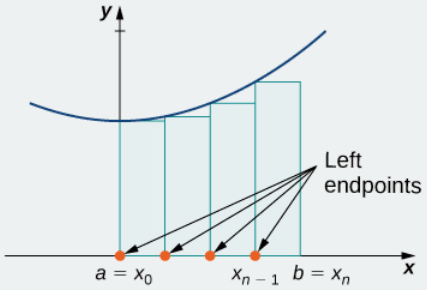


Figure 5.1.2: In the left-endpoint approximation of area under a curve, the height of each rectangle is determined by the function value at the left of each subinterval.

The second method for approximating area under a curve is the right-endpoint approximation. It is almost the same as the left-endpoint approximation, but now the heights of the rectangles are determined by the function values at the right of each subinterval.

Rule: Right-Endpoint Approximation

Construct a rectangle on each subinterval $[x_{i-1}, x_i]$, only this time the height of the rectangle is determined by the function value $f(x_i)$ at the right endpoint of the subinterval. Then, the area of each rectangle is $f(x_i)\,\Delta x$ and the approximation for A is given by

$$A \approx R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x.$$

The notation R_n indicates this is a **right-endpoint approximation** for A (Figure 5.1.3).

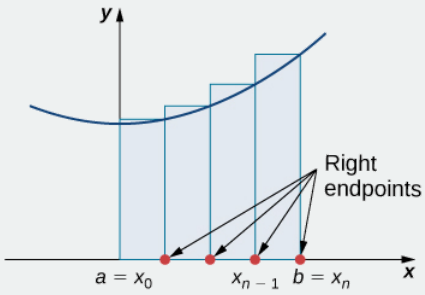


Figure 5.1.3: In the right-endpoint approximation of area under a curve, the height of each rectangle is determined by the function value at the right of each subinterval. Note that the right-endpoint approximation differs from the left-endpoint approximation in Figure 5.1.2.

The graphs in Figure 5.1.4 represent the curve $f(x) = \frac{x^2}{2}$. In Figure 5.1.4*b* we divide the region represented by the interval $[0, 3]$ into six subintervals, each of width 0.5. Thus, $\Delta x = 0.5$. We then form six rectangles by drawing vertical lines perpendicular to x_{i-1}, the left endpoint of each subinterval. We determine the height of each rectangle by calculating $f(x_{i-1})$ for $i = 1, 2, 3, 4, 5, 6$. The intervals are $[0, 0.5], [0.5, 1], [1, 1.5], [1.5, 2], [2, 2.5], [2.5, 3]$. We find the area of each rectangle by multiplying the height by the width. Then, the sum of the rectangular areas approximates the area between $f(x)$ and the x-axis. When the left endpoints are used to calculate height, we have a left-endpoint approximation. Thus,

$$\begin{aligned} A \approx L_6 &= \sum_{i=1}^{6} f(x_{i-1})\Delta x = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x \\ &= f(0)0.5 + f(0.5)0.5 + f(1)0.5 + f(1.5)0.5 + f(2)0.5 + f(2.5)0.5 \\ &= (0)0.5 + (0.125)0.5 + (0.5)0.5 + (1.125)0.5 + (2)0.5 + (3.125)0.5 \\ &= 0 + 0.0625 + 0.25 + 0.5625 + 1 + 1.5625 \\ &= 3.4375 \text{ units}^2 \end{aligned}$$

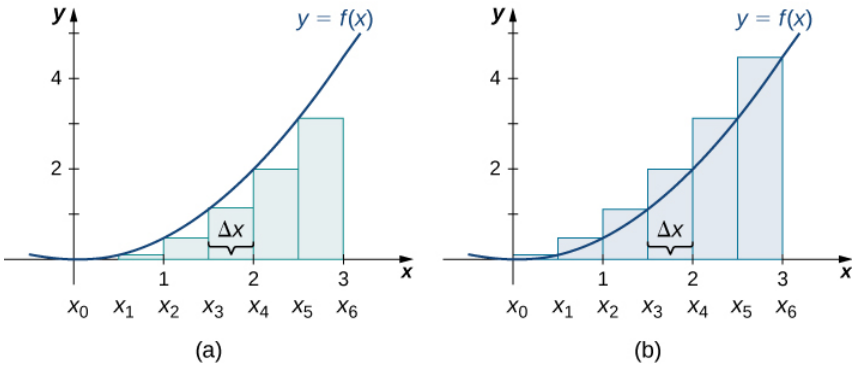


Figure 5.1.4: Methods of approximating the area under a curve by using (a) the left endpoints and (b) the right endpoints.

In Figure 5.1.4*b*, we draw vertical lines perpendicular to x_i such that x_i is the right endpoint of each subinterval, and calculate $f(x_i)$ for $i = 1, 2, 3, 4, 5, 6$. We multiply each $f(x_i)$ by Δx to find the rectangular areas, and then add them. This is a right-endpoint approximation of the area under $f(x)$. Thus,

$$\begin{aligned} A \approx R_6 &= \sum_{i=1}^{6} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x \\ &= f(0.5)0.5 + f(1)0.5 + f(1.5)0.5 + f(2)0.5 + f(2.5)0.5 + f(3)0.5 \\ &= (0.125)0.5 + (0.5)0.5 + (1.125)0.5 + (2)0.5 + (3.125)0.5 + (4.5)0.5 \\ &= 0.0625 + 0.25 + 0.5625 + 1 + 1.5625 + 2.25 \\ &= 5.6875 \text{ units}^2. \end{aligned}$$

Example 5.1.4: Approximating the Area Under a Curve

Use both left-endpoint and right-endpoint approximations to approximate the area under the curve of $f(x) = x^2$ on the interval $[0, 2]$; use $n = 4$.

Solution

First, divide the interval $[0, 2]$ into n equal subintervals. Using $n = 4$, $\Delta x = \dfrac{(2-0)}{4} = 0.5$. This is the width of each rectangle. The intervals $[0, 0.5], [0.5, 1], [1, 1.5], [1.5, 2]$ are shown in Figure 5.1.5. Using a left-endpoint approximation, the heights are $f(0) = 0$, $f(0.5) = 0.25$, $f(1) = 1$, and $f(1.5) = 2.25$. Then,

$$\begin{aligned} L_4 &= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x \\ &= 0(0.5) + 0.25(0.5) + 1(0.5) + 2.25(0.5) \\ &= 1.75 \text{ units}^2 \end{aligned}$$

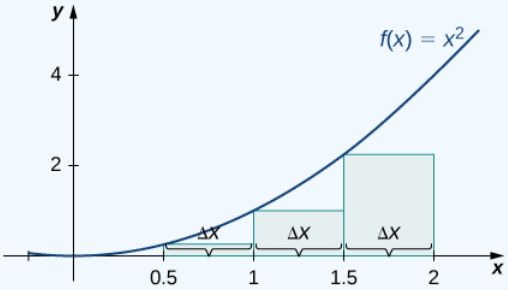


Figure 5.1.5: The graph shows the left-endpoint approximation of the area under $f(x) = x^2$ from 0 to 2.

The right-endpoint approximation is shown in Figure 5.1.6. The intervals are the same, $\Delta x = 0.5$, but now use the right endpoint to calculate the height of the rectangles. We have

$$\begin{aligned} R_4 &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \\ &= 0.25(0.5) + 1(0.5) + 2.25(0.5) + 4(0.5) \\ &= 3.75 \text{ units}^2 \end{aligned}$$

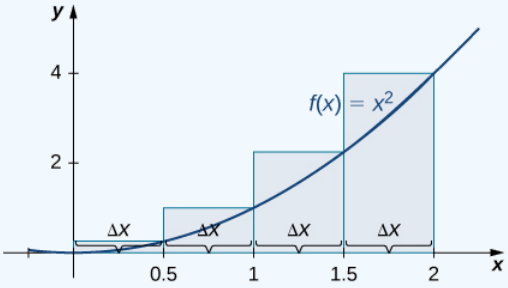


Figure 5.1.6: The graph shows the right-endpoint approximation of the area under $f(x) = x^2$ from 0 to 2.

The left-endpoint approximation is 1.75 units^2; the right-endpoint approximation is 3.75 units^2.

Exercise 5.1.4

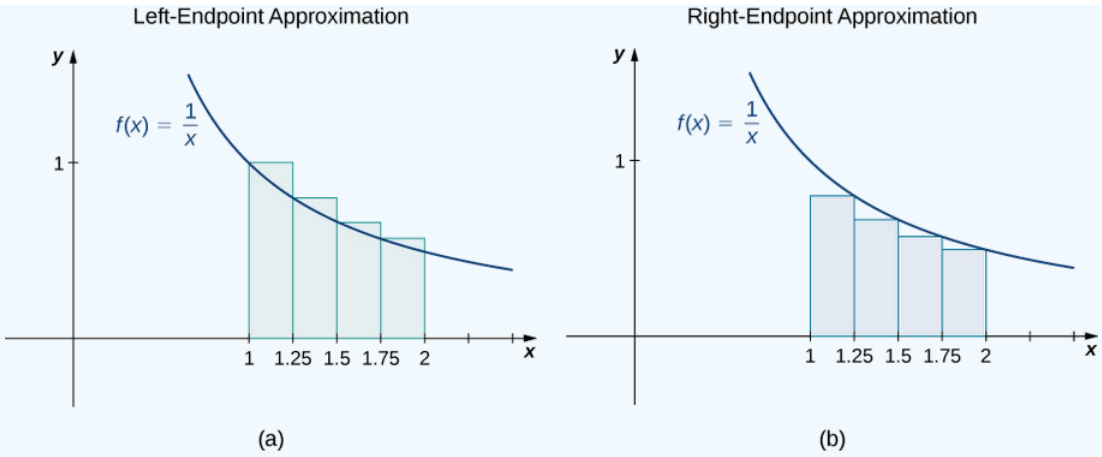
Sketch left-endpoint and right-endpoint approximations for $f(x) = \dfrac{1}{x}$ on $[1, 2]$; use $n = 4$. Approximate the area using both methods.

Hint

Follow the solving strategy in Example 5.1.4 step-by-step.

Answer

The left-endpoint approximation is 0.7595 units^2. The right-endpoint approximation is 0.6345 units^2. See the below Media.



Looking at Figure 5.1.4 and the graphs in Example 5.1.4, we can see that when we use a small number of intervals, neither the left-endpoint approximation nor the right-endpoint approximation is a particularly accurate estimate of the area under the curve. However, it seems logical that if we increase the number of points in our partition, our estimate of A will improve. We will have more rectangles, but each rectangle will be thinner, so we will be able to fit the rectangles to the curve more precisely.

We can demonstrate the improved approximation obtained through smaller intervals with an example. Let's explore the idea of increasing n, first in a left-endpoint approximation with four rectangles, then eight rectangles, and finally 32 rectangles. Then, let's do the same thing in a right-endpoint approximation, using the same sets of intervals, of the same curved region. Figure 5.1.7 shows the area of the region under the curve $f(x) = (x-1)^3 + 4$ on the interval $[0, 2]$ using a left-endpoint approximation where $n = 4$. The width of each rectangle is

$$\Delta x = \frac{2-0}{4} = \frac{1}{2}.$$

The area is approximated by the summed areas of the rectangles, or

$$L_4 = f(0)(0.5) + f(0.5)(0.5) + f(1)(0.5) + f(1.5)0.5 = 7.5 \text{ units}^2$$

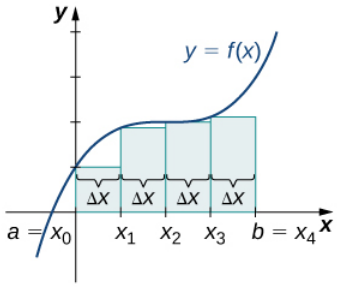


Figure 5.1.7: With a left-endpoint approximation and dividing the region from a to b into four equal intervals, the area under the curve is approximately equal to the sum of the areas of the rectangles.

Figure 5.1.8 shows the same curve divided into eight subintervals. Comparing the graph with four rectangles in Figure 5.1.7 with this graph with eight rectangles, we can see there appears to be less white space under the curve when $n = 8$. This white space is area under the curve we are unable to include using our approximation. The area of the rectangles is

$$L_8 = f(0)(0.25) + f(0.25)(0.25) + f(0.5)(0.25) + f(0.75)(0.25) + f(1)(0.25) + f(1.25)(0.25) + f(1.5)(0.25) + f(1.75)(0.25) = 7.75 \text{ units}^2$$

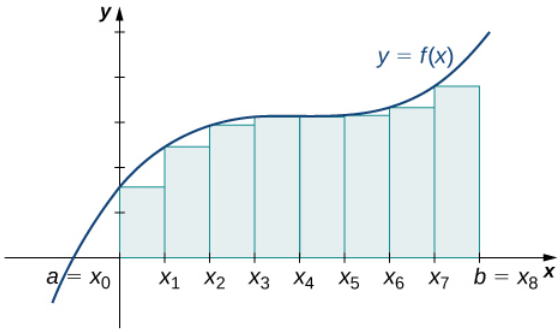


Figure 5.1.8: The region under the curve is divided into $n = 8$ rectangular areas of equal width for a left-endpoint approximation.

The graph in Figure 5.1.9 shows the same function with 32 rectangles inscribed under the curve. There appears to be little white space left. The area occupied by the rectangles is

$$L_{32} = f(0)(0.0625) + f(0.0625)(0.0625) + f(0.125)(0.0625) + \cdots + f(1.9375)(0.0625) = 7.9375 \text{ units}^2.$$

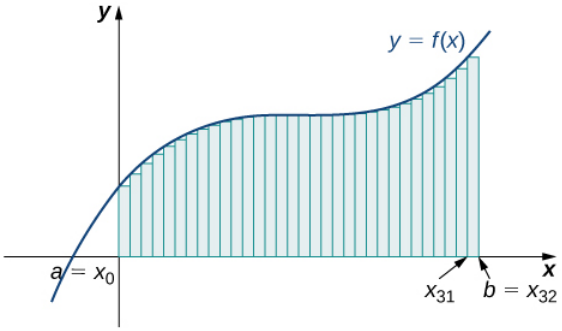


Figure 5.1.9: Here, 32 rectangles are inscribed under the curve for a left-endpoint approximation.

We can carry out a similar process for the right-endpoint approximation method. A right-endpoint approximation of the same curve, using four rectangles (Figure 5.1.10), yields an area

$$R_4 = f(0.5)(0.5) + f(1)(0.5) + f(1.5)(0.5) + f(2)(0.5) = 8.5 \text{ units}^2.$$

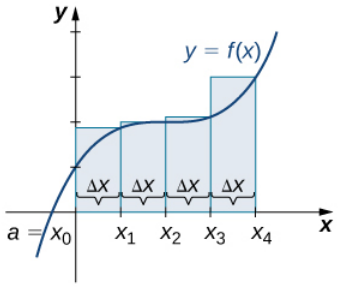


Figure 5.1.10: Now we divide the area under the curve into four equal subintervals for a right-endpoint approximation.

Dividing the region over the interval $[0, 2]$ into eight rectangles results in $\Delta x = \dfrac{2-0}{8} = 0.25$. The graph is shown in Figure 5.1.11. The area is

$$R_8 = f(0.25)(0.25) + f(0.5)(0.25) + f(0.75)(0.25) + f(1)(0.25) + f(1.25)(0.25) + f(1.5)(0.25) + f(1.75)(0.25) + f(2)(0.25) = 8.25 \text{ units}^2$$

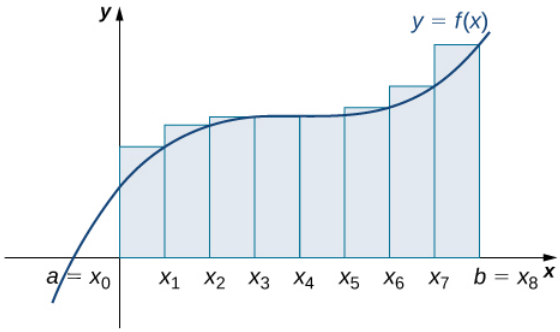


Figure 5.1.11: Here we use right-endpoint approximation for a region divided into eight equal subintervals.

Last, the right-endpoint approximation with $n = 32$ is close to the actual area (Figure 5.1.12). The area is approximately

$$R_{32}$$
$$= f(0.0625)(0.0625) + f(0.125)(0.0625) + f(0.1875)(0.0625) + \cdots + f(2)(0.0625) = 8.0625 \text{ units}^2$$

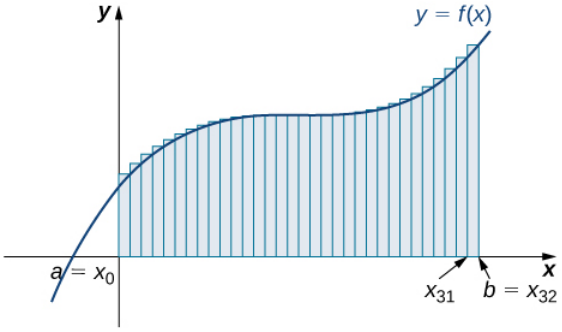


Figure 5.1.12: The region is divided into 32 equal subintervals for a right-endpoint approximation.

Based on these figures and calculations, it appears we are on the right track; the rectangles appear to approximate the area under the curve better as n gets larger. Furthermore, as n increases, both the left-endpoint and right-endpoint approximations appear to approach an area of 8 square units. Table 5.1.15 shows a numerical comparison of the left- and right-endpoint methods. The idea that the approximations of the area under the curve get better and better as n gets larger and larger is very important, and we now explore this idea in more detail.

Table 5.1.15: Converging Values of Left- and Right-Endpoint Approximations as n Increases

Value of n	Approximate Area L_n	Approximate Area R_n
$n = 4$	7.5	8.5
$n = 8$	7.75	8.25
$n = 32$	7.94	8.06

Forming Riemann Sums

So far we have been using rectangles to approximate the area under a curve. The heights of these rectangles have been determined by evaluating the function at either the right or left endpoints of the subinterval $[x_{i-1}, x_i]$. In reality, there is no reason to restrict evaluation of the function to one of these two points only. We could evaluate the function at any point x_i^* in the subinterval $[x_{i-1}, x_i]$, and use $f(x_i^*)$ as the height of our rectangle. This gives us an estimate for the area of the form

$$A \approx \sum_{i=1}^{n} f(x_i^*)\,\Delta x.$$

A sum of this form is called a Riemann sum, named for the 19th-century mathematician Bernhard Riemann, who developed the idea.

Definition: Riemann sum

Let $f(x)$ be defined on a closed interval $[a, b]$ and let P be any partition of $[a, b]$. Let Δx_i be the width of each subinterval $[x_{i-1}, x_i]$ and for each i, let x_i^* be any point in $[x_{i-1}, x_i]$. A Riemann sum is defined for $f(x)$ as

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i.$$

At this point, we'll choose a regular partition P, as we have in our examples above. This forces all Δx_i to be equal to $\Delta x = \dfrac{b-a}{n}$ for any natural number of intervals n.

Recall that with the left- and right-endpoint approximations, the estimates seem to get better and better as n get larger and larger. The same thing happens with Riemann sums. Riemann sums give better approximations for larger values of n. We are now ready to define the area under a curve in terms of Riemann sums.

Definition: Area Under the Curve

Let $f(x)$ be a continuous, nonnegative function on an interval $[a, b]$, and let $\sum_{i=1}^{n} f(x_i^*)\,\Delta x$ be a Riemann sum for $f(x)$ with a regular partition P. Then, the **area under the curve** $y = f(x)$ on $[a, b]$ is given by

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x.$$

See a graphical demonstration of the construction of a Riemann sum.

Some subtleties here are worth discussing. First, note that taking the limit of a sum is a little different from taking the limit of a function $f(x)$ as x goes to infinity. Limits of sums are discussed in detail in the chapter on Sequences and Series; however, for now we can assume that the computational techniques we used to compute limits of functions can also be used to calculate limits of sums.

Second, we must consider what to do if the expression converges to different limits for different choices of x_i^*. Fortunately, this does not happen. Although the proof is beyond the scope of this text, it can be shown that if $f(x)$ is continuous on the closed interval $[a, b]$, then $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$ exists and is unique (in other words, it does not depend on the choice of x_i^*).

We look at some examples shortly. But, before we do, let's take a moment and talk about some specific choices for x_i^*. Although any choice for x_i^* gives us an estimate of the area under the curve, we don't necessarily know whether that estimate is too high (overestimate) or too low (underestimate). If it is important to know whether our estimate is high or low, we can select our value for x_i^* to guarantee one result or the other.

If we want an overestimate, for example, we can choose x_i^* such that for $i = 1, 2, 3, \ldots, n$, $f(x_i^*) \geq f(x)$ for all $x \in [x_i - 1, x_i]$. In other words, we choose x_i^* so that for $i = 1, 2, 3, \ldots, n$, $f(x_i^*)$ is the maximum function value on the interval $[x_{i-1}, x_i]$. If we select x_i^* in this way, then the Riemann sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called an **upper sum**. Similarly, if we want an underestimate, we can choose x_i^* so that for $i = 1, 2, 3, \ldots, n$, $f(x_i^*)$ is the minimum function value on the interval $[x_{i-1}, x_i]$. In this case, the associated Riemann sum is called a **lower sum**. Note that if $f(x)$ is either increasing or decreasing throughout the interval $[a, b]$, then the maximum and minimum values of the function occur at the endpoints of the subintervals, so the upper and lower sums are just the same as the left- and right-endpoint approximations.

Example 5.1.5: Finding Lower and Upper Sums

Find a lower sum for $f(x) = 10 - x^2$ on $[1, 2]$; let $n = 4$ subintervals.

Solution

With $n = 4$ over the interval $[1, 2]$, $\Delta x = \dfrac{1}{4}$. We can list the intervals as $[1, 1.25]$, $[1.25, 1.5]$, $[1.5, 1.75]$, and $[1.75, 2]$. Because the function is decreasing over the interval $[1, 2]$, Figure shows that a lower sum is obtained by using the right endpoints.

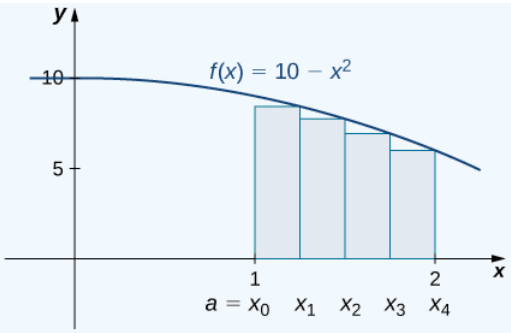


Figure 5.1.13: The graph of $f(x) = 10 - x^2$ is set up for a right-endpoint approximation of the area bounded by the curve and the x-axis on $[1, 2]$, and it shows a lower sum.

The Riemann sum is

$$\begin{aligned}\sum_{k=1}^{4}(10 - x^2)(0.25) &= 0.25[10 - (1.25)^2 + 10 - (1.5)^2 + 10 - (1.75)^2 + 10 - (2)^2] \\ &= 0.25[8.4375 + 7.75 + 6.9375 + 6] \\ &= 7.28 \text{ units}^2.\end{aligned}$$

The area of 7.28 units^2 is a lower sum and an underestimate.

? Exercise 5.1.5

a. Find an upper sum for $f(x) = 10 - x^2$ on $[1, 2]$; let $n = 4$.
b. Sketch the approximation.

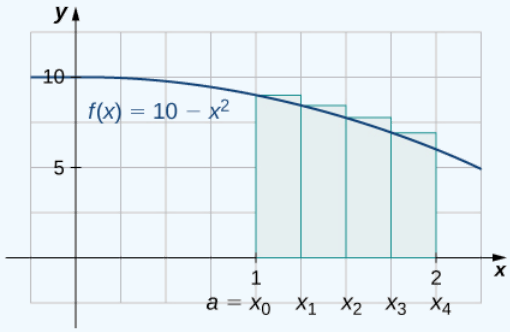
Hint

$f(x)$ is decreasing on $[1, 2]$, so the maximum function values occur at the left endpoints of the subintervals.

Answer

a. Upper sum=8.0313 units^2.

b.



✓ Example 5.1.6: Finding Lower and Upper Sums for $f(x) = \sin x$

Find a lower sum for $f(x) = \sin x$ over the interval $[a, b] = [0, \frac{\pi}{2}]$; let $n = 6$.

Solution

Let's first look at the graph in Figure 5.1.14 to get a better idea of the area of interest.

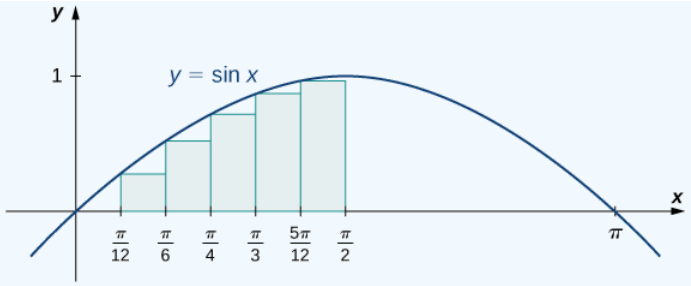


Figure 5.1.14: The graph of $y = \sin x$ is divided into six regions: $\Delta x = \frac{\pi/2}{6} = \frac{\pi}{12}$.

The intervals are $[0, \frac{\pi}{12}]$, $[\frac{\pi}{12}, \frac{\pi}{6}]$, $[\frac{\pi}{6}, \frac{\pi}{4}]$, $[\frac{\pi}{4}, \frac{\pi}{3}]$, $[\frac{\pi}{3}, \frac{5\pi}{12}]$, and $[\frac{5\pi}{12}, \frac{\pi}{2}]$. Note that $f(x) = \sin x$ is increasing on the interval $[0, \frac{\pi}{2}]$, so a left-endpoint approximation gives us the lower sum. A left-endpoint approximation is the Riemann sum $\sum_{i=0}^{5} \sin x_i \, (\frac{\pi}{12})$.We have

$$A \approx \sin(0)\,(\tfrac{\pi}{12}) + \sin(\tfrac{\pi}{12})\,(\tfrac{\pi}{12}) + \sin(\tfrac{\pi}{6})\,(\tfrac{\pi}{12}) + \sin(\tfrac{\pi}{4})\,(\tfrac{\pi}{12}) + \sin(\tfrac{\pi}{3})\,(\tfrac{\pi}{12}) + \sin(\tfrac{5\pi}{12})\,(\tfrac{\pi}{12}) \approx 0.863 \text{ units}^2.$$

Exercise 5.1.6

Using the function $f(x) = \sin x$ over the interval $[0, \frac{\pi}{2}]$, find an upper sum; let $n = 6$.

Hint

Follow the steps from Example 5.1.6.

Answer

$A \approx 1.125 \text{ units}^2$

Key Concepts

- The use of sigma (summation) notation of the form $\sum_{i=1}^{n} a_i$ is useful for expressing long sums of values in compact form.
- For a continuous function defined over an interval $[a, b]$, the process of dividing the interval into n equal parts, extending a rectangle to the graph of the function, calculating the areas of the series of rectangles, and then summing the areas yields an approximation of the area of that region.
- When using a regular partition, the width of each rectangle is $\Delta x = \frac{b-a}{n}$.
- Riemann sums are expressions of the form $\sum_{i=1}^{n} f(x_i^*)\Delta x$, and can be used to estimate the area under the curve $y = f(x)$. Left- and right-endpoint approximations are special kinds of Riemann sums where the values of x_i^* are chosen to be the left or right endpoints of the subintervals, respectively.
- Riemann sums allow for much flexibility in choosing the set of points x_i^* at which the function is evaluated, often with an eye to obtaining a lower sum or an upper sum.

Key Equations

- **Properties of Sigma Notation**

$$\sum_{i=1}^{n} c = nc$$

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n}(a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i$$

- **Sums and Powers of Integers**

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=0}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

- **Left-Endpoint Approximation**

$$A \approx L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x = \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

- **Right-Endpoint Approximation**

$$A \approx R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x$$

Glossary

left-endpoint approximation

an approximation of the area under a curve computed by using the left endpoint of each subinterval to calculate the height of the vertical sides of each rectangle

lower sum

a sum obtained by using the minimum value of $f(x)$ on each subinterval

partition

a set of points that divides an interval into subintervals

regular partition

a partition in which the subintervals all have the same width

riemann sum

an estimate of the area under the curve of the form $A \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$

right-endpoint approximation

the right-endpoint approximation is an approximation of the area of the rectangles under a curve using the right endpoint of each subinterval to construct the vertical sides of each rectangle

sigma notation

(also, **summation notation**) the Greek letter sigma (Σ) indicates addition of the values; the values of the index above and below the sigma indicate where to begin the summation and where to end it

upper sum

a sum obtained by using the maximum value of $f(x)$ on each subinterval

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5.1E: EXERCISES FOR SECTION 5.1

1.) State whether the given sums are equal or unequal.

a. $\sum_{i=1}^{10} i$ and $\sum_{k=1}^{10} k$

b. $\sum_{i=1}^{10} i$ and $\sum_{i=6}^{15} (i-5)$

c. $\sum_{i=1}^{10} i(i-1)$ and $\sum_{j=0}^{9} (j+1)j$

d. $\sum_{i=1}^{10} i(i-1)$ and $\sum_{k=1}^{10} (k^2-k)$

Answer

a. They are equal; both represent the sum of the first 10 whole numbers.
b. They are equal; both represent the sum of the first 10 whole numbers.
c. They are equal by substituting $j = i-1$.
d. They are equal; the first sum factors the terms of the second.

In exercises 2 - 3, use the rules for sums of powers of integers to compute the sums.

2) $\sum_{i=5}^{10} i$

3) $\sum_{i=5}^{10} i^2$

Answer

$$\sum_{i=5}^{10} i^2 = 385 - 30 = 355$$

Suppose that $\sum_{i=1}^{100} a_i = 15$ and $\sum_{i=1}^{100} b_i = -12$.

In exercises 4 - 7, compute the sums.

4) $\sum_{i=1}^{100} (a_i + b_i)$

5) $\sum_{i=1}^{100} (a_i - b_i)$

Answer

$$\sum_{i=1}^{100} (a_i - b_i) = 15 - (-12) = 27$$

6) $\sum_{i=1}^{100} (3a_i - 4b_i)$

7) $\sum_{i=1}^{100} (5a_i + 4b_i)$

Answer

$$\sum_{i=1}^{100} (5a_i + 4b_i) = 5(15) + 4(-12) = 27$$

In exercises 8 - 11, use summation properties and formulas to rewrite and evaluate the sums.

8) $\sum_{k=1}^{20} 100(k^2 - 5k + 1)$

9) $\sum_{j=1}^{50} (j^2 - 2j)$

Answer

$$\sum_{j=1}^{50} j^2 - 2\sum_{j=1}^{50} j = \frac{(50)(51)(101)}{6} - \frac{2(50)(51)}{2} = 40,375$$

10) $\sum_{j=11}^{20}(j^2 - 10j)$

11) $\sum_{k=1}^{25}[(2k)^2 - 100k]$

Answer

$$4\sum_{k=1}^{25} k^2 - 100\sum_{k=1}^{25} k = \frac{4(25)(26)(51)}{9} - 50(25)(26) = -10,400$$

Let L_n denote the left-endpoint sum using n subintervals and let R_n denote the corresponding right-endpoint sum.

In exercises 12 - 19, compute the indicated left and right sums for the given functions on the indicated interval.

12) L_4 for $f(x) = \frac{1}{x-1}$ on $[2,3]$

13) R_4 for $g(x) = \cos(\pi x)$ on $[0,1]$

Answer

$R_4 = 0.25$

14) L_6 for $f(x) = \frac{1}{x(x-1)}$ on $[2,5]$

15) R_6 for $f(x) = \frac{1}{x(x-1)}$ on $[2,5]$

Answer

$R_6 = 0.372$

16) R_4 for $\frac{1}{x^2+1}$ on $[-2,2]$

17) L_4 for $\frac{1}{x^2+1}$ on $[-2,2]$

Answer

$L_4 = 2.20$

18) R_4 for $x^2 - 2x + 1$ on $[0,2]$

19) L_8 for $x^2 - 2x + 1$ on $[0,2]$

Answer

$L_8 = 0.6875$

20) Compute the left and right Riemann sums— L_4 and R_4, respectively—for $f(x) = (2 - |x|)$ on $[-2,2]$. Compute their average value and compare it with the area under the graph of f.

21) Compute the left and right Riemann sums— L_6 and R_6, respectively—for $f(x) = (3 - |3 - x|)$ on $[0,6]$. Compute their average value and compare it with the area under the graph of f.

Answer

$L_6 = 9.000 = R_6$. The graph of f is a triangle with area $9\,\text{units}^2$.

22) Compute the left and right Riemann sums— L_4 and R_4, respectively—for $f(x) = \sqrt{4 - x^2}$ on $[-2,2]$ and compare their values.

23) Compute the left and right Riemann sums— L_6 and R_6, respectively—for $f(x) = \sqrt{9 - (x-3)^2}$ on $[0,6]$ and compare their values.

Answer

$L_6 = 13.12899 = R_6$. They are equal.

For exercises 24 - 27, express the following endpoint sums in sigma notation but do not evaluate them.

24) L_{30} for $f(x) = x^2$ on $[1, 2]$

25) L_{10} for $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$

Answer

$$L_{10} = \frac{4}{10} \sum_{i=1}^{10} \sqrt{4 - (-2 + 4\frac{(i-1)}{10})}$$

26) R_{20} for $f(x) = \sin x$ on $[0, \pi]$

27) R_{100} for $\ln x$ on $[1, e]$

Answer

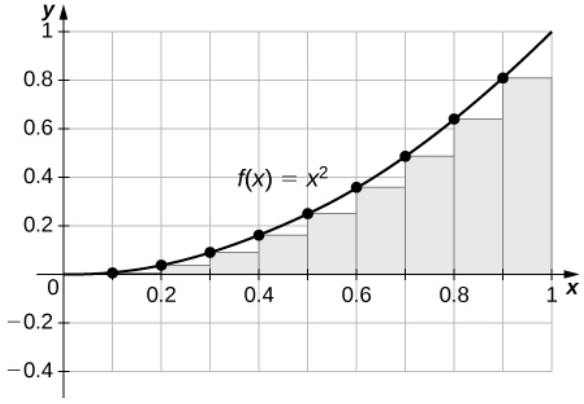
$$R_{100} = \frac{e-1}{100} \sum_{i=1}^{100} ln(1 + (e-1)\frac{i}{100})$$

In exercises 28 - 33, graph the function then use a calculator or a computer program to evaluate the following left and right endpoint sums. Is the area under the curve between the left and right endpoint sums?

28) [T] L_{100} and R_{100} for $y = x^2 - 3x + 1$ on the interval $[-1, 1]$

29) [T] L_{100} and R_{100} for $y = x^2$ on the interval $[0, 1]$

Answer



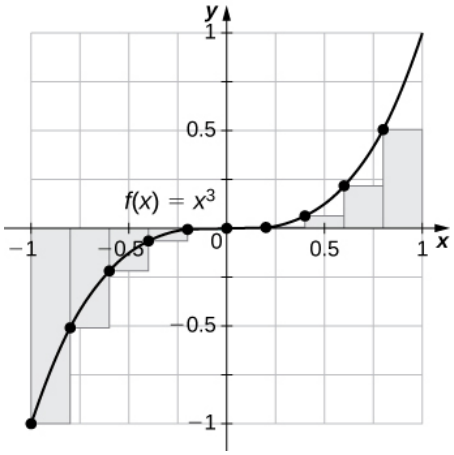
$R_{100} = 0.33835, \quad L_{100} = 0.32835.$

The plot shows that the left Riemann sum is an underestimate because the function is increasing. Similarly, the right Riemann sum is an overestimate. The area lies between the left and right Riemann sums. Ten rectangles are shown for visual clarity. This behavior persists for more rectangles.

30) [T] L_{50} and R_{50} for $y = \dfrac{x+1}{x^2-1}$ on the interval $[2, 4]$

31) [T] L_{100} and R_{100} for $y = x^3$ on the interval $[-1, 1]$

Answer



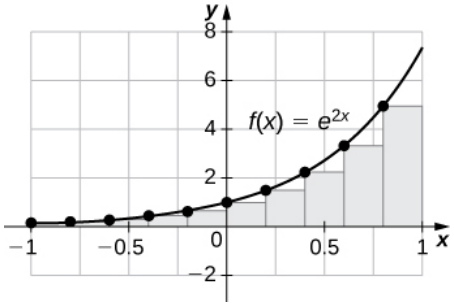
$L_{100} = -0.02, \quad R_{100} = 0.02.$

The left endpoint sum is an underestimate because the function is increasing. Similarly, a right endpoint approximation is an overestimate. The area lies between the left and right endpoint estimates.

32) [T] L_{50} and R_{50} for $y = \tan(x)$ on the interval $[0, \frac{\pi}{4}]$

33) [T] L_{100} and R_{100} for $y = e^{2x}$ on the interval $[-1, 1]$

Answer



$L_{100} = 3.555, \quad R_{100} = 3.670.$

The plot shows that the left Riemann sum is an underestimate because the function is increasing. Ten rectangles are shown for visual clarity. This behavior persists for more rectangles.

34) Let t_j denote the time that it took Tejay van Garteren to ride the jth stage of the Tour de France in 2014. If there were a total of 21 stages, interpret $\sum_{j=1}^{21} t_j$.

35) Let r_j denote the total rainfall in Portland on the jth day of the year in 2009. Interpret $\sum_{j=1}^{31} r_j$.

Answer

The sum represents the cumulative rainfall in January 2009.

36) Let d_j denote the hours of daylight and δ_j denote the increase in the hours of daylight from day $j-1$ to day j in Fargo, North Dakota, on the jth day of the year. Interpret $d1 + \sum_{j=2}^{365} \delta_j$.

37) To help get in shape, Joe gets a new pair of running shoes. If Joe runs 1 mi each day in week 1 and adds $\frac{1}{10}$ mi to his daily routine each week, what is the total mileage on Joe's shoes after 25 weeks?

Answer

The total mileage is $7 \times \sum_{i=1}^{25}(1 + \frac{(i-1)}{10}) = 7 \times 25 + \frac{7}{10} \times 12 \times 25 = 385$ mi.

38) The following table gives approximate values of the average annual atmospheric rate of increase in carbon dioxide (CO_2) each decade since 1960, in parts per million (ppm). Estimate the total increase in atmospheric CO_2 between 1964 and 2013.

Decade	Ppm/y
1964-1973	1.07
1976-1983	1.34
1984-1993	1.40
1994-2003	1.87
2004-2013	2.07

Average Annual Atmospheric CO2 Increase, 1964–2013 Source: http://www.esrl.noaa.gov/gmd/ccgg/trends/.

39) The following table gives the approximate increase in sea level in inches over 20 years starting in the given year. Estimate the net change in mean sea level from 1870 to 2010.

Starting Year	20- Year Change
1870	0.3
1890	1.5
1910	0.2
1930	2.8
1950	0.7
1970	1.1
1990	1.5

Approximate 20-Year Sea Level Increases, 1870–1990

Source: http://link.springer.com/article/10....712-011-9119-1

Answer

Add the numbers to get 8.1-in. net increase.

40) The following table gives the approximate increase in dollars in the average price of a gallon of gas per decade since 1950. If the average price of a gallon of gas in 2010 was $2.60, what was the average price of a gallon of gas in 1950?

Starting Year	10- Year Change
1950	0.03
1960	0.05
1970	0.86
1980	−0.03
1990	0.29
2000	1.12

Approximate 10-Year Gas Price Increases, 1950–2000

Source: epb.lbl.gov/homepages/Rick_Di...011-trends.pdf.

41) The following table gives the percent growth of the U.S. population beginning in July of the year indicated. If the U.S. population was 281,421,906 in July 2000, estimate the U.S. population in July 2010.

Year	% Change/Year
2000	1.12
2001	0.99
2002	0.93
2003	0.86
2004	0.93
2005	0.93
2006	0.97
2007	0.96
2008	0.95
2009	0.88

Annual Percentage Growth of U.S. Population, 2000–2009

Source: www.census.gov/popest/data.

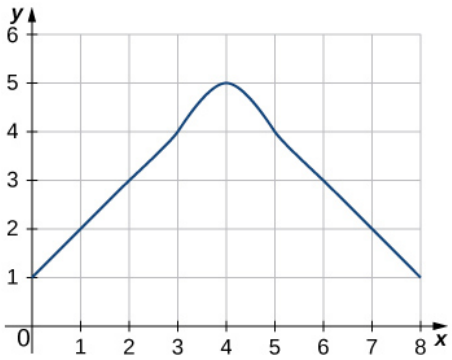
(Hint: To obtain the population in July 2001, multiply the population in July 2000 by 1.0112 to get 284,573,831.)

Answer

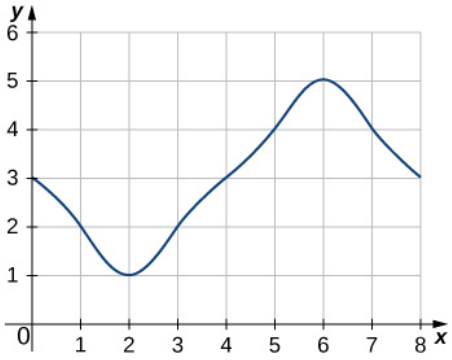
309,389,957

In exercises 42 - 45, estimate the areas under the curves by computing the left Riemann sums, L_8.

42)



43)



Answer

$L_8 = 3+2+1+2+3+4+5+4 = 24$

44)

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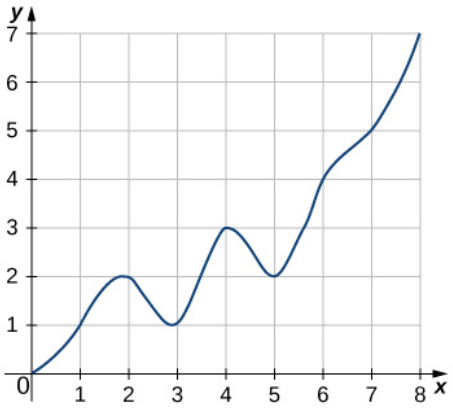
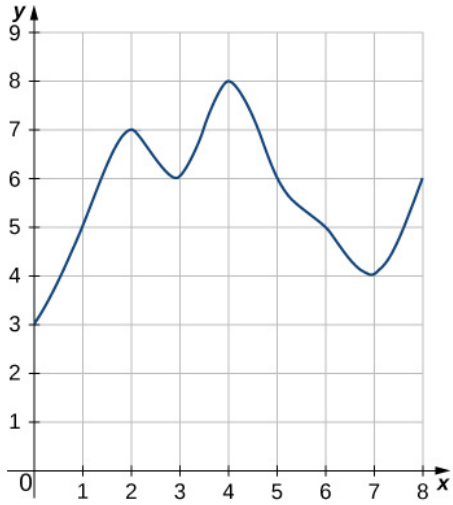


45)



Answer

$L_8 = 3 + 5 + 7 + 6 + 8 + 6 + 5 + 4 = 44$

46) [T] Use a computer algebra system to compute the Riemann sum, L_N, for $N = 10, 30, 50$ for $f(x) = \sqrt{1 - x^2}$ on $[-1, 1]$.

47) [T] Use a computer algebra system to compute the Riemann sum, L_N, for $N = 10, 30, 50$ for $f(x) = \dfrac{1}{\sqrt{1 + x^2}}$ on $[-1, 1]$.

Answer

$L_{10} \approx 1.7604, \quad L_{30} \approx 1.7625, \quad L_{50} \approx 1.76265$

48) [T] Use a computer algebra system to compute the Riemann sum, L_N, for $N = 10, 30, 50$ for $f(x) = \sin^2 x$ on $[0, 2\pi]$. Compare these estimates with π.

In exercises 49-50, use a calculator or a computer program to evaluate the endpoint sums R_N and L_N for $N = 1, 10, 100$. How do these estimates compare with the exact answers, which you can find via geometry?

49) [T] $y = \cos(\pi x)$ on the interval $[0, 1]$

Answer

$R_1 = -1, \quad L_1 = 1,$
$R_{10} = -0.1, \quad L_{10} = 0.1,$
$L_{100} = 0.01, \quad R_{100} = -0.01.$
By symmetry of the graph, the exact area is zero.

50) [T] $y = 3x + 2$ on the interval $[3, 5]$

In exercises 51 - 52, use a calculator or a computer program to evaluate the endpoint sums R_N and L_N for $N = 1, 10, 100$.

51) [T] $y = x^4 - 5x^2 + 4$ on the interval $[-2, 2]$, which has an exact area of $\frac{32}{15}$

Answer

$R_1 = 0, \quad L_1 = 0,$
$R_{10} = 2.4499, \quad L_{10} = 2.4499,$
$R_{100} = 2.1365, \quad L_{100} = 2.1365$

52) [T] $y = \ln x$ on the interval $[1, 2]$, which has an exact area of $2\ln(2) - 1$

53) Explain why, if $f(a) \geq 0$ and f is increasing on $[a, b]$, that the left endpoint estimate is a lower bound for the area below the graph of f on $[a, b]$.

Answer

If $[c, d]$is a subinterval of $[a, b]$ under one of the left-endpoint sum rectangles, then the area of the rectangle contributing to the left-endpoint estimate is $f(c)(d - c)$. But, $f(c) \leq f(x)$ for $c \leq x \leq d$, so the area under the graph of f between c and d is $f(c)(d - c)$ plus the area below the graph of f but above the horizontal line segment at height $f(c)$, which is positive. As this is true for each left-endpoint sum interval, it follows that the left Riemann sum is less than or equal to the area below the graph of f on $[a, b]$.

54) Explain why, if $f(b) \geq 0$ and f is decreasing on $[a, b]$, that the left endpoint estimate is an upper bound for the area below the graph of f on $[a, b]$.

55) Show that, in general, $R_N - L_N = (b - a) \times \frac{f(b) - f(a)}{N}$.

Answer

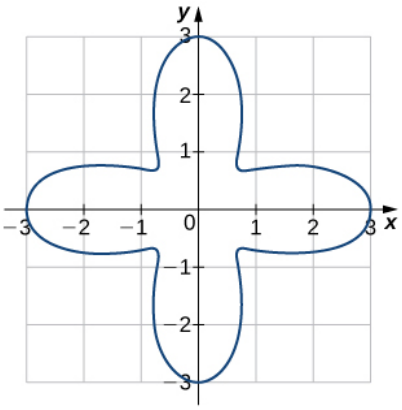
$L_N = \frac{b-a}{N} \sum_{i=1}^{N} f(a + (b-a)\frac{i-1}{N}) = \frac{b-a}{N} \sum_{i=0}^{N-1} f(a + (b-a)\frac{i}{N})$ and $R_N = \frac{b-a}{N} \sum_{i=1}^{N} f(a + (b-a)\frac{i}{N})$. The left sum has a term corresponding to $i = 0$ and the right sum has a term corresponding to $i = N$. In $R_N - L_N$, any term corresponding to $i = 1, 2, \ldots, N - 1$ occurs once with a plus sign and once with a minus sign, so each such term cancels and one is left with $R_N - L_N = \frac{b-a}{N}(f(a + (b-a))\frac{N}{N}) - (f(a) + (b-a)\frac{0}{N}) = \frac{b-a}{N}(f(b) - f(a))$.

56) Explain why, if f is increasing on $[a, b]$, the error between either L_N or R_N and the area A below the graph of f is at most $(b - a)\frac{f(b) - f(a)}{N}$.

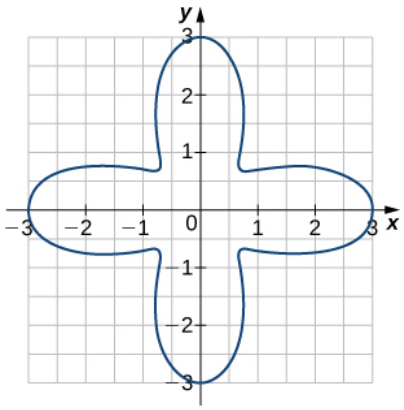
57) For each of the three graphs:

a. Obtain a lower bound $L(A)$ for the area enclosed by the curve by adding the areas of the squares enclosed completely by the curve.

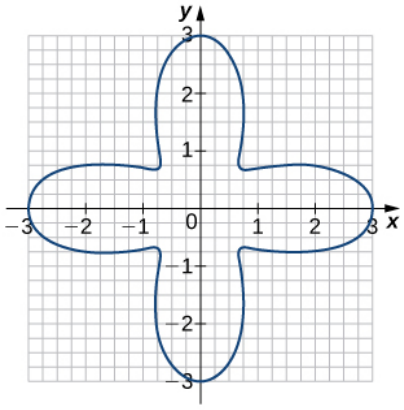
b. Obtain an upper bound $U(A)$ for the area by adding to $L(A)$ the areas $B(A)$ of the squares enclosed partially by the curve.



Graph 1



Graph 2



Graph 3

Answer

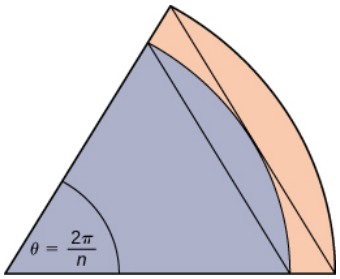
Graph 1: a. $L(A) = 0, \quad B(A) = 20; \quad b. U(A) = 20.$

Graph 2: $a. L(A) = 9; \quad b. B(A) = 11, \quad U(A) = 20.$

Graph 3: a. $L(A) = 11.0; \quad b. B(A) = 4.5, \quad U(A) = 15.5.$

58) In the previous exercise, explain why $L(A)$ gets no smaller while $U(A)$ gets no larger as the squares are subdivided into four boxes of equal area.

59) A unit circle is made up of n wedges equivalent to the inner wedge in the figure. The base of the inner triangle is 1 unit and its height is $\sin(\frac{\pi}{n})$. The base of the outer triangle is $B = \cos(\frac{\pi}{n}) + \sin(\frac{\pi}{n})\tan(\frac{\pi}{n})$ and the height is $H = B\sin(\frac{2\pi}{n})$. Use this information to argue that the area of a unit circle is equal to π.



Answer

Let A be the area of the unit circle. The circle encloses n congruent triangles each of area $\frac{\sin(\frac{2\pi}{n})}{2}$, so $\frac{n}{2}\sin(\frac{2\pi}{n}) \leq A$. Similarly, the circle is contained inside n congruent triangles each of area $\frac{BH}{2} = \frac{1}{2}(\cos(\frac{\pi}{n}) + \sin(\frac{\pi}{n})\tan(\frac{\pi}{n}))\sin(\frac{2\pi}{n})$, so $A \leq \frac{n}{2}\sin(\frac{2\pi}{n})(\cos(\frac{\pi}{n}))$ $+ \sin(\frac{\pi}{n})\tan(\frac{\pi}{n})$. As $n \to \infty, \quad \frac{n}{2}\sin(\frac{2\pi}{n}) = \frac{\pi\sin(\frac{2\pi}{n})}{(\frac{2\pi}{n})} \to \pi$, so we conclude $\pi \leq A$. Also, as $n \to \infty, \quad \cos(\frac{\pi}{n}) + \sin(\frac{\pi}{n})\tan(\frac{\pi}{n}) \to 1$, so we also have $A \leq \pi$. By the squeeze theorem for limits, we conclude that $A = \pi$.

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5.2: The Definite Integral

Learning Objectives

- State the definition of the definite integral.
- Explain the terms integrand, limits of integration, and variable of integration.
- Explain when a function is integrable.
- Describe the relationship between the definite integral and net area.
- Use geometry and the properties of definite integrals to evaluate them.
- Calculate the average value of a function.

In the preceding section we defined the area under a curve in terms of Riemann sums:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

However, this definition came with restrictions. We required $f(x)$ to be continuous and nonnegative. Unfortunately, real-world problems don't always meet these restrictions. In this section, we look at how to apply the concept of the area under the curve to a broader set of functions through the use of the definite integral.

Definition and Notation

The definite integral generalizes the concept of the area under a curve. We lift the requirements that $f(x)$ be continuous and nonnegative, and define the definite integral as follows.

Definition: Definite Integral

If $f(x)$ is a function defined on an interval $[a, b]$, the ***definite integral*** of f from a to b is given by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

provided the limit exists. If this limit exists, the function $f(x)$ is said to be integrable on $[a, b]$, or is an integrable function.

The integral symbol in the previous definition should look familiar. We have seen similar notation in the chapter on Applications of Derivatives, where we used the indefinite integral symbol (without the a and b above and below) to represent an antiderivative. Although the notation for indefinite integrals may look similar to the notation for a definite integral, they are not the same. A definite integral is a number. An indefinite integral is a family of functions. Later in this chapter we examine how these concepts are related. However, close attention should always be paid to notation so we know whether we're working with a definite integral or an indefinite integral.

Integral notation goes back to the late seventeenth century and is one of the contributions of Gottfried Wilhelm **Leibniz**, who is often considered to be the codiscoverer of calculus, along with Isaac Newton. The integration symbol $\int$ is an elongated S, suggesting sigma or summation. On a definite integral, above and below the summation symbol are the boundaries of the interval, $[a, b]$. The numbers a and b are x-values and are called the **limits of integration**; specifically, a is the lower limit and b is the upper limit. To clarify, we are using the word limit in two different ways in the context of the definite integral. First, we talk about the limit of a sum as $n \to \infty$. Second, the boundaries of the region are called the *limits of integration.*

We call the function $f(x)$ the **integrand**, and the dx indicates that $f(x)$ is a function with respect to x, called the **variable of integration**. Note that, like the index in a sum, the variable of integration is a **dummy variable**, and has no impact on the computation of the integral. We could use any variable we like as the variable of integration:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\,du$$

Previously, we discussed the fact that if $f(x)$ is continuous on $[a, b]$, then the limit $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$ exists and is unique. This leads to the following theorem, which we state without proof.

Continuous Functions Are Integrable

If $f(x)$ is continuous on $[a, b]$, then f is integrable on $[a, b]$.

Functions that are not continuous on $[a, b]$ may still be integrable, depending on the nature of the discontinuities. For example, functions with a finite number of jump discontinuities or removable discontinuities on a closed interval are integrable.

It is also worth noting here that we have retained the use of a regular partition in the Riemann sums. This restriction is not strictly necessary. Any partition can be used to form a Riemann sum. However, if a nonregular partition is used to define the definite integral, it is not sufficient to take the limit as the number of subintervals goes to infinity. Instead, we must take the limit as the width of the largest subinterval goes to zero. This introduces a little more complex notation in our limits and makes the calculations more difficult without really gaining much additional insight, so we stick with regular partitions for the Riemann sums.

Example 5.2.1: Evaluating an Integral Using the Definition

Use the definition of the definite integral to evaluate $\int_0^2 x^2\,dx$. Use a right-endpoint approximation to generate the Riemann sum.

Solution

We first want to set up a Riemann sum. Based on the limits of integration, we have $a = 0$ and $b = 2$. For $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of $[0, 2]$. Then

$$\Delta x = \frac{b-a}{n} = \frac{2}{n}.$$

Since we are using a right-endpoint approximation to generate Riemann sums, for each i, we need to calculate the function value at the right endpoint of the interval $[x_{i-1}, x_i]$. The right endpoint of the interval is x_i, and since P is a regular partition,

$$x_i = x_0 + i\Delta x = 0 + i\left[\frac{2}{n}\right] = \frac{2i}{n}.$$

Thus, the function value at the right endpoint of the interval is

$$f(x_i) = x_i^2 = \left(\frac{2i}{n}\right)^2 = \frac{4i^2}{n^2}.$$

Then the Riemann sum takes the form

$$\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\left(\frac{4i^2}{n^2}\right)\frac{2}{n} = \sum_{i=1}^{n}\frac{8i^2}{n^3} = \frac{8}{n^3}\sum_{i=1}^{n} i^2.$$

Using the summation formula for $\sum_{i=1}^{n} i^2$, we have

$$\begin{aligned}\sum_{i=1}^{n} f(x_i)\Delta x &= \frac{8}{n^3}\sum_{i=1}^{n} i^2 \\ &= \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] \\ &= \frac{8}{n^3}\left[\frac{2n^3+3n^2+n}{6}\right] \\ &= \frac{16n^3+24n^2+n}{6n^3} \\ &= \frac{8}{3}+\frac{4}{n}+\frac{1}{6n^2}.\end{aligned}$$

Now, to calculate the definite integral, we need to take the limit as $n \to \infty$. We get

$$\begin{aligned}\int_0^2 x^2 dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \\ &= \lim_{n\to\infty}\left(\frac{8}{3}+\frac{4}{n}+\frac{1}{6n^2}\right) \\ &= \lim_{n\to\infty}\left(\frac{8}{3}\right)+\lim_{n\to\infty}\left(\frac{4}{n}\right)+\lim_{n\to\infty}\left(\frac{1}{6n^2}\right) \\ &= \frac{8}{3}+0+0=\frac{8}{3}.\end{aligned}$$

? Exercise 5.2.1

Use the definition of the definite integral to evaluate $\int_0^3 (2x-1)\,dx$.

Use a right-endpoint approximation to generate the Riemann sum.

Hint

Use the solving strategy from Example 5.2.1.

Answer

6

Evaluating Definite Integrals

Evaluating definite integrals this way can be quite tedious because of the complexity of the calculations. Later in this chapter we develop techniques for evaluating definite integrals *without* taking limits of Riemann sums. However, for now, we can rely on the fact that definite integrals represent the area under the curve, and we can evaluate definite integrals by using geometric formulas to calculate that area. We do this to confirm that definite integrals do, indeed, represent areas, so we can then discuss what to do in the case of a curve of a function dropping below the x-axis.

✓ Example 5.2.2: Using Geometric Formulas to Calculate Definite Integrals

Use the formula for the area of a circle to evaluate $\int_3^6 \sqrt{9-(x-3)^2}\,dx$.

Solution

The function describes a semicircle with radius 3. To find

$$\int_3^6 \sqrt{9-(x-3)^2}\,dx$$

we want to find the area under the curve over the interval $[3,6]$. The formula for the area of a circle is $A=\pi r^2$. The area of a semicircle is just one-half the area of a circle, or $A=\frac{1}{2}\pi r^2$. The shaded area in Figure 5.2.1 covers one-half of the semicircle, or $A=\frac{1}{4}\pi r^2$. Thus,

$$\int_3^6 \sqrt{9-(x-3)^2}\,dx = \frac{1}{4}\pi(3)^2 = \frac{9}{4}\pi \approx 7.069.$$

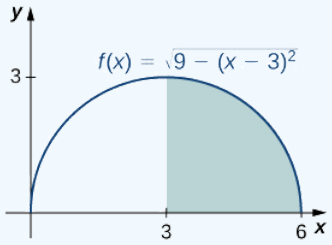


Figure 5.2.1: The value of the integral of the function $f(x)$ over the interval $[3,6]$ is the area of the shaded region.

Exercise 5.2.2

Use the formula for the area of a trapezoid to evaluate $\int_2^4 (2x+3)\,dx$.

Hint

Graph the function $f(x)$ and calculate the area under the function on the interval $[2,4]$.

Answer

18 square units

Area and the Definite Integral

When we defined the definite integral, we lifted the requirement that $f(x)$ be nonnegative. But how do we interpret "the area under the curve" when $f(x)$ is negative?

Net Signed Area

Let us return to the Riemann sum. Consider, for example, the function $f(x) = 2 - 2x^2$ (shown in Figure 5.2.2) on the interval $[0,2]$. Use $n = 8$ and choose $\{x_i^*\}$ as the left endpoint of each interval. Construct a rectangle on each subinterval of height $f(x_i^*)$ and width Δx. When $f(x_i^*)$ is positive, the product $f(x_i^*)\Delta x$ represents the area of the rectangle, as before. When $f(x_i^*)$ is negative, however, the product $f(x_i^*)\Delta x$ represents the negative of the area of the rectangle. The Riemann sum then becomes

$$\sum_{i=1}^{8} f(x_i^*)\Delta x = (\text{Area of rectangles above the } x\text{-axis}) - (\text{Area of rectangles below the } x\text{-axis})$$

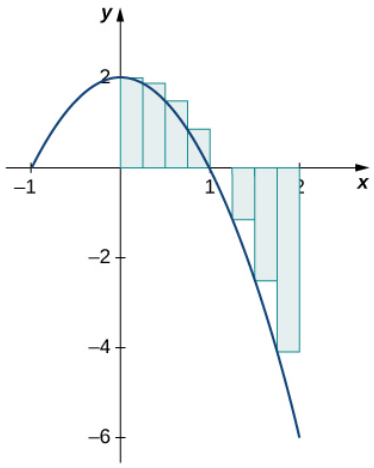


Figure 5.2.2: For a function that is partly negative, the Riemann sum is the area of the rectangles above the x-axis less the area of the rectangles below the x-axis.

Taking the limit as $n \to \infty$, the Riemann sum approaches the area between the curve above the x-axis and the x-axis, less the area between the curve below the x-axis and the x-axis, as shown in Figure 5.2.3. Then,

$$\int_0^2 f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = A_1 - A_2.$$

The quantity $A_1 - A_2$ is called the **net signed area.**

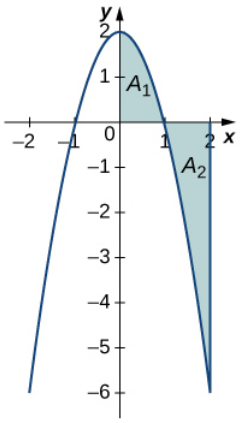


Figure 5.2.3: In the limit, the definite integral equals area A_1 less area A_2, or the net signed area.

Notice that net signed area can be positive, negative, or zero. If the area above the x-axis is larger, the net signed area is positive. If the area below the x-axis is larger, the net signed area is negative. If the areas above and below the x-axis are equal, the net signed area is zero.

Example 5.2.3: Finding the Net Signed Area

Find the net signed area between the curve of the function $f(x) = 2x$ and the x-axis over the interval $[-3, 3]$.

Solution

The function produces a straight line that forms two triangles: one from $x = -3$ to $x = 0$ and the other from $x = 0$ to $x = 3$ (Figure 5.2.4). Using the geometric formula for the area of a triangle, $A = \frac{1}{2}bh$, the area of triangle A_1, above the axis, is

$$A_1 = \frac{1}{2}3(6) = 9,$$

where 3 is the base and $2(3) = 6$ is the height. The area of triangle A_2, below the axis, is

$$A_2 = \frac{1}{2}(3)(6) = 9,$$

where 3 is the base and 6 is the height. Thus, the net area is

$$\int_{-3}^{3} 2x\,dx = A_1 - A_2 = 9 - 9 = 0.$$

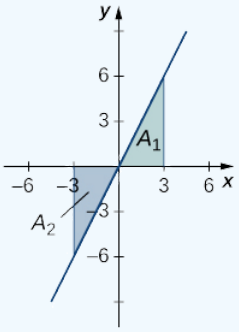


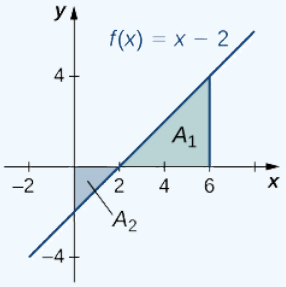
Figure 5.2.4: The area above the curve and below the x-axis equals the area below the curve and above the x-axis.

Analysis

If A_1 is the area above the x-axis and A_2 is the area below the x-axis, then the net area is $A_1 - A_2$. Since the areas of the two triangles are equal, the net area is zero.

Exercise 5.2.3

Find the net signed area of $f(x) = x - 2$ over the interval $[0, 6]$, illustrated in the following image.



Hint

Use the solving method described in Example 5.2.3.

Answer

6

Total Area

One application of the definite integral is finding **displacement** when given a velocity function. If $v(t)$ represents the velocity of an object as a function of time, then the area under the curve tells us how far the object is from its original position. This is a very important application of the definite integral, and we examine it in more detail later in the chapter. For now, we're just going to look at some basics to get a feel for how this works by studying constant velocities.

When velocity is a constant, the area under the curve is just velocity times time. This idea is already very familiar. If a car travels away from its starting position in a straight line at a speed of 70 mph for 2 hours, then it is 140 miles away from its original position (Figure 5.2.5). Using integral notation, we have

$$\int_0^2 70\,dt = 140 \text{ miles.}$$

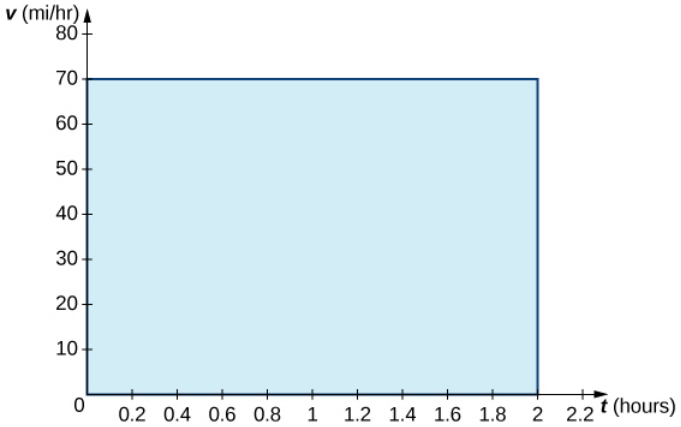


Figure 5.2.5: The area under the curve $v(t) = 70$ tells us how far the car is from its starting point at a given time.

In the context of displacement, net signed area allows us to take direction into account. If a car travels straight north at a speed of 60 mph for 2 hours, it is 120 miles north of its starting position. If the car then turns around and travels south at a speed of 40 mph for 3 hours, it will be back at it starting position (Figure 5.2.6). Again, using integral notation, we have

$$\int_0^2 60\,dt + \int_2^5 -40\,dt = 120 - 120 = 0.$$

In this case the displacement is zero.

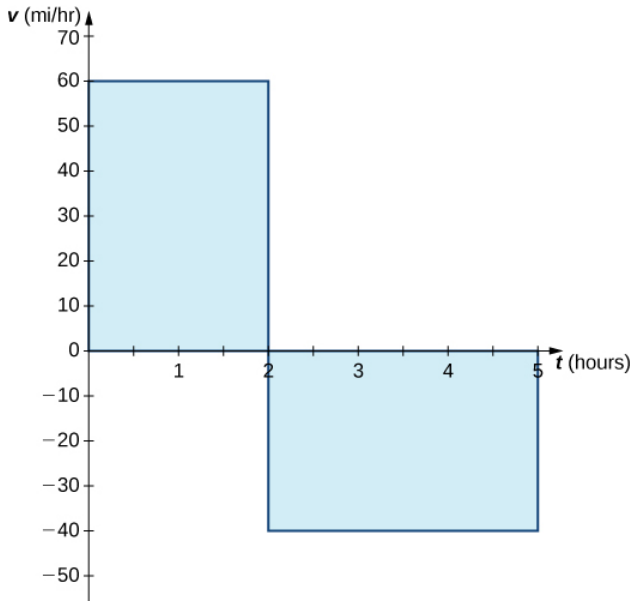


Figure 5.2.6: The area above the axis and the area below the axis are equal, so the net signed area is zero.

Suppose we want to know how far the car travels overall, regardless of direction. In this case, we want to know the area between the curve and the t-axis, regardless of whether that area is above or below the axis. This is called the **total area.**

Graphically, it is easiest to think of calculating total area by adding the areas above the axis and the areas below the axis (rather than subtracting the areas below the axis, as we did with net signed area). To accomplish this mathematically, we use the absolute value function. Thus, the total distance traveled by the car is

$$\int_0^2 |60|\,dt + \int_2^5 |-40|\,dt = \int_0^2 60\,dt + \int_2^5 40\,dt = 120 + 120 = 240.$$

Bringing these ideas together formally, we state the following definitions.

Definition: Net Signed Area

Let $f(x)$ be an integrable function defined on an interval $[a, b]$. Let A_1 represent the area between $f(x)$ and the x-axis that lies above the axis and let A_2 represent the area between $f(x)$ and the x-axis that lies below the axis. Then, the *net signed area* between $f(x)$ and the x-axis is given by

$$\int_a^b f(x)\,dx = A_1 - A_2.$$

The total area between $f(x)$ and the x-axis is given by

$$\int_a^b |f(x)|\,dx = A_1 + A_2.$$

Example 5.2.4: Finding the Total Area

Find the total area between $f(x) = x - 2$ and the x-axis over the interval $[0, 6]$.

Solution

Calculate the x-intercept as $(2, 0)$ (set $y = 0$, solve for x). To find the total area, take the area below the x-axis over the subinterval $[0, 2]$ and add it to the area above the x-axis on the subinterval $[2, 6]$ (Figure 5.2.7).

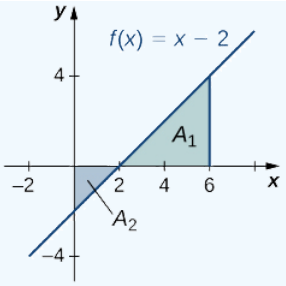


Figure 5.2.7: The total area between the line and the x-axis over $[0, 6]$ is A_2 plus A_1.

We have

$$\int_0^6 |(x-2)|\,dx = A_2 + A_1.$$

Then, using the formula for the area of a triangle, we obtain

$$A_2 = \frac{1}{2}bh = \frac{1}{2}\cdot 2\cdot 2 = 2$$

$$A_1 = \frac{1}{2}bh = \frac{1}{2}\cdot 4\cdot 4 = 8.$$

The total area, then, is

$$A_1 + A_2 = 8 + 2 = 10\,\text{units}^2.$$

? Exercise 5.2.4

Find the total area between the function $f(x) = 2x$ and the x-axis over the interval $[-3, 3]$.

Hint

Review the solving strategy in Example 5.2.4.

Answer

$18\,\text{units}^2$

Properties of the Definite Integral

The properties of indefinite integrals apply to definite integrals as well. Definite integrals also have properties that relate to the limits of integration. These properties, along with the rules of integration that we examine later in this chapter, help us manipulate expressions to evaluate definite integrals.

Rule: Properties of the Definite Integral

1.

$$\int_a^a f(x)\,dx = 0 \tag{5.2.1}$$

If the limits of integration are the same, the integral is just a line and contains no area.

2.

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

If the limits are reversed, then place a negative sign in front of the integral.

3.

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

The integral of a sum is the sum of the integrals.

4.

$$\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

The integral of a difference is the difference of the integrals.

5.

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

for constant c. The integral of the product of a constant and a function is equal to the constant multiplied by the integral of the function.

6.

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Although this formula normally applies when c is between a and b, the formula holds for all values of a, b, and c, provided $f(x)$ is integrable on the largest interval.

Example 5.2.5: Using the Properties of the Definite Integral

Use the properties of the definite integral to express the definite integral of $f(x) = -3x^3 + 2x + 2$ over the interval $[-2, 1]$ as the sum of three definite integrals.

Solution

Using integral notation, we have $\int_{-2}^{1} (-3x^3 + 2x + 2)\,dx$. We apply properties 3. and 5. to get

$$\begin{aligned}\int_{-2}^{1} (-3x^3 + 2x + 2)\,dx &= \int_{-2}^{1} -3x^3\,dx + \int_{-2}^{1} 2x\,dx + \int_{-2}^{1} 2\,dx \\ &= -3\int_{-2}^{1} x^3\,dx + 2\int_{-2}^{1} x\,dx + \int_{-2}^{1} 2\,dx.\end{aligned}$$

Exercise 5.2.5

Use the properties of the definite integral to express the definite integral of $f(x) = 6x^3 - 4x^2 + 2x - 3$ over the interval $[1, 3]$ as the sum of four definite integrals.

Hint

Use the solving strategy from Example 5.2.5 and the properties of definite integrals.

Answer

$$6\int_1^3 x^3\,dx - 4\int_1^3 x^2\,dx + 2\int_1^3 x\,dx - \int_1^3 3\,dx$$

Example 5.2.6: Using the Properties of the Definite Integral

If it is known that $\int_0^8 f(x)\,dx = 10$ and $\int_0^5 f(x)\,dx = 5$, find the value of $\int_5^8 f(x)\,dx$.

Solution

By property 6,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Thus,

$$\int_0^8 f(x)\,dx = \int_0^5 f(x)\,dx + \int_5^8 f(x)\,dx$$
$$10 = 5 + \int_5^8 f(x)\,dx$$
$$5 = \int_5^8 f(x)\,dx.$$

Exercise 5.2.6

If it is known that $\int_1^5 f(x)\,dx = -3$ and $\int_2^5 f(x)\,dx = 4$, find the value of $\int_1^2 f(x)\,dx$.

Hint

Use the solving strategy from Example 5.2.6 and the rule on properties of definite integrals.

Answer

-7

Comparison Properties of Integrals

A picture can sometimes tell us more about a function than the results of computations. Comparing functions by their graphs as well as by their algebraic expressions can often give new insight into the process of integration. Intuitively, we might say that if a function $f(x)$ is above another function $g(x)$, then the area between $f(x)$ and the x-axis is greater than the area between $g(x)$ and the x-axis. This is true depending on the interval over which the comparison is made. The properties of definite integrals are valid whether $a < b$, $a = b$, or $a > b$. The following properties, however, concern only the case $a \le b$, and are used when we want to compare the sizes of integrals.

Comparison Theorem

i. If $f(x) \ge 0$ for $a \le x \le b$, then

$$\int_a^b f(x)\,dx \ge 0.$$

ii. If $f(x) \ge g(x)$ for $a \le x \le b$, then

$$\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx.$$

iii. If m and M are constants such that $m \le f(x) \le M$ for $a \le x \le b$, then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

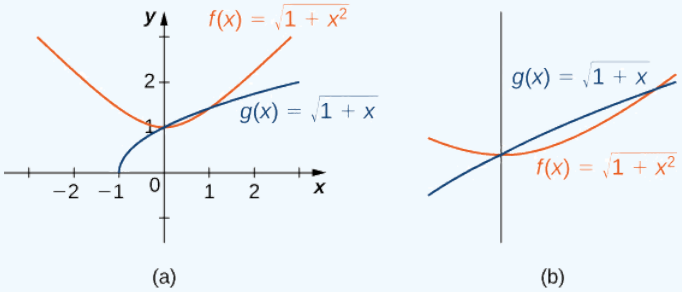
Example 5.2.7: Comparing Two Functions over a Given Interval

Compare $f(x) = \sqrt{1+x^2}$ and $g(x) = \sqrt{1+x}$ over the interval $[0,1]$.

Solution

Graphing these functions is necessary to understand how they compare over the interval $[0,1]$. Initially, when graphed on a graphing calculator, $f(x)$ appears to be above $g(x)$ everywhere. However, on the interval $[0,1]$, the graphs appear to be on top of

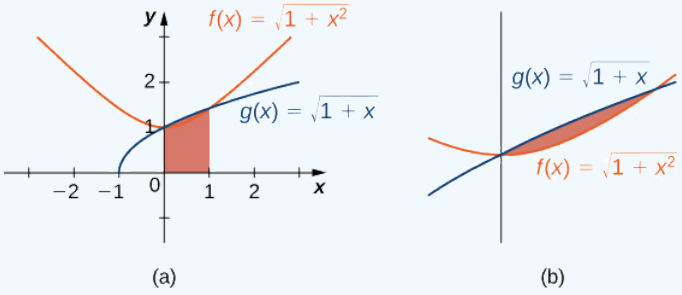
each other. We need to zoom in to see that, on the interval $[0,1]$, $g(x)$ is above $f(x)$. The two functions intersect at $x=0$ and $x=1$ (Figure 5.2.8).



(a) (b)

Figure 5.2.8: (a) The function $f(x)$ appears above the function $g(x)$ except over the interval $[0,1]$ (b) Viewing the same graph with a greater zoom shows this more clearly.

We can see from the graph that over the interval $[0,1]$, $g(x) \geq f(x)$. Comparing the integrals over the specified interval $[0,1]$, we also see that $\int_0^1 g(x)\,dx \geq \int_0^1 f(x)\,dx$ (Figure 5.2.9). The thin, red-shaded area shows just how much difference there is between these two integrals over the interval $[0,1]$.



(a) (b)

Figure 5.2.9: (a) The graph shows that over the interval $[0,1], g(x) \geq f(x)$, where equality holds only at the endpoints of the interval. (b) Viewing the same graph with a greater zoom shows this more clearly.

Average Value of a Function

We often need to find the average of a set of numbers, such as an average test grade. Suppose you received the following test scores in your algebra class: 89, 90, 56, 78, 100, and 69. Your semester grade is your average of test scores and you want to know what grade to expect. We can find the average by adding all the scores and dividing by the number of scores. In this case, there are six test scores. Thus,

$$\frac{89+90+56+78+100+69}{6} = \frac{482}{6} \approx 80.33.$$

Therefore, your average test grade is approximately 80.33, which translates to a B− at most schools.

Suppose, however, that we have a function $v(t)$ that gives us the speed of an object at any time t, and we want to find the object's average speed. The function $v(t)$ takes on an infinite number of values, so we can't use the process just described. Fortunately, we can use a definite integral to find the average value of a function such as this.

Let $f(x)$ be continuous over the interval $[a,b]$ and let $[a,b]$ be divided into n subintervals of width $\Delta x = (b-a)/n$. Choose a representative x_i^* in each subinterval and calculate $f(x_i^*)$ for $i = 1, 2, \ldots, n$. In other words, consider each $f(x_i^*)$ as a sampling of the function over each subinterval. The average value of the function may then be approximated as

$$f_{ave} \approx \frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n},$$

which is basically the same expression used to calculate the average of discrete values.

But we know $\Delta x = \frac{b-a}{n}$, so $n = \frac{b-a}{\Delta x}$, and we get

$$f_{ave} \approx \frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n} = \frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{\left(\frac{b-a}{\Delta x}\right)}.$$

Following through with the algebra, the numerator is a sum that is represented as $\sum_{i=1}^{n} f(x_i^*)$, and we are dividing by a fraction. To divide by a fraction, invert the denominator and multiply. Thus, an approximate value for the average value of the function is given by

$$\frac{\sum_{i=1}^{n} f(x_i^*)}{\left(\frac{b-a}{\Delta x}\right)} = \left(\frac{\Delta x}{b-a}\right)\sum_{i=1}^{n} f(x_i^*) = \left(\frac{1}{b-a}\right)\sum_{i=1}^{n} f(x_i^*)\Delta x.$$

This is a Riemann sum. Then, to get the exact average value, take the limit as n goes to infinity. Thus, the average value of a function is given by

$$\frac{1}{b-a}\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \frac{1}{b-a}\int_a^b f(x)dx.$$

Definition: Average Value of a Function

Let $f(x)$ be continuous over the interval $[a, b]$. Then, the *average value of the function* $f(x)$ (or f_{ave}) on $[a, b]$ is given by

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{5.2.2}$$

Example 5.2.8: Finding the Average Value of a Linear Function

Find the average value of $f(x) = x + 1$ over the interval $[0, 5]$.

Solution

First, graph the function on the stated interval, as shown in Figure 5.2.10.

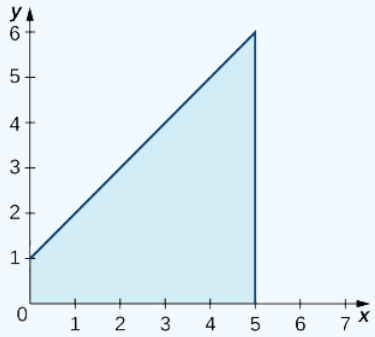


Figure 5.2.10:The graph shows the area under the function $(x) = x + 1$ over $[0, 5]$.

The region is a trapezoid lying on its side, so we can use the area formula for a trapezoid $A = \frac{1}{2}h(a + b)$, where h represents height, and a and b represent the two parallel sides. Then,

$$\int_0^5 x + 1\,dx = \frac{1}{2}h(a + b) = \frac{1}{2}\cdot 5\cdot(1 + 6) = \frac{35}{2}.$$

Thus the average value of the function is

$$\frac{1}{5-0}\int_0^5 x + 1\,dx = \frac{1}{5}\cdot\frac{35}{2} = \frac{7}{2}.$$

Exercise 5.2.7

Find the average value of $f(x) = 6 - 2x$ over the interval $[0, 3]$.

Hint

Use the average value formula (Equation 5.2.2), and use geometry to evaluate the integral.

Answer

3

Key Concepts

- The definite integral can be used to calculate net signed area, which is the area above the x-axis less the area below the x-axis. Net signed area can be positive, negative, or zero.
- The component parts of the definite integral are the integrand, the variable of integration, and the limits of integration.
- Continuous functions on a closed interval are integrable. Functions that are not continuous may still be integrable, depending on the nature of the discontinuities.
- The properties of definite integrals can be used to evaluate integrals.
- The area under the curve of many functions can be calculated using geometric formulas.
- The average value of a function can be calculated using definite integrals.

Key Equations

- **Definite Integral**

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

- **Properties of the Definite Integral**

$$\int_a^a f(x)\,dx = 0$$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx, \text{ for constant } c$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Glossary

average value of a function

(or f_{ave}) the average value of a function on an interval can be found by calculating the definite integral of the function and dividing that value by the length of the interval

definite integral

a primary operation of calculus; the area between the curve and the x-axis over a given interval is a definite integral

integrable function

a function is integrable if the limit defining the integral exists; in other words, if the limit of the Riemann sums as n goes to infinity exists

integrand

the function to the right of the integration symbol; the integrand includes the function being integrated

limits of integration

these values appear near the top and bottom of the integral sign and define the interval over which the function should be integrated

net signed area

the area between a function and the x-axis such that the area below the x-axis is subtracted from the area above the x-axis; the result is the same as the definite integral of the function

total area

total area between a function and the x-axis is calculated by adding the area above the x-axis and the area below the x-axis; the result is the same as the definite integral of the absolute value of the function

variable of integration

indicates which variable you are integrating with respect to; if it is x, then the function in the integrand is followed by dx

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5.2E: EXERCISES FOR SECTION 5.2

In exercises 1 - 4, express the limits as integrals.

1) $\lim_{n\to\infty}\sum_{i=1}^{n}(x_i^*)\Delta x$ over $[1,3]$

2) $\lim_{n\to\infty}\sum_{i=1}^{n}(5(x_i^*)^2-3(x_i^*)^3)\Delta x$ over $[0,2]$

Answer

$$\int_0^2 (5x^2-3x^3)\,dx$$

3) $\lim_{n\to\infty}\sum_{i=1}^{n}\sin^2(2\pi x_i^*)\Delta x$ over $[0,1]$

4) $\lim_{n\to\infty}\sum_{i=1}^{n}\cos^2(2\pi x_i^*)\Delta x$ over $[0,1]$

Answer

$$\int_0^1 \cos^2(2\pi x)\,dx$$

In exercises 5 - 10, given L_n or R_n as indicated, express their limits as $n\to\infty$ as definite integrals, identifying the correct intervals.

5) $L_n=\frac{1}{n}\sum_{i=1}^{n}\frac{i-1}{n}$

6) $R_n=\frac{1}{n}\sum_{i=1}^{n}\frac{i}{n}$

Answer

$$\int_0^1 x\,dx$$

7) $Ln=\frac{2}{n}\sum_{i=1}^{n}(1+2\frac{i-1}{n})$

8) $R_n=\frac{3}{n}\sum_{i=1}^{n}(3+3\frac{i}{n})$

Answer

$$\int_3^6 x\,dx$$

9) $L_n=\frac{2\pi}{n}\sum_{i=1}^{n}2\pi\frac{i-1}{n}\cos(2\pi\frac{i-1}{n})$

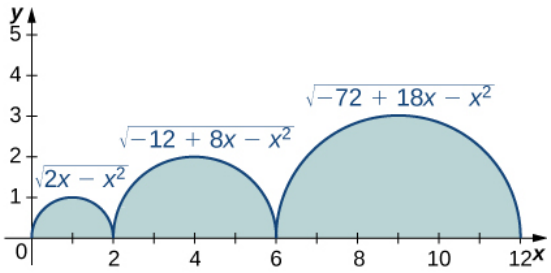
10 $R_n=\frac{1}{n}\sum_{i=1}^{n}(1+\frac{i}{n})\log((1+\frac{i}{n})^2)$

Answer

$$\int_1^2 x\log(x^2)\,dx$$

In exercises 11 - 16, evaluate the integrals of the functions graphed using the formulas for areas of triangles and circles, and subtracting the areas below the x-axis.

11)



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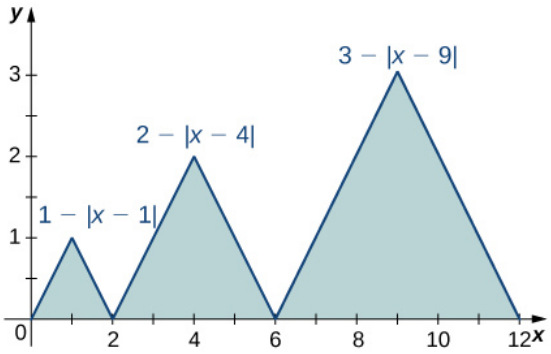
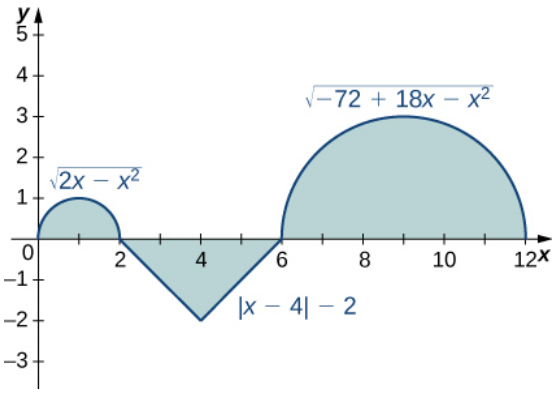


Answer

$1 + 2 \cdot 2 + 3 \cdot 3 = 14$

13)



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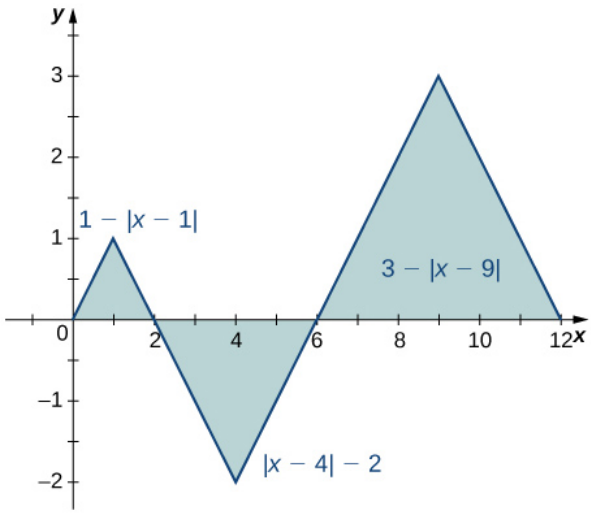
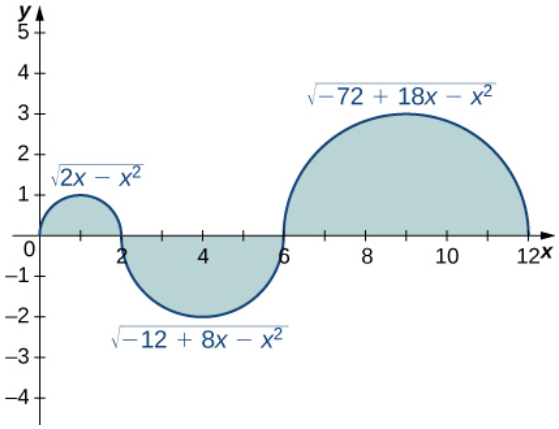


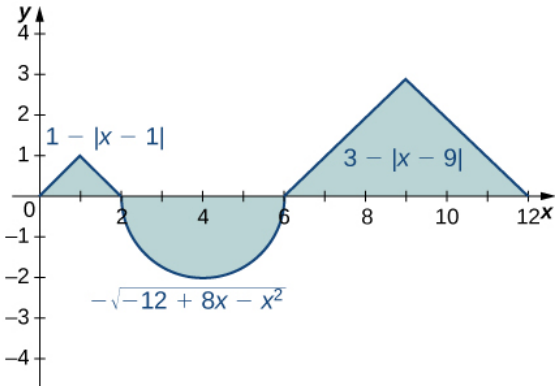
Answer

$1-4+9=6$

15)



16)



Answer

$1-2\pi+9=10-2\pi$

In exercises 17 - 24, evaluate the integral using area formulas.

17) $\int_0^3 (3-x)\,dx$

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18) $\int_2^3 (3-x)\,dx$

Answer

The integral is the area of the triangle, $\frac{1}{2}$.

19) $\int_{-3}^3 (3-|x|)\,dx$

20) $\int_0^6 (3-|x-3|)\,dx$

Answer

The integral is the area of the triangle, 9.

21) $\int_{-2}^2 \sqrt{4-x^2}\,dx$

22) $\int_1^5 \sqrt{4-(x-3)^2}\,dx$

Answer

The integral is the area $\frac{1}{2}\pi r^2 = 2\pi$.

23) $\int_0^{12} \sqrt{36-(x-6)^2}\,dx$

24) $\int_{-2}^3 (3-|x|)\,dx$

Answer

The integral is the area of the "big" triangle less the "missing" triangle, $9-\frac{1}{2}$.

In exercises 25 - 28, use averages of values at the left (L) and right (R) endpoints to compute the integrals of the piecewise linear functions with graphs that pass through the given list of points over the indicated intervals.

25) $(0,0),(2,1),(4,3),(5,0),(6,0),(8,3)$ over $[0,8]$

26) $(0,2),(1,0),(3,5),(5,5),(6,2),(8,0)$ over $[0,8]$

Answer

$$L = 2+0+10+5+4 = 21,\ R = 0+10+10+2+0 = 22,\ \frac{L+R}{2} = 21.5$$

27) $(-4,-4),(-2,0),(0,-2),(3,3),(4,3)$ over $[-4,4]$

28) $(-4,0),(-2,2),(0,0),(1,2),(3,2),(4,0)$ over $[-4,4]$

Answer

$$L = 0+4+0+4+2 = 10,\ R = 4+0+2+4+0 = 10,\ \frac{L+R}{2} = 10$$

Suppose that $\int_0^4 f(x)\,dx = 5$ and $\int_0^2 f(x)\,dx = -3$, and $\int_0^4 g(x)\,dx = -1$ and $\int_0^2 g(x)\,dx = 2$. In exercises 29 - 34, compute the integrals.

29) $\int_0^4 (f(x)+g(x))\,dx$

30) $\int_2^4 (f(x)+g(x))\,dx$

Answer

$$\int_2^4 f(x)\,dx + \int_2^4 g(x)\,dx = 8-3 = 5$$

31) $\int_0^2 (f(x)-g(x))\,dx$

32) $\int_2^4 (f(x)-g(x))\,dx$

Answer

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$$\int_2^4 f(x)\,dx - \int_2^4 g(x)\,dx = 8 + 3 = 11$$

33) $\int_0^2 (3f(x) - 4g(x))\,dx$

34) $\int_2^4 (4f(x) - 3g(x))\,dx$

Answer

$$4\int_2^4 f(x)\,dx - 3\int_2^4 g(x)\,dx = 32 + 9 = 41$$

In exercises 35 - 38, use the identity $\int_{-A}^{A} f(x)\,dx = \int_{-A}^{0} f(x)\,dx + \int_0^A f(x)\,dx$ to compute the integrals.

35) $\int_{-\pi}^{\pi} \frac{\sin t}{1+t^2}\,dt$ (Hint: $\sin(-t) = -\sin(t)$)

36) $\int_{\sqrt{-\pi}}^{\sqrt{\pi}} \frac{t}{1+\cos t}\,dt$

Answer

The integrand is odd; the integral is zero.

37) $\int_1^3 (2-x)\,dx$ (Hint: Look at the graph of f.)

38) $\int_2^4 (x-3)^3\,dx$ (Hint: Look at the graph of f.)

Answer

The integrand is antisymmetric with respect to $x = 3$. The integral is zero.

In exercises 39 - 44, given that $\int_0^1 x\,dx = \frac{1}{2}$, $\int_0^1 x^2\,dx = \frac{1}{3}$, and $\int_0^1 x^3\,dx = \frac{1}{4}$, compute the integrals.

39) $\int_0^1 (1 + x + x^2 + x^3)\,dx$

40) $\int_0^1 (1 - x + x^2 - x^3)\,dx$

Answer

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12}$$

41) $\int_0^1 (1-x)^2\,dx$

42) $\int_0^1 (1-2x)^3\,dx$

Answer

$$\int_0^1 (1 - 6x + 12x^2 - 8x^3)\,dx = 1 - 6\left(\frac{1}{2}\right) + 12\left(\frac{1}{3}\right) - 8\left(\frac{1}{4}\right) = 1 - 3 + 4 - 2 = 0$$

43) $\int_0^1 \left(6x - \frac{4}{3}x^2\right)\,dx$

44) $\int_0^1 (7 - 5x^3)\,dx$

Answer

$7 - \frac{5}{4} = \frac{23}{4}$

In exercises 45 - 50, use the comparison theorem.

45) Show that $\int_0^3 (x^2 - 6x + 9)\,dx \geq 0$.

46) Show that $\int_{-2}^3 (x-3)(x+2)\,dx \leq 0$.

Answer

The integrand is negative over $[-2, 3]$.

47) Show that $\int_0^1 \sqrt{1+x^3}\,dx \le \int_0^1 \sqrt{1+x^2}\,dx$.

48) Show that $\int_1^2 \sqrt{1+x}\,dx \le \int_1^2 \sqrt{1+x^2}\,dx$.

Answer

$x \le x^2$ over $[1, 2]$, so $\sqrt{1+x} \le \sqrt{1+x^2}$ over $[1, 2]$.

49) Show that $\int_0^{\pi/2} \sin t dt \ge \frac{\pi}{4}$ (Hint: $\sin t \ge \frac{2t}{\pi}$ over $[0, \frac{\pi}{2}]$)

50) Show that $\int_{-\pi/4}^{\pi/4} \cos t\,dt \ge \pi\sqrt{2}/4$.

Answer

$\cos(t) \ge \frac{\sqrt{2}}{2}$. Multiply by the length of the interval to get the inequality.

In exercises 51 - 56, find the average value f_{ave} of f between a and b, and find a point c, where $f(c) = f_{ave}$

51) $f(x) = x^2,\ a = -1,\ b = 1$

52) $f(x) = x^5,\ a = -1,\ b = 1$

Answer

$f_{ave} = 0;\ c = 0$

53) $f(x) = \sqrt{4-x^2},\ a = 0,\ b = 2$

54) $f(x) = 3 - |x|,\ a = -3,\ b = 3$

Answer

$\frac{3}{2}$ when $c = \pm\frac{3}{2}$

55) $f(x) = \sin x,\ a = 0,\ b = 2\pi$

56) $f(x) = \cos x,\ a = 0,\ b = 2\pi$

Answer

$f_{ave} = 0;\ c = \frac{\pi}{2},\ \frac{3\pi}{2}$

In exercises 57 - 60, approximate the average value using Riemann sums L_{100} and R_{100}. How does your answer compare with the exact given answer?

57) [T] $y = \ln(x)$ over the interval $[1, 4]$; the exact solution is $\frac{\ln(256)}{3} - 1$.

58) [T] $y = e^{x/2}$ over the interval $[0, 1]$; the exact solution is $2(\sqrt{e} - 1)$.

Answer

$L_{100} = 1.294,\ R_{100} = 1.301$; the exact average is between these values.

59) [T] $y = \tan x$ over the interval $[0, \frac{\pi}{4}]$; the exact solution is $\frac{2\ln(2)}{\pi}$.

60) [T] $y = \frac{x+1}{\sqrt{4-x^2}}$ over the interval $[-1, 1]$; the exact solution is $\frac{\pi}{6}$.

Answer

$L_{100} \times (\frac{1}{2}) = 0.5178,\ R_{100} \times (\frac{1}{2}) = 0.5294$

In exercises 61 - 64, compute the average value using the left Riemann sums L_N for $N = 1, 10, 100$. How does the accuracy compare with the given exact value?

61) [T] $y = x^2 - 4$ over the interval $[0, 2]$; the exact solution is $-\frac{8}{3}$.

62) [T] $y = xe^{x^2}$ over the interval $[0, 2]$; the exact solution is $\frac{1}{4}(e^4 - 1)$.

Answer

$L_1 = 0,\ L_{10} \times (\frac{1}{2}) = 8.743493,\ L_{100} \times (\frac{1}{2}) = 12.861728.$ The exact answer $\approx 26.799,$ so L_{100} is not accurate.

63) [T] $y = \left(\frac{1}{2}\right)^x$ over the interval $[0, 4]$; the exact solution is $\frac{15}{64\ln(2)}$.

64) [T] $y = x\sin(x^2)$ over the interval $[-\pi, 0]$; the exact solution is $\frac{\cos(\pi^2) - 1}{2\pi.}$

Answer

$L_1 \times (\frac{1}{\pi}) = 1.352, L_{10} \times (\frac{1}{\pi}) = -0.1837, L_{100} \times (1\pi) = -0.2956.$ The exact answer $\approx -0.303,$ so L_{100} is not accurate to the first decimal.

65) Suppose that $A = \int_0^{2\pi} \sin^2 t\, dt$ and $B = \int_0^{2\pi} \cos^2 t\, dt$. Show that $A + B = 2\pi$ and $A = B$.

66) Suppose that $A = \int_{-\pi/4}^{\pi/4} \sec^2 t\, dt = \pi$ and $B = \int_{-\pi/}^{\pi/4} 4\tan^2 t\, dt$. Show that $A - B = \frac{\pi}{2}$.

Answer

Use $\tan^2\theta + 1 = \sec^2\theta$. Then, $B - A = \int_{-\pi/4}^{\pi/4} 1\, dx = \frac{\pi}{2}$.

67) Show that the average value of $\sin^2 t$ over $[0, 2\pi]$ is equal to $1/2$. Without further calculation, determine whether the average value of $\sin^2 t$ over $[0, \pi]$ is also equal to $1/2$.

68) Show that the average value of $\cos^2 t$ over $[0, 2\pi]$ is equal to $1/2$. Without further calculation, determine whether the average value of $\cos^2(t)$ over $[0, \pi]$ is also equal to $1/2$.

Answer

$\int_0^{2\pi} \cos^2 t\, dt = \pi,$ so divide by the length 2π of the interval. $\cos^2 t$ has period π, so yes, it is true.

69) Explain why the graphs of a quadratic function (parabola) $p(x)$ and a linear function $\ell(x)$ can intersect in at most two points. Suppose that $p(a) = \ell(a)$ and $p(b) = \ell(b)$, and that $\int_a^b p(t)\, dt > \int_a^b \ell(t)\, dt$. Explain why $\int_c^d p(t)\, dt > \int_c^d \ell(t)\, dt$ whenever $a \le c < d \le b$.

70) Suppose that parabola $p(x) = ax^2 + bx + c$ opens downward $(a < 0)$ and has a vertex of $y = \frac{-b}{2a} > 0$. For which interval $[A, B]$ is $\int_A^B (ax^2 + bx + c)\, dx$ as large as possible?

Answer

The integral is maximized when one uses the largest interval on which p is nonnegative. Thus, $A = \frac{-b-\sqrt{b^2-4ac}}{2a}$ and $B = \frac{-b+\sqrt{b^2-4ac}}{2a}$.

71) Suppose $[a, b]$ can be subdivided into subintervals $a = a_0 < a_1 < a_2 < \cdots < a_N = b$ such that either $f \ge 0$ over $[a_{i-1}, a_i]$ or $f \le 0$ over $[a_{i-1}, a_i]$. Set $A_i = \int_{a_{i-1}}^{a_i} f(t)\, dt$.

a. Explain why $\int_a^b f(t)\, dt = A_1 + A_2 + \cdots + A_N$.

b. Then, explain why $\int_a^b f(t)\, dt \le \int_a^b |f(t)|\, dt$.

72) Suppose f and g are continuous functions such that $\int_c^d f(t)\, dt \le \int_c^d g(t)\, dt$ for every subinterval $[c, d]$ of $[a, b]$. Explain why $f(x) \le g(x)$ for all values of x.

Answer

If $f(t_0) > g(t_0)$ for some $t_0 \in [a, b]$, then since $f - g$ is continuous, there is an interval containing t_0 such that $f(t) > g(t)$ over the interval $[c, d]$, and then $\int_d^d f(t)\, dt > \int_c^d g(t)\, dt$over this interval.

73) Suppose the average value of f over $[a, b]$ is 1 and the average value of f over $[b, c]$ is 1 where $a < c < b$. Show that the average value of f over $[a, c]$ is also 1.

74) Suppose that $[a, b]$ can be partitioned. taking $a = a_0 < a_1 < \cdots < a_N = b$ such that the average value of f over each subinterval $[a_{i-1}, a_i] = 1$ is equal to 1 for each $i = 1, \ldots, N$. Explain why the average value of f over $[a, b]$ is also equal to 1.

Answer

The integral of f over an interval is the same as the integral of the average of f over that interval. Thus, $\int_a^b f(t)\,dt = \int_{a_0}^{a_1} f(t)\,dt$
$+ \int_a^{a_2} 1_f(t)\,dt + \cdots + \int_{a_{N+1}}^{a_N} f(t)\,dt = \int_{a_0}^{a_1} 1\,dt + \int_{a_1}^{a_2} 1\,dt + \cdots + \int_{a_{N+1}}^{a_N} 1\,dt$
$= (a_1 - a_0) + (a_2 - a_1) + \cdots + (a_N - a_{N-1}) = a_N - a_0 = b - a.$
Dividing through by $b - a$ gives the desired identity.

75) Suppose that for each i such that $1 \le i \le N$ one has $\int_{i-1}^{i} f(t)\,dt = i$. Show that $\int_0^N f(t)\,dt = \frac{N(N+1)}{2}$.

76) Suppose that for each i such that $1 \le i \le N$ one has $\int_{i-1}^{i} f(t)\,dt = i^2$. Show that $\int_0^N f(t)\,dt = \frac{N(N+1)(2N+1)}{6}$.

Answer

$$\int_0^N f(t)\,dt = \sum_{i=1}^{N} \int_{i-1}^{i} f(t)\,dt = \sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6}$$

77) [T] Compute the left and right Riemann sums L_{10} and R_{10} and their average $\frac{L_{10} + R_{10}}{2}$ for $f(t) = t^2$over $[0, 1]$. Given that $\int_0^1 t^2\,dt = 1/3$, to how many decimal places is $\frac{L_{10} + R_{10}}{2}$ accurate?

78) [T] Compute the left and right Riemann sums, L_10 and R_{10}, and their average $\frac{L_{10} + R_{10}}{2}$ for $f(t) = (4 - t^2)$ over $[1, 2]$. Given that $\int_1^2 (4 - t^2)\,dt = 1.66$, to how many decimal places is $\frac{L_{10} + R_{10}}{2}$ accurate?

Answer

$L_{10} = 1.815,\ R_{10} = 1.515,\ \frac{L_{10}+R_{10}}{2} = 1.665,$ so the estimate is accurate to two decimal places.

79) If $\int_1^5 \sqrt{1 + t^4}\,dt = 41.7133\ldots$, what is $\int_1^5 \sqrt{1 + u^4}\,du$?

80) Estimate $\int_0^1 t\,dt$ using the left and right endpoint sums, each with a single rectangle. How does the average of these left and right endpoint sums compare with the actual value $\int_0^1 t\,dt$?

Answer

The average is $1/2$, which is equal to the integral in this case.

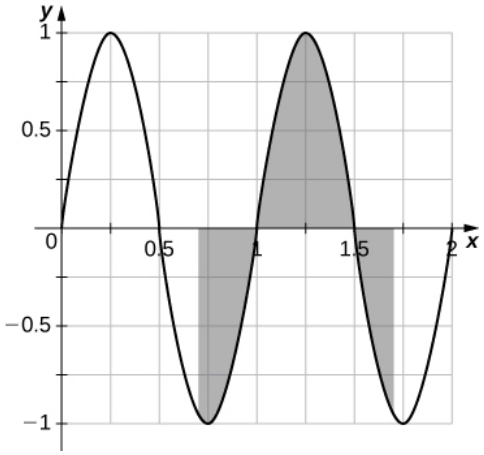
81) Estimate $\int_0^1 t\,dt$ by comparison with the area of a single rectangle with height equal to the value of t at the midpoint $t = \frac{1}{2}$. How does this midpoint estimate compare with the actual value $\int_0^1 t\,dt$?

82) From the graph of $\sin(2\pi x)$ shown:

a. Explain why $\int_0^1 \sin(2\pi t)\,dt = 0$.

b. Explain why, in general, $\int_a^{a+1} \sin(2\pi t)\,dt = 0$ for any value of a.

Answer



a. The graph is antisymmetric with respect to $t = \frac{1}{2}$ over $[0, 1]$, so the average value is zero.
b. For any value of a, the graph between $[a, a + 1]$ is a shift of the graph over $[0, 1]$, so the net areas above and below the axis do not change and the average remains zero.

83) If f is 1-periodic $(f(t + 1) = f(t))$, odd, and integrable over $[0, 1]$, is it always true that $\int_0^1 f(t)\,dt = 0$?

84) If f is 1-periodic and $\int 10 f(t)\,dt = A$, is it necessarily true that $\int_a^{1+a} f(t)\,dt = A$ for all A?

Answer

Yes, the integral over any interval of length 1 is the same.

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5.3: The Fundamental Theorem of Calculus

Learning Objectives

- Describe the meaning of the Mean Value Theorem for Integrals.
- State the meaning of the Fundamental Theorem of Calculus, Part 1.
- Use the Fundamental Theorem of Calculus, Part 1, to evaluate derivatives of integrals.
- State the meaning of the Fundamental Theorem of Calculus, Part 2.
- Use the Fundamental Theorem of Calculus, Part 2, to evaluate definite integrals.
- Explain the relationship between differentiation and integration.

In the previous two sections, we looked at the definite integral and its relationship to the area under the curve of a function. Unfortunately, so far, the only tools we have available to calculate the value of a definite integral are geometric area formulas and limits of Riemann sums, and both approaches are extremely cumbersome. In this section we look at some more powerful and useful techniques for evaluating definite integrals.

These new techniques rely on the relationship between differentiation and integration. This relationship was discovered and explored by both Sir Isaac Newton and Gottfried Wilhelm Leibniz (among others) during the late 1600s and early 1700s, and it is codified in what we now call the **Fundamental Theorem of Calculus**, which has two parts that we examine in this section. Its very name indicates how central this theorem is to the entire development of calculus.

Isaac Newton's contributions to mathematics and physics changed the way we look at the world. The relationships he discovered, codified as Newton's laws and the law of universal gravitation, are still taught as foundational material in physics today, and his calculus has spawned entire fields of mathematics.

Before we get to this crucial theorem, however, let's examine another important theorem, the Mean Value Theorem for Integrals, which is needed to prove the Fundamental Theorem of Calculus.

The Mean Value Theorem for Integrals

The Mean Value Theorem for Integrals states that a continuous function on a closed interval takes on its average value at some point in that interval. The theorem guarantees that if $f(x)$ is continuous, a point c exists in an interval $[a, b]$ such that the value of the function at c is equal to the average value of $f(x)$ over $[a, b]$. We state this theorem mathematically with the help of the formula for the average value of a function that we presented at the end of the preceding section.

Theorem 5.3.1: The Mean Value Theorem for Integrals

If $f(x)$ is continuous over an interval $[a, b]$, then there is at least one point $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

This formula can also be stated as

$$\int_a^b f(x)\, dx = f(c)(b-a). \tag{5.3.1}$$

Since $f(x)$ is continuous on $[a, b]$, by the extreme value theorem (see section on Maxima and Minima), it assumes minimum and maximum values—m and M, respectively—on $[a, b]$. Then, for all x in $[a, b]$, we have $m \leq f(x) \leq M$. Therefore, by the comparison theorem (see Section on The Definite Integral), we have

$$m(b-a) \leq \int_a^b f(x)\, dx \leq M(b-a).$$

Proof

Since $f(x)$ is continuous on $[a,b]$, by the extreme value theorem (see section on Maxima and Minima), it assumes minimum and maximum values—m and M, respectively—on $[a,b]$. Then, for all x in $[a,b]$, we have $m \le f(x) \le M$. Therefore, by the comparison theorem (see Section on The Definite Integral), we have

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

Dividing by $b-a$ gives us

$$m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M.$$

Since $\frac{1}{b-a}\int_a^b f(x)\,dx$ is a number between m and M, and since $f(x)$ is continuous and assumes the values m and M over $[a,b]$, by the Intermediate Value Theorem, there is a number c over $[a,b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx,$$

and the proof is complete.

□

Example 5.3.1: Finding the Average Value of a Function

Find the average value of the function $f(x) = 8 - 2x$ over the interval $[0,4]$ and find c such that $f(c)$ equals the average value of the function over $[0,4]$.

Solution

The formula states the mean value of $f(x)$ is given by

$$\frac{1}{4-0}\int_0^4 (8-2x)\,dx.$$

We can see in Figure 5.3.1 that the function represents a straight line and forms a right triangle bounded by the x- and y-axes. The area of the triangle is $A = \frac{1}{2}(base)(height)$. We have

$$A = \frac{1}{2}(4)(8) = 16.$$

The average value is found by multiplying the area by $1/(4-0)$. Thus, the average value of the function is

$$\frac{1}{4}(16) = 4$$

Set the average value equal to $f(c)$ and solve for c.

$$\begin{aligned} 8 - 2c &= 4 \\ c &= 2 \end{aligned}$$

At $c = 2$, $f(2) = 4$.

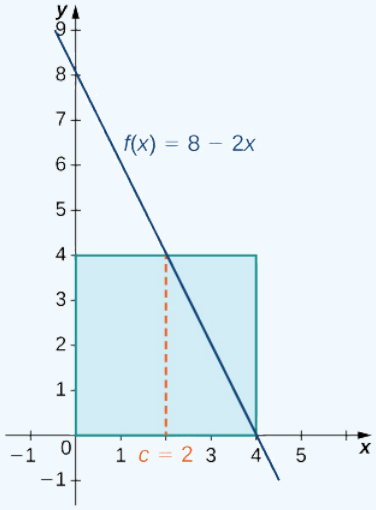

Figure 5.3.1: By the Mean Value Theorem, the continuous function $f(x)$ takes on its average value at c at least once over a closed interval.

Exercise 5.3.1

Find the average value of the function $f(x) = \frac{x}{2}$ over the interval $[0, 6]$ and find c such that $f(c)$ equals the average value of the function over $[0, 6]$.

Hint

Use the procedures from Example 5.3.1 to solve the problem

Answer

The average value is 1.5 and $c = 3$.

Example 5.3.2: Finding the Point Where a Function Takes on Its Average Value

Given $\int_0^3 x^2\,dx = 9$, find c such that $f(c)$ equals the average value of $f(x) = x^2$ over $[0, 3]$.

Solution

We are looking for the value of c such that

$$f(c) = \frac{1}{3-0}\int_0^3 x^2\,dx = \frac{1}{3}(9) = 3.$$

Replacing $f(c)$ with c^2, we have

$$c^2 = 3$$
$$c = \pm\sqrt{3}.$$

Since $-\sqrt{3}$ is outside the interval, take only the positive value. Thus, $c = \sqrt{3}$ (Figure 5.3.2).

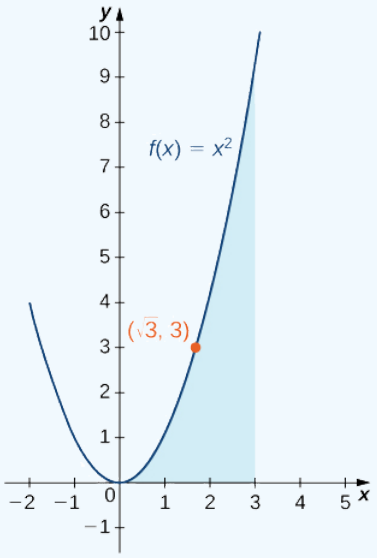

Figure 5.3.2: Over the interval $[0, 3]$, the function $f(x) = x^2$ takes on its average value at $c = \sqrt{3}$.

? Exercise 5.3.2

Given $\int_0^3 (2x^2 - 1)\,dx = 15$, find c such that $f(c)$ equals the average value of $f(x) = 2x^2 - 1$ over $[0, 3]$.

Hint

Use the procedures from Example 5.3.2 to solve the problem.

Answer

$c = \sqrt{3}$

Fundamental Theorem of Calculus Part 1: Integrals and Antiderivatives

As mentioned earlier, the Fundamental Theorem of Calculus is an extremely powerful theorem that establishes the relationship between differentiation and integration, and gives us a way to evaluate definite integrals without using Riemann sums or calculating areas. The theorem is comprised of two parts, the first of which, the **Fundamental Theorem of Calculus, Part 1,** is stated here. Part 1 establishes the relationship between differentiation and integration.

Theorem 5.3.2: The Fundamental Theorem of Calculus, Part 1

If $f(x)$ is continuous over an interval $[a, b]$, and the function $F(x)$ is defined by

$$F(x) = \int_a^x f(t)\,dt,$$

then $F'(x) = f(x)$ over $[a, b]$.

Before we delve into the proof, a couple of subtleties are worth mentioning here. First, a comment on the notation. Note that we have defined a function, $F(x)$, as the definite integral of another function, $f(t)$, from the point a to the point x. At first glance, this is confusing, because we have said several times that a definite integral is a number, and here it looks like it's a function. The key here is to notice that for any particular value of x, the definite integral is a number. So the function $F(x)$ returns a number (the value of the definite integral) for each value of x.

Second, it is worth commenting on some of the key implications of this theorem. There is a reason it is called the *Fundamental Theorem of Calculus*. Not only does it establish a relationship between integration and differentiation, but also it guarantees that any integrable function has an antiderivative. Specifically, it guarantees that any continuous function has an antiderivative.

Proof: Fundamental Theorem of Calculus, Part 1

Applying the definition of the derivative, we have

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)dt - \int_a^x f(t)\,dt\right] \\ &= \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)\,dt + \int_x^a f(t)\,dt\right] \\ &= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt. \end{aligned}$$

Looking carefully at this last expression, we see $\frac{1}{h}\int_x^{x+h} f(t)\,dt$ is just the average value of the function $f(x)$ over the interval $[x, x+h]$. Therefore, by Equation 5.3.1, there is some number c in $[x, x+h]$ such that

$$\frac{1}{h}\int_x^{x+h} f(t)\,dt = f(c).$$

In addition, since c is between x and h, c approaches x as h approaches zero. Also, since $f(x)$ is continuous, we have

$$\lim_{h\to 0} f(c) = \lim_{c\to x} f(c) = f(x)$$

Putting all these pieces together, we have

$$F'(x) = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt = \lim_{h\to 0} f(c) = f(x),$$

and the proof is complete.

□

Example 5.3.3: Finding a Derivative with the Fundamental Theorem of Calculus

Use the Fundamental Theorem of Calculus, Part 1 to find the derivative of

$$g(x) = \int_1^x \frac{1}{t^3+1}\,dt.$$

Solution

According to the Fundamental Theorem of Calculus, the derivative is given by

$$g'(x) = \frac{1}{x^3+1}.$$

Exercise 5.3.3

Use the Fundamental Theorem of Calculus, Part 1 to find the derivative of $g(r) = \int_0^r \sqrt{x^2+4}\,dx$.

Hint

Follow the procedures from Example 5.3.3 to solve the problem.

Answer

$g'(r) = \sqrt{r^2+4}$

Example 5.3.4: Using the Fundamental Theorem and the Chain Rule to Calculate Derivatives

Let $F(x) = \int_1^{\sqrt{x}} \sin t \, dt$. Find $F'(x)$.

Solution

Letting $u(x) = \sqrt{x}$, we have $F(x) = \int_1^{u(x)} \sin t \, dt$.

Thus, by the Fundamental Theorem of Calculus and the chain rule,

$$F'(x) = \sin(u(x))\frac{du}{dx} = \sin(u(x)) \cdot \left(\frac{1}{2}x^{-1/2}\right) = \frac{\sin\sqrt{x}}{2\sqrt{x}}.$$

Exercise 5.3.4

Let $F(x) = \int_1^{x^3} \cos t \, dt$. Find $F'(x)$.

Hint

Use the chain rule to solve the problem.

Answer

$F'(x) = 3x^2 \cos x^3$

Example 5.3.5: Using the Fundamental Theorem of Calculus with Two Variable Limits of Integration

Let $F(x) = \int_x^{2x} t^3 \, dt$. Find $F'(x)$.

Solution

We have $F(x) = \int_x^{2x} t^3 \, dt$. Both limits of integration are variable, so we need to split this into two integrals. We get

$$\begin{aligned} F(x) &= \int_x^{2x} t^3 \, dt = \int_x^0 t^3 \, dt + \int_0^{2x} t^3 \, dt \\ &= -\int_0^x t^3 \, dt + \int_0^{2x} t^3 \, dt. \end{aligned}$$

Differentiating the first term, we obtain

$$\frac{d}{dx}\left[-\int_0^x t^3 \, dt\right] = -x^3.$$

Differentiating the second term, we first let $u(x) = 2x$. Then,

$$\begin{aligned} \frac{d}{dx}\left[\int_0^{2x} t^3 \, dt\right] &= \frac{d}{dx}\left[\int_0^{u(x)} t^3 \, dt\right] \\ &= (u(x))^3 \frac{du}{dx} \\ &= (2x)^3 \cdot 2 = 16x^3. \end{aligned}$$

Thus,

$$\begin{aligned} F'(x) &= \frac{d}{dx}\left[-\int_0^x t^3 \, dt\right] + \frac{d}{dx}\left[\int_0^{2x} t^3 \, dt\right] \\ &= -x^3 + 16x^3 = 15x^3 \end{aligned}$$

Exercise 5.3.5

Let $F(x) = \int_x^{x^2} \cos t \, dt$. Find $F'(x)$.

Hint

Use the procedures from Example 5.3.5 to solve the problem

Answer

$F'(x) = 2x \cos x^2 - \cos x$

Fundamental Theorem of Calculus, Part 2: The Evaluation Theorem

The Fundamental Theorem of Calculus, Part 2, is perhaps the most important theorem in calculus. After tireless efforts by mathematicians for approximately 500 years, new techniques emerged that provided scientists with the necessary tools to explain many phenomena. Using calculus, astronomers could finally determine distances in space and map planetary orbits. Everyday financial problems such as calculating marginal costs or predicting total profit could now be handled with simplicity and accuracy. Engineers could calculate the bending strength of materials or the three-dimensional motion of objects. Our view of the world was forever changed with calculus.

After finding approximate areas by adding the areas of n rectangles, the application of this theorem is straightforward by comparison. It almost seems too simple that the area of an entire curved region can be calculated by just evaluating an antiderivative at the first and last endpoints of an interval.

Theorem 5.3.3: The Fundamental Theorem of Calculus, Part 2

If $f(x)$ is continuous over the interval $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x) \, dx = F(b) - F(a). \tag{5.3.2}$$

We often see the notation $F(x)|_a^b$ to denote the expression $F(b) - F(a)$. We use this vertical bar and associated limits a and b to indicate that we should evaluate the function $F(x)$ at the upper limit (in this case, b), and subtract the value of the function $F(x)$ evaluated at the lower limit (in this case, a).

The **Fundamental Theorem of Calculus, Part 2** (also known as the **evaluation theorem**) states that if we can find an antiderivative for the integrand, then we can evaluate the definite integral by evaluating the antiderivative at the endpoints of the interval and subtracting.

Proof

Let $P = x_i, i = 0, 1, \ldots, n$ be a regular partition of $[a, b]$. Then, we can write

$$\begin{aligned} F(b) - F(a) &= F(x_n) - F(x_0) \\ &= [F(x_n) - F(x_{n-1})] + [F(x_{n-1}) - F(x_{n-2})] + \ldots + [F(x_1) - F(x_0)] \\ &= \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})]. \end{aligned}$$

Now, we know F is an antiderivative of f over $[a, b]$, so by the Mean Value Theorem for derivatives (see The Mean Value Theorem) for $i = 0, 1, \ldots, n$ we can find c_i in $[x_{i-1}, x_i]$ such that

$$F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}) = f(c_i) \, \Delta x.$$

Then, substituting into the previous equation, we have

$$F(b) - F(a) = \sum_{i=1}^{n} f(c_i) \, \Delta x.$$

Taking the limit of both sides as $n \to \infty$, we obtain

$$F(b) - F(a) = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \int_a^b f(x)\,dx.$$

□

Example 5.3.6: Evaluating an Integral with the Fundamental Theorem of Calculus

Use Equation 5.3.2 to evaluate

$$\int_{-2}^{2} (t^2 - 4)\,dt.$$

Solution

Recall the power rule for Antiderivatives:

If $y = x^n$,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C.$$

Use this rule to find the antiderivative of the function and then apply the theorem. We have

$$\begin{aligned}
\int_{-2}^{2} (t^2 - 4)dt &= \left(\frac{t^3}{3} - 4t\right)\Big|_{-2}^{2} \\
&= \left[\frac{(2)^3}{3} - 4(2)\right] - \left[\frac{(-2)^3}{3} - 4(-2)\right] \\
&= \left[\frac{8}{3} - 8\right] - \left[-\frac{8}{3} + 8\right] \\
&= \frac{8}{3} - 8 + \frac{8}{3} - 8 \\
&= \frac{16}{3} - 16 = -\frac{32}{3}.
\end{aligned}$$

Analysis

Notice that we did not include the "$+C$" term when we wrote the antiderivative. The reason is that, according to the Fundamental Theorem of Calculus, Part 2 (Equation 5.3.2), any antiderivative works. So, for convenience, we chose the antiderivative with $C = 0$. If we had chosen another antiderivative, the constant term would have canceled out. This always happens when evaluating a definite integral.

The region of the area we just calculated is depicted in Figure 5.3.3. Note that the region between the curve and the x-axis is all below the x-axis. Area is always positive, but a definite integral can still produce a negative number (a net signed area). For example, if this were a profit function, a negative number indicates the company is operating at a loss over the given interval.

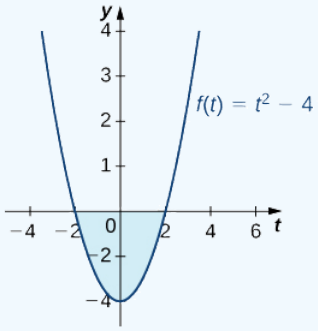

Figure 5.3.3: The evaluation of a definite integral can produce a negative value, even though area is always positive.

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Example 5.3.7: Evaluating a Definite Integral Using the Fundamental Theorem of Calculus, Part 2

Evaluate the following integral using the Fundamental Theorem of Calculus, Part 2 (Equation 5.3.2):

$$\int_1^9 \frac{x-1}{\sqrt{x}}dx.$$

Solution

First, eliminate the radical by rewriting the integral using rational exponents. Then, separate the numerator terms by writing each one over the denominator:

$$\int_1^9 \frac{x-1}{x^{1/2}}\,dx = \int_1^9 \left(\frac{x}{x^{1/2}} - \frac{1}{x^{1/2}}\right)\,dx.$$

Use the properties of exponents to simplify:

$$\int_1^9 \left(\frac{x}{x^{1/2}} - \frac{1}{x^{1/2}}\right)\,dx = \int_1^9 (x^{1/2} - x^{-1/2})\,dx.$$

Now, integrate using the power rule:

$$\begin{aligned}\int_1^9 (x^{1/2} - x^{-1/2})\,dx &= \left(\frac{x^{3/2}}{\frac{3}{2}} - \frac{x^{1/2}}{\frac{1}{2}}\right)\Big|_1^9 \\ &= \left[\frac{(9)^{3/2}}{\frac{3}{2}} - \frac{(9)^{1/2}}{\frac{1}{2}}\right] - \left[\frac{(1)^{3/2}}{\frac{3}{2}} - \frac{(1)^{1/2}}{\frac{1}{2}}\right] \\ &= \left[\frac{2}{3}(27) - 2(3)\right] - \left[\frac{2}{3}(1) - 2(1)\right] \\ &= 18 - 6 - \frac{2}{3} + 2 = \frac{40}{3}.\end{aligned}$$

See Figure 5.3.4.

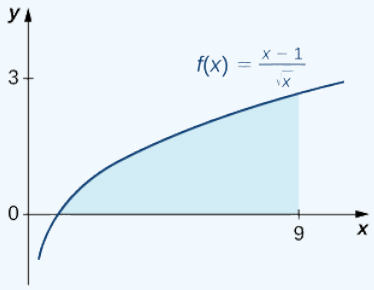

Figure 5.3.4: The area under the curve from $x = 1$ to $x = 9$ can be calculated by evaluating a definite integral.

Exercise 5.3.6

Use Note to evaluate $\int_1^2 x^{-4}\,dx$.

Hint

Use the power rule.

Answer

$\frac{7}{24}$

Applications of Definite Integrals

Example 5.3.8: A Roller-Skating Race

James and Kathy are racing on roller skates. They race along a long, straight track, and whoever has gone the farthest after 5 sec wins a prize. If James can skate at a velocity of $f(t) = 5 + 2t$ ft/sec and Kathy can skate at a velocity of $g(t) = 10 + \cos\left(\frac{\pi}{2}t\right)$ ft/sec, who is going to win the race?

Solution

We need to integrate both functions over the interval $[0, 5]$ and see which value is bigger. For James, we want to calculate

$$\int_0^5 (5 + 2t)\, dt.$$

Using the power rule, we have

$$\begin{aligned}\int_0^5 (5 + 2t)\, dt &= \left(5t + t^2\right)\Big|_0^5 \\ &= (25 + 25) \\ &= 50.\end{aligned}$$

Thus, James has skated 50 ft after 5 sec. Turning now to Kathy, we want to calculate

$$\int_0^5 10 + \cos\left(\frac{\pi}{2}t\right)\, dt.$$

We know $\sin t$ is an antiderivative of $\cos t$, so it is reasonable to expect that an antiderivative of $\cos\left(\frac{\pi}{2}t\right)$ would involve $\sin\left(\frac{\pi}{2}t\right)$. However, when we differentiate $\sin\left(\pi^2 t\right)$, we get $\pi^2 \cos\left(\pi^2 t\right)$ as a result of the chain rule, so we have to account for this additional coefficient when we integrate. We obtain

$$\begin{aligned}\int_0^5 10 + \cos\left(\frac{\pi}{2}t\right)\, dt &= \left(10t + \frac{2}{\pi}\sin\left(\frac{\pi}{2}t\right)\right)\Big|_0^5 \\ &= \left(50 + \frac{2}{\pi}\right) - \left(0 - \frac{2}{\pi}\sin 0\right) \approx 50.6.\end{aligned}$$

Kathy has skated approximately 50.6 ft after 5 sec. Kathy wins, but not by much!

Exercise 5.3.7

Suppose James and Kathy have a rematch, but this time the official stops the contest after only 3 sec. Does this change the outcome?

Hint

Change the limits of integration from those in Example 5.3.7.

Answer

Kathy still wins, but by a much larger margin: James skates 24 ft in 3 sec, but Kathy skates 29.3634 ft in 3 sec.

A Parachutist in Free Fall

Julie is an avid skydiver with more than 300 jumps under her belt and has mastered the art of making adjustments to her body position in the air to control how fast she falls. If she arches her back and points her belly toward the ground, she reaches a terminal velocity of approximately 120 mph (176 ft/sec). If, instead, she orients her body with her head straight down, she falls faster, reaching a terminal velocity of 150 mph (220 ft/sec).

Figure 5.3.5: Skydivers can adjust the velocity of their dive by changing the position of their body during the free fall. (credit: Jeremy T. Lock)

Since Julie will be moving (falling) in a downward direction, we assume the downward direction is positive to simplify our calculations. Julie executes her jumps from an altitude of 12,500 ft. After she exits the aircraft, she immediately starts falling at a velocity given by $v(t) = 32t$.

She continues to accelerate according to this velocity function until she reaches terminal velocity. After she reaches terminal velocity, her speed remains constant until she pulls her ripcord and slows down to land.

On her first jump of the day, Julie orients herself in the slower "belly down" position (terminal velocity is 176 ft/sec). Using this information, answer the following questions.

1. How long after she exits the aircraft does Julie reach terminal velocity?
2. Based on your answer to question 1, set up an expression involving one or more integrals that represents the distance Julie falls after 30 sec.
3. If Julie pulls her ripcord at an altitude of 3000 ft, how long does she spend in a free fall?
4. Julie pulls her ripcord at 3000 ft. It takes 5 sec for her parachute to open completely and for her to slow down, during which time she falls another 400 ft. After her canopy is fully open, her speed is reduced to 16 ft/sec. Find the total time Julie spends in the air, from the time she leaves the airplane until the time her feet touch the ground. On Julie's second jump of the day, she decides she wants to fall a little faster and orients herself in the "head down" position. Her terminal velocity in this position is 220 ft/sec. Answer these questions based on this velocity:
5. How long does it take Julie to reach terminal velocity in this case?
6. Before pulling her ripcord, Julie reorients her body in the "belly down" position so she is not moving quite as fast when her parachute opens. If she begins this maneuver at an altitude of 4000 ft, how long does she spend in a free fall before beginning the reorientation?

Some jumpers wear "**wingsuits**" (Figure 5.3.6). These suits have fabric panels between the arms and legs and allow the wearer to glide around in a free fall, much like a flying squirrel. (Indeed, the suits are sometimes called "flying squirrel suits.") When wearing these suits, terminal velocity can be reduced to about 30 mph (44 ft/sec), allowing the wearers a much longer time in the air. Wingsuit flyers still use parachutes to land; although the vertical velocities are within the margin of safety, horizontal velocities can exceed 70 mph, much too fast to land safely.

Figure 5.3.6: The fabric panels on the arms and legs of a wingsuit work to reduce the vertical velocity of a skydiver's fall. (credit: Richard Schneider)

Answer the following question based on the velocity in a wingsuit.

7. If Julie dons a wingsuit before her third jump of the day, and she pulls her ripcord at an altitude of 3000 ft, how long does she get to spend gliding around in the air

Key Concepts

- The Mean Value Theorem for Integrals states that for a continuous function over a closed interval, there is a value c such that $f(c)$ equals the average value of the function.
- The Fundamental Theorem of Calculus, Part 1 shows the relationship between the derivative and the integral.
- The Fundamental Theorem of Calculus, Part 2 is a formula for evaluating a definite integral in terms of an antiderivative of its integrand. The total area under a curve can be found using this formula.

Key Equations

- **Mean Value Theorem for Integrals**

  If $f(x)$ is continuous over an interval $[a, b]$, then there is at least one point $c \in [a, b]$ such that

  $$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

- **Fundamental Theorem of Calculus, Part 1**

  If $f(x)$ is continuous over an interval $[a, b]$, and the function $F(x)$ is defined by

  $$F(x) = \int_a^x f(t)\, dt,$$

  then

  $$F'(x) = f(x).$$

- **Fundamental Theorem of Calculus, Part 2**

  If f is continuous over the interval $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then

  $$\int_a^b f(x)\, dx = F(b) - F(a).$$

Glossary

fundamental theorem of calculus

the theorem, central to the entire development of calculus, that establishes the relationship between differentiation and integration

fundamental theorem of calculus, part 1

uses a definite integral to define an antiderivative of a function

fundamental theorem of calculus, part 2

(also, **evaluation theorem**) we can evaluate a definite integral by evaluating the antiderivative of the integrand at the endpoints of the interval and subtracting

mean value theorem for integrals

guarantees that a point c exists such that $f(c)$ is equal to the average value of the function

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5.3E: EXERCISES FOR SECTION 5.3

1) Consider two athletes running at variable speeds $v_1(t)$ and $v_2(t)$. The runners start and finish a race at exactly the same time. Explain why the two runners must be going the same speed at some point.

2) Two mountain climbers start their climb at base camp, taking two different routes, one steeper than the other, and arrive at the peak at exactly the same time. Is it necessarily true that, at some point, both climbers increased in altitude at the same rate?

Answer

Yes. It is implied by the Mean Value Theorem for Integrals.

3) To get on a certain toll road a driver has to take a card that lists the mile entrance point. The card also has a timestamp. When going to pay the toll at the exit, the driver is surprised to receive a speeding ticket along with the toll. Explain how this can happen.

4) Set $F(x) = \int_1^x (1-t)\,dt$. Find $F'(2)$ and the average value of F' over $[1,2]$.

Answer

$F'(2) = -1$; average value of F' over $[1,2]$ is $-1/2$.

In exercises 5 - 16, use the Fundamental Theorem of Calculus, Part 1, to find each derivative.

5) $\frac{d}{dx}\left[\int_1^x e^{-t^2}\,dt\right]$

6) $\frac{d}{dx}\left[\int_1^x e^{\cos t}\,dt\right]$

Answer

$\frac{d}{dx}\left[\int_1^x e^{\cos t}\,dt\right] = e^{\cos x}$

7) $\frac{d}{dx}\left[\int_3^x \sqrt{9-y^2}\,dy\right]$

8) $\frac{d}{dx}\left[\int_3^x \frac{ds}{\sqrt{16-s^2}}\right]$

Answer

$\frac{d}{dx}\left[\int_3^x \frac{ds}{\sqrt{16-s^2}}\right] = \frac{1}{\sqrt{16-x^2}}$

9) $\frac{d}{dx}\left[\int^{2x} x_t\,dt\right]$

10) $\frac{d}{dx}\left[\int_0^{\sqrt{x}} t\,dt\right]$

Answer

$\frac{d}{dx}\left[\int_0^{\sqrt{x}} t\,dt\right] = \sqrt{x}\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2}$

11) $\frac{d}{dx}\left[\int_0^{\sin x} \sqrt{1-t^2}\,dt\right]$

12) $\frac{d}{dx}\left[\int_{\cos x}^1 \sqrt{1-t^2}\,dt\right]$

Answer

$\frac{d}{dx}\left[\int_{\cos x}^1 \sqrt{1-t^2}\,dt\right] = -\sqrt{1-\cos^2 x}\frac{d}{dx}\left(\cos x\right) = |\sin x|\sin x$

13) $\frac{d}{dx}\left[\int_1^{\sqrt{x}} \frac{t^2}{1+t^4}\,dt\right]$

14) $\frac{d}{dx}\left[\int_1^{x^2} \frac{\sqrt{t}}{1+t}\,dt\right]$

Answer

$$\frac{d}{dx}\left[\int_1^{x^2}\frac{\sqrt{t}}{1+t}\,dt\right]=2x\frac{|x|}{1+x^2}$$

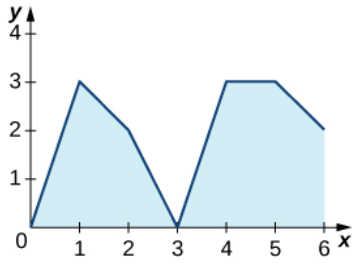
15) $\frac{d}{dx}\left[\int_0^{\ln x} e^t\,dt\right]$

16) $\frac{d}{dx}\left[\int_1^{e^x}\ln u^2\,du\right]$

Answer

$$\frac{d}{dx}\left[\int_1^{e^x}\ln u^2\,du\right]=\ln(e^{2x})\frac{d}{dx}\left(e^x\right)=2xe^x$$

17) The graph of $y=\int_0^x f(t)\,dt$, where f is a piecewise constant function, is shown here.

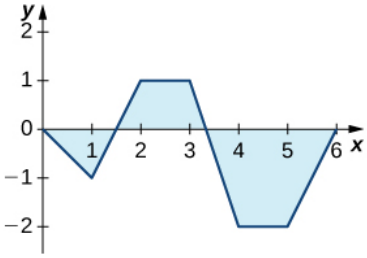


a. Over which intervals is f positive? Over which intervals is it negative? Over which intervals, if any, is it equal to zero?

b. What are the maximum and minimum values of f?

c. What is the average value of f?

18) The graph of $y=\int_0^x f(t)\,dt$, where f is a piecewise constant function, is shown here.



a. Over which intervals is f positive? Over which intervals is it negative? Over which intervals, if any, is it equal to zero?

b. What are the maximum and minimum values of f?

c. What is the average value of f?

Answer

a. f is positive over $[1,2]$ and $[5,6]$, negative over $[0,1]$ and $[3,4]$, and zero over $[2,3]$ and $[4,5]$.

b. The maximum value is 2 and the minimum is -3.

c. The average value is 0.

19) The graph of $y=\int_0^x \ell(t)\,dt$, where ℓ is a piecewise linear function, is shown here.

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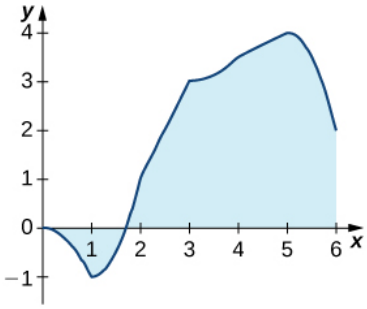
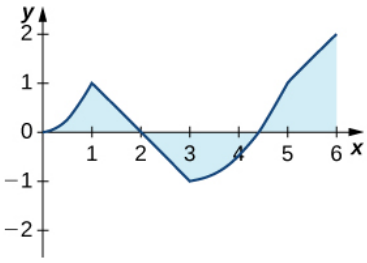


a. Over which intervals is ℓ positive? Over which intervals is it negative? Over which, if any, is it zero?

b. Over which intervals is ℓ increasing? Over which is it decreasing? Over which, if any, is it constant?

c. What is the average value of ℓ?

20) The graph of $y = \int_0^x \ell(t)\,dt$, where ℓ is a piecewise linear function, is shown here.



a. Over which intervals is ℓ positive? Over which intervals is it negative? Over which, if any, is it zero?

b. Over which intervals is ℓ increasing? Over which is it decreasing? Over which intervals, if any, is it constant?

c. What is the average value of ℓ?

Answer

a. ℓ is positive over $[0,1]$ and $[3,6]$, and negative over $[1,3]$.
b. It is increasing over $[0,1]$ and $[3,5]$, and it is constant over $[1,3]$ and $[5,6]$.
c. Its average value is $\frac{1}{3}$.

In exercises 21 - 26, use a calculator to estimate the area under the curve by computing T_{10}, the average of the left- and right-endpoint Riemann sums using $N = 10$ rectangles. Then, using the Fundamental Theorem of Calculus, Part 2, determine the exact area.

21) [T] $y = x^2$ over $[0,4]$

22) [T] $y = x^3 + 6x^2 + x - 5$ over $[-4,2]$

Answer

$T_{10} = 49.08, \quad \int_{-2}^{3} \left(x^3 + 6x^2 + x - 5\right)\,dx = 48$

23) [T] $y = \sqrt{x^3}$ over $[0,6]$

24) [T] $y = \sqrt{x} + x^2$ over $[1,9]$

Answer

$T_{10} = 260.836, \quad \int_1^9 (\sqrt{x} + x^2)\,dx = 260$

25) [T] $\int (\cos x - \sin x)\,dx$ over $[0,\pi]$

26) [T] $\int \frac{4}{x^2}\,dx$ over $[1,4]$

Answer

$$T_{10} = 3.058, \quad \int_1^4 \frac{4}{x^2}\,dx = 3$$

In exercises 27 - 46, evaluate each definite integral using the Fundamental Theorem of Calculus, Part 2.

27) $\int_{-1}^{2} (x^2 - 3x)\,dx$

28) $\int_{-2}^{3} (x^2 + 3x - 5)\,dx$

Answer

$$F(x) = \frac{x^3}{3} + \frac{3x^2}{2} - 5x, \quad F(3) - F(-2) = -\frac{35}{6}$$

29) $\int_{-2}^{3} (t+2)(t-3)dt$

30) $\int_{2}^{3} (t^2 - 9)(4 - t^2)dt$

Answer

$$F(x) = -\frac{t^5}{5} + \frac{13t^3}{3} - 36t, \quad F(3) - F(2) = \tfrac{62}{15}$$

31) $\int_1^2 x^9\,dx$

32) $\int_0^1 x^{99}\,dx$

Answer

$$F(x) = \frac{x^{100}}{100}, \quad F(1) - F(0) = \frac{1}{100}$$

33) $\int_4^8 (4t^{5/2} - 3t^{3/2})dt$

34) $\int_{1/4}^{4} \left(x^2 - \frac{1}{x^2}\right)\,dx$

Answer

$$F(x) = \frac{x^3}{3} + \frac{1}{x}, \quad F(4) - F\left(\tfrac{1}{4}\right) = \tfrac{1125}{64}$$

35) $\int_1^2 \frac{2}{x^3}\,dx$

36) $\int_1^4 \frac{1}{2\sqrt{x}}\,dx$

Answer

$$F(x) = \sqrt{x}, \quad F(4) - F(1) = 1$$

37) $\int_1^4 \frac{2 - \sqrt{t}}{t^2}\,dt$

38) $\int_1^{16} \frac{dt}{t^{1/4}}$

Answer

$$F(t) = \tfrac{4}{3}t^{3/4}, \quad F(16) - F(1) = \tfrac{28}{3}$$

39) $\int_0^{2\pi} \cos\theta\,d\theta$

40) $\int_0^{\pi/2} \sin\theta\,d\theta$

Answer

$$F(\theta) = -\cos\theta, \quad F(\tfrac{\pi}{2}) - F(0) = 1$$

41) $\int_0^{\pi/4} \sec^2\theta\, d\theta$

42) $\int_0^{\pi/4} \sec\theta\tan\theta\, d\theta$

Answer

$F(\theta) = \sec\theta, \quad F(\frac{\pi}{4}) - F(0) = \sqrt{2} - 1$

43) $\int_{\pi/3}^{\pi/4} \csc\theta\cot\theta\, d\theta$

44) $\int_{\pi/4}^{\pi/2} \csc^2\theta\, d\theta$

Answer

$F(\theta) = -\cot\theta, \quad F(\frac{\pi}{2}) - F(\frac{\pi}{4}) = 1$

45) $\int_1^2 \left(\frac{1}{t^2} - \frac{1}{t^3}\right) dt$

46) $\int_{-2}^{-1} \left(\frac{1}{t^2} - \frac{1}{t^3}\right) dt$

Answer

$F(t) = -\frac{1}{t} + \frac{1}{2t^2}, \quad F(-1) - F(-2) = \frac{7}{8}$

In exercises 47 - 50, use the evaluation theorem to express the integral as a function $F(x)$.

47) $\int_a^x t^2\, dt$

48) $\int_1^x e^t\, dt$

Answer

$F(x) = e^x - e$

49) $\int_0^x \cos t\, dt$

50) $\int_{-x}^x \sin t\, dt$

Answer

$F(x) = 0$

In exercises 51 - 54, identify the roots of the integrand to remove absolute values, then evaluate using the Fundamental Theorem of Calculus, Part 2.

51) $\int_{-2}^3 |x|\, dx$

52) $\int_{-2}^4 |\, t^2 - 2t - 3\,|\, dt$

Answer

$\int_{-2}^{-1} (t^2 - 2t - 3)\, dt - \int_{-1}^3 (t^2 - 2t - 3)\, dt + \int_3^4 (t^2 - 2t - 3)\, dt = \frac{46}{3}$

53) $\int_0^{\pi} |\cos t|\, dt$

54) $\int_{-\pi/2}^{\pi/2} |\sin t|\, dt$

Answer

$-\int_{-\pi/2}^0 \sin t\, dt + \int_0^{\pi/2} \sin t\, dt = 2$

55) Suppose that the number of hours of daylight on a given day in Seattle is modeled by the function $-3.75\cos(\frac{\pi t}{6})+12.25$, with t given in months and $t=0$ corresponding to the winter solstice.

a. What is the average number of daylight hours in a year?

b. At which times t_1 and t_2, where $0 \le t_1 < t_2 < 12$, do the number of daylight hours equal the average number?

c. Write an integral that expresses the total number of daylight hours in Seattle between t_1 and t_2

d. Compute the mean hours of daylight in Seattle between t_1 and t_2, where $0 \le t_1 < t_2 < 12$, and then between t_2 and t_1, and show that the average of the two is equal to the average day length.

56) Suppose the rate of gasoline consumption in the United States can be modeled by a sinusoidal function of the form $(11.21-\cos(\frac{\pi t}{6}))\times 10^9$ gal/mo.

a. What is the average monthly consumption, and for which values of t is the rate at time t equal to the average rate?

b. What is the number of gallons of gasoline consumed in the United States in a year?

c. Write an integral that expresses the average monthly U.S. gas consumption during the part of the year between the beginning of April $(t=3)$ and the end of September $(t=9)$.

Answer

a. The average is 11.21×10^9 since $\cos(\frac{\pi t}{6})$ has period 12 and integral 0 over any period. Consumption is equal to the average when $\cos(\frac{\pi t}{6})=0$, when $t=3$, and when $t=9$.

b. Total consumption is the average rate times duration: $11.21\times 12\times 10^9 = 1.35\times 10^{11}$

c. $10^9(11.21-\frac{1}{6}\int_3^9 \cos(\frac{\pi t}{6})\,dt) = 10^9(11.21+2\pi) = 11.84x10^9$

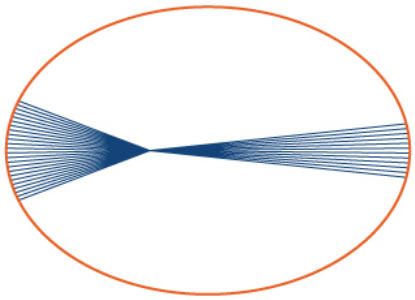
57) Explain why, if f is continuous over $[a,b]$, there is at least one point $c\in[a,b]$ such that $f(c)=\frac{1}{b-a}\int_a^b f(t)\,dt$.

58) Explain why, if f is continuous over $[a,b]$ and is not equal to a constant, there is at least one point $M\in[a,b]$ such that $f(M)=\frac{1}{b-a}\int_a^b f(t)\,dt$ and at least one point $m\in[a,b]$ such that $f(m)<\frac{1}{b-a}\int_a^b f(t)\,dt$.

Answer

If f is not constant, then its average is strictly smaller than the maximum and larger than the minimum, which are attained over $[a,b]$ by the extreme value theorem.

59) Kepler's first law states that the planets move in elliptical orbits with the Sun at one focus. The closest point of a planetary orbit to the Sun is called the **perihelion** (for Earth, it currently occurs around January 3) and the farthest point is called the **aphelion** (for Earth, it currently occurs around July 4). Kepler's second law states that planets sweep out equal areas of their elliptical orbits in equal times. Thus, the two arcs indicated in the following figure are swept out in equal times. At what time of year is Earth moving fastest in its orbit? When is it moving slowest? Kepler's first law states that the planets move in elliptical orbits with the Sun at one focus. The closest point of a planetary orbit to the Sun is called the perihelion (for Earth, it currently occurs around January 3) and the farthest point is called the aphelion (for Earth, it currently occurs around July 4). Kepler's second law states that planets sweep out equal areas of their elliptical orbits in equal times. Thus, the two arcs indicated in the following figure are swept out in equal times. At what time of year is Earth moving fastest in its orbit? When is it moving slowest?



60) A point on an ellipse with major axis length $2a$ and minor axis length $2b$ has the coordinates $(a\cos\theta, b\sin\theta),\quad 0\le\theta\le 2\pi$.

a. Show that the distance from this point to the focus at $(-c,0)$ is $d(\theta)=a+c\cos\theta$, where $c=\sqrt{a^2-b^2}$.

Use these coordinates to show that the average distance $\bar{d}$ from a point on the ellipse to the focus at $(-c,0)$, with respect to angle θ, is a.

Answer

a. $d^2\theta = (a\cos\theta + c)^2 + b^2\sin^2\theta = a^2 + c^2\cos^2\theta + 2ac\cos\theta = (a + c\cos\theta)^2$;

b. $\bar{d} = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}(a + 2c\cos\theta)\,d\theta = a$

61) As implied earlier, according to Kepler's laws, Earth's orbit is an ellipse with the Sun at one focus. The perihelion for Earth's orbit around the Sun is 147,098,290 km and the aphelion is 152,098,232 km.

a. By placing the major axis along the x-axis, find the average distance from Earth to the Sun.

b. The classic definition of an astronomical unit (AU) is the distance from Earth to the Sun, and its value was computed as the average of the perihelion and aphelion distances. Is this definition justified?

62) The force of gravitational attraction between the Sun and a planet is $F(\theta) = \dfrac{GmM}{r^2(\theta)}$, where m is the mass of the planet, M is the mass of the Sun, G is a universal constant, and $r(\theta)$ is the distance between the Sun and the planet when the planet is at an angle θ with the major axis of its orbit. Assuming that M, m, and the ellipse parameters a and b (half-lengths of the major and minor axes) are given, set up—but do not evaluate—an integral that expresses in terms of G, m, M, a, b the average gravitational force between the Sun and the planet.

Answer

Mean gravitational force $= \dfrac{GmM}{2}\displaystyle\int_0^{2\pi}\frac{1}{(a + 2\sqrt{a^2 - b^2}\cos\theta)^2}\,d\theta.$

The displacement from rest of a mass attached to a spring satisfies the simple harmonic motion equation $x(t) = A\cos(\omega t - \phi)$, where ϕ is a phase constant, ω is the angular frequency, and A is the amplitude. Find the average velocity, the average speed (magnitude of velocity), the average displacement, and the average distance from rest (magnitude of displacement) of the mass.

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5.4: Integration Formulas and the Net Change Theorem

Learning Objectives

- Apply the basic integration formulas.
- Explain the significance of the net change theorem.
- Use the net change theorem to solve applied problems.
- Apply the integrals of odd and even functions.

In this section, we use some basic integration formulas studied previously to solve some key applied problems. It is important to note that these formulas are presented in terms of indefinite integrals. Although definite and indefinite integrals are closely related, there are some key differences to keep in mind. A definite integral is either a number (when the limits of integration are constants) or a single function (when one or both of the limits of integration are variables). An indefinite integral represents a family of functions, all of which differ by a constant. As you become more familiar with integration, you will get a feel for when to use definite integrals and when to use indefinite integrals. You will naturally select the correct approach for a given problem without thinking too much about it. However, until these concepts are cemented in your mind, think carefully about whether you need a definite integral or an indefinite integral and make sure you are using the proper notation based on your choice.

Basic Integration Formulas

Recall the integration formulas given in the section on Antiderivatives and the properties of definite integrals. Let's look at a few examples of how to apply these formulas and properties.

Example 5.4.1: Integrating a Function Using the Power Rule

Use the power rule to integrate the function $\int_1^4 \sqrt{t}(1+t)\,dt$.

Solution

The first step is to rewrite the function and simplify it so we can apply the power rule:

$$\begin{aligned}\int_1^4 \sqrt{t}(1+t)\,dt &= \int_1^4 t^{1/2}(1+t)\,dt \\ &= \int_1^4 (t^{1/2}+t^{3/2})\,dt.\end{aligned}$$

Now apply the power rule:

$$\begin{aligned}\int_1^4 (t^{1/2}+t^{3/2})\,dt &= \left(\frac{2}{3}t^{3/2}+\frac{2}{5}t^{5/2}\right)\Big|_1^4 \\ &= \left[\frac{2}{3}(4)^{3/2}+\frac{2}{5}(4)^{5/2}\right]-\left[\frac{2}{3}(1)^{3/2}+\frac{2}{5}(1)^{5/2}\right] \\ &= \frac{256}{15}.\end{aligned}$$

Exercise 5.4.1

Find the definite integral of $f(x)=x^2-3x$ over the interval $[1,3]$.

Hint

Follow the process from Example 5.4.1 to solve the problem.

Answer

$$\int_1^3 (x^2-3x)\,dx = -\frac{10}{3}$$

The Net Change Theorem

The **net change theorem** considers the integral of a **rate of change**. It says that when a quantity changes, the new value equals the initial value plus the integral of the rate of change of that quantity. The formula can be expressed in two ways. The second is more familiar; it is simply the definite integral.

Net Change Theorem

The new value of a changing quantity equals the initial value plus the integral of the rate of change:

$$F(b) = F(a) + \int_a^b F'(x)dx \tag{5.4.1}$$

or

$$\int_a^b F'(x)dx = F(b) - F(a). \tag{5.4.2}$$

Subtracting $F(a)$ from both sides of the Equation 5.4.1 yields Equation 5.4.2. Since they are equivalent formulas, which one we use depends on the application.

The significance of the net change theorem lies in the results. Net change can be applied to area, distance, and volume, to name only a few applications. Net change accounts for negative quantities automatically without having to write more than one integral. To illustrate, let's apply the net change theorem to a **velocity** function in which the result is **displacement.**

We looked at a simple example of this in The Definite Integral section. Suppose a car is moving due north (the positive direction) at 40 mph between 2 p.m. and 4 p.m., then the car moves south at 30 mph between 4 p.m. and 5 p.m. We can graph this motion as shown in Figure 5.4.1.

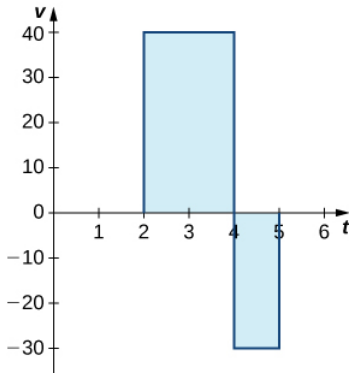

Figure 5.4.1: The graph shows speed versus time for the given motion of a car.

Just as we did before, we can use definite integrals to calculate the net displacement as well as the total distance traveled. The **net displacement** is given by

$$\int_2^5 v(t)\,dt = \int_2^4 40\,dt + \int_4^5 -30\,dt = 80 - 30 = 50.$$

Thus, at 5 p.m. the car is 50 mi north of its starting position. The **total distance** traveled is given by

$$\int_2^5 |v(t)|\,dt = \int_2^4 40\,dt + \int_4^5 30\,dt = 80 + 30 = 110.$$

Therefore, between 2 p.m. and 5 p.m., the car traveled a total of 110 mi.

To summarize, net displacement may include both positive and negative values. In other words, the velocity function accounts for both forward distance and backward distance. To find net displacement, integrate the velocity function over the interval. Total distance traveled, on the other hand, is always positive. To find the total distance traveled by an object, regardless of direction, we need to integrate the absolute value of the velocity function.

Example 5.4.2: Finding Net Displacement

Given a velocity function $v(t) = 3t - 5$ (in meters per second) for a particle in motion from time $t = 0$ to time $t = 3$, find the net displacement of the particle.

Solution

Applying the net change theorem, we have

$$\int_0^3 (3t - 5)\, dt = \left(\frac{3t^2}{2} - 5t\right)\Big|_0^3 = \left[\frac{3(3)^2}{2} - 5(3)\right] - 0 = \frac{27}{2} - 15 = \frac{27}{2} - \frac{30}{2} = -\frac{3}{2}.$$

The net displacement is $-\frac{3}{2}$ m (Figure 5.4.2).

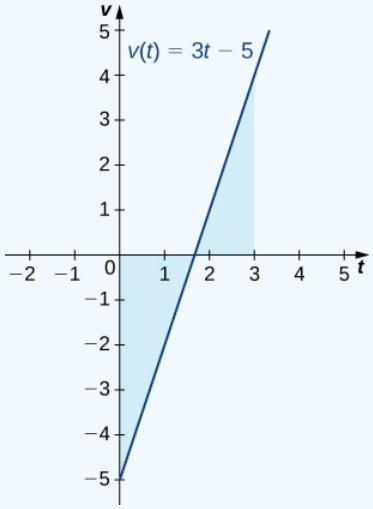

Figure 5.4.2: The graph shows velocity versus time for a particle moving with a linear velocity function.

Example 5.4.3: Finding the Total Distance Traveled

Use Example 5.4.2 to find the total distance traveled by a particle according to the velocity function $v(t) = 3t - 5$ m/sec over a time interval $[0, 3]$.

Solution

The total distance traveled includes both the positive and the negative values. Therefore, we must integrate the absolute value of the velocity function to find the total distance traveled.

To continue with the example, use two integrals to find the total distance. First, find the t-intercept of the function, since that is where the division of the interval occurs. Set the equation equal to zero and solve for t. Thus,

$$\begin{aligned} 3t - 5 &= 0 \\ 3t &= 5 \\ t &= \frac{5}{3}. \end{aligned}$$

The two subintervals are $[0, \frac{5}{3}]$ and $[\frac{5}{3}, 3]$. To find the total distance traveled, integrate the absolute value of the function. Since the function is negative over the interval $[0, \frac{5}{3}]$, we have $|v(t)| = -v(t)$ over that interval. Over $[\frac{5}{3}, 3]$, the function is positive, so $|v(t)| = v(t)$. Thus, we have

$$\begin{aligned}\int_0^3 |v(t)|\,dt &= \int_0^{5/3} -v(t)\,dt + \int_{5/3}^3 v(t)\,dt \\ &= \int_0^{5/3} 5-3t\,dt + \int_{5/3}^3 3t-5\,dt \\ &= \left(5t-\frac{3t^2}{2}\right)\Big|_0^{5/3} + \left(\frac{3t^2}{2}-5t\right)\Big|_{5/3}^3 \\ &= \left[5(\frac{5}{3})-\frac{3(5/3)^2}{2}\right]-0+\left[\frac{27}{2}-15\right]-\left[\frac{3(5/3)^2}{2}-\frac{25}{3}\right] \\ &= \frac{25}{3}-\frac{25}{6}+\frac{27}{2}-15-\frac{25}{6}+\frac{25}{3} \\ &= \frac{41}{6}\end{aligned}$$

So, the total distance traveled is $\frac{41}{6}$ m.

? Exercise 5.4.2

Find the net displacement and total distance traveled in meters given the velocity function $f(t)=\frac{1}{2}e^t-2$ over the interval $[0,2]$.

Hint

Follow the procedures from Examples 5.4.2 and 5.4.3. Note that $f(t)\le 0$ for $t\le \ln 4$ and $f(t)\ge 0$ for $t\ge \ln 4$.

Answer

Net displacement: $\frac{e^2-9}{2}\approx -0.8055$ m; total distance traveled: $4\ln 4-7.5+\frac{e^2}{2}\approx 1.740$ m.

Applying the Net Change Theorem

The net change theorem can be applied to the flow and consumption of fluids, as shown in Example 5.4.4.

✓ Example 5.4.4: How Many Gallons of Gasoline Are Consumed?

If the motor on a motorboat is started at $t=0$ and the boat consumes gasoline at the rate of $5-t^3$ gal/hr, how much gasoline is used in the first 2 hours?

Solution

Express the problem as a definite integral, integrate, and evaluate using the Fundamental Theorem of Calculus. The limits of integration are the endpoints of the interval [0,2]. We have

$$\begin{aligned}\int_0^2 (5-t^3)\,dt &= \left(5t-\frac{t^4}{4}\right)|_0^2 \\ &= \left[5(2)-\frac{(2)^4}{4}\right]-0 \\ &= 10-\frac{16}{4} \\ &= 6.\end{aligned}$$

Thus, the motorboat uses 6 gal of gas in 2 hours.

✓ Example 5.4.5: Chapter Opener: Iceboats

As we saw at the beginning of the chapter, top **iceboat** racers can attain speeds of up to five times the wind speed. Andrew is an intermediate iceboater, though, so he attains speeds equal to only twice the wind speed.

Figure 5.4.3: (credit: modification of work by Carter Brown, Flickr)

Suppose Andrew takes his iceboat out one morning when a light 5-mph breeze has been blowing all morning. As Andrew gets his iceboat set up, though, the wind begins to pick up. During his first half hour of iceboating, the wind speed increases according to the function $v(t) = 20t + 5$. For the second half hour of Andrew's outing, the wind remains steady at 15 mph. In other words, the wind speed is given by

$$v(t) = \begin{cases} 20t + 5, & \text{for } 0 \le t \le \frac{1}{2} \\ 15, & \text{for } \frac{1}{2} \le t \le 1 \end{cases}$$

Recalling that Andrew's iceboat travels at twice the wind speed, and assuming he moves in a straight line away from his starting point, how far is Andrew from his starting point after 1 hour?

Solution

To figure out how far Andrew has traveled, we need to integrate his velocity, which is twice the wind speed. Then

$$\text{Distance} = \int_0^1 2v(t)\,dt.$$

Substituting the expressions we were given for $v(t)$, we get

$$\begin{aligned} \int_0^1 2v(t)\,dt &= \int_0^{1/2} 2v(t)\,dt + \int_{1/2}^1 2v(t)\,dt \\ &= \int_0^{1/2} 2(20t+5)\,dt + \int_{1/3}^1 2(15)\,dt \\ &= \int_0^{1/2} (40t+10)\,dt + \int_{1/2}^1 30\,dt \\ &= [20t^2 + 10t]\Big|_0^{1/2} + [30t]\Big|_{1/2}^1 \\ &= \left(\frac{20}{4} + 5\right) - 0 + (30 - 15) \\ &= 25. \end{aligned}$$

Andrew is 25 mi from his starting point after 1 hour.

? Exercise 5.4.3

Suppose that, instead of remaining steady during the second half hour of Andrew's outing, the wind starts to die down according to the function $v(t) = -10t + 15$. In other words, the wind speed is given by

$$v(t) = \begin{cases} 20t + 5, & \text{for } 0 \le t \le \frac{1}{2} \\ -10t + 15, & \text{for } \frac{1}{2} \le t \le 1 \end{cases}.$$

Under these conditions, how far from his starting point is Andrew after 1 hour?

Hint

Don't forget that Andrew's iceboat moves twice as fast as the wind.

Answer

17.5 mi

Integrating Even and Odd Functions

We saw in Functions and Graphs that an **even function** is a function in which $f(-x) = f(x)$ for all x in the domain—that is, the graph of the curve is unchanged when x is replaced with $-x$. The graphs of even functions are symmetric about the y-axis. An **odd function** is one in which $f(-x) = -f(x)$ for all x in the domain, and the graph of the function is symmetric about the origin.

Integrals of even functions, when the limits of integration are from $-a$ to a, involve two equal areas, because they are symmetric about the y-axis. Integrals of odd functions, when the limits of integration are similarly $[-a, a]$, evaluate to zero because the areas above and below the x-axis are equal.

Integrals of Even and Odd Functions

For continuous even functions such that $f(-x) = f(x)$,

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

For continuous odd functions such that $f(-x) = -f(x)$,

$$\int_{-a}^{a} f(x)\,dx = 0.$$

Example 5.4.6: Integrating an Even Function

Integrate the even function $\int_{-2}^{2}(3x^8 - 2)\,dx$ and verify that the integration formula for even functions holds.

Solution

The symmetry appears in the graphs in Figure 5.4.4. Graph (a) shows the region below the curve and above the x-axis. We have to zoom in to this graph by a huge amount to see the region. Graph (b) shows the region above the curve and below the x-axis. The signed area of this region is negative. Both views illustrate the symmetry about the y-axis of an even function. We have

$$\begin{aligned}\int_{-2}^{2}(3x^8 - 2)\,dx &= \left(\frac{x^9}{3} - 2x\right)\Big|_{-2}^{2} \\ &= \left[\frac{(2)^9}{3} - 2(2)\right] - \left[\frac{(-2)^9}{3} - 2(-2)\right] \\ &= \left(\frac{512}{3} - 4\right) - \left(-\frac{512}{3} + 4\right) \\ &= \frac{1000}{3}.\end{aligned}$$

To verify the integration formula for even functions, we can calculate the integral from 0 to 2 and double it, then check to make sure we get the same answer.

$$\int_{0}^{2}(3x^8 - 2)\,dx = \left(\frac{x^9}{3} - 2x\right)\Big|_{0}^{2} = \frac{512}{3} - 4 = \frac{500}{3}$$

Since $2 \cdot \frac{500}{3} = \frac{1000}{3}$, we have verified the formula for even functions in this particular example.

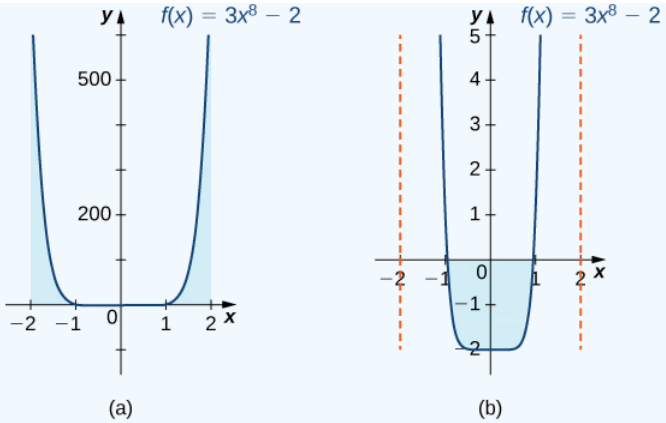

Figure 5.4.4: Graph (a) shows the positive area between the curve and the x-axis, whereas graph (b) shows the negative area between the curve and the x-axis. Both views show the symmetry about the y-axis.

Example 5.4.7: Integrating an Odd Function

Evaluate the definite integral of the odd function $-5\sin x$ over the interval $[-\pi, \pi]$.

Solution

The graph is shown in Figure 5.4.5. We can see the symmetry about the origin by the positive area above the x-axis over $[-\pi, 0]$, and the negative area below the x-axis over $[0, \pi]$. we have

$$\begin{aligned}\int_{-\pi}^{\pi} -5\sin x\, dx &= -5(-\cos x)\Big|_{-\pi}^{\pi} \\ &= 5\cos x\Big|_{-\pi}^{\pi} \\ &= [5\cos\pi] - [5\cos(-\pi)] \\ &= -5 - (-5) = 0.\end{aligned}$$

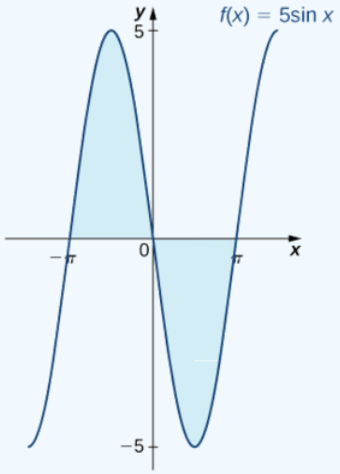

Figure 5.4.5:The graph shows areas between a curve and the x-axis for an odd function.

Exercise 5.4.4

Integrate the function $\int_{-2}^{2} x^4\, dx$.

Hint

Integrate an even function.

Answer

$\frac{64}{5}$

Key Concepts

- The net change theorem states that when a quantity changes, the final value equals the initial value plus the integral of the rate of change. Net change can be a positive number, a negative number, or zero.
- The area under an even function over a symmetric interval can be calculated by doubling the area over the positive x-axis. For an odd function, the integral over a symmetric interval equals zero, because half the area is negative.

Key Equations

- Net Change Theorem

$$F(b) = F(a) + \int_a^b F'(x)\,dx$$

or

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Glossary

net change theorem

if we know the rate of change of a quantity, the net change theorem says the future quantity is equal to the initial quantity plus the integral of the rate of change of the quantity

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5.4E: EXERCISES FOR SECTION 5.4

In exercises 1 - 6, use basic integration formulas to compute the following antiderivatives.

1) $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx$

Answer

$$\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx = \int x^{1/2}\, dx - \int x^{-1/2}\, dx = \frac{2}{3}x^{3/2} + C_1 - 2x^{1/2} + C_2 = \frac{2}{3}x^{3/2} - 2x^{1/2} + C$$

2) $\int \left(e^{2x} - \frac{1}{2}e^{x/2}\right) dx$

3) $\int \frac{dx}{2x}$

Answer

$$\int \frac{dx}{2x} = \frac{1}{2}\ln|x| + C$$

4) $\int \frac{x-1}{x^2}\, dx$

5) $\int_0^{\pi} (\sin x - \cos x)\, dx$

Answer

$$\int_0^{\pi} \sin x\, dx - \int_0^{\pi} \cos x\, dx = -\cos x\big|_0^{\pi} - (\sin x)\big|_0^{\pi} = (-(-1)+1) - (0-0) = 2$$

6) $\int_0^{\pi/2} (x - \sin x)\, dx$

7) Write an integral that expresses the increase in the perimeter $P(s)$ of a square when its side length s increases from 2 units to 4 units and evaluate the integral.

Answer

$P(s) = 4s$, so $\frac{dP}{ds} = 4$ and $\int_2^4 4\, ds = 8$.

8) Write an integral that quantifies the change in the area $A(s) = s^2$ of a square when the side length doubles from S units to $2S$ units and evaluate the integral.

9) A regular N-gon (an N-sided polygon with sides that have equal length s, such as a pentagon or hexagon) has perimeter Ns. Write an integral that expresses the increase in perimeter of a regular N-gon when the length of each side increases from 1 unit to 2 units and evaluate the integral.

Answer

$$\int_1^2 N\, ds = N$$

10) The area of a regular pentagon with side length $a > 0$ is pa^2 with $p = \frac{1}{4}\sqrt{5 + \sqrt{5 + 2\sqrt{5}}}$. The Pentagon in Washington, DC, has inner sides of length 360 ft and outer sides of length 920 ft. Write an integral to express the area of the roof of the Pentagon according to these dimensions and evaluate this area.

11) A dodecahedron is a Platonic solid with a surface that consists of 12 pentagons, each of equal area. By how much does the surface area of a dodecahedron increase as the side length of each pentagon doubles from 1 unit to 2 units?

Answer

With p as in the previous exercise, each of the 12 pentagons increases in area from $2p$ to $4p$ units so the net increase in the area of the dodecahedron is $36p$ units.

12) An icosahedron is a Platonic solid with a surface that consists of 20 equilateral triangles. By how much does the surface area of an icosahedron increase as the side length of each triangle doubles from a unit to $2a$ units?

13) Write an integral that quantifies the change in the area of the surface of a cube when its side length doubles from s unit to $2s$ units and evaluate the integral.

Answer

$18s^2 = 6\int_s^{2s} 2x\,dx$

14) Write an integral that quantifies the increase in the volume of a cube when the side length doubles from s unit to $2s$ units and evaluate the integral.

15) Write an integral that quantifies the increase in the surface area of a sphere as its radius doubles from R unit to $2R$ units and evaluate the integral.

Answer

$12\pi R^2 = 8\pi\int_R^{2R} r\,dr$

16) Write an integral that quantifies the increase in the volume of a sphere as its radius doubles from R unit to $2R$ units and evaluate the integral.

17) Suppose that a particle moves along a straight line with velocity $v(t) = 4 - 2t$, where $0 \le t \le 2$ (in meters per second). Find the displacement at time t and the total distance traveled up to $t = 2$.

Answer

$d(t) = \int_0^t v(s)\,ds = 4t - t^2$. The total distance is $d(2) = 4$ m.

18) Suppose that a particle moves along a straight line with velocity defined by $v(t) = t^2 - 3t - 18$, where $0 \le t \le 6$ (in meters per second). Find the displacement at time t and the total distance traveled up to $t = 6$.

19) Suppose that a particle moves along a straight line with velocity defined by $v(t) = |2t - 6|$, where $0 \le t \le 6$ (in meters per second). Find the displacement at time t and the total distance traveled up to $t = 6$.

Answer

$d(t) = \int_0^t v(s)\,ds$. For $t < 3$, $d(t) = \int_0^t (6 - 2t)\,dt = 6t - t^2$. For $t > 3$, $d(t) = d(3) + \int_3^t (2t - 6)\,dt = 9 + (t^2 - 6t)$. The total distance is $d(6) = 9$ m.

20) Suppose that a particle moves along a straight line with acceleration defined by $a(t) = t - 3$, where $0 \le t \le 6$ (in meters per second). Find the velocity and displacement at time t and the total distance traveled up to $t = 6$ if $v(0) = 3$ and $d(0) = 0$.

21) A ball is thrown upward from a height of 1.5 m at an initial speed of 40 m/sec. Acceleration resulting from gravity is $-9.8\ \text{m/sec}^2$. Neglecting air resistance, solve for the velocity $v(t)$ and the height $h(t)$ of the ball t seconds after it is thrown and before it returns to the ground.

Answer

$v(t) = 40 - 9.8t;\ h(t) = 1.5 + 40t - 4.9t^2$ m/s

22) A ball is thrown upward from a height of 3 m at an initial speed of 60 m/sec. Acceleration resulting from gravity is $-9.8\ \text{m/sec}^2$. Neglecting air resistance, solve for the velocity $v(t)$ and the height $h(t)$ of the ball t seconds after it is thrown and before it returns to the ground.

23) The area $A(t)$ of a circular shape is growing at a constant rate. If the area increases from 4π units to 9π units between times $t = 2$ and $t = 3$, find the net change in the radius during that time.

Answer

The net increase is 1 unit.

24) A spherical balloon is being inflated at a constant rate. If the volume of the balloon changes from $36\pi\ \text{in}^3$ to $288\pi\ \text{in}^3$ between time $t = 30$ and $t = 60$ seconds, find the net change in the radius of the balloon during that time.

25) Water flows into a conical tank with cross-sectional area πx^2 at height x and volume $\frac{\pi x^3}{3}$ up to height x. If water flows into the tank at a rate of $1\ \text{m}^3/\text{min}$, find the height of water in the tank after 5 min. Find the change in height between 5 min and 10 min.

Answer

At $t = 5$, the height of water is $x = \left(\frac{15}{\pi}\right)^{1/3}$ m. The net change in height from $t = 5$ to $t = 10$ is $\left(\left(\frac{30}{\pi}\right)^{1/3} - \left(\frac{15}{\pi}\right)^{1/3}\right)$ m.

26) A horizontal cylindrical tank has cross-sectional area $A(x) = 4(6x - x^2)\,\text{m}^2$ at height x meters above the bottom when $x \leq 3$.

a. The volume V between heights a and b is $\int_a^b A(x)\,dx$. Find the volume at heights between 2 m and 3 m.

b. Suppose that oil is being pumped into the tank at a rate of 50 L/min. Using the chain rule, $\frac{dx}{dt} = \frac{dx}{dV}\frac{dV}{dt}$, at how many meters per minute is the height of oil in the tank changing, expressed in terms of x, when the height is at x meters?

c. How long does it take to fill the tank to 3 m starting from a fill level of 2 m?

27) The following table lists the electrical power in gigawatts—the rate at which energy is consumed—used in a certain city for different hours of the day, in a typical 24-hour period, with hour 1 corresponding to midnight to 1 a.m.

Hour	Power	Hour	Power
1	28	13	48
2	25	14	49
3	24	15	49
4	23	16	50
5	24	17	50
6	27	18	50
7	29	19	46
8	32	20	43
9	34	21	42
10	39	22	40
11	42	23	37
12	46	24	34

Find the total amount of power in gigawatt-hours (gW-h) consumed by the city in a typical 24-hour period.

Answer

The total daily power consumption is estimated as the sum of the hourly power rates, or 911 gW-h.

28) The average residential electrical power use (in hundreds of watts) per hour is given in the following table.

Hour	Power	Hour	Power
1	8	13	12
2	6	14	13
3	5	15	14
4	4	16	15
5	5	17	17
6	6	18	19
7	7	19	18
8	8	20	17
9	9	21	16
10	10	22	16
11	10	23	13
12	11	24	11

a. Compute the average total energy used in a day in kilowatt-hours (kWh).

b. If a ton of coal generates 1842 kWh, how long does it take for an average residence to burn a ton of coal?

c. Explain why the data might fit a plot of the form $p(t) = 11.5 - 7.5\sin\left(\frac{\pi t}{12}\right)$.

29) The data in the following table are used to estimate the average power output produced by Peter Sagan for each of the last 18 sec of Stage 1 of the 2012 **Tour de France.**

Second	Watts	Second	Watts
1	600	10	1200
2	500	11	1170
3	575	12	1125
4	1050	13	1100
5	925	14	1075
6	950	15	1000
7	1050	16	950
8	950	17	900
9	1100	18	780

Average Power OutputSource: sportsexercisengineering.com

Estimate the net energy used in kilojoules (kJ), noting that 1W = 1 J/s, and the average power output by Sagan during this time interval.

Answer

17 kJ

30) The data in the following table are used to estimate the average power output produced by Peter Sagan for each 15-min interval of Stage 1 of the 2012 Tour de France.

Minutes	Watts	Minutes	Watts
15	200	165	170
30	180	180	220
45	190	195	140
60	230	210	225
75	240	225	170
90	210	240	10
105	210	255	200
1120	220	270	220
135	210	285	250
150	150	300	400

Average Power Output*Source*: sportsexercisengineering.com

Estimate the net energy used in kilojoules, noting that 1W = 1 J/s.

31) The distribution of incomes as of 2012 in the United States in $5000 increments is given in the following table. The kth row denotes the percentage of households with incomes between $\$5000xk$ and $5000xk + 4999$. The row $k = 40$ contains all households with income between $200,000 and $250,000 and $k = 41$ accounts for all households with income exceeding $250,000.

0	3.5	21	1.5
1	4.1	22	1.4
2	5.9	23	1.3
3	5.7	24	1.3
4	5.9	25	1.1
5	5.4	26	1.0
6	5.5	27	0.75
7	5.1	28	0.8
8	4.8	29	1.0
9	4.1	30	0.6
10	4.3	31	0.6
11	3.5	32	.5
12	3.7	33	0.5
13	3.2	34	0.4
14	3.0	35	0.3
15	2.8	36	0.3
16	2.5	37	0.3
17	2.2	38	0.2
18	2.2	39	1.8
19	1.8	40	2.3
20	2.1	41	

Income DistributionsSource: www.census.gov/prod/2013pubs/p60-245.pdf

a. Estimate the percentage of U.S. households in 2012 with incomes less than $55,000.

b. What percentage of households had incomes exceeding $85,000?

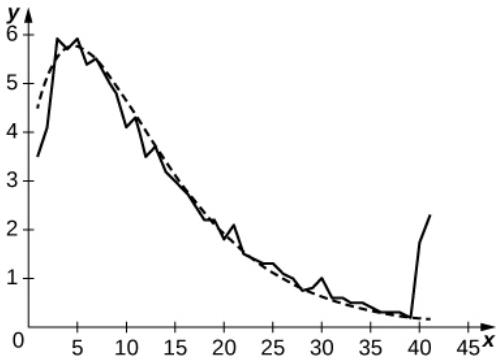
c. Plot the data and try to fit its shape to that of a graph of the form $a(x+c)e^{-b(x+e)}$ for suitable a, b, c.

Answer

a. 54.3%;

b. 27.00%;

c. The curve in the following plot is $2.35(t+3)e^{-0.15(t+3)}$.



32) Newton's law of gravity states that the gravitational force exerted by an object of mass M and one of mass m with centers that are separated by a distance r is $F = G\frac{mM}{r^2}$, with G an empirical constant $G = 6.67x10^{-11}\ \mathrm{m}^3/(\mathrm{kg}\cdot\mathrm{s}^2)$. The work done by a variable force over an interval $[a,b]$ is defined as $W = \int_a^b F(x)\,dx$. If Earth has mass 5.97219×10^{24} and radius 6371 km, compute the amount of work to elevate a polar weather satellite of mass 1400 kg to its orbiting altitude of 850 km above Earth.

33) For a given motor vehicle, the maximum achievable **deceleration** from braking is approximately $7\ \mathrm{m/sec}^2$ on dry concrete. On wet asphalt, it is approximately $2.5\ \mathrm{m/sec}^2$. Given that 1 mph corresponds to 0.447 m/sec, find the total distance that a car travels in meters on dry concrete after the brakes are applied until it comes to a complete stop if the initial velocity is 67 mph (30 m/sec) or if the initial braking velocity is 56 mph (25 m/sec). Find the corresponding distances if the surface is slippery wet asphalt.

Answer

In dry conditions, with initial velocity $v_0 = 30$ m/s, $D = 64.3$ and, if $v_0 = 25,\ D = 44.64$. In wet conditions, if $v_0 = 30$, and $D = 180$ and if $v_0 = 25,\ D = 125$.

34) John is a 25-year old man who weighs 160 lb. He burns $500 - 50t$ calories/hr while riding his bike for t hours. If an oatmeal cookie has 55 cal and John eats $4t$ cookies during the tth hour, how many net calories has he lost after 3 hours riding his bike?

35) Sandra is a 25-year old woman who weighs 120 lb. She burns $300 - 50t$ cal/hr while walking on her treadmill for t hours. Her caloric intake from drinking Gatorade is $100t$ calories during the tth hour. What is her net decrease in calories after walking for 3 hours?

Answer

225 cal

36) A motor vehicle has a maximum efficiency of 33 mpg at a cruising speed of 40 mph. The efficiency drops at a rate of 0.1 mpg/mph between 40 mph and 50 mph, and at a rate of 0.4 mpg/mph between 50 mph and 80 mph. What is the efficiency in miles per gallon if the car is cruising at 50 mph? What is the efficiency in miles per gallon if the car is cruising at 80 mph? If gasoline costs $3.50/gal, what is the cost of fuel to drive 50 mi at 40 mph, at 50 mph, and at 80 mph?

37) Although some engines are more efficient at given a horsepower than others, on average, fuel efficiency decreases with horsepower at a rate of $1/25$ mpg/horsepower. If a typical 50-horsepower engine has an average fuel efficiency of 32 mpg, what is the average fuel efficiency of an engine with the following horsepower: 150, 300, 450?

Answer

$E(150) = 28,\ E(300) = 22,\ E(450) = 16$

38) [T] The following table lists the 2013 schedule of **federal income tax** versus taxable income.

Taxable Income Range	The Tax Is ...	... Of the Amount Over
$0–$8925	10%	$0
$8925–$36,250	$892.50 + 15%	$8925
$36,250–$87,850	$4,991.25 + 25%	$36,250
$87,850–$183,250	$17,891.25 + 28%	$87,850
$183,250–$398,350	$44,603.25 + 33%	$183,250
$398,350–$400,000	$115,586.25 + 35%	$398,350
> $400,000	$116,163.75 + 39.6%	$400,000

Federal Income Tax Versus Taxable IncomeSource: http://www.irs.gov/pub/irs-prior/i1040tt--2013.pdf.

Suppose that Steve just received a $10,000 raise. How much of this raise is left after federal taxes if Steve's salary before receiving the raise was $40,000? If it was $90,000? If it was $385,000?

39) [T] The following table provides hypothetical data regarding the level of service for a certain highway.

Highway Speed Range (mph)	Vehicles per Hour per Lane	Density Range (vehicles/mi)
>60	<600	<10
60-57	300-1000	10-20
57-54	1000-1500	20-30
57-54	1500-1900	30-45
46-30	1900-2100	48-70
<30	Unstable	70-200

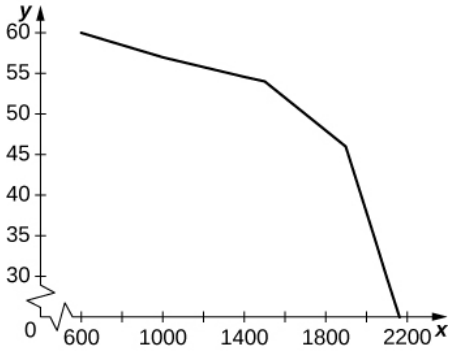
a. Plot vehicles per hour per lane on the x-axis and highway speed on the y-axis.

b. Compute the average decrease in speed (in miles per hour) per unit increase in congestion (vehicles per hour per lane) as the latter increases from 600 to 1000, from 1000 to 1500, and from 1500 to 2100. Does the decrease in miles per hour depend linearly on the increase in vehicles per hour per lane?

c. Plot minutes per mile (60 times the reciprocal of miles per hour) as a function of vehicles per hour per lane. Is this function linear?

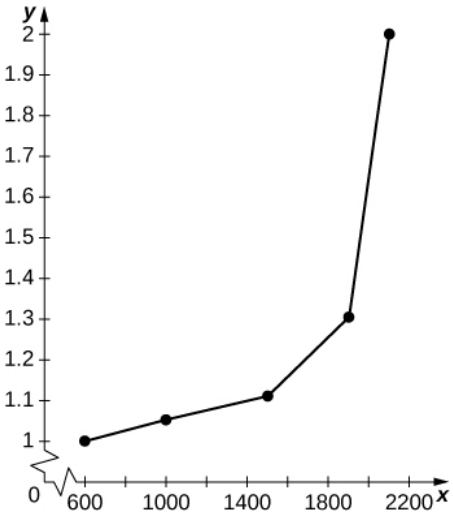
Answer

a.



b. Between 600 and 1000 the average decrease in vehicles per hour per lane is −0.0075. Between 1000 and 1500 it is −0.006 per vehicles per hour per lane, and between 1500 and 2100 it is −0.04 vehicles per hour per lane.

c.



The graph is nonlinear, with minutes per mile increasing dramatically as vehicles per hour per lane reach 2000.

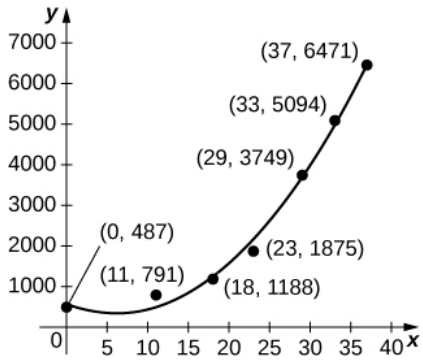
For the next two exercises use the data in the following table, which displays bald eagle populations from 1963 to 2000 in the continental United States.

Year	Population of Breeding Pairs of Bald Eagles
1963	487
1974	791
1981	1188
1986	1875
1992	3749
1996	5094
2000	6471

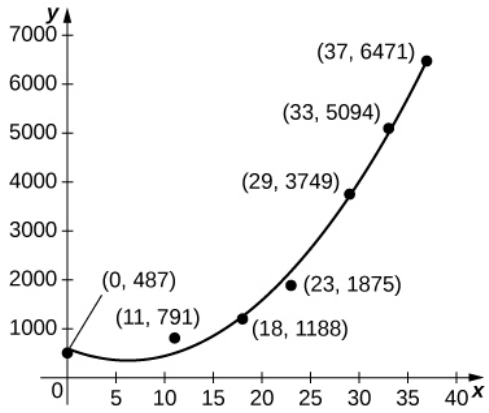
Population of Breeding Bald Eagle Pairs*Source*: www.fws.gov/Midwest/eagle/pop.../chtofprs.html.

40) [T] The graph below plots the quadratic $p(t) = 6.48t^2 - 80.31t + 585.69$ against the data in preceding table, normalized so that $t = 0$ corresponds to 1963. Estimate the average number of bald eagles per year present for the 37 years by computing the average value of p over $[0, 37]$.

https://math.libretexts.org/@go/page/53520



41) [T] The graph below plots the cubic $p(t) = 0.07t^3 + 2.42t^2 - 25.63t + 521.23$ against the data in the preceding table, normalized so that $t = 0$ corresponds to 1963. Estimate the average number of bald eagles per year present for the 37 years by computing the average value of p over $[0, 37]$.



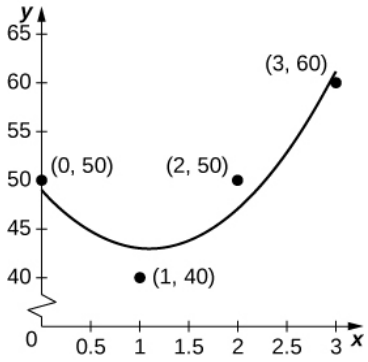
Answer

$$\frac{1}{37}\int_0^{37} p(t)\,dt = \frac{0.07(37)^3}{4} + \frac{2.42(37)^2}{3} - \frac{25.63(37)}{2} + 521.23 \approx 2037$$

Solution:

42) [T] Suppose you go on a road trip and record your speed at every half hour, as compiled in the following table. The best quadratic fit to the data is $q(t) = 5x^2 - 11x + 49$, shown in the accompanying graph. Integrate q to estimate the total distance driven over the 3 hours.

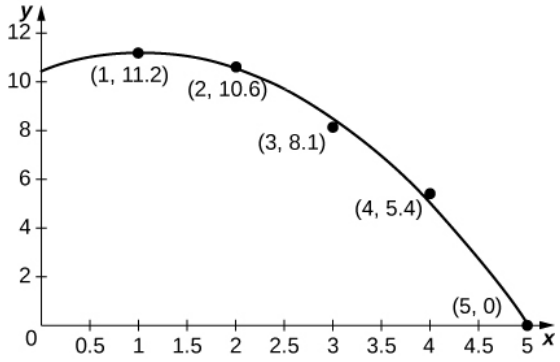
Time (hr)	Speed (m[h)
0 (start)	50
1	40
2	50
3	60



As a car accelerates, it does not accelerate at a constant rate; rather, the acceleration is variable. For the following exercises, use the following table, which contains the acceleration measured at every second as a driver merges onto a freeway.

Time (sec)	Acceleration (mph/sec)
1	11.2
2	10.6
3	8.1
4	5.4
5	0

43) [T] The accompanying graph plots the best quadratic fit, $a(t) = -0.70t^2 + 1.44t + 10.44$, to the data from the preceding table. Compute the average value of $a(t)$ to estimate the average acceleration between $t = 0$ and $t = 5$.



Answer

Average acceleration is $A = \frac{1}{5}\int_0^5 a(t)\,dt = -\frac{0.7(5^2)}{3} + \frac{1.44(5)}{2} + 10.44 \approx 8.2$ mph/s

44) [T] Using your acceleration equation from the previous exercise, find the corresponding velocity equation. Assuming the final velocity is 0 mph, find the velocity at time $t = 0$.

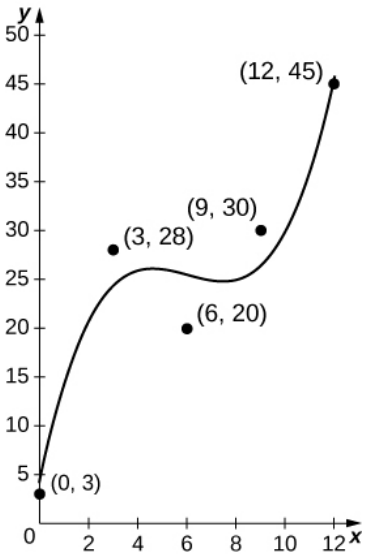
45) [T] Using your velocity equation from the previous exercise, find the corresponding distance equation, assuming your initial distance is 0 mi. How far did you travel while you accelerated your car? (Hint: You will need to convert time units.)

Answer

$d(t) = \int_0^1 |v(t)|\,dt = \int_0^t \left(\frac{7}{30}t^3 - 0.72t^2 - 10.44t + 41.033\right)\,dt = \frac{7}{120}t^4 - 0.24t^3 - 5.22t^3 + 41.033t$. Then, $d(5) \approx 81.12\,\text{mph} \times \text{sec} \approx 119$ feet.

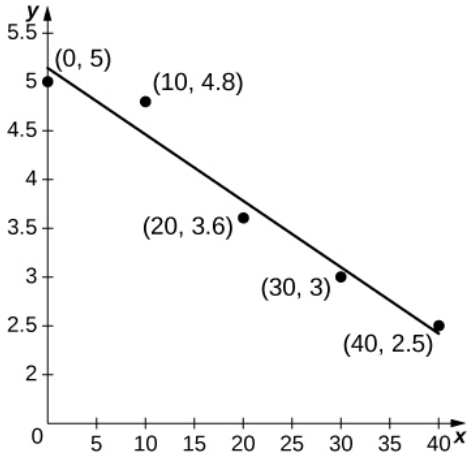
46) [T] The number of hamburgers sold at a restaurant throughout the day is given in the following table, with the accompanying graph plotting the best cubic fit to the data, $b(t) = 0.12t^3 - 2.13t^3 + 12.13t + 3.91$, with $t = 0$ corresponding to 9 a.m. and $t = 12$ corresponding to 9 p.m. Compute the average value of $b(t)$ to estimate the average number of hamburgers sold per hour.

Hours Past Midnight	No. of Burgers Sold
9	3
12	28
15	20
18	30
21	45



47) [T] An athlete runs by a motion detector, which records her speed, as displayed in the following table. The best linear fit to this data, $\ell(t) = -0.068t + 5.14$, is shown in the accompanying graph. Use the average value of $\ell(t)$ between $t = 0$ and $t = 40$ to estimate the runner's average speed.

Minutes	Speed (m/sec)
0	5
10	4.8
20	3.6
30	3.0
40	2.5



Answer

$$\frac{1}{40}\int_0^{40}(-0.068t + 5.14)\,dt = -\frac{0.068(40)}{2} + 5.14 = 3.78 \text{ m/sec}$$

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5.5: Substitution

Learning Objectives

- Use substitution to evaluate indefinite integrals.
- Use substitution to evaluate definite integrals.

The Fundamental Theorem of Calculus gave us a method to evaluate integrals without using Riemann sums. The drawback of this method, though, is that we must be able to find an antiderivative, and this is not always easy. In this section we examine a technique, called *integration by substitution*, to help us find antiderivatives. Specifically, this method helps us find antiderivatives when the integrand is the result of a chain-rule derivative.

At first, the approach to the substitution procedure may not appear very obvious. However, it is primarily a visual task—that is, the integrand shows you what to do; it is a matter of recognizing the form of the function. So, what are we supposed to see? We are looking for an integrand of the form $f[g(x)]g'(x)\,dx$. For example, in the integral

$$\int (x^2-3)^3\,2x\,dx. \tag{5.5.1}$$

we have

$$f(x)=x^3$$

and

$$g(x)=x^2-3.$$

Then

$$g'(x)=2x.$$

and

$$f[g(x)]g'(x)=(x^2-3)^3(2x),$$

and we see that our integrand is in the correct form. The method is called substitution because we substitute part of the integrand with the variable u and part of the integrand with du. It is also referred to as *change of variables* because we are changing variables to obtain an expression that is easier to work with for applying the integration rules.

Substitution with Indefinite Integrals

Let $u=g(x)$, where $g'(x)$ is continuous over an interval, let $f(x)$ be continuous over the corresponding range of g, and let $F(x)$ be an antiderivative of $f(x)$. Then,

$$\begin{aligned}\int f[g(x)]g'(x)\,dx &= \int f(u)\,du\\ &= F(u)+C\\ &= F(g(x))+C\end{aligned}$$

Proof

Let f, g, u, and F be as specified in the theorem. Then

$$\frac{d}{dx}\left[F(g(x))\right]=F'(g(x))g'(x)=f[g(x)]g'(x).$$

Integrating both sides with respect to x, we see that

$$\int f[g(x)]g'(x)\,dx=F(g(x))+C.$$

If we now substitute $u=g(x)$, and $du=g'(x)\,dx$, we get

$$\int f[g(x)]g'(x)\,dx = \int f(u)\,du = F(u) + C = F(g(x)) + C.$$

□

Returning to the problem we looked at originally, we let $u = x^2 - 3$ and then $du = 2x\,dx$.

Rewrite the integral (Equation 5.5.1) in terms of u:

$$\int (x^2 - 3)^3 (2x\,dx) = \int u^3\,du.$$

Using the power rule for integrals, we have

$$\int u^3\,du = \frac{u^4}{4} + C.$$

Substitute the original expression for x back into the solution:

$$\frac{u^4}{4} + C = \frac{(x^2 - 3)^4}{4} + C.$$

We can generalize the procedure in the following Problem-Solving Strategy.

Problem-Solving Strategy: Integration by Substitution

1. Look carefully at the integrand and select an expression $g(x)$ within the integrand to set equal to u. Let's select $g(x)$. such that $g'(x)$ is also part of the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)dx$. into the integral.
3. We should now be able to evaluate the integral with respect to u. If the integral can't be evaluated we need to go back and select a different expression to use as u.
4. Evaluate the integral in terms of u.
5. Write the result in terms of x and the expression $g(x)$.

Example 5.5.1: Using Substitution to Find an Antiderivative

Use substitution to find the antiderivative of $\int 6x(3x^2 + 4)^4\,dx$.

Solution

The first step is to choose an expression for u. We choose $u = 3x^2 + 4$ because then $du = 6x\,dx$ and we already have du in the integrand. Write the integral in terms of u:

$$\int 6x(3x^2 + 4)^4\,dx = \int u^4\,du.$$

Remember that du is the derivative of the expression chosen for u, regardless of what is inside the integrand. Now we can evaluate the integral with respect to u:

$$\int u^4\,du = \frac{u^5}{5} + C = \frac{(3x^2 + 4)^5}{5} + C.$$

Analysis

We can check our answer by taking the derivative of the result of integration. We should obtain the integrand. Picking a value for C of 1, we let $y = \frac{1}{5}(3x^2 + 4)^5 + 1$. We have

$$y = \frac{1}{5}(3x^2 + 4)^5 + 1,$$

so

$$y' = \left(\frac{1}{5}\right) 5(3x^2+4)^4 6x$$
$$= 6x(3x^2+4)^4.$$

This is exactly the expression we started with inside the integrand.

? Exercise 5.5.1

Use substitution to find the antiderivative of $\int 3x^2(x^3-3)^2\,dx$.

Hint

Let $u = x^3 - 3$.

Answer

$$\int 3x^2(x^3-3)^2\,dx = \frac{1}{3}(x^3-3)^3 + C$$

Sometimes we need to adjust the constants in our integral if they don't match up exactly with the expressions we are substituting.

✓ Example 5.5.2: Using Substitution with Alteration

Use substitution to find the antiderivative of

$$\int z\sqrt{z^2-5}\,dz.$$

Solution

Rewrite the integral as $\int z(z^2-5)^{1/2}\,dz$. Let $u = z^2 - 5$ and $du = 2z\,dz$. Now we have a problem because $du = 2z\,dz$ and the original expression has only $z\,dz$. We have to alter our expression for du or the integral in u will be twice as large as it should be. If we multiply both sides of the du equation by $\frac{1}{2}$. we can solve this problem. Thus,

$$u = z^2 - 5$$
$$du = 2z\,dz$$
$$\frac{1}{2}du = \frac{1}{2}(2z)\,dz = z\,dz.$$

Write the integral in terms of u, but pull the $\frac{1}{2}$ outside the integration symbol:

$$\int z(z^2-5)^{1/2}\,dz = \frac{1}{2}\int u^{1/2}\,du.$$

Integrate the expression in u:

$$\frac{1}{2}\int u^{1/2}\,du = \left(\frac{1}{2}\right)\frac{u^{3/2}}{\frac{3}{2}} + C$$
$$= \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)u^{3/2} + C$$
$$= \frac{1}{3}u^{3/2} + C$$
$$= \frac{1}{3}(z^2-5)^{3/2} + C$$

Exercise 5.5.2

Use substitution to find the antiderivative of $\int x^2(x^3+5)^9\,dx$.

Hint

Multiply the du equation by $\frac{1}{3}$.

Answer

$$\int x^2(x^3+5)^9\,dx = \frac{(x^3+5)^{10}}{30} + C$$

Example 5.5.3: Using Substitution with Integrals of Trigonometric Functions

Use substitution to evaluate the integral $\int \frac{\sin t}{\cos^3 t}\,dt$.

Solution

We know the derivative of $\cos t$ is $-\sin t$, so we set $u = \cos t$. Then $du = -\sin t\,dt$.

Substituting into the integral, we have

$$\int \frac{\sin t}{\cos^3 t}\,dt = -\int \frac{du}{u^3}.$$

Evaluating the integral, we get

$$-\int \frac{du}{u^3} = -\int u^{-3}\,du = -\left(-\frac{1}{2}\right)u^{-2} + C.$$

Putting the answer back in terms of t, we get

$$\int \frac{\sin t}{\cos^3 t}\,dt = \frac{1}{2u^2} + C = \frac{1}{2\cos^2 t} + C.$$

Exercise 5.5.3

Use substitution to evaluate the integral $\int \frac{\cos t}{\sin^2 t}\,dt$.

Hint

Use the process from Example 5.5.3 to solve the problem.

Answer

$$\int \frac{\cos t}{\sin^2 t}\,dt = -\frac{1}{\sin t} + C$$

Exercise 5.5.4

Use substitution to evaluate the indefinite integral $\int \cos^3 t \sin t\,dt$.

Hint

Use the process from Example 5.5.3 to solve the problem.

Answer

$$\int \cos^3 t \sin t\,dt = -\frac{\cos^4 t}{4} + C$$

Sometimes we need to manipulate an integral in ways that are more complicated than just multiplying or dividing by a constant. We need to eliminate all the expressions within the integrand that are in terms of the original variable. When we are done, u should be the only variable in the integrand. In some cases, this means solving for the original variable in terms of u. This technique should become clear in the next example.

Example 5.5.4: Finding an Antiderivative Using u-Substitution

Use substitution to find the antiderivative of

$$\int \frac{x}{\sqrt{x-1}}\,dx.$$

Solution

If we let $u = x - 1$, then $du = dx$. But this does not account for the x in the numerator of the integrand. We need to express x in terms of u. If $u = x - 1$, then $x = u + 1$. Now we can rewrite the integral in terms of u :

$$\int \frac{x}{\sqrt{x-1}}\,dx = \int \frac{u+1}{\sqrt{u}}\,du = \int \left(\sqrt{u} + \frac{1}{\sqrt{u}}\right)\,du = \int \left(u^{1/2} + u^{-1/2}\right)\,du.$$

Then we integrate in the usual way, replace u with the original expression, and factor and simplify the result. Thus,

$$\begin{aligned}\int (u^{1/2} + u^{-1/2})\,du &= \frac{2}{3}u^{3/2} + 2u^{1/2} + C \\ &= \frac{2}{3}(x-1)^{3/2} + 2(x-1)^{1/2} + C \\ &= (x-1)^{1/2}\left[\frac{2}{3}(x-1) + 2\right] + C \\ &= (x-1)^{1/2}\left(\frac{2}{3}x - \frac{2}{3} + \frac{6}{3}\right) \\ &= (x-1)^{1/2}\left(\frac{2}{3}x + \frac{4}{3}\right) \\ &= \frac{2}{3}(x-1)^{1/2}(x+2) + C.\end{aligned}$$

Substitution for Definite Integrals

Substitution can be used with definite integrals, too. However, using substitution to evaluate a definite integral requires a change to the limits of integration. If we change variables in the integrand, the limits of integration change as well.

Substitution with Definite Integrals

Let $u = g(x)$ and let g' be continuous over an interval $[a, b]$, and let f be continuous over the range of $u = g(x)$. Then,

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Although we will not formally prove this theorem, we justify it with some calculations here. From the substitution rule for indefinite integrals, if $F(x)$ is an antiderivative of $f(x)$, we have

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

Then

$$\begin{aligned}\int_a^b f[g(x)]g'(x)\,dx &= F(g(x))\Big|_{x=a}^{x=b}\\ &= F(g(b)) - F(g(a))\\ &= F(u)\Big|_{u=g(a)}^{u=g(b)}\\ &= \int_{g(a)}^{g(b)} f(u)\,du\end{aligned}$$

and we have the desired result.

Example 5.5.5: Using Substitution to Evaluate a Definite Integral

Use substitution to evaluate

$$\int_0^1 x^2(1+2x^3)^5\,dx.$$

Solution

Let $u = 1 + 2x^3$, so $du = 6x^2\,dx$. Since the original function includes one factor of x^2 and $du = 6x^2\,dx$, multiply both sides of the du equation by $1/6$. Then,

$$du = 6x^2\,dx$$

$$\text{becomes} \quad \frac{1}{6}du = x^2\,dx.$$

To adjust the limits of integration, note that when $x = 0, u = 1 + 2(0) = 1$, and when $x = 1,\ u = 1 + 2(1) = 3$.

Then

$$\int_0^1 x^2(1+2x^3)^5 dx = \frac{1}{6}\int_1^3 u^5\,du.$$

Evaluating this expression, we get

$$\begin{aligned}\frac{1}{6}\int_1^3 u^5\,du &= \left(\frac{1}{6}\right)\left(\frac{u^6}{6}\right)\Big|_1^3\\ &= \frac{1}{36}\left[(3)^6 - (1)^6\right]\\ &= \frac{182}{9}.\end{aligned}$$

Exercise 5.5.5

Use substitution to evaluate the definite integral $\int_{-1}^0 y(2y^2-3)^5\,dy$.

Hint

Use the steps from Example 5.5.5 to solve the problem.

Answer

$$\int_{-1}^0 y(2y^2-3)^5\,dy = \frac{91}{3}$$

Exercise 5.5.6

Use substitution to evaluate $\int_0^1 x^2\cos\left(\frac{\pi}{2}x^3\right)\,dx$.

Hint

Use the process from Example 5.5.5 to solve the problem.

Answer

$$\int_0^1 x^2 \cos\left(\frac{\pi}{2}x^3\right)\,dx = \frac{2}{3\pi} \approx 0.2122$$

Example 5.5.6: Using Substitution with an Exponential Function

Use substitution to evaluate

$$\int_0^1 xe^{4x^2+3}\,dx.$$

Solution

Let $u = 4x^3 + 3$. Then, $du = 8x\,dx$. To adjust the limits of integration, we note that when $x = 0$, $u = 3$, and when $x = 1$, $u = 7$. So our substitution gives

$$\begin{aligned}\int_0^1 xe^{4x^2+3}\,dx &= \frac{1}{8}\int_3^7 e^u\,du \\ &= \frac{1}{8}e^u\Big|_3^7 \\ &= \frac{e^7 - e^3}{8} \\ &\approx 134.568\end{aligned}$$

Substitution may be only one of the techniques needed to evaluate a definite integral. All of the properties and rules of integration apply independently, and trigonometric functions may need to be rewritten using a trigonometric identity before we can apply substitution. Also, we have the option of replacing the original expression for u after we find the antiderivative, which means that we do not have to change the limits of integration. These two approaches are shown in Example 5.5.7.

Example 5.5.7: Using Substitution to Evaluate a Trigonometric Integral

Use substitution to evaluate

$$\int_0^{\pi/2} \cos^2\theta\,d\theta.$$

Solution

Let us first use a trigonometric identity to rewrite the integral. The trig identity $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$ allows us to rewrite the integral as

$$\int_0^{\pi/2} \cos^2\theta\,d\theta = \int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\,d\theta.$$

Then,

$$\begin{aligned}\int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right)\,d\theta &= \int_0^{\pi/2} \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right)\,d\theta \\ &= \frac{1}{2}\int_0^{\pi/2} d\theta + \int_0^{\pi/2} \cos 2\theta\,d\theta.\end{aligned}$$

We can evaluate the first integral as it is, but we need to make a substitution to evaluate the second integral. Let $u = 2\theta$. Then, $du = 2\,d\theta$, or $\frac{1}{2}\,du = d\theta$. Also, when $\theta = 0$, $u = 0$, and when $\theta = \pi/2$, $u = \pi$. Expressing the second integral in terms of u, we have

$$\frac{1}{2}\int_0^{\pi/2} d\theta + \frac{1}{2}\int_0^{\pi/2} \cos 2\theta\, d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta + \frac{1}{2}\left(\frac{1}{2}\right)\int_0^{\pi} \cos u\, du$$
$$= \frac{\theta}{2}\Big|_{\theta=0}^{\theta=\pi/2} + \frac{1}{4}\sin u\Big|_{u=0}^{u=\pi}$$
$$= \left(\frac{\pi}{4} - 0\right) + (0 - 0) = \frac{\pi}{4}$$

Key Concepts

- Substitution is a technique that simplifies the integration of functions that are the result of a chain-rule derivative. The term 'substitution' refers to changing variables or substituting the variable u and du for appropriate expressions in the integrand.
- When using substitution for a definite integral, we also have to change the limits of integration.

Key Equations

- **Substitution with Indefinite Integrals**

$$\int f[g(x)]g'(x)\, dx = \int f(u)\, du = F(u) + C = F(g(x)) + C$$

- **Substitution with Definite Integrals**

$$\int_a^b f(g(x))g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

Glossary

change of variables
the substitution of a variable, such as u, for an expression in the integrand

integration by substitution
a technique for integration that allows integration of functions that are the result of a chain-rule derivative

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5.5E: EXERCISES FOR SECTION 5.5

1) Why is u-substitution referred to as a *change of variable*?

2) If $f = g \circ h$, when reversing the chain rule, $\frac{d}{dx}(g \circ h)(x) = g'(h(x))h'(x)$, should you take $u = g(x)$ or $u = h(x)$?

Answer

$u = h(x)$

In exercises 3 - 7, verify each identity using differentiation. Then, using the indicated u-substitution, identify f such that the integral takes the form $\int f(u)\,du$.

3) $\int x\sqrt{x+1}\,dx = \frac{2}{15}(x+1)^{3/2}(3x-2)+C; \quad u = x+1$

4) $\int \frac{x^2}{\sqrt{x-1}}\,dx = \frac{2}{15}\sqrt{x-1}(3x^2+4x+8)+C, \quad (x>1); \quad u = x-1$

Answer

$f(u) = \frac{(u+1)^2}{\sqrt{u}}$

5) $\int x\sqrt{4x^2+9}\,dx = \frac{1}{12}(4x^2+9)^{3/2}+C; \quad u = 4x^2+9$

6) $\int \frac{x}{\sqrt{4x^2+9}}\,dx = \frac{1}{4}\sqrt{4x^2+9}+C; \quad u = 4x^2+9$

Answer

$du = 8x\,dx; \quad f(u) = \frac{1}{8\sqrt{u}}$

7) $\int \frac{x}{(4x^2+9)^2}\,dx = -\frac{1}{8(4x^2+9)}+C; \quad u = 4x^2+9$

In exercises 8 - 17, find the antiderivative using the indicated substitution.

8) $\int (x+1)^4\,dx; \quad u = x+1$

Answer

$\int (x+1)^4\,dx = \frac{1}{5}(x+1)^5+C$

9) $\int (x-1)^5\,dx; \quad u = x-1$

10) $\int (2x-3)^{-7}\,dx; \quad u = 2x-3$

Answer

$\int (2x-3)^{-7}\,dx = -\frac{1}{12(2x-3)^6}+C$

11) $\int (3x-2)^{-11}\,dx; \quad u = 3x-2$

12) $\int \frac{x}{\sqrt{x^2+1}}\,dx; \quad u = x^2+1$

Answer

$\int \frac{x}{\sqrt{x^2+1}}\,dx = \sqrt{x^2+1}+C$

13) $\int \frac{x}{\sqrt{1-x^2}}\,dx; \quad u = 1-x^2$

14) $\int (x-1)(x^2-2x)^3\,dx; \quad u = x^2-2x$

Answer

$\int (x-1)(x^2-2x)^3\,dx = \frac{1}{8}(x^2-2x)^4+C$

15) $\int (x^2-2x)(x^3-3x^2)^2\,dx; \quad u=x^3-3x^2$

16) $\int \cos^3\theta\,d\theta; \quad u=\sin\theta$ (Hint: $\cos^2\theta=1-\sin^2\theta$)

Answer

$$\int \cos^3\theta\,d\theta=\sin\theta-\frac{\sin^3\theta}{3}+C$$

17) $\int \sin^3\theta\,d\theta; \quad u=\cos\theta$ (Hint: $\sin^2\theta=1-\cos^2\theta$)

In exercises 18 - 34, use a suitable change of variables to determine the indefinite integral.

18) $\int x(1-x)^{99}\,dx$

Answer

$$\int x(1-x)^{99}\,dx=\frac{(1-x)^{101}}{101}-\frac{(1-x)^{100}}{100}+C$$
$$=-\frac{(1-x)^{100}}{10100}\left[100x+1\right]+C$$

19) $\int t(1-t^2)^{10}dt$

20) $\int (11x-7)^{-3}\,dx$

Answer

$$\int (11x-7)^{-3}\,dx=-\frac{1}{22(11x-7)^2}+C$$

21) $\int (7x-11)^4\,dx$

22) $\int \cos^3\theta\sin\theta\,d\theta$

Answer

$$\int \cos^3\theta\sin\theta\,d\theta=-\frac{\cos^4\theta}{4}+C$$

23) $\int \sin^7\theta\cos\theta\,d\theta$

24) $\int \cos^2(\pi t)\sin(\pi t)\,dt$

Answer

$$\int \cos^2(\pi t)\sin(\pi t)\,dt=-\frac{cos^3(\pi t)}{3\pi}+C$$

25) $\int \sin^2 x\cos^3 x\,dx$ (Hint: $\sin^2 x+\cos^2 x=1$)

26) $\int t\sin(t^2)\cos(t^2)\,dt$

Answer

$$\int t\sin(t^2)\cos(t^2)\,dt=-\frac{1}{4}\cos^2(t^2)+C$$

27) $\int t^2\cos^2(t^3)\sin(t^3)\,dt$

28) $\int \frac{x^2}{(x^3-3)^2}\,dx$

Answer

$$\int \frac{x^2}{(x^3-3)^2}\,dx=-\frac{1}{3(x^3-3)}+C$$

29) $\int \frac{x^3}{\sqrt{1-x^2}}\,dx$

30) $\int \frac{y^5}{(1-y^3)^{3/2}}\,dy$

Answer

$$\int \frac{y^5}{(1-y^3)^{3/2}}\,dy = -\frac{2(y^3-2)}{3\sqrt{1-y^3}} + C$$

31) $\int \cos\theta(1-\cos\theta)^{99}\sin\theta\,d\theta$

32) $\int (1-\cos^3\theta)^{10}\cos^2\theta\sin\theta\,d\theta$

Answer

$$\int (1-\cos^3\theta)^{10}\cos^2\theta\sin\theta\,d\theta = \frac{1}{33}(1-\cos^3\theta)^{11} + C$$

33) $\int (\cos\theta - 1)(\cos^2\theta - 2\cos\theta)^3\sin\theta\,d\theta$

34) $\int (\sin^2\theta - 2\sin\theta)(\sin^3\theta - 3\sin^2\theta)^3\cos\theta\,d\theta$

Answer

$$\int (\sin^2\theta - 2\sin\theta)(\sin^3\theta - 3\sin^2\theta)^3\cos\theta\,d\theta = \frac{1}{12}(\sin^3\theta - 3\sin^2\theta)^4 + C$$

In exercises 35 - 38, use a calculator to estimate the area under the curve using left Riemann sums with 50 terms, then use substitution to solve for the exact answer.

35) [T] $y = 3(1-x)^2$ over $[0, 2]$

36) [T] $y = x(1-x^2)^3$ over $[-1, 2]$

Answer

$L_{50} = -8.5779$. The exact area is $\frac{-81}{8}$ units^2.

37) [T] $y = \sin x(1-\cos x)^2$ over $[0, \pi]$

38) [T] $y = \frac{x}{(x^2+1)^2}$ over $[-1, 1]$

Answer

$L_{50} = -0.006399$. The exact area is 0.

In exercises 39 - 44, use a change of variables to evaluate the definite integral.

39) $\int_0^1 x\sqrt{1-x^2}\,dx$

40) $\int_0^1 \frac{x}{\sqrt{1+x^2}}\,dx$

Answer

$$u = 1+x^2, \quad du = 2x\,dx, \quad \int_0^1 \frac{x}{\sqrt{1+x^2}}\,dx = \frac{1}{2}\int_1^2 u^{-1/2}du = \sqrt{2}-1$$

41) $\int_0^2 \frac{t}{\sqrt{5+t^2}}\,dt$

42) $\int_0^1 \frac{t^2}{\sqrt{1+t^3}}\,dt$

Answer

$$u = 1+t^3, \quad du = 3t^2, \quad \int_0^1 \frac{t^2}{\sqrt{1+t^3}}\,dt = \frac{1}{3}\int_1^2 u^{-1/2}du = \frac{2}{3}(\sqrt{2}-1)$$

43) $\int_0^{\pi/4} \sec^2\theta\tan\theta\,d\theta$

44) $\int_0^{\pi/4} \frac{\sin\theta}{\cos^4\theta}\,d\theta$

Answer

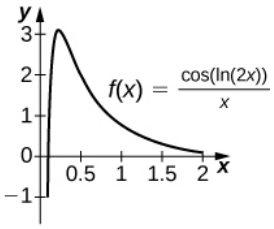
$$u = \cos\theta, \quad du = -\sin\theta\, d\theta, \quad \int_0^{\pi/4} \frac{\sin\theta}{\cos^4\theta}\, d\theta = -\int_1^{\sqrt{2}/2} u^{-4}\, du = \int_{\sqrt{2}/2}^1 u^{-4}\, du = \frac{1}{3}(2\sqrt{2} - 1)$$

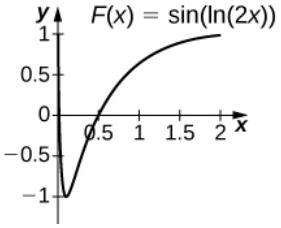
In exercises 45 - 50, evaluate the indefinite integral $\int f(x)\, dx$ with constant $C = 0$ using u-substitution. Then, graph the function and the antiderivative over the indicated interval. If possible, estimate a value of C that would need to be added to the antiderivative to make it equal to the definite integral $F(x) = \int_a^x f(t)\, dt$, with a the left endpoint of the given interval.

45) [T] $\int (2x+1)e^{x^2+x-6}\, dx$ over $[-3, 2]$

46) [T] $\int \frac{\cos(\ln(2x))}{x}\, dx$ on $[0, 2]$

Answer





The antiderivative is $y = \sin(\ln(2x))$. Since the antiderivative is not continuous at $x = 0$, one cannot find a value of C that would make $y = \sin(\ln(2x)) - C$ work as a definite integral.

47) [T] $\int \frac{3x^2 + 2x + 1}{\sqrt{x^3 + x^2 + x + 4}}\, dx$ over $[-1, 2]$

48) [T] $\int \frac{\sin x}{\cos^3 x}\, dx$ over $[-\frac{\pi}{3}, \frac{\pi}{3}]$

Answer

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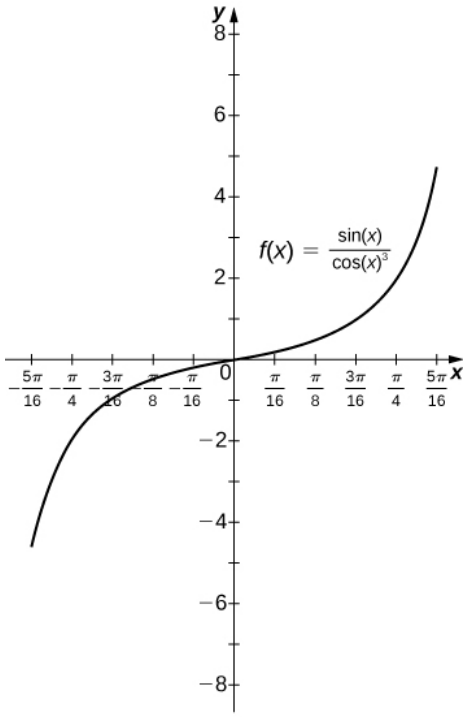

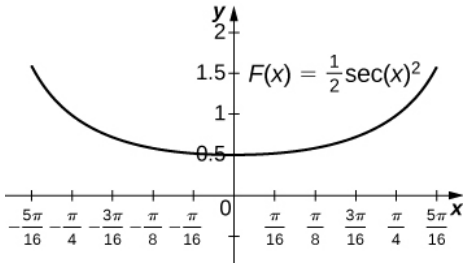

The antiderivative is $y = \frac{1}{2}\sec^2 x$. You should take $C = -2$ so that $F(-\frac{\pi}{3}) = 0$.

49) [T] $\int (x+2)e^{-x^2-4x+3}\,dx$ over $[-5, 1]$

50) [T] $\int 3x^2\sqrt{2x^3+1}\,dx$ over $[0, 1]$

Answer

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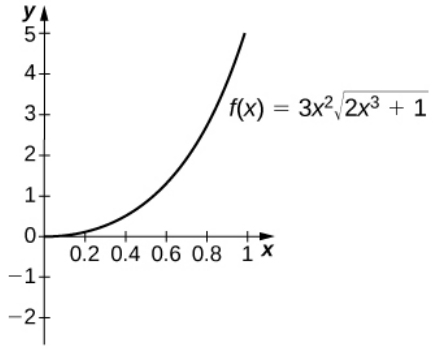
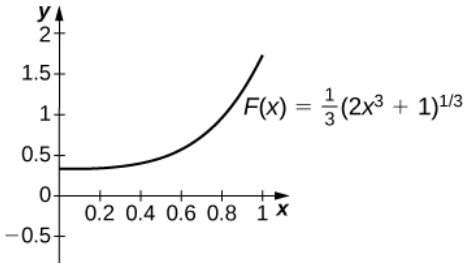




The antiderivative is $y = \frac{1}{3}(2x^3+1)^{3/2}$. One should take $C = -\frac{1}{3}$.

51) If $h(a) = h(b)$ in $\int_a^b g'(h(x))h(x)\,dx$, what can you say about the value of the integral?

52) Is the substitution $u = 1 - x^2$ in the definite integral $\int_0^2 \frac{x}{1-x^2}\,dx$ okay? If not, why not?

Answer

No, because the integrand is discontinuous at $x = 1$.

In exercises 53 - 59, use a change of variables to show that each definite integral is equal to zero.

53) $\int_0^{\pi} \cos^2(2\theta)\sin(2\theta)\,d\theta$

54) $\int_0^{\sqrt{\pi}} t\cos(t^2)\sin(t^2)\,dt$

Answer

$u = \sin(t^2)$; the integral becomes $\frac{1}{2}\int_0^0 u\,du$.

55) $\int_0^1 (1-2t)\,dt$

56) $\int_0^1 \frac{1-2t}{1+(t-\frac{1}{2})^2}\,dt$

Answer

$u = 1 + (t-\frac{1}{2})^2$; the integral becomes $-\int_{5/4}^{5/4} \frac{1}{u}\,du$.

57) $\int_0^{\pi} \sin\left((t-\frac{\pi}{2})^3\right)\cos\left(t-\frac{\pi}{2}\right)\,dt$

58) $\int_0^2 (1-t)\cos(\pi t)\,dt$

Answer

$u = 1 - t$; Since the integrand is odd, the integral becomes

$$\int_1^{-1} u\cos\left(\pi(1-u)\right)\,du = \int_1^{-1} u[\cos\pi\cos u - \sin\pi\sin u]\,du = -\int_1^{-1} u\cos u\,du = \int_{-1}^{1} u\cos u\,du = 0$$

59) $\int_{\pi/4}^{3\pi/4} \sin^2 t \cos t \, dt$

60) Show that the average value of $f(x)$ over an interval $[a, b]$ is the same as the average value of $f(cx)$ over the interval $\left[\frac{a}{c}, \frac{b}{c}\right]$ for $c > 0$.

Answer

Setting $u = cx$ and $du = c\,dx$ gets you $\frac{1}{\frac{b}{c} - \frac{a}{c}} \int_{a/c}^{b/c} f(cx)\,dx = \frac{c}{b-a} \int_{u=a}^{u=b} f(u) \frac{du}{c} = \frac{1}{b-a} \int_a^b f(u)\,du.$

61) Find the area under the graph of $f(t) = \frac{t}{(1+t^2)^a}$ between $t = 0$ and $t = x$ where $a > 0$ and $a \neq 1$ is fixed, and evaluate the limit as $x \to \infty$.

62) Find the area under the graph of $g(t) = \frac{t}{(1-t^2)^a}$ between $t = 0$ and $t = x$, where $0 < x < 1$ and $a > 0$ is fixed. Evaluate the limit as $x \to 1$.

Answer

$\int_0^x g(t)\,dt = \frac{1}{2} \int_{u=1-x^2}^{1} \frac{du}{u^a} = \frac{1}{2(1-a)} u^{1-a} \,|\, 1u = \frac{1}{2(1-a)} (1 - (1-x^2)^{1-a})$ As $x \to 1$ the limit is $\frac{1}{2(1-a)}$ if $a < 1$, and the limit diverges to $+\infty$ if $a > 1$.

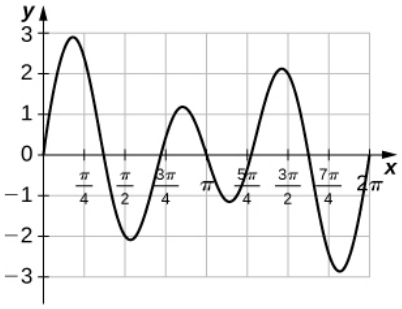
63) The area of a semicircle of radius 1 can be expressed as $\int_{-1}^{1} \sqrt{1-x^2}\,dx$. Use the substitution $x = \cos t$ to express the area of a semicircle as the integral of a trigonometric function. You do not need to compute the integral.

64) The area of the top half of an ellipse with a major axis that is the x-axis from $x = -1$ to a and with a minor axis that is the y-axis from $y = -b$ to $y = b$ can be written as $\int_{-a}^{a} b\sqrt{1 - \frac{x^2}{a^2}}\,dx$. Use the substitution $x = a \cos t$ to express this area in terms of an integral of a trigonometric function. You do not need to compute the integral.

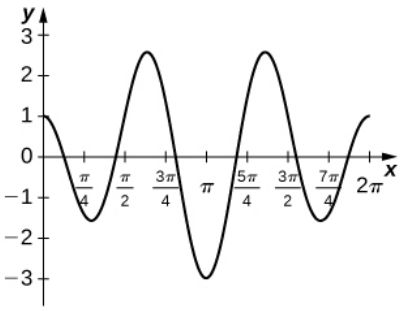
Answer

$\int_{t=\pi}^{t=0} b\sqrt{1-\cos^2 t} \times (-a \sin t)\,dt = \int_{t=0}^{t=\pi} ab \sin^2 t\,dt$

65) [T] The following graph is of a function of the form $f(t) = a \sin(nt) + b \sin(mt)$. Estimate the coefficients a and b and the frequency parameters n and m. Use these estimates to approximate $\int_0^{\pi} f(t)\,dt$.



66) [T] The following graph is of a function of the form $f(x) = a \cos(nt) + b \cos(mt)$. Estimate the coefficients a and b and the frequency parameters n and m. Use these estimates to approximate $\int_0^{\pi} f(t)\,dt$.



Answer

$$f(t) = 2\cos(3t) - \cos(2t); \quad \int_0^{\pi/2} (2\cos(3t) - \cos(2t))\, dt = -\frac{2}{3}$$

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5.6: Integrals Involving Exponential and Logarithmic Functions

Learning Objectives

- Integrate functions involving exponential functions.
- Integrate functions involving logarithmic functions.

Exponential and logarithmic functions are used to model population growth, cell growth, and financial growth, as well as depreciation, radioactive decay, and resource consumption, to name only a few applications. In this section, we explore integration involving exponential and logarithmic functions.

Integrals of Exponential Functions

The exponential function is perhaps the most efficient function in terms of the operations of calculus. The exponential function, $y = e^x$, is its own derivative and its own integral.

Rule: Integrals of Exponential Functions

Exponential functions can be integrated using the following formulas.

$$\int e^x \, dx = e^x + C \tag{5.6.1}$$

$$\int a^x \, dx = \frac{a^x}{\ln a} + C \tag{5.6.2}$$

Example 5.6.1: Finding an Antiderivative of an Exponential Function

Find the antiderivative of the exponential function e^{-x}.

Solution

Use substitution, setting $u = -x$, and then $du = -1 \, dx$. Multiply the du equation by -1, so you now have $-du = dx$. Then,

$$\int e^{-x} \, dx = -\int e^u \, du = -e^u + C = -e^{-x} + C.$$

Exercise 5.6.1

Find the antiderivative of the function using substitution: $x^2 e^{-2x^3}$.

Hint

Let u equal the exponent on e.

Answer

$$\int x^2 e^{-2x^3} \, dx = -\frac{1}{6} e^{-2x^3} + C$$

A common mistake when dealing with exponential expressions is treating the exponent on e the same way we treat exponents in polynomial expressions. We cannot use the power rule for the exponent on e. This can be especially confusing when we have both exponentials and polynomials in the same expression, as in the previous checkpoint. In these cases, we should always double-check to make sure we're using the right rules for the functions we're integrating.

Example 5.6.2: Square Root of an Exponential Function

Find the antiderivative of the exponential function $e^x \sqrt{1 + e^x}$.

Solution

First rewrite the problem using a rational exponent:

$$\int e^x \sqrt{1+e^x}\,dx = \int e^x (1+e^x)^{1/2}\,dx.$$

Using substitution, choose $u = 1 + e^x$. Then, $du = e^x\,dx$. We have

$$\int e^x (1+e^x)^{1/2}\,dx = \int u^{1/2}\,du.$$

Then

$$\int u^{1/2}\,du = \frac{u^{3/2}}{3/2} + C = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1+e^x)^{3/2} + C$$

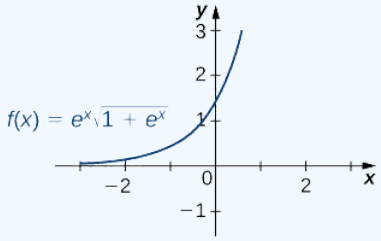

Figure 5.6.1: The graph shows an exponential function times the square root of an exponential function.

Exercise 5.6.2

Find the antiderivative of $e^x(3e^x - 2)^2$.

Hint

Let $u = 3e^x - 2$.

Answer

$$\int e^x (3e^x - 2)^2\,dx = \frac{1}{9}(3e^x - 2)^3 + C$$

Example 5.6.3: Using Substitution with an Exponential Function

Use substitution to evaluate the indefinite integral $\int 3x^2 e^{2x^3}\,dx$.

Solution

Here we choose to let u equal the expression in the exponent on e. Let $u = 2x^3$ and $du = 6x^2\,dx$. Again, du is off by a constant multiplier; the original function contains a factor of $3x^2$, not $6x^2$. Multiply both sides of the equation by $\frac{1}{2}$ so that the integrand in u equals the integrand in x. Thus,

$$\int 3x^2 e^{2x^3}\,dx = \frac{1}{2}\int e^u\,du.$$

Integrate the expression in u and then substitute the original expression in x back into the u-integral:

$$\frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C = \frac{1}{2}e^{2x^3} + C.$$

Exercise 5.6.3

Evaluate the indefinite integral $\int 2x^3 e^{x^4}\,dx$.

Hint

Let $u = x^4$.

Answer

$$\int 2x^3 e^{x^4}\, dx = \frac{1}{2}e^{x^4} + C$$

As mentioned at the beginning of this section, exponential functions are used in many real-life applications. The number e is often associated with compounded or accelerating growth, as we have seen in earlier sections about the derivative. Although the derivative represents a rate of change or a growth rate, the integral represents the total change or the total growth. Let's look at an example in which integration of an exponential function solves a common business application.

A price–demand function tells us the relationship between the quantity of a product demanded and the price of the product. In general, price decreases as quantity demanded increases. The marginal price–demand function is the derivative of the price–demand function and it tells us how fast the price changes at a given level of production. These functions are used in business to determine the price–elasticity of demand, and to help companies determine whether changing production levels would be profitable.

Example 5.6.4: Finding a Price–Demand Equation

Find the price–demand equation for a particular brand of toothpaste at a supermarket chain when the demand is 50 tubes per week at $2.35 per tube, given that the marginal price—demand function, $p'(x)$, for x number of tubes per week, is given as

$$p'(x) = -0.015e^{-0.01x}.$$

If the supermarket chain sells 100 tubes per week, what price should it set?

Solution

To find the price–demand equation, integrate the marginal price–demand function. First find the antiderivative, then look at the particulars. Thus,

$$p(x) = \int -0.015e^{-0.01x}\, dx = -0.015 \int e^{-0.01x}\, dx.$$

Using substitution, let $u = -0.01x$ and $du = -0.01\, dx$. Then, divide both sides of the du equation by -0.01. This gives

$$\frac{-0.015}{-0.01} \int e^u\, du = 1.5 \int e^u\, du = 1.5e^u + C = 1.5e^{-0.01x} + C.$$

The next step is to solve for C. We know that when the price is $2.35 per tube, the demand is 50 tubes per week. This means

$$p(50) = 1.5e^{-0.01(50)} + C = 2.35.$$

Now, just solve for C:

$$C = 2.35 - 1.5e^{-0.5} = 2.35 - 0.91 = 1.44.$$

Thus,

$$p(x) = 1.5e^{-0.01x} + 1.44.$$

If the supermarket sells 100 tubes of toothpaste per week, the price would be

$$p(100) = 1.5e^{-0.01(100)} + 1.44 = 1.5e^{-1} + 1.44 \approx 1.99.$$

The supermarket should charge $1.99 per tube if it is selling 100 tubes per week.

Example 5.6.5: Evaluating a Definite Integral Involving an Exponential Function

Evaluate the definite integral $\int_1^2 e^{1-x}\, dx$.

Solution

Again, substitution is the method to use. Let $u = 1 - x$, so $du = -1\,dx$ or $-du = dx$. Then $\int e^{1-x}\,dx = -\int e^u\,du$.

Next, change the limits of integration. Using the equation $u = 1 - x$, we have:

$$\text{When } x = 1, \quad u = 1 - (1) = 0,$$

$$\text{and when } x = 2, \quad u = 1 - (2) = -1.$$

The integral then becomes

$$\begin{aligned}\int_1^2 e^{1-x}\,dx &= -\int_0^{-1} e^u\,du \\ &= \int_{-1}^0 e^u\,du \\ &= e^u\Big|_{-1}^0 = e^0 - (e^{-1}) \\ &= -e^{-1} + 1.\end{aligned}$$

See Figure 5.6.2.

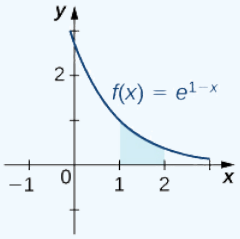

Figure 5.6.2: The indicated area can be calculated by evaluating a definite integral using substitution.

? Exercise 5.6.4

Evaluate $\int_0^2 e^{2x}\,dx$.

Hint

Let $u = 2x$.

Answer

$\frac{1}{2}\int_0^4 e^u\,du = \frac{1}{2}(e^4 - 1)$

✓ Example 5.6.6: Growth of Bacteria in a Culture

Suppose the rate of growth of bacteria in a Petri dish is given by $q(t) = 3^t$, where t is given in hours and $q(t)$ is given in thousands of bacteria per hour. If a culture starts with $10,000$ bacteria, find a function $Q(t)$ that gives the number of bacteria in the Petri dish at any time t. How many bacteria are in the dish after 2 hours?

Solution

We have

$$Q(t) = \int 3^t dt = \frac{3^t}{\ln 3} + C.$$

Then, at $t = 0$ we have $Q(0) = 10 = \frac{1}{\ln 3} + C$, so $C \approx 9.090$ and we get

$$Q(t) = \frac{3^t}{\ln 3} + 9.090.$$

At time $t = 2$, we have

$$Q(2) = \frac{3^2}{\ln 3} + 9.090$$
$$\approx 17.282.$$

After 2 hours, there are 17,282 bacteria in the dish.

Exercise 5.6.5

From Example, suppose the bacteria grow at a rate of $q(t) = 2^t$. Assume the culture still starts with $10,000$ bacteria. Find $Q(t)$. How many bacteria are in the dish after 3 hours?

Hint

Use the procedure from Example $5.6.6$ to solve the problem

Answer

$$Q(t) = \frac{2^t}{\ln 2} + 8.557.$$
$$Q(3) \approx 20,099$$

So there are $20,099$ bacteria in the dish after 3 hours.

Example 5.6.7: Fruit Fly Population Growth

Suppose a population of **fruit flies** increases at a rate of $g(t) = 2e^{0.02t}$, in flies per day. If the initial population of fruit flies is 100 flies, how many flies are in the population after 10 days?

Solution

Let $G(t)$ represent the number of flies in the population at time t. Applying the net change theorem, we have

$$\begin{aligned} G(10) = G(0) + \int_0^{10} 2e^{0.02t}\, dt \\ &= 100 + \left[\frac{2}{0.02} e^{0.02t}\right] \Big|_0^{10} \\ &= 100 + \left[100e^{0.02t}\right] \Big|_0^{10} \\ &= 100 + 100e^{0.2} - 100 \\ &\approx 122. \end{aligned}$$

There are 122 flies in the population after 10 days.

Exercise 5.6.6

Suppose the rate of growth of the fly population is given by $g(t) = e^{0.01t}$, and the initial fly population is 100 flies. How many flies are in the population after 15 days?

Hint

Use the process from Example $5.6.7$ to solve the problem.

Answer

There are 116 flies.

Example 5.6.8: Evaluating a Definite Integral Using Substitution

Evaluate the definite integral using substitution:

$$\int_1^2 \frac{e^{1/x}}{x^2}\,dx.$$

Solution

This problem requires some rewriting to simplify applying the properties. First, rewrite the exponent on e as a power of x, then bring the x^2 in the denominator up to the numerator using a negative exponent. We have

$$\int_1^2 \frac{e^{1/x}}{x^2}\,dx = \int_1^2 e^{x^{-1}}x^{-2}\,dx.$$

Let $u = x^{-1}$, the exponent on e. Then

$$du = -x^{-2}\,dx$$

$$-du = x^{-2}\,dx.$$

Bringing the negative sign outside the integral sign, the problem now reads

$$-\int e^u\,du.$$

Next, change the limits of integration:

$$u = (1)^{-1} = 1$$

$$u = (2)^{-1} = \frac{1}{2}.$$

Notice that now the limits begin with the larger number, meaning we can multiply by -1 and interchange the limits. Thus,

$$-\int_1^{1/2} e^u\,du = \int_{1/2}^1 e^u\,du = e^u\big|_{1/2}^1 = e - e^{1/2} = e - \sqrt{e}.$$

Exercise 5.6.7

Evaluate the definite integral using substitution:

$$\int_1^2 \frac{1}{x^3}e^{4x^{-2}}\,dx.$$

Hint

Let $u = 4x^{-2}$.

Answer

$$\int_1^2 \frac{1}{x^3}e^{4x^{-2}}\,dx = \frac{1}{8}[e^4 - e].$$

Integrals Involving Logarithmic Functions

Integrating functions of the form $f(x) = x^{-1}$ result in the absolute value of the natural log function, as shown in the following rule. Integral formulas for other logarithmic functions, such as $f(x) = \ln x$ and $f(x) = \log_a x$, are also included in the rule.

Rule: Integration Formulas Involving Logarithmic Functions

The following formulas can be used to evaluate integrals involving logarithmic functions.

$$\int x^{-1}\,dx = \ln|x| + C$$

$$\int \ln x\;dx = x\ln x - x + C = x(\ln x - 1) + C$$

$$\int \log_a x\,dx = \frac{x}{\ln a}(\ln x - 1) + C$$

Example 5.6.9: Finding an Antiderivative Involving $\ln x$

Find the antiderivative of the function $\dfrac{3}{x-10}$.

Solution

First factor the 3 outside the integral symbol. Then use the u^{-1} rule. Thus,

$$\int \frac{3}{x-10}\,dx = 3\int \frac{1}{x-10}\,dx = 3\int \frac{du}{u} = 3\ln|u| + C = 3\ln|x-10| + C, \quad x \neq 10.$$

See Figure 5.6.3.

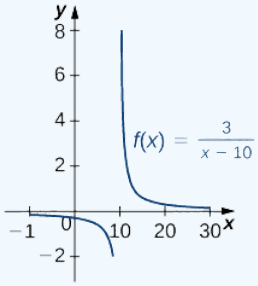

Figure 5.6.3: The domain of this function is $x \neq 10$.

Exercise 5.6.8

Find the antiderivative of $\dfrac{1}{x+2}$.

Hint

Follow the pattern from Example 5.6.9 to solve the problem.

Answer

$$\int \frac{1}{x+2}\,dx = \ln|x+2| + C$$

Example 5.6.10: Finding an Antiderivative of a Rational Function

Find the antiderivative of $\dfrac{2x^3+3x}{x^4+3x^2}$.

Solution

This can be rewritten as $\int (2x^3+3x)(x^4+3x^2)^{-1}\,dx$. Use substitution.

Let $u = x^4+3x^2$, then $du = (4x^3+6x)\,dx$. Alter du by factoring out the 2. Thus,

$$du = (4x^3+6x)\,dx = 2(2x^3+3x)\,dx$$

$$\frac{1}{2}\,du = (2x^3+3x)\,dx.$$

Rewrite the integrand in u:

$$\int (2x^3 + 3x)(x^4 + 3x^2)^{-1}\, dx = \frac{1}{2}\int u^{-1}\, du.$$

Then we have

$$\frac{1}{2}\int u^{-1}\, du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|\, x^4 + 3x^2\,| + C.$$

Example 5.6.11: Finding an Antiderivative of a Logarithmic Function

Find the antiderivative of the log function $\log_2 x$.

Solution

Follow the format in the formula listed in the rule on integration formulas involving logarithmic functions. Based on this format, we have

$$\int \log_2 x\, dx = \frac{x}{\ln 2}(\ln x - 1) + C.$$

Exercise 5.6.9

Find the antiderivative of $\log_3 x$.

Hint

Follow Example 5.6.11 and refer to the rule on integration formulas involving logarithmic functions.

Answer

$$\int \log_3 x\, dx = \frac{x}{\ln 3}(\ln x - 1) + C$$

Example 5.6.12 is a definite integral of a trigonometric function. With trigonometric functions, we often have to apply a trigonometric property or an identity before we can move forward. Finding the right form of the integrand is usually the key to a smooth integration.

Example 5.6.12: Evaluating a Definite Integral

Evaluate the definite integral

$$\int_0^{\pi/2} \frac{\sin x}{1 + \cos x}\, dx.$$

Solution

We need substitution to evaluate this problem. Let $u = 1 + \cos x$ so $du = -\sin x\ dx$.

Rewrite the integral in terms of u, changing the limits of integration as well. Thus,

$$u = 1 + \cos(0) = 2$$
$$u = 1 + \cos\left(\frac{\pi}{2}\right) = 1.$$

Then

$$\begin{aligned}\int_0^{\pi/2} \frac{\sin x}{1 + \cos x} &= -\int_2^1 u^{-1}\, du \\ &= \int_1^2 u^{-1}\, du \\ &= \ln|u|\Big|_1^2 \\ &= [\ln 2 - \ln 1] = \ln 2\end{aligned}$$

Key Concepts

- Exponential and logarithmic functions arise in many real-world applications, especially those involving growth and decay.
- Substitution is often used to evaluate integrals involving exponential functions or logarithms.

Key Equations

- **Integrals of Exponential Functions**

$$\int e^x \, dx = e^x + C$$

$$\int a^x \, dx = \frac{a^x}{\ln a} + C$$

- **Integration Formulas Involving Logarithmic Functions**

$$\int x^{-1} \, dx = \ln |x| + C$$

$$\int \ln x \, dx = x \ln x - x + C = x(\ln x - 1) + C$$

$$\int \log_a x \, dx = \frac{x}{\ln a}(\ln x - 1) + C$$

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5.6E: EXERCISES FOR SECTION 5.6

For exercises 1 - 8, compute each indefinite integral.

1) $\int e^{2x}\,dx$

2) $\int e^{-3x}\,dx$

Answer

$$\int e^{-3x}\,dx \quad = \quad \frac{-1}{3}e^{-3x}+C$$

3) $\int 2^{x}\,dx$

4) $\int 3^{-x}\,dx$

Answer

$$\int 3^{-x}\,dx \quad = \quad -\frac{3^{-x}}{\ln 3}+C$$

5) $\int \frac{1}{2x}\,dx$

6) $\int \frac{2}{x}\,dx$

Answer

$$\int \frac{2}{x}\,dx \quad = \quad 2\ln|x|+C \quad = \quad \ln(x^2)+C$$

7) $\int \frac{1}{x^2}\,dx$

8) $\int \frac{1}{\sqrt{x}}\,dx$

Answer

$$\int \frac{1}{\sqrt{x}}\,dx \quad = \quad 2\sqrt{x}+C$$

In exercises 9 - 16, find each indefinite integral by using appropriate substitutions.

9) $\int \frac{\ln x}{x}\,dx$

10) $\int \frac{dx}{x(\ln x)^2}$

Answer

$$\int \frac{dx}{x(\ln x)^2} \quad = \quad -\frac{1}{\ln x}+C$$

11) $\int \frac{dx}{x\ln x} \quad (x>1)$

12) $\int \frac{dx}{x\ln x\ln(\ln x)}$

Answer

$$\int \frac{dx}{x\ln x\ln(\ln x)} \quad = \quad \ln(\ln(\ln x))+C$$

13) $\int \tan\theta\,d\theta$

14) $\int \frac{\cos x - x\sin x}{x\cos x}\,dx$

Answer

$$\int \frac{\cos x - x\sin x}{x\cos x}\,dx \quad = \quad \ln(x\cos x)+C$$

15) $\int \frac{\ln(\sin x)}{\tan x}\,dx$

16) $\int \ln(\cos x)\tan x\,dx$

Answer

$$\int \ln(\cos x)\tan x\,dx \quad = \quad -\frac{1}{2}(\ln(\cos(x)))^2 + C$$

17) $\int xe^{-x^2}\,dx$

18) $\int x^2 e^{-x^3}\,dx$

Answer

$$\int x^2 e^{-x^3}\,dx \quad = \quad \frac{-e^{-x^3}}{3} + C$$

19) $\int e^{\sin x}\cos x\,dx$

20) $\int e^{\tan x}\sec^2 x\,dx$

Answer

$$\int e^{\tan x}\sec^2 x\,dx \quad = \quad e^{\tan x} + C$$

21) $\int \frac{e^{\ln x}}{x}\,dx$

22) $\int \frac{e^{\ln(1-t)}}{1-t}\,dt$

Answer

$$\int \frac{e^{\ln(1-t)}}{1-t}\,dt = \int \frac{1-t}{1-t}\,dt = \int 1\,dt \quad = \quad t + C$$

In exercises 23 - 28, verify by differentiation that $\int \ln x\,dx = x(\ln x - 1) + C$, then use appropriate changes of variables to compute the integral.

23) $\int \ln x\,dx$ (Hint: $\int \ln x\,dx = \frac{1}{2}\int x\ln(x^2)\,dx$)

24) $\int x^2 \ln^2 x\,dx$

Answer

$$\int x^2 \ln^2 x\,dx \quad = \quad \frac{1}{9}x^3(\ln(x^3) - 1) + C$$

25) $\int \frac{\ln x}{x^2}\,dx$ (Hint: Set $u = \frac{1}{x}$.)

26) $\int \frac{\ln x}{\sqrt{x}}\,dx$ (Hint: Set $u = \sqrt{x}$.)

Answer

$$\int \frac{\ln x}{\sqrt{x}}\,dx \quad = \quad 2\sqrt{x}(\ln x - 2) + C$$

27) Write an integral to express the area under the graph of $y = \frac{1}{t}$ from $t = 1$ to e^x and evaluate the integral.

28) Write an integral to express the area under the graph of $y = e^t$ between $t = 0$ and $t = \ln x$, and evaluate the integral.

Answer

$$\int_0^{\ln x} e^t\,dt = e^t\Big|_0^{\ln x} = e^{\ln x} - e^0 = x - 1$$

In exercises 29 - 35, use appropriate substitutions to express the trigonometric integrals in terms of compositions with logarithms.

29) $\int \tan(2x)\,dx$

30) $\int \dfrac{\sin(3x)-\cos(3x)}{\sin(3x)+\cos(3x)}\,dx$

Answer

$$\int \frac{\sin(3x)-\cos(3x)}{\sin(3x)+\cos(3x)}\,dx \quad = \quad -\frac{1}{3}\ln|\sin(3x)+\cos(3x)|+C$$

31) $\int \dfrac{x\sin(x^2)}{\cos(x^2)}\,dx$

32) $\int x\csc(x^2)\,dx$

Answer

$$\int x\csc(x^2)\,dx \quad = \quad -\frac{1}{2}\ln|\csc(x^2)+\cot(x^2)|+C$$

33) $\int \ln(\cos x)\tan x\,dx$

34) $\int \ln(\csc x)\cot x\,dx$

Answer

$$\int \ln(\csc x)\cot x\,dx \quad = \quad -\frac{1}{2}(\ln(\csc x))^2+C$$

35) $\int \dfrac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$

In exercises 36 - 40, evaluate the definite integral.

36) $\int_1^2 \dfrac{1+2x+x^2}{3x+3x^2+x^3}\,dx$

Answer

$$\int_1^2 \frac{1+2x+x^2}{3x+3x^2+x^3}\,dx \quad = \quad \frac{1}{3}\ln\left(\tfrac{26}{7}\right)$$

37) $\int_0^{\pi/4} \tan x\,dx$

38) $\int_0^{\pi/3} \dfrac{\sin x-\cos x}{\sin x+\cos x}\,dx$

Answer

$$\int_0^{\pi/3} \frac{\sin x-\cos x}{\sin x+\cos x}\,dx \quad = \quad \ln(\sqrt{3}-1)$$

39) $\int_{\pi/6}^{\pi/2} \csc x\,dx$

40) $\int_{\pi/4}^{\pi/3} \cot x\,dx$

Answer

$$\int_{\pi/4}^{\pi/3} \cot x\,dx \quad = \quad \frac{1}{2}\ln\frac{3}{2}$$

In exercises 41 - 46, integrate using the indicated substitution.

41) $\int \dfrac{x}{x-100}\,dx; \quad u=x-100$

42) $\int \dfrac{y-1}{y+1}\,dy; \quad u=y+1$

Answer

$$\int \frac{y-1}{y+1}\,dy \quad = \quad y-2\ln|y+1|+C$$

43) $\int \frac{1-x^2}{3x-x^3}\,dx; \quad u=3x-x^3$

44) $\int \frac{\sin x+\cos x}{\sin x-\cos x}\,dx; \quad u=\sin x-\cos x$

Answer

$$\int \frac{\sin x+\cos x}{\sin x-\cos x}\,dx \quad = \quad \ln|\sin x-\cos x|+C$$

45) $\int e^{2x}\sqrt{1-e^{2x}}\,dx; \quad u=1-e^{2x}$

46) $\int \ln(x)\frac{\sqrt{1-(\ln x)^2}}{x}\,dx; \quad u=1-(\ln x)^2$

Answer

$$\int \ln(x)\frac{\sqrt{1-(\ln x)^2}}{x}\,dx \quad = \quad -\frac{1}{3}(1-(\ln x)^2)^{3/2}+C$$

47) $\int \frac{\sqrt{x}}{\sqrt{x}+2}\,dx; \quad u=\sqrt{x}+2$

Answer

$$\int \frac{\sqrt{x}}{\sqrt{x}+2}\,dx \quad = \quad (\sqrt{x}+2)^2-8(\sqrt{x}+2)+8\ln(\sqrt{x}+2)+C$$

48) $\int e^x \sec(e^x+1)\tan(e^x+1)\,dx; \quad u=e^x+1$

Answer

$$\int e^x \sec(e^x+1)\tan(e^x+1)\,dx \quad = \quad \sec(e^x+1)+C$$

In exercises 49 - 54, state whether the right-endpoint approximation overestimates or underestimates the exact area. Then calculate the right endpoint estimate R_{50} and solve for the exact area.

49) [T] $y=e^x$ over $[0,1]$

50) [T] $y=e^{-x}$ over $[0,1]$

Answer

Since f is decreasing, the right endpoint estimate underestimates the area.

Exact solution: $\frac{e-1}{e}$, $R_{50}=0.6258$.

51) [T] $y=\ln(x)$ over $[1,2]$

52) [T] $y=\frac{x+1}{x^2+2x+6}$ over $[0,1]$

Answer

Since f is increasing, the right endpoint estimate overestimates the area.

Exact solution: $\frac{2\ln(3)-\ln(6)}{2}$, $R_{50}=0.2033$.

53) [T] $y=2^x$ over $[-1,0]$

54) [T] $y=-2^{-x}$ over $[0,1]$

Answer

Since f is increasing, the right endpoint estimate overestimates the area (the actual area is a larger negative number).

Exact solution: $-\frac{1}{\ln(4)}$, $R_{50}=-0.7164$.

In exercises 55 - 58, $f(x)\geq 0$ for $a\leq x\leq b$. Find the area under the graph of $f(x)$ between the given values a and b by integrating.

55) $f(x)=\frac{\log_{10}(x)}{x}; \quad a=10, b=100$

56) $f(x)=\frac{\log_2(x)}{x}; \quad a=32, b=64$

Answer

$\frac{11}{2}\ln 2$

57) $f(x)=2^{-x}; \quad a=1, b=2$

58) $f(x)=2^{-x}; \quad a=3, b=4$

Answer

$\frac{1}{\ln(65,536)}$

59) Find the area under the graph of the function $f(x)=xe^{-x^2}$ between $x=0$ and $x=5$.

60) Compute the integral of $f(x)=xe^{-x^2}$ and find the smallest value of N such that the area under the graph $f(x)=xe^{-x^2}$ between $x=N$ and $x=N+1$ is, at most, 0.01.

Answer

$\int_N^{N+1} xe^{-x^2}\,dx=\frac{1}{2}(e^{-N^2}-e^{-(N+1)^2})$. The quantity is less than 0.01 when $N=2$.

61) Find the limit, as N tends to infinity, of the area under the graph of $f(x)=xe^{-x^2}$ between $x=0$ and $x=5$.

62) Show that $\int_a^b \frac{dt}{t}=\int_{1/b}^{1/a}\frac{dt}{t}$ when $0<a\le b$.

Answer

$\int_a^b \frac{dx}{x}=\ln(b)-\ln(a)=\ln(\frac{1}{a})-\ln(\frac{1}{b})=\int_{1/b}^{1/a}\frac{dx}{x}$

63) Suppose that $f(x)>0$ for all x and that f and g are differentiable. Use the identity $f^g=e^{g\ln f}$ and the chain rule to find the derivative of f^g.

64) Use the previous exercise to find the antiderivative of $h(x)=x^x(1+\ln x)$ and evaluate $\int_2^3 x^x(1+\ln x)\,dx$.

Answer

23

65) Show that if $c>0$, then the integral of $\frac{1}{x}$ from ac to bc (for $0<a<b$) is the same as the integral of $\frac{1}{x}$ from a to b.

The following exercises are intended to derive the fundamental properties of the natural log starting from the definition $\ln(x)=\int_1^x \frac{dt}{t}$, using properties of the definite integral and making no further assumptions.

66) Use the identity $\ln(x)=\int_1^x \frac{dt}{t}$ to derive the identity $\ln\left(\frac{1}{x}\right)=-\ln x$.

Answer

We may assume that $x>1$,so $\frac{1}{x}<1$. Then, $\int_1^{1/x}\frac{dt}{t}$. Now make the substitution $u=\frac{1}{t}$, so $du=-\frac{dt}{t^2}$ and $\frac{du}{u}=-\frac{dt}{t}$, and change endpoints: $\int_1^{1/x}\frac{dt}{t}=-\int_1^x\frac{du}{u}=-\ln x$.

67) Use a change of variable in the integral $\int_1^{xy}\frac{1}{t}\,dt$ to show that $\ln xy=\ln x+\ln y$ for $x,y>0$.

68) Use the identity $\ln x=\int_1^x\frac{dt}{t}$ to show that $\ln(x)$ is an increasing function of x on $[0,\infty)$, and use the previous exercises to show that the range of $\ln(x)$ is $(-\infty,\infty)$. Without any further assumptions, conclude that $\ln(x)$ has an inverse function defined on $(-\infty,\infty)$.

69) Pretend, for the moment, that we do not know that e^x is the inverse function of $\ln(x)$, but keep in mind that $\ln(x)$ has an inverse function defined on $(-\infty,\infty)$. Call it E. Use the identity $\ln xy=\ln x+\ln y$ to deduce that $E(a+b)=E(a)E(b)$ for any real numbers a, b.

70) Pretend, for the moment, that we do not know that e^x is the inverse function of $\ln x$, but keep in mind that $\ln x$ has an inverse function defined on $(-\infty,\infty)$. Call it E. Show that $E'(t)=E(t)$.

Answer

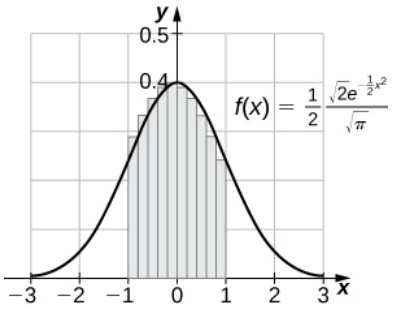
$x=E(\ln(x))$. Then, $1=\frac{E'(\ln x)}{x}$ or $x=E'(\ln x)$. Since any number t can be written $t=\ln x$ for some x, and for such t we have $x=E(t)$, it follows that for any t, $E'(t)=E(t)$.

71) The sine integral, defined as $S(x) = \int_0^x \frac{\sin t}{t}\, dt$ is an important quantity in engineering. Although it does not have a simple closed formula, it is possible to estimate its behavior for large x. Show that for $k \geq 1, \quad |S(2\pi k) - S(2\pi(k+1))| \leq \frac{1}{k(2k+1)\pi}$. (Hint: $\sin(t+\pi) = -\sin t$)

72) [T] The normal distribution in probability is given by $p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$, where σ is the standard deviation and μ is the average. The standard normal distribution in probability, p_s, corresponds to $\mu = 0$ and $\sigma = 1$. Compute the left endpoint estimates R_{10} and R_{100} of $\int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$.

Answer

$R_{10} = 0.6811, \quad R_{100} = 0.6827$



73) [T] Compute the right endpoint estimates R_{50} and R_{100} of $\int_{-3}^{5} \frac{1}{2\sqrt{2\pi}} e^{-(x-1)^2/8}$.

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- Paul Seeburger (Monroe Community College) added problems 47-48 to Section 5.6 exercises.

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5.7: Integrals Resulting in Inverse Trigonometric Functions

Learning Objectives

- Integrate functions resulting in inverse trigonometric functions

In this section we focus on integrals that result in inverse trigonometric functions. We have worked with these functions before. Recall, that trigonometric functions are not one-to-one unless the domains are restricted. When working with inverses of trigonometric functions, we always need to be careful to take these restrictions into account. Also, we previously developed formulas for derivatives of inverse trigonometric functions. The formulas developed there give rise directly to integration formulas involving inverse trigonometric functions.

Integrals that Result in Inverse Trigonometric Functions

Let us begin this last section of the chapter with the three formulas. Along with these formulas, we use substitution to evaluate the integrals. We prove the formula for the inverse sine integral.

Rule: Integration Formulas Resulting in Inverse Trigonometric Functions

The following integration formulas yield inverse trigonometric functions:

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C \tag{5.7.1}$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C \tag{5.7.2}$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{|u|}{a}\right) + C \tag{5.7.3}$$

Proof of the first formula

Let $y = \sin^{-1}\frac{x}{a}$. Then $a\sin y = x$. Now using implicit differentiation, we obtain

$$\frac{d}{dx}(a\sin y) = \frac{d}{dx}(x)$$

$$a\cos y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{a\cos y}.$$

For $-\frac{\pi}{2} \le y \le \frac{\pi}{2}, \cos y \ge 0$. Thus, applying the Pythagorean identity $\sin^2 y + \cos^2 y = 1$, we have $\cos y = \sqrt{1-\sin^2 y}$. This gives

$$\frac{1}{a\cos y} = \frac{1}{a\sqrt{1-\sin^2 y}} \tag{5.7.4}$$

$$= \frac{1}{\sqrt{a^2-a^2\sin^2 y}} \tag{5.7.5}$$

$$= \frac{1}{\sqrt{a^2-x^2}}. \tag{5.7.6}$$

Then for $-a \le x \le a$, we have

$$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\left(\frac{u}{a}\right) + C.$$

□

Example 5.7.1: Evaluating a Definite Integral Using Inverse Trigonometric Functions

Evaluate the definite integral

$$\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}}.$$

Solution

We can go directly to the formula for the antiderivative in the rule on integration formulas resulting in inverse trigonometric functions, and then evaluate the definite integral. We have

$$\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \Big|_0^{1/2} = \sin^{-1} \tfrac{1}{2} - \sin^{-1} 0 = \frac{\pi}{6} - 0 = \frac{\pi}{6}.$$

Note that since the integrand is simply the derivative of $\sin^{-1} x$, we are really just using this fact to find the antiderivative here.

Exercise 5.7.1

Find the indefinite integral using an inverse trigonometric function and substitution for $\int \frac{dx}{\sqrt{9-x^2}}$.

Hint

Use the formula in the rule on integration formulas resulting in inverse trigonometric functions.

Answer

$$\int \frac{dx}{\sqrt{9-x^2}} = \sin^{-1}\left(\frac{x}{3}\right) + C$$

In many integrals that result in inverse trigonometric functions in the antiderivative, we may need to use substitution to see how to use the integration formulas provided above.

Example 5.7.2: Finding an Antiderivative Involving an Inverse Trigonometric Function using substitution

Evaluate the integral

$$\int \frac{dx}{\sqrt{4-9x^2}}.$$

Solution

Substitute $u = 3x$. Then $du = 3\,dx$ and we have

$$\int \frac{dx}{\sqrt{4-9x^2}} = \frac{1}{3}\int \frac{du}{\sqrt{4-u^2}}.$$

Applying the formula with $a = 2$, we obtain

$$\int \frac{dx}{\sqrt{4-9x^2}} = \frac{1}{3}\int \frac{du}{\sqrt{4-u^2}} = \frac{1}{3}\sin^{-1}\left(\frac{u}{2}\right) + C = \frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right) + C.$$

Exercise 5.7.2

Find the antiderivative of $\int \frac{dx}{\sqrt{1-16x^2}}$.

Hint

Substitute $u = 4x$.

Answer

$$\int \frac{dx}{\sqrt{1-16x^2}} = \frac{1}{4}\sin^{-1}(4x) + C$$

Example 5.7.3: Evaluating a Definite Integral

Evaluate the definite integral

$$\int_0^{\sqrt{3}/2} \frac{du}{\sqrt{1-u^2}}.$$

Solution

The format of the problem matches the inverse sine formula. Thus,

$$\int_0^{\sqrt{3}/2} \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u \Big|_0^{\sqrt{3}/2} = [\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)] - [\sin^{-1}(0)] = \frac{\pi}{3}.$$

Integrals Resulting in Other Inverse Trigonometric Functions

There are six inverse trigonometric functions. However, only three integration formulas are noted in the rule on integration formulas resulting in inverse trigonometric functions because the remaining three are negative versions of the ones we use. The only difference is whether the integrand is positive or negative. Rather than memorizing three more formulas, if the integrand is negative, simply factor out −1 and evaluate the integral using one of the formulas already provided. To close this section, we examine one more formula: the integral resulting in the inverse tangent function.

Example 5.7.4: Finding an Antiderivative Involving the Inverse Tangent Function

Find the antiderivative of $\int \frac{1}{9+x^2}\,dx$.

Solution

Apply the formula with $a = 3$. Then,

$$\int \frac{dx}{9+x^2} = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C.$$

Exercise 5.7.3

Find the antiderivative of $\int \frac{dx}{16+x^2}$.

Hint

Follow the steps in Example 5.7.4.

Answer

$$\int \frac{dx}{16+x^2} = \frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right) + C$$

Example 5.7.5: Applying the Integration Formulas WITH SUBSTITUTION

Find an antiderivative of $\int \frac{1}{1+4x^2}\,dx$.

Solution

Comparing this problem with the formulas stated in the rule on integration formulas resulting in inverse trigonometric functions, the integrand looks similar to the formula for $\tan^{-1} u + C$. So we use substitution, letting $u = 2x$, then $du = 2\,dx$ and $\frac{1}{2}\,du = dx$.Then, we have

$$\frac{1}{2}\int \frac{1}{\quad} du = \frac{1}{2}\tan^{-1} u + C = \frac{1}{2}\tan^{-1}(2x) + C$$

$$2\int \frac{du}{1+u^2} \qquad 2\tan^{-1} u + C \qquad 2\tan^{-1}(2x) + C.$$

Exercise 5.7.4

Use substitution to find the antiderivative of $\int \frac{dx}{25+4x^2}$.

Hint

Use the solving strategy from Example 5.7.5 and the rule on integration formulas resulting in inverse trigonometric functions.

Answer

$$\int \frac{dx}{25+4x^2} = \frac{1}{10}\tan^{-1}\left(\frac{2x}{5}\right) + C$$

Example 5.7.6: Evaluating a Definite Integral

Evaluate the definite integral $\int_{\sqrt{3}/3}^{\sqrt{3}} \frac{dx}{1+x^2}$.

Solution

Use the formula for the inverse tangent. We have

$$\int_{\sqrt{3}/3}^{\sqrt{3}} \frac{dx}{1+x^2} = \tan^{-1} x \Big|_{\sqrt{3}/3}^{\sqrt{3}} = [\tan^{-1}(\sqrt{3})] - [\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)] = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}.$$

Exercise 5.7.5

Evaluate the definite integral $\int_0^2 \frac{dx}{4+x^2}$.

Hint

Follow the procedures from Example 5.7.6 to solve the problem.

Answer

$$\int_0^2 \frac{dx}{4+x^2} = \frac{\pi}{8}$$

Key Concepts

- Formulas for derivatives of inverse trigonometric functions developed in Derivatives of Exponential and Logarithmic Functions lead directly to integration formulas involving inverse trigonometric functions.
- Use the formulas listed in the rule on integration formulas resulting in inverse trigonometric functions to match up the correct format and make alterations as necessary to solve the problem.
- Substitution is often required to put the integrand in the correct form.

Key Equations

- **Integrals That Produce Inverse Trigonometric Functions**

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{|u|}{a}\right) + C$$

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- Includes some added textual clarifications and edits by Paul Seeburger (Monroe Community College)

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5.7E: EXERCISES FOR SECTION 5.7

In exercises 1 - 6, evaluate each integral in terms of an inverse trigonometric function.

1) $\int_0^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}}$

Answer

$$\int_0^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x\Big|_0^{\sqrt{3}/2} = \frac{\pi}{3}$$

2) $\int_{-1/2}^{1/2} \frac{dx}{\sqrt{1-x^2}}$

3) $\int_{\sqrt{3}}^{1} \frac{dx}{\sqrt{1+x^2}}$

Answer

$$\int_{\sqrt{3}}^{1} \frac{dx}{\sqrt{1+x^2}} = \tan^{-1} x\Big|_{\sqrt{3}}^{1} = -\frac{\pi}{12}$$

4) $\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{dx}{1+x^2}$

5) $\int_1^{\sqrt{2}} \frac{dx}{|x|\sqrt{x^2-1}}$

Answer

$$\int_1^{\sqrt{2}} \frac{dx}{|x|\sqrt{x^2-1}} = \sec^{-1} x\Big|_1^{\sqrt{2}} = \frac{\pi}{4}$$

6) $\int_1^{\frac{2}{\sqrt{3}}} \frac{dx}{|x|\sqrt{x^2-1}}$

In exercises 7 - 12, find each indefinite integral, using appropriate substitutions.

7) $\int \frac{dx}{\sqrt{9-x^2}}$

Answer

$$\int \frac{dx}{\sqrt{9-x^2}} = \sin^{-1}\left(\frac{x}{3}\right) + C$$

8) $\int \frac{dx}{\sqrt{1-16x^2}}$

9) $\int \frac{dx}{9+x^2}$

Answer

$$\int \frac{dx}{9+x^2} = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$$

10) $\int \frac{dx}{25+16x^2}$

11) $\int \frac{dx}{x\sqrt{x^2-9}}$

Answer

$$\int \frac{dx}{x\sqrt{x^2-9}} = \frac{1}{3}\sec^{-1}\left(\frac{|x|}{3}\right) + C$$

12) $\int \frac{dx}{x\sqrt{4x^2-16}}$

13) Explain the relationship $-\cos^{-1} t + C = \int \frac{dt}{\sqrt{1-t^2}} = \sin^{-1} t + C$. Is it true, in general, that $\cos^{-1} t = -\sin^{-1} t$?

Answer

$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$. So, $\sin^{-1}t=\frac{\pi}{2}-\cos^{-1}t$. They differ by a constant.

14) Explain the relationship $\sec^{-1}t+C=\int\frac{dt}{|t|\sqrt{t^2-1}}=-\csc^{-1}t+C$. Is it true, in general, that $\sec^{-1}t=-\csc^{-1}t$?

15) Explain what is wrong with the following integral: $\int_1^2\frac{dt}{\sqrt{1-t^2}}$.

Answer

$\sqrt{1-t^2}$ is not defined as a real number when $t>1$.

16) Explain what is wrong with the following integral: $\int_{-1}^1\frac{dt}{|t|\sqrt{t^2-1}}$.

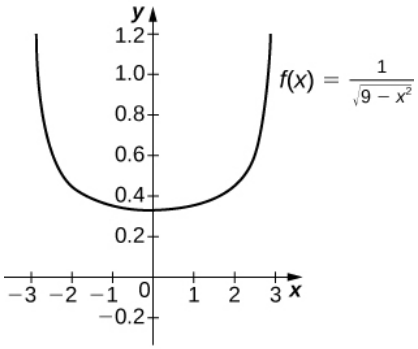
Answer

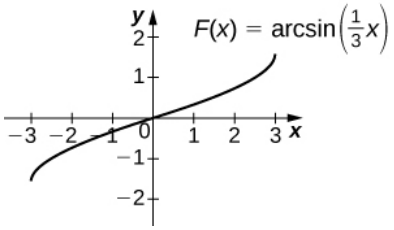
$\sqrt{t^2-1}$ is not defined as a real number when $-1<t<1$, and the integrand is undefined when $t=-1$ or $t=1$.

In exercises 17 - 20, solve for the antiderivative of f with $C=0$, then use a calculator to graph f and the antiderivative over the given interval $[a,b]$. Identify a value of C such that adding C to the antiderivative recovers the definite integral $F(x)=\int_a^x f(t)\,dt$.

17) [T] $\int\frac{1}{\sqrt{9-x^2}}\,dx$ over $[-3,3]$

Answer



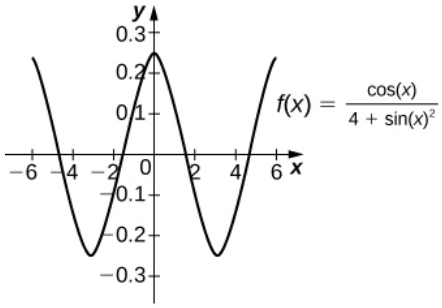


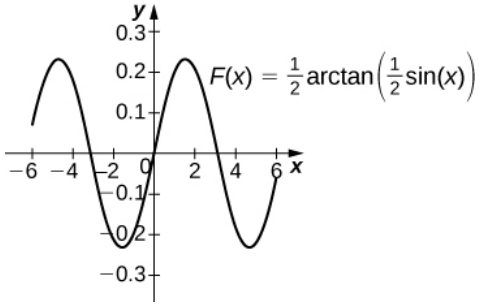
The antiderivative is $\sin^{-1}\left(\frac{x}{3}\right)+C$. Taking $C=\frac{\pi}{2}$ recovers the definite integral.

18) [T] $\int\frac{9}{9+x^2}\,dx$ over $[-6,6]$

19) [T] $\int\frac{\cos x}{4+\sin^2 x}\,dx$ over $[-6,6]$

Answer





The antiderivative is $\frac{1}{2}\tan^{-1}(\frac{\sin x}{2})+C$. Taking $C=\frac{1}{2}\tan^{-1}(\frac{\sin(6)}{2})$ recovers the definite integral.

20) [T] $\int \frac{e^x}{1+e^{2x}}\,dx$ over $[-6,6]$

In exercises 21 - 26, compute the antiderivative using appropriate substitutions.

21) $\int \frac{\sin^{-1} t}{\sqrt{1-t^2}}\,dt$

Answer

$$\int \frac{\sin^{-1} t\,dt}{\sqrt{1-t^2}} \quad = \quad \tfrac{1}{2}(\sin^{-1} t)^2+C$$

22) $\int \frac{dt}{\sin^{-1} t\sqrt{1-t^2}}$

23) $\int \frac{\tan^{-1}(2t)}{1+4t^2}\,dt$

Answer

$$\int \frac{\tan^{-1}(2t)}{1+4t^2}\,dt \quad = \quad \frac{1}{4}(\tan^{-1}(2t))^2+C$$

24) $\int \frac{t\tan^{-1}(t^2)}{1+t^4}\,dt$

25) $\int \frac{\sec^{-1}(\frac{t}{2})}{|t|\sqrt{t^2-4}}\,dt$

Answer

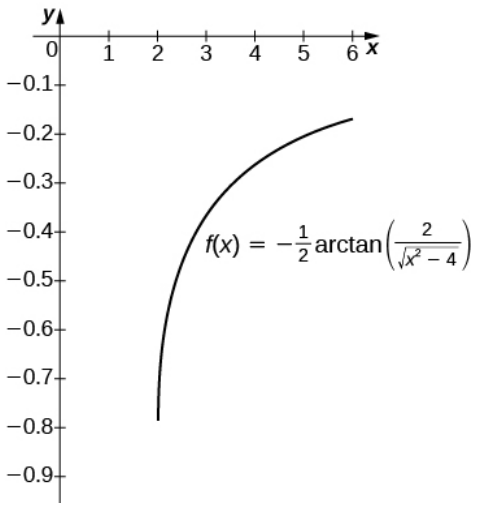
$$\int \frac{\sec^{-1}(\frac{t}{2})}{|t|\sqrt{t^2-4}}\,dt \quad = \quad \tfrac{1}{4}(\sec^{-1}(\tfrac{t}{2}))^2+C$$

26) $\int \frac{t\sec^{-1}(t^2)}{t^2\sqrt{t^4-1}}\,dt$

In exercises 27 - 32, use a calculator to graph the antiderivative of f with $C=0$ over the given interval $[a,b]$. Approximate a value of C, if possible, such that adding C to the antiderivative gives the same value as the definite integral $F(x)=\int_a^x f(t)\,dt$.

27) [T] $\int \frac{1}{x\sqrt{x^2-4}}\,dx$ over $[2,6]$

Answer

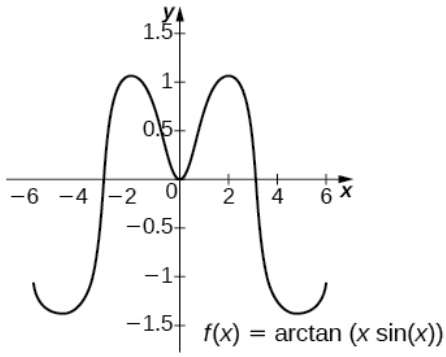


The antiderivative is $\frac{1}{2}\sec^{-1}(\frac{x}{2}) + C$. Taking $C = 0$ recovers the definite integral over $[2, 6]$.

28) [T] $\int \frac{1}{(2x+2)\sqrt{x}}\,dx$ over $[0, 6]$

29) [T] $\int \frac{(\sin x + x\cos x)}{1 + x^2\sin^2 x\,dx}$ over $[-6, 6]$

Answer

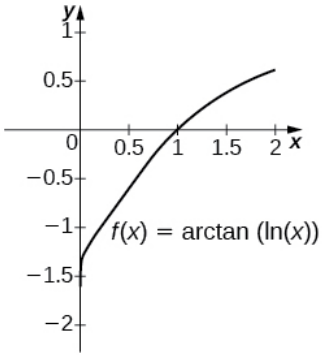


The general antiderivative is $\tan^{-1}(x\sin x) + C$. Taking $C = -\tan^{-1}(6\sin(6))$ recovers the definite integral.

30) [T] $\int \frac{2e^{-2x}}{\sqrt{1-e^{-4x}}}\,dx$ over $[0, 2]$

31) [T] $\int \frac{1}{x + x\ln 2x}$ over $[0, 2]$

Answer



The general antiderivative is $\tan^{-1}(\ln x) + C$. Taking $C = \frac{\pi}{2} = \lim_{t\to\infty} \tan^{-1} t$ recovers the definite integral.

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32) [T] $\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}$ over $[-1,1]$

In exercises 33 - 38, compute each integral using appropriate substitutions.

33) $\int \frac{e^t}{\sqrt{1-e^{2t}}}\,dt$

Answer

$$\int \frac{e^t}{\sqrt{1-e^{2t}}}\,dt = \sin^{-1}(e^t)+C$$

34) $\int \frac{e^t}{1+e^{2t}}\,dt$

35) $\int \frac{dt}{t\sqrt{1-\ln^2 t}}$

Answer

$$\int \frac{dt}{t\sqrt{1-\ln^2 t}} = \sin^{-1}(\ln t)+C$$

36) $\int \frac{dt}{t(1+\ln^2 t)}$

37) $\int \frac{\cos^{-1}(2t)}{\sqrt{1-4t^2}}\,dt$

Answer

$$\int \frac{\cos^{-1}(2t)}{\sqrt{1-4t^2}}\,dt = -\frac{1}{2}(\cos^{-1}(2t))^2+C$$

38) $\int \frac{e^t\cos^{-1}(e^t)}{\sqrt{1-e^{2t}}}\,dt$

In exercises 39 - 42, compute each definite integral.

39) $\int_0^{1/2} \frac{\tan(\sin^{-1} t)}{\sqrt{1-t^2}}\,dt$

Answer

$$\int_0^{1/2} \frac{\tan(\sin^{-1} t)}{\sqrt{1-t^2}}\,dt = \frac{1}{2}\ln\left(\frac{4}{3}\right)$$

40) $\int_{1/4}^{1/2} \frac{\tan(\cos^{-1} t)}{\sqrt{1-t^2}}\,dt$

41) $\int_0^{1/2} \frac{\sin(\tan^{-1} t)}{1+t^2}\,dt$

Answer

$$\int_0^{1/2} \frac{\sin(\tan^{-1} t)}{1+t^2}\,dt = 1-\frac{2}{\sqrt{5}}$$

42) $\int_0^{1/2} \frac{\cos(\tan^{-1} t)}{1+t^2}\,dt$

43) For $A>0$, compute $I(A)=\int_{-A}^{A} \frac{dt}{1+t^2}$ and evaluate $\lim_{a\to\infty} I(A)$, the area under the graph of $\frac{1}{1+t^2}$ on $[-\infty,\infty]$.

Answer

$2\tan^{-1}(A)\to\pi$ as $A\to\infty$

44) For $1<B<\infty$, compute $I(B)=\int_1^B \frac{dt}{t\sqrt{t^2-1}}$ and evaluate $\lim_{B\to\infty} I(B)$, the area under the graph of $\frac{1}{t\sqrt{t^2-1}}$ over $[1,\infty)$.

45) Use the substitution $u=\sqrt{2}\cot x$ and the identity $1+\cot^2 x=\csc^2 x$ to evaluate $\int \frac{dx}{1+\cos^2 x}$. (Hint: Multiply the top and bottom of the integrand by $\csc^2 x$.)

Answer

Using the hint, one has $\int \frac{\csc^2 x}{\csc^2 x + \cot^2 x}\,dx = \int \frac{\csc^2 x}{1 + 2\cot^2 x}\,dx$. Set $u = \sqrt{2}\cot x$. Then, $du = -\sqrt{2}\csc^2 x$ and the integral is $-\frac{1}{\sqrt{2}}\int \frac{du}{1+u^2} = -\frac{\sqrt{2}}{2}\tan^{-1} u + C = \frac{\sqrt{2}}{2}\tan^{-1}(\sqrt{2}\cot x) + C$. If one uses the identity $\tan^{-1} s + \tan^{-1}(\frac{1}{s}) = \frac{\pi}{2}$, then this can also be written $\frac{\sqrt{2}}{2}\tan^{-1}(\frac{\tan x}{\sqrt{2}}) + C$.

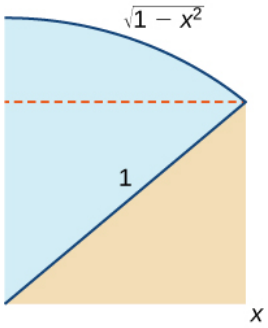
46) [T] Approximate the points at which the graphs of $f(x) = 2x^2 - 1$ and $g(x) = (1 + 4x^2)^{-3/2}$ intersect, and approximate the area between their graphs accurate to three decimal places.

47) [T] Approximate the points at which the graphs of $f(x) = x^2 - 1$ and $f(x) = x^2 - 1$ intersect, and approximate the area between their graphs accurate to three decimal places.

Answer

$x \approx \pm 1.13525$. The left endpoint estimate with $N = 100$ is 2.796 and these decimals persist for $N = 500$.

48) Use the following graph to prove that $\int_0^x \sqrt{1 - t^2}\,dt = \frac{1}{2}x\sqrt{1 - x^2} + \frac{1}{2}\sin^{-1} x$.



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5.7: Integrals Resulting in Inverse Trigonometric Functions

Learning Objectives

- Integrate functions resulting in inverse trigonometric functions

In this section we focus on integrals that result in inverse trigonometric functions. We have worked with these functions before. Recall, that trigonometric functions are not one-to-one unless the domains are restricted. When working with inverses of trigonometric functions, we always need to be careful to take these restrictions into account. Also, we previously developed formulas for derivatives of inverse trigonometric functions. The formulas developed there give rise directly to integration formulas involving inverse trigonometric functions.

Integrals that Result in Inverse Trigonometric Functions

Let us begin this last section of the chapter with the three formulas. Along with these formulas, we use substitution to evaluate the integrals. We prove the formula for the inverse sine integral.

Rule: Integration Formulas Resulting in Inverse Trigonometric Functions

The following integration formulas yield inverse trigonometric functions:

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C \tag{5.7.1}$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C \tag{5.7.2}$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{|u|}{a}\right) + C \tag{5.7.3}$$

Proof of the first formula

Let $y = \sin^{-1}\frac{x}{a}$. Then $a\sin y = x$. Now using implicit differentiation, we obtain

$$\frac{d}{dx}(a\sin y) = \frac{d}{dx}(x)$$

$$a\cos y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{a\cos y}.$$

For $-\frac{\pi}{2} \le y \le \frac{\pi}{2}, \cos y \ge 0$. Thus, applying the Pythagorean identity $\sin^2 y + \cos^2 y = 1$, we have $\cos y = \sqrt{1-\sin^2 y}$. This gives

$$\frac{1}{a\cos y} = \frac{1}{a\sqrt{1-\sin^2 y}} \tag{5.7.4}$$

$$= \frac{1}{\sqrt{a^2-a^2\sin^2 y}} \tag{5.7.5}$$

$$= \frac{1}{\sqrt{a^2-x^2}}. \tag{5.7.6}$$

Then for $-a \le x \le a$, we have

$$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\left(\frac{u}{a}\right) + C.$$

□

Example 5.7.1: Evaluating a Definite Integral Using Inverse Trigonometric Functions

Evaluate the definite integral

$$\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}}.$$

Solution

We can go directly to the formula for the antiderivative in the rule on integration formulas resulting in inverse trigonometric functions, and then evaluate the definite integral. We have

$$\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \Big|_0^{1/2} = \sin^{-1} \tfrac{1}{2} - \sin^{-1} 0 = \frac{\pi}{6} - 0 = \frac{\pi}{6}.$$

Note that since the integrand is simply the derivative of $\sin^{-1} x$, we are really just using this fact to find the antiderivative here.

Exercise 5.7.1

Find the indefinite integral using an inverse trigonometric function and substitution for $\int \frac{dx}{\sqrt{9-x^2}}$.

Hint

Use the formula in the rule on integration formulas resulting in inverse trigonometric functions.

Answer

$$\int \frac{dx}{\sqrt{9-x^2}} = \sin^{-1}\left(\frac{x}{3}\right) + C$$

In many integrals that result in inverse trigonometric functions in the antiderivative, we may need to use substitution to see how to use the integration formulas provided above.

Example 5.7.2: Finding an Antiderivative Involving an Inverse Trigonometric Function using substitution

Evaluate the integral

$$\int \frac{dx}{\sqrt{4-9x^2}}.$$

Solution

Substitute $u = 3x$. Then $du = 3\,dx$ and we have

$$\int \frac{dx}{\sqrt{4-9x^2}} = \frac{1}{3}\int \frac{du}{\sqrt{4-u^2}}.$$

Applying the formula with $a = 2$, we obtain

$$\int \frac{dx}{\sqrt{4-9x^2}} = \frac{1}{3}\int \frac{du}{\sqrt{4-u^2}} = \frac{1}{3}\sin^{-1}\left(\frac{u}{2}\right) + C = \frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right) + C.$$

Exercise 5.7.2

Find the antiderivative of $\int \frac{dx}{\sqrt{1-16x^2}}$.

Hint

Substitute $u = 4x$.

Answer

$$\int \frac{dx}{\sqrt{1-16x^2}} = \frac{1}{4}\sin^{-1}(4x) + C$$

Example 5.7.3: Evaluating a Definite Integral

Evaluate the definite integral

$$\int_0^{\sqrt{3}/2} \frac{du}{\sqrt{1-u^2}}.$$

Solution

The format of the problem matches the inverse sine formula. Thus,

$$\int_0^{\sqrt{3}/2} \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u \Big|_0^{\sqrt{3}/2} = [\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)] - [\sin^{-1}(0)] = \frac{\pi}{3}.$$

Integrals Resulting in Other Inverse Trigonometric Functions

There are six inverse trigonometric functions. However, only three integration formulas are noted in the rule on integration formulas resulting in inverse trigonometric functions because the remaining three are negative versions of the ones we use. The only difference is whether the integrand is positive or negative. Rather than memorizing three more formulas, if the integrand is negative, simply factor out −1 and evaluate the integral using one of the formulas already provided. To close this section, we examine one more formula: the integral resulting in the inverse tangent function.

Example 5.7.4: Finding an Antiderivative Involving the Inverse Tangent Function

Find the antiderivative of $\int \frac{1}{9+x^2}\,dx$.

Solution

Apply the formula with $a = 3$. Then,

$$\int \frac{dx}{9+x^2} = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C.$$

Exercise 5.7.3

Find the antiderivative of $\int \frac{dx}{16+x^2}$.

Hint

Follow the steps in Example 5.7.4.

Answer

$$\int \frac{dx}{16+x^2} = \frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right) + C$$

Example 5.7.5: Applying the Integration Formulas WITH SUBSTITUTION

Find an antiderivative of $\int \frac{1}{1+4x^2}\,dx$.

Solution

Comparing this problem with the formulas stated in the rule on integration formulas resulting in inverse trigonometric functions, the integrand looks similar to the formula for $\tan^{-1} u + C$. So we use substitution, letting $u = 2x$, then $du = 2\,dx$ and $\frac{1}{2}\,du = dx$.Then, we have

$$\frac{1}{2}\int \frac{1}{\quad} du = \frac{1}{2}\tan^{-1} u + C = \frac{1}{2}\tan^{-1}(2x) + C$$

$2\int \frac{du}{1+u^2} \quad 2\tan^{-1} u + C \quad 2\tan^{-1}(2x) + C.$

Exercise 5.7.4

Use substitution to find the antiderivative of $\int \frac{dx}{25+4x^2}$.

Hint

Use the solving strategy from Example 5.7.5 and the rule on integration formulas resulting in inverse trigonometric functions.

Answer

$$\int \frac{dx}{25+4x^2} = \frac{1}{10}\tan^{-1}\left(\frac{2x}{5}\right) + C$$

Example 5.7.6: Evaluating a Definite Integral

Evaluate the definite integral $\int_{\sqrt{3}/3}^{\sqrt{3}} \frac{dx}{1+x^2}$.

Solution

Use the formula for the inverse tangent. We have

$$\int_{\sqrt{3}/3}^{\sqrt{3}} \frac{dx}{1+x^2} = \tan^{-1} x \Big|_{\sqrt{3}/3}^{\sqrt{3}} = [\tan^{-1}(\sqrt{3})] - [\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)] = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}.$$

Exercise 5.7.5

Evaluate the definite integral $\int_0^2 \frac{dx}{4+x^2}$.

Hint

Follow the procedures from Example 5.7.6 to solve the problem.

Answer

$$\int_0^2 \frac{dx}{4+x^2} = \frac{\pi}{8}$$

Key Concepts

- Formulas for derivatives of inverse trigonometric functions developed in Derivatives of Exponential and Logarithmic Functions lead directly to integration formulas involving inverse trigonometric functions.
- Use the formulas listed in the rule on integration formulas resulting in inverse trigonometric functions to match up the correct format and make alterations as necessary to solve the problem.
- Substitution is often required to put the integrand in the correct form.

Key Equations

- **Integrals That Produce Inverse Trigonometric Functions**

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$
$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$
$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{|u|}{a}\right) + C$$

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- Includes some added textual clarifications and edits by Paul Seeburger (Monroe Community College)

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5.R: CHAPTER 5 REVIEW EXERCISES

In exercises 1 - 4, answer *True or False.* Justify your answer with a proof or a counterexample. Assume all functions f and g are continuous over their domains.

1) If $f(x) > 0,\ f'(x) > 0$ for all x, then the right-hand rule underestimates the integral $\int_a^b f(x)\,dx$. Use a graph to justify your answer.

Answer

False

2) $\int_a^b f(x)^2\,dx = \int_a^b f(x)\,dx$

3) If $f(x) \le g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

Answer

True

4) All continuous functions have an antiderivative.

In exercises 5 - 8, evaluate the Riemann sums L_4 and R_4 for the given functions over the specified interval. Compare your answer with the exact answer, when possible, or use a calculator to determine the answer.

5) $y = 3x^2 - 2x + 1)$ over $[-1, 1]$

Answer

$L_4 = 5.25,\ R_4 = 3.25,$ exact answer: 4

6) $y = \ln(x^2 + 1)$ over $[0, e]$

7) $y = x^2 \sin x$ over $[0, \pi]$

Answer

$L_4 = 5.364,\ R_4 = 5.364,$ exact answer: 5.870

8) $y = \sqrt{x} + \frac{1}{x}$ over $[1, 4]$

In exercises 9 - 12, evaluate the integrals.

9) $\int_{-1}^{1} (x^3 - 2x^2 + 4x)\,dx$

Answer

$-\frac{4}{3}$

10) $\int_0^4 \frac{3t}{\sqrt{1 + 6t^2}}\,dt$

11) $\int_{\pi/3}^{\pi/2} 2\sec(2\theta)\tan(2\theta)\,d\theta$

Answer

1

12) $\int_0^{\pi/4} e^{\cos^2 x} \sin x \cos x\,dx$

In exercises 13 - 16, find the antiderivative.

13) $\int \frac{dx}{(x+4)^3}$

Answer

$-\frac{1}{2(x+4)^2} + C$

14) $\int x \ln(x^2)\,dx$

15) $\int \frac{4x^2}{\sqrt{1-x^6}}\,dx$

Answer

$\frac{4}{3}\sin^{-1}(x^3)+C$

16) $\int \frac{e^{2x}}{1+e^{4x}}\,dx$

In exercises 17 - 20, find the derivative.

17) $\frac{d}{dt}\int_0^t \frac{\sin x}{\sqrt{1+x^2}}\,dx$

Answer

$\frac{\sin t}{\sqrt{1+t^2}}$

18) $\frac{d}{dx}\int_1^{x^3} \sqrt{4-t^2}\,dt$

19) $\frac{d}{dx}\int_1^{\ln(x)} (4t+e^t)\,dt$

Answer

$4\frac{\ln x}{x}+1$

20) $\frac{d}{dx}\int_0^{\cos x} e^{t^2}\,dt$

In exercises 21 - 23, consider the historic average cost per gigabyte of RAM on a computer.

Year	5-Year Change ($)
1980	0
1985	−5,468,750
1990	−755,495
1995	−73,005
2000	−29,768
2005	−918
2010	−177

21) If the average cost per gigabyte of RAM in 2010 is $12, find the average cost per gigabyte of RAM in 1980.

Answer

$6,328,113

Solution: $6,328,113

22) The average cost per gigabyte of RAM can be approximated by the function $C(t) = 8,500,000(0.65)^t$, where t is measured in years since 1980, and C is cost in US dollars. Find the average cost per gigabyte of RAM for the period from 1980 to 2010.

23) Find the average cost of 1 GB RAM from 2005 to 2010.

Answer

$73.36

24) The velocity of a bullet from a rifle can be approximated by $v(t) = 6400t^2 - 6505t + 2686$, where t is seconds after the shot and v is the velocity measured in feet per second. This equation only models the velocity for the first half-second after the shot: $0 \le t \le 0.5$. What is the total distance the bullet travels in 0.5 sec?

25) What is the average velocity of the bullet for the first half-second?

Answer

$\frac{19117}{12}$ ft/sec, or about 1593 ft/sec

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CHAPTER OVERVIEW

6: Applications of Integration

In this chapter, we use definite integrals to calculate the force exerted on the dam when the reservoir is full and we examine how changing water levels affect that force. Hydrostatic force is only one of the many applications of definite integrals we explore in this chapter. From geometric applications such as surface area and volume, to physical applications such as mass and work, to growth and decay models, definite integrals are a powerful tool to help us understand and model the world around us.

6.0: Prelude to Applications of Integration
6.1: Areas between Curves
6.1E: Exercises for Section 6.1
6.2: Determining Volumes by Slicing
6.2E: Exercises for Section 6.2
6.3: Volumes of Revolution - Cylindrical Shells
6.3E: Exercises for Section 6.3
6.4: Arc Length of a Curve and Surface Area
6.4E: Exercises for Section 6.4
6.5: Physical Applications of Integration
6.5E: Exercises for Section 6.5
6.6: Moments and Centers of Mass
6.6E: Exercises for Section 6.6
6.7: Integrals, Exponential Functions, and Logarithms
6.7E: Exercises for Section 6.7
6.8: Exponential Growth and Decay
6.8E: Exercises for Section 6.8
6.9: Calculus of the Hyperbolic Functions
6.9E: Exercises for Section 6.9
6.R: Chapter 6 Review Exercises

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6.0: Prelude to Applications of Integration

The Hoover Dam is an engineering marvel. When Lake Mead, the reservoir behind the dam, is full, the dam withstands a great deal of force. However, water levels in the lake vary considerably as a result of droughts and varying water demands. Later in this chapter, we use definite integrals to calculate the force exerted on the dam when the reservoir is full and we examine how changing water levels affect that force. Hydrostatic force is only one of the many applications of definite integrals we explore in this chapter. From geometric applications such as surface area and volume, to physical applications such as mass and work, to growth and decay models, definite integrals are a powerful tool to help us understand and model the world around us.

Figure 6.0.1: Hoover Dam is one of the United States' iconic landmarks, and provides irrigation and hydroelectric power for millions of people in the southwest United States. (credit: modification of work by Lynn Betts, Wikimedia).

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6.1: Areas between Curves

Learning Objectives

- Determine the area of a region between two curves by integrating with respect to the independent variable.
- Find the area of a compound region.
- Determine the area of a region between two curves by integrating with respect to the dependent variable.

In Introduction to Integration, we developed the concept of the definite integral to calculate the area below a curve on a given interval. In this section, we expand that idea to calculate the area of more complex regions. We start by finding the area between two curves that are functions of x, beginning with the simple case in which one function value is always greater than the other. We then look at cases when the graphs of the functions cross. Last, we consider how to calculate the area between two curves that are functions of y.

Area of a Region between Two Curves

Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a, b]$ such that $f(x) \geq g(x)$ on $[a, b]$. We want to find the area between the graphs of the functions, as shown in Figure 6.1.1.

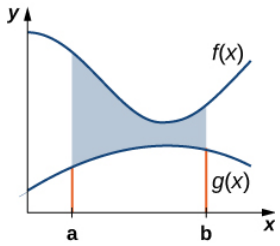


Figure 6.1.1: The area between the graphs of two functions, $f(x)$ and $g(x)$, on the interval $[a, b]$

As we did before, we are going to partition the interval on the **x-axis** and approximate the area between the graphs of the functions with rectangles. So, for $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of $[a, b]$. Then, for $i = 1, 2, \ldots, n$, choose a point x_i^* $\in [x_{i-1}, x_i]$, and on each interval $[x_{i-1}, x_i]$ construct a rectangle that extends vertically from $g(x_i^*)$ to $f(x_i^*)$. Figure 6.1.2a shows the rectangles when x_i^* is selected to be the left endpoint of the interval and $n = 10$. Figure 6.1.2b shows a representative rectangle in detail.

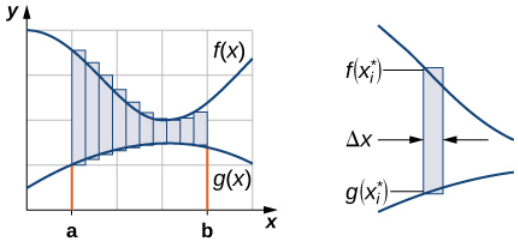


Figure 6.1.2: (a)We can approximate the area between the graphs of two functions, $f(x)$ and $g(x)$, with rectangles. (b) The area of a typical rectangle goes from one curve to the other.

The height of each individual rectangle is $f(x_i^*) - g(x_i^*)$ and the width of each rectangle is Δx. Adding the areas of all the rectangles, we see that the area between the curves is approximated by

$$A \approx \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)] \Delta x.$$

This is a Riemann sum, so we take the limit as $n \to \infty$ and we get

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)] \Delta x = \int_a^b [f(x) - g(x)] dx.$$

These findings are summarized in the following theorem.

Finding the Area between Two Curves

Let $f(x)$ and $g(x)$ be continuous functions such that $f(x) \geq g(x)$ over an interval $[a, b]$. Let R denote the region bounded above by the graph of $f(x)$, below by the graph of $g(x)$, and on the left and right by the lines $x = a$ and $x = b$, respectively. Then, the area of R is given by

$$A = \int_a^b [f(x) - g(x)]dx.$$

We apply this theorem in the following example.

Example 6.1.1: Finding the Area of a Region between Two Curves I

If $\mathbf{R}$ is the region bounded above by the graph of the function $f(x) = x + 4$ and below by the graph of the function $g(x) = 3 - \frac{x}{2}$ over the interval $[1, 4]$, find the area of region $\mathbf{R}$.

Solution

The region is depicted in the following figure.

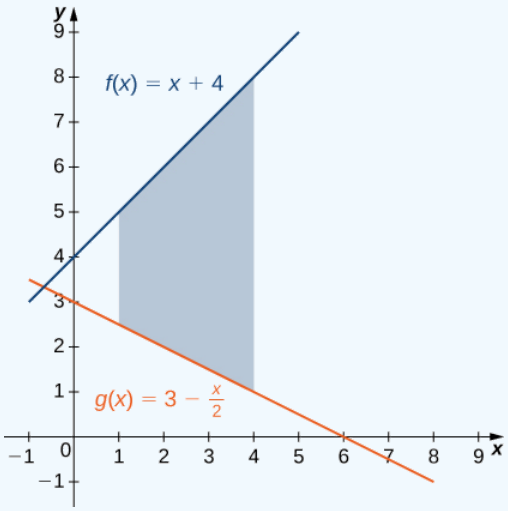


Figure 6.1.3: A region between two curves is shown where one curve is always greater than the other.

We have

$$A = \int_a^b [f(x) - g(x)]\, dx$$

$$= \int_1^4 [(x + 4) - (3 - \frac{x}{2})]\, dx = \int_1^4 \left[\frac{3x}{2} + 1\right]\, dx$$

$$= [\frac{3x^2}{4} + x]\Big|_1^4 = (16 - \frac{7}{4}) = \frac{57}{4}.$$

The area of the region is $\frac{57}{4} units^2$.

Exercise 6.1.1

If $\mathbf{R}$ is the region bounded by the graphs of the functions $f(x) = \frac{x}{2} + 5$ and $g(x) = x + \frac{1}{2}$ over the interval $[1, 5]$, find the area of region $\mathbf{R}$.

Hint

Graph the functions to determine which function's graph forms the upper bound and which forms the lower bound, then follow the process used in Example.

Answer

12 units^2

In Example 6.1.1, we defined the interval of interest as part of the problem statement. Quite often, though, we want to define our interval of interest based on where the graphs of the two functions intersect. This is illustrated in the following example.

Example 6.1.2: Finding the Area of a Region between Two Curves II

If $\mathbf{R}$ is the region bounded above by the graph of the function $f(x) = 9 - (x/2)^2$ and below by the graph of the function $g(x) = 6 - x$, find the area of region $\mathbf{R}$.

Solution

The region is depicted in the following figure.

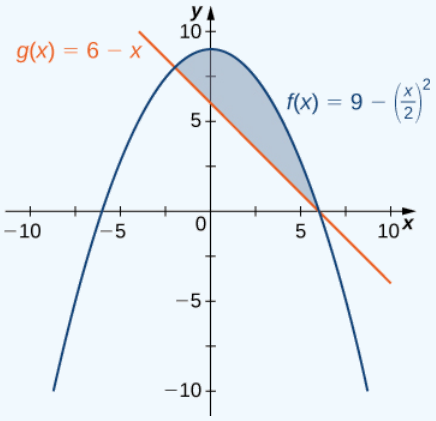


Figure 6.1.4: This graph shows the region below the graph of $f(x)$ and above the graph of $g(x)$.

We first need to compute where the graphs of the functions intersect. Setting $f(x) = g(x)$, we get

$$\begin{aligned} f(x) &= g(x) \\ 9 - (\frac{x}{2})^2 &= 6 - x \\ 9 - \frac{x^2}{4} &= 6 - x \\ 36 - x^2 &= 24 - 4x \\ x^2 - 4x - 12 &= 0 \\ (x-6)(x+2) &= 0. \end{aligned}$$

The graphs of the functions intersect when $x = 6$ or $x = -2$, so we want to integrate from -2 to 6. Since $f(x) \geq g(x)$ for $-2 \leq x \leq 6$, we obtain

$$\begin{aligned} A &= \int_a^b [f(x) - g(x)]\, dx \\ &= \int_{-2}^{6} \left[9 - (\frac{x}{2})^2 - (6 - x)\right] dx \\ &= \int_{-2}^{6} \left[3 - \frac{x^2}{4} + x\right] dx \\ &= \left[3x - \frac{x^3}{12} + \frac{x^2}{2}\right]\Big|_{-2}^{6} = \frac{64}{3}. \end{aligned}$$

The area of the region is $64/3$ units^2.

Exercise 6.1.2

If $\mathbf{R}$ is the region bounded above by the graph of the function $f(x) = x$ and below by the graph of the function $g(x) = x^4$, find the area of region $\mathbf{R}$.

Hint

Use the process from Example 6.1.2.

Answer

$\frac{3}{10}$ unit^2

Areas of Compound Regions

So far, we have required $f(x) \geq g(x)$ over the entire interval of interest, but what if we want to look at regions bounded by the graphs of functions that cross one another? In that case, we modify the process we just developed by using the absolute value function.

Finding the Area of a Region between Curves That Cross

Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a, b]$. Let $\mathbf{R}$ denote the region between the graphs of $f(x)$ and $g(x)$, and be bounded on the left and right by the lines $x = a$ and $x = b$, respectively. Then, the area of $\mathbf{R}$ is given by

$$A = \int_a^b |f(x) - g(x)|dx.$$

In practice, applying this theorem requires us to break up the interval $[a, b]$ and evaluate several integrals, depending on which of the function values is greater over a given part of the interval. We study this process in the following example.

Example 6.1.3: Finding the Area of a Region Bounded by Functions That Cross

If $\mathbf{R}$ is the region between the graphs of the functions $f(x) = \sin x$ and $g(x) = \cos x$ over the interval $[0, \pi]$, find the area of region $\mathbf{R}$.

Solution

The region is depicted in the following figure.

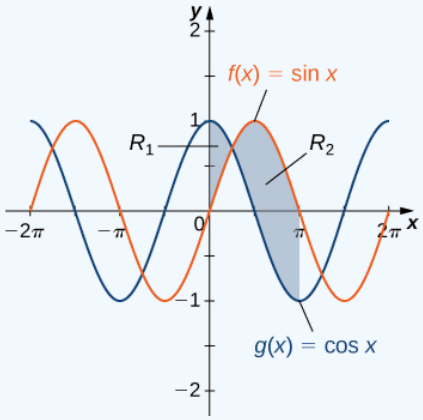


Figure 6.1.5: The region between two curves can be broken into two sub-regions.

The graphs of the functions intersect at $x = \pi/4$. For $x \in [0, \pi/4], \cos x \geq \sin x$, so

$$|f(x) - g(x)| = |\sin x - \cos x| = \cos x - \sin x.$$

On the other hand, for $x \in [\pi/4, \pi], \sin x \geq \cos x$, so

$$|f(x) - g(x)| = |\sin x - \cos x| = \sin x - \cos x.$$

Then

$$A = \int_a^b |f(x) - g(x)|dx$$

$$= \int_0^{\pi} |\sin x - \cos x|dx = \int_0^{\pi/4} (\cos x - \sin x)dx + \int_{\pi/4}^{\pi} (\sin x - \cos x)dx$$

$$= [\sin x + \cos x]|_0^{\pi/4} + [-\cos x - \sin x]|_{\pi/4}^{\pi}$$

$$= (\sqrt{2} - 1) + (1 + \sqrt{2}) = 2\sqrt{2}.$$

The area of the region is $2\sqrt{2}$ units^2.

Exercise 6.1.3

If $\mathbf{R}$ is the region between the graphs of the functions $f(x) = \sin x$ and $g(x) = \cos x$ over the interval $[\pi/2, 2\pi]$, find the area of region $\mathbf{R}$.

Hint

The two curves intersect at $x = (5\pi)/4$.

Answer

$2 + 2\sqrt{2}$ units^2

Example 6.1.4: Finding the Area of a Complex Region

Consider the region depicted in Figure 6.1.6. Find the area of $\mathbf{R}$.

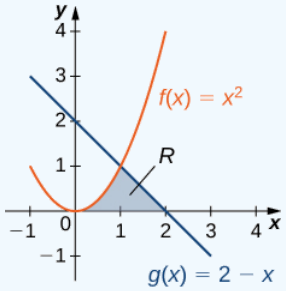


Figure 6.1.6: Two integrals are required to calculate the area of this region.

Solution

As with Example 6.1.3, we need to divide the interval into two pieces. The graphs of the functions intersect at $x = 1$ (set $f(x) = g(x)$ and solve for x), so we evaluate two separate integrals: one over the interval $[0, 1]$ and one over the interval $[1, 2]$.

Over the interval $[0, 1]$, the region is bounded above by $f(x) = x^2$ and below by the x-axis, so we have

$$A_1 = \int_0^1 x^2 dx = \frac{x^3}{3} |_0^1 = \frac{1}{3}.$$

Over the interval $[1, 2]$, the region is bounded above by $g(x) = 2 - x$ and below by the **x-axis**, so we have

$$A_2 = \int_1^2 (2 - x)dx = [2x - \frac{x^2}{2}] |_1^2 = \frac{1}{2}.$$

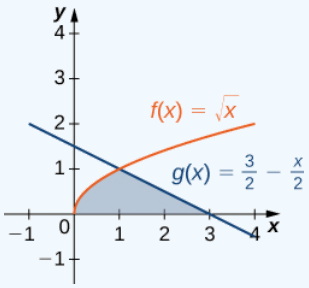
Adding these areas together, we obtain

$$A = A_1 + A_2 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$

The area of the region is $5/6$ units^2.

Exercise 6.1.4

Consider the region depicted in the following figure. Find the area of $\mathbf{R}$.



Hint

The two curves intersect at **x=1**

Answer

$\frac{5}{3}$ units^2

Regions Defined with Respect to y

In Example 6.1.4, we had to evaluate two separate integrals to calculate the area of the region. However, there is another approach that requires only one integral. What if we treat the curves as functions of y, instead of as functions of x? Review Figure. Note that the left graph, shown in red, is represented by the function $y = f(x) = x^2$. We could just as easily solve this for x and represent the curve by the function $x = v(y) = \sqrt{y}$. (Note that $x = -\sqrt{y}$ is also a valid representation of the function $y = f(x) = x^2$ as a function of y. However, based on the graph, it is clear we are interested in the positive square root.) Similarly, the right graph is represented by the function $y = g(x) = 2 - x$, but could just as easily be represented by the function $x = u(y) = 2 - y$. When the graphs are represented as functions of y, we see the region is bounded on the left by the graph of one function and on the right by the graph of the other function. Therefore, if we integrate with respect to y, we need to evaluate one integral only. Let's develop a formula for this type of integration.

Let $u(y)$ and $v(y)$ be continuous functions over an interval $[c, d]$ such that $u(y) \geq v(y)$ for all $y \in [c, d]$. We want to find the area between the graphs of the functions, as shown in Figure 6.1.7.

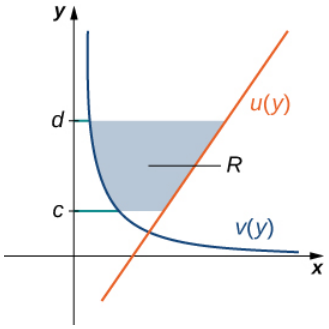
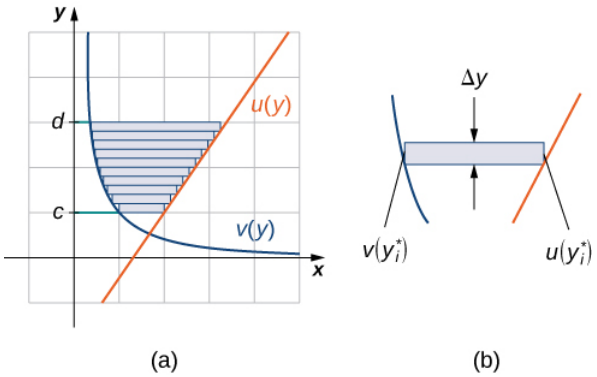


Figure 6.1.7: We can find the area between the graphs of two functions, $u(y)$ and $v(y)$.

This time, we are going to partition the interval on the **y-axis** and use horizontal rectangles to approximate the area between the functions. So, for $i = 0, 1, 2, \ldots, n$, let $Q = y_i$ be a regular partition of $[c, d]$. Then, for $i = 1, 2, \ldots, n$, choose a point $y_i^* \in [y_{i-1}, y_i]$, then over each interval $[y_{i-1}, y_i]$ construct a rectangle that extends horizontally from $v(y^0 *_i)$ to $u(y_i^*)$. Figure 6.1.8$a$shows the rectangles when y_i^* is selected to be the lower endpoint of the interval and $n = 10$. Figure 6.1.8b shows a representative rectangle in detail.



(a) (b)

Figure 6.1.8: (a) Approximating the area between the graphs of two functions, $u(y)$ and $v(y)$, with rectangles. (b) The area of a typical rectangle.

The height of each individual rectangle is Δy and the width of each rectangle is $u(y_i^*) - v(y_i^*)$. Therefore, the area between the curves is approximately

$$A \approx \sum_{i=1}^{n} [u(y_i^*) - v(y_i^*)]\Delta y.$$

This is a Riemann sum, so we take the limit as $n \to \infty$, obtaining

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{i=1}^{n} [u(y_i^*) - v(y_i^*)]\Delta y \\ &= \int_c^d [u(y) - v(y)]dy. \end{aligned}$$

These findings are summarized in the following theorem.

Finding the Area between Two Curves, Integrating along the y-axis

Let $u(y)$ and $v(y)$ be continuous functions such that $u(y) \geq v(y)$for all $y \in [c, d]$. Let $\mathbf{R}$ denote the region bounded on the right by the graph of $u(y)$, on the left by the graph of $v(y)$, and above and below by the lines $y = d$ and $y = c$, respectively. Then, the area of $\mathbf{R}$ is given by

$$A = \int_c^d [u(y) - v(y)]dy.$$

Example 6.1.5: Integrating with Respect to y

Let's revisit Example 6.1.4, only this time let's integrate with respect to y. Let $\mathbf{R}$ be the region depicted in Figure 6.1.9. Find the area of $\mathbf{R}$ by integrating with respect to y.

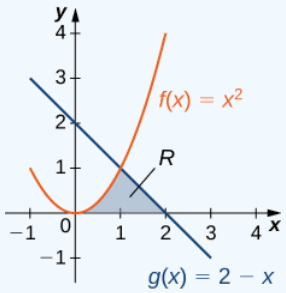


Figure 6.1.9: The area of region $\mathbf{R}$ can be calculated using one integral only when the curves are treated as functions of y.

Solution

We must first express the graphs as functions of y. As we saw at the beginning of this section, the curve on the left can be represented by the function $x = v(y) = \sqrt{y}$, and the curve on the right can be represented by the function $x = u(y) = 2 - y$.

Now we have to determine the limits of integration. The region is bounded below by the x-axis, so the lower limit of integration is $y = 0$. The upper limit of integration is determined by the point where the two graphs intersect, which is the point $(1, 1)$, so the upper limit of integration is $y = 1$. Thus, we have $[c, d] = [0, 1]$.

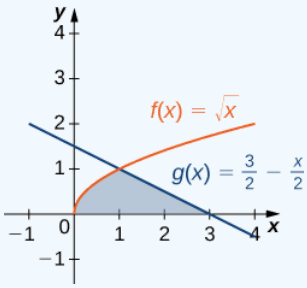
Calculating the area of the region, we get

$$\begin{aligned} A &= \int_c^d [u(y) - v(y)]dy \\ &= \int_0^1 [(2 - y) - \sqrt{y}]dy \\ &= [2y - \frac{y^2}{2} - \frac{2}{3}y^{3/2}] \,|_0^1 \\ &= \frac{5}{6}. \end{aligned}$$

The area of the region is $5/6$ units^2.

Exercise 6.1.5

Let's revisit the checkpoint associated with Example 6.1.4, only this time, let's integrate with respect to y. Let $\mathbf{R}$ be the region depicted in the following figure. Find the area of $\mathbf{R}$ by integrating with respect to y.



Hint

Follow the process from the previous example.

Answer

$\frac{5}{3}$ units^2

Key Concepts

- Just as definite integrals can be used to find the area under a curve, they can also be used to find the area between two curves.
- To find the area between two curves defined by functions, integrate the difference of the functions.
- If the graphs of the functions cross, or if the region is complex, use the absolute value of the difference of the functions. In this case, it may be necessary to evaluate two or more integrals and add the results to find the area of the region.
- Sometimes it can be easier to integrate with respect to y to find the area. The principles are the same regardless of which variable is used as the variable of integration.

Key Equations

- **Area between two curves, integrating on the x-axis**

  $$A = \int_a^b [f(x) - g(x)]dx$$

- **Area between two curves, integrating on the y-axis**

  $$A = \int_c^d [u(y) - v(y)]dy$$

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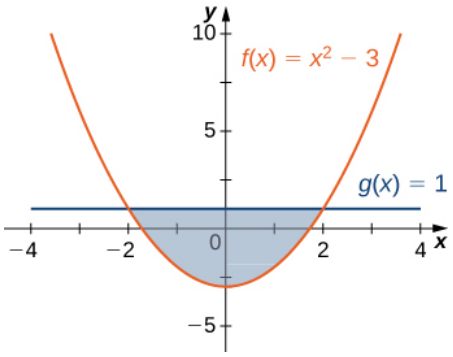
6.1E: EXERCISES FOR SECTION 6.1

For exercises 1 - 2, determine the area of the region between the two curves in the given figure by integrating over the x-axis.

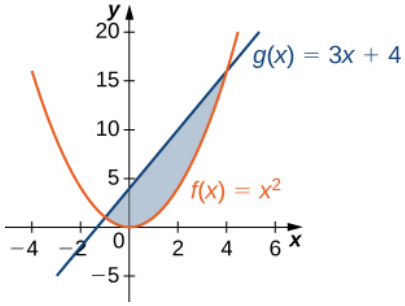
1) $y = x^2 - 3$ and $y = 1$



Answer

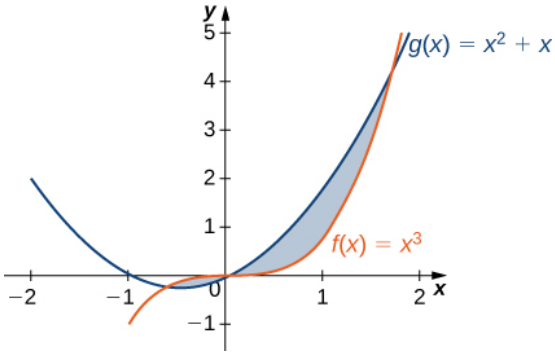
$\frac{32}{3}$ units^2

2) $y = x^2$ and $y = 3x + 4$



For exercises 3 - 4, split the region between the two curves into two smaller regions, then determine the area by integrating over the x-axis. Note that you will have two integrals to solve.

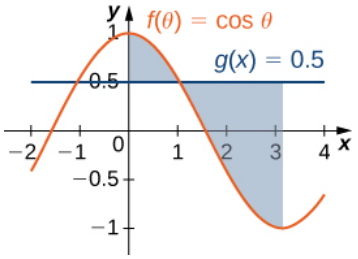
3) $y = x^3$ and $y = x^2 + x$



Answer

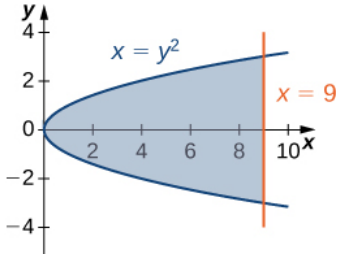
$\frac{13}{12}$ units^2

4) $y = \cos\theta$ and $y = 0.5$, for $0 \le \theta \le \pi$



For exercises 5-6, determine the area of the region between the two curves by integrating over the y-axis.

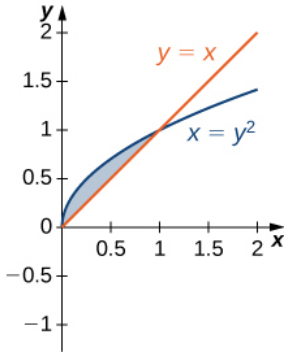
5) $x = y^2$ and $x = 9$



Answer

36 units^2

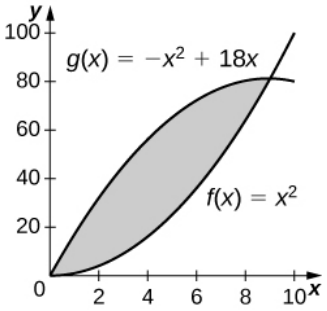
6) $y = x$ and $x = y^2$



For exercises 7 - 13, graph the equations and shade the area of the region between the curves. Determine its area by integrating over the x-axis.

7) $y = x^2$ and $y = -x^2 + 18x$

Answer

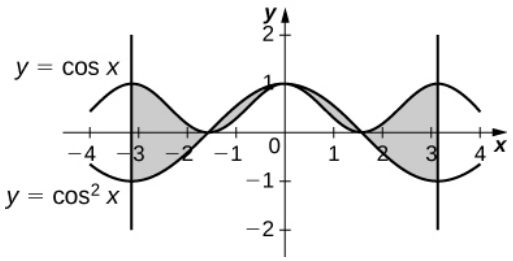


243 square units

8) $y = \frac{1}{x}$, $y = \frac{1}{x^2}$, and $x = 3$

9) $y = \cos x$ and $y = \cos^2 x$ on $x \in [-\pi, \pi]$

Answer

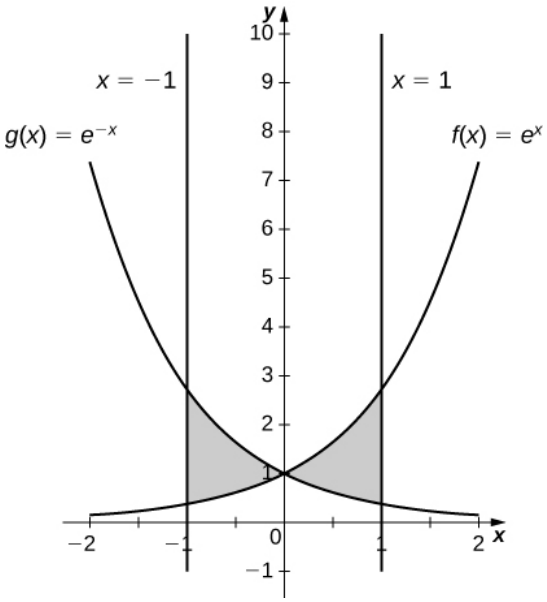


4 square units

10) $y = e^x$, $y = e^{2x-1}$, and $x = 0$

11) $y = e^x$, $y = e^{-x}$, $x = -1$ and $x = 1$

Answer

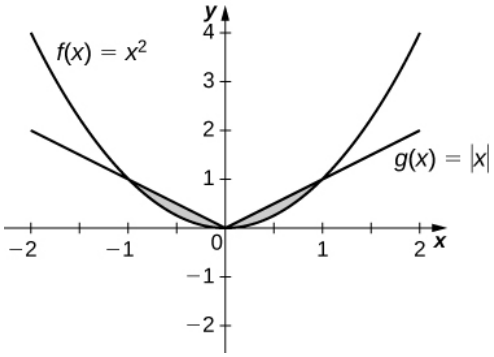


$\frac{2(e-1)^2}{e}$ units^2

12) $y = e$, $y = e^x$, and $y = e^{-x}$

13) $y = |x|$ and $y = x^2$

Answer



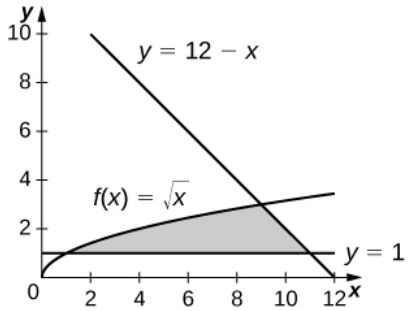
$\frac{1}{3}$ units^2

For exercises 14 - 19, graph the equations and shade the area of the region between the curves. If necessary, break the region into sub-regions to determine its entire area.

14) $y = \sin(\pi x)$, $y = 2x$, and $x > 0$

15) $y = 12 - x$, $y = \sqrt{x}$, and $y = 1$

Answer

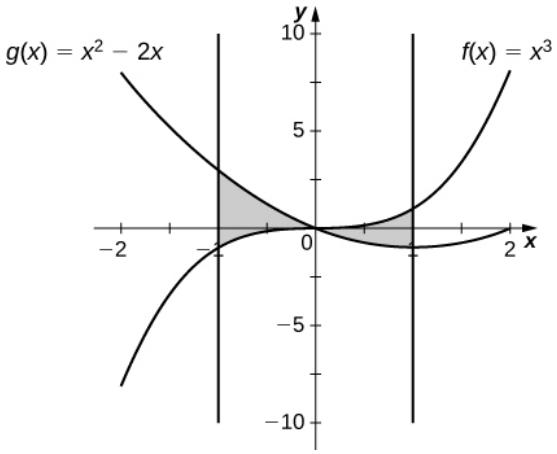


$\frac{34}{3}$ units^2

16) $y = \sin x$ and $y = \cos x$ over $x \in [-\pi, \pi]$

17) $y = x^3$ and $y = x^2 - 2x$ over $x \in [-1, 1]$

Answer



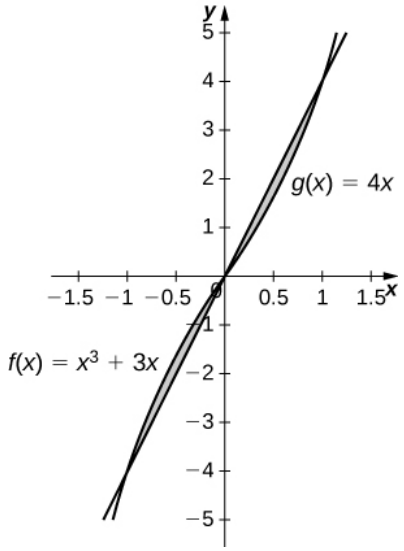
$\frac{5}{2}$ units^2

18) $y = x^2 + 9$ and $y = 10 + 2x$ over $x \in [-1, 3]$

19) $y = x^3 + 3x$ and $y = 4x$

Answer

https://math.libretexts.org/@go/page/69808



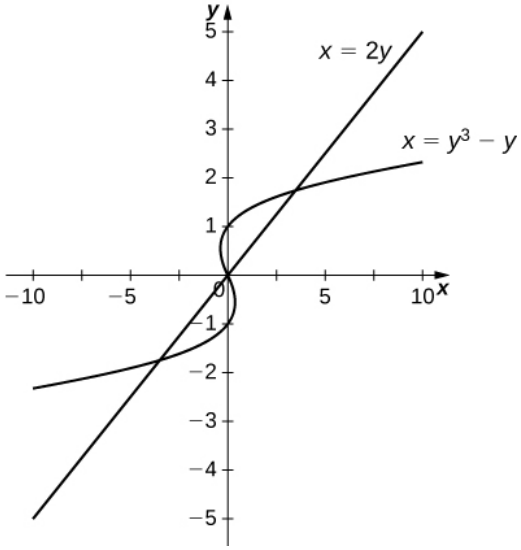
$\frac{1}{2}$ units^2

For exercises 20 -25, graph the equations and shade the area of the region between the curves. Determine its area by integrating over the y-axis.

20) $x = y^3$ and $x = 3y - 2$

21) $x = 2y$ and $x = y^3 - y$

Answer



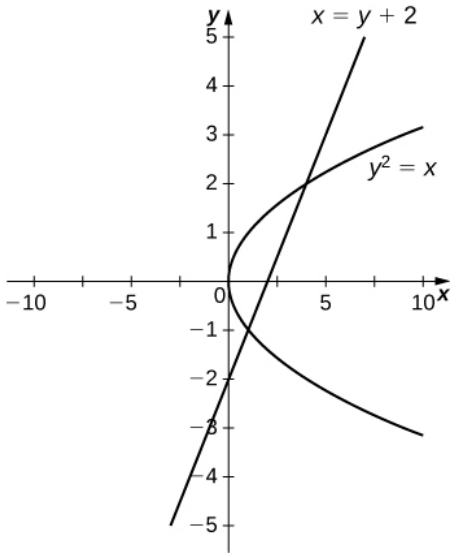
$\frac{9}{2}$ units^2

22) $x = -3 + y^2$ and $x = y - y^2$

23) $y^2 = x$ and $x = y + 2$

Answer

https://math.libretexts.org/@go/page/69808

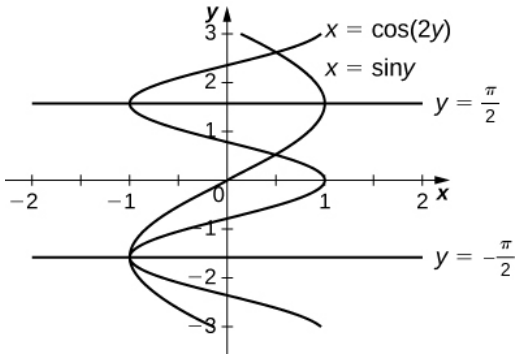


$\frac{9}{2}$ units^2

24) $x = |y|$ and $2x = -y^2 + 2$

25) $x = \sin y, \quad x = \cos(2y), \quad y = \pi/2$, and $y = -\pi/2$

Answer



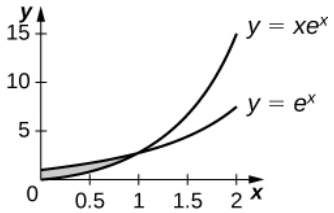
$\frac{3\sqrt{3}}{2}$ units^2

For exercises 26 - 37, graph the equations and shade the area of the region between the curves. Determine its area by integrating over the x-axis or y-axis, whichever seems more convenient.

26) $x = y^4$ and $x = y^5$

27) $y = xe^x, \quad y = e^x, \quad x = 0$, and $x = 1$.

Answer

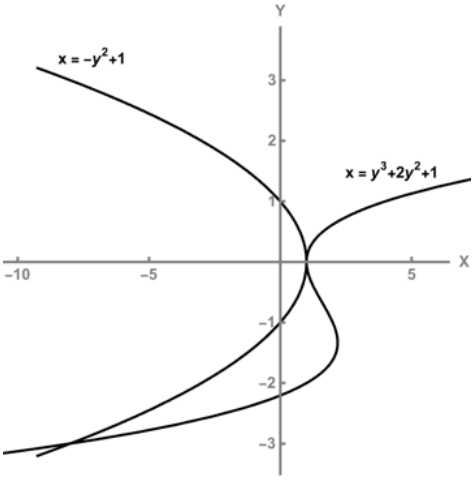


$(e - 2)$ units^2

28) $y = x^6$ and $y = x^4$

29) $x = y^3 + 2y^2 + 1$ and $x = -y^2 + 1$

Answer

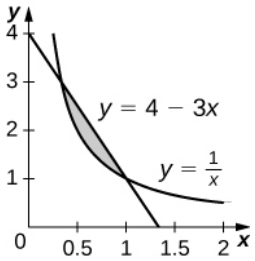


$\frac{27}{4}$ units^2

30) $y = |x|$ and $y = x^2 - 1$

31) $y = 4 - 3x$ and $y = \frac{1}{x}$

Answer

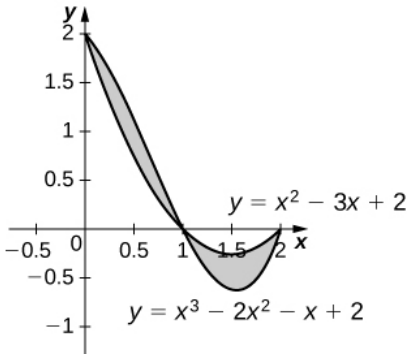


$\left(\frac{4}{3} - \ln(3)\right)$ units^2

32) $y = \sin x, \quad x = -\pi/6, \quad x = \pi/6$, and $y = \cos^3 x$

33) $y = x^2 - 3x + 2$ and $y = x^3 - 2x^2 - x + 2$

Answer

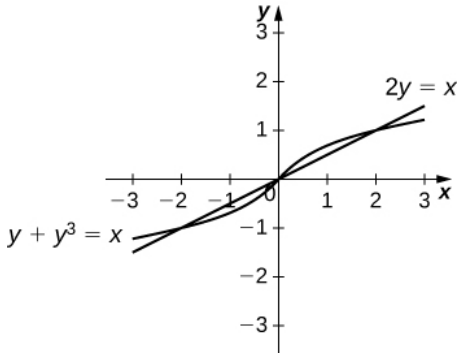


$\frac{1}{2}$ square units

34) $y = 2\cos^3(3x), \quad y = -1, \quad x = \frac{\pi}{4}$, and $x = -\frac{\pi}{4}$

35) $y + y^3 = x$ and $2y = x$

Answer

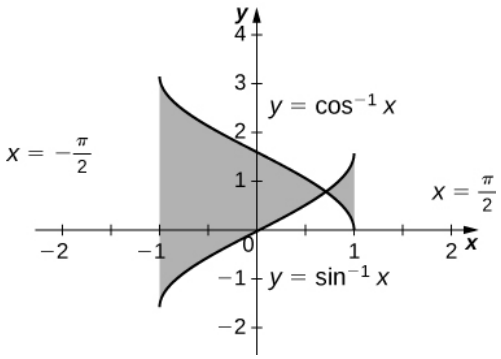


$\frac{1}{2}$ square units

36) $y = \sqrt{1 - x^2}$ and $y = x^2 - 1$

37) $y = \cos^{-1} x, \quad y = \sin^{-1} x, \quad x = -1,$ and $x = 1$

Answer



$-2(\sqrt{2} - \pi)$ square units

For exercises 38 - 47, find the exact area of the region bounded by the given equations if possible. If you are unable to determine the intersection points analytically, use a calculator to approximate the intersection points with three decimal places and determine the approximate area of the region.

38) [T] $x = e^y$ and $y = x - 2$

39) [T] $y = x^2$ and $y = \sqrt{1 - x^2}$

Answer

1.067 square units

40) [T] $y = 3x^2 + 8x + 9$ and $3y = x + 24$

41) [T] $x = \sqrt{4 - y^2}$ and $y^2 = 1 + x^2$

Answer

0.852 square units

42) [T] $x^2 = y^3$ and $x = 3y$

43) [T] $y = \sin^3 x + 2, \quad y = \tan x, \quad x = -1.5,$ and $x = 1.5$

Answer

7.523 square units

44) [T] $y = \sqrt{1 - x^2}$ and $y^2 = x^2$

45) [T] $y = \sqrt{1 - x^2}$ and $y = x^2 + 2x + 1$

Answer

$\frac{3\pi - 4}{12}$ square units

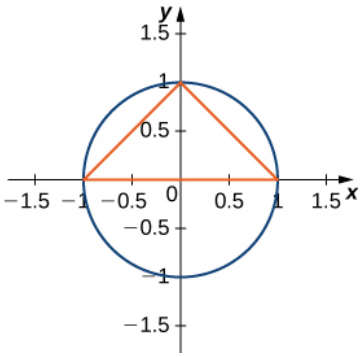
46) [T] $x = 4 - y^2$ and $x = 1 + 3y + y^2$

47) [T] $y = \cos x, \quad y = e^x, \quad x = -\pi, \quad$ and $\quad x = 0$

Answer

1.429 square units

48) The largest triangle with a base on the x-axis that fits inside the upper half of the unit circle $y^2 + x^2 = 1$ is given by $y = 1 + x$ and $y = 1 - x$. See the following figure. What is the area inside the semicircle but outside the triangle?



49) A factory selling cell phones has a marginal cost function $C(x) = 0.01x^2 - 3x + 229$, where x represents the number of cell phones, and a marginal revenue function given by $R(x) = 429 - 2x$. Find the area between the graphs of these curves and $x = 0$. What does this area represent?

Answer

$\$33{,}333.33$ total profit for 200 cell phones sold

50) An amusement park has a marginal cost function $C(x) = 1000e - x + 5$, where x represents the number of tickets sold, and a marginal revenue function given by $R(x) = 60 - 0.1x$. Find the total profit generated when selling 550 tickets. Use a calculator to determine intersection points, if necessary, to two decimal places.

51) The tortoise versus the hare: The speed of the hare is given by the sinusoidal function $H(t) = 1 - \cos((\pi t)/2)$ whereas the speed of the tortoise is $T(t) = (1/2)\tan^{-1}(t/4)$, where t is time measured in hours and the speed is measured in miles per hour. Find the area between the curves from time $t = 0$ to the first time after one hour when the tortoise and hare are traveling at the same speed. What does it represent? Use a calculator to determine the intersection points, if necessary, accurate to three decimal places.

Answer

3.263 mi represents how far ahead the hare is from the tortoise

52) The tortoise versus the hare: The speed of the hare is given by the sinusoidal function $H(t) = (1/2) - (1/2)\cos(2\pi t)$ whereas the speed of the tortoise is $T(t) = \sqrt{t}$, where t is time measured in hours and speed is measured in kilometers per hour. If the race is over in 1 hour, who won the race and by how much? Use a calculator to determine the intersection points, if necessary, accurate to three decimal places.

For exercises 53 - 55, find the area between the curves by integrating with respect to x and then with respect to y. Is one method easier than the other? Do you obtain the same answer?

53) $y = x^2 + 2x + 1$ and $y = -x^2 - 3x + 4$

Answer

$\frac{343}{24}$ square units

54) $y = x^4$ and $x = y^5$

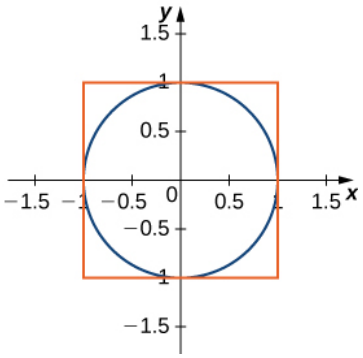
55) $x = y^2 - 2$ and $x = 2y$

Answer

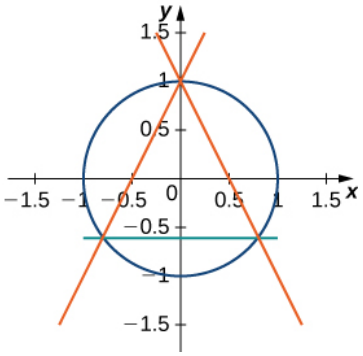
$4\sqrt{3}$ square units

For exercises 56 - 57, solve using calculus, then check your answer with geometry.

56) Determine the equations for the sides of the square that touches the unit circle on all four sides, as seen in the following figure. Find the area between the perimeter of this square and the unit circle. Is there another way to solve this without using calculus?



57) Find the area between the perimeter of the unit circle and the triangle created from $y = 2x + 1$, $y = 1 - 2x$ and $y = -\frac{3}{5}$, as seen in the following figure. Is there a way to solve this without using calculus?



Answer

$\left(\pi - \frac{32}{25}\right)$ square units

CONTRIBUTORS

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6.2: Determining Volumes by Slicing

Learning Objectives

- Determine the volume of a solid by integrating a cross-section (the slicing method).
- Find the volume of a solid of revolution using the disk method.
- Find the volume of a solid of revolution with a cavity using the washer method.

In the preceding section, we used definite integrals to find the area between two curves. In this section, we use definite integrals to find volumes of three-dimensional solids. We consider three approaches—slicing, disks, and washers—for finding these volumes, depending on the characteristics of the solid.

Volume and the Slicing Method

Just as area is the numerical measure of a two-dimensional region, volume is the numerical measure of a three-dimensional solid. Most of us have computed volumes of solids by using basic geometric formulas. The volume of a rectangular solid, for example, can be computed by multiplying length, width, and height: $V = lwh.$ The formulas for the volumes of:

- a sphere

$$V_{sphere} = \frac{4}{3}\pi r^3,$$

- a cone

$$V_{cone} = \frac{1}{3}\pi r^2 h$$

- and a pyramid

$$V_{pyramid} = \frac{1}{3}Ah$$

have also been introduced. Although some of these formulas were derived using geometry alone, all these formulas can be obtained by using integration.

We can also calculate the volume of a cylinder. Although most of us think of a cylinder as having a circular base, such as a soup can or a metal rod, in mathematics the word cylinder has a more general meaning. To discuss cylinders in this more general context, we first need to define some vocabulary.

We define the *cross-section* of a solid to be the intersection of a plane with the solid. A cylinder is defined as any solid that can be generated by translating a plane region along a line perpendicular to the region, called the axis of the cylinder. Thus, all cross-sections perpendicular to the axis of a cylinder are identical. The solid shown in Figure 6.2.1 is an example of a cylinder with a noncircular base. To calculate the volume of a cylinder, then, we simply multiply the area of the cross-section by the height of the cylinder: $V = A \cdot h.$ In the case of a right circular cylinder (soup can), this becomes $V = \pi r^2 h.$

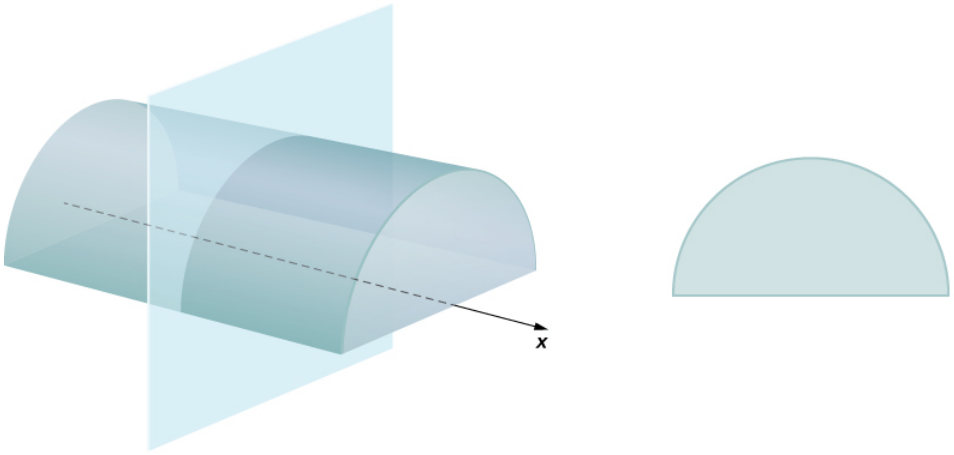


Figure 6.2.1*: Each cross-section of a particular cylinder is identical to the others.*

If a solid does not have a constant cross-section (and it is not one of the other basic solids), we may not have a formula for its volume. In this case, we can use a definite integral to calculate the volume of the solid. We do this by slicing the solid into pieces, estimating the volume of each slice, and then adding those estimated volumes together. The slices should all be parallel to one another, and when we put all the slices together, we should get the whole solid. Consider, for example, the solid S shown in Figure 6.2.2, extending along the x**-axis.**

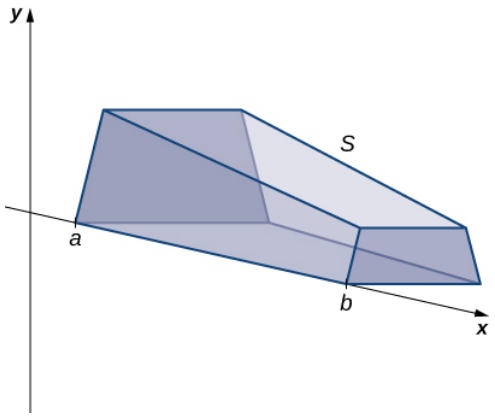


Figure 6.2.2: A solid with a varying cross-section.

We want to divide S into slices perpendicular to the x**-axis**. As we see later in the chapter, there may be times when we want to slice the solid in some other direction—say, with slices perpendicular to the y-axis. The decision of which way to slice the solid is very important. If we make the wrong choice, the computations can get quite messy. Later in the chapter, we examine some of these situations in detail and look at how to decide which way to slice the solid. For the purposes of this section, however, we use slices perpendicular to the x**-axis.**

Because the cross-sectional area is not constant, we let $A(x)$ represent the area of the cross-section at point x. Now let $P = x_0, x_1 \ldots, X_n$ be a regular partition of $[a, b]$, and for $i = 1, 2, \ldots n$, let S_i represent the slice of S stretching from x_{i-1} to x_i. The following figure shows the sliced solid with $n = 3$.

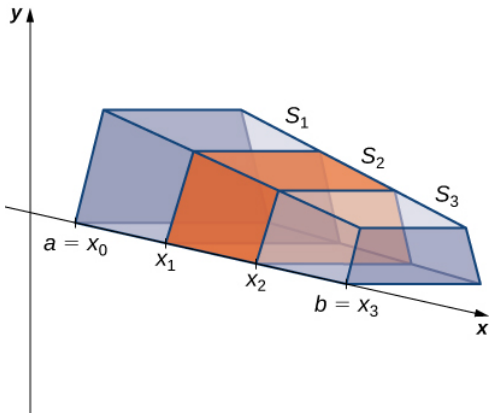


Figure 6.2.3: The solid S has been divided into three slices perpendicular to the x-**axis.**

Finally, for $i = 1, 2, \ldots n$, let x_i^* be an arbitrary point in $[x_{i-1}, x_i]$. Then the volume of slice S_i can be estimated by $V(S_i) \approx A(x_i^*)\,\Delta x$. Adding these approximations together, we see the volume of the entire solid S can be approximated by

$$V(S) \approx \sum_{i=1}^{n} A(x_i^*)\,\Delta x.$$

By now, we can recognize this as a Riemann sum, and our next step is to take the limit as $n \to \infty$. Then we have

$$V(S) = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\,\Delta x = \int_a^b A(x)\,dx.$$

The technique we have just described is called the slicing method. To apply it, we use the following strategy.

Problem-Solving Strategy: Finding Volumes by the Slicing Method

1. Examine the solid and determine the shape of a cross-section of the solid. It is often helpful to draw a picture if one is not provided.
2. Determine a formula for the area of the cross-section.
3. Integrate the area formula over the appropriate interval to get the volume.

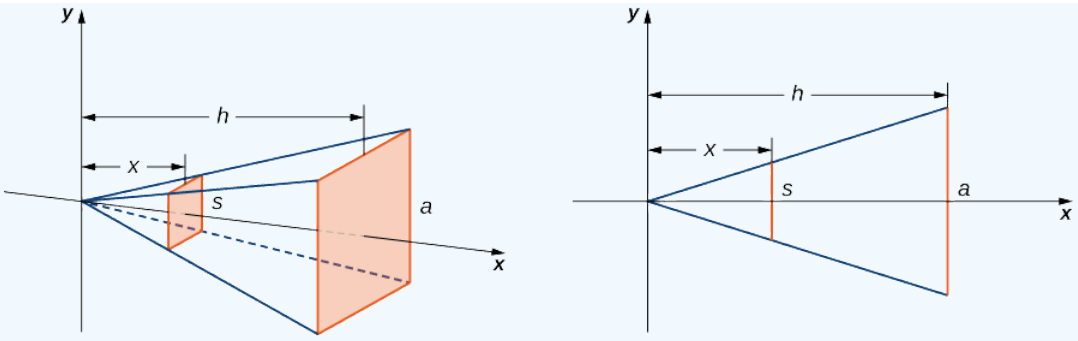
Recall that in this section, we assume the slices are perpendicular to the x-**axis**. Therefore, the area formula is in terms of x and the limits of integration lie on the x-**axis.** However, the problem-solving strategy shown here is valid regardless of how we choose to slice the solid.

Example 6.2.1: Deriving the Formula for the Volume of a Pyramid

We know from geometry that the formula for the volume of a pyramid is $V = \frac{1}{3}Ah$. If the pyramid has a square base, this becomes $V = \frac{1}{3}a^2h$, where a denotes the length of one side of the base. We are going to use the slicing method to derive this formula.

Solution

We want to apply the slicing method to a pyramid with a square base. To set up the integral, consider the pyramid shown in Figure 6.2.4, oriented along the x-**axis.**



(a) (b)

Figure 6.2.4: (a) A pyramid with a square base is oriented along the x-axis. (b) A two-dimensional view of the pyramid is seen from the side.

We first want to determine the shape of a cross-section of the pyramid. We know the base is a square, so the cross-sections are squares as well (step 1). Now we want to determine a formula for the area of one of these cross-sectional squares. Looking at Figure 6.2.4 (b), and using a proportion, since these are similar triangles, we have

$$\frac{s}{a} = \frac{x}{h}$$

or

$$s = \frac{ax}{h}.$$

Therefore, the area of one of the cross-sectional squares is

$$A(x) = s^2 = \left(\frac{ax}{h}\right)^2 \qquad \text{(step 2)}$$

Then we find the volume of the pyramid by integrating from 0 to h (step 3):

$$V = \int_0^h A(x)\,dx = \int_0^h \left(\frac{ax}{h}\right)^2 dx = \frac{a^2}{h^2}\int_0^h x^2\,dx = \left[\frac{a^2}{h^2}\left(\frac{1}{3}x^3\right)\right]\Bigg|_0^h = \frac{1}{3}a^2h.$$

This is the formula we were looking for.

? Exercise 6.2.1

Use the slicing method to derive the formula

$$V = \frac{1}{3}\pi r^2 h$$

for the volume of a circular cone.

Hint

Use similar triangles, as in Example 6.2.1.

Solids of Revolution

If a region in a plane is revolved around a line in that plane, the resulting solid is called a **solid of revolution**, as shown in the following figure.

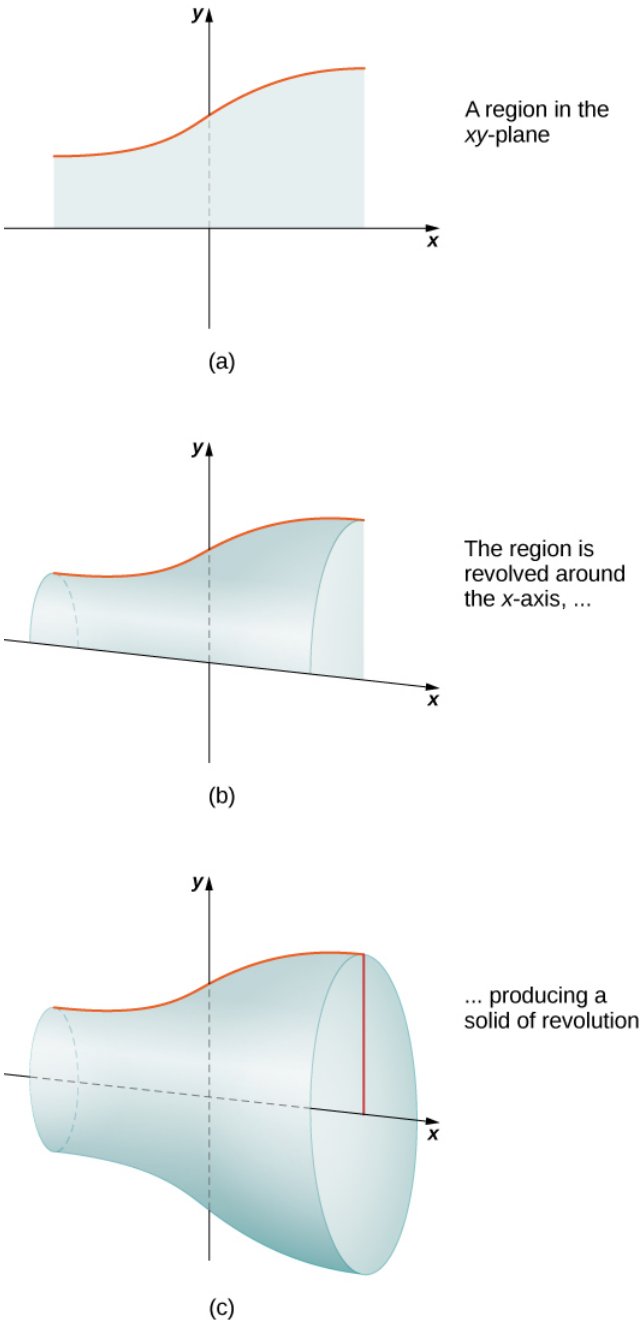


Figure 6.2.5: (a) This is the region that is revolved around the x-axis. (b) As the region begins to revolve around the axis, it sweeps out a solid of revolution. (c) This is the solid that results when the revolution is complete.

Solids of revolution are common in mechanical applications, such as machine parts produced by a lathe. We spend the rest of this section looking at solids of this type. The next example uses the slicing method to calculate the volume of a solid of revolution.

Example 6.2.2: Using the Slicing Method to find the Volume of a Solid of Revolution

Use the slicing method to find the volume of the solid of revolution bounded by the graphs of $f(x) = x^2 - 4x + 5, x = 1$,and $x = 4$, and rotated about the x-axis.

Solution

Using the problem-solving strategy, we first sketch the graph of the quadratic function over the interval $[1, 4]$ as shown in the following figure.

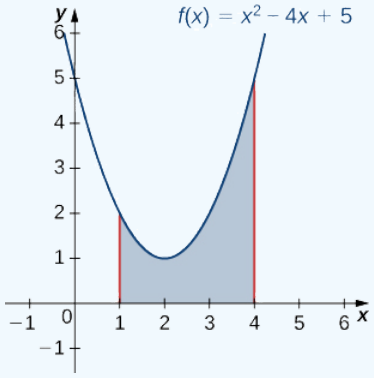
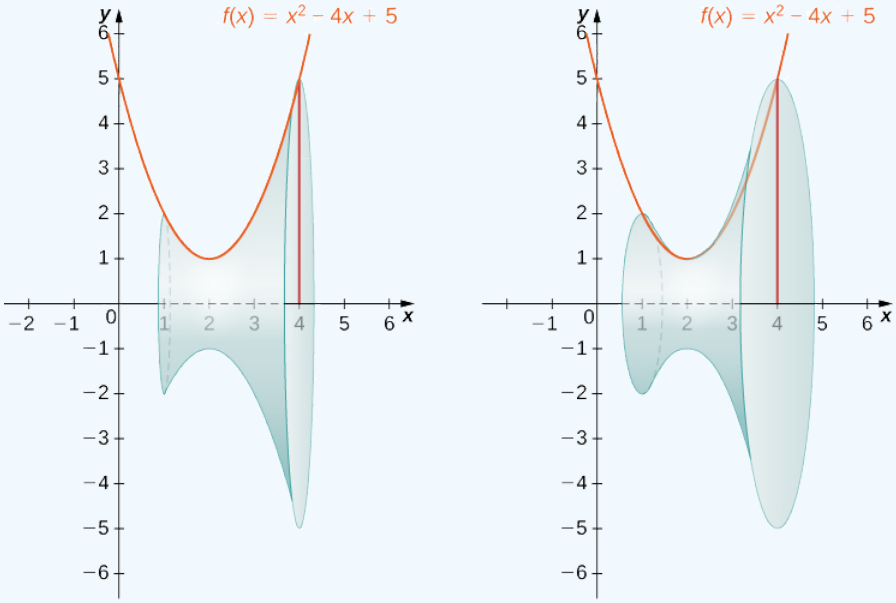


Figure 6.2.6: A region used to produce a solid of revolution.

Next, revolve the region around the x-axis, as shown in the following figure.



(a) (b)

Figure 6.2.7: Two views, (a) and (b), of the solid of revolution produced by revolving the region in Figure 6.2.6 about the x-**axis.**

Since the solid was formed by revolving the region around the x-**axis,** the cross-sections are circles (step 1). The area of the cross-section, then, is the area of a circle, and the radius of the circle is given by $f(x)$. Use the formula for the area of the circle:

$$A(x) = \pi r^2 = \pi[f(x)]^2 = \pi(x^2 - 4x + 5)^2 \qquad \text{(step 2)}.$$

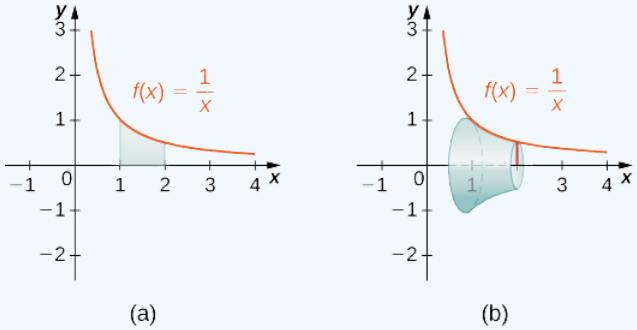
The volume, then, is (step 3)

$$\begin{aligned} V &= \int_a^b A(x)\,dx \\ &= \int_1^4 \pi(x^2 - 4x + 5)^2\,dx \\ &= \pi \int_1^4 (x^4 - 8x^3 + 26x^2 - 40x + 25)\,dx \\ &= \pi \left(\frac{x^5}{5} - 2x^4 + \frac{26x^3}{3} - 20x^2 + 25x \right) \Bigg|_1^4 \\ &= \frac{78}{5}\pi \end{aligned}$$

The volume is $78\pi/5\ \text{units}^3$.

Exercise 6.2.2

Use the method of slicing to find the volume of the solid of revolution formed by revolving the region between the graph of the function $f(x) = 1/x$ and the x**-axis** over the interval $[1, 2]$ around the x**-axis.** See the following figure.



Hint

Use the problem-solving strategy presented earlier and follow Example 6.2.2 to help with step 2.

Answer

$\frac{\pi}{2}$ units^3

The Disk Method

When we use the slicing method with solids of revolution, it is often called the disk method because, for solids of revolution, the slices used to over approximate the volume of the solid are disks. To see this, consider the solid of revolution generated by revolving the region between the graph of the function $f(x) = (x - 1)^2 + 1$ and the x**-axis** over the interval $[-1, 3]$ around the x**-axis**. The graph of the function and a representative disk are shown in Figure 6.2.8 (a) and (b). The region of revolution and the resulting solid are shown in Figure 6.2.8 (c) and (d).

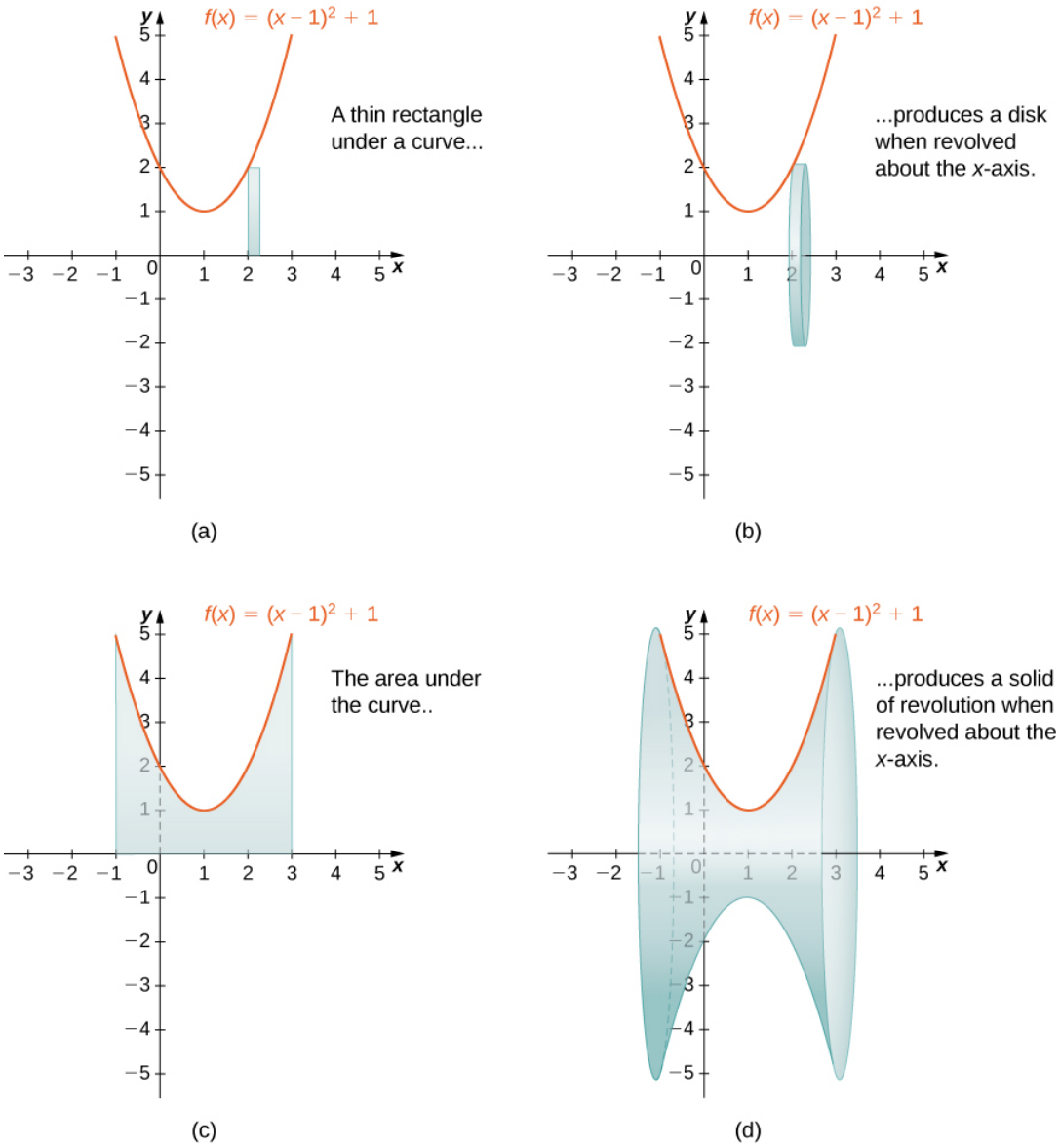


Figure 6.2.8: (a) A thin rectangle for approximating the area under a curve. (b) A representative disk formed by revolving the rectangle about the x**-axis.** (c) The region under the curve is revolved about the x**-axis,** resulting in (d) the solid of revolution.

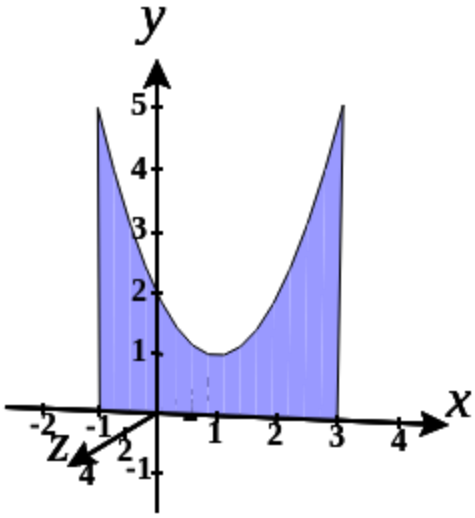


Figure 6.2.8: (e) A dynamic version of this solid of revolution generated using CalcPlot3D.

We already used the formal Riemann sum development of the volume formula when we developed the slicing method. We know that

$$\int_a^b A(x)\,dx.$$

The only difference with the disk method is that we know the formula for the cross-sectional area ahead of time; it is the area of a circle. This gives the following rule.

The Disk Method

Let $f(x)$ be continuous and nonnegative. Define R as the region bounded above by the graph of $f(x)$, below by the x-**axis**, on the left by the line $x = a$, and on the right by the line $x = b$. Then, the volume of the solid of revolution formed by revolving R around the x-**axis** is given by

$$V = \int_a^b \pi[f(x)]^2\,dx.$$

The volume of the solid we have been studying (Figure 6.2.8) is given by

$$
\begin{aligned}
V &= \int_a^b \pi [f(x)]^2 \, dx \\
&= \int_{-1}^{3} \pi \left[(x-1)^2 + 1\right]^2 dx = \pi \int_{-1}^{3} \left[(x-1)^4 + 2(x-1)^2 + 1\right]^2 dx \\
&= \pi \left[\frac{1}{5}(x-1)^5 + \frac{2}{3}(x-1)^3 + x\right]\Big|_{-1}^{3} \\
&= \pi \left[\left(\frac{32}{5} + \frac{16}{3} + 3\right) - \left(-\frac{32}{5} - \frac{16}{3} - 1\right)\right] \\
&= \frac{412\pi}{15} \text{ units}^3.
\end{aligned}
$$

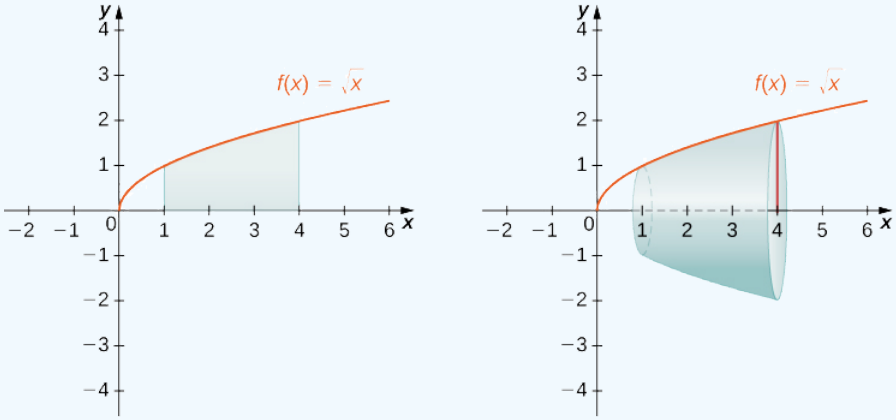
Let's look at some examples.

Example 6.2.3: Using the Disk Method to Find the Volume of a Solid of Revolution 1

Use the disk method to find the volume of the solid of revolution generated by rotating the region between the graph of $f(x) = \sqrt{x}$ and the x-**axis** over the interval $[1, 4]$ around the x-**axis.**

Solution

The graphs of the function and the solid of revolution are shown in the following figure.



(a) (b)

Figure 6.2.9: (a) The function $f(x) = \sqrt{x}$ over the interval $[1, 4]$. (b) The solid of revolution obtained by revolving the region under the graph of $f(x)$ about the x-**axis**.

We have

$$
\begin{aligned}
V &= \int_a^b \pi \left[f(x)\right]^2 dx \\
&= \int_1^4 \pi \left[\sqrt{x}\right]^2 dx = \pi \int_1^4 x \, dx \\
&= \frac{\pi}{2} x^2 \Big|_1^4 = \frac{15\pi}{2}
\end{aligned}
$$

The volume is $(15\pi)/2 \text{ units}^3$.

Exercise 6.2.3

Use the disk method to find the volume of the solid of revolution generated by rotating the region between the graph of $f(x) = \sqrt{4 - x}$ and the x-**axis** over the interval $[0, 4]$ around the x-**axis.**

Hint

Use the procedure from Example 6.2.3.

Answer

8π units^3

So far, our examples have all concerned regions revolved around the x-**axis**, but we can generate a solid of revolution by revolving a plane region around any horizontal or vertical line. In the next example, we look at a solid of revolution that has been generated by revolving a region around the y-**axis**. The mechanics of the disk method are nearly the same as when the x-**axis** is the axis of revolution, but we express the function in terms of y and we integrate with respect to y as well. This is summarized in the following rule.

Rule: The Disk Method for Solids of Revolution around the y-axis

Let $g(y)$ be continuous and nonnegative. Define Q as the region bounded on the right by the graph of $g(y)$, on the left by the y-**axis**, below by the line $y = c$, and above by the line $y = d$. Then, the volume of the solid of revolution formed by revolving Q around the y-**axis** is given by

$$V = \int_c^d \pi [g(y)]^2 \, dy.$$

The next example shows how this rule works in practice.

Example 6.2.4: Using the Disk Method to Find the Volume of a Solid of Revolution 2

Let R be the region bounded by the graph of $g(y) = \sqrt{4 - y}$ and the y-**axis** over the y-**axis** interval $[0, 4]$. Use the disk method to find the volume of the solid of revolution generated by rotating R around the y-**axis.**

Solution

Figure 6.2.10 shows the function and a representative disk that can be used to estimate the volume. Notice that since we are revolving the function around the y-**axis,** the disks are horizontal, rather than vertical.

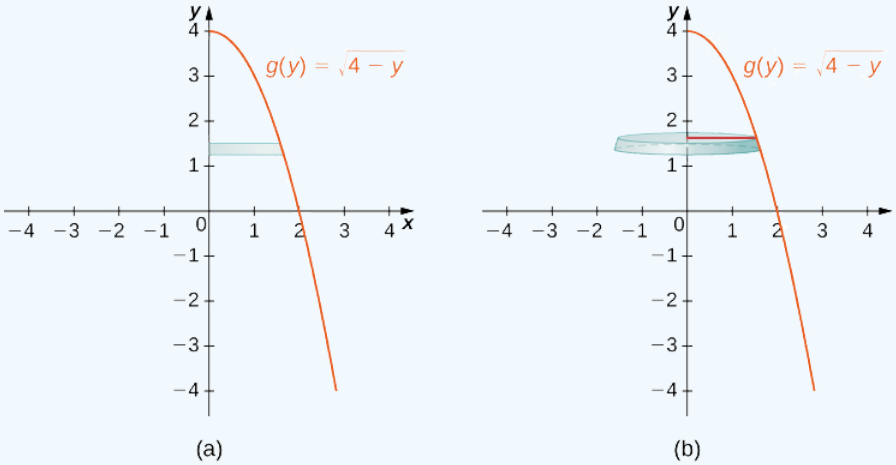


Figure 6.2.10: (a) Shown is a thin rectangle between the curve of the function $g(y) = \sqrt{4 - y}$ and the y-**axis.** (b) The rectangle forms a representative disk after revolution around the y-**axis.**

The region to be revolved and the full solid of revolution are depicted in the following figure.

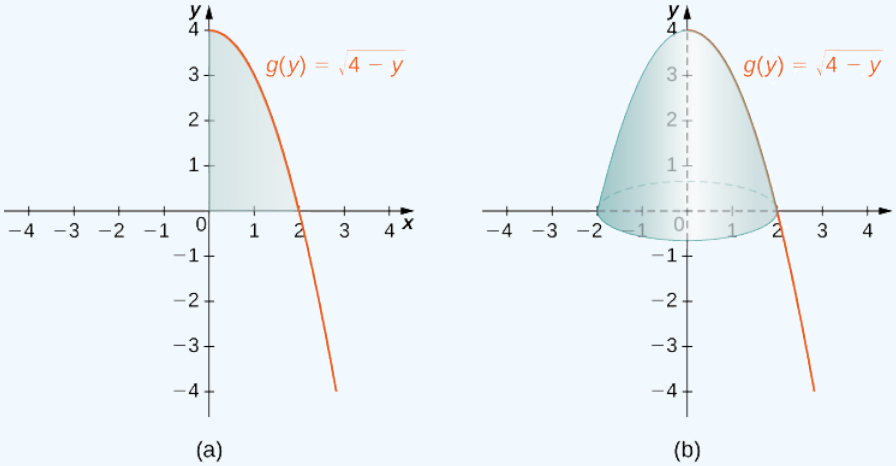


Figure 6.2.11: (a) The region to the left of the function $g(y) = \sqrt{4-y}$ over the y-axis interval $[0, 4]$. (b) The solid of revolution formed by revolving the region about the y-axis.

To find the volume, we integrate with respect to y. We obtain

$$V = \int_c^d \pi \big[g(y)\big]^2\, dy = \int_0^4 \pi \left[\sqrt{4-y}\right]^2\, dy = \pi \int_0^4 (4-y)\, dy = \pi \left[4y - \frac{y^2}{2}\right]\Big|_0^4 = 8\pi.$$

The volume is 8π units^3.

Exercise 6.2.4

Use the disk method to find the volume of the solid of revolution generated by rotating the region between the graph of $g(y) = y$ and the y-**axis** over the interval $[1, 4]$ around the y-**axis.**

Hint

Use the procedure from Example 6.2.4.

Answer

21π units^3

The Washer Method

Some solids of revolution have cavities in the middle; they are not solid all the way to the axis of revolution. Sometimes, this is just a result of the way the region of revolution is shaped with respect to the axis of revolution. In other cases, cavities arise when the region of revolution is defined as the region between the graphs of two functions. A third way this can happen is when an axis of revolution other than the x-**axis** or y-**axis** is selected.

When the solid of revolution has a cavity in the middle, the slices used to approximate the volume are not disks, but washers (disks with holes in the center). For example, consider the region bounded above by the graph of the function $f(x) = \sqrt{x}$ and below by the graph of the function $g(x) = 1$ over the interval $[1, 4]$. When this region is revolved around the x-**axis**, the result is a solid with a cavity in the middle, and the slices are washers. The graph of the function and a representative washer are shown in Figure 6.2.12 (a) and (b). The region of revolution and the resulting solid are shown in Figure 6.2.12 (c) and (d).

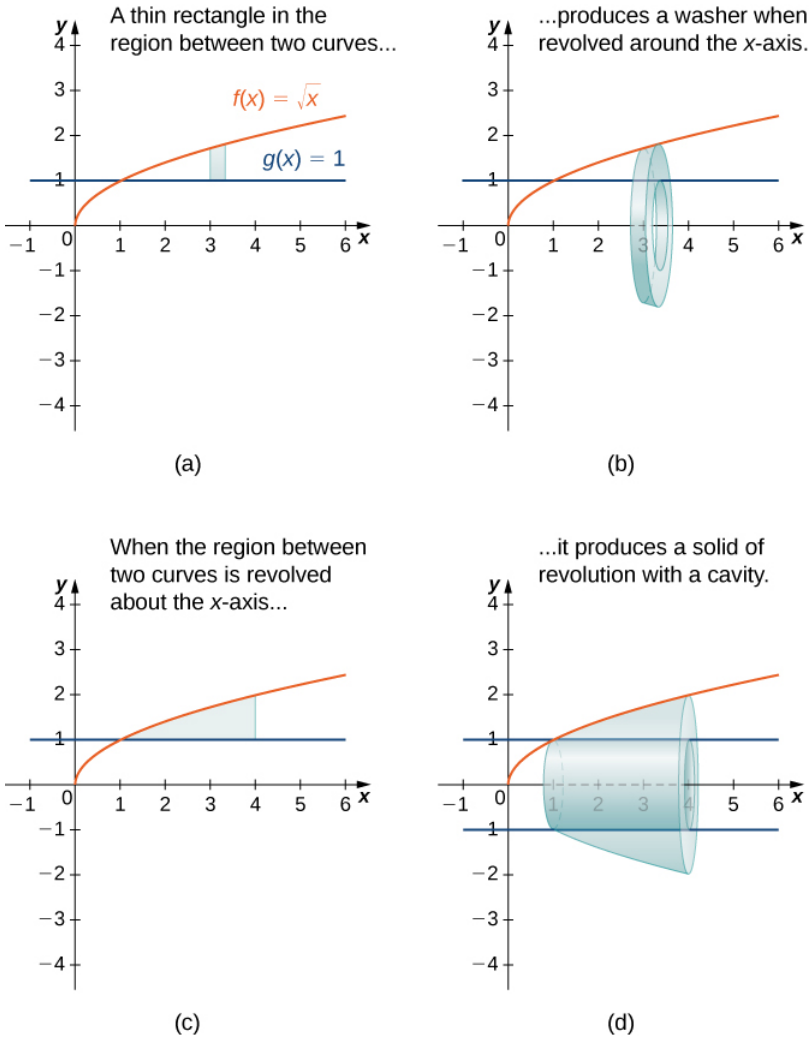


Figure 6.2.12: (a) A thin rectangle in the region between two curves. (b) A representative disk formed by revolving the rectangle about the x**-axis.** (c) The region between the curves over the given interval. (d) The resulting solid of revolution.

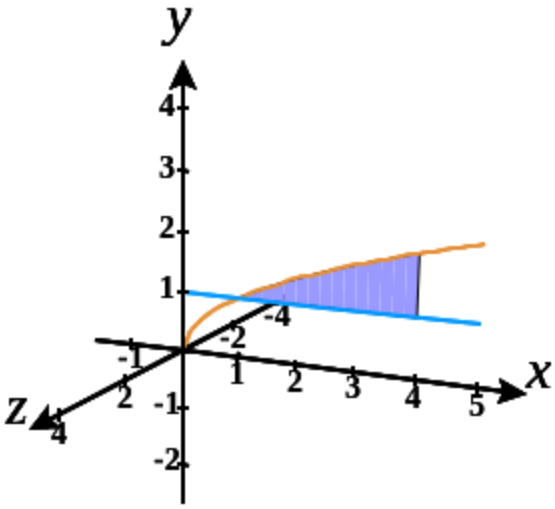


Figure 6.2.12: (e) A dynamic version of this solid of revolution generated using CalcPlot3D.

The cross-sectional area, then, is the area of the outer circle less the area of the inner circle. In this case,

$$A(x) = \pi\left(\sqrt{x}\right)^2 - \pi(1)^2 = \pi(x-1).$$

Then the volume of the solid is

$$V = \int_a^b A(x)\,dx = \int_1^4 \pi(x-1)\,dx = \pi\left[\frac{x^2}{2} - x\right]\Big|_1^4 = \frac{9}{2}\pi \text{ units}^3.$$

Generalizing this process gives the **washer method.**

Rule: The Washer Method

Suppose $f(x)$ and $g(x)$ are continuous, nonnegative functions such that $f(x) \geq g(x)$ over $[a, b]$. Let R denote the region bounded above by the graph of $f(x)$, below by the graph of $g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$. Then, the volume of the solid of revolution formed by revolving R around the x**-axis** is given by

$$V = \int_a^b \pi\left[(f(x))^2 - (g(x))^2\right]\,dx.$$

Example 6.2.5: Using the Washer Method

Find the volume of a solid of revolution formed by revolving the region bounded above by the graph of $f(x) = x$ and below by the graph of $g(x) = 1/x$ over the interval $[1, 4]$ around the x-**axis.**

Solution

The graphs of the functions and the solid of revolution are shown in the following figure.

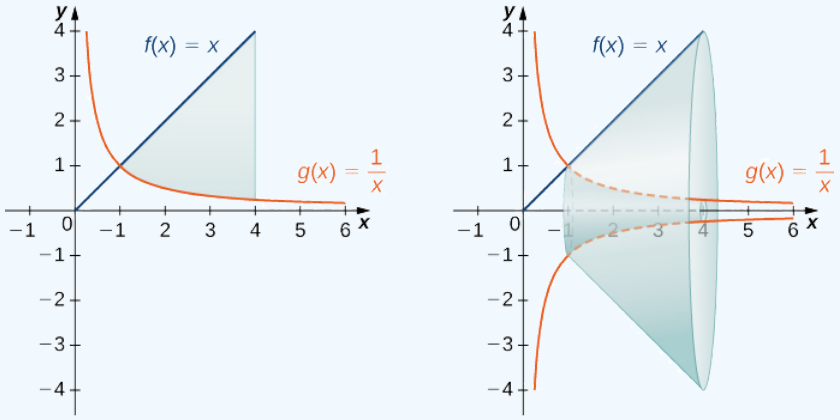


Figure 6.2.13: (a) The region between the graphs of the functions $f(x) = x$ and $g(x) = 1/x$ over the interval $[1, 4]$. (b) Revolving the region about the x-**axis** generates a solid of revolution with a cavity in the middle.

We have

$$V = \int_a^b \pi\left[(f(x))^2 - (g(x))^2\right] dx = \pi \int_1^4 \left[x^2 - \left(\frac{1}{x}\right)^2\right] dx$$
$$= \pi \left[\frac{x^3}{3} + \frac{1}{x}\right]\Big|_1^4$$
$$= \frac{81\pi}{4} \text{ units}^3.$$

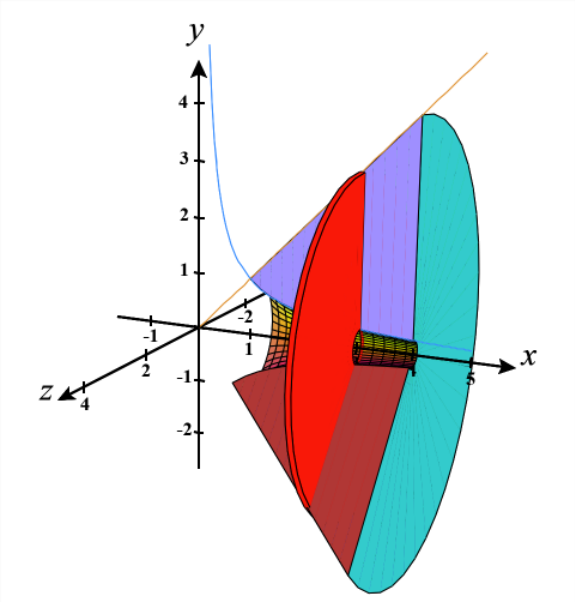


Figure 6.2.13: (c) A dynamic version of this solid of revolution generated using CalcPlot3D.

Exercise 6.2.5

Find the volume of a solid of revolution formed by revolving the region bounded by the graphs of $f(x) = \sqrt{x}$ and $g(x) = 1/x$ over the interval $[1, 3]$ around the x**-axis.**

Hint

Graph the functions to determine which graph forms the upper bound and which graph forms the lower bound, then use the procedure from Example 6.2.5.

Answer

$\frac{10\pi}{3}$ units^3

As with the disk method, we can also apply the washer method to solids of revolution that result from revolving a region around the y-axis. In this case, the following rule applies.

Rule: The Washer Method for Solids of Revolution around the y-axis

Suppose $u(y)$ and $v(y)$ are continuous, nonnegative functions such that $v(y) \leq u(y)$ for $y \in [c, d]$. Let Q denote the region bounded on the right by the graph of $u(y)$, on the left by the graph of $v(y)$, below by the line $y = c$, and above by the line $y = d$. Then, the volume of the solid of revolution formed by revolving Q around the y**-axis** is given by

$$V = \int_c^d \pi \left[(u(y))^2 - (v(y))^2 \right] \, dy.$$

Rather than looking at an example of the washer method with the y**-axis** as the axis of revolution, we now consider an example in which the axis of revolution is a line other than one of the two coordinate axes. The same general method applies, but you may have to visualize just how to describe the cross-sectional area of the volume.

Example 6.2.6:

Find the volume of a solid of revolution formed by revolving the region bounded above by $f(x) = 4 - x$ and below by the x**-axis** over the interval $[0, 4]$ around the line $y = -2$.

Solution

The graph of the region and the solid of revolution are shown in the following figure.

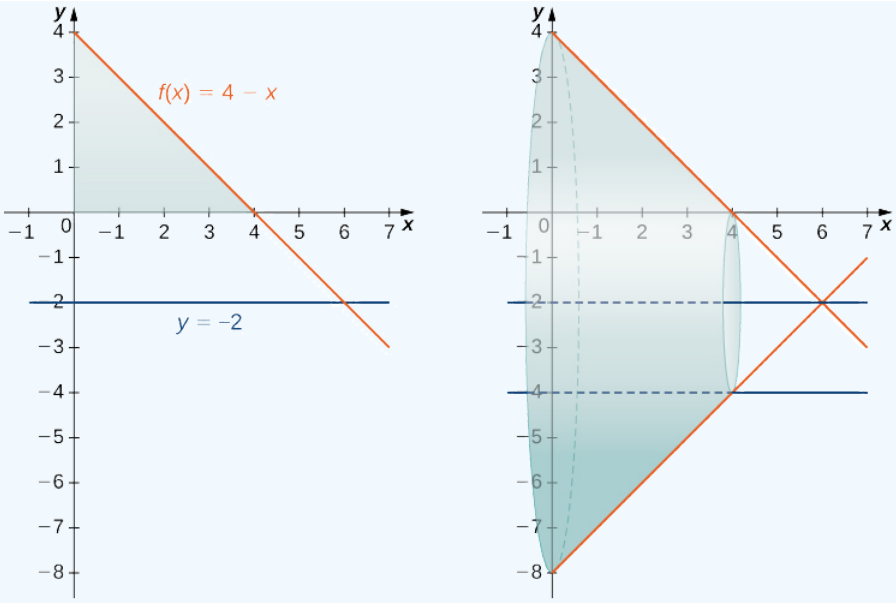


Figure 6.2.14: (a) The region between the graph of the function $f(x) = 4 - x$ and the x-**axis** over the interval $[0, 4]$. (b) Revolving the region about the line $y = -2$ generates a solid of revolution with a cylindrical hole through its middle.

We can't apply the volume formula to this problem directly because the axis of revolution is not one of the coordinate axes. However, we still know that the area of the cross-section is the area of the outer circle less the area of the inner circle. Looking at the graph of the function, we see the radius of the outer circle is given by $f(x) + 2$, which simplifies to

$$f(x) + 2 = (4 - x) + 2 = 6 - x.$$

The radius of the inner circle is $g(x) = 2$. Therefore, we have

$$\begin{aligned} V &= \int_0^4 \pi \left[(6-x)^2 - (2)^2\right] \, dx \\ &= \pi \int_0^4 (x^2 - 12x + 32) \, dx = \pi \left[\frac{x^3}{3} - 6x^2 + 32x\right]\Big|_0^4 \\ &= \frac{160\pi}{3} \text{ units}^3. \end{aligned}$$

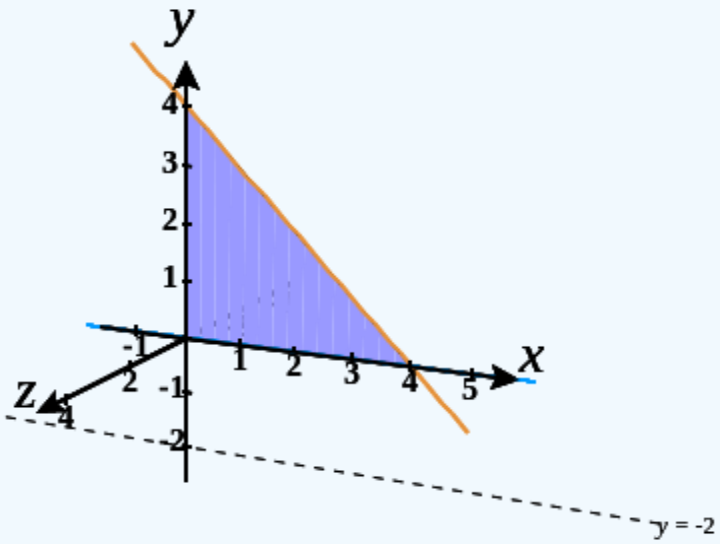


Figure 6.2.14: (c) A dynamic version of this solid of revolution generated using CalcPlot3D.

? Exercise 6.2.6

Find the volume of a solid of revolution formed by revolving the region bounded above by the graph of $f(x) = x + 2$ and below by the x-**axis** over the interval $[0, 3]$ around the line $y = -1$.

Hint

Use the procedure from Example 6.2.6.

Answer

60π units^3

Key Concepts

- Definite integrals can be used to find the volumes of solids. Using the slicing method, we can find a volume by integrating the cross-sectional area.
- For solids of revolution, the volume slices are often disks and the cross-sections are circles. The method of disks involves applying the method of slicing in the particular case in which the cross-sections are circles, and using the formula for the area of a circle.
- If a solid of revolution has a cavity in the center, the volume slices are washers. With the method of washers, the area of the inner circle is subtracted from the area of the outer circle before integrating.

Key Equations

- **Disk Method along the x-axis**

$$V = \int_a^b \pi \left[f(x) \right]^2 dx$$

- **Disk Method along the y-axis**

$$V = \int_c^d \pi \left[g(y) \right]^2 dy$$

- **Washer Method**

$$V = \int_a^b \pi \left[(f(x))^2 - (g(x))^2 \right] \, dx$$

Glossary

cross-section

the intersection of a plane and a solid object

disk method

a special case of the slicing method used with solids of revolution when the slices are disks

slicing method

a method of calculating the volume of a solid that involves cutting the solid into pieces, estimating the volume of each piece, then adding these estimates to arrive at an estimate of the total volume; as the number of slices goes to infinity, this estimate becomes an integral that gives the exact value of the volume

solid of revolution

a solid generated by revolving a region in a plane around a line in that plane

washer method

a special case of the slicing method used with solids of revolution when the slices are washers

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6.2E: Exercises for Section 6.2

1) Derive the formula for the volume of a sphere using the slicing method.

2) Use the slicing method to derive the formula for the volume of a cone.

3) Use the slicing method to derive the formula for the volume of a tetrahedron with side length a.

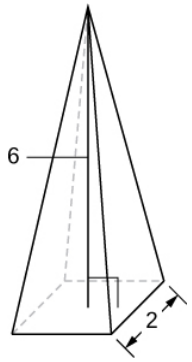
4) Use the disk method to derive the formula for the volume of a trapezoidal cylinder.

5) Explain when you would use the disk method versus the washer method. When are they interchangeable?

Volumes by Slicing

For exercises 6 - 10, draw a typical slice and find the volume using the slicing method for the given volume.

6) A pyramid with height 6 units and square base of side 2 units, as pictured here.



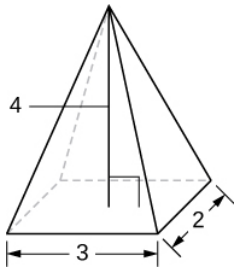
Solution:

Here the cross-sections are squares taken perpendicular to the y-axis.
We use the vertical cross-section of the pyramid through its center to obtain an equation relating x and y.
Here this would be the equation, $y = 6 - 6x$. Since we need the dimensions of the square at each y-level, we solve this equation for x to get, $x = 1 - \frac{y}{6}$.
This is half the distance across the square cross-section at the y-level, so the side length of the square cross-section is, $s = 2\left(1 - \frac{y}{6}\right)$.
Thus, we have the area of a cross-section is,

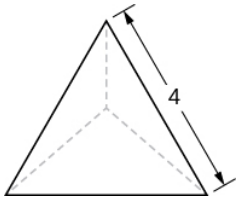
$$A(y) = \left[2\left(1 - \tfrac{y}{6}\right)\right]^2 = 4\left(1 - \tfrac{y}{6}\right)^2.$$

$$\begin{aligned}
\text{Then,}\quad V &= \int_0^6 4\left(1 - \tfrac{y}{6}\right)^2 dy \\
&= -24\int_1^0 u^2\, du, \quad \text{where } u = 1 - \tfrac{y}{6},\ \text{so } du = -\tfrac{1}{6}\, dy, \quad \Longrightarrow \quad -6\, du = dy \\
&= 24\int_0^1 u^2\, du = 24\frac{u^3}{3}\Big|_0^1 \\
&= 8u^3\Big|_0^1 \\
&= 8\left(1^3 - 0^3\right) \quad = \quad 8 \text{ units}^3
\end{aligned}$$

7) A pyramid with height 4 units and a rectangular base with length 2 units and width 3 units, as pictured here.



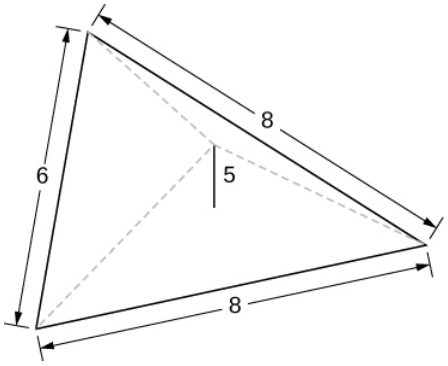
8) A tetrahedron with a base side of 4 units,as seen here.



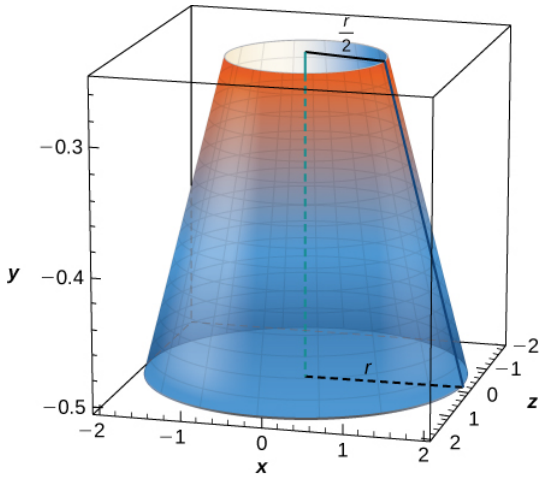
Answer

$V = \frac{32}{3\sqrt{2}} = \frac{16\sqrt{2}}{3}$ units^3

9) A pyramid with height 5 units, and an isosceles triangular base with lengths of 6 units and 8 units, as seen here.



10) A cone of radius r and height h has a smaller cone of radius $r/2$ and height $h/2$ removed from the top, as seen here. The resulting solid is called a frustum.



Answer

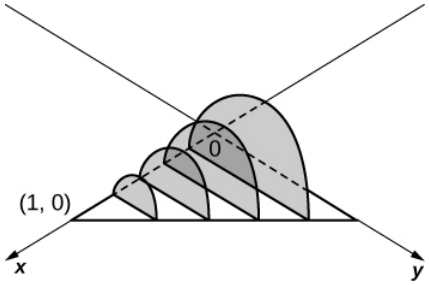
$V = \frac{7\pi}{12} hr^2$ units^3

For exercises 11 - 16, draw an outline of the solid and find the volume using the slicing method.

11) The base is a circle of radius a. The slices perpendicular to the base are squares.

12) The base is a triangle with vertices $(0,0), (1,0)$, and $(0,1)$. Slices perpendicular to the xy-plane are semicircles.

Answer

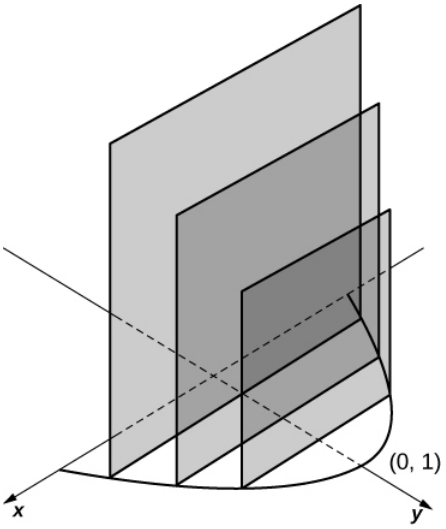


$$V = \int_0^1 \frac{\pi(1-x)^2}{8}\,dx \quad = \quad \frac{\pi}{24} \text{ units}^3$$

13) The base is the region under the parabola $y = 1 - x^2$ in the first quadrant. Slices perpendicular to the xy-plane are squares.

14) The base is the region under the parabola $y = 1 - x^2$ and above the x-**axis**. Slices perpendicular to the y-**axis** are squares.

Answer

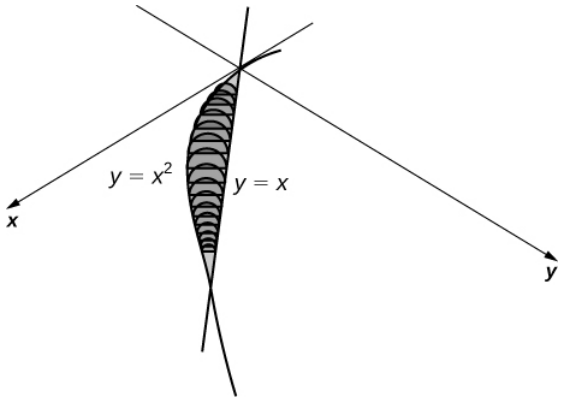


$$V = \int_0^1 4(1-y)\,dy \quad = \quad 2 \text{ units}^3$$

15) The base is the region enclosed by $y = x^2$ and $y = 9$. Slices perpendicular to the x-axis are right isosceles triangles.

16) The base is the area between $y = x$ and $y = x^2$. Slices perpendicular to the x-axis are semicircles.

Answer



$$V = \int_0^1 \frac{\pi}{8}(x - x^2)^2 \, dx \quad = \quad \frac{\pi}{240} \text{ units}^3$$

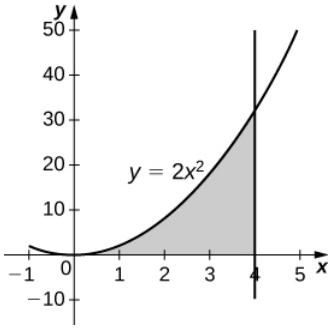
Disk and Washer Method

For exercises 17 - 24, draw the region bounded by the curves. Then, use the disk or washer method to find the volume when the region is rotated around the x-axis.

17) $x + y = 8, \quad x = 0$, and $y = 0$

18) $y = 2x^2, \quad x = 0, \quad x = 4$, and $y = 0$

Answer

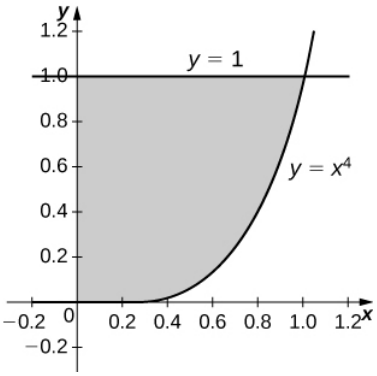


$$V = \int_0^4 4\pi x^4 \, dx \quad = \quad \frac{4096\pi}{5} \text{ units}^3$$

19) $y = e^x + 1, \quad x = 0, \quad x = 1$, and $y = 0$

20) $y = x^4, \quad x = 0$, and $y = 1$

Answer



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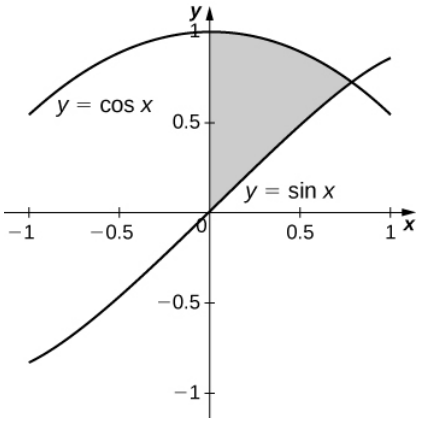
$$V = \int_0^1 \pi\left(1^2 - \left(x^4\right)^2\right)\,dx = \int_0^1 \pi\left(1 - x^8\right)\,dx \quad = \quad \frac{8\pi}{9} \text{ units}^3$$

21) $y = \sqrt{x}, \quad x = 0, \quad x = 4$, and $y = 0$

22) $y = \sin x, \quad y = \cos x$, and $x = 0$

Answer

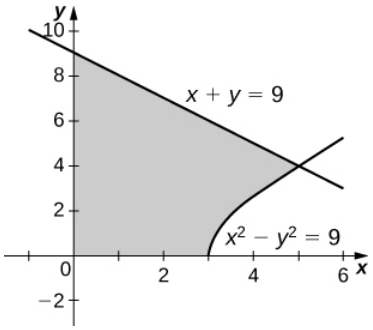


$$V = \int_0^{\pi/4} \pi\left(\cos^2 x - \sin^2 x\right)\,dx = \int_0^{\pi/4} \pi \cos 2x\,dx \quad = \quad \frac{\pi}{2} \text{ units}^3$$

23) $y = \dfrac{1}{x}, \quad x = 2$, and $y = 3$

24) $x^2 - y^2 = 9$ and $x + y = 9, \quad y = 0$ and $x = 0$

Answer



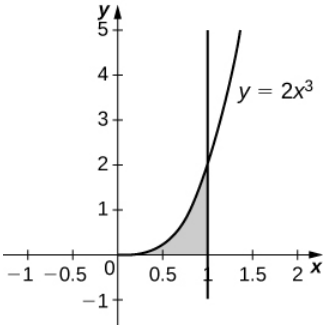
$V = 207\pi$ units^3

For exercises 25 - 32, draw the region bounded by the curves. Then, find the volume when the region is rotated around the y -axis.

25) $y = 4 - \dfrac{1}{2}x, \quad x = 0$, and $y = 0$

26) $y = 2x^3, \quad x = 0, \quad x = 1$, and $y = 0$

Answer

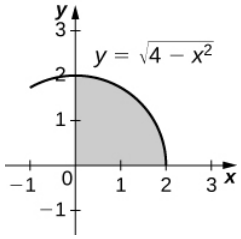


$V = \frac{4\pi}{5}$ units^3

27) $y = 3x^2$, $x = 0$, and $y = 3$

28) $y = \sqrt{4 - x^2}$, $y = 0$, and $x = 0$

Answer

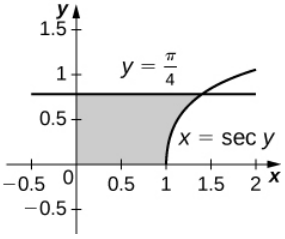


$V = \frac{16\pi}{3}$ units^3

29) $y = \dfrac{1}{\sqrt{x+1}}$, $x = 0$, and $x = 3$

30) $x = \sec(y)$ and $y = \dfrac{\pi}{4}$, $y = 0$ and $x = 0$

Answer

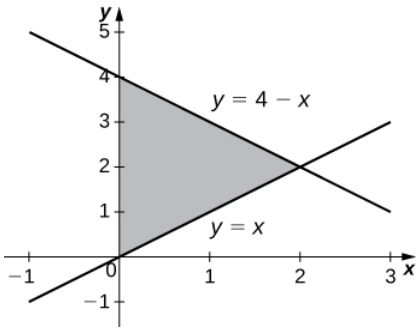


$V = \pi$ units^3

31) $y = \dfrac{1}{x+1}$, $x = 0$, and $x = 2$

32) $y = 4 - x$, $y = x$, and $x = 0$

Answer



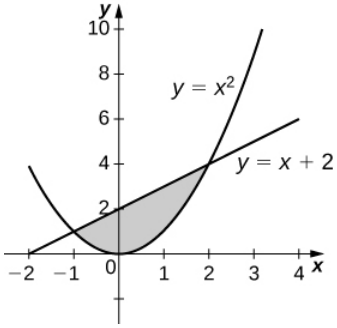
$V = \frac{16\pi}{3}$ units^3

For exercises 33 - 40, draw the region bounded by the curves. Then, find the volume when the region is rotated around the x -axis.

33) $y = x + 2, \quad y = x + 6, \quad x = 0$, and $x = 5$

34) $y = x^2$ and $y = x + 2$

Answer

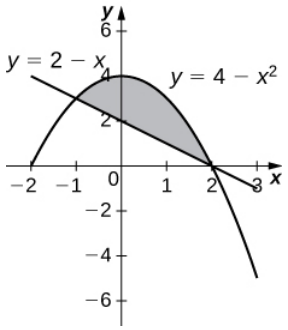


$V = \frac{72\pi}{5}$ units^3

35) $x^2 = y^3$ and $x^3 = y^2$

36) $y = 4 - x^2$ and $y = 2 - x$

Answer

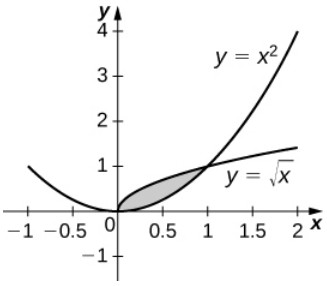


$V = \frac{108\pi}{5}$ units^3

37) [T] $y = \cos x, \quad y = e^{-x}, \quad x = 0$, and $x = 1.2927$

38) $y = \sqrt{x}$ and $y = x^2$

Answer

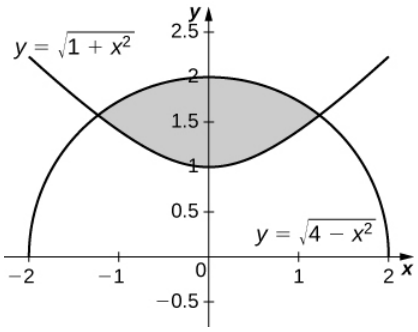


$V = \frac{3\pi}{10}$ units^3

39) $y = \sin x, \quad y = 5\sin x, \quad x = 0$ and $x = \pi$

40) $y = \sqrt{1 + x^2}$ and $y = \sqrt{4 - x^2}$

Answer



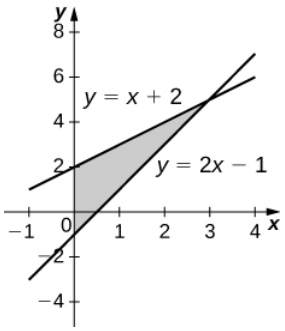
$V = 2\sqrt{6}\pi$ units^3

For exercises 41 - 45, draw the region bounded by the curves. Then, use the washer method to find the volume when the region is revolved around the y-axis.

41) $y = \sqrt{x}, \quad x = 4$, and $y = 0$

42) $y = x + 2, \quad y = 2x - 1$, and $x = 0$

Answer

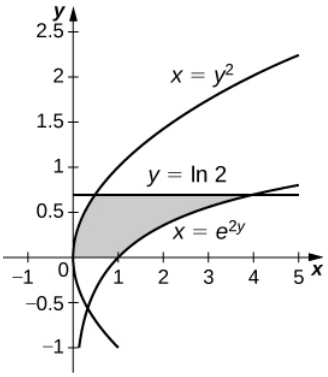


$V = 9\pi$ units^3

43) $y = \dfrac{3}{x}$ and $y = x^3$

44) $x = e^{2y}, \quad x = y^2, \quad y = 0$, and $y = \ln(2)$

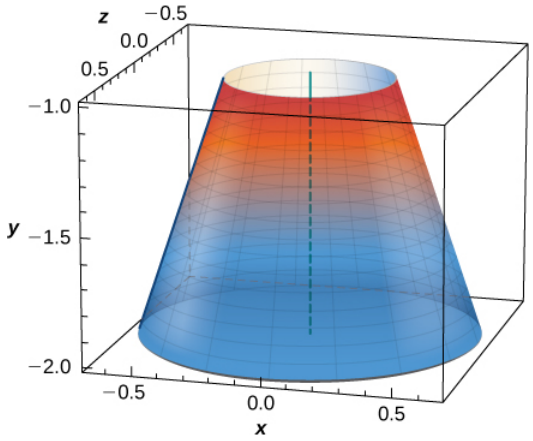
Answer



$$V = \frac{\pi}{20}(75 - 4\ln^5(2)) \text{ units}^3$$

45) $x = \sqrt{9 - y^2}$, $x = e^{-y}$, $y = 0$, and $y = 3$

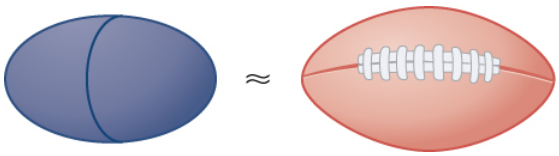
46) Yogurt containers can be shaped like frustums. Rotate the line $y = \left(\frac{1}{m}\right)x$ around the y-axis to find the volume between $y = a$ and $y = b$.



Answer

$$V = \frac{m^2\pi}{3}(b^3 - a^3) \text{ units}^3$$

47) Rotate the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ around the x-axis to approximate the volume of a football, as seen here.



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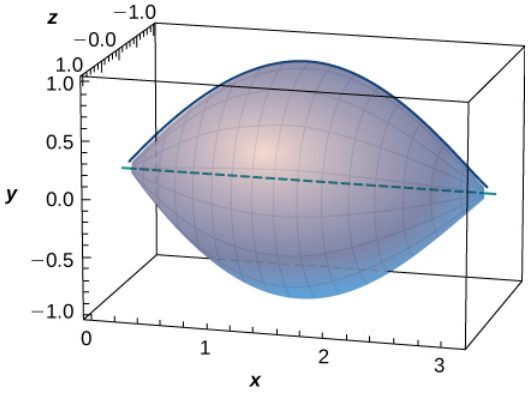
48) Rotate the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ around the y-axis to approximate the volume of a football.

Answer

$V=\frac{4a^2b\pi}{3}$ units^3

49) A better approximation of the volume of a football is given by the solid that comes from rotating $y=\sin x$ around the x-axis from $x=0$ to $x=\pi$. What is the volume of this football approximation, as seen here?



For exercises 51 - 56, find the volume of the solid described.

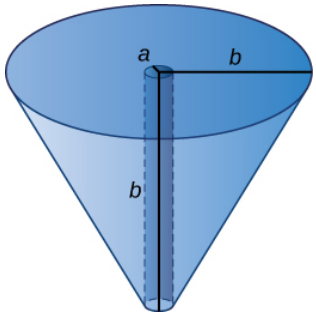
51) The base is the region between $y=x$ and $y=x^2$. Slices perpendicular to the x-axis are semicircles.

52) The base is the region enclosed by the generic ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Slices perpendicular to the x-axis are semicircles.

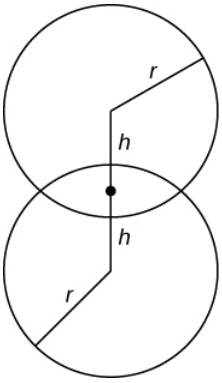
Answer

$V=\frac{2ab^2\pi}{3}$ units^3

53) Bore a hole of radius a down the axis of a right cone and through the base of radius b, as seen here.



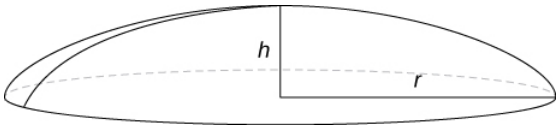
54) Find the volume common to two spheres of radius r with centers that are $2h$ apart, as shown here.



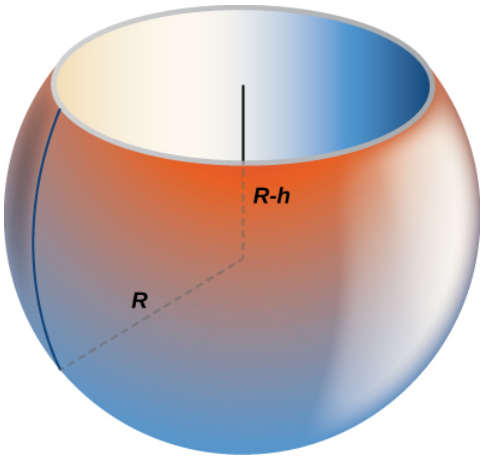
Answer

$V = \frac{\pi}{12}(r+h)^2(6r-h)$ units^3

55) Find the volume of a spherical cap of height h and radius r where $h < r$, as seen here.



56) Find the volume of a sphere of radius R with a cap of height h removed from the top, as seen here.



Answer

$V = \frac{\pi}{3}(h+R)(h-2R)^2$ units^3

Contributors

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6.3: Volumes of Revolution - Cylindrical Shells

Learning Objectives

- Calculate the volume of a solid of revolution by using the method of cylindrical shells.
- Compare the different methods for calculating a volume of revolution.

In this section, we examine the method of cylindrical shells, the final method for finding the volume of a solid of revolution. We can use this method on the same kinds of solids as the disk method or the washer method; however, with the disk and washer methods, we integrate along the coordinate axis parallel to the axis of revolution. With the method of cylindrical shells, we integrate along the coordinate *axis perpendicular* to the axis of revolution. The ability to choose which variable of integration we want to use can be a significant advantage with more complicated functions. Also, the specific geometry of the solid sometimes makes the method of using cylindrical shells more appealing than using the washer method. In the last part of this section, we review all the methods for finding volume that we have studied and lay out some guidelines to help you determine which method to use in a given situation.

The Method of Cylindrical Shells

Again, we are working with a solid of revolution. As before, we define a region R, bounded above by the graph of a function $y = f(x)$, below by the x**-axis**, and on the left and right by the lines $x = a$ and $x = b$, respectively, as shown in Figure 6.3.1a. We then revolve this region around the y-axis, as shown in Figure 6.3.1b. Note that this is different from what we have done before. Previously, regions defined in terms of functions of x were revolved around the x**-axis** or a line parallel to it.

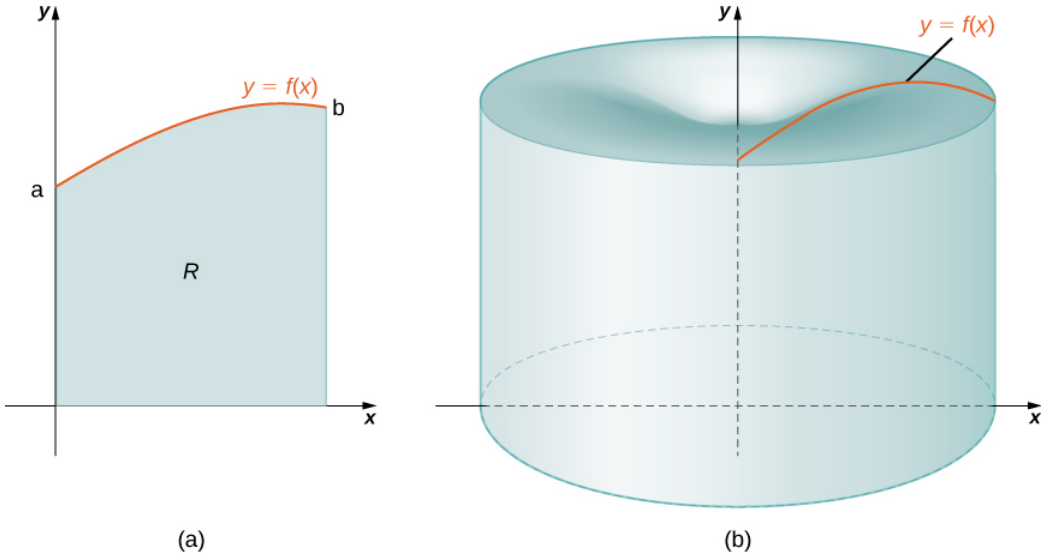


Figure 6.3.1: (a) A region bounded by the graph of a function of x. (b) The solid of revolution formed when the region is revolved around the y**-axis.**

As we have done many times before, partition the interval $[a, b]$ using a regular partition, $P = x_0, x_1, \ldots, x_n$ and, for $i = 1, 2, \ldots, n$, choose a point $x_i^* \in [x_{i-1}, x_i]$. Then, construct a rectangle over the interval $[x_{i-1}, x_i]$ of height $f(x_i^*)$ and width Δx. A representative rectangle is shown in Figure 6.3.2a. When that rectangle is revolved around the y-axis, instead of a disk or a washer, we get a cylindrical shell, as shown in Figure 6.3.2.

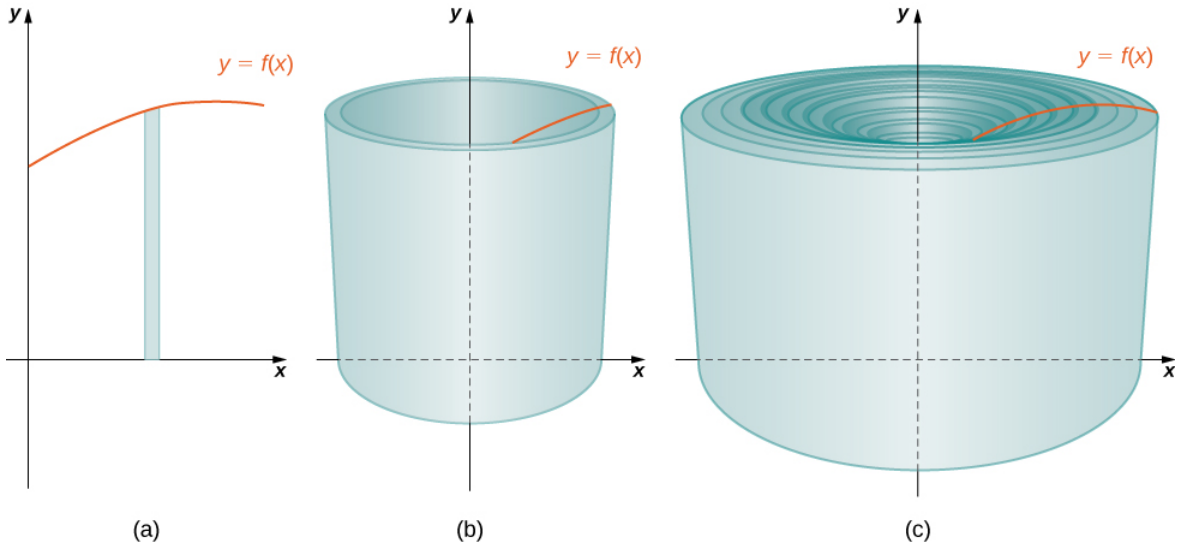


Figure 6.3.2: (a) A representative rectangle. (b) When this rectangle is revolved around the y-**axis**, the result is a cylindrical shell. (c) When we put all the shells together, we get an approximation of the original solid.

To calculate the volume of this shell, consider Figure 6.3.3.

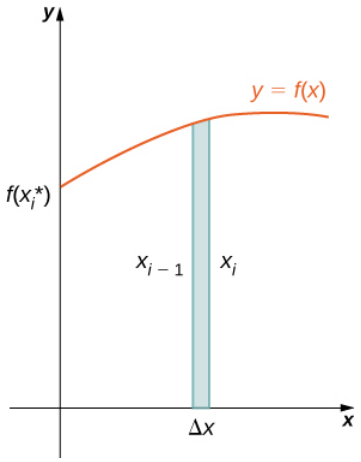


Figure 6.3.3: Calculating the volume of the shell.

The shell is a cylinder, so its volume is the cross-sectional area multiplied by the height of the cylinder. The cross-sections are annuli (ring-shaped regions—essentially, circles with a hole in the center), with outer radius x_i and inner radius x_{i-1}. Thus, the cross-sectional area is $\pi x_i^2 - \pi x_{i-1}^2$. The height of the cylinder is $f(x_i^*)$. Then the volume of the shell is

$$\begin{aligned} V_{shell} &= f(x_i^*)(\pi\, x_i^2 - \pi\, x_{i-1}^2) \\ &= \pi\, f(x_i^*)(x_i^2 - x_{i-1}^2) \\ &= \pi\, f(x_i^*)(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= 2\pi\, f(x_i^*)\left(\frac{x_i + x_{i-1}}{2}\right)(x_i - x_{i-1}). \end{aligned}$$

Note that $x_i - x_{i-1} = \Delta x$, so we have

$$V_{shell} = 2\pi\, f(x_i^*)\left(\frac{x_i + x_{i-1}}{2}\right)\Delta x.$$

Furthermore, $\dfrac{x_i + x_{i-1}}{2}$ is both the midpoint of the interval $[x_{i-1}, x_i]$ and the average radius of the shell, and we can approximate this by x_i^*. We then have

$$V_{shell} \approx 2\pi\, f(x_i^*) x_i^*\, \Delta x.$$

Another way to think of this is to think of making a vertical cut in the shell and then opening it up to form a flat plate (Figure 6.3.4).

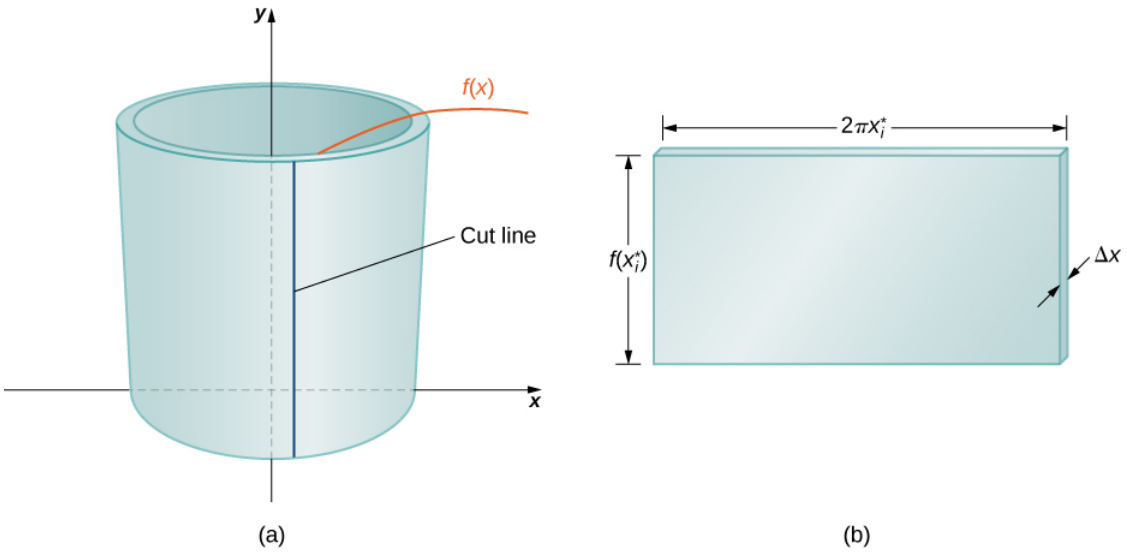


Figure 6.3.4: (a) Make a vertical cut in a representative shell. (b) Open the shell up to form a flat plate.

In reality, the outer radius of the shell is greater than the inner radius, and hence the back edge of the plate would be slightly longer than the front edge of the plate. However, we can approximate the flattened shell by a flat plate of height $f(x_i^*)$, width $2\pi x_i^*$, and thickness Δx (Figure). The volume of the shell, then, is approximately the volume of the flat plate. Multiplying the height, width, and depth of the plate, we get

$$V_{shell} \approx f(x_i^*)(2\pi\, x_i^*)\,\Delta x,$$

which is the same formula we had before.

To calculate the volume of the entire solid, we then add the volumes of all the shells and obtain

$$V \approx \sum_{i=1}^{n} (2\pi\, x_i^* f(x_i^*)\,\Delta x).$$

Here we have another Riemann sum, this time for the function $2\pi\, x\, f(x)$. Taking the limit as $n \to \infty$ gives us

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} (2\pi\, x_i^* f(x_i^*)\,\Delta x) = \int_a^b (2\pi\, x\, f(x))\, dx.$$

This leads to the following rule for the *method of cylindrical shells*.

Rule: The Method of Cylindrical Shells

Let $f(x)$ be continuous and nonnegative. Define R as the region bounded above by the graph of $f(x)$, below by the x-**axis,** on the left by the line $x = a$, and on the right by the line $x = b$. Then the volume of the solid of revolution formed by revolving R around the y-axis is given by

$$V = \int_a^b (2\pi\, x\, f(x))\, dx.$$

Now let's consider an example.

Example 6.3.1: The Method of Cylindrical Shells I

Define R as the region bounded above by the graph of $f(x) = 1/x$ and below by the x-**axis** over the interval $[1, 3]$. Find the volume of the solid of revolution formed by revolving R around the y-axis.

Solution

First we must graph the region R and the associated solid of revolution, as shown in Figure 6.3.5.

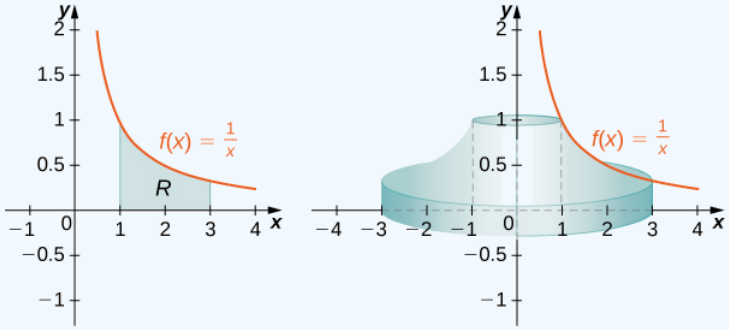


Figure 6.3.5: (a) The region R under the graph of $f(x) = 1/x$ over the interval $[1, 3]$. (b) The solid of revolution generated by revolving R about the y-axis.

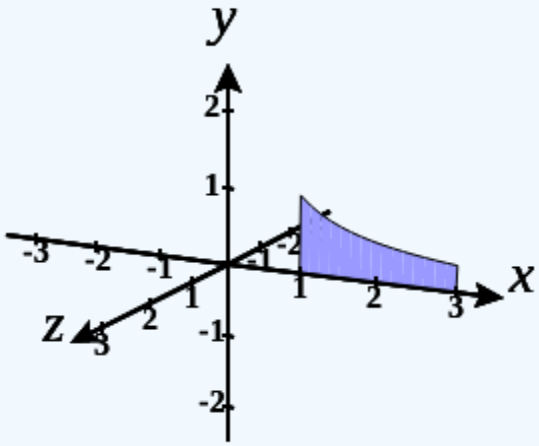


Figure 6.3.5 (c) Visualizing the solid of revolution with CalcPlot3D.

Then the volume of the solid is given by

$$V = \int_a^b (2\pi\, x\, f(x))\, dx$$
$$= \int_1^3 \left(2\pi\, x \left(\frac{1}{x}\right)\right) dx$$
$$= \int_1^3 2\pi\, dx$$
$$= 2\pi\, x\Big|_1^3 = 4\pi \text{ units}^3.$$

Exercise 6.3.1

Define R as the region bounded above by the graph of $f(x) = x^2$ and below by the x-axis over the interval $[1, 2]$. Find the volume of the solid of revolution formed by revolving R around the y-axis.

Hint

Use the procedure from Example 6.3.1.

Answer

$\frac{15\pi}{2}$ units^3

Example 6.3.2: The Method of Cylindrical Shells II

Define R as the region bounded above by the graph of $f(x) = 2x - x^2$ and below by the x-axis over the interval $[0, 2]$. Find the volume of the solid of revolution formed by revolving R around the y-**axis.**

Solution

First graph the region R and the associated solid of revolution, as shown in Figure 6.3.6.

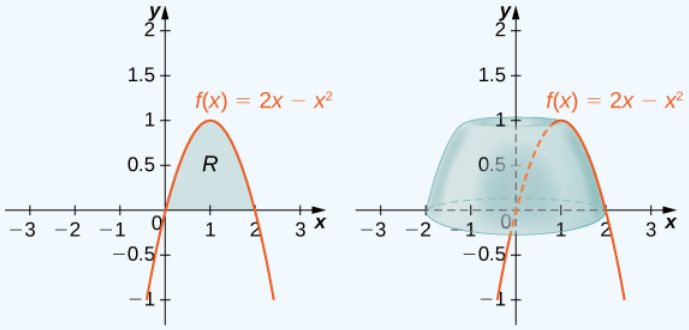


Figure 6.3.6: (a) The region R *under the graph of* $f(x) = 2x - x^2$ *over the interval* $[0, 2]$. *(b) The volume of revolution obtained by revolving* R *about the* y*-axis.*

Then the volume of the solid is given by

$$V = \int_a^b (2\pi\, x\, f(x))\, dx$$
$$= \int_0^2 (2\pi\, x(2x - x^2))\, dx$$
$$= 2\pi \int_0^2 (2x^2 - x^3)\, dx$$
$$= 2\pi \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]\Big|_0^2$$
$$= \frac{8\pi}{3} \text{ units}^3$$

Exercise 6.3.2

Define R as the region bounded above by the graph of $f(x) = 3x - x^2$ and below by the x-axis over the interval $[0, 2]$. Find the volume of the solid of revolution formed by revolving R around the y-axis.

Hint

Use the process from Example 6.3.2.

Answer

8π units^3

As with the disk method and the washer method, we can use the method of cylindrical shells with solids of revolution, revolved around the x-axis, when we want to integrate with respect to y. The analogous rule for this type of solid is given here.

Rule: The Method of Cylindrical Shells for Solids of Revolution around the x-axis

Let $g(y)$ be continuous and nonnegative. Define Q as the region bounded on the right by the graph of $g(y)$, on the left by the y-axis, below by the line $y = c$, and above by the line $y = d$. Then, the volume of the solid of revolution formed by revolving Q around the x-axis is given by

$$V = \int_c^d (2\pi\, y\, g(y))\, dy.$$

Example 6.3.3: The Method of Cylindrical Shells for a Solid Revolved around the x-axis

Define Q as the region bounded on the right by the graph of $g(y) = 2\sqrt{y}$ and on the left by the y-axis for $y \in [0, 4]$. Find the volume of the solid of revolution formed by revolving Q around the x-axis.

Solution

First, we need to graph the region Q and the associated solid of revolution, as shown in Figure 6.3.7.

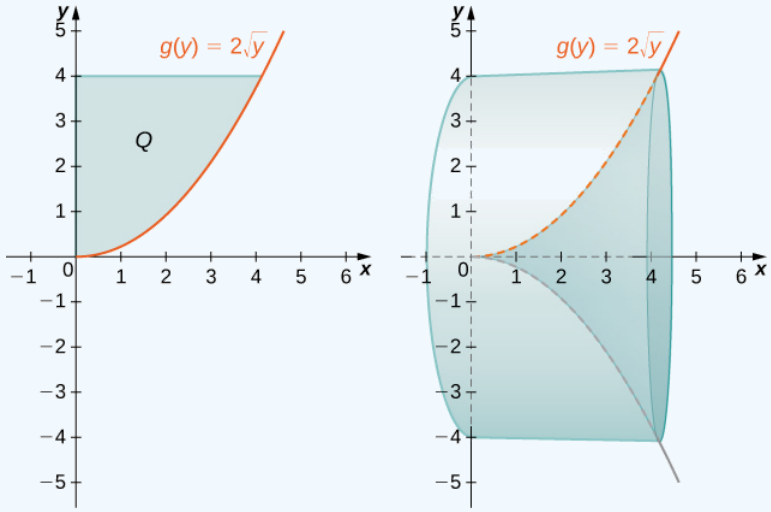


Figure 6.3.7: (a) The region Q to the left of the function $g(y)$ over the interval $[0, 4]$. (b) The solid of revolution generated by revolving Q around the x-axis.

Label the shaded region Q. Then the volume of the solid is given by

$$\begin{aligned} V &= \int_c^d (2\pi\, y\, g(y))\, dy \\ &= \int_0^4 (2\pi\, y(2\sqrt{y}))\, dy \\ &= 4\pi \int_0^4 y^{3/2}\, dy \\ &= 4\pi \left[\frac{2y^{5/2}}{5} \right] |_0^4 \\ &= \frac{256\pi}{5} \text{ units}^3 \end{aligned}$$

Exercise 6.3.3

Define Q as the region bounded on the right by the graph of $g(y) = 3/y$ and on the left by the y-axis for $y \in [1, 3]$. Find the volume of the solid of revolution formed by revolving Q around the x-axis.

Hint

Use the process from Example 6.3.3.

Answer

12π units^3

For the next example, we look at a solid of revolution for which the graph of a function is revolved around a line other than one of the two coordinate axes. To set this up, we need to revisit the development of the method of cylindrical shells. Recall that we found the volume of one of the shells to be given by

$$\begin{aligned} V_{shell} &= f(x_i^*)(\pi\, x_i^2 - \pi\, x_{i-1}^2) \\ &= \pi\, f(x_i^*)(x_i^2 - x_{i-1}^2) \\ &= \pi\, f(x_i^*)(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= 2\pi\, f(x_i^*) \left(\frac{x_i + x_{i-1}}{2} \right) (x_i - x_{i-1}). \end{aligned}$$

This was based on a shell with an outer radius of x_i and an inner radius of x_{i-1}. If, however, we rotate the region around a line other than the y-axis, we have a different outer and inner radius. Suppose, for example, that we rotate the region around the line $x = -k$, where k is some positive constant. Then, the outer radius of the shell is $x_i + k$ and the inner radius of the shell is $x_{i-1} + k$. Substituting these terms into the expression for volume, we see that when a plane region is rotated around the line $x = -k$, the volume of a shell is given by

$$\begin{aligned} V_{shell} &= 2\pi\, f(x_i^*)(\frac{(x_i + k) + (x_{i-1} + k)}{2})((x_i + k) - (x_{i-1} + k)) \\ &= 2\pi\, f(x_i^*) \left(\left(\frac{x_i + x_{i-2}}{2} \right) + k \right) \Delta x. \end{aligned}$$

As before, we notice that $\frac{x_i + x_{i-1}}{2}$ is the midpoint of the interval $[x_{i-1}, x_i]$ and can be approximated by x_i^*. Then, the approximate volume of the shell is

$$V_{shell} \approx 2\pi(x_i^* + k) f(x_i^*) \Delta x.$$

The remainder of the development proceeds as before, and we see that

$$V = \int_a^b (2\pi(x + k) f(x)) dx.$$

We could also rotate the region around other horizontal or vertical lines, such as a vertical line in the right half plane. In each case, the volume formula must be adjusted accordingly. Specifically, the x-term in the integral must be replaced with an expression representing the radius of a shell. To see how this works, consider the following example.

Example 6.3.4: A Region of Revolution Revolved around a Line

Define R as the region bounded above by the graph of $f(x) = x$ and below by the x-axis over the interval $[1, 2]$. Find the volume of the solid of revolution formed by revolving R around the line $x = -1$.

Solution

First, graph the region R and the associated solid of revolution, as shown in Figure 6.3.8.

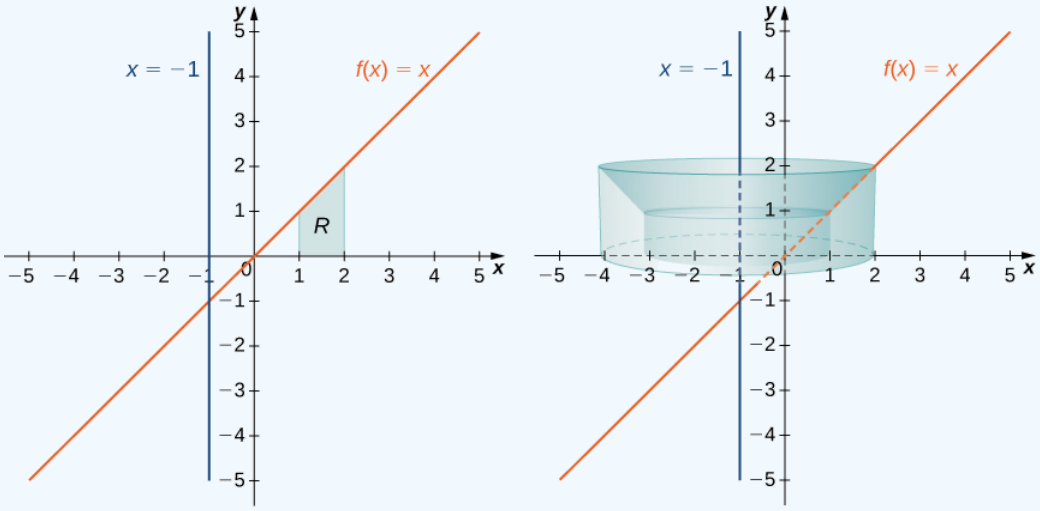


Figure 6.3.8: (a) The region R between the graph of $f(x)$ and the x-axis over the interval $[1, 2]$. (b) The solid of revolution generated by revolving R around the line $x = -1$.

Note that the radius of a shell is given by $x + 1$. Then the volume of the solid is given by

$$\begin{aligned} V &= \int_1^2 2\pi(x+1)f(x)\,dx \\ &= \int_1^2 2\pi(x+1)x\,dx = 2\pi\int_1^2 x^2 + x\,dx \\ &= 2\pi\left[\frac{x^3}{3} + \frac{x^2}{2}\right]\Big|_1^2 \\ &= \frac{23\pi}{3}\ \text{units}^3 \end{aligned}$$

Exercise 6.3.4

Define R as the region bounded above by the graph of $f(x) = x^2$ and below by the x-axis over the interval $[0, 1]$. Find the volume of the solid of revolution formed by revolving R around the line $x = -2$.

Hint

Use the process from Example 6.3.4.

Answer

$\frac{11\pi}{6}\ \text{units}^3$

For our final example in this section, let's look at the volume of a solid of revolution for which the region of revolution is bounded by the graphs of two functions.

Example 6.3.5: A Region of Revolution Bounded by the Graphs of Two Functions

Define R as the region bounded above by the graph of the function $f(x) = \sqrt{x}$ and below by the graph of the function $g(x) = 1/x$ over the interval $[1, 4]$. Find the volume of the solid of revolution generated by revolving R around the y-axis.

Solution

First, graph the region R and the associated solid of revolution, as shown in Figure 6.3.9.

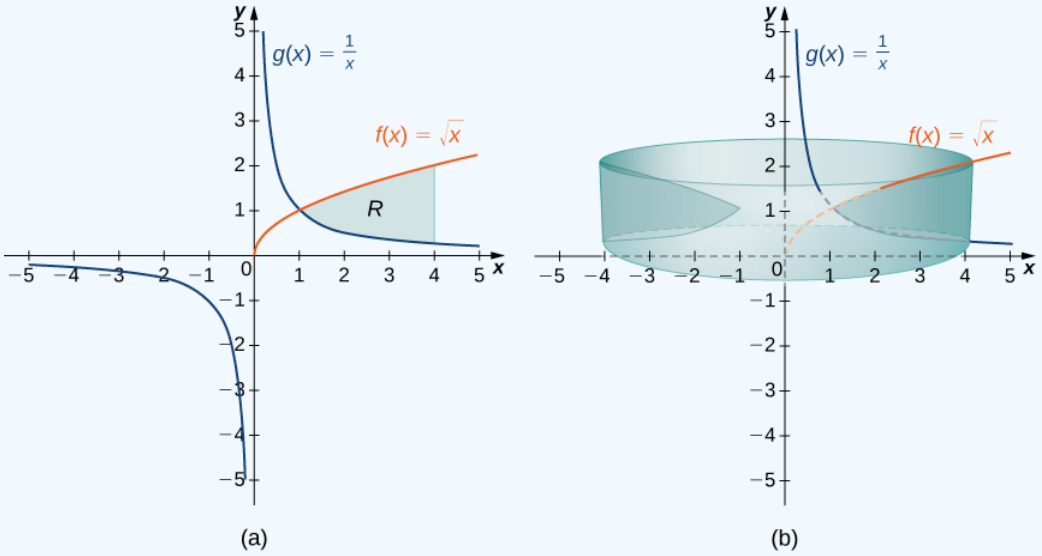


Figure 6.3.9: (a) The region R between the graph of $f(x)$ and the graph of $g(x)$ over the interval $[1, 4]$. (b) The solid of revolution generated by revolving R around the y-axis.

Note that the axis of revolution is the y-axis, so the radius of a shell is given simply by x. We don't need to make any adjustments to the x-term of our integrand. The height of a shell, though, is given by $f(x) - g(x)$, so in this case we need to adjust the $f(x)$ term of the integrand. Then the volume of the solid is given by

$$V = \int_1^4 (2\pi x(f(x) - g(x)))\, dx$$

$$= \int_1^4 (2\pi x(\sqrt{x} - \frac{1}{x}))\, dx = 2\pi \int_1^4 (x^{3/2} - 1) dx$$

$$= 2\pi \left[\frac{2x^{5/2}}{5} - x\right]\Big|_1^4 = \frac{94\pi}{5} \text{ units}^3.$$

Exercise 6.3.5

Define R as the region bounded above by the graph of $f(x) = x$ and below by the graph of $g(x) = x^2$ over the interval $[0, 1]$. Find the volume of the solid of revolution formed by revolving R around the y-axis.

Hint

Hint: Use the process from Example 6.3.5.

Answer

$\frac{\pi}{6}$ units^3

Which Method Should We Use?

We have studied several methods for finding the volume of a solid of revolution, but how do we know which method to use? It often comes down to a choice of which integral is easiest to evaluate. Figure 6.3.10 describes the different approaches for solids of revolution around the x-axis. It's up to you to develop the analogous table for solids of revolution around the y-axis.

Comparing the Methods for Finding the Volume of a Solid Revolution around the x-axis

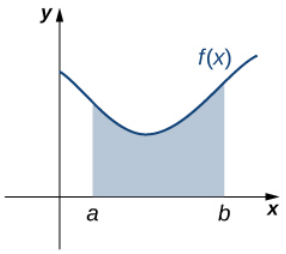
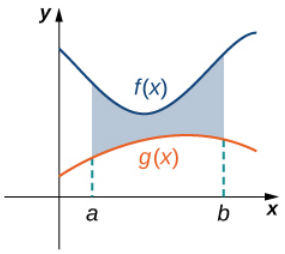
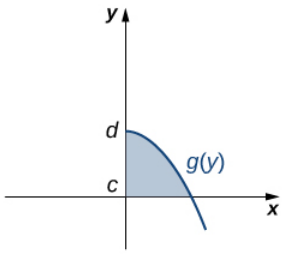
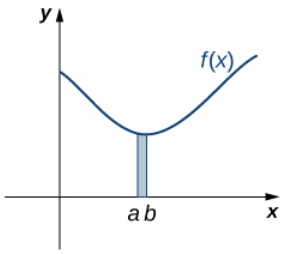
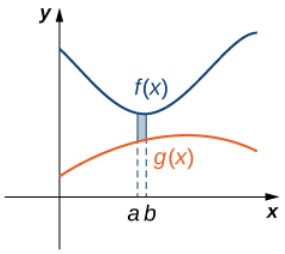
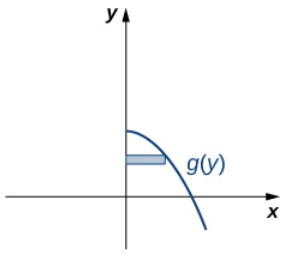
Compare	Disk Method	Washer Method	Shell Method
Volume formula	$V = \int_a^b \pi[f(x)]^2\,dx$	$V = \int_a^b \pi\left[(f(x))^2 - (g(x))^2\right]dx$	$V = \int_c^d 2\pi y\, g(y)\,dy$
Solid	No cavity in the center	Cavity in the center	With or without a cavity in the center
Interval to partition	$[a, b]$ on x-axis	$[a, b]$ on x-axis	$[c, d]$ on y-axis
Rectangle	Vertical	Vertical	Horizontal
Typical region			
Typical element			

Figure 6.3.10

Let's take a look at a couple of additional problems and decide on the best approach to take for solving them.

Example 6.3.6: Selecting the Best Method

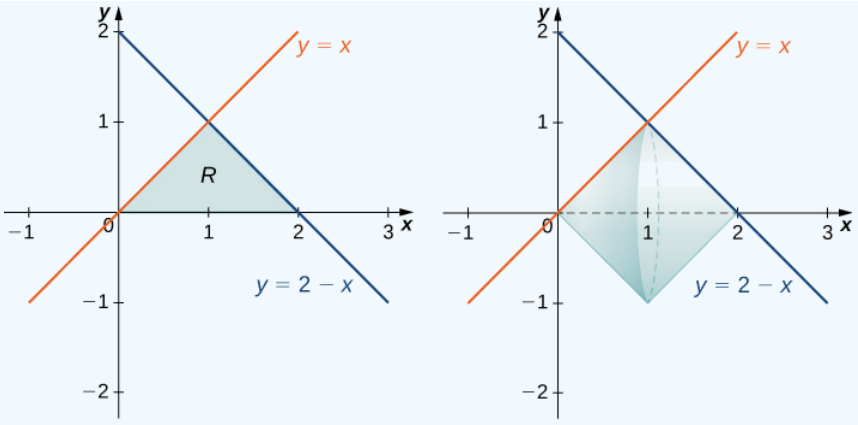
For each of the following problems, select the best method to find the volume of a solid of revolution generated by revolving the given region around the x-axis, and set up the integral to find the volume (do not evaluate the integral).

a. The region bounded by the graphs of $y = x, y = 2 - x$, and the x-axis.
b. The region bounded by the graphs of $y = 4x - x^2$ and the x-axis.

Solution

a.

First, sketch the region and the solid of revolution as shown.



(a) (b)

Figure 6.3.11: (a) The region R bounded by two lines and the x-axis. (b) The solid of revolution generated by revolving R about the x-axis.

Looking at the region, if we want to integrate with respect to x, we would have to break the integral into two pieces, because we have different functions bounding the region over $[0, 1]$ and $[1, 2]$. In this case, using the disk method, we would have

$$V = \int_0^1 \pi x^2 \, dx + \int_1^2 \pi (2 - x)^2 \, dx.$$

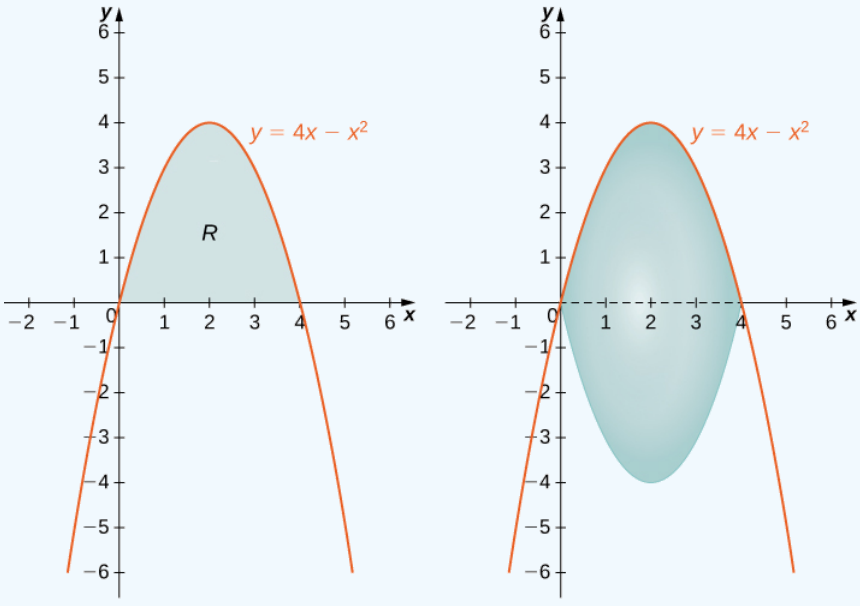
If we used the shell method instead, we would use functions of y to represent the curves, producing

$$V = \int_0^1 2\pi y[(2 - y) - y] \, dy = \int_0^1 2\pi y[2 - 2y] \, dy.$$

Neither of these integrals is particularly onerous, but since the shell method requires only one integral, and the integrand requires less simplification, we should probably go with the shell method in this case.

b.

First, sketch the region and the solid of revolution as shown.



(a) (b)

Figure 6.3.12: (a) The region R between the curve and the x-axis. (b) The solid of revolution generated by revolving R about the x-axis.

Looking at the region, it would be problematic to define a horizontal rectangle; the region is bounded on the left and right by the same function. Therefore, we can dismiss the method of shells. The solid has no cavity in the middle, so we can use the

method of disks. Then

$$V = \int_0^4 \pi \left(4x - x^2\right)^2 \, dx$$

? Exercise 6.3.6

Select the best method to find the volume of a solid of revolution generated by revolving the given region around the x-axis, and set up the integral to find the volume (do not evaluate the integral): the region bounded by the graphs of $y = 2 - x^2$ and $y = x^2$.

Hint

Sketch the region and use Figure 6.3.12 to decide which integral is easiest to evaluate.

Answer

Use the method of washers;

$$V = \int_{-1}^1 \pi \left[\left(2 - x^2\right)^2 - \left(x^2\right)^2\right] \, dx$$

Key Concepts

- The method of cylindrical shells is another method for using a definite integral to calculate the volume of a solid of revolution. This method is sometimes preferable to either the method of disks or the method of washers because we integrate with respect to the other variable. In some cases, one integral is substantially more complicated than the other.
- The geometry of the functions and the difficulty of the integration are the main factors in deciding which integration method to use.

Key Equations

- **Method of Cylindrical Shells**

$$V = \int_a^b (2\pi x \, f(x)) \, dx$$

Glossary

method of cylindrical shells

a method of calculating the volume of a solid of revolution by dividing the solid into nested cylindrical shells; this method is different from the methods of disks or washers in that we integrate with respect to the opposite variable

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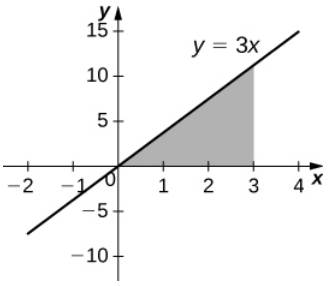
6.3E: Exercises for Section 6.3

For exercises 1 - 6, find the volume generated when the region between the two curves is rotated around the given axis. Use both the shell method and the washer method. Use technology to graph the functions and draw a typical slice by hand.

1) [T] Over the curve of $y = 3x$, $x = 0$, and $y = 3$ rotated around the y-axis.

2) [T] Under the curve of $y = 3x$, $x = 0$, and $x = 3$ rotated around the y-axis.

Answer

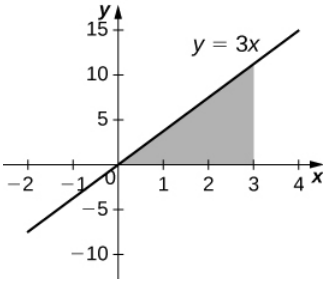


$V = 54\pi$ units^3

3) [T] Over the curve of $y = 3x$, $x = 0$, and $y = 3$ rotated around the x-axis.

4) [T] Under the curve of $y = 3x$, $x = 0$, and $x = 3$ rotated around the x-axis.

Answer

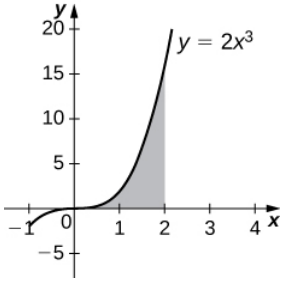


$V = 81\pi$ units^3

5) [T] Under the curve of $y = 2x^3,\ x = 0$, and $x = 2$ rotated around the y-axis.

6) [T] Under the curve of $y = 2x^3,\ x = 0$, and $x = 2$ rotated around the x-axis.

Answer



$V = \frac{512\pi}{7}$ units^3

For exercises 7 - 16, use shells to find the volumes of the given solids. Note that the rotated regions lie between the curve and the x-axis and are rotated around the y-axis.

7) $y = 1 - x^2$, $x = 0$, and $x = 1$

8) $y = 5x^3$, $x = 0$, and $x = 1$

Answer

$V = 2\pi$ units^3

9) $y = \dfrac{1}{x}$, $x = 1$, and $x = 100$

10) $y = \sqrt{1 - x^2}$, $x = 0$, and $x = 1$

Answer

$V = \frac{2\pi}{3}$ units^3

11) $y = \dfrac{1}{1 + x^2}$, $x = 0$,and $x = 3$

12) $y = \sin x^2$, $x = 0$, and $x = \sqrt{\pi}$

Answer

$V = 2\pi$ units^3

13) $y = \dfrac{1}{\sqrt{1 - x^2}}$, $x = 0$, and $x = \frac{1}{2}$

14) $y = \sqrt{x}$, $x = 0$, and $x = 1$

Answer

$V = \frac{4\pi}{5}$ units^3

15) $y = (1 + x^2)^3$, $x = 0$, and $x = 1$

16) $y = 5x^3 - 2x^4$, $x = 0$, and $x = 2$

Answer

$V = \frac{64\pi}{3}$ units^3

For exercises 17 - 26, use shells to find the volume generated by rotating the regions between the given curve and $y = 0$ around the x-axis.

17) $y = \sqrt{1 - x^2}$, $x = 0$, and $x = 1$

18) $y = x^2$, $x = 0$, and $x = 2$

Answer

$V = \frac{32\pi}{5}$ units^3

19) $y = e^x$, $x = 0$, and $x = 1$

20) $y = \ln(x)$, $x = 1$, and $x = e$

Answer

$V = \pi(e - 2)$ units^3

21) $x = \dfrac{1}{1 + y^2}$, $y = 1$, and $y = 4$

22) $x = \dfrac{1 + y^2}{y}$, $y = 0$, and $y = 2$

Answer

$V = \frac{28\pi}{3}$ units^3

23) $x = \cos y$, $y = 0$, and $y = \pi$

24) $x = y^3 - 4y^2$, $x = -1$, and $x = 2$

Answer

$V = \frac{84\pi}{5}$ units^3

25) $x = ye^y$, $x = -1$, and $x = 2$

26) $x = e^y \cos y$, $x = 0$, and $x = \pi$

Answer

$V = e^{\pi}\pi^2$ units^3

For exercises 27 - 36, find the volume generated when the region between the curves is rotated around the given axis.

27) $y = 3 - x$, $y = 0$, $x = 0$, and $x = 2$ rotated around the y-axis.

28) $y = x^3$, $y = 0$, $x = 0$, and $y = 8$ rotated around the y-axis.

Answer

$V = \frac{64\pi}{5}$ units^3

29) $y = x^2$, $y = x$, rotated around the y-axis.

30) $y = \sqrt{x}$, $x = 0$, and $x = 1$ rotated around the line $x = 2$.

Answer

$V = \frac{28\pi}{15}$ units^3

31) $y = \dfrac{1}{4 - x}$, $x = 1$, and $x = 2$ rotated around the line $x = 4$.

32) $y = \sqrt{x}$ and $y = x^2$ rotated around the y-axis.

Answer

$V = \frac{3\pi}{10}$ units^3

33) $y = \sqrt{x}$ and $y = x^2$ rotated around the line $x = 2$.

34) $x = y^3$, $y = \dfrac{1}{x}$, $x = 1$, and $y = 2$ rotated around the x-axis.

Answer

$\frac{52\pi}{5}$ units^3

35) $x = y^2$ and $y = x$ rotated around the line $y = 2$.

36) [T] Left of $x = \sin(\pi y)$, right of $y = x$, around the y-axis.

Answer

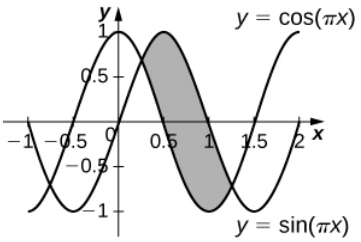
$V \approx 0.9876$ units^3

For exercises 37 - 44, use technology to graph the region. Determine which method you think would be easiest to use to calculate the volume generated when the function is rotated around the specified axis. Then, use your chosen method to find the volume.

37) [T] $y = x^2$ and $y = 4x$ rotated around the y-axis.

38) [T] $y = \cos(\pi x), y = \sin(\pi x), x = \frac{1}{4}$, and $x = \frac{5}{4}$ rotated around the y-axis.

Answer

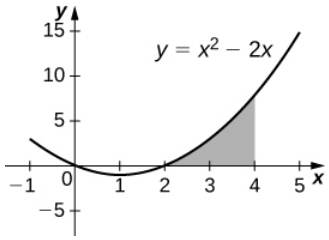


$V = 3\sqrt{2}$ units^3

39) [T] $y = x^2 - 2x,\ x = 2$, and $x = 4$ rotated around the y-axis.

40) [T] $y = x^2 - 2x,\ x = 2$, and $x = 4$ rotated around the x-axis.

Answer

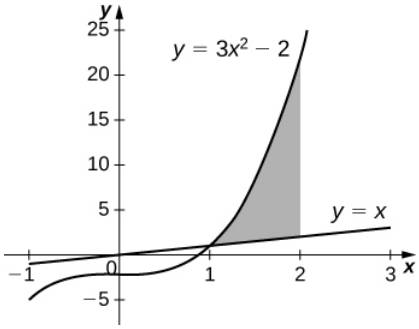


$V = \frac{496\pi}{15}$ units^3

41) [T] $y = 3x^3 - 2,\ y = x$, and $x = 2$ rotated around the x-axis.

42) [T] $y = 3x^3 - 2,\ y = x$, and $x = 2$ rotated around the y-axis.

Answer

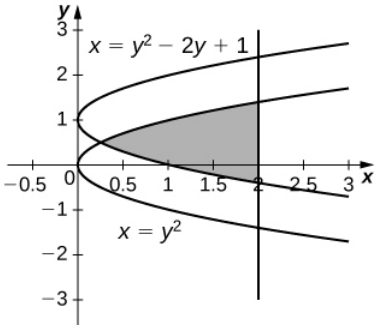


$V = \frac{398\pi}{15}$ units^3

43) [T] $x = \sin(\pi y^2)$ and $x = \sqrt{2}y$ rotated around the x-axis.

44) [T] $x = y^2,\ x = y^2 - 2y + 1$, and $x = 2$ rotated around the y-axis.

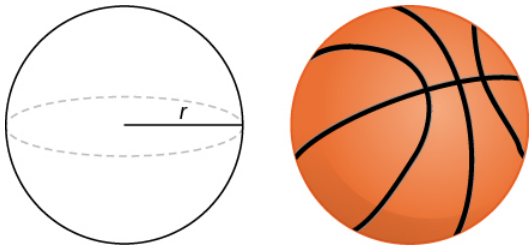
Answer



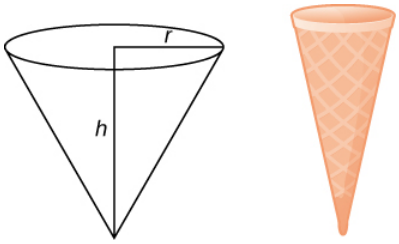
$V = 15.9074$ units^3

For exercises 45 - 51, use the method of shells to approximate the volumes of some common objects, which are pictured in accompanying figures.

45) Use the method of shells to find the volume of a sphere of radius r.



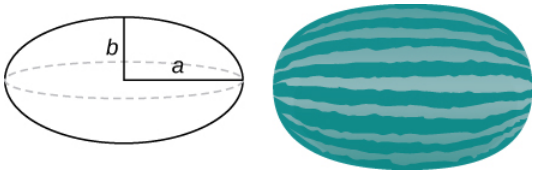
46) Use the method of shells to find the volume of a cone with radius r and height h.



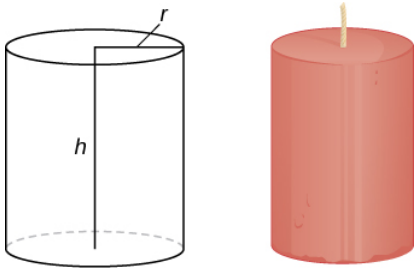
Answer

$V = \frac{1}{3}\pi r^2 h$ units^3

47) Use the method of shells to find the volume of an ellipse $(x^2/a^2) + (y^2/b^2) = 1$ rotated around the x-axis.



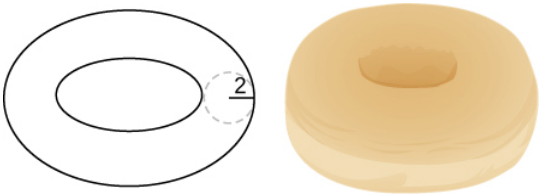
48) Use the method of shells to find the volume of a cylinder with radius r and height h.



Answer

$V = \pi r^2 h$ units^3

49) Use the method of shells to find the volume of the donut created when the circle $x^2 + y^2 = 4$ is rotated around the line $x = 4$.



50) Consider the region enclosed by the graphs of $y = f(x)$, $y = 1 + f(x)$, $x = 0$, $y = 0$, and $x = a > 0$. What is the volume of the solid generated when this region is rotated around the y-axis? Assume that the function is defined over the interval $[0, a]$.

Answer

$V = \pi a^2$ units^3

51) Consider the function $y = f(x)$, which decreases from $f(0) = b$ to $f(1) = 0$. Set up the integrals for determining the volume, using both the shell method and the disk method, of the solid generated when this region, with $x = 0$ and $y = 0$, is rotated around the y-axis. Prove that both methods approximate the same volume. Which method is easier to apply? (Hint: Since $f(x)$ is one-to-one, there exists an inverse $f^{-1}(y)$.)

Contributors

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6.4: Arc Length of a Curve and Surface Area

Learning Objectives

- Determine the length of a curve, $y = f(x)$, between two points.
- Determine the length of a curve, $x = g(y)$, between two points.
- Find the surface area of a solid of revolution.

In this section, we use definite integrals to find the arc length of a curve. We can think of **arc length** as the distance you would travel if you were walking along the path of the curve. Many real-world applications involve arc length. If a rocket is launched along a parabolic path, we might want to know how far the rocket travels. Or, if a curve on a map represents a road, we might want to know how far we have to drive to reach our destination.

We begin by calculating the arc length of curves defined as functions of x, then we examine the same process for curves defined as functions of y. (The process is identical, with the roles of x and y reversed.) The techniques we use to find arc length can be extended to find the surface area of a surface of revolution, and we close the section with an examination of this concept.

Arc Length of the Curve y = f(x)

In previous applications of integration, we required the function $f(x)$ to be integrable, or at most continuous. However, for calculating arc length we have a more stringent requirement for $f(x)$. Here, we require $f(x)$ to be differentiable, and furthermore we require its derivative, $f'(x)$, to be continuous. Functions like this, which have continuous derivatives, are called **smooth.** (This property comes up again in later chapters.)

Let $f(x)$ be a smooth function defined over $[a, b]$. We want to calculate the length of the curve from the point $(a, f(a))$ to the point $(b, f(b))$. We start by using line segments to approximate the length of the curve. For $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of $[a, b]$. Then, for $i = 1, 2, \ldots, n$, construct a line segment from the point $(x_{i-1}, f(x_{i-1}))$ to the point $(x_i, f(x_i))$. Although it might seem logical to use either horizontal or vertical line segments, we want our line segments to approximate the curve as closely as possible. Figure 6.4.1 depicts this construct for $n = 5$.

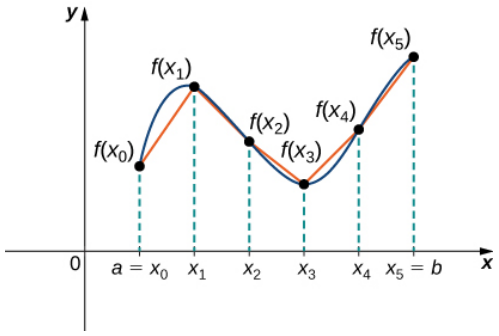

Figure 6.4.1: We can approximate the length of a curve by adding line segments.

To help us find the length of each line segment, we look at the change in vertical distance as well as the change in horizontal distance over each interval. Because we have used a regular partition, the change in horizontal distance over each interval is given by Δx. The change in vertical distance varies from interval to interval, though, so we use $\Delta y_i = f(x_i) - f(x_{i-1})$ to represent the change in vertical distance over the interval $[x_{i-1}, x_i]$, as shown in Figure 6.4.2. Note that some (or all) Δy_i may be negative.

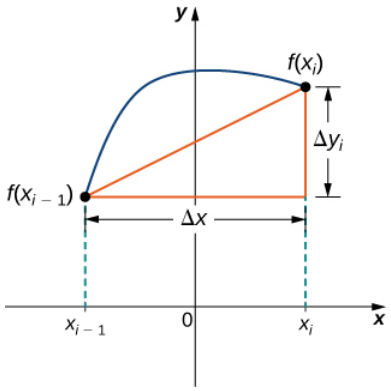

Figure 6.4.2: A representative line segment approximates the curve over the interval $[x_{i-1}, x_i]$.

By the Pythagorean theorem, the length of the line segment is

$$\sqrt{(\Delta x)^2 + (\Delta y_i)^2}.$$

We can also write this as

$$\Delta x\sqrt{1 + ((\Delta y_i)/(\Delta x))^2}.$$

Now, by the Mean Value Theorem, there is a point $x_i^* \in [x_{i-1}, x_i]$ such that $f'(x_i^*) = (\Delta y_i)/(\Delta x)$. Then the length of the line segment is given by

$$\Delta x\sqrt{1 + [f'(x_i^*)]^2}.$$

Adding up the lengths of all the line segments, we get

$$\text{Arc Length} \approx \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\Delta x.$$

This is a Riemann sum. Taking the limit as $n \to \infty$, we have

$$\begin{aligned}\text{Arc Length} &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\Delta x \\ &= \int_a^b \sqrt{1 + [f'(x)]^2}dx.\end{aligned}$$

We summarize these findings in the following theorem.

Arc Length for $y = f(x)$

Let $f(x)$ be a smooth function over the interval $[a, b]$. Then the arc length of the portion of the graph of $f(x)$ from the point $(a, f(a))$ to the point $(b, f(b))$ is given by

$$\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

Note that we are integrating an expression involving $f'(x)$, so we need to be sure $f'(x)$ is integrable. This is why we require $f(x)$ to be smooth. The following example shows how to apply the theorem.

Example 6.4.1: Calculating the Arc Length of a Function of x

Let $f(x) = 2x^{3/2}$. Calculate the arc length of the graph of $f(x)$ over the interval $[0, 1]$. Round the answer to three decimal places.

Solution

We have $f'(x) = 3x^{1/2}$, so $[f'(x)]^2 = 9x$. Then, the arc length is

$$\text{Arc Length} = \int_a^b \sqrt{1+[f'(x)]^2}dx$$
$$= \int_0^1 \sqrt{1+9x}\,dx.$$

Substitute $u = 1 + 9x$. Then, $du = 9\,dx$. When $x = 0$, then $u = 1$, and when $x = 1$, then $u = 10$. Thus,

$$\text{Arc Length} = \int_0^1 \sqrt{1+9x}\,dx$$
$$= \frac{1}{9}\int_0^1 \sqrt{1+9x}\cdot 9\,dx$$
$$= \frac{1}{9}\int_1^{10} \sqrt{u}\,du$$
$$= \frac{1}{9}\cdot\frac{2}{3}u^{3/2}\Big|_1^{10} = \frac{2}{27}[10\sqrt{10}-1]$$
$$\approx 2.268 \text{ units}.$$

Exercise 6.4.1

Let $f(x) = (4/3)x^{3/2}$. Calculate the arc length of the graph of $f(x)$ over the interval $[0, 1]$. Round the answer to three decimal places.

Hint

Use the process from the previous example. Don't forget to change the limits of integration.

Answer

$$\frac{1}{6}(5\sqrt{5}-1) \approx 1.697 \text{ units}$$

Although it is nice to have a formula for calculating arc length, this particular theorem can generate expressions that are difficult to integrate. We study some techniques for integration in Introduction to Techniques of Integration. In some cases, we may have to use a computer or calculator to approximate the value of the integral.

Example 6.4.2: Using a Computer or Calculator to Determine the Arc Length of a Function of x

Let $f(x) = x^2$. Calculate the arc length of the graph of $f(x)$ over the interval $[1, 3]$.

Solution

We have $f'(x) = 2x$, so $[f'(x)]^2 = 4x^2$. Then the arc length is given by

$$\text{Arc Length} = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$$
$$= \int_1^3 \sqrt{1+4x^2}\,dx.$$

Using a computer to approximate the value of this integral, we get

$$\int_1^3 \sqrt{1+4x^2}\,dx \approx 8.26815 \text{ units}.$$

Exercise 6.4.2

Let $f(x) = \sin x$. Calculate the arc length of the graph of $f(x)$ over the interval $[0, \pi]$. Use a computer or calculator to approximate the value of the integral.

Hint

Use the process from the previous example.

Answer

$$\text{Arc Length} \approx 3.8202 \text{ units}$$

Arc Length of the Curve x = g(y)

We have just seen how to approximate the length of a curve with line segments. If we want to find the arc length of the graph of a function of y, we can repeat the same process, except we partition the y-axis instead of the x-axis. Figure 6.4.3 shows a representative line segment.

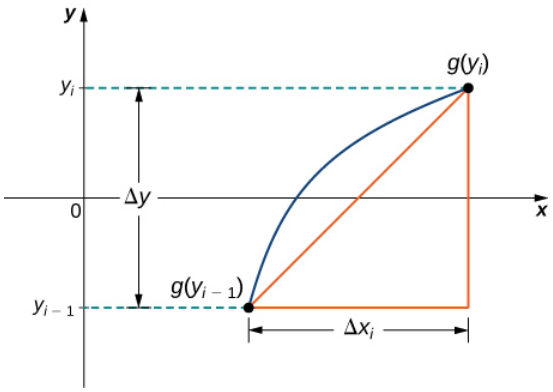

Figure 6.4.3: A representative line segment over the interval $[y_{i-1}, y_i]$.

Then the length of the line segment is

$$\sqrt{(\Delta y)^2 + (\Delta x_i)^2},$$

which can also be written as

$$\Delta y \sqrt{1 + \left(\frac{\Delta x_i}{\Delta y}\right)^2}.$$

If we now follow the same development we did earlier, we get a formula for arc length of a function $x = g(y)$.

Arc Length for $x = g(y)$

Let $g(y)$ be a smooth function over an interval $[c, d]$. Then, the arc length of the graph of $g(y)$ from the point $(c, g(c))$ to the point $(d, g(d))$ is given by

$$\text{Arc Length} = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy.$$

Example 6.4.3: Calculating the Arc Length of a Function of y

Let $g(y) = 3y^3$. Calculate the arc length of the graph of $g(y)$ over the interval $[1, 2]$.

Solution

We have $g'(y) = 9y^2$, so $[g'(y)]^2 = 81y^4$. Then the arc length is

$$\begin{aligned} \text{Arc Length} &= \int_c^d \sqrt{1 + [g'(y)]^2}\, dy \\ &= \int_1^2 \sqrt{1 + 81y^4}\, dy. \end{aligned}$$

Using a computer to approximate the value of this integral, we obtain

$$\int_1^2 \sqrt{}$$

$$\int_1 \sqrt{1+81y^4}dy \approx 21.0277 \text{ units.}$$

? Exercise 6.4.3

Let $g(y) = 1/y$. Calculate the arc length of the graph of $g(y)$ over the interval $[1, 4]$. Use a computer or calculator to approximate the value of the integral.

Hint

Use the process from the previous example.

Answer

$$\text{Arc Length} = 3.15018 \text{ units}$$

Area of a Surface of Revolution

The concepts we used to find the arc length of a curve can be extended to find the surface area of a surface of revolution. **Surface area** is the total area of the outer layer of an object. For objects such as cubes or bricks, the surface area of the object is the sum of the areas of all of its faces. For curved surfaces, the situation is a little more complex. Let $f(x)$ be a nonnegative smooth function over the interval $[a, b]$. We wish to find the surface area of the surface of revolution created by revolving the graph of $y = f(x)$ around the x-axis as shown in the following figure.

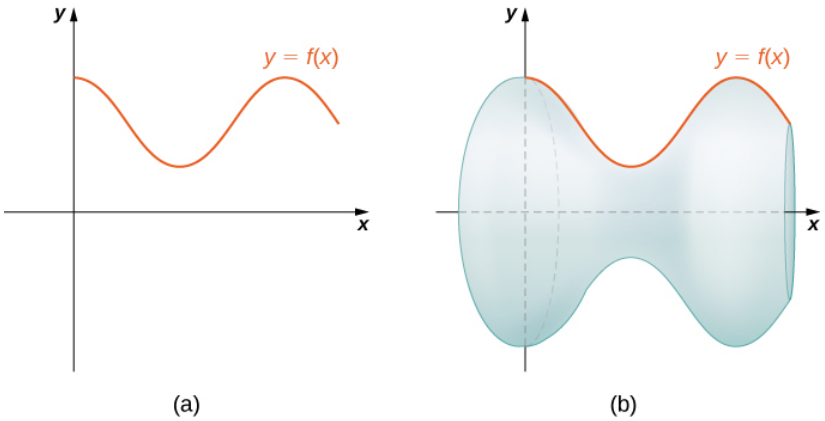

Figure 6.4.4: (a) A curve representing the function $f(x)$. (b) The surface of revolution formed by revolving the graph of $f(x)$ around the $x - axis$.

As we have done many times before, we are going to partition the interval $[a, b]$ and approximate the surface area by calculating the surface area of simpler shapes. We start by using line segments to approximate the curve, as we did earlier in this section. For $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of $[a, b]$. Then, for $i = 1, 2, \ldots, n$, construct a line segment from the point $(x_{i-1}, f(x_{i-1}))$ to the point $(x_i, f(x_i))$. Now, revolve these line segments around the x-axis to generate an approximation of the surface of revolution as shown in the following figure.

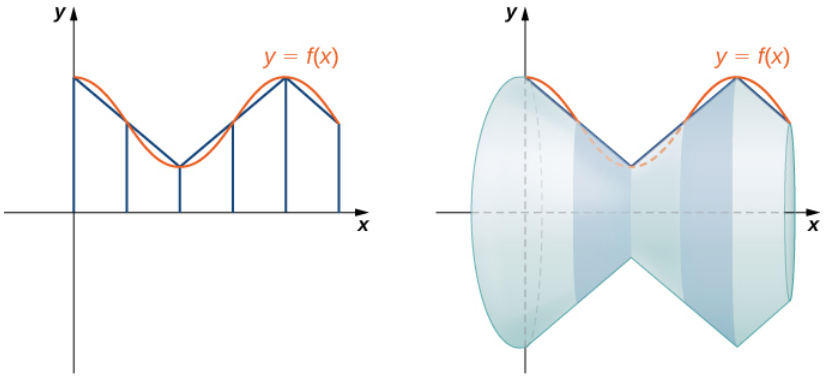

Figure 6.4.5: (a) Approximating $f(x)$ with line segments. (b) The surface of revolution formed by revolving the line segments around the x-axis.

Notice that when each line segment is revolved around the axis, it produces a band. These bands are actually pieces of cones (think of an ice cream cone with the pointy end cut off). A piece of a cone like this is called a **frustum** of a cone.

To find the surface area of the band, we need to find the lateral surface area, S, of the frustum (the area of just the slanted outside surface of the frustum, not including the areas of the top or bottom faces). Let r_1 and r_2 be the radii of the wide end and the narrow end of the frustum, respectively, and let l be the slant height of the frustum as shown in the following figure.

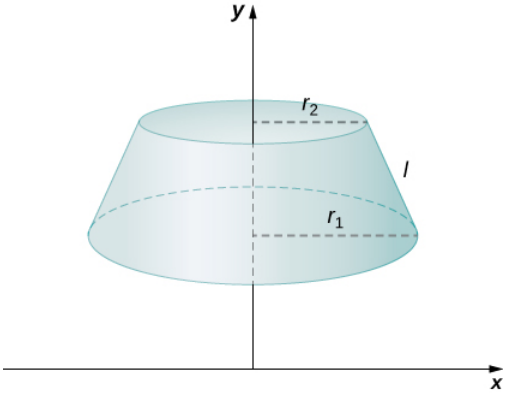

Figure 6.4.6: A frustum of a cone can approximate a small part of surface area.

We know the lateral surface area of a cone is given by

$$\text{Lateral Surface Area } = \pi rs,$$

where r is the radius of the base of the cone and s is the **slant height** (Figure 6.4.7).

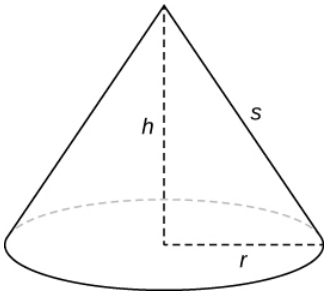

Figure 6.4.7: The lateral surface area of the cone is given by πrs.

Since a frustum can be thought of as a piece of a cone, the lateral surface area of the frustum is given by the lateral surface area of the whole cone less the lateral surface area of the smaller cone (the pointy tip) that was cut off (Figure 6.4.8).

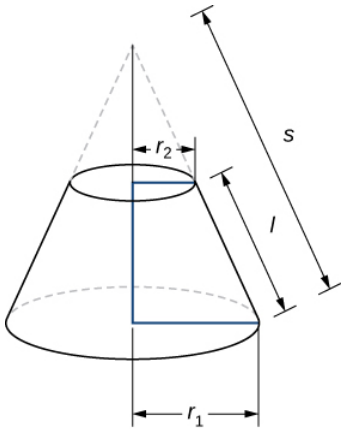

Figure 6.4.8: Calculating the lateral surface area of a frustum of a cone.

The cross-sections of the small cone and the large cone are similar triangles, so we see that

$$\frac{r_2}{r_1} = \frac{s-l}{s}$$

Solving for s, we get

$$\begin{aligned} \frac{r_2}{r_1} &= \frac{s-l}{s} \\ r_2 s &= r_1(s-l) \\ r_2 s &= r_1 s - r_1 l \\ r_1 l &= r_1 s - r_2 s \\ r_1 l &= (r_1 - r_2)s \\ \frac{r_1 l}{r_1 - r_2} &= s \end{aligned}$$

Then the lateral surface area (SA) of the frustum is

$$\begin{aligned} S &= (\text{Lateral SA of large cone}) - (\text{Lateral SA of small cone}) \\ &= \pi r_1 s - \pi r_2 (s-l) \\ &= \pi r_1 \left(\frac{r_1 l}{r_1 - r_2}\right) - \pi r_2 \left(\frac{r_1 l}{r_1 - r_2} - l\right) \\ &= \frac{\pi r_1^2 l}{r^1 - r^2} - \frac{\pi r_1 r_2 l}{r_1 - r_2} + \pi r_2 l \\ &= \frac{\pi r_1^2 l}{r_1 - r_2} - \frac{\pi r_1 r 2_l}{r_1 - r_2} + \frac{\pi r_2 l(r_1 - r_2)}{r_1 - r_2} \\ &= \frac{\pi r_1^2}{l r_1 - r_2} - \frac{\pi r_1 r_2 l}{r_1 - r_2} + \frac{\pi r_1 r_2 l}{r_1 - r_2} - \frac{\pi r_2^2 l}{r_1 - r_3} \\ &= \frac{\pi(r_1^2 - r_2^2)l}{r_1 - r_2} = \frac{\pi(r_1 - r + 2)(r1 + r2)l}{r_1 - r_2} \\ &= \pi(r_1 + r_2)l. \end{aligned}$$

Let's now use this formula to calculate the surface area of each of the bands formed by revolving the line segments around the x-axis. A representative band is shown in the following figure.

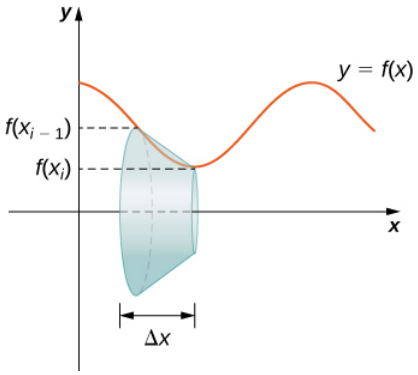

Figure 6.4.9: A representative band used for determining surface area.

Note that the slant height of this frustum is just the length of the line segment used to generate it. So, applying the surface area formula, we have

$$\begin{aligned} S &= \pi(r_1 + r_2)l \\ &= \pi\big(f(x_{i-1}) + f(x_i)\big)\sqrt{\Delta x^2 + (\Delta y_i)^2} \\ &= \pi\big(f(x_{i-1}) + f(x_i)\big)\Delta x\sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2} \end{aligned}$$

Now, as we did in the development of the arc length formula, we apply the Mean Value Theorem to select $x_i^* \in [x_{i-1}, x_i]$ such that $f'(x_i^*) = (\Delta y_i)/\Delta x$. This gives us

$$S = \pi\big(f(x_{i-1}) + f(x_i)\big)\Delta x\sqrt{1 + \big(f'(x_i^*)\big)^2}$$

Furthermore, since $f(x)$ is continuous, by the *Intermediate Value Theorem*, there is a point $x_i^{**} \in [x_{i-1}, x_i]$ such that $f(x_i^{**}) = \frac{1}{2}\left[f(x_{i-1}) + f(x_i)\right]$,

so we get

$$S = 2\pi f(x_i^{**})\Delta x\sqrt{1 + \big(f'(x_i^*)\big)^2}.$$

Then the approximate surface area of the whole surface of revolution is given by

$$\text{Surface Area} \approx \sum_{i=1}^{n} 2\pi f(x_i^{**})\Delta x\sqrt{1 + \big(f'(x_i^*)\big)^2}.$$

This almost looks like a Riemann sum, except we have functions evaluated at two different points, x_i^* and x_i^{**}, over the interval $[x_{i-1}, x_i]$. Although we do not examine the details here, it turns out that because $f(x)$ is smooth, if we let $n \to \infty$, the limit works the same as a Riemann sum even with the two different evaluation points. This makes sense intuitively. Both x_i^* and x_i^{**} are in the interval $[x_{i-1}, x_i]$, so it makes sense that as $n \to \infty$, both x_i^* and x_i^{**} approach x Those of you who are interested in the details should consult an advanced calculus text.

Taking the limit as $n \to \infty$, we get

$$\begin{aligned} \text{Surface Area} &= \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi f(x_i^{**})\Delta x\sqrt{1 + \big(f'(x_i^*)\big)^2} \\ &= \int_a^b \left(2\pi f(x)\sqrt{1 + \big(f'(x)\big)^2}\right) dx \end{aligned}$$

As with arc length, we can conduct a similar development for functions of y to get a formula for the surface area of surfaces of revolution about the y-axis. These findings are summarized in the following theorem.

Surface Area of a Surface of Revolution

Let $f(x)$ be a nonnegative smooth function over the interval $[a, b]$. Then, the surface area of the surface of revolution formed by revolving the graph of $f(x)$ around the x-axis is given by

$$\text{Surface Area} = \int_a^b \left(2\pi f(x)\sqrt{1+\left(f'(x)\right)^2}\right) dx$$

Similarly, let $g(y)$ be a nonnegative smooth function over the interval $[c, d]$. Then, the surface area of the surface of revolution formed by revolving the graph of $g(y)$ around the y-axis is given by

$$\text{Surface Area} = \int_c^d \left(2\pi g(y)\sqrt{1+\left(g'(y)\right)^2}\right) dy$$

Example 6.4.4: Calculating the Surface Area of a Surface of Revolution 1.

Let $f(x) = \sqrt{x}$ over the interval $[1, 4]$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the x-axis. Round the answer to three decimal places.

Solution

The graph of $f(x)$ and the surface of rotation are shown in Figure 6.4.10.

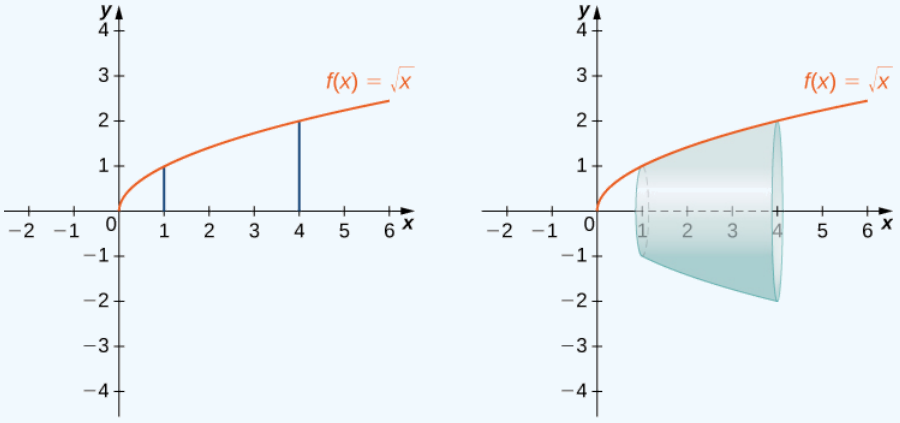

Figure 6.4.10: (a) The graph of $f(x)$. (b) The surface of revolution.

We have $f(x) = \sqrt{x}$. Then, $f'(x) = \frac{1}{2\sqrt{x}}$ and $\left(f'(x)\right)^2 = \frac{1}{4x}$. Then,

$$\begin{aligned}\text{Surface Area} &= \int_a^b \left(2\pi f(x)\sqrt{1+\left(f'(x)\right)^2}\right) dx \\ &= \int_1^4 \left(2\pi\sqrt{x}\sqrt{1+\frac{1}{4x}}\right) dx \\ &= \int_1^4 \left(2\pi\sqrt{x+\frac{1}{4}}\right) dx.\end{aligned}$$

Let $u = x + 1/4$. Then, $du = dx$. When $x = 1$, $u = 5/4$, and when $x = 4$, $u = 17/4$. This gives us

$$\begin{aligned}\int_0^1 \left(2\pi\sqrt{x+\frac{1}{4}}\right) dx &= \int_{5/4}^{17/4} 2\pi\sqrt{u}\, du \\ &= 2\pi\left[\frac{2}{3}u^{3/2}\right]\Big|_{5/4}^{17/4} \\ &= \frac{\pi}{6}[17\sqrt{17} - 5\sqrt{5}] \approx 30.846 \text{ units}^2\end{aligned}$$

Exercise 6.4.4

Let $f(x) = \sqrt{1-x}$ over the interval $[0, 1/2]$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the x-axis. Round the answer to three decimal places.

Hint

Use the process from the previous example.

Answer

$$\frac{\pi}{6}\left(5\sqrt{5} - 3\sqrt{3}\right) \approx 3.133 \text{ units}^2$$

Example 6.4.5: Calculating the Surface Area of a Surface of Revolution 2

Let $f(x) = y = \sqrt[3]{3x}$. Consider the portion of the curve where $0 \le y \le 2$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the y-axis.

Solution

Notice that we are revolving the curve around the y-axis, and the interval is in terms of y, so we want to rewrite the function as a function of y. We get $x = g(y) = (1/3)y^3$. The graph of $g(y)$ and the surface of rotation are shown in the following figure.

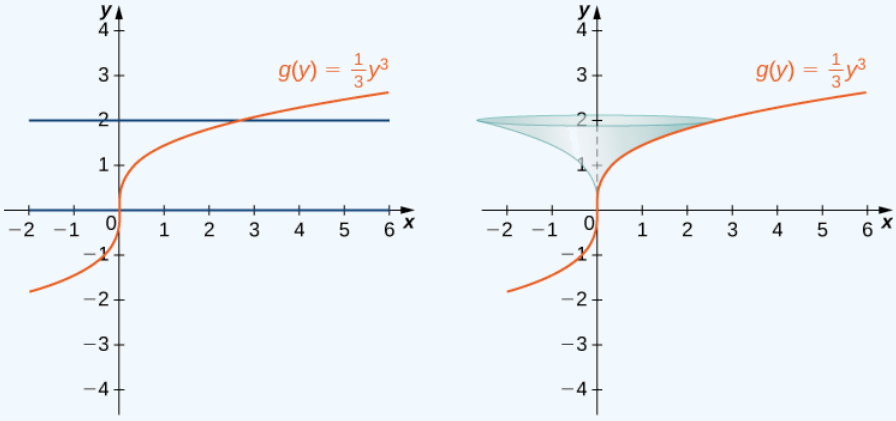

Figure 6.4.11: (a) The graph of $g(y)$. (b) The surface of revolution.

We have $g(y) = (1/3)y^3$, so $g'(y) = y^2$ and $\left(g'(y)\right)^2 = y^4$. Then

$$\begin{aligned}\text{Surface Area} &= \int_c^d \left(2\pi g(y)\sqrt{1+\left(g'(y)\right)^2}\right) dy \\ &= \int_0^2 \left(2\pi\left(\frac{1}{3}y^3\right)\sqrt{1+y^4}\right) dy \\ &= \frac{2\pi}{3}\int_0^2 \left(y^3\sqrt{1+y^4}\right) dy.\end{aligned}$$

Let $u = y^4 + 1$. Then $du = 4y^3\, dy$. When $y = 0,\ u = 1$, and when $y = 2,\ u = 17$. Then

$$\begin{aligned}\frac{2\pi}{3}\int_0^2 \left(y^3\sqrt{1+y^4}\right) dy &= \frac{2\pi}{3}\int_1^{17} \frac{1}{4}\sqrt{u}\, du \\ &= \frac{\pi}{6}\left[\frac{2}{3}u^{3/2}\right]\Big|_1^{17} = \frac{\pi}{9}\left[(17)^{3/2} - 1\right] \approx 24.118 \text{ units}^2.\end{aligned}$$

Exercise 6.4.5

Let $g(y) = \sqrt{9 - y^2}$ over the interval $y \in [0, 2]$. Find the surface area of the surface generated by revolving the graph of $g(y)$ around the y-axis.

Hint

Use the process from the previous example.

Answer

$12\pi \text{ units}^2$

Key Concepts

- The arc length of a curve can be calculated using a definite integral.
- The arc length is first approximated using line segments, which generates a Riemann sum. Taking a limit then gives us the definite integral formula. The same process can be applied to functions of y.
- The concepts used to calculate the arc length can be generalized to find the surface area of a surface of revolution.
- The integrals generated by both the arc length and surface area formulas are often difficult to evaluate. It may be necessary to use a computer or calculator to approximate the values of the integrals.

Key Equations

- **Arc Length of a Function of x**

$$\textbf{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

- **Arc Length of a Function of y**

$$\textbf{Arc Length} = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy$$

- **Surface Area of a Function of x**

$$\textbf{Surface Area} = \int_a^b \left(2\pi f(x)\sqrt{1 + \left(f'(x)\right)^2}\right)\, dx$$

Glossary

arc length

the arc length of a curve can be thought of as the distance a person would travel along the path of the curve

frustum

a portion of a cone; a frustum is constructed by cutting the cone with a plane parallel to the base

surface area

the surface area of a solid is the total area of the outer layer of the object; for objects such as cubes or bricks, the surface area of the object is the sum of the areas of all of its faces

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6.4E: Exercises for Section 6.4

For exercises 1 - 3, find the length of the functions over the given interval.

1) $y = 5x$ from $x = 0$ to $x = 2$

Answer

$s = 2\sqrt{26}$ units

2) $y = -\frac{1}{2}x + 25$ from $x = 1$ to $x = 4$

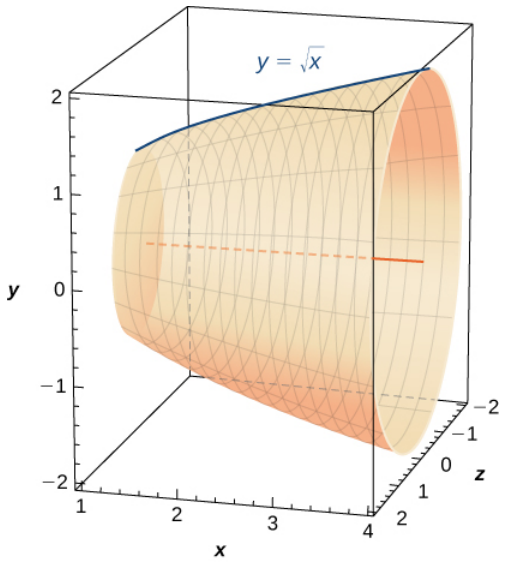
3) $x = 4y$ from $y = -1$ to $y = 1$

Answer

$s = 2\sqrt{17}$ units

4) Pick an arbitrary linear function $x = g(y)$ over any interval of your choice (y_1, y_2). Determine the length of the function and then prove the length is correct by using geometry.

5) Find the surface area of the volume generated when the curve $y = \sqrt{x}$ revolves around the x-axis from $(1, 1)$ to $(4, 2)$, as seen here.



Answer

$A = \frac{\pi}{6}(17\sqrt{17} - 5\sqrt{5})$ units^2

6) Find the surface area of the volume generated when the curve $y = x^2$ revolves around the y-axis from $(1, 1)$ to $(3, 9)$.

For exercises 7 - 16, find the lengths of the functions of x over the given interval. If you cannot evaluate the integral exactly, use technology to approximate it.

7) $y = x^{3/2}$ from $(0, 0)$ to $(1, 1)$

Answer

$s = \frac{13\sqrt{13}-8}{27}$ units

8) $y = x^{2/3}$ from $(1, 1)$ to $(8, 4)$

9) $y = \frac{1}{3}(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 1$

Answer

$s = \frac{4}{3}$ units

10) $y = \frac{1}{3}(x^2 - 2)^{3/2}$ from $x = 2$ to $x = 4$

11) [T] $y = e^x$ on $x = 0$ to $x = 1$

Answer

$s \approx 2.0035$ units

12) $y = \dfrac{x^3}{3} + \dfrac{1}{4x}$ from $x = 1$ to $x = 3$

13) $y = \dfrac{x^4}{4} + \dfrac{1}{8x^2}$ from $x = 1$ to $x = 2$

Answer

$s = \frac{123}{32}$ units

14) $y = \dfrac{2x^{3/2}}{3} - \dfrac{x^{1/2}}{2}$ from $x = 1$ to $x = 4$

15) $y = \frac{1}{27}(9x^2 + 6)^{3/2}$ from $x = 0$ to $x = 2$

Answer

$s = 10$ units

16) [T] $y = \sin x$ on $x = 0$ to $x = \pi$

For exercises 17 - 26, find the lengths of the functions of y over the given interval. If you cannot evaluate the integral exactly, use technology to approximate it.

17) $y = \dfrac{5 - 3x}{4}$ from $y = 0$ to $y = 4$

Answer

$s = \frac{20}{3}$ units

18) $x = \frac{1}{2}(e^y + e^{-y})$ from $y = -1$ to $y = 1$

19) $x = 5y^{3/2}$ from $y = 0$ to $y = 1$

Answer

$s = \frac{1}{675}(229\sqrt{229} - 8)$ units

20) [T] $x = y^2$ from $y = 0$ to $y = 1$

21) $x = \sqrt{y}$ from $y = 0$ to $y = 1$

Answer

$s = \frac{1}{8}(4\sqrt{5} + \ln(9 + 4\sqrt{5}))$ units

22) $x = \frac{2}{3}(y^2 + 1)^{3/2}$ from $y = 1$ to $y = 3$

23) [T] $x = \tan y$ from $y = 0$ to $y = \frac{3}{4}$

Answer

$s \approx 1.201$ units

24) [T] $x = \cos^2 y$ from $y = -\frac{\pi}{2}$ to $y = \frac{\pi}{2}$

25) [T] $x = 4^y$ from $y = 0$ to $y = 2$

Answer

$s \approx 15.2341$ units

26) [T] $x = \ln(y)$ on $y = \dfrac{1}{e}$ to $y = e$

For exercises 27 - 34, find the surface area of the volume generated when the following curves revolve around the x-axis. If you cannot evaluate the integral exactly, use your calculator to approximate it.

27) $y = \sqrt{x}$ from $x = 2$ to $x = 6$

Answer

$A = \frac{49\pi}{3}$ units^2

28) $y = x^3$ from $x = 0$ to $x = 1$

29) $y = 7x$ from $x = -1$ to $x = 1$

Answer

$A = 70\pi\sqrt{2}$ units^2

30) [T] $y = \frac{1}{x^2}$ from $x = 1$ to $x = 3$

31) $y = \sqrt{4 - x^2}$ from $x = 0$ to $x = 2$

Answer

$A = 8\pi$ units^2

32) $y = \sqrt{4 - x^2}$ from $x = -1$ to $x = 1$

33) $y = 5x$ from $x = 1$ to $x = 5$

Answer

$A = 120\pi\sqrt{26}$ units^2

34) [T] $y = \tan x$ from $x = -\frac{\pi}{4}$ to $x = \frac{\pi}{4}$

For exercises 35 - 42, find the surface area of the volume generated when the following curves revolve around the y-axis. If you cannot evaluate the integral exactly, use your calculator to approximate it.

35) $y = x^2$ from $x = 0$ to $x = 2$

Answer

$A = \frac{\pi}{6}(17\sqrt{17} - 1)$ units^2

36) $y = \frac{1}{2}x^2 + \frac{1}{2}$ from $x = 0$ to $x = 1$

37) $y = x + 1$ from $x = 0$ to $x = 3$

Answer

$A = 9\sqrt{2}\pi$ units^2

38) [T] $y = \dfrac{1}{x}$ from $x = \dfrac{1}{2}$ to $x = 1$

39) $y = \sqrt[3]{x}$ from $x = 1$ to $x = 27$

Answer

$A = \frac{10\sqrt{10}\pi}{27}(73\sqrt{73} - 1)$ units^2

40) [T] $y = 3x^4$ from $x = 0$ to $x = 1$

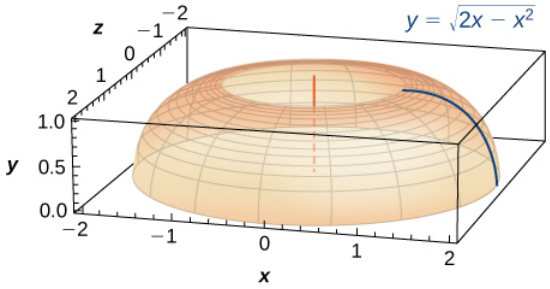
41) [T] $y = \dfrac{1}{\sqrt{x}}$ from $x = 1$ to $x = 3$

Answer

$A \approx 25.645$ units^2

42) [T] $y = \cos x$ from $x = 0$ to $x = \frac{\pi}{2}$

43) The base of a lamp is constructed by revolving a quarter circle $y = \sqrt{2x - x^2}$ around the y-axis from $x = 1$ to $x = 2$, as seen here. Create an integral for the surface area of this curve and compute it.

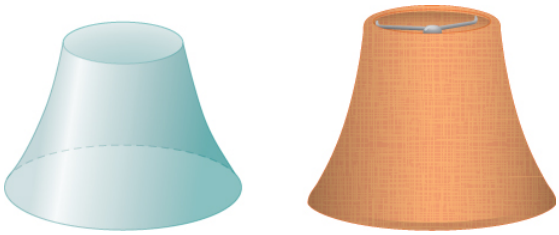


Answer

$A = 2\pi$ units^2

44) A light bulb is a sphere with radius $1/2$ in. with the bottom sliced off to fit exactly onto a cylinder of radius $1/4$ in. and length $1/3$ in., as seen here. The sphere is cut off at the bottom to fit exactly onto the cylinder, so the radius of the cut is $1/4$ in. Find the surface area (not including the top or bottom of the cylinder).

45) [T] A lampshade is constructed by rotating $y = 1/x$ around the x-axis from $y = 1$ to $y = 2$, as seen here. Determine how much material you would need to construct this lampshade—that is, the surface area—accurate to four decimal places.



Answer

10.5017 units^2

46) [T] An anchor drags behind a boat according to the function $y = 24e^{-x/2} - 24$, where y represents the depth beneath the boat and x is the horizontal distance of the anchor from the back of the boat. If the anchor is 23 ft below the boat, how much rope do you have to pull to reach the anchor? Round your answer to three decimal places.

47) [T] You are building a bridge that will span 10 ft. You intend to add decorative rope in the shape of $y = 5|\sin((x\pi)/5)|$, where x is the distance in feet from one end of the bridge. Find out how much rope you need to buy, rounded to the nearest foot.

Answer

23 ft

For exercise 48, find the exact arc length for the following problems over the given interval.

48) $y = \ln(\sin x)$ from $x = \frac{\pi}{4}$ to $x = \frac{3\pi}{4}$. (Hint: Recall trigonometric identities.)

49) Draw graphs of $y = x^2, y = x^6$, and $y = x^{10}$. For $y = x^n$, as n increases, formulate a prediction on the arc length from $(0, 0)$ to $(1, 1)$. Now, compute the lengths of these three functions and determine whether your prediction is correct.

Answer

2

50) Compare the lengths of the parabola $x = y^2$ and the line $x = by$ from $(0, 0)$ to (b^2, b) as b increases. What do you notice?

51) Solve for the length of $x = y^2$ from $(0, 0)$ to $(1, 1)$. Show that $x = \dfrac{y^2}{2}$ from $(0, 0)$ to $(2, 2)$ is twice as long. Graph both functions and explain why this is so.

Answer

Answers may vary

52) [T] Which is longer between $(1, 1)$ and $(2, \frac{1}{2})$: the hyperbola $y = \dfrac{1}{x}$ or the graph of $x + 2y = 3$?

53) Explain why the surface area is infinite when $y = 1/x$ is rotated around the x-axis for $1 \leq x < \infty$, but the volume is finite.

Answer

For more information, look up Gabriel's Horn.

Contributors

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6.5: Physical Applications of Integration

Learning Objectives

- Determine the mass of a one-dimensional object from its linear density function.
- Determine the mass of a two-dimensional circular object from its radial density function.
- Calculate the work done by a variable force acting along a line.
- Calculate the work done in pumping a liquid from one height to another.
- Find the hydrostatic force against a submerged vertical plate.

In this section, we examine some physical applications of integration. Let's begin with a look at calculating mass from a density function. We then turn our attention to work, and close the section with a study of hydrostatic force.

Mass and Density

We can use integration to develop a formula for calculating mass based on a density function. First we consider a thin rod or wire. Orient the rod so it aligns with the x-axis, with the left end of the rod at $x = a$ and the right end of the rod at $x = b$ (Figure 6.5.1). Note that although we depict the rod with some thickness in the figures, for mathematical purposes we assume the rod is thin enough to be treated as a one-dimensional object.

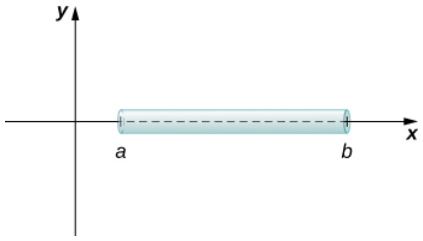


Figure 6.5.1: We can calculate the mass of a thin rod oriented along the x-axis by integrating its density function.

If the rod has constant density ρ, given in terms of mass per unit length, then the mass of the rod is just the product of the density and the length of the rod: $(b - a)\rho$. If the density of the rod is not constant, however, the problem becomes a little more challenging. When the density of the rod varies from point to point, we use a linear density function, $\rho(x)$, to denote the density of the rod at any point, x. Let $\rho(x)$ be an integrable linear density function. Now, for $i = 0, 1, 2, \ldots, n$ let $P = x_i$ be a regular partition of the interval $[a, b]$, and for $i = 1, 2, \ldots, n$ choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. Figure 6.5.2 shows a representative segment of the rod.

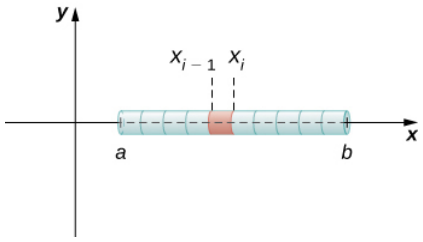


Figure 6.5.2: A representative segment of the rod.

The mass m_i of the segment of the rod from x_{i-1} to x_i is approximated by

$$\begin{aligned} m_i &\approx \rho(x_i^*)(x_i - x_{i-1}) \\ &= \rho(x_i^*)\Delta x. \end{aligned}$$

Adding the masses of all the segments gives us an approximation for the mass of the entire rod:

$$\begin{aligned} m &= \sum_{i=1}^{n} m_i \\ &\approx \sum_{i=1}^{n} \rho(x_i^*)\Delta x. \end{aligned}$$

This is a Riemann sum. Taking the limit as $n \to \infty$, we get an expression for the exact mass of the rod:

$$\begin{aligned} m &= \lim_{n\to\infty} \sum_{i=1}^{n} \rho(x_i^*)\Delta x \\ &= \int_a^b \rho(x)dx. \end{aligned}$$

We state this result in the following theorem.

Mass–Density Formula of a One-Dimensional Object

Given a thin rod oriented along the x-axis over the interval $[a, b]$, let $\rho(x)$ denote a linear density function giving the density of the rod at a point x in the interval. Then the mass of the rod is given by

$$m = \int_a^b \rho(x)dx. \tag{6.5.1}$$

We apply this theorem in the next example.

Example 6.5.1: Calculating Mass from Linear Density

Consider a thin rod oriented on the x-axis over the interval $[\pi/2, \pi]$. If the density of the rod is given by $\rho(x) = \sin x$, what is the mass of the rod?

Solution

Applying Equation 6.5.1 directly, we have

$$\begin{aligned} m &= \int_a^b \rho(x)dx \\ &= \int_{\pi/2}^{\pi} \sin x\, dx \\ &= -\cos x\Big|_{\pi/2}^{\pi} \\ &= 1. \end{aligned}$$

Exercise 6.5.1

Consider a thin rod oriented on the x-axis over the interval $[1, 3]$. If the density of the rod is given by $\rho(x) = 2x^2 + 3$, what is the mass of the rod?

Hint

Use the process from the previous example.

Solution

$70/3$

We now extend this concept to find the mass of a two-dimensional disk of radius r. As with the rod we looked at in the one-dimensional case, here we assume the disk is thin enough that, for mathematical purposes, we can treat it as a two-dimensional object. We assume the density is given in terms of mass per unit area (called **area density**), and further assume the density varies only along the disk's radius (called **radial density**). We orient the disk in the $xy - plane$, with the center at the origin. Then, the density of the disk can be treated as a function of x, denoted $\rho(x)$. We assume $\rho(x)$ is integrable. Because density is a function of x, we partition the interval from $[0, r]$ along the x-axis. For $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of the interval $[0, r]$, and for $i = 1, 2, \ldots, n$, choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. Now, use the partition to break up the disk into thin (two-dimensional) washers. A disk and a representative washer are depicted in the following figure.

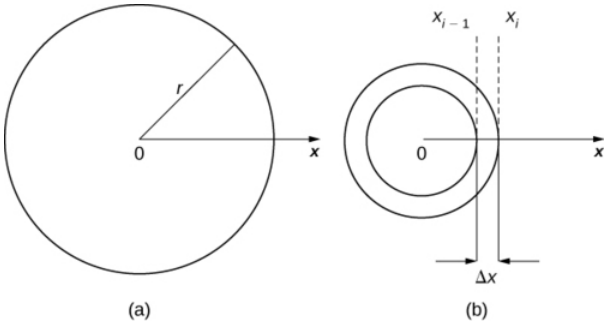


Figure 6.5.3: (a) A thin disk in the *xy*-plane. (b) A representative washer.

We now approximate the density and area of the washer to calculate an approximate mass, m_i. Note that the area of the washer is given by

$$\begin{aligned} A_i &= \pi(x_i)^2 - \pi(x_{i-1})^2 \\ &= \pi[x_i^2 - x_{i-1}^2] \\ &= \pi(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= \pi(x_i + x_{i-1})\Delta x. \end{aligned}$$

You may recall that we had an expression similar to this when we were computing volumes by shells. As we did there, we use $x_i^* \approx (x_i + x_{i-1})/2$ to approximate the average radius of the washer. We obtain

$$A_i = \pi(x_i + x_{i-1})\Delta x \approx 2\pi x_i^* \Delta x.$$

Using $\rho(x_i^*)$ to approximate the density of the washer, we approximate the mass of the washer by

$$m_i \approx 2\pi x_i^* \rho(x_i^*)\Delta x.$$

Adding up the masses of the washers, we see the mass m of the entire disk is approximated by

$$m = \sum_{i=1}^{n} m_i \approx \sum_{i=1}^{n} 2\pi x_i^* \rho(x_i^*)\Delta x.$$

We again recognize this as a Riemann sum, and take the limit as $n \to \infty$. This gives us

$$\begin{aligned} m &= \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi x_i^* \rho(x_i^*)\Delta x \\ &= \int_0^r 2\pi x \rho(x) dx. \end{aligned}$$

We summarize these findings in the following theorem.

Mass–Density Formula of a Circular Object

Let $\rho(x)$ be an integrable function representing the radial density of a disk of radius r. Then the mass of the disk is given by

$$m = \int_0^r 2\pi x \rho(x) dx. \tag{6.5.2}$$

Example 6.5.2: Calculating Mass from Radial Density

Let $\rho(x) = \sqrt{x}$ represent the radial density of a disk. Calculate the mass of a disk of radius 4.

Solution

Applying Equation 6.5.2, we find

$$m = \int_0^r 2\pi x \rho(x) dx$$
$$= \int_0^4 2\pi x\sqrt{x} dx = 2\pi \int_0^4 x^{3/2} dx$$
$$= 2\pi \frac{2}{5} x^{5/2} \,|_0^4 = \frac{4\pi}{5}[32]$$
$$= \frac{128\pi}{5}.$$

? Exercise 6.5.2

Let $\rho(x) = 3x + 2$ represent the radial density of a disk. Calculate the mass of a disk of radius 2.

Hint

Use the process from the previous example.

Solution

24π

Work Done by a Force

We now consider work. In physics, work is related to force, which is often intuitively defined as a push or pull on an object. When a force moves an object, we say the force does work on the object. In other words, work can be thought of as the amount of energy it takes to move an object. According to physics, when we have a constant force, work can be expressed as the product of force and distance.

In the English system, the unit of force is the pound and the unit of distance is the foot, so work is given in foot-pounds. In the metric system, kilograms and meters are used. One newton is the force needed to accelerate 1 kilogram of mass at the rate of 1 m/sec^2. Thus, the most common unit of work is the newton-meter. This same unit is also called the joule. Both are defined as kilograms times meters squared over seconds squared $(kg \cdot m^2/s^2)$.

When we have a constant force, things are pretty easy. It is rare, however, for a force to be constant. The work done to compress (or elongate) a spring, for example, varies depending on how far the spring has already been compressed (or stretched). We look at springs in more detail later in this section.

Suppose we have a variable force $F(x)$ that moves an object in a positive direction along the x-axis from point a to point b. To calculate the work done, we partition the interval $[a, b]$ and estimate the work done over each subinterval. So, for $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of the interval $[a, b]$, and for $i = 1, 2, \ldots, n$, choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. To calculate the work done to move an object from point x_{i-1} to point x_i, we assume the force is roughly constant over the interval, and use $F(x_i^*)$ to approximate the force. The work done over the interval $[x_{i-1}, x_i]$, then, is given by

$$W_i \approx F(x_i^*)(x_i - x_{i-1}) = F(x_i^*)\Delta x.$$

Therefore, the work done over the interval $[a, b]$ is approximately

$$W = \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} F(x_i^*)\Delta x.$$

Taking the limit of this expression as $n \to \infty$ gives us the exact value for work:

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} F(x_i^*)\Delta x$$
$$= \int_a^b F(x) dx.$$

Thus, we can define work as follows.

Definition: Work

If a variable force $F(x)$ moves an object in a positive direction along the x-axis from point a to point b, then the work done on the object is

$$W = \int_a^b F(x)dx. \tag{6.5.3}$$

Note that if F is constant, the integral evaluates to $F \cdot (b - a) = F \cdot d$, which is the formula we stated at the beginning of this section.

Now let's look at the specific example of the work done to compress or elongate a spring. Consider a block attached to a horizontal spring. The block moves back and forth as the spring stretches and compresses. Although in the real world we would have to account for the force of friction between the block and the surface on which it is resting, we ignore friction here and assume the block is resting on a frictionless surface. When the spring is at its natural length (at rest), the system is said to be at equilibrium. In this state, the spring is neither elongated nor compressed, and in this equilibrium position the block does not move until some force is introduced. We orient the system such that $x = 0$ corresponds to the equilibrium position (Figure 6.5.4).

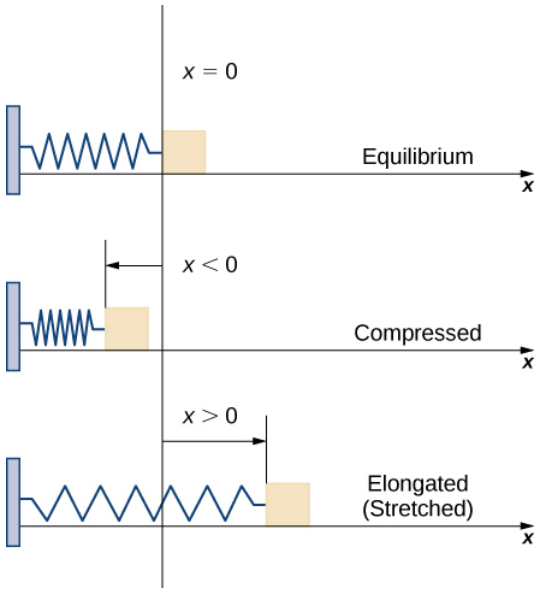


Figure 6.5.4: A block attached to a horizontal spring at equilibrium, compressed, and elongated.

According to Hooke's law, the force required to compress or stretch a spring from an equilibrium position is given by $F(x) = kx$, for some constant k. The value of k depends on the physical characteristics of the spring. The constant k is called the spring constant and is always positive. We can use this information to calculate the work done to compress or elongate a spring, as shown in the following example.

Example 6.5.3: The Work Required to Stretch or Compress a Spring

Suppose it takes a force of 10 N (in the negative direction) to compress a spring 0.2 m from the equilibrium position. How much work is done to stretch the spring 0.5 m from the equilibrium position?

Solution

First find the spring constant, k. When $x = -0.2$, we know $F(x) = -10$, so

$$\begin{aligned} F(x) &= kx \\ -10 &= k(-0.2) \\ k &= 50 \end{aligned}$$

and $F(x) = 50x$. Then, to calculate work, we integrate the force function, obtaining

$$W = \int_a^b F(x)dx$$
$$= \int_0^{0.5} 50x\,dx$$
$$= 25x^2\big|_0^{0.5}$$
$$= 6.25.$$

The work done to stretch the spring is 6.25 J.

? Exercise 6.5.3

Suppose it takes a force of 8 lb to stretch a spring 6 in. from the equilibrium position. How much work is done to stretch the spring 1 ft from the equilibrium position?

Hint

Use the process from the previous example. Be careful with units.

Solution

8 ft-lb

Work Done in Pumping

Consider the work done to pump water (or some other liquid) out of a tank. Pumping problems are a little more complicated than spring problems because many of the calculations depend on the shape and size of the tank. In addition, instead of being concerned about the work done to move a single mass, we are looking at the work done to move a volume of water, and it takes more work to move the water from the bottom of the tank than it does to move the water from the top of the tank.

We examine the process in the context of a cylindrical tank, then look at a couple of examples using tanks of different shapes. Assume a cylindrical tank of radius 4 m and height 10 m is filled to a depth of 8 m. How much work does it take to pump all the water over the top edge of the tank?

The first thing we need to do is define a frame of reference. We let x represent the vertical distance below the top of the tank. That is, we orient the x-axis vertically, with the origin at the top of the tank and the downward direction being positive (Figure $6.5.5$).

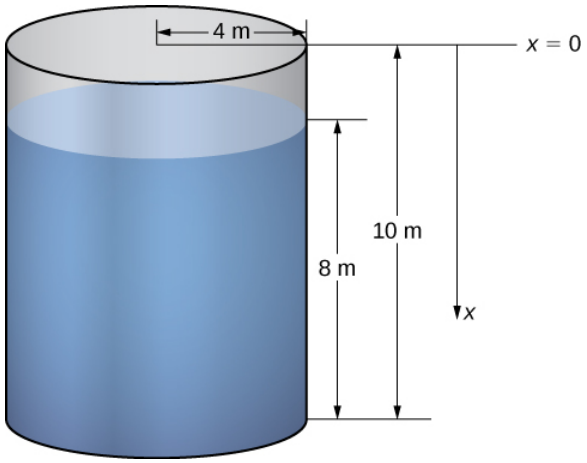


Figure $6.5.5$: How much work is needed to empty a tank partially filled with water?

Using this coordinate system, the water extends from $x = 2$ to $x = 10$. Therefore, we partition the interval $[2, 10]$ and look at the work required to lift each individual "layer" of water. So, for $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of the interval $[2, 10]$, and for $i = 1, 2, \ldots, n$, choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. Figure $6.5.6$ shows a representative layer.

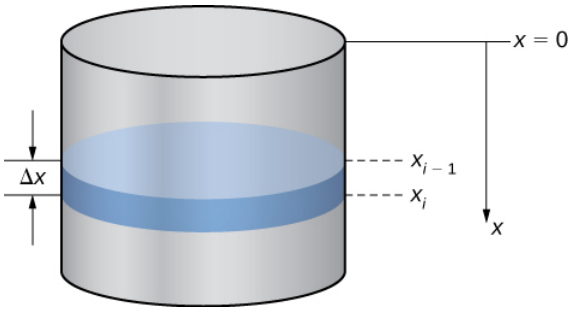


Figure 6.5.6: A representative layer of water.

In pumping problems, the force required to lift the water to the top of the tank is the force required to overcome gravity, so it is equal to the weight of the water. Given that the weight-density of water is $9800\ \text{N/m}^3$, or $62.4\ \text{lb/ft}^3$, calculating the volume of each layer gives us the weight. In this case, we have

$$V = \pi(4)^2\Delta x = 16\pi\Delta x.$$

Then, the force needed to lift each layer is

$$F = 9800 \cdot 16\pi\Delta x = 156,800\pi\Delta x.$$

Note that this step becomes a little more difficult if we have a noncylindrical tank. We look at a noncylindrical tank in the next example.

We also need to know the distance the water must be lifted. Based on our choice of coordinate systems, we can use x_i^* as an approximation of the distance the layer must be lifted. Then the work to lift the ith layer of water W_i is approximately

$$W_i \approx 156,800\pi x_i^*\Delta x.$$

Adding the work for each layer, we see the approximate work to empty the tank is given by

$$\begin{aligned} W &= \sum_{i=1}^{n} W_i \\ &\approx \sum_{i=1}^{n} 156,800\pi x_i^*\Delta x. \end{aligned}$$

This is a Riemann sum, so taking the limit as $n \to \infty$, we get

$$\begin{aligned} W &= \lim_{n\to\infty} \sum_{i=1}^{n} 156,800\pi x_i^*\Delta x \\ &= 156,800\pi \int_2^{10} x dx \\ &= 156,800\pi \left(\frac{x^2}{2}\right)\Big|_2^{10} = 7,526,400\pi \approx 23,644,883. \end{aligned}$$

The work required to empty the tank is approximately 23,650,000 J.

For pumping problems, the calculations vary depending on the shape of the tank or container. The following problem-solving strategy lays out a step-by-step process for solving pumping problems.

Problem-Solving Strategy: Solving Pumping Problems

1. Sketch a picture of the tank and select an appropriate frame of reference.
2. Calculate the volume of a representative layer of water.
3. Multiply the volume by the weight-density of water to get the force.
4. Calculate the distance the layer of water must be lifted.
5. Multiply the force and distance to get an estimate of the work needed to lift the layer of water.

6. Sum the work required to lift all the layers. This expression is an estimate of the work required to pump out the desired amount of water, and it is in the form of a Riemann sum.
7. Take the limit as $n \to \infty$ and evaluate the resulting integral to get the exact work required to pump out the desired amount of water.

We now apply this problem-solving strategy in an example with a noncylindrical tank.

Example 6.5.4: A Pumping Problem with a Noncylindrical Tank

Assume a tank in the shape of an inverted cone, with height 12 ft and base radius 4 ft. The tank is full to start with, and water is pumped over the upper edge of the tank until the height of the water remaining in the tank is 4 ft. How much work is required to pump out that amount of water?

Solution

The tank is depicted in Figure 6.5.7. As we did in the example with the cylindrical tank, we orient the x-axis vertically, with the origin at the top of the tank and the downward direction being positive (step 1).

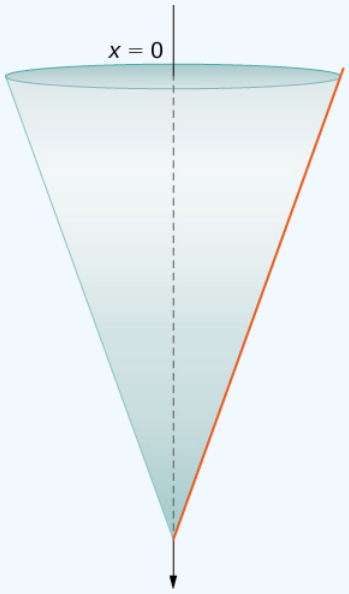


Figure 6.5.7: A water tank in the shape of an inverted cone.

The tank starts out full and ends with 4 ft of water left, so, based on our chosen frame of reference, we need to partition the interval $[0, 8]$. Then, for $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of the interval $[0, 8]$, and for $i = 1, 2, \ldots, n$, choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. We can approximate the volume of a layer by using a disk, then use similar triangles to find the radius of the disk (Figure 6.5.8).

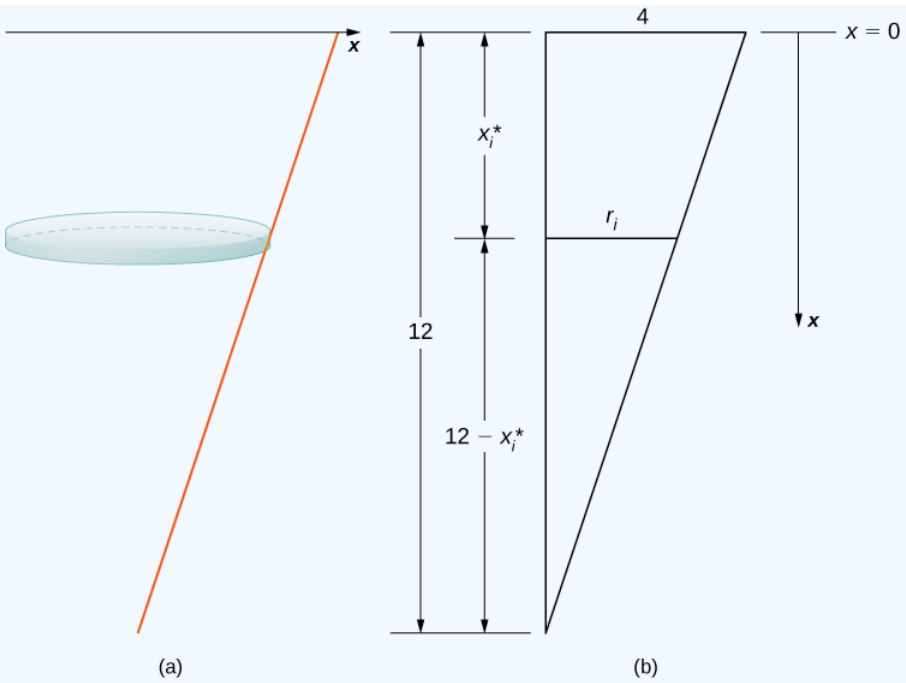


Figure 6.5.8: Using similar triangles to express the radius of a disk of water.

From properties of similar triangles, we have

$$\begin{aligned} \frac{r_i}{12 - x_i^*} &= \frac{4}{12} = \frac{1}{3} \qquad \text{(step 1)} \\ 3r_i &= 12 - x_i^* \\ r_i &= \frac{12 - x_i^*}{3} \\ &= 4 - \frac{x_i^*}{3}. \end{aligned}$$

Then the volume of the disk is

$$V_i = \pi \left(4 - \frac{x_i^*}{3}\right)^2 \Delta x. \qquad \text{(step 2)}$$

The weight-density of water is 62.4lb/ft^3, so the force needed to lift each layer is approximately

$$F_i \approx 62.4\pi \left(4 - \frac{x_i^*}{3}\right)^2 \Delta x \qquad \text{(step 3)}$$

Based on the diagram, the distance the water must be lifted is approximately x_i^* feet (step 4), so the approximate work needed to lift the layer is

$$W_i \approx 62.4\pi x_i^* \left(4 - \frac{x_i^*}{3}\right)^2 \Delta x. \qquad \text{(step 5)}$$

Summing the work required to lift all the layers, we get an approximate value of the total work:

$$W = \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} 62.4\pi x_i^* \left(4 - \frac{x_i^*}{3}\right)^2 \Delta x. \qquad \text{(step 6)}$$

Taking the limit as $n \to \infty$, we obtain

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} 62.4\pi x_i^*(4 - \frac{x_i^*}{3})^2 \Delta x$$

$$= \int_0^8 62.4\pi x \left(4 - \frac{x}{3}\right)^2 dx$$

$$= 62.4\pi \int_0^8 x\left(16 - \frac{8x}{3} + \frac{x^2}{9}\right) dx = 62.4\pi \int_0^8 \left(16x - \frac{8x^2}{3} + \frac{x^3}{9}\right) dx$$

$$= 62.4\pi \left[8x^2 - \frac{8x^3}{9} + \frac{x^4}{36}\right]\Big|_0^8 = 10,649.6\pi \approx 33,456.7.$$

It takes approximately $33,450$ ft-lb of work to empty the tank to the desired level.

? Exercise 6.5.4

A tank is in the shape of an inverted cone, with height 10 ft and base radius 6 ft. The tank is filled to a depth of 8 ft to start with, and water is pumped over the upper edge of the tank until 3 ft of water remain in the tank. How much work is required to pump out that amount of water?

Hint

Use the process from the previous example.

Solution

Approximately $43,255.2$ ft-lb

Hydrostatic Force and Pressure

In this last section, we look at the force and pressure exerted on an object submerged in a liquid. In the English system, force is measured in pounds. In the metric system, it is measured in newtons. Pressure is force per unit area, so in the English system we have pounds per square foot (or, perhaps more commonly, pounds per square inch, denoted psi). In the metric system we have newtons per square meter, also called **pascals.**

Let's begin with the simple case of a plate of area A submerged horizontally in water at a depth s (Figure 6.5.9). Then, the force exerted on the plate is simply the weight of the water above it, which is given by $F = \rho As$, where ρ is the weight density of water (weight per unit volume). To find the **hydrostatic pressure**—that is, the pressure exerted by water on a submerged object—we divide the force by the area. So the pressure is $p = F/A = \rho s$.

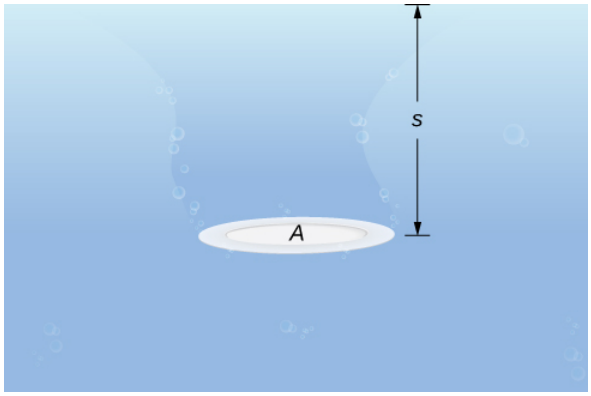


Figure 6.5.9: A plate submerged horizontally in water.

By Pascal's principle, the pressure at a given depth is the same in all directions, so it does not matter if the plate is submerged horizontally or vertically. So, as long as we know the depth, we know the pressure. We can apply Pascal's principle to find the force exerted on surfaces, such as dams, that are oriented vertically. We cannot apply the formula $F = \rho As$ directly, because the depth varies from point to point on a vertically oriented surface. So, as we have done many times before, we form a partition, a Riemann sum, and, ultimately, a definite integral to calculate the force.

Suppose a thin plate is submerged in water. We choose our frame of reference such that the x-axis is oriented vertically, with the downward direction being positive, and point $x = 0$ corresponding to a logical reference point. Let $s(x)$ denote the depth at point x. Note we often let $x = 0$ correspond to the surface of the water. In this case, depth at any point is simply given by $s(x) = x$. However, in some cases we may want to select a different reference point for $x = 0$, so we proceed with the development in the more general case. Last, let $w(x)$ denote the width of the plate at the point x.

Assume the top edge of the plate is at point $x = a$ and the bottom edge of the plate is at point $x = b$. Then, for $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of the interval $[a, b]$, and for $i = 1, 2, \ldots, n$, choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. The partition divides the plate into several thin, rectangular strips (Figure 6.5.10).

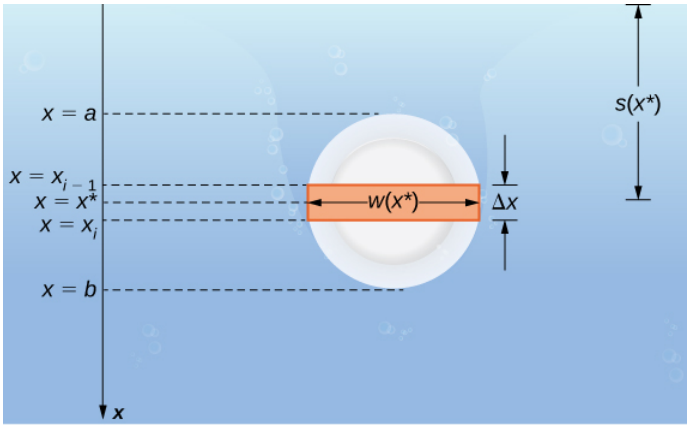


Figure 6.5.10: A thin plate submerged vertically in water.

Let's now estimate the force on a representative strip. If the strip is thin enough, we can treat it as if it is at a constant depth, $s(x_i^*)$. We then have

$$F_i = \rho A s = \rho[w(x_i^*)\Delta x]s(x_i^*).$$

Adding the forces, we get an estimate for the force on the plate:

$$F \approx \sum_{i=1}^{n} F_i = \sum_{i=1}^{n} \rho[w(x_i^*)\Delta x]s(x_i^*).$$

This is a Riemann sum, so taking the limit gives us the exact force. We obtain

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} \rho[w(x_i^*)\Delta x]s(x_i^*) = \int_a^b \rho w(x)s(x)dx. \tag{6.5.4}$$

Evaluating this integral gives us the force on the plate. We summarize this in the following problem-solving strategy.

Problem-Solving Strategy: Finding Hydrostatic Force

1. Sketch a picture and select an appropriate frame of reference. (Note that if we select a frame of reference other than the one used earlier, we may have to adjust Equation 6.5.4 accordingly.)
2. Determine the depth and width functions, $s(x)$ and $w(x)$.
3. Determine the weight-density of whatever liquid with which you are working. The weight-density of water is $62.4\,\text{lb/ft}^3$, or $9800\,\text{N/m}^3$.
4. Use the equation to calculate the total force.

Example 6.5.5: Finding Hydrostatic Force

A water trough 15 ft long has ends shaped like inverted isosceles triangles, with base 8 ft and height 3 ft. Find the force on one end of the trough if the trough is full of water.

Solution

Figure 6.5.11 shows the trough and a more detailed view of one end.

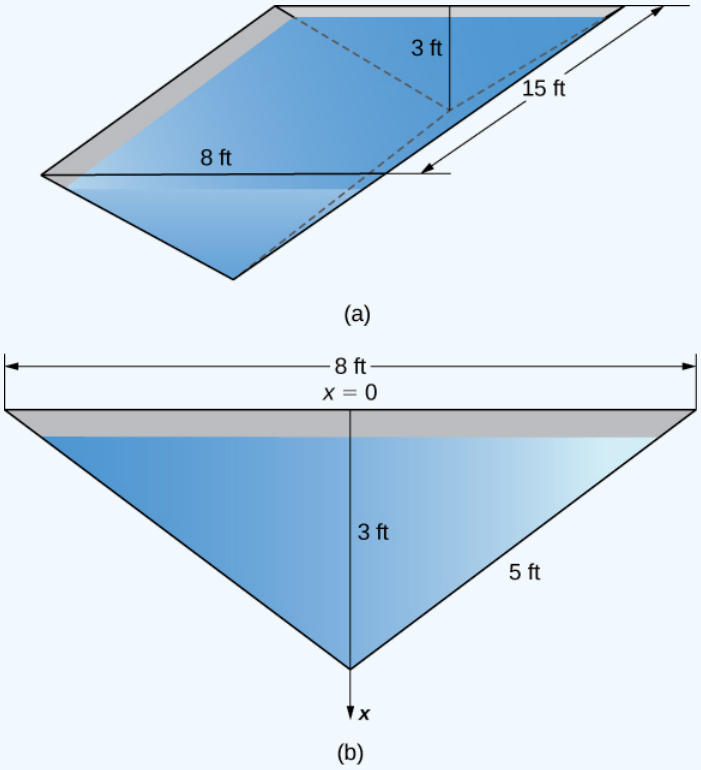


Figure 6.5.11: (a) A water trough with a triangular cross-section. (b) Dimensions of one end of the water trough.

Select a frame of reference with the x-axis oriented vertically and the downward direction being positive. Select the top of the trough as the point corresponding to $x = 0$ (step 1). The depth function, then, is $s(x) = x$. Using similar triangles, we see that $w(x) = 8 - (8/3)x$ (step 2). Now, the weight density of water is $62.4\,\text{lb/ft}^3$ (step 3), so applying Equation 6.5.4, we obtain

$$F = \int_a^b \rho w(x)s(x)dx$$

$$= \int_0^3 62.4\left(8 - \frac{8}{3}x\right)x\,dx = 62.4\int_0^3 \left(8x - \frac{8}{3}x^2\right)dx$$

$$= 62.4\left[4x^2 - \frac{8}{9}x^3\right]\Big|_0^3 = 748.8.$$

The water exerts a force of 748.8 lb on the end of the trough (step 4).

? Exercise 6.5.5

A water trough 12 m long has ends shaped like inverted isosceles triangles, with base 6 m and height 4 m. Find the force on one end of the trough if the trough is full of water.

Hint

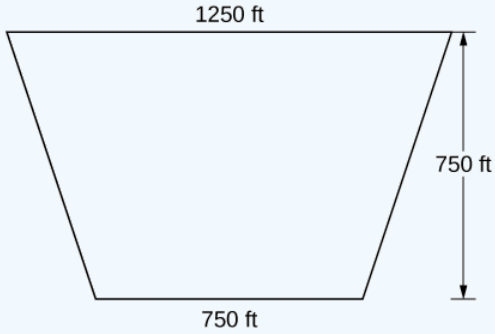
Follow the problem-solving strategy and the process from the previous example.

Solution

$156,800\ \text{N}$

Example 6.5.6: Finding Hydrostatic Force

We now return our attention to the Hoover Dam, mentioned at the beginning of this chapter. The actual dam is arched, rather than flat, but we are going to make some simplifying assumptions to help us with the calculations. Assume the face of the Hoover Dam is shaped like an isosceles trapezoid with lower base 750 ft, upper base 1250 ft, and height 750 ft (see the following figure).



When the reservoir is full, Lake Mead's maximum depth is about 530 ft, and the surface of the lake is about 10 ft below the top of the dam (see the following figure).

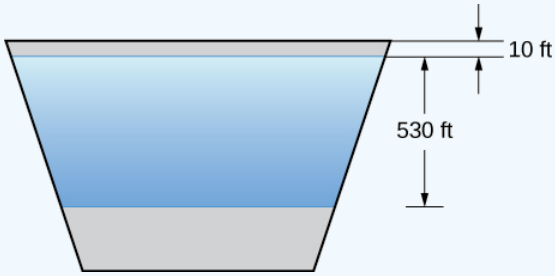


Figure 6.5.12: A simplified model of the Hoover Dam with assumed dimensions.

a. Find the force on the face of the dam when the reservoir is full.
b. The southwest United States has been experiencing a drought, and the surface of Lake Mead is about 125 ft below where it would be if the reservoir were full. What is the force on the face of the dam under these circumstances?

Solution:

a.

We begin by establishing a frame of reference. As usual, we choose to orient the x-axis vertically, with the downward direction being positive. This time, however, we are going to let $x = 0$ represent the top of the dam, rather than the surface of the water. When the reservoir is full, the surface of the water is 10 ft below the top of the dam, so $s(x) = x - 10$ (see the following figure).

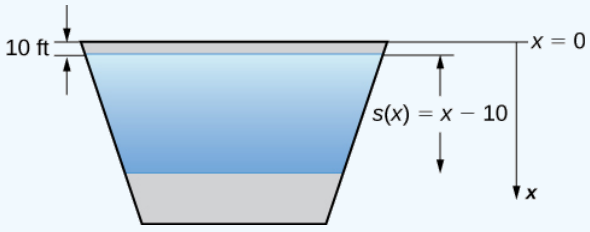


Figure 6.5.13: We first choose a frame of reference.

To find the width function, we again turn to similar triangles as shown in the figure below.

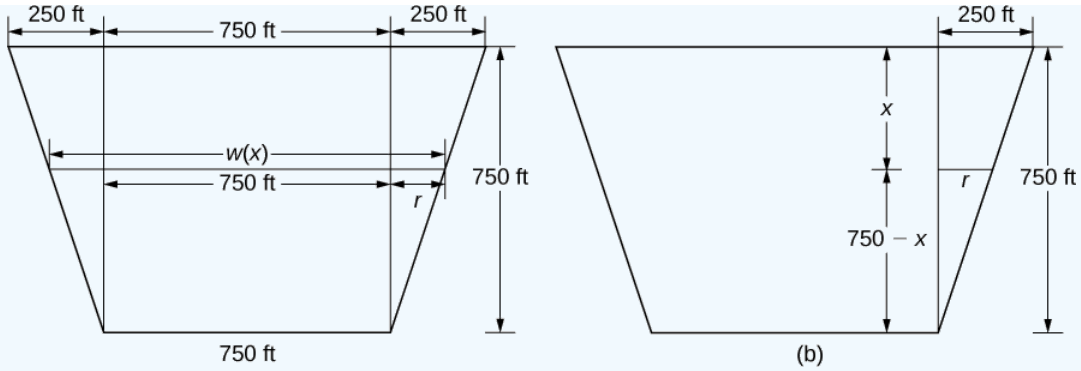


Figure 6.5.14: We use similar triangles to determine a function for the width of the dam. (a) Assumed dimensions of the dam; (b) highlighting the similar triangles.

From the figure, we see that $w(x) = 750 + 2r$. Using properties of similar triangles, we get $r = 250 - (1/3)x$. Thus,

$$w(x) = 1250 - \frac{2}{3}x \qquad \text{(step 2)}$$

Using a weight-density of 62.4lb/ft^3 (step 3) and applying Equation 6.5.4, we get

$$\begin{aligned} F &= \int_a^b \rho w(x) s(x)\, dx \\ &= \int_{10}^{540} 62.4\left(1250 - \frac{2}{3}x\right)(x - 10)\, dx \\ &= 62.4\int_{10}^{540} -\frac{2}{3}[x^2 - 1885x + 18750]\, dx \\ &= -62.4\left(\frac{2}{3}\right)\left[\frac{x^3}{3} - \frac{1885x^2}{2} + 18750x\right]\Big|_{10}^{540} \approx 8,832,245,000\,\text{lb} = 4,416,122.5\,\text{t}. \end{aligned}$$

Note the change from pounds to tons ($2000\text{lb} = 1$ ton) (step 4). This changes our depth function, $s(x)$, and our limits of integration. We have $s(x) = x - 135$. The lower limit of integration is 135. The upper limit remains 540. Evaluating the integral, we get

$$\begin{aligned} F &= \int_a^b \rho w(x) s(x)\, dx \\ &= \int_{135}^{540} 62.4\left(1250 - \frac{2}{3}x\right)(x - 135)\, dx \\ &= -62.4(\tfrac{2}{3})\int_{135}^{540}(x - 1875)(x - 135)\, dx = -62.4\left(\frac{2}{3}\right)\int_{135}^{540}(x^2 - 2010x + 253125)\, dx \\ &= -62.4\left(\frac{2}{3}\right)\left[\frac{x^3}{3} - 1005x^2 + 253125x\right]\Big|_{135}^{540} \approx 5,015,230,000\,\text{lb} = 2,507,615\,\text{t}. \end{aligned}$$

? Exercise 6.5.6

When the reservoir is at its average level, the surface of the water is about 50 ft below where it would be if the reservoir were full. What is the force on the face of the dam under these circumstances?

Hint

Change the depth function, $s(x)$, and the limits of integration.

Solution

Approximately 7,164,520,000 lb or 3,582,260 t

Key Concepts

- Several physical applications of the definite integral are common in engineering and physics.
- Definite integrals can be used to determine the mass of an object if its density function is known.
- Work can also be calculated from integrating a force function, or when counteracting the force of gravity, as in a pumping problem.
- Definite integrals can also be used to calculate the force exerted on an object submerged in a liquid.

Key Equations

- **Mass of a one-dimensional object**

$$m = \int_a^b \rho(x)dx$$

- **Mass of a circular object**

$$m = \int_0^r 2\pi x\rho(x)dx$$

- **Work done on an object**

$$W = \int_a^b F(x)dx$$

- **Hydrostatic force on a plate**

$$F = \int_a^b \rho w(x)s(x)dx$$

Glossary

density function
: a density function describes how mass is distributed throughout an object; it can be a linear density, expressed in terms of mass per unit length; an area density, expressed in terms of mass per unit area; or a volume density, expressed in terms of mass per unit volume; weight-density is also used to describe weight (rather than mass) per unit volume

Hooke's law
: this law states that the force required to compress (or elongate) a spring is proportional to the distance the spring has been compressed (or stretched) from equilibrium; in other words, $F = kx$, where k is a constant

hydrostatic pressure
: the pressure exerted by water on a submerged object

work
: the amount of energy it takes to move an object; in physics, when a force is constant, work is expressed as the product of force and distance

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6.5E: Exercises for Section 6.5

Basic Work Problems

For exercises 1 - 6, find the work done.

1) Find the work done when a constant force $F = 12$ lb moves a chair from $x = 0.9$ to $x = 1.1$ ft.

2) How much work is done when a person lifts a 50 lb box of comics onto a truck that is 3 ft off the ground?

Answer

$W = 150$ ft-lb

3) What is the work done lifting a 20 kg child from the floor to a height of 2 m? (Note that 1 kg equates to 9.8 N)

4) Find the work done when you push a box along the floor 2 m, when you apply a constant force of $F = 100$ N.

Answer

$W = 200$ J

5) Compute the work done for a force $F = \dfrac{12}{x^2}$ N from $x = 1$ to $x = 2$ m.

6) What is the work done moving a particle from $x = 0$ to $x = 1$ m if the force acting on it is $F = 3x^2$ N?

Answer

$W = 1$ J

Density Problems

In exercises 7 - 11, find the mass of the one-dimensional object.

7) A wire that is 2 ft long (starting at $x = 0$) and has a density function of $\rho(x) = x^2 + 2x$ lb/ft

8) A car antenna that is 3 ft long (starting at $x = 0$) and has a density function of $\rho(x) = 3x + 2$ lb/ft

Answer

$\frac{39}{2}$

9) A metal rod that is 8 in. long (starting at $x = 0$) and has a density function of $\rho(x) = e^{1/2x}$ lb/in.

10) A pencil that is 4 in. long (starting at $x = 2$) and has a density function of $\rho(x) = \dfrac{5}{x}$ oz/in.

Answer

$\ln(243)$

11) A ruler that is 12 in. long (starting at $x = 5$) and has a density function of $\rho(x) = \ln(x) + (1/2)x^2$ oz/in.

In exercises 12 - 16, find the mass of the two-dimensional object that is centered at the origin.

12) An oversized hockey puck of radius 2 in. with density function $\rho(x) = x^3 - 2x + 5$

Answer

$\frac{332\pi}{15}$

13) A frisbee of radius 6 in. with density function $\rho(x) = e^{-x}$

14) A plate of radius 10 in. with density function $\rho(x) = 1 + \cos(\pi x)$

Answer

100π

15) A jar lid of radius 3 in. with density function $\rho(x) = \ln(x + 1)$

16) A disk of radius 5 cm with density function $\rho(x) = \sqrt{3x}$

Answer

$20\pi\sqrt{15}$

Spring Work Problems

17) A 12-in. spring is stretched to 15 in. by a force of 75 lb. What is the spring constant?

18) A spring has a natural length of 10 cm. It takes 2 J to stretch the spring to 15 cm. How much work would it take to stretch the spring from 15 cm to 20 cm?

Answer

$W = 6$ J

19) A 1-m spring requires 10 J to stretch the spring to 1.1 m. How much work would it take to stretch the spring from 1 m to 1.2 m?

20) A spring requires 5 J to stretch the spring from 8 cm to 12 cm, and an additional 4 J to stretch the spring from 12 cm to 14 cm. What is the natural length of the spring?

Answer

The natural length is5 cm.

21) A shock absorber is compressed 1 in. by a weight of 1 ton. What is the spring constant?

22) A force of $F = (20x - x^3)$ N stretches a nonlinear spring by x meters. What work is required to stretch the spring from $x = 0$ to $x = 2$ m?

Answer

$W = 36$ J

Cable and Chain Work Problems

23) Find the work done by winding up a hanging cable of length 100 ft and weight-density 5 lb/ft.

24) For the cable in the preceding exercise, how much work is done to lift the cable 50 ft?

Answer

$W = 18,750$ ft-lb

25) For the cable in the preceding exercise, how much additional work is done by hanging a 200 lb weight at the end of the cable?

Pyramid & Satellite/Rocket Work Problems

26) [T] A pyramid of height 500 ft has a square base 800 ft by 800 ft. Find the area A at height h. If the rock used to build the pyramid weighs approximately $w = 100\,\text{lb/ft}^3$, how much work did it take to lift all the rock?

Answer

$W = \frac{32}{3} \times 10^9$ ft-lb

27) [T] For the pyramid in the preceding exercise, assume there were 1000 workers each working 10 hours a day, 5 days a week, 50 weeks a year. If each of the workers, on average, lifted ten 100-lb rocks 2 ft/hr, how long did it take to build the pyramid?

28) [T] The force of gravity on a mass m is $F = -((GMm)/x^2)$ newtons. For a rocket of mass $m = 1000$ kg, compute the work to lift the rocket from $x = 6400$ to $x = 6500$ km. (Note: $G = 6 \times 10^{-17}$ N m^2/kg^2 and $M = 6 \times 10^{24}$ kg.)

Answer

$W = 8.65 \times 10^5$ J

29) [T] For the rocket in the preceding exercise, find the work to lift the rocket from $x = 6400$ to $x = \infty$.

Hydrostatic Force and Pressure

30) [T] A rectangular dam is 40 ft high and 60 ft wide. Compute the total force F on the dam when

a. the surface of the water is at the top of the dam and

b. the surface of the water is halfway down the dam.

Answer

a. $3,000,000$ lb,
b. $749,000$ lb

Pumping Work Problems

31) [T] Find the work required to pump all the water out of a cylinder that has a circular base of radius 5ft and height 200 ft. Use the fact that the density of water is 62 lb/ft^3.

32) [T] Find the work required to pump all the water out of the cylinder in the preceding exercise if the cylinder is only half full.

Answer

$W = 23.25\pi$ million ft-lb

33) [T] How much work is required to pump out a swimming pool if the area of the base is $800\,\text{ft}^2$, the water is 4 ft deep, and the top is 1 ft above the water level? Assume that the density of water is 62 lb/ft^3.

34) A cylinder of depth H and cross-sectional area A stands full of water at density ρ. Compute the work to pump all the water to the top.

Answer

$W = \dfrac{A\rho H^2}{2}$

35) For the cylinder in the preceding exercise, compute the work to pump all the water to the top if the cylinder is only half full.

36) A cone-shaped tank has a cross-sectional area that increases with its depth: $A = \dfrac{\pi r^2 h^2}{H^3}$. Show that the work to empty it is half the work for a cylinder with the same height and base.

Answer

Answers may vary.

Contributors

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6.6: Moments and Centers of Mass

Learning Objectives

- Find the center of mass of objects distributed along a line.
- Locate the center of mass of a thin plate.
- Use symmetry to help locate the centroid of a thin plate.
- Apply the theorem of Pappus for volume.

In this section, we consider centers of mass (also called *centroids*, under certain conditions) and moments. The basic idea of the center of mass is the notion of a balancing point. Many of us have seen performers who spin plates on the ends of sticks. The performers try to keep several of them spinning without allowing any of them to drop. If we look at a single plate (without spinning it), there is a sweet spot on the plate where it balances perfectly on the stick. If we put the stick anywhere other than that sweet spot, the plate does not balance and it falls to the ground. (That is why performers spin the plates; the spin helps keep the plates from falling even if the stick is not exactly in the right place.) Mathematically, that sweet spot is called the *center of mass of the plate*.

In this section, we first examine these concepts in a one-dimensional context, then expand our development to consider centers of mass of two-dimensional regions and symmetry. Last, we use centroids to find the volume of certain solids by applying the theorem of Pappus.

Center of Mass and Moments

Let's begin by looking at the center of mass in a one-dimensional context. Consider a long, thin wire or rod of negligible mass resting on a fulcrum, as shown in Figure 6.6.1a. Now suppose we place objects having masses m_1 and m_2 at distances d_1 and d_2 from the fulcrum, respectively, as shown in Figure 6.6.1b.

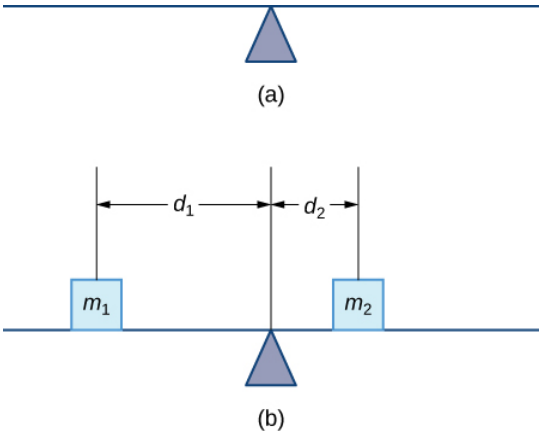


Figure 6.6.1: (a) A thin rod rests on a fulcrum. (b) Masses are placed on the rod.

The most common real-life example of a system like this is a playground seesaw, or teeter-totter, with children of different weights sitting at different distances from the center. On a seesaw, if one child sits at each end, the heavier child sinks down and the lighter child is lifted into the air. If the heavier child slides in toward the center, though, the seesaw balances. Applying this concept to the masses on the rod, we note that the masses balance each other if and only if

$$m_1 d_1 = m_2 d_2.$$

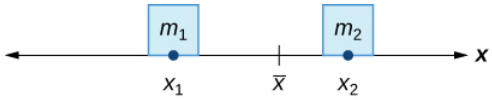


Figure 6.6.2: The center of mass $\bar{x}$ is the balance point of the system.

In the seesaw example, we balanced the system by moving the masses (children) with respect to the fulcrum. However, we are really interested in systems in which the masses are not allowed to move, and instead we balance the system by moving the

fulcrum. Suppose we have two point masses, m_1 and m_2, located on a number line at points x_1 and x_2, respectively (Figure 6.6.2). The center of mass, $\bar{x}$, is the point where the fulcrum should be placed to make the system balance.

Thus, we have

$$\begin{aligned} m_1|x_1 - \bar{x}| &= m_2|x_2 - \bar{x}| \\ m_1(\bar{x} - x_1) &= m_2(x_2 - \bar{x}) \\ m_1\bar{x} - m_1x_1 &= m_2x_2 - m_2\bar{x} \\ \bar{x}(m_1 + m_2) &= m_1x_1 + m_2x_2 \end{aligned}$$

or

$$\bar{x} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} \tag{6.6.1}$$

The expression in the numerator of Equation 6.6.1, $m_1x_1 + m_2x_2$, is called the *first moment of the system* with respect to the origin. If the context is clear, we often drop the word first and just refer to this expression as the **moment** of the system. The expression in the denominator, $m_1 + m_2$, is the total mass of the system. Thus, the **center of mass** of the system is the point at which the total mass of the system could be concentrated without changing the moment.

This idea is not limited just to two point masses. In general, if n masses, $m_1, m_2, \ldots, m_n$, are placed on a number line at points $x_1, x_2, \ldots, x_n$, respectively, then the center of mass of the system is given by

$$\bar{x} = \frac{\sum_{i=1}^{n} m_ix_i}{\sum_{i=1}^{n} m_i}$$

Center of Mass of Objects on a Line

Let $m_1, m_2, \ldots, m_n$ be point masses placed on a number line at points $x_1, x_2, \ldots, x_n$, respectively, and let $m = \sum_{i=1}^{n} m_i$ denote the total mass of the system. Then, the *moment of the system* with respect to the origin is given by

$$M = \sum_{i=1}^{n} m_ix_i \tag{6.6.2}$$

and the *center of mass of the system* is given by

$$\bar{x} = \frac{M}{m}. \tag{6.6.3}$$

We apply this theorem in the following example.

Example 6.6.1: Finding the Center of Mass of Objects along a Line

Suppose four point masses are placed on a number line as follows:

- $m_1 = 30\,kg$, placed at $x_1 = -2m$
- $m_2 = 5\,kg$, placed at $x_2 = 3m$
- $m_3 = 10\,kg$, placed at $x_3 = 6m$
- $m_4 = 15\,kg$, placed at $x_4 = -3m$.

Solution

Find the moment of the system with respect to the origin and find the center of mass of the system.

First, we need to calculate the moment of the system (Equation 6.6.2):

$$\begin{aligned} M &= \sum_{i=1}^{4} m_i x_i \\ &= -60 + 15 + 60 - 45 \\ &= -30. \end{aligned}$$

Now, to find the center of mass, we need the total mass of the system:

$$\begin{aligned} m &= \sum_{i=1}^{4} m_i \\ &= 30 + 5 + 10 + 15 \\ &= 60\, kg \end{aligned}$$

Then we have (from Equation 6.6.3)

$$\bar{x} - = \frac{M}{m} = -\frac{30}{60} = -\frac{1}{2}.$$

The center of mass is located 1/2 m to the left of the origin.

? Exercise 6.6.1

Suppose four point masses are placed on a number line as follows:

- $m_1 = 12\, kg$ placed at $x_1 = -4m$
- $m_2 = 12\, kg$ placed at $x_2 = 4m$
- $m_3 = 30\, kg$ placed at $x_3 = 2m$
- $m_4 = 6\, kg$, placed at $x_4 = -6m$.

Find the moment of the system with respect to the origin and find the center of mass of the system.

Hint

Use the process from the previous example.

Answer

$M = 24, \bar{x} = \frac{2}{5}m$

We can generalize this concept to find the center of mass of a system of point masses in a plane. Let m_1 be a point mass located at point (x_1, y_1) in the plane. Then the moment M_x of the mass with respect to the x-axis is given by $M_x = m_1 y_1$. Similarly, the moment M_y with respect to the y-axis is given by

$$M_y = m_1 x_1.$$

Notice that the x-coordinate of the point is used to calculate the moment with respect to the y-axis, and vice versa. The reason is that the x-coordinate gives the distance from the point mass to the y-axis, and the y-coordinate gives the distance to the x-axis (see the following figure).

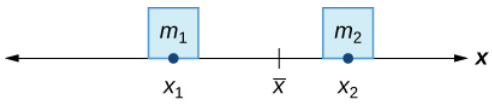


Figure 6.6.3: Point mass m_1 is located at point (x_1, y_1) in the plane.

If we have several point masses in the xy-plane, we can use the moments with respect to the x- and y-axes to calculate the x- and y-coordinates of the center of mass of the system.

Center of Mass of Objects in a Plane

Let m_1, m_2, …, m_n be point masses located in the xy-plane at points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, respectively, and let $m = \sum_{i=1}^{n} m_i$ denote the total mass of the system. Then the moments M_x and M_y of the system with respect to the x- and y-axes, respectively, are given by

$$M_x = \sum_{i=1}^{n} m_i y_i \tag{6.6.4}$$

and

$$M_y = \sum_{i=1}^{n} m_i x_i. \tag{6.6.5}$$

Also, the coordinates of the center of mass $(\bar{x}, \bar{y})$ of the system are

$$\bar{x} = \frac{M_y}{m} \tag{6.6.6}$$

and

$$\bar{y} = \frac{M_x}{m}. \tag{6.6.7}$$

The next example demonstrates how to the center of mass formulas (Equations 6.6.4 - 6.6.7) may be applied.

Example 6.6.2: Finding the Center of Mass of Objects in a Plane

Suppose three point masses are placed in the xy-plane as follows (assume coordinates are given in meters):

- $m_1 = 2\,kg$ placed at $(-1, 3)$,
- $m_2 = 6\,kg$ placed at $(1, 1)$,
- $m_3 = 4\,kg$ placed at $(2, -2)$.

Find the center of mass of the system.

Solution

First we calculate the total mass of the system:

$$m = \sum_{i=1}^{3} m_i = 2 + 6 + 4 = 12\,kg.$$

Next we find the moments with respect to the x- and y-axes:

$$M_y = \sum_{i=1}^{3} m_i x_i = -2 + 6 + 8 = 12,$$

$$M_x = \sum_{i=1}^{3} m_i y_i = 6 + 6 - 8 = 4.$$

Then we have

$$\bar{x} = \frac{M_y}{m} = \frac{12}{12} = 1$$

and

$$\bar{y} = \frac{M_x}{m} = \frac{4}{12} = \frac{1}{3}.$$

The center of mass of the system is $(1, 1/3)$, in meters.

Exercise 6.6.2

Suppose three point masses are placed on a number line as follows (assume coordinates are given in meters):

- $m_1 = 5\,kg$, placed at $(-2, -3)$,
- $m_2 = 3\,kg$, placed at $(2, 3)$,
- $m_3 = 2\,kg$, placed at $(-3, -2)$.

Find the center of mass of the system.

Hint

Use the process from the previous example.

Answer

$(-1, -1)$ m

Center of Mass of Thin Plates

So far we have looked at systems of point masses on a line and in a plane. Now, instead of having the mass of a system concentrated at discrete points, we want to look at systems in which the mass of the system is distributed continuously across a thin sheet of material. For our purposes, we assume the sheet is thin enough that it can be treated as if it is two-dimensional. Such a sheet is called a **lamina**. Next we develop techniques to find the center of mass of a lamina. In this section, we also assume the density of the lamina is constant.

Laminas are often represented by a two-dimensional region in a plane. The geometric center of such a region is called its **centroid**. Since we have assumed the density of the lamina is constant, the center of mass of the lamina depends only on the shape of the corresponding region in the plane; it does not depend on the density. In this case, the center of mass of the lamina corresponds to the centroid of the delineated region in the plane. As with systems of point masses, we need to find the total mass of the lamina, as well as the moments of the lamina with respect to the x- and y-axes.

We first consider a lamina in the shape of a rectangle. Recall that the center of mass of a lamina is the point where the lamina balances. For a rectangle, that point is both the horizontal and vertical center of the rectangle. Based on this understanding, it is clear that the center of mass of a rectangular lamina is the point where the diagonals intersect, which is a result of the **symmetry principle**, and it is stated here without proof.

The Symmetry Principle

If a region R is symmetric about a line l, then the centroid of R lies on l.

Let's turn to more general laminas. Suppose we have a lamina bounded above by the graph of a continuous function $f(x)$, below by the x-axis, and on the left and right by the lines $x = a$ and $x = b$, respectively, as shown in the following figure.

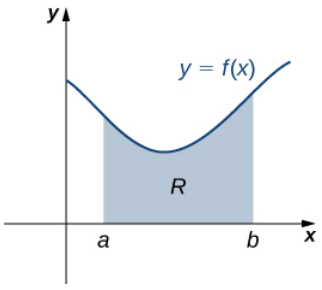


Figure 6.6.4: A region in the plane representing a lamina.

As with systems of point masses, to find the center of mass of the lamina, we need to find the total mass of the lamina, as well as the moments of the lamina with respect to the x- and y-axes. As we have done many times before, we approximate these quantities by partitioning the interval $[a, b]$ and constructing rectangles.

For $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of $[a, b]$. Recall that we can choose any point within the interval $[x_{i-1}, x_i]$ as our x_i^*. In this case, we want x_i^* to be the x-coordinate of the centroid of our rectangles. Thus, for $i = 1, 2, \ldots, n$, we select $x_i^* \in [x_{i-1}, x_i]$ such that x_i^* is the midpoint of the interval. That is, $x_i^* = (x_{i-1} + x_i)/2$. Now, for $i = 1, 2, \ldots, n$, construct a rectangle of height $f(x_i^*)$ on $[x_{i-1}, x_i]$. The center of mass of this rectangle is $(x_i^*, (f(x_i^*))/2)$, as shown in the following figure.

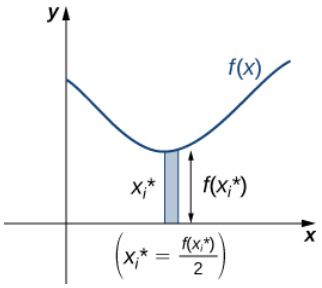


Figure 6.6.5: A representative rectangle of the lamina.

Next, we need to find the total mass of the rectangle. Let ρ represent the density of the lamina (note that ρ is a constant). In this case, ρ is expressed in terms of mass per unit area. Thus, to find the total mass of the rectangle, we multiply the area of the rectangle by ρ. Then, the mass of the rectangle is given by $\rho f(x_i^*)\Delta x$.

To get the approximate mass of the lamina, we add the masses of all the rectangles to get

$$m \approx \sum_{i=1}^{n} \rho f(x_i^*)\Delta x. \tag{6.6.8}$$

Equation 6.6.8 is a Riemann sum. Taking the limit as $n \to \infty$ gives the exact mass of the lamina:

$$\begin{aligned} m &= \lim_{n\to\infty} \sum_{i=1}^{n} \rho f(x_i^*)\Delta x \\ &= \rho \int_a^b f(x)dx. \end{aligned}$$

Next, we calculate the moment of the lamina with respect to the x-axis. Returning to the representative rectangle, recall its center of mass is $(x_i^*, (f(x_i^*))/2)$. Recall also that treating the rectangle as if it is a point mass located at the center of mass does not change the moment. Thus, the moment of the rectangle with respect to the x-axis is given by the mass of the rectangle, $\rho f(x_i^*)\Delta x$, multiplied by the distance from the center of mass to the x-axis: $(f(x_i^*))/2$. Therefore, the moment with respect to the x-axis of the rectangle is $\rho([f(x_i^*)]^2/2)\Delta x$. Adding the moments of the rectangles and taking the limit of the resulting Riemann sum, we see that the moment of the lamina with respect to the x-axis is

$$\begin{aligned} M_x &= \lim_{n\to\infty} \sum_{i=1}^{n} \rho \frac{[f(x_i^*)]^2}{2}\Delta x \\ &= \rho \int_a^b \frac{[f(x)]^2}{2}dx. \end{aligned}$$

We derive the moment with respect to the y-axis similarly, noting that the distance from the center of mass of the rectangle to the y-axis is x_i^*. Then the moment of the lamina with respect to the y-axis is given by

$$\begin{aligned} M_y &= \lim_{n\to\infty} \sum_{i=1}^{n} \rho x_i^* f(x^*)i)\Delta x \\ &= \rho \int_a^b x f(x)dx. \end{aligned}$$

We find the coordinates of the center of mass by dividing the moments by the total mass to give $\bar{x} = M_y/m$ and $\bar{y} = M_x/m$. If we look closely at the expressions for M_x, M_y, and m, we notice that the constant ρ cancels out when $\bar{x}$ and $\bar{y}$ are calculated.

We summarize these findings in the following theorem.

Center of Mass of a Thin Plate in the xy-Plane

Let R denote a region bounded above by the graph of a continuous function $f(x)$, below by the x-axis, and on the left and right by the lines $x = a$ and $x = b$, respectively. Let ρ denote the density of the associated lamina. Then we can make the following statements:

i. The mass of the lamina is

$$m = \rho \int_a^b f(x)dx. \tag{6.6.9}$$

ii. The moments M_x and M_y of the lamina with respect to the x- and y-axes, respectively, are

$$M_x = \rho \int_a^b \frac{[f(x)]^2}{2} dx \tag{6.6.10}$$

and

$$M_y = \rho \int_a^b xf(x)dx. \tag{6.6.11}$$

iii. The coordinates of the center of mass $(\bar{x}, \bar{y})$ are

$$\bar{x} = \frac{M_y}{m} \tag{6.6.12}$$

and

$$\bar{y} = \frac{M_x}{m}. \tag{6.6.13}$$

In the next example, we use this theorem to find the center of mass of a lamina.

Example 6.6.3: Finding the Center of Mass of a Lamina

Let R be the region bounded above by the graph of the function $f(x) = \sqrt{x}$ and below by the x-axis over the interval $[0, 4]$. Find the centroid of the region.

Solution

The region is depicted in the following figure.

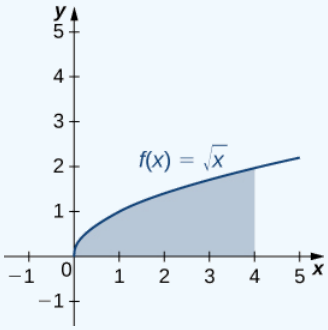


Figure 6.6.6: Finding the center of mass of a lamina.

Since we are only asked for the centroid of the region, rather than the mass or moments of the associated lamina, we know the density constant ρ cancels out of the calculations eventually. Therefore, for the sake of convenience, let's assume $\rho = 1$.

First, we need to calculate the total mass (Equation 6.6.9):

$$
\begin{aligned}
m &= \rho \int_a^b f(x)dx \\
&= \int_0^4 \sqrt{x}dx \\
&= \frac{2}{3}x^{3/2}\,|_0^4 \\
&= \frac{2}{3}[8-0] \\
&= \frac{16}{3}.
\end{aligned}
$$

Next, we compute the moments (Equation 6.6.12):

$$
\begin{aligned}
M_x &= \rho \int_a^b \frac{[f(x)]^2}{2}dx \\
&= \int_0^4 \frac{x}{2}dx \\
&= \frac{1}{4}x^2\,|_0^4 \\
&= 4
\end{aligned}
$$

and (Equation 6.6.11):

$$
\begin{aligned}
M_y &= \rho \int_a^b xf(x)dx \\
&= \int_0^4 x\sqrt{x}dx \\
&= \int_0^4 x^{3/2}dx \\
&= \frac{2}{5}x^{5/2}\,|_0^4 \\
&= \frac{2}{5}[32-0] \\
&= \frac{64}{5}.
\end{aligned}
$$

Thus, we have (Equation 6.6.12):

$$
\begin{aligned}
\bar{x} &= \frac{M_y}{m} \\
&= \frac{64/5}{16/3} \\
&= \frac{64}{5}\cdot\frac{3}{16} \\
&= \frac{12}{5}
\end{aligned}
$$

and (Equation 6.6.13):

$$
\begin{aligned}
\bar{y} &= \frac{M_x}{y} \\
&= \frac{4}{16/3} \\
&= 4\cdot\frac{3}{16} \\
&= \frac{3}{4}.
\end{aligned}
$$

The centroid of the region is $(12/5, 3/4)$.

Exercise 6.6.3

Let R be the region bounded above by the graph of the function $f(x) = x^2$ and below by the x-axis over the interval $[0, 2]$. Find the centroid of the region.

Hint

Use the process from the previous example.

Answer

The centroid of the region is $(3/2, 6/5)$.

We can adapt this approach to find centroids of more complex regions as well. Suppose our region is bounded above by the graph of a continuous function $f(x)$, as before, but now, instead of having the lower bound for the region be the x-axis, suppose the region is bounded below by the graph of a second continuous function, $g(x)$, as shown in Figure 6.6.7.

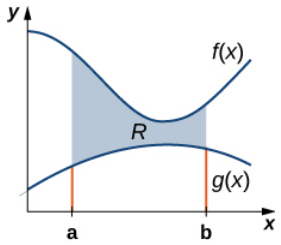


Figure 6.6.7: A region between two functions.

Again, we partition the interval $[a, b]$ and construct rectangles. A representative rectangle is shown in Figure 6.6.8.

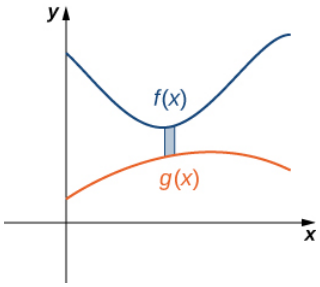


Figure 6.6.8: A representative rectangle of the region between two functions.

Note that the centroid of this rectangle is $(x_i^*, (f(x_i^*) + g(x_i^*))/2)$. We won't go through all the details of the Riemann sum development, but let's look at some of the key steps. In the development of the formulas for the mass of the lamina and the moment with respect to the y-axis, the height of each rectangle is given by $f(x_i^*) - g(x_i^*)$, which leads to the expression $f(x) - g(x)$ in the integrands.

In the development of the formula for the moment with respect to the x-axis, the moment of each rectangle is found by multiplying the area of the rectangle, $\rho[f(x_i^*) - g(x_i^*)]\Delta x$, by the distance of the centroid from the x-axis, $(f(x_i^*) + g(x_i^*))/2$, which gives $\rho(1/2)[f(x_i^*)]^2 - [g(x_i^*)]^2\Delta x$. Summarizing these findings, we arrive at the following theorem.

Center of Mass of a Lamina Bounded by Two Functions

Let R denote a region bounded above by the graph of a continuous function $f(x)$, below by the graph of the continuous function $g(x)$, and on the left and right by the lines $x = a$ and $x = b$, respectively. Let ρ denote the density of the associated lamina. Then we can make the following statements:

i. The mass of the lamina is

$$m = \rho \int_a^b [f(x) - g(x)]dx.$$

ii. The moments M_x and M_y of the lamina with respect to the x- and y-axes, respectively, are

$$M_x = \rho \int_a^b \frac{1}{2}([f(x)]^2 - [g(x)]^2)dx$$

and

$$M_y = \rho \int_a^b x[f(x) - g(x)]dx.$$

iii. The coordinates of the center of mass $\bar{x}, \bar{y})$ are

$$\bar{x} = \frac{M_y}{m}$$

and

$$\bar{y} = \frac{M_x}{m}$$

We illustrate this theorem in the following example.

Example 6.6.4: Finding the Centroid of a Region Bounded by Two Functions

Let *R* be the region bounded above by the graph of the function $f(x) = 1 - x^2$ and below by the graph of the function $g(x) = x - 1$. Find the centroid of the region.

Solution

The region is depicted in the following figure.

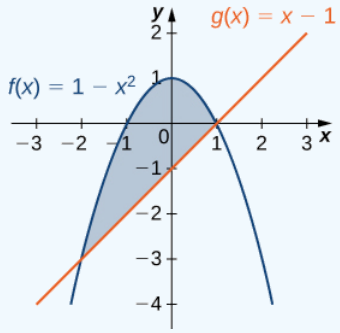


Figure 6.6.9: Finding the centroid of a region between two curves.

The graphs of the functions intersect at $(-2, -3)$ and $(1, 0)$, so we integrate from −2 to 1. Once again, for the sake of convenience, assume $\rho = 1$.

First, we need to calculate the total mass:

$$\begin{aligned} m &= \rho \int_a^b [f(x) - g(x)]dx \\ &= \int_{-2}^1 [1 - x^2 - (x - 1)]dx \\ &= \int_{-2}^1 (2 - x^2 - x)dx \\ &= \left[2x - \frac{1}{3}x^3 - \frac{1}{2}x^2\right] |_{-2}^1 \\ &= \left[2 - \frac{1}{3} - \frac{1}{2}\right] - \left[-4 + \frac{8}{3} - 2\right] \\ &= \frac{9}{2}. \end{aligned}$$

Next, we compute the moments:

$$\begin{aligned} M_x &= \rho \int_a^b \frac{1}{2}([f(x)]^2 - [g(x)]^2)dx \\ &= \frac{1}{2}\int_{-2}^{1}((1-x^2)^2 - (x-1)^2)dx \\ &= \frac{1}{2}\int_{-2}^{1}(x^4 - 3x^2 + 2x)dx \\ &= \frac{1}{2}\left[\frac{x^5}{5} - x^3 + x^2\right]|_{-2}^{1} \\ &= -\frac{27}{10} \end{aligned}$$

and

$$\begin{aligned} M_y &= \rho \int_a^b x[f(x) - g(x)]dx \\ &= \int_{-2}^{1} x[(1-x^2) - (x-1)]dx \\ &= \int_{-2}^{1} x[2 - x^2 - x]dx \\ &= \int_{-2}^{1} (2x - x^3 - x^2)dx \\ &= \left[x^2 - \frac{x^4}{4} - \frac{x^3}{3}\right]|_{-2}^{1} \\ &= -\frac{9}{4}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \bar{x} &= \frac{M_y}{m} \\ &= -\frac{9}{4}\cdot\frac{2}{9} \\ &= -\frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} \bar{y} &= \frac{M_x}{y} \\ &= -\frac{27}{10}\cdot\frac{2}{9} \\ &= -\frac{3}{5}. \end{aligned}$$

The centroid of the region is $(-(1/2), -(3/5))$.

? Exercise 6.6.4

Let R be the region bounded above by the graph of the function $f(x) = 6 - x^2$ and below by the graph of the function $g(x) = 3 - 2x$. Find the centroid of the region.

Hint

Use the process from the previous example.

Answer

The centroid of the region is $(1, 13/5)$.

The Symmetry Principle

We stated the symmetry principle earlier, when we were looking at the centroid of a rectangle. The symmetry principle can be a great help when finding centroids of regions that are symmetric. Consider the following example.

Example 6.6.5: Finding the Centroid of a Symmetric Region

Let R be the region bounded above by the graph of the function $f(x) = 4 - x^2$ and below by the x-axis. Find the centroid of the region.

Solution

The region is depicted in the following figure

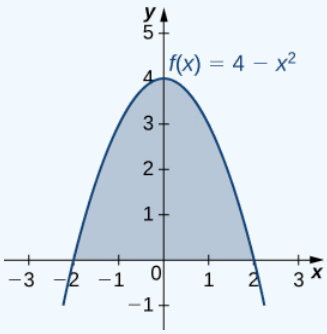


Figure 6.6.10: We can use the symmetry principle to help find the centroid of a symmetric region.

The region is symmetric with respect to the y-axis. Therefore, the x-coordinate of the centroid is zero. We need only calculate $\bar{y}$. Once again, for the sake of convenience, assume $\rho = 1$.

First, we calculate the total mass:

$$\begin{aligned} m &= \rho \int_a^b f(x)dx \\ &= \int_{-2}^{2} (4 - x^2)dx \\ &= \left[4x - \frac{x^3}{3}\right] |_{-2}^{2} \\ &= \frac{32}{3}. \end{aligned}$$

Next, we calculate the moments. We only need M_x:

$$\begin{aligned} M_x &= \rho \int_a^b \frac{[f(x)]^2}{2} dx \\ &= \frac{1}{2} \int_{-2}^{2} \left[4 - x^2\right]^2 dx = \frac{1}{2} \int_{-2}^{2} (16 - 8x^2 + x^4)dx \\ &= \frac{1}{2} \left[\frac{x^5}{5} - \frac{8x^3}{3} + 16x\right] |_{-2}^{2} = \frac{256}{15} \end{aligned}$$

Then we have

$$\bar{y} = \frac{M_x}{y} = \frac{256}{15} \cdot \frac{3}{32} = \frac{8}{5}.$$

The centroid of the region is $(0, 8/5)$.

Exercise 6.6.5

Let R be the region bounded above by the graph of the function $f(x) = 1 - x^2$ and below by x-axis. Find the centroid of the region.

Hint

Use the process from the previous example.

Answer

The centroid of the region is $(0, 2/5)$.

The Grand Canyon Skywalk

The Grand Canyon Skywalk opened to the public on March 28, 2007. This engineering marvel is a horseshoe-shaped observation platform suspended 4000 ft above the Colorado River on the West Rim of the Grand Canyon. Its crystal-clear glass floor allows stunning views of the canyon below (see the following figure).

Figure 6.6.11: The Grand Canyon Skywalk offers magnificent views of the canyon. (credit: 10da_ralta, Wikimedia Commons)

The Skywalk is a cantilever design, meaning that the observation platform extends over the rim of the canyon, with no visible means of support below it. Despite the lack of visible support posts or struts, cantilever structures are engineered to be very stable and the Skywalk is no exception. The observation platform is attached firmly to support posts that extend 46 ft down into bedrock. The structure was built to withstand 100-mph winds and an 8.0-magnitude earthquake within 50 mi, and is capable of supporting more than 70,000,000 lb.

One factor affecting the stability of the Skywalk is the center of gravity of the structure. We are going to calculate the center of gravity of the Skywalk, and examine how the center of gravity changes when tourists walk out onto the observation platform.

The observation platform is U-shaped. The legs of the U are 10 ft wide and begin on land, under the visitors' center, 48 ft from the edge of the canyon. The platform extends 70 ft over the edge of the canyon.

To calculate the center of mass of the structure, we treat it as a lamina and use a two-dimensional region in the xy-plane to represent the platform. We begin by dividing the region into three subregions so we can consider each subregion separately. The first region, denoted R_1, consists of the curved part of the U. We model R_1 as a semicircular annulus, with inner radius 25 ft and outer radius 35 ft, centered at the origin (Figure 6.6.12).

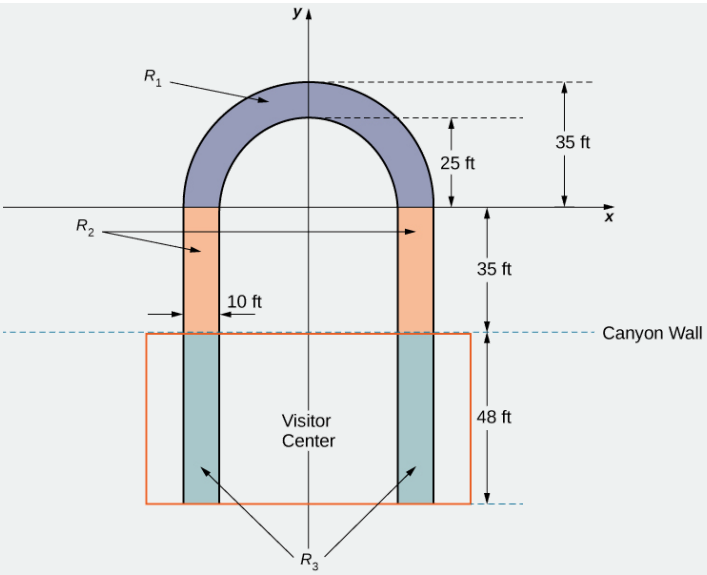


Figure 6.6.12: We model the Skywalk with three sub-regions.

The legs of the platform, extending 35 ft between R_1 and the canyon wall, comprise the second sub-region, R_2. Last, the ends of the legs, which extend 48 ft under the visitor center, comprise the third sub-region, R_3. Assume the density of the lamina is constant and assume the total weight of the platform is 1,200,000 lb (not including the weight of the visitor center; we will consider that later). Use $g = 32\ ft/sec^2$.

1. Compute the area of each of the three sub-regions. Note that the areas of regions R_2 and R_3 should include the areas of the legs only, not the open space between them. Round answers to the nearest square foot.
2. Determine the mass associated with each of the three sub-regions.
3. Calculate the center of mass of each of the three sub-regions.
4. Now, treat each of the three sub-regions as a point mass located at the center of mass of the corresponding sub-region. Using this representation, calculate the center of mass of the entire platform.
5. Assume the visitor center weighs 2,200,000 lb, with a center of mass corresponding to the center of mass of R_3.Treating the visitor center as a point mass, recalculate the center of mass of the system. How does the center of mass change?
6. Although the Skywalk was built to limit the number of people on the observation platform to 120, the platform is capable of supporting up to 800 people weighing 200 lb each. If all 800 people were allowed on the platform, and all of them went to the farthest end of the platform, how would the center of gravity of the system be affected? (Include the visitor center in the calculations and represent the people by a point mass located at the farthest edge of the platform, 70 ft from the canyon wall.)

Theorem of Pappus

This section ends with a discussion of the *theorem of Pappus for volume*, which allows us to find the volume of particular kinds of solids by using the centroid. (There is also a theorem of Pappus for surface area, but it is much less useful than the theorem for volume.)

Theorem of Pappus for Volume

Let R be a region in the plane and let l be a line in the plane that does not intersect R. Then the volume of the solid of revolution formed by revolving R around l is equal to the area of R multiplied by the distance d traveled by the centroid of R.

Proof

We can prove the case when the region is bounded above by the graph of a function $f(x)$ and below by the graph of a function $g(x)$ over an interval $[a, b]$, and for which the axis of revolution is the y-axis. In this case, the area of the region is $A = \int_a^b [f(x)$

$-g(x)]\,dx$. Since the axis of rotation is the y-axis, the distance traveled by the centroid of the region depends only on the x-coordinate of the centroid, $\bar{x}$, which is

$$x = \frac{M_y}{m},$$

where

$$m = \rho \int_a^b [f(x) - g(x)]dx$$

and

$$M_y = \rho \int_a^b x[f(x) - g(x)]dx.$$

Then,

$$d = 2\pi \frac{\rho \int_a^b x[f(x) - g(x)]dx}{\rho \int_a^b [f(x) - g(x)]dx}$$

and thus

$$d \cdot A = 2\pi \int_a^b x[f(x) - g(x)]dx.$$

However, using the method of cylindrical shells, we have

$$V = 2\pi \int_a^b x[f(x) - g(x)]dx.$$

So,

$$V = d \cdot A$$

and the proof is complete.

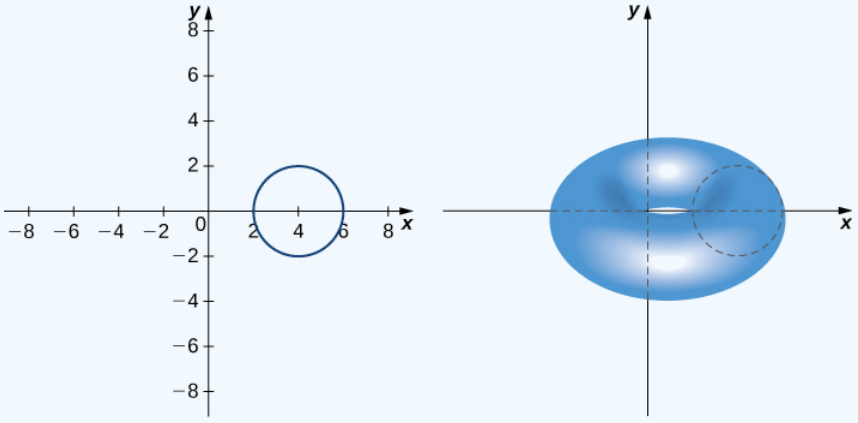
□

Example 6.6.6: Using the Theorem of Pappus for Volume

Let R be a circle of radius 2 centered at $(4, 0)$. Use the theorem of Pappus for volume to find the volume of the torus generated by revolving R around the y-axis.

Solution

The region and torus are depicted in the following figure.



(a) (b)

Figure 6.6.13: Determining the volume of a torus by using the theorem of Pappus. (a) A circular region R in the plane; (b) the torus generated by revolving R about the y-axis.

The region R is a circle of radius 2, so the area of R is $A = 4\pi \text{ units}^2$. By the symmetry principle, the centroid of R is the center of the circle. The centroid travels around the y-axis in a circular path of radius 4, so the centroid travels $d = 8\pi$ units. Then, the volume of the torus is $A \cdot d = 32\pi^2$ units^3.

Exercise 6.6.6

Let R be a circle of radius 1 centered at $(3, 0)$. Use the theorem of Pappus for volume to find the volume of the torus generated by revolving R around the y-axis.

Hint

Use the process from the previous example.

Answer

$6\pi^2$ units^3

Key Concepts

- Mathematically, the center of mass of a system is the point at which the total mass of the system could be concentrated without changing the moment. Loosely speaking, the center of mass can be thought of as the balancing point of the system.
- For point masses distributed along a number line, the moment of the system with respect to the origin is $M = \sum_{i=1}^{n} m_i x_i$. For point masses distributed in a plane, the moments of the system with respect to the x- and y-axes, respectively, are $M_x = \sum_{i=1}^{n} m_i y_i$ and $M_y = \sum_{i=}^{n} m_i x_i$, respectively.
- For a lamina bounded above by a function $f(x)$, the moments of the system with respect to the x- and y-axes, respectively, are $M_x = \rho \int_a^b \frac{[f(x)]^2}{2}\, dx$ and $M_y = \rho \int_a^b x f(x)\, dx$.
- The x- and y-coordinates of the center of mass can be found by dividing the moments around the y-axis and around the x-axis, respectively, by the total mass. The symmetry principle says that if a region is symmetric with respect to a line, then the centroid of the region lies on the line.
- The theorem of Pappus for volume says that if a region is revolved around an external axis, the volume of the resulting solid is equal to the area of the region multiplied by the distance traveled by the centroid of the region.

Key Equations

- **Mass of a lamina**

$$m = \rho \int_a^b f(x) dx$$

- **Moments of a lamina**

  $$M_x = \rho \int_a^b \frac{[f(x)]^2}{2}\, dx \text{ and } M_y = \rho \int_a^b x f(x)\, dx$$

- **Center of mass of a lamina**

  $$\bar{x} = \frac{M_y}{m} \text{ and } \bar{y} = \frac{M_x}{m}$$

Glossary

center of mass

the point at which the total mass of the system could be concentrated without changing the moment

centroid

the centroid of a region is the geometric center of the region; laminas are often represented by regions in the plane; if the lamina has a constant density, the center of mass of the lamina depends only on the shape of the corresponding planar region; in this case, the center of mass of the lamina corresponds to the centroid of the representative region

lamina

a thin sheet of material; laminas are thin enough that, for mathematical purposes, they can be treated as if they are two-dimensional

moment

if n masses are arranged on a number line, the moment of the system with respect to the origin is given by $M = \sum_{i=1}^{n} m_i x_i$; if, instead, we consider a region in the plane, bounded above by a function $f(x)$ over an interval $[a, b]$, then the moments of the region with respect to the x- and y-axes are given by $M_x = \rho \int_a^b \frac{[f(x)]^2}{2}\, dx$ and $M_y = \rho \int_a^b x f(x)\, dx$, respectively

symmetry principle

the symmetry principle states that if a region R is symmetric about a line I, then the centroid of R lies on I

theorem of Pappus for volume

this theorem states that the volume of a solid of revolution formed by revolving a region around an external axis is equal to the area of the region multiplied by the distance traveled by the centroid of the region

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6.6E: Exercises for Section 6.6

In exercises 1 - 6, calculate the center of mass for the collection of masses given.

1) $m_1 = 2$ at $x_1 = 1$ and $m_2 = 4$ at $x_2 = 2$

2) $m_1 = 1$ at $x_1 = -1$ and $m_2 = 3$ at $x_2 = 2$

Answer

$x = \frac{5}{4}$

3) $m = 3$ at $x = 0, 1, 2, 6$

4) Unit masses at $(x, y) = (1, 0), (0, 1), (1, 1)$

Answer

$\left(\frac{2}{3}, \frac{2}{3}\right)$

5) $m_1 = 1$ at $(1, 0)$ and $m_2 = 4$ at $(0, 1)$

6) $m_1 = 1$ at $(1, 0)$ and $m_2 = 3$ at $(2, 2)$

Answer

$\left(\frac{7}{4}, \frac{3}{2}\right)$

In exercises 7 - 16, compute the center of mass $\bar{x}$.

7) $\rho = 1$ for $x \in (-1, 3)$

8) $\rho = x^2$ for $x \in (0, L)$

Answer

$$\frac{3L}{4}$$

9) $\rho = 1$ for $x \in (0, 1)$ and $\rho = 2$ for $x \in (1, 2)$

10) $\rho = \sin x$ for $x \in (0, \pi)$

Answer

$\frac{\pi}{2}$

11) $\rho = \cos x$ for $x \in (0, \frac{\pi}{2})$

12) $\rho = e^x$ for $x \in (0, 2)$

Answer

$$\frac{e^2 + 1}{e^2 - 1}$$

13) $\rho = x^3 + xe^{-x}$ for $x \in (0, 1)$

14) $\rho = x \sin x$ for $x \in (0, \pi)$

Answer

$$\frac{\pi^2 - 4}{\pi}$$

15) $\rho = \sqrt{x}$ for $x \in (1, 4)$

16) $\rho = \ln x$ for $x \in (1, e)$

Answer

Answer

$\frac{1}{4}(1+e^2)$

In exercises 17 - 19, compute the center of mass $(\bar{x},\bar{y})$. Use symmetry to help locate the center of mass whenever possible.

17) $\rho=7$ in the square $0\le x\le 1,\ 0\le y\le 1$

18) $\rho=3$ in the triangle with vertices $(0,0),\ (a,0)$, and $(0,b)$

Answer

$\left(\frac{a}{3},\ \frac{b}{3}\right)$

19) $\rho=2$ for the region bounded by $y=\cos(x),\ y=-\cos(x),\ x=-\frac{\pi}{2}$, and $x=\frac{\pi}{2}$

In exercises 20 - 26, use a calculator to draw the region, then compute the center of mass $(\bar{x},\bar{y})$. Use symmetry to help locate the center of mass whenever possible.

20) [T] The region bounded by $y=\cos(2x),\ x=-\frac{\pi}{4}$, and $x=\frac{\pi}{4}$

Answer

$\left(0,\frac{\pi}{8}\right)$

21) [T] The region between $y=2x^2,\ y=0,\ x=0$, and $x=1$

22) [T] The region between $y=\frac{5}{4}x^2$ and $y=5$

Answer

$(0,3)$

23) [T] Region between $y=\sqrt{x},\ y=\ln x,\ x=1$, and $x=4$

24) [T] The region bounded by $y=0$ and $\dfrac{x^2}{4}+\dfrac{y^2}{9}=1$

Answer

$\left(0,\frac{4}{\pi}\right)$

25) [T] The region bounded by $y=0,\ x=0$, and $\dfrac{x^2}{4}+\dfrac{y^2}{9}=1$

26) [T] The region bounded by $y=x^2$ and $y=x^4$ in the first quadrant

Answer

$\left(\frac{5}{8},\ \frac{1}{3}\right)$

In exercises 27 - 31, use the theorem of Pappus to determine the volume of the shape.

27) Rotating $y=mx$ around the x-axis between $x=0$ and $x=1$

28) Rotating $y=mx$ around the y-axis between $x=0$ and $x=1$

Answer

$V=\frac{m\pi}{3}$ units^3

29) A general cone created by rotating a triangle with vertices $(0,0),\ (a,0)$, and $(0,b)$ around the y-axis. Does your answer agree with the volume of a cone?

30) A general cylinder created by rotating a rectangle with vertices $(0,0),\ (a,0),\ (0,b)$, and (a,b) around the y-axis. Does your answer agree with the volume of a cylinder?

Answer

$V=\pi a^2 b$ units^3

31) A sphere created by rotating a semicircle with radius a around the y-axis. Does your answer agree with the volume of a sphere?

In exercises 32 - 36, use a calculator to draw the region enclosed by the curve. Find the area M and the centroid $(\bar{x}, \bar{y})$ for the given shapes. Use symmetry to help locate the center of mass whenever possible.

32) [T] Quarter-circle: $y = \sqrt{1 - x^2},\ y = 0$, and $x = 0$

Answer

$\left(\frac{4}{3\pi}, \frac{4}{3\pi}\right)$

33) [T] Triangle: $y = x,\ y = 2 - x$, and $y = 0$

34) [T] Lens: $y = x^2$ and $y = x$

Answer

$\left(\frac{1}{2}, \frac{2}{5}\right)$

35) [T] Ring: $y^2 + x^2 = 1$ and $y^2 + x^2 = 4$

36) [T] Half-ring: $y^2 + x^2 = 1,\ y^2 + x^2 = 4$, and $y = 0$

Answer

$\left(0, \frac{28}{9\pi}\right)$

37) Find the generalized center of mass in the sliver between $y = x^a$ and $y = x^b$ with $a > b$. Then, use the Pappus theorem to find the volume of the solid generated when revolving around the y-axis.

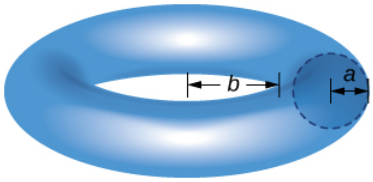
38) Find the generalized center of mass between $y = a^2 - x^2,\ x = 0$, and $y = 0$. Then, use the Pappus theorem to find the volume of the solid generated when revolving around the y-axis.

Answer

Center of mass: $\left(\frac{a}{6}, \frac{4a^2}{5}\right)$,
Volume: $\frac{2\pi a^4}{9}$ units^3

39) Find the generalized center of mass between $y = b\sin(ax),\ x = 0$, and $x = \frac{\pi}{a}$. Then, use the Pappus theorem to find the volume of the solid generated when revolving around the y-axis.

40) Use the theorem of Pappus to find the volume of a torus (pictured here). Assume that a disk of radius a is positioned with the left end of the circle at $x = b$, $b > 0$, and is rotated around the y-axis.



Answer

Volume: $V = 2\pi^2 a^2 (b + a)$

41) Find the center of mass $(\bar{x}, \bar{y})$ for a thin wire along the semicircle $y = \sqrt{1 - x^2}$ with unit mass. (Hint: Use the theorem of Pappus.)

Contributors

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6.7: Integrals, Exponential Functions, and Logarithms

Learning Objectives

- Write the definition of the natural logarithm as an integral.
- Recognize the derivative of the natural logarithm.
- Integrate functions involving the natural logarithmic function.
- Define the number e through an integral.
- Recognize the derivative and integral of the exponential function.
- Prove properties of logarithms and exponential functions using integrals.
- Express general logarithmic and exponential functions in terms of natural logarithms and exponentials.

We already examined exponential functions and logarithms in earlier chapters. However, we glossed over some key details in the previous discussions. For example, we did not study how to treat exponential functions with exponents that are irrational. The definition of the number e is another area where the previous development was somewhat incomplete. We now have the tools to deal with these concepts in a more mathematically rigorous way, and we do so in this section.

For purposes of this section, assume we have not yet defined the natural logarithm, the number e, or any of the integration and differentiation formulas associated with these functions. By the end of the section, we will have studied these concepts in a mathematically rigorous way (and we will see they are consistent with the concepts we learned earlier). We begin the section by defining the natural logarithm in terms of an integral. This definition forms the foundation for the section. From this definition, we derive differentiation formulas, define the number e, and expand these concepts to logarithms and exponential functions of any base.

The Natural Logarithm as an Integral

Recall the power rule for integrals:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1.$$

Clearly, this does not work when $n = -1$, as it would force us to divide by zero. So, what do we do with $\int \frac{1}{x}\,dx$? Recall from the Fundamental Theorem of Calculus that $\int_1^x \frac{1}{t}dt$ is an antiderivative of $\frac{1}{x}$. Therefore, we can make the following definition.

Definition: The Natural Logarithm

For $x > 0$, define the natural logarithm function by

$$\ln x = \int_1^x \frac{1}{t}\,dt.$$

For $x > 1$, this is just the area under the curve $y = \frac{1}{t}$ from 1 to x. For $x < 1$, we have

$$\int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt,$$

so in this case it is the negative of the area under the curve from x to 1 (see the following figure).

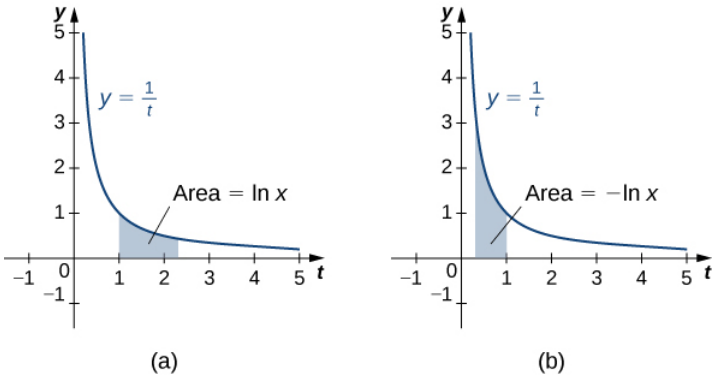

Figure 6.7.1: (a) When $x > 1$, the natural logarithm is the area under the curve $y = 1/t$ from 1 to x. (b) When $x < 1$, the natural logarithm is the negative of the area under the curve from x to 1.

Notice that $\ln 1 = 0$. Furthermore, the function $y = \dfrac{1}{t} > 0$ for $x > 0$. Therefore, by the properties of integrals, it is clear that $\ln x$ is increasing for $x > 0$.

Properties of the Natural Logarithm

Because of the way we defined the natural logarithm, the following differentiation formula falls out immediately as a result of to the Fundamental Theorem of Calculus.

Definition: Derivative of the Natural Logarithm

For $x > 0$, the derivative of the natural logarithm is given by

$$\frac{d}{dx}\left(\ln x\right) = \frac{1}{x}.$$

Corollary to the Derivative of the Natural Logarithm

The function $\ln x$ is differentiable; therefore, it is continuous.

A graph of $\ln x$ is shown in Figure. Notice that it is continuous throughout its domain of $(0, \infty)$.

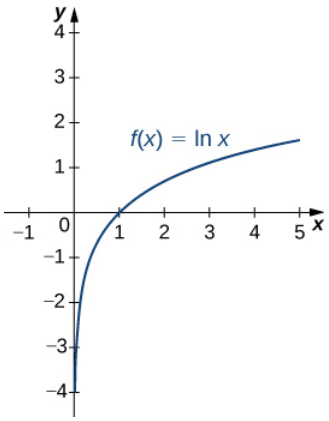

Figure 6.7.2: The graph of $f(x) = \ln x$ shows that it is a continuous function.

Example 6.7.1: Calculating Derivatives of Natural Logarithms

Calculate the following derivatives:

a. $\dfrac{d}{dx}\left(\ln(5x^3 - 2)\right)$

b. $\dfrac{d}{dx}\left((\ln(3x))^2\right)$

Solution

We need to apply the chain rule in both cases.

a. $\frac{d}{dx}\Big(\ln(5x^3-2)\Big) = \frac{15x^2}{5x^3-2}$

b. $\frac{d}{dx}\Big((\ln(3x))^2\Big) = \frac{2(\ln(3x))\cdot 3}{3x} = \frac{2(\ln(3x))}{x}$

Exercise 6.7.1

Calculate the following derivatives:

a. $\frac{d}{dx}\Big(\ln(2x^2+x)\Big)$

b. $\frac{d}{dx}\Big((\ln(x^3))^2\Big)$

Hint

Apply the differentiation formula just provided and use the chain rule as necessary.

Answer

a. $\frac{d}{dx}\Big(\ln(2x^2+x)\Big) = \frac{4x+1}{2x^2+x}$

b. $\frac{d}{dx}\Big((\ln(x^3))^2\Big) = \frac{6\ln(x^3)}{x}$

Note that if we use the absolute value function and create a new function $\ln|x|$, we can extend the domain of the natural logarithm to include $x<0$. Then $\frac{d}{dx}\Big(\ln x\Big) = \frac{1}{x}$. This gives rise to the familiar integration formula.

Integral of $\frac{1}{u}\,du$

The natural logarithm is the antiderivative of the function $f(u)=\frac{1}{u}$:

$$\int \frac{1}{u}\,du = \ln|u| + C.$$

Example 6.7.2: Calculating Integrals Involving Natural Logarithms

Calculate the integral $\int \frac{x}{x^2+4}\,dx$.

Solution

Using u-substitution, let $u=x^2+4$. Then $du = 2x\,dx$ and we have

$$\int \frac{x}{x^2+4}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+4| + C = \frac{1}{2}\ln(x^2+4) + C.$$

Exercise 6.7.2

Calculate the integral $\int \frac{x^2}{x^3+6}\,dx$.

Hint

Apply the integration formula provided earlier and use u-substitution as necessary.

Answer

$$\int \frac{x^2}{x^3+6}\,dx = \frac{1}{3}\ln|\,x^3+6\,| + C$$

Although we have called our function a "logarithm," we have not actually proved that any of the properties of logarithms hold for this function. We do so here.

Properties of the Natural Logarithm

If $a,\ b > 0$ and r is a rational number, then

i. $\ln 1 = 0$
ii. $\ln(ab) = \ln a + \ln b$
iii. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$
iv. $\ln\left(a^r\right) = r \ln a$

Proof

i. By definition, $\ln 1 = \int_1^1 \frac{1}{t}\, dt = 0.$

ii. We have

$$\ln(ab) = \int_1^{ab} \frac{1}{t}\, dt = \int_1^a \frac{1}{t}\, dt + \int_a^{ab} \frac{1}{t}\, dt.$$

Use $u - substitution$ on the last integral in this expression. Let $u = t/a$. Then $du = (1/a)dt.$ Furthermore, when $t = a,\ u = 1$, and when $t = ab,\ u = b.$ So we get

$$\ln(ab) = \int_1^a \frac{1}{t}\, dt + \int_a^{ab} \frac{1}{t}\, dt = \int_1^a \frac{1}{t}\, dt + \int_1^{ab} \frac{a}{t} \cdot \frac{1}{a}\, dt = \int_1^a \frac{1}{t}\, dt + \int_1^b \frac{1}{u}\, du = \ln a + \ln b.$$

iii. Note that

$$\frac{d}{dx}\left(\ln(x^r)\right) = \frac{rx^{r-1}}{x^r} = \frac{r}{x}.$$

Furthermore,

$$\frac{d}{dx}\Big((r \ln x)\Big) = \frac{r}{x}.$$

Since the derivatives of these two functions are the same, by the Fundamental Theorem of Calculus, they must differ by a constant. So we have

$$\ln(x^r) = r \ln x + C$$

for some constant C. Taking $x = 1$, we get

$$\ln(1^r) = r \ln(1) + C$$

$$0 = r(0) + C$$

$$C = 0.$$

Thus $\ln(x^r) = r \ln x$ and the proof is complete. Note that we can extend this property to irrational values of r later in this section.

Part iii. follows from parts ii. and iv. and the proof is left to you.

□

Example 6.7.3: Using Properties of Logarithms

Use properties of logarithms to simplify the following expression into a single logarithm:

$$\ln 9 - 2 \ln 3 + \ln\left(\tfrac{1}{3}\right).$$

Solution

We have

$$\ln 9 - 2\ln 3 + \ln\left(\tfrac{1}{3}\right) = \ln(3^2) - 2\ln 3 + \ln(3^{-1}) = 2\ln 3 - 2\ln 3 - \ln 3 = -\ln 3.$$

Exercise 6.7.3

Use properties of logarithms to simplify the following expression into a single logarithm:

$$\ln 8 - \ln 2 - \ln\left(\tfrac{1}{4}\right)$$

Hint

Apply the properties of logarithms.

Answer

$4\ln 2$

Defining the Number e

Now that we have the natural logarithm defined, we can use that function to define the number e.

Definition: e

The number e is defined to be the real number such that

$$\ln e = 1$$

To put it another way, the area under the curve $y = 1/t$ between $t = 1$ and $t = e$ is 1 (Figure). The proof that such a number exists and is unique is left to you. (Hint: Use the Intermediate Value Theorem to prove existence and the fact that $\ln x$ is increasing to prove uniqueness.)

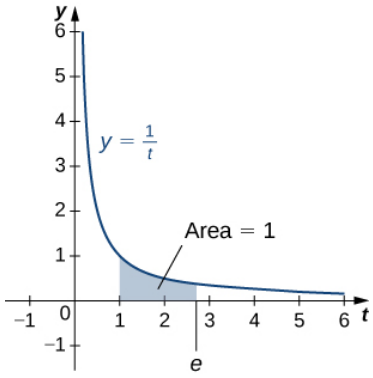

Figure 6.7.3:The area under the curve from 1 to e is equal to one.

The number e can be shown to be *irrational*, although we won't do so here (see the Student Project in Taylor and Maclaurin Series). Its approximate value is given by

$$e \approx 2.71828182846.$$

The Exponential Function

We now turn our attention to the function e^x. Note that the natural logarithm is one-to-one and therefore has an inverse function. For now, we denote this inverse function by $\exp x$. Then,

$$\exp(\ln x) = x$$

for $x > 0$ and

$$\ln(\exp x) = x$$

for all x.

The following figure shows the graphs of $\exp x$ and $\ln x$.

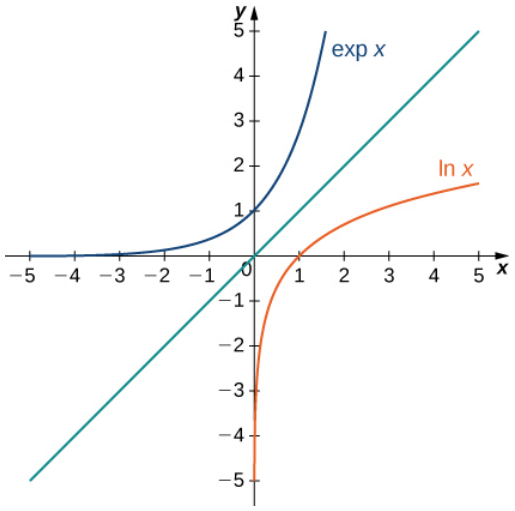

Figure 6.7.4: The graphs of $\ln x$ and $\exp x$.

We hypothesize that $\exp x = e^x$. For rational values of x, this is easy to show. If x is rational, then we have $\ln(e^x) = x \ln e = x$. Thus, when x is rational, $e^x = \exp x$. For irrational values of x, we simply define e^x as the inverse function of $\ln x$.

Definition

For any real number x, define $y = e^x$ to be the number for which

$$\ln y = \ln(e^x) = x.$$

Then we have $e^x = \exp x$ for all x, and thus

$$e^{\ln x} = x \text{ for } x > 0 \text{ and } \ln(e^x) = x$$

for all x.

Properties of the Exponential Function

Since the exponential function was defined in terms of an inverse function, and not in terms of a power of e we must verify that the usual laws of exponents hold for the function e^x.

Properties of the Exponential Function

If p and q are any real numbers and r is a rational number, then

i. $e^p e^q = e^{p+q}$
ii. $\dfrac{e^p}{e^q} = e^{p-q}$
iii. $(e^p)^r = e^{pr}$

Proof

Note that if p and q are rational, the properties hold. However, if p or q are irrational, we must apply the inverse function definition of e^x and verify the properties. Only the first property is verified here; the other two are left to you. We have

$$\ln(e^p e^q) = \ln(e^p) + \ln(e^q) = p + q = \ln(e^{p+q}).$$

Since $\ln x$ is one-to-one, then

$$e^p e^q = e^{p+q}.$$

□

As with part iv. of the logarithm properties, we can extend property iii. to irrational values of r, and we do so by the end of the section.

We also want to verify the differentiation formula for the function $y = e^x$. To do this, we need to use implicit differentiation. Let $y = e^x$. Then

$$\begin{aligned} \ln y &= x \\ \frac{d}{dx}\left(\ln y\right) &= \frac{d}{dx}\left(x\right) \\ \frac{1}{y}\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= y. \end{aligned}$$

Thus, we see

$$\frac{d}{dx}\left(e^x\right) = e^x$$

as desired, which leads immediately to the integration formula

$$\int e^x\,dx = e^x + C.$$

We apply these formulas in the following examples.

Example 6.7.4: Using Properties of Exponential Functions

Evaluate the following derivatives:

a. $\frac{d}{dt}\left(e^{3t}e^{t^2}\right)$
b. $\frac{d}{dx}\left(e^{3x^2}\right)$

Solution

We apply the chain rule as necessary.

a. $\frac{d}{dt}\left(e^{3t}e^{t^2}\right) = \frac{d}{dt}\left(e^{3t+t^2}\right) = e^{3t+t^2}(3+2t)$
b. $\frac{d}{dx}\left(e^{3x^2}\right) = e^{3x^2}6x$

Exercise 6.7.4

Evaluate the following derivatives:

a. $\frac{d}{dx}\left(\frac{e^{x^2}}{e^{5x}}\right)$
b. $\frac{d}{dt}\left((e^{2t})^3\right)$

Hint

Use the properties of exponential functions and the chain rule as necessary.

Answer

a. $\frac{d}{dx}\left(\frac{e^{x^2}}{e^{5x}}\right) = e^{x^{2-5x}}(2x-5)$
b. $\frac{d}{dt}\left((e^{2t})^3\right) = 6e^{6t}$

Example 6.7.5: Using Properties of Exponential Functions

Evaluate the following integral: $\int 2xe^{-x^2}\,dx$.

Solution

Using u-substitution, let $u = -x^2$. Then $du = -2x\,dx$, and we have

$$\int 2xe^{-x^2}\,dx = -\int e^u\,du = -e^u + C = -e^{-x^2} + C.$$

Exercise 6.7.5

Evaluate the following integral: $\int \frac{4}{e^{3x}}\,dx$.

Hint

Use the properties of exponential functions and $u - substitution$ as necessary.

Answer

$$\int \frac{4}{e^{3x}}\,dx = -\frac{4}{3}e^{-3x} + C$$

General Logarithmic and Exponential Functions

We close this section by looking at exponential functions and logarithms with bases other than e. Exponential functions are functions of the form $f(x) = a^x$. Note that unless $a = e$, we still do not have a mathematically rigorous definition of these functions for irrational exponents. Let's rectify that here by defining the function $f(x) = a^x$ in terms of the exponential function e^x. We then examine logarithms with bases other than e as inverse functions of exponential functions.

Definition: Exponential Function

For any $a > 0$, and for any real number x, define $y = a^x$ as follows:

$$y = a^x = e^{x \ln a}.$$

Now a^x is defined rigorously for all values of x. This definition also allows us to generalize property iv. of logarithms and property iii. of exponential functions to apply to both rational and irrational values of r. It is straightforward to show that properties of exponents hold for general exponential functions defined in this way.

Let's now apply this definition to calculate a differentiation formula for a^x. We have

$$\frac{d}{dx}\left(a^x\right) = \frac{d}{dx}\left(e^{x \ln a}\right) = e^{x \ln a} \ln a = a^x \ln a.$$

The corresponding integration formula follows immediately.

Derivatives and Integrals Involving General Exponential Functions

Let $a > 0$. Then,

$$\frac{d}{dx}\left(a^x\right) = a^x \ln a$$

and

$$\int a^x\,dx = \frac{1}{\ln a}a^x + C.$$

If $a \neq 1$, then the function a^x is one-to-one and has a well-defined inverse. Its inverse is denoted by $\log_a x$. Then,

$$y = \log_a x \text{ if and only if } x = a^y.$$

Note that general logarithm functions can be written in terms of the natural logarithm. Let $y = \log_a x$. Then, $x = a^y$. Taking the natural logarithm of both sides of this second equation, we get

$$\begin{aligned} \ln x &= \ln(a^y) \\ \ln x &= y \ln a \\ y &= \frac{\ln x}{\ln a} \\ \log_a x &= \frac{\ln x}{\ln a}. \end{aligned}$$

Thus, we see that all logarithmic functions are constant multiples of one another. Next, we use this formula to find a differentiation formula for a logarithm with base a. Again, let $y = \log_a x$. Then,

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(\log_a x\right) \\ &= \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) \\ &= (\frac{1}{\ln a})\frac{d}{dx}\left(\ln x\right) \\ &= \frac{1}{\ln a} \cdot \frac{1}{x} = \frac{1}{x \ln a} \end{aligned}$$

Derivatives of General Logarithm Functions

Let $a > 0$. Then,

$$\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x \ln a}.$$

Example 6.7.6: Calculating Derivatives of General Exponential and Logarithm Functions

Evaluate the following derivatives:

a. $\frac{d}{dt}\left(4^t \cdot 2^{t^2}\right)$
b. $\frac{d}{dx}\left(\log_8(7x^2 + 4)\right)$

Solution: We need to apply the chain rule as necessary.

a. $\frac{d}{dt}\left(4^t \cdot 2^{t^2}\right) = \frac{d}{dt}\left(2^{2t} \cdot 2^{t^2}\right) = \frac{d}{dt}\left(2^{2t+t^2}\right) = 2^{2t+t^2} \ln(2)(2 + 2t)$
b. $\frac{d}{dx}\left(\log_8(7x^2 + 4)\right) = \frac{1}{(7x^2 + 4)(\ln 8)}(14x)$

Exercise 6.7.6

Evaluate the following derivatives:

a. $\frac{d}{dt}\left(4^{t^4}\right)$
b. $\frac{d}{dx}\left(\log_3(\sqrt{x^2 + 1})\right)$

Hint

Use the formulas and apply the chain rule as necessary.

Answer

a. $\frac{d}{dt}\left(4^{t^4}\right) = 4^{t^4}(\ln 4)(4t^3)$

b. $\frac{d}{dx}\left(\log_3(\sqrt{x^2+1})\right) = \frac{x}{(\ln 3)(x^2+1)}$

Example 6.7.7: Integrating General Exponential Functions

Evaluate the following integral: $\int \frac{3}{2^{3x}}\,dx.$

Solution

Use $u-substitution$ and let $u=-3x$. Then $du=-3\,dx$ and we have

$$\int \frac{3}{2^{3x}}\,dx = \int 3\cdot 2^{-3x}\,dx = -\int 2^u\,du = -\frac{1}{\ln 2}2^u + C = -\frac{1}{\ln 2}2^{-3x} + C.$$

Exercise 6.7.7

Evaluate the following integral: $\int x^2 2^{x^3}\,dx.$

Hint

Use the properties of exponential functions and u-substitution

Answer

$$\int x^2 2^{x^3}\,dx = \frac{1}{3\ln 2}2^{x^3} + C$$

Key Concepts

- The earlier treatment of logarithms and exponential functions did not define the functions precisely and formally. This section develops the concepts in a mathematically rigorous way.
- The cornerstone of the development is the definition of the natural logarithm in terms of an integral.
- The function e^x is then defined as the inverse of the natural logarithm. General exponential functions are defined in terms of e^x, and the corresponding inverse functions are general logarithms.
- Familiar properties of logarithms and exponents still hold in this more rigorous context.

Key Equations

- **Natural logarithm function**
- $\ln x = \int_1^x \frac{1}{t}\,dt$
- **Exponential function** $y = e^x$
- $\ln y = \ln(e^x) = x$

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6.7E: Exercises for Section 6.7

In exercises 1 - 3, find the derivative $\frac{dy}{dx}$.

1) $y = \ln(2x)$

Answer

$$\frac{dy}{dx} = \frac{1}{x}$$

2) $y = \ln(2x + 1)$

3) $y = \frac{1}{\ln x}$

Answer

$$\frac{dy}{dx} = -\frac{1}{x(\ln x)^2}$$

In exercises 4 - 5, find the indefinite integral.

4) $\int \frac{dt}{3t}$

5) $\int \frac{dx}{1+x}$

Answer

$$\int \frac{dx}{1+x} = \ln|x+1| + C$$

In exercises 6 - 15, find the derivative $\frac{dy}{dx}$. (You can use a calculator to plot the function and the derivative to confirm that it is correct.)

6) [T] $y = \frac{\ln x}{x}$

7) [T] $y = x \ln x$

Answer

$$\frac{dy}{dx} = \ln(x) + 1$$

8) [T] $y = \log_{10} x$

9) [T] $y = \ln(\sin x)$

Answer

$$\frac{dy}{dx} = \cot x$$

10) [T] $y = \ln(\ln x)$

11) [T] $y = 7\ln(4x)$

Answer

$$\frac{dy}{dx} = \frac{7}{x}$$

12) [T] $y = \ln\left((4x)^7\right)$

13) [T] $y = \ln(\tan x)$

Answer

$$\frac{dy}{dx} = \csc x \sec x$$

14) [T] $y = \ln(\tan 3x)$

15) [T] $y = \ln(\cos^2 x)$

Answer

$$\frac{dy}{dx} = -2\tan x$$

In exercises 16 - 25, find the definite or indefinite integral.

16) $\int_0^1 \frac{dx}{3+x}$

17) $\int_0^1 \frac{dt}{3+2t}$

Answer

$$\int_0^1 \frac{dt}{3+2t} = \tfrac{1}{2}\ln\left(\tfrac{5}{3}\right)$$

18) $\int_0^2 \frac{x}{x^2+1}\,dx$

19) $\int_0^2 \frac{x^3}{x^2+1}\,dx$

Answer

$$\int_0^2 \frac{x^3}{x^2+1}\,dx = 2 - \tfrac{1}{2}\ln(5)$$

20) $\int_2^e \frac{dx}{x\ln x}$

21) $\int_2^e \frac{dx}{(x\ln x)^2}$

Answer

$$\int_2^e \frac{dx}{(x\ln x)^2} = \frac{1}{\ln(2)} - 1$$

22) $\int \frac{\cos x}{\sin x}\,dx$

23) $\int_0^{\pi/4} \tan x\,dx$

Answer

$$\int_0^{\pi/4} \tan x\,dx = \tfrac{1}{2}\ln(2)$$

24) $\int \cot(3x)\,dx$

25) $\int \frac{(\ln x)^2}{x}\,dx$

Answer

$$\int \frac{(\ln x)^2}{x}\,dx = \tfrac{1}{3}(\ln x)^3$$

In exercises 26 - 35, compute $\frac{dy}{dx}$ by differentiating $\ln y$.

26) $y = \sqrt{x^2+1}$

27) $y = \sqrt{x^2+1}\sqrt{x^2-1}$

Answer

$$\frac{dy}{dx} = \frac{2x^3}{\sqrt{x^2+1}\sqrt{x^2-1}}$$

28) $y = e^{\sin x}$

29) $y = x^{-1/x}$

Answer

$$\frac{dy}{dx} = x^{-2-(1/x)}(\ln x - 1)$$

30) $y = e^{ex}$

31) $y = x^e$

Answer

$$\frac{dy}{dx} = ex^{e-1}$$

32) $y = x^{(ex)}$

33) $y = \sqrt{x}\sqrt[3]{x}\sqrt[6]{x}$

Answer

$$\frac{dy}{dx} = 1$$

34) $y = x^{-1/\ln x}$

35) $y = e^{-\ln x}$

Answer

$$\frac{dy}{dx} = -\frac{1}{x^2}$$

In exercises 36 - 40, evaluate by any method.

36) $\int_5^{10} \frac{dt}{t} - \int_{5x}^{10x} \frac{dt}{t}$

37) $\int_1^{e^\pi} \frac{dx}{x} + \int_{-2}^{-1} \frac{dx}{x}$

Answer

$$\pi - \ln(2)$$

38) $\frac{d}{dx}\left[\int_x^1 \frac{dt}{t}\right]$

39) $\frac{d}{dx}\left[\int_x^{x^2} \frac{dt}{t}\right]$

Answer

$$\frac{1}{x}$$

40) $\frac{d}{dx}\left[\ln(\sec x + \tan x)\right]$

In exercises 41 - 44, use the function $\ln x$. If you are unable to find intersection points analytically, use a calculator.

41) Find the area of the region enclosed by $x = 1$ and $y = 5$ above $y = \ln x$.

Answer

$(e^5 - 6)$ units^2

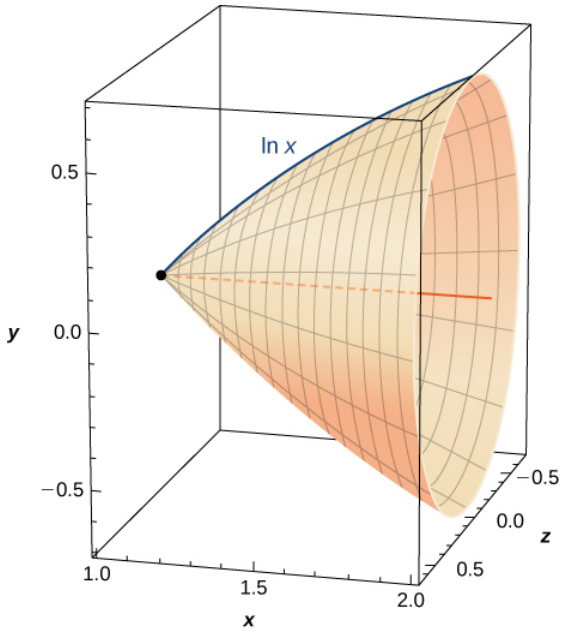
42) [T] Find the arc length of $\ln x$ from $x = 1$ to $x = 2$.

43) Find the area between $\ln x$ and the x-axis from $x = 1$ to $x = 2$.

Answer

$\ln(4) - 1)$ units^2

44) Find the volume of the shape created when rotating this curve from $x = 1$ to $x = 2$ around the x-axis, as pictured here.



45) [T] Find the surface area of the shape created when rotating the curve in the previous exercise from $x = 1$ to $x = 2$ around the x-axis.

Answer

2.8656 units^2

If you are unable to find intersection points analytically in the following exercises, use a calculator.

46) Find the area of the hyperbolic quarter-circle enclosed by $x = 2$ and $y = 2$ above $y = 1/x$.

47) [T] Find the arc length of $y = 1/x$ from $x = 1$ to $x = 4$.

Answer

$s = 3.1502$ units

48) Find the area under $y = 1/x$ and above the x-axis from $x = 1$ to $x = 4$.

In exercises 49 - 53, verify the derivatives and antiderivatives.

49) $\frac{d}{dx}\left[\ln(x + \sqrt{x^2 + 1})\right] = \frac{1}{\sqrt{1 + x^2}}$

50) $\frac{d}{dx}\left[\ln\left(\frac{x-a}{x+a}\right)\right] = \frac{2a}{(x^2 - a^2)}$

51) $\frac{d}{dx}\left[\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)\right] = -\frac{1}{x\sqrt{1 - x^2}}$

52) $\frac{d}{dx}\left[\ln(x+\sqrt{x^2-a^2})\right]=\frac{1}{\sqrt{x^2-a^2}}$

53) $\int \frac{dx}{x\ln(x)\ln(\ln x)}=\ln|\ln(\ln x)|+C$

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6.8: Exponential Growth and Decay

Learning Objectives

- Use the exponential growth model in applications, including population growth and compound interest.
- Explain the concept of doubling time.
- Use the exponential decay model in applications, including radioactive decay and Newton's law of cooling.
- Explain the concept of half-life.

One of the most prevalent applications of exponential functions involves growth and decay models. Exponential growth and decay show up in a host of natural applications. From population growth and continuously compounded interest to radioactive decay and Newton's law of cooling, exponential functions are ubiquitous in nature. In this section, we examine exponential growth and decay in the context of some of these applications.

Exponential Growth Model

Many systems exhibit exponential growth. These systems follow a model of the form $y = y_0 e^{kt}$, where y_0 represents the initial state of the system and k is a positive constant, called the growth constant. Notice that in an exponential growth model, we have

$$y' = ky_0 e^{kt} = ky. \tag{6.8.1}$$

That is, the rate of growth is proportional to the current function value. This is a key feature of exponential growth. Equation 6.8.1 involves derivatives and is called *a differential equation*.

Exponential Growth

Systems that exhibit *exponential growth* increase according to the mathematical model

$$y = y_0 e^{kt}$$

where y_0 represents the initial state of the system and $k > 0$ is a constant, called the growth constant.

Population growth is a common example of exponential growth. Consider a population of bacteria, for instance. It seems plausible that the rate of population growth would be proportional to the size of the population. After all, the more bacteria there are to reproduce, the faster the population grows. Figure 6.8.1 and Table 6.8.1 represent the growth of a population of bacteria with an initial population of 200 bacteria and a growth constant of 0.02. Notice that after only 2 hours (120 minutes), the population is 10 times its original size!

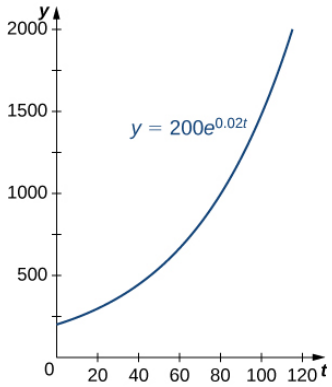

Figure 6.8.1: An example of exponential growth for bacteria.

Table 6.8.1: Exponential Growth of a Bacterial Population

Time(min)	Population Size (no. of bacteria)
10	244
20	298
30	364
40	445
50	544
60	664
70	811
80	991
90	1210
100	1478
110	1805
120	2205

Note that we are using a continuous function to model what is inherently discrete behavior. At any given time, the real-world population contains a whole number of bacteria, although the model takes on noninteger values. When using exponential growth models, we must always be careful to interpret the function values in the context of the phenomenon we are modeling.

Example 6.8.1: Population Growth

Consider the population of bacteria described earlier. This population grows according to the function $f(t) = 200e^{0.02t}$, where t is measured in minutes. How many bacteria are present in the population after 5 hours (300 minutes)? When does the population reach $100,000$ bacteria?

Solution

We have $f(t) = 200e^{0.02t}$. Then

$$f(300) = 200e^{0.02(300)} \approx 80,686.$$

There are $80,686$ bacteria in the population after 5 hours.

To find when the population reaches $100,000$ bacteria, we solve the equation

$$\begin{aligned} 100,000 &= 200e^{0.02t} \\ 500 &= e^{0.02t} \\ \ln 500 &= 0.02t \\ t &= \frac{\ln 500}{0.02} \approx 310.73. \end{aligned}$$

The population reaches $100,000$ bacteria after 310.73 minutes.

Exercise 6.8.1

Consider a population of bacteria that grows according to the function $f(t) = 500e^{0.05t}$, where t is measured in minutes. How many bacteria are present in the population after 4 hours? When does the population reach 100 million bacteria?

Answer

Use the process from the previous example.

Answer

There are $81,377,396$ bacteria in the population after 4 hours. The population reaches 100 million bacteria after 244.12 minutes.

Let's now turn our attention to a financial application: **compound interest**. Interest that is not compounded is called simple interest. ***Simple interest*** is paid once, at the end of the specified time period (usually 1 year). So, if we put $\$1000$ in a savings account earning 2 simple interest per year, then at the end of the year we have

$$1000(1+0.02)=\$1020.$$

Compound interest is paid multiple times per year, depending on the compounding period. Therefore, if the bank compounds the interest every 6 months, it credits half of the year's interest to the account after 6 months. During the second half of the year, the account earns interest not only on the initial $\$1000$, but also on the interest earned during the first half of the year. Mathematically speaking, at the end of the year, we have

$$1000\left(1+\frac{0.02}{2}\right)^2=\$1020.10.$$

Similarly, if the interest is compounded every 4 months, we have

$$1000\left(1+\frac{0.02}{3}\right)^3=\$1020.13,$$

and if the interest is compounded daily (365 times per year), we have $\$1020.20$. If we extend this concept, so that the interest is compounded continuously, after t years we have

$$1000\lim_{n\to\infty}\left(1+\frac{0.02}{n}\right)^{nt}.$$

Now let's manipulate this expression so that we have an exponential growth function. Recall that the number e can be expressed as a limit:

$$e=\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m.$$

Based on this, we want the expression inside the parentheses to have the form $(1+1/m)$. Let $n=0.02m$. Note that as $n\to\infty, m\to\infty$ as well. Then we get

$$1000\lim_{n\to\infty}\left(1+\frac{0.02}{n}\right)^{nt}=1000\lim_{m\to\infty}\left(1+\frac{0.02}{0.02m}\right)^{0.02mt}=1000\left[\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right]^{0.02t}.$$

We recognize the limit inside the brackets as the number e. So, the balance in our bank account after t years is given by $1000e^{0.02t}$. Generalizing this concept, we see that if a bank account with an initial balance of $\$P$ earns interest at a rate of r, compounded continuously, then the balance of the account after t years is

$$\text{Balance }=Pe^{rt}.$$

Example 6.8.2: Compound Interest

A 25-year-old student is offered an opportunity to invest some money in a retirement account that pays 5 annual interest compounded continuously. How much does the student need to invest today to have $\$1$ million when she retires at age 65? What if she could earn 6 annual interest compounded continuously instead?

Solution

We have

$$1,000,000=Pe^{0.05(40)}$$

$$P=135,335.28.$$

She must invest $\$135,335.28$ at 5 interest.

If, instead, she is able to earn 6 then the equation becomes

$$1,000,000 = Pe^{0.06(40)}$$

$$P = 90,717.95.$$

In this case, she needs to invest only $\$90,717.95$. This is roughly two-thirds the amount she needs to invest at 5. The fact that the interest is compounded continuously greatly magnifies the effect of the 1 increase in interest rate.

Exercise 6.8.2

Suppose instead of investing at age $25\sqrt{b^2-4ac}$, the student waits until age 35. How much would she have to invest at 5? At 6?

Hint

Use the process from the previous example.

Answer

At 5 interest, she must invest $\$223,130.16$. At 6 interest, she must invest $\$165,298.89$.

If a quantity grows exponentially, the time it takes for the quantity to double remains constant. In other words, it takes the same amount of time for a population of bacteria to grow from 100 to 200 bacteria as it does to grow from $10,000$ to $20,000$ bacteria. This time is called the doubling time. To calculate the doubling time, we want to know when the quantity reaches twice its original size. So we have

$$\begin{aligned} 2y_0 &= y_0 e^{kt} \\ 2 &= e^{kt} \\ \ln 2 &= kt \\ t &= \frac{\ln 2}{k}. \end{aligned}$$

Definition: Doubling Time

If a quantity grows exponentially, the **doubling time** is the amount of time it takes the quantity to double. It is given by

$$\text{Doubling time} = \frac{\ln 2}{k}.$$

Example 6.8.3: Using the Doubling Time

Assume a population of fish grows exponentially. A pond is stocked initially with 500 fish. After 6 months, there are 1000 fish in the pond. The owner will allow his friends and neighbors to fish on his pond after the fish population reaches $10,000$. When will the owner's friends be allowed to fish?

Solution

We know it takes the population of fish 6 months to double in size. So, if t represents time in months, by the doubling-time formula, we have $6 = (\ln 2)/k$. Then, $k = (\ln 2)/6$. Thus, the population is given by $y = 500e^{((\ln 2)/6)t}$. To figure out when the population reaches $10,000$ fish, we must solve the following equation:

$$\begin{aligned} 10,000 &= 500e^{(\ln 2/6)t} \\ 20 &= e^{(\ln 2/6)t} \\ \ln 20 &= \left(\frac{\ln 2}{6}\right) t \\ t &= \frac{6(\ln 20)}{\ln 2} \\ &\approx 25.93. \end{aligned}$$

The owner's friends have to wait 25.93 months (a little more than 2 years) to fish in the pond.

Exercise $6.8.3$

Suppose it takes 9 months for the fish population in Example $6.8.3$ to reach 1000 fish. Under these circumstances, how long do the owner's friends have to wait?

Hint

Use the process from the previous example.

Answer

38.90 months

Exponential Decay Model

Exponential functions can also be used to model populations that shrink (from disease, for example), or chemical compounds that break down over time. We say that such systems exhibit exponential decay, rather than exponential growth. The model is nearly the same, except there is a negative sign in the exponent. Thus, for some positive constant k, we have

$$y = y_0 e^{-kt}.$$

As with exponential growth, there is a differential equation associated with exponential decay. We have

$$y' = -ky_0 e^{-kt} = -ky.$$

Exponential Decay

Systems that exhibit exponential decay behave according to the model

$$y = y_0 e^{-kt},$$

where y_0 represents the initial state of the system and $k > 0$ is a constant, called the decay constant.

Figure $6.8.2$ shows a graph of a representative exponential decay function.

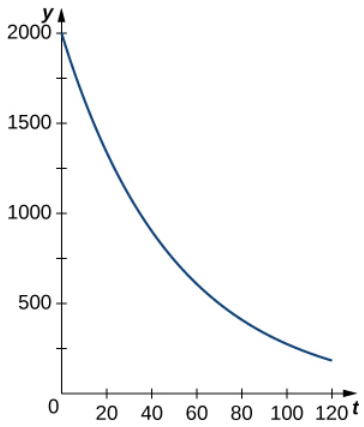

Figure $6.8.2$: An example of exponential decay.

Let's look at a physical application of exponential decay. *Newton's law of cooling* says that an object cools at a rate proportional to the difference between the temperature of the object and the temperature of the surroundings. In other words, if T represents the temperature of the object and T_a represents the ambient temperature in a room, then

$$T' = -k(T - T_a).$$

Note that this is not quite the right model for exponential decay. We want the derivative to be proportional to the function, and this expression has the additional T_a term. Fortunately, we can make a change of variables that resolves this issue. Let $y(t) = T(t) - T_a$. Then $y'(t) = T'(t) - 0 = T'(t)$, and our equation becomes

$$y' = -ky.$$

From our previous work, we know this relationship between y and its derivative leads to exponential decay. Thus,

$$y = y_0 e^{-kt},$$

and we see that

$$T - T_a = (T_0 - T_a)e^{-kt}$$

$$T = (T_0 - T_a)e^{-kt} + T_a$$

where T_0 represents the initial temperature. Let's apply this formula in the following example.

Example 6.8.4: Newton's Law of Cooling

According to experienced baristas, the optimal temperature to serve coffee is between $155°F$ and $175°F$. Suppose coffee is poured at a temperature of $200°F$, and after 2 minutes in a $70°F$ room it has cooled to $180°F$. When is the coffee first cool enough to serve? When is the coffee too cold to serve? Round answers to the nearest half minute.

Solution

We have

$$\begin{aligned} T &= (T_0 - T_a)e^{-kt} + T_a \\ 180 &= (200 - 70)e^{-k(2)} + 70 \\ 110 &= 130e^{-2k} \\ \frac{11}{13} &= e^{-2k} \\ \ln\frac{11}{13} &= -2k \\ \ln 11 - \ln 13 &= -2k \\ k &= \frac{\ln 13 - \ln 11}{2} \end{aligned}$$

Then, the model is

$$T = 130e^{(\ln 11 - \ln 13/2)t} + 70.$$

The coffee reaches $175°F$ when

$$\begin{aligned} 175 &= 130e^{(\ln 11 - \ln 13/2)t} + 70 \\ 105 &= 130e^{(\ln 11 - \ln 13/2)t} \\ \frac{21}{26} &= e^{(\ln 11 - \ln 13/2)t} \\ \ln\frac{21}{26} &= \frac{\ln 11 - \ln 13}{2}t \\ \ln 21 - \ln 26 &= \left(\frac{\ln 11 - \ln 13}{2}\right)t \\ t &= \frac{2(\ln 21 - \ln 26)}{\ln 11 - \ln 13} \\ &\approx 2.56. \end{aligned}$$

The coffee can be served about 2.5 minutes after it is poured. The coffee reaches $155°F$ at

$$155 = 130e^{(\ln 11 - \ln 13/2)t} + 70$$
$$85 = 130e^{(\ln 11 - \ln 13)t}$$
$$\frac{17}{26} = e^{(\ln 11 - \ln 13)t}$$
$$\ln 17 - \ln 26 = \left(\frac{\ln 11 - \ln 13}{2}\right) t$$
$$t = \frac{2(\ln 17 - \ln 26)}{\ln 11 - \ln 13}$$
$$\approx 5.09.$$

The coffee is too cold to be served about 5 minutes after it is poured.

Exercise 6.8.4

Suppose the room is warmer $(75°F)$ and, after 2 minutes, the coffee has cooled only to $185°F$. When is the coffee first cool enough to serve? When is the coffee be too cold to serve? Round answers to the nearest half minute.

Hint

Use the process from the previous example.

Answer

The coffee is first cool enough to serve about 3.5 minutes after it is poured. The coffee is too cold to serve about 7 minutes after it is poured.

Just as systems exhibiting exponential growth have a constant doubling time, systems exhibiting exponential decay have a constant half-life. To calculate the half-life, we want to know when the quantity reaches half its original size. Therefore, we have

$$\frac{y_0}{2} = y_0 e^{-kt}$$
$$\frac{1}{2} = e^{-kt}$$
$$-\ln 2 = -kt$$
$$t = \frac{\ln 2}{k}.$$

Note: This is the same expression we came up with for doubling time.

Definition: Half-Life

If a quantity decays exponentially, the half-life is the amount of time it takes the quantity to be reduced by half. It is given by

$$\text{Half-life} = \frac{\ln 2}{k}.$$

Example 6.8.5: Radiocarbon Dating

One of the most common applications of an exponential decay model is carbon dating. Carbon-14 decays (emits a radioactive particle) at a regular and consistent exponential rate. Therefore, if we know how much carbon-14 was originally present in an object and how much carbon-14 remains, we can determine the age of the object. The half-life of carbon-14 is approximately 5730 years—meaning, after that many years, half the material has converted from the original carbon-14 to the new nonradioactive nitrogen-14. If we have 100 g carbon-14 today, how much is left in 50 years? If an artifact that originally contained 100 g of carbon-14 now contains 10 g of carbon-14, how old is it? Round the answer to the nearest hundred years.

Solution

We have

$$5730 = \frac{\ln 2}{k}$$

$$k = \frac{\ln 2}{5730}.$$

So, the model says

$$y = 100e^{-(\ln 2/5730)t}.$$

In 50 years, we have

$$y = 100e^{-(\ln 2/5730)(50)} \approx 99.40$$

Therefore, in 50 years, 99.40 g of carbon-14 remains.

To determine the age of the artifact, we must solve

$$10 = 100e^{-(\ln 2/5730)t}$$

$$\frac{1}{10} = e^{-(\ln 2/5730)t}$$

$$t \approx 19035.$$

The artifact is about $19,000$ years old.

? Exercise 6.8.5: Carbon-14 Decay

If we have 100 g of carbon-14 , how much is left after 500 years? If an artifact that originally contained 100 g of carbon-14 now contains 20 g of carbon-14, how old is it? Round the answer to the nearest hundred years.

Hint

Use the process from the previous example.

Answer

A total of 94.13 g of carbon-14 remains after 500 years. The artifact is approximately 13,300 years old.

Key Concepts

- Exponential growth and exponential decay are two of the most common applications of exponential functions.
- Systems that exhibit exponential growth follow a model of the form $y = y_0e^{kt}$.
- In exponential growth, the rate of growth is proportional to the quantity present. In other words, $y' = ky$.
- Systems that exhibit exponential growth have a constant doubling time, which is given by $(\ln 2)/k$.
- Systems that exhibit exponential decay follow a model of the form $y = y_0e^{-kt}$.
- Systems that exhibit exponential decay have a constant half-life, which is given by $(\ln 2)/k$.

Glossary

doubling time

if a quantity grows exponentially, the doubling time is the amount of time it takes the quantity to double, and is given by $(\ln 2)/k$

exponential decay

systems that exhibit exponential decay follow a model of the form $y = y_0e^{-kt}$

exponential growth

systems that exhibit exponential growth follow a model of the form $y = y_0e^{kt}$

half-life

if a quantity decays exponentially, the half-life is the amount of time it takes the quantity to be reduced by half. It is given by $(\ln 2)/k$

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6.8E: Exercises for Section 6.8

In exercises 1 - 2, answer *True or False*? If true, prove it. If false, find the true answer.

1) The doubling time for $y = e^{ct}$ is $\frac{\ln 2}{\ln c}$.

2) If you invest $\$500$, an annual rate of interest of 3% yields more money in the first year than a 2.5% continuous rate of interest.

Answer

True

3) If you leave a $100°C$ pot of tea at room temperature ($25°C$) and an identical pot in the refrigerator $(5°C)$, with $k = 0.02$, the tea in the refrigerator reaches a drinkable temperature $(70°C)$ more than 5 minutes before the tea at room temperature.

4) If given a half-life of t years, the constant k for $y = e^{kt}$ is calculated by $k = \frac{\ln(\frac{1}{2})}{t}$.

Answer

False; $k = \frac{\ln 2}{t}$

In exercises 5 - 18, use $y = y_0 e^{kt}$.

5) If a culture of bacteria doubles in 3 hours, how many hours does it take to multiply by 10?

6) If bacteria increase by a factor of 10 in 10 hours, how many hours does it take to increase by 100?

Answer

20 hours

7) How old is a skull that contains one-fifth as much radiocarbon as a modern skull? Note that the half-life of radiocarbon is 5730 years.

8) If a relic contains 90% as much radiocarbon as new material, can it have come from the time of Christ (approximately 2000 years ago)? Note that the half-life of radiocarbon is 5730 years.

Answer

No. The relic is approximately 871 years old.

9) The population of Cairo grew from 5 million to 10 million in 20 years. Use an exponential model to find when the population was 8 million.

10) The populations of New York and Los Angeles are growing at 1% and 1.4% a year, respectively. Starting from 8 million (New York) and 6 million (Los Angeles), when are the populations equal?

Answer

71.92 years

11) Suppose the value of $\$1$ in Japanese yen decreases at 2% per year. Starting from $\$1 = ¥250$, when will $\$1 = ¥1$?

12) The effect of advertising decays exponentially. If 40% of the population remembers a new product after 3 days, how long will 20%remember it?

Answer

5 days 6 hours 27minutes

13) If $y = 1000$ at $t = 3$ and $y = 3000$ at $t = 4$, what was y_0 at $t = 0$?

14) If $y = 100$ at $t = 4$ and $y = 10$ at $t = 8$, when does $y = 1$?

Answer

At $t = 12$

15) If a bank offers annual interest of 7.5% or continuous interest of 7.25%, which has a better annual yield?

16) What continuous interest rate has the same yield as an annual rate of 9%?

Answer

8.618%

17) If you deposit $\$5000$at 8% annual interest, how many years can you withdraw $\$500$ (starting after the first year) without running out of money?

18) You are trying to save $\$50,000$ in 20 years for college tuition for your child. If interest is a continuous 10%, how much do you need to invest initially?

Answer

$6766.76

19) You are cooling a turkey that was taken out of the oven with an internal temperature of $165°F$. After 10 minutes of resting the turkey in a $70°F$ apartment, the temperature has reached $155°F$. What is the temperature of the turkey 20 minutes after taking it out of the oven?

20) You are trying to thaw some vegetables that are at a temperature of $1°F$. To thaw vegetables safely, you must put them in the refrigerator, which has an ambient temperature of $44°F$. You check on your vegetables 2 hours after putting them in the refrigerator to find that they are now $12°F$. Plot the resulting temperature curve and use it to determine when the vegetables reach $33°$.

Answer

9hours 13minutes

21) You are an archeologist and are given a bone that is claimed to be from a Tyrannosaurus Rex. You know these dinosaurs lived during the Cretaceous Era (146 million years to 65 million years ago), and you find by radiocarbon dating that there is 0.000001% the amount of radiocarbon. Is this bone from the Cretaceous?

22) The spent fuel of a nuclear reactor contains plutonium-239, which has a half-life of $24,000$ years. If 1 barrel containing 10 kg of plutonium-239 is sealed, how many years must pass until only 10 g of plutonium-239 is left?

Answer

$239,179$ years

For exercises 23 - 26, use the following table, which features the world population by decade.

Years since 1950	Population (millions)
0	2,556
10	3,039
20	3,706
30	4,453
40	5,279
50	6,083
60	6,849

Source: http:/www.factmonster.com/ipka/A0762181.html.

23) [T] The best-fit exponential curve to the data of the form $P(t) = ae^{bt}$ is given by $P(t) = 2686e^{0.01604t}$. Use a graphing calculator to graph the data and the exponential curve together.

24) [T] Find and graph the derivative y' of your equation. Where is it increasing and what is the meaning of this increase?

Answer

$P'(t) = 43e^{0.01604t}$. The population is always increasing.

25) [T] Find and graph the second derivative of your equation. Where is it increasing and what is the meaning of this increase?

26) [T] Find the predicted date when the population reaches 10 billion. Using your previous answers about the first and second derivatives, explain why exponential growth is unsuccessful in predicting the future.

Answer

The population reaches 10 billion people in 2027.

For exercises 27 - 29, use the following table, which shows the population of San Francisco during the 19th century.

Years since 1850	Population (thousands)
0	21.00
10	56.80
20	149.5
30	234.0

Source: http:/www.sfgenealogy.com/sf/history/hgpop.htm.

27) [T] The best-fit exponential curve to the data of the form $P(t) = ae^{bt}$ is given by $P(t) = 35.26e^{0.06407t}$. Use a graphing calculator to graph the data and the exponential curve together.

28) [T] Find and graph the derivative y' of your equation. Where is it increasing? What is the meaning of this increase? Is there a value where the increase is maximal?

Answer

$P'(t) = 2.259e^{0.06407t}$. The population is always increasing.

29) [T] Find and graph the second derivative of your equation. Where is it increasing? What is the meaning of this increase?

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6.9: Calculus of the Hyperbolic Functions

Learning Objectives

- Apply the formulas for derivatives and integrals of the hyperbolic functions.
- Apply the formulas for the derivatives of the inverse hyperbolic functions and their associated integrals.
- Describe the common applied conditions of a catenary curve.

We were introduced to hyperbolic functions previously, along with some of their basic properties. In this section, we look at differentiation and integration formulas for the hyperbolic functions and their inverses.

Derivatives and Integrals of the Hyperbolic Functions

Recall that the hyperbolic sine and hyperbolic cosine are defined as

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

and

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

The other hyperbolic functions are then defined in terms of $\sinh x$ and $\cosh x$. The graphs of the hyperbolic functions are shown in Figure 6.9.1.

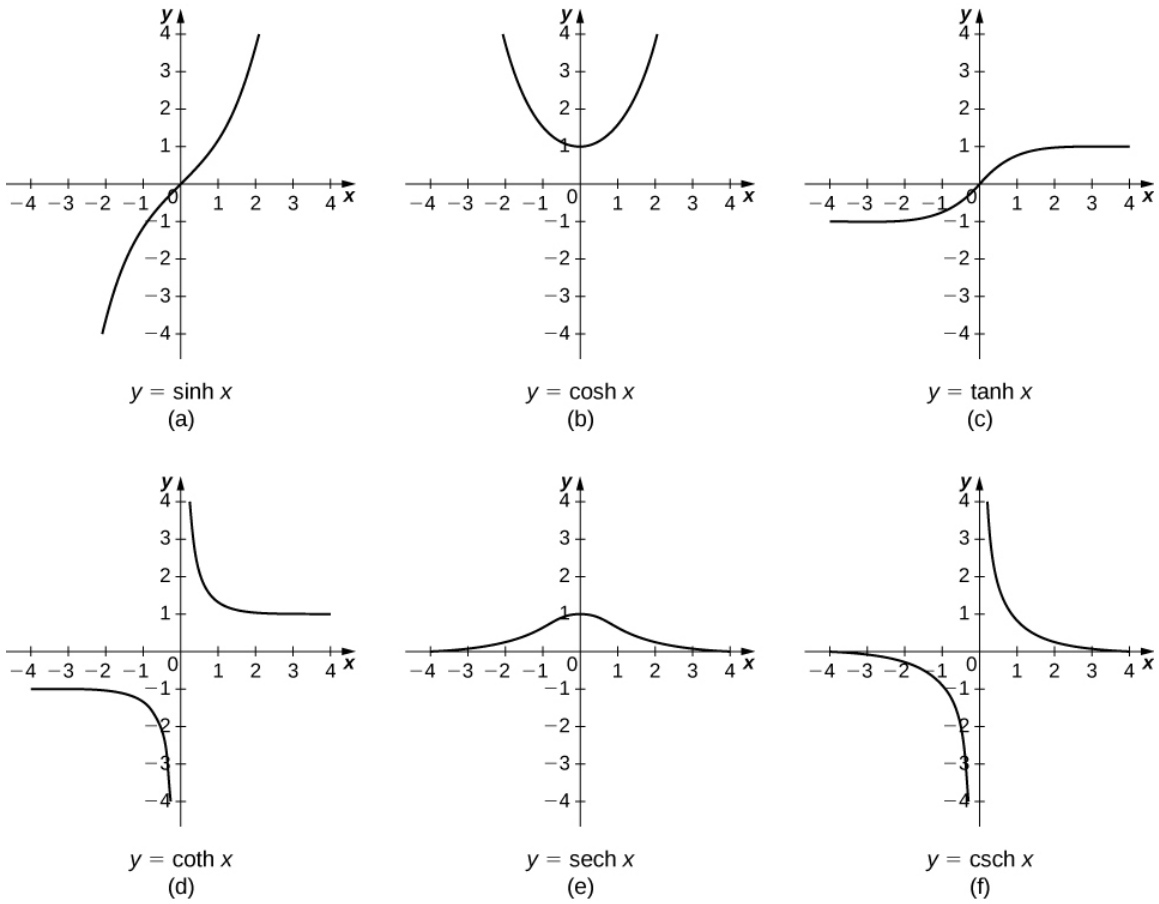

Figure 6.9.1: Graphs of the hyperbolic functions.

It is easy to develop differentiation formulas for the hyperbolic functions. For example, looking at $\sinh x$ we have

$$\begin{aligned}\frac{d}{dx}(\sinh x) &= \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right)\\ &= \frac{1}{2}\left[\frac{d}{dx}(e^x) - \frac{d}{dx}(e^{-x})\right]\\ &= \frac{1}{2}[e^x + e^{-x}]\\ &= \cosh x.\end{aligned}$$

Similarly,

$$\frac{d}{dx}\cosh x = \sinh x.$$

We summarize the differentiation formulas for the hyperbolic functions in Table 6.9.1.

Table 6.9.1: Derivatives of the Hyperbolic Functions

$f(x)$	$\frac{d}{dx}f(x)$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\coth\ x$	$-\operatorname{csch}^2 x$
$\operatorname{sech}\ x$	$-\operatorname{sech} x \tanh x$
$\operatorname{csch}\ x$	$-\operatorname{csch} x \coth x$

Let's take a moment to compare the derivatives of the hyperbolic functions with the derivatives of the standard trigonometric functions. There are a lot of similarities, but differences as well. For example, the derivatives of the sine functions match:

$$\frac{d}{dx}\sin x = \cos x$$

and

$$\frac{d}{dx}\sinh x = \cosh x.$$

The derivatives of the cosine functions, however, differ in sign:

$$\frac{d}{dx}\cos x = -\sin x,$$

but

$$\frac{d}{dx}\cosh x = \sinh x.$$

As we continue our examination of the hyperbolic functions, we must be mindful of their similarities and differences to the standard trigonometric functions. These differentiation formulas for the hyperbolic functions lead directly to the following integral formulas.

$$\int \sinh u\, du = \cosh u + C \tag{6.9.1}$$

$$\int \operatorname{csch}^2 u\, du = -\coth u + C \tag{6.9.2}$$

$$\int \cosh u\, du = \sinh u + C \tag{6.9.3}$$

$$\int \operatorname{sech} u \tanh u\, du = -\operatorname{sech}\ u + C - \operatorname{csch} u + C \tag{6.9.4}$$

$$\int \operatorname{sech}\ ^2 u\, du = \tanh u + C \tag{6.9.5}$$

$$\int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + C \tag{6.9.6}$$

Example 6.9.1: Differentiating Hyperbolic Functions

Evaluate the following derivatives:

a. $\frac{d}{dx}(\sinh(x^2))$
b. $\frac{d}{dx}(\cosh x)^2$

Solution:

Using the formulas in Table 6.9.1 and the chain rule, we get

a. $\frac{d}{dx}(\sinh(x^2)) = \cosh(x^2) \cdot 2x$
b. $\frac{d}{dx}(\cosh x)^2 = 2 \cosh x \sinh x$

Exercise 6.9.1

Evaluate the following derivatives:

a. $\frac{d}{dx}(\tanh(x^2 + 3x))$
b. $\frac{d}{dx}\left(\frac{1}{(\sinh x)^2}\right)$

Hint

Use the formulas in Table 6.9.1 and apply the chain rule as necessary.

Answer a

$$\frac{d}{dx}(\tanh(x^2 + 3x)) = (\operatorname{sech}^2(x^2 + 3x))(2x + 3)$$

Answer b

$$\frac{d}{dx}\left(\frac{1}{(\sinh x)^2}\right) = \frac{d}{dx}(\sinh x)^{-2} = -2(\sinh x)^{-3} \cosh x$$

Example 6.9.2: Integrals Involving Hyperbolic Functions

Evaluate the following integrals:

a. $\int x \cosh(x^2) dx$
b. $\int \tanh x \, dx$

Solution

We can use u-substitution in both cases.

a. Let $u = x^2$. Then, $du = 2x \, dx$ and

$$\begin{aligned} \int x \cosh(x^2) dx &= \int \frac{1}{2} \cosh u \, du \\ &= \frac{1}{2} \sinh u + C \\ &= \frac{1}{2} \sinh(x^2) + C. \end{aligned}$$

b. Let $u = \cosh x$. Then, $du = \sinh x \, dx$ and

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$$\begin{aligned}\int \tanh x\,dx &= \int \frac{\sinh x}{\cosh x}\,dx \\ &= \int \frac{1}{u}\,du \\ &= \ln|u| + C \\ &= \ln|\cosh x| + C.\end{aligned}$$

Note that $\cosh x > 0$ for all x, so we can eliminate the absolute value signs and obtain

$$\int \tanh x\,dx = \ln(\cosh x) + C.$$

? Exercise 6.9.2

Evaluate the following integrals:

a. $\int \sinh^3 x \cosh x\,dx$

b. $\int \text{sech}^{\,2}(3x)\,dx$

Hint

Use the formulas above and apply u-substitution as necessary.

Answer a

$$\int \sinh^3 x \cosh x\,dx = \frac{\sinh^4 x}{4} + C$$

Answer b

$$\int \text{sech}^{\,2}(3x)\,dx = \frac{\tanh(3x)}{3} + C$$

Calculus of Inverse Hyperbolic Functions

Looking at the graphs of the hyperbolic functions, we see that with appropriate range restrictions, they all have inverses. Most of the necessary range restrictions can be discerned by close examination of the graphs. The domains and ranges of the inverse hyperbolic functions are summarized in Table 6.9.2.

Table 6.9.2: Domains and Ranges of the Inverse Hyperbolic Functions

Function	Domain	Range
$\sinh^{-1} x$	$(-\infty,\infty)$	$(-\infty,\infty)$
$\cosh^{-1} x$	$(1,\infty)$	$[0,\infty)$
$\tanh^{-1} x$	$(-1,1)$	$(-\infty,\infty)$
$\coth^{-1} x$	$(-\infty,1)\cup(1,\infty)$	$(-\infty,0)\cup(0,\infty)$
$\text{sech}^{-1} x$	$(0,1)$	$[0,\infty)$
$\text{csch}^{-1} x$	$(-\infty,0)\cup(0,\infty)$	$(-\infty,0)\cup(0,\infty)$

The graphs of the inverse hyperbolic functions are shown in the following figure.

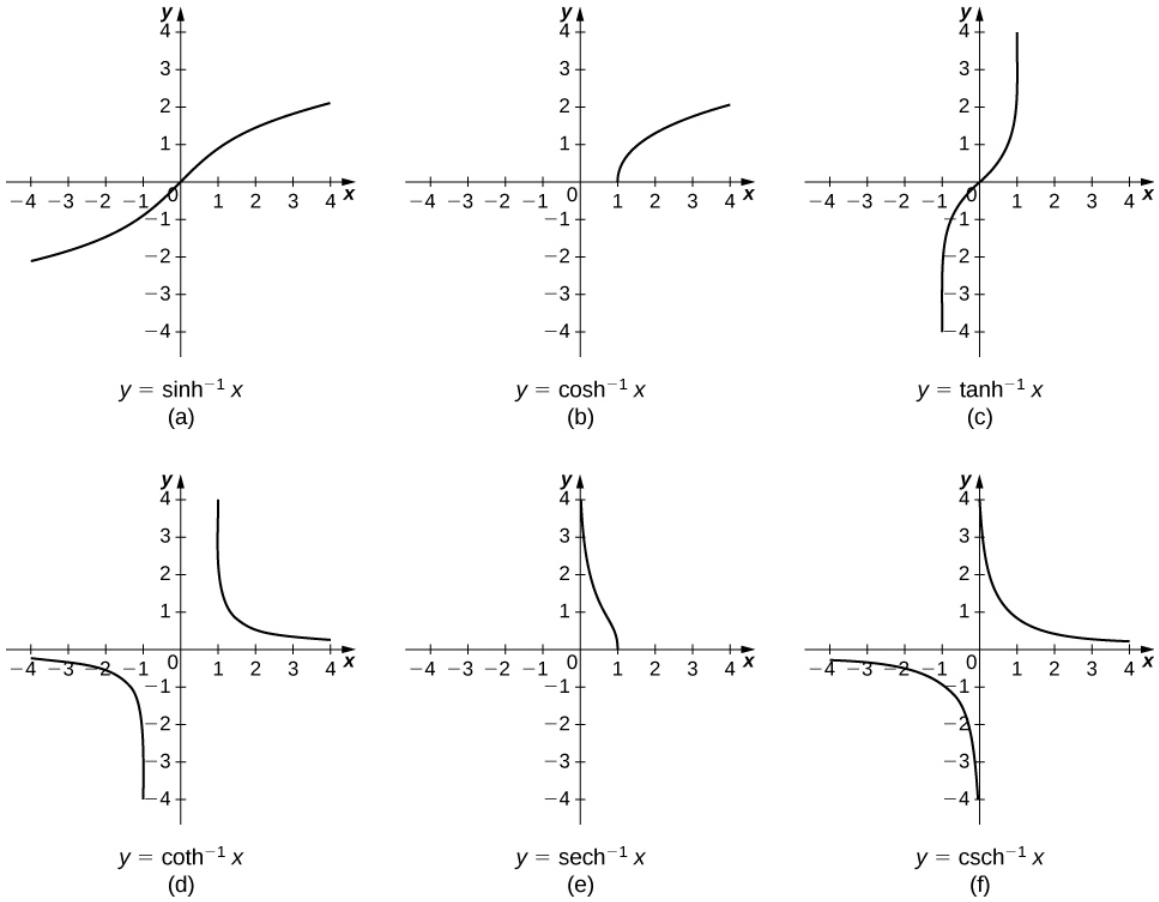

Figure 6.9.3: Graphs of the inverse hyperbolic functions.

To find the derivatives of the inverse functions, we use implicit differentiation. We have

$$y = \sinh^{-1} x \tag{6.9.7}$$

$$\sinh y = x \tag{6.9.8}$$

$$\frac{d}{dx}\sinh y = \frac{d}{dx}x \tag{6.9.9}$$

$$\cosh y \frac{dy}{dx} = 1. \tag{6.9.10}$$

Recall that $\cosh^2 y - \sinh^2 y = 1$, so $\cosh y = \sqrt{1 + \sinh^2 y}$. Then,

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1 + x^2}}.$$

We can derive differentiation formulas for the other inverse hyperbolic functions in a similar fashion. These differentiation formulas are summarized in Table 6.9.3.

Table 6.9.3: Derivatives of the Inverse Hyperbolic Functions

$f(x)$	$\frac{d}{dx}f(x)$
$\sinh^{-1} x$	$\frac{1}{\sqrt{1+x^2}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}$
$\coth^{-1} x$	$\frac{1}{1-x^2}$

$f(x)$	$\frac{d}{dx}f(x)$
$\operatorname{sech}^{-1}x$	$\frac{-1}{x\sqrt{1-x^2}}$
$\operatorname{csch}^{-1}x$	$\frac{-1}{\lvert x\rvert\sqrt{1+x^2}}$

Note that the derivatives of $\tanh^{-1}x$ and $\coth^{-1}x$ are the same. Thus, when we integrate $1/(1-x^2)$, we need to select the proper antiderivative based on the domain of the functions and the values of x. Integration formulas involving the inverse hyperbolic functions are summarized as follows.

$$\int \frac{1}{\sqrt{1+u^2}}du = \sinh^{-1}u + C$$

$$\int \frac{1}{u\sqrt{1-u^2}}du = -\operatorname{sech}^{-1}|u| + C$$

$$\int \frac{1}{\sqrt{u^2-1}}du = \cosh^{-1}u + C$$

$$\int \frac{1}{u\sqrt{1+u^2}}du = -\operatorname{csch}^{-1}|u| + C$$

$$\int \frac{1}{1-u^2}du = \begin{cases}\tanh^{-1}u + C & \text{if } |u|<1\\ \coth^{-1}u + C & \text{if } |u|>1\end{cases}$$

Example 6.9.3: Differentiating Inverse Hyperbolic Functions

Evaluate the following derivatives:

a. $\frac{d}{dx}\left(\sinh^{-1}\left(\frac{x}{3}\right)\right)$
b. $\frac{d}{dx}\left(\tanh^{-1}x\right)^2$

Solution

Using the formulas in Table 6.9.3 and the chain rule, we obtain the following results:

a. $\frac{d}{dx}(\sinh^{-1}(\frac{x}{3})) = \frac{1}{3\sqrt{1+\frac{x^2}{9}}} = \frac{1}{\sqrt{9+x^2}}$
b. $\frac{d}{dx}(\tanh^{-1}x)^2 = \frac{2(\tanh^{-1}x)}{1-x^2}$

Exercise 6.9.3

Evaluate the following derivatives:

a. $\frac{d}{dx}(\cosh^{-1}(3x))$
b. $\frac{d}{dx}(\coth^{-1}x)^3$

Hint

Use the formulas in Table 6.9.3 and apply the chain rule as necessary.

Answer a

$$\frac{d}{dx}(\cosh^{-1}(3x)) = \frac{3}{\sqrt{9x^2-1}}$$

Answer b

$$\frac{d}{dx}(\coth^{-1}x)^3 = \frac{3(\coth^{-1}x)^2}{1-x^2}$$

Example 6.9.4: Integrals Involving Inverse Hyperbolic Functions

Evaluate the following integrals:

a. $\int \frac{1}{\sqrt{4x^2-1}}dx$

b. $\int \frac{1}{2x\sqrt{1-9x^2}}dx$

Solution

We can use u-**substitution** in both cases.

Let $u = 2x$. Then, $du = 2\,dx$ and we have

$$\begin{aligned}\int \frac{1}{\sqrt{4x^2-1}}\,dx &= \int \frac{1}{2\sqrt{u^2-1}}\,du \\ &= \frac{1}{2}\cosh^{-1} u + C \\ &= \frac{1}{2}\cosh^{-1}(2x) + C.\end{aligned}$$

Let $u = 3x$. Then, $du = 3\,dx$ and we obtain

$$\begin{aligned}\int \frac{1}{2x\sqrt{1-9x^2}}dx &= \frac{1}{2}\int \frac{1}{u\sqrt{1-u^2}}du \\ &= -\frac{1}{2}\operatorname{sech}^{-1}|u| + C \\ &= -\frac{1}{2}\operatorname{sech}^{-1}|3x| + C\end{aligned}$$

Exercise 6.9.4

Evaluate the following integrals:

a. $\int \frac{1}{\sqrt{x^2-4}}dx, x > 2$

b. $\int \frac{1}{\sqrt{1-e^{2x}}}dx$

Hint

Use the formulas above and apply u-substitution as necessary.

Answer a

$$\int \frac{1}{\sqrt{x^2-4}}dx = \cosh^{-1}(\frac{x}{2}) + C$$

Answer b

$$\int \frac{1}{\sqrt{1-e^{2x}}}dx = -\operatorname{sech}^{-1}(e^x) + C$$

Applications

One physical application of hyperbolic functions involves hanging cables. If a cable of uniform density is suspended between two supports without any load other than its own weight, the cable forms a curve called a **catenary**. High-voltage power lines, chains hanging between two posts, and strands of a spider's web all form catenaries. The following figure shows chains hanging from a row of posts.

Figure 6.9.3: Chains between these posts take the shape of a catenary. (credit: modification of work by OKFoundryCompany, Flickr)

Hyperbolic functions can be used to model catenaries. Specifically, functions of the form $y = a \cdot \cosh(x/a)$ are catenaries. Figure 6.9.4 shows the graph of $y = 2\cosh(x/2)$.

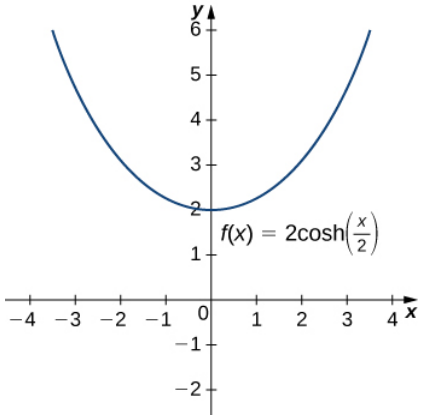

Figure 6.9.4: A hyperbolic cosine function forms the shape of a catenary.

Example 6.9.5: Using a Catenary to Find the Length of a Cable

Assume a hanging cable has the shape $10\cosh(x/10)$ for $-15 \le x \le 15$, where x is measured in feet. Determine the length of the cable (in feet).

Solution

Recall from Section 6.4 that the formula for arc length is

$$\underbrace{\int_a^b \sqrt{1+[f'(x)]^2}dx}_{\text{Arc Length}}.$$

We have $f(x) = 10\cosh(x/10)$, so $f'(x) = \sinh(x/10)$. Then the arc length is

$$\int_a^b \sqrt{1+[f'(x)]^2}dx = \int_{-15}^{15} \sqrt{1+\sinh^2\left(\frac{x}{10}\right)}dx.$$

Now recall that

$$1+\sinh^2 x = \cosh^2 x,$$

so we have

$$\begin{aligned}\text{Arc Length} &= \int_{-15}^{15} \sqrt{1+\sinh^2\left(\frac{x}{10}\right)}dx \\ &= \int_{-15}^{15} \cosh\left(\frac{x}{10}\right)dx \\ &= 10\sinh\left(\frac{x}{10}\right)\Big|_{-15}^{15} \\ &= 10\left[\sinh\left(\frac{3}{2}\right)-\sinh\left(-\frac{3}{2}\right)\right] \\ &= 20\sinh\left(\frac{3}{2}\right) \\ &\approx 42.586\,\text{ft}.\end{aligned}$$

? Exercise 6.9.5:

Assume a hanging cable has the shape $15\cosh(x/15)$ for $-20 \le x \le 20$. Determine the length of the cable (in feet).

Answer

52.95 ft

Key Concepts

- Hyperbolic functions are defined in terms of exponential functions.
- Term-by-term differentiation yields differentiation formulas for the hyperbolic functions. These differentiation formulas give rise, in turn, to integration formulas.
- With appropriate range restrictions, the hyperbolic functions all have inverses.
- Implicit differentiation yields differentiation formulas for the inverse hyperbolic functions, which in turn give rise to integration formulas.
- The most common physical applications of hyperbolic functions are calculations involving catenaries.

Glossary

catenary

a curve in the shape of the function $y = a \cdot \cosh(x/a)$ is a catenary; a cable of uniform density suspended between two supports assumes the shape of a catenary

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6.9E: Exercises for Section 6.9

1) [T] Find expressions for $\cosh x + \sinh x$ and $\cosh x - \sinh x$. Use a calculator to graph these functions and ensure your expression is correct.

Answer

e^x and e^{-x}

2) From the definitions of $\cosh(x)$ and $\sinh(x)$, find their antiderivatives.

3) Show that $\cosh(x)$ and $\sinh(x)$ satisfy $y'' = y$.

Answer

Answers may vary

4) Use the quotient rule to verify that $\frac{d}{dx}\big(\tanh(x)\big) = \operatorname{sech}^2(x)$.

5) Derive $\cosh^2(x) + \sinh^2(x) = \cosh(2x)$ from the definition.

Answer

Answers may vary

6) Take the derivative of the previous expression to find an expression for $\sinh(2x)$.

7) Prove $\sinh(x + y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$ by changing the expression to exponentials.

Answer

Answers may vary

8) Take the derivative of the previous expression to find an expression for $\cosh(x + y)$.

In exercises 9 - 18, find the derivatives of the given functions and graph along with the function to ensure your answer is correct.

9) [T] $\cosh(3x + 1)$

Answer

$3\sinh(3x + 1)$

10) [T] $\sinh(x^2)$

11) [T] $\frac{1}{\cosh(x)}$

Answer

$-\tanh(x)\operatorname{sech}(x)$

12) [T] $\sinh(\ln(x))$

13) [T] $\cosh^2(x) + \sinh^2(x)$

Answer

$4\cosh(x)\sinh(x)$

14) [T] $\cosh^2(x) - \sinh^2(x)$

15) [T] $\tanh(\sqrt{x^2 + 1})$

Answer

$$\frac{x\text{sech}^2(\sqrt{x^2+1})}{\sqrt{x^2+1}}$$

16) [T] $\frac{1+\tanh(x)}{1-\tanh(x)}$

17) [T] $\sinh^6(x)$

Answer

$6\sinh^5(x)\cosh(x)$

18) [T] $\ln(\text{sech}(x)+\tanh(x))$

In exercises 19 - 28, find the antiderivatives for the given functions.

19) $\cosh(2x+1)$

Answer

$\frac{1}{2}\sinh(2x+1)+C$

20) $\tanh(3x+2)$

21) $x\cosh(x^2)$

Answer

$\frac{1}{2}\sinh^2(x^2)+C$

22) $3x^3\tanh(x^4)$

23) $\cosh^2(x)\sinh(x)$

Answer

$\frac{1}{3}\cosh^3(x)+C$

24) $\tanh^2(x)\text{sech}^2(x)$

25) $\frac{\sinh(x)}{1+\cosh(x)}$

Answer

$\ln(1+\cosh(x))+C$

26) $\coth(x)$

27) $\cosh(x)+\sinh(x)$

Answer

$\cosh(x)+\sinh(x)+C$

28) $(\cosh(x)+\sinh(x))^n$

In exercises 29 - 35, find the derivatives for the functions.

29) $\tanh^{-1}(4x)$

Answer

$\frac{4}{1-16x^2}$

30) $\sinh^{-1}(x^2)$

31) $\sinh^{-1}(\cosh(x))$

Answer

$$\frac{\sinh(x)}{\sqrt{\cosh^2(x)+1}}$$

32) $\cosh^{-1}(x^3)$

33) $\tanh^{-1}(\cos(x))$

Answer

$$-\csc(x)$$

34) $e^{\sinh^{-1}(x)}$

35) $\ln(\tanh^{-1}(x))$

Answer

$$-\frac{1}{(x^2-1)\tanh^{-1}(x)}$$

In exercises 36 - 42, find the antiderivatives for the functions.

36) $\int \frac{dx}{4-x^2}$

37) $\int \frac{dx}{a^2-x^2}$

Answer

$$\frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right)+C$$

38) $\int \frac{dx}{\sqrt{x^2+1}}$

39) $\int \frac{xdx}{\sqrt{x^2+1}}$

Answer

$$\sqrt{x^2+1}+C$$

40) $\int -\frac{dx}{x\sqrt{1-x^2}}$

41) $\int \frac{e^x}{\sqrt{e^{2x}-1}}$

Answer

$$\cosh^{-1}(e^x)+C$$

42) $\int -\frac{2x}{x^4-1}$

In exercises 43 - 45, use the fact that a falling body with friction equal to velocity squared obeys the equation $\frac{dv}{dt}=g-v^2$.

43) Show that $v(t)=\sqrt{g}\tanh(\sqrt{g}t)$ satisfies this equation.

Answer

Answers may vary

44) Derive the previous expression for $v(t)$ by integrating $\frac{dv}{g-v^2}=dt$.

45) [T] Estimate how far a body has fallen in 12seconds by finding the area underneath the curve of $v(t)$.

Answer

37.30

In exercises 46 - 48, use this scenario: A cable hanging under its own weight has a slope $S = \frac{dy}{dx}$ that satisfies $\frac{dS}{dx} = c\sqrt{1+S^2}$. The constant c is the ratio of cable density to tension.

46) Show that $S = \sinh(cx)$ satisfies this equation.

47) Integrate $\frac{dy}{dx} = \sinh(cx)$ to find the cable height $y(x)$ if $y(0) = 1/c$.

Answer

$y = \frac{1}{c}\cosh(cx)$

48) Sketch the cable and determine how far down it sags at $x = 0$.

In exercises 49 - 52, solve each problem.

49) [T] A chain hangs from two posts 2m apart to form a catenary described by the equation $y = 2\cosh(x/2) - 1$. Find the slope of the catenary at the left fence post.

Answer

-0.521095

50) [T] A chain hangs from two posts four meters apart to form a catenary described by the equation $y = 4\cosh(x/4) - 3$. Find the total length of the catenary (arc length).

51) [T] A high-voltage power line is a catenary described by $y = 10\cosh(x/10)$. Find the ratio of the area under the catenary to its arc length. What do you notice?

Answer

10

52) A telephone line is a catenary described by $y = a\cosh(x/a)$. Find the ratio of the area under the catenary to its arc length. Does this confirm your answer for the previous question?

53) Prove the formula for the derivative of $y = \sinh^{-1}(x)$ by differentiating $x = \sinh(y)$.

(Hint: Use hyperbolic trigonometric identities.)

54) Prove the formula for the derivative of $y = \cosh^{-1}(x)$ by differentiating $x = \cosh(y)$.

(Hint: Use hyperbolic trigonometric identities.)

55) Prove the formula for the derivative of $y = \operatorname{sech}^{-1}(x)$ by differentiating $x = \operatorname{sech}(y)$.

(Hint: Use hyperbolic trigonometric identities.)

56) Prove that $\cosh(x) + \sinh(x))^n = \cosh(nx) + \sinh(nx)$.

57) Prove the expression for $\sinh^{-1}(x)$. Multiply $x = \sinh(y) = \frac{e^y - e^{-y}}{2}$ by $2e^y$ and solve for y. Does your expression match the textbook?

58) Prove the expression for $\cosh^{-1}(x)$. Multiply $x = \cosh(y) = \frac{e^y + e^{-y}}{2}$ by $2e^y$ and solve for y. Does your expression match the textbook?

Contributors

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6.R: CHAPTER 6 REVIEW EXERCISES

***True or False?* Justify your answer with a proof or a counterexample.**

1) The amount of work to pump the water out of a half-full cylinder is half the amount of work to pump the water out of the full cylinder.

Answer

False

2) If the force is constant, the amount of work to move an object from $x = a$ to $x = b$ is $F(b - a)$.

3) The disk method can be used in any situation in which the washer method is successful at finding the volume of a solid of revolution.

Answer

False

4) If the half-life of $seaborgium - 266$ is 360 ms, then $k = \dfrac{\ln 2}{360}$.

For exercises 5 - 8, use the requested method to determine the volume of the solid.

5) The volume that has a base of the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ and cross-sections of an equilateral triangle perpendicular to the y-axis. Use the method of slicing.

Answer

$V = 32\sqrt{3}\,\text{units}^3$

6) $y = x^2 - x$, from $x = 1$ to $x = 4$, rotated around the y-axis using the washer method

7) $x = y^2$ and $x = 3y$ rotated around the y-axis using the washer method

Answer

$V = \frac{162\pi}{5}\,\text{units}^3$

8) $x = 2y^2 - y^3,\ x = 0$,and $y = 0$ rotated around the x-axis using cylindrical shells

For exercises 9 - 14, find

a. the area of the region,

b.the volume of the solid when rotated around the x-axis, and

c. the volume of the solid when rotated around the y-axis. Use whichever method seems most appropriate to you.

9) $y = x^3, x = 0, y = 0$, and $x = 2$

Answer

a. $A = 4$ units^2
b. $V = \frac{128\pi}{7}$ units^3
c. $V = \frac{64\pi}{5}$ units^3

10) $y = x^2 - x$ and $x = 0$

11) [T] $y = \ln(x) + 2$ and $y = x$

Answer

a. $A \approx 1.949$ units^2
b. $V \approx 21.952$ units^3
c. $V = \approx 17.099$ units^3

12) $y = x^2$ and $y = \sqrt{x}$

13) $y = 5 + x, y = x^2, x = 0$, and $x = 1$

Answer

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a. $A = \frac{31}{6}$ units^2
b. $V = \frac{452\pi}{15}$ units^3
c. $V = \frac{31\pi}{6}$ units^3

14) Below $x^2 + y^2 = 1$ and above $y = 1 - x$

15) Find the mass of $\rho = e^{-x}$ on a disk centered at the origin with radius 4.

Answer

$m \approx 245.282$

16) Find the center of mass for $\rho = \tan^2 x$ on $x \in (-\frac{\pi}{4}, \frac{\pi}{4})$.

17) Find the mass and the center of mass of $\rho = 1$ on the region bounded by $y = x^5$ and $y = \sqrt{x}$.

Answer

Mass: $\frac{1}{2}$,
Center of mass: $(\frac{18}{35}, \frac{9}{11})$

For exercises 18 - 19, find the requested arc lengths.

18) The length of x for $y = \cosh(x)$ from $x = 0$ to $x = 2$.

19) The length of y for $x = 3 - \sqrt{y}$ from $y = 0$ to $y = 4$

Answer

$s = [\sqrt{17} + \frac{1}{8}\ln(33 + 8\sqrt{17})]$ units

For exercises 20 - 21, find the surface area and volume when the given curves are revolved around the specified axis.

20) The shape created by revolving the region between $y = 4 + x$, $y = 3 - x$, $x = 0$, and $x = 2$ rotated around the y-axis.

21) The loudspeaker created by revolving $y = \dfrac{1}{x}$ from $x = 1$ to $x = 4$ around the x-axis.

Answer

Volume: $V = \frac{3\pi}{4}$ units^3
Surface area: $A = \pi\left(\sqrt{2} - \sinh^{-1}(1) + \sinh^{-1}(16) - \frac{\sqrt{257}}{16}\right)$ units^2

For exercise 22, consider the Karun-3 dam in Iran. Its shape can be approximated as an isosceles triangle with height 205 m and width 388 m. Assume the current depth of the water is 180 m. The density of water is 1000 kg/m^3.

22) Find the total force on the wall of the dam.

23) You are a crime scene investigator attempting to determine the time of death of a victim. It is noon and 45 °F outside and the temperature of the body is 78 °F. You know the cooling constant is $k = 0.00824$ °F/min. When did the victim die, assuming that a human's temperature is 98 °F?

Answer

11:02 a.m.

For the following exercise, consider the stock market crash in 1929 in the United States. The table lists the Dow Jones industrial average per year leading up to the crash.

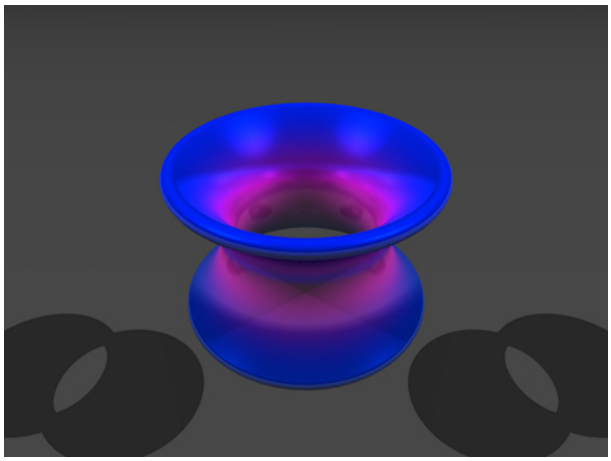
Year after 1920	Value ($)
1	63.90
3	100
5	110
7	160
9	381.17

Source: http:/stockcharts.com/freecharts/hi...a19201940.html

24) [T] The best-fit exponential curve to these data is given by $y = 40.71 + 1.224^x$. Why do you think the gains of the market were unsustainable? Use first and second derivatives to help justify your answer. What would this model predict the Dow Jones industrial average to be in **2014** ?

For exercises 25 - 26, consider the catenoid, the only solid of revolution that has a minimal surface, or zero mean curvature. A catenoid in nature can be found when stretching soap between two rings.

25) Find the volume of the catenoid $y = \cosh(x)$ from $x = -1$ to $x = 1$ that is created by rotating this curve around the x-**axis,** as shown here.



Answer

$V = \pi\left(1 + \sinh(1)\cosh(1)\right)$ units^3

26) Find surface area of the catenoid $y = \cosh(x)$ from $x = -1$ to $x = 1$ that is created by rotating this curve around the x-**axis.**

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CHAPTER OVERVIEW

7: Techniques of Integration

We saw in the previous chapter how important integration can be for all kinds of different topics—from calculations of volumes to flow rates, and from using a velocity function to determine a position to locating centers of mass. It is no surprise, then, that techniques for finding antiderivatives (or indefinite integrals) are important to know for everyone who uses them. We have already discussed some basic integration formulas and the method of integration by substitution. In this chapter, we study some additional techniques, including some ways of approximating definite integrals when normal techniques do not work.

7.0: Prelude to Techniques of Integration
7.1: Integration by Parts
7.1E: Exercises for Section 7.1
7.2: Trigonometric Integrals
7.2E: Exercises for Section 7.2
7.3: Trigonometric Substitution
7.3E: Exercises for Section 7.3
7.4: Partial Fractions
7.4E: Exercises for Section 7.4
7.5: Other Strategies for Integration
7.5E: Exercises for Section 7.5
7.6: Numerical Integration
7.6E: Exercises for Section 7.6
7.7: Improper Integrals
7.7E: Exercises for Section 7.7
7.R: Chapter 7 Review Exercises

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7.0: Prelude to Techniques of Integration

In a large city, accidents occurred at an average rate of one every three months at a particularly busy intersection. After residents complained, changes were made to the traffic lights at the intersection. It has now been eight months since the changes were made and there have been no accidents. Were the changes effective or is the eight-month interval without an accident a result of chance? We explore this question later in this chapter and see that integration is an essential part of determining the answer.

Figure 7.0.1: Careful planning of traffic signals can prevent or reduce the number of accidents at busy intersections. (credit: modification of work by David McKelvey, Flickr)

We saw in the previous chapter how important integration can be for all kinds of different topics—from calculations of volumes to flow rates, and from using a velocity function to determine a position to locating centers of mass. It is no surprise, then, that techniques for finding antiderivatives (or indefinite integrals) are important to know for everyone who uses them. We have already discussed some basic integration formulas and the method of integration by substitution. In this chapter, we study some additional techniques, including some ways of approximating definite integrals when normal techniques do not work.

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7.1: Integration by Parts

Learning Objectives

- Recognize when to use integration by parts.
- Use the integration-by-parts formula to solve integration problems.
- Use the integration-by-parts formula for definite integrals.

By now we have a fairly thorough procedure for how to evaluate many basic integrals. However, although we can integrate $\int x\sin(x^2)\,dx$ by using the substitution, $u = x^2$, something as simple looking as $\int x\sin x\;dx$ defies us. Many students want to know whether there is a product rule for integration. There is not, but there is a technique based on the product rule for differentiation that allows us to exchange one integral for another. We call this technique **integration by parts.**

The Integration-by-Parts Formula

If, $h(x) = f(x)g(x)$, then by using the product rule, we obtain

$$h'(x) = f'(x)g(x) + g'(x)f(x). \tag{7.1.1}$$

Although at first it may seem counterproductive, let's now integrate both sides of Equation 7.1.1:

$$\int h'(x)\;dx = \int (g(x)f'(x) + f(x)g'(x))\;dx.$$

This gives us

$$h(x) = f(x)g(x) = \int g(x)f'(x)\,dx + \int f(x)g'(x)\;dx.$$

Now we solve for $\int f(x)g'(x)\;dx$:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\;dx.$$

By making the substitutions $u = f(x)$ and $v = g(x)$, which in turn make $du = f'(x)\,dx$ and $dv = g'(x)\,dx$, we have the more compact form

$$\int u\,dv = uv - \int v\,du.$$

Integration by Parts

Let $u = f(x)$ and $v = g(x)$ be functions with continuous derivatives. Then, the integration-by-parts formula for the integral involving these two functions is:

$$\int u\,dv = uv - \int v\,du. \tag{7.1.2}$$

The advantage of using the integration-by-parts formula is that we can use it to exchange one integral for another, possibly easier, integral. The following example illustrates its use.

Example 7.1.1: Using Integration by Parts

Use integration by parts with $u = x$ and $dv = \sin x\;dx$ to evaluate

$$\int x\sin x\;dx.$$

Solution

By choosing $u = x$, we have $du = 1\;dx$. Since $dv = \sin x\;dx$, we get

$$v = \int \sin x \; dx = -\cos x.$$

It is handy to keep track of these values as follows:

- $u = x$
- $dv = \sin x \; dx$
- $du = 1 \, dx$
- $v = \int \sin x \; dx = -\cos x.$

Applying the integration-by-parts formula (Equation 7.1.2) results in

$$\begin{aligned} \int x \sin x \; dx &= (x)(-\cos x) - \int (-\cos x)(1 \; dx) && \text{(Substitute)} \\ &= -x \cos x + \int \cos x \; dx && \text{(Simplify)} \end{aligned}$$

Then use

$$\int \cos x \; dx = \sin x + C.$$

to obtain

$$\int x \sin x \; dx = -x \cos x + \sin x + C.$$

Analysis

At this point, there are probably a few items that need clarification. First of all, you may be curious about what would have happened if we had chosen $u = \sin x$ and $dv = x \, dx$. If we had done so, then we would have $du = \cos x$ and $v = \frac{1}{2}x^2$. Thus, after applying integration by parts (Equation 7.1.2), we have

$$\int x \sin x \; dx = \frac{1}{2}x^2 \sin x - \int \frac{1}{2}x^2 \cos x \; dx.$$

Unfortunately, with the new integral, we are in no better position than before. It is important to keep in mind that when we apply integration by parts, we may need to try several choices for u and dv before finding a choice that works.

Second, you may wonder why, when we find $v = \int \sin x \; dx = -\cos x$, we do not use $v = -\cos x + K$. To see that it makes no difference, we can rework the problem using $v = -\cos x + K$:

$$\begin{aligned} \int x \sin x \; dx &= (x)(-\cos x + K) - \int (-\cos x + K)(1 \; dx) \\ &= -x \cos x + Kx + \int \cos x \; dx - \int K \; dx \\ &= -x \cos x + Kx + \sin x - Kx + C \\ &= -x \cos x + \sin x + C. \end{aligned}$$

As you can see, it makes no difference in the final solution.

Last, we can check to make sure that our antiderivative is correct by differentiating $-x \cos x + \sin x + C$:

$$\begin{aligned} \frac{d}{dx}(-x \cos x + \sin x + C) &= \cancel{(-1)\cos x} + (-x)(-\sin x) + \cancel{\cos x} \\ &= x \sin x \end{aligned}$$

Therefore, the antiderivative checks out.

? Exercise 7.1.1

Evaluate $\int xe^{2x}\,dx$ using the integration-by-parts formula (Equation 7.1.2) with $u = x$ and $dv = e^{2x}\,dx$.

Hint

Find du and v, and use the previous example as a guide.

Answer

$$\int xe^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

The natural question to ask at this point is: How do we know how to choose u and dv? Sometimes it is a matter of trial and error; however, the acronym **LIATE** can often help to take some of the guesswork out of our choices. This acronym stands for **L**ogarithmic Functions, **I**nverse Trigonometric Functions, **A**lgebraic Functions, **T**rigonometric Functions, and **E**xponential Functions. This mnemonic serves as an aid in determining an appropriate choice for u. The type of function in the integral that appears first in the list should be our first choice of u.

For example, if an integral contains a **logarithmic function** and an **algebraic function,** we should choose u to be the logarithmic function, because L comes before A in LIATE. The integral in Example 7.1.1 has a trigonometric function ($\sin x$) and an algebraic function (x). Because A comes before T in LIATE, we chose u to be the algebraic function. When we have chosen u, dv is selected to be the remaining part of the function to be integrated, together with dx.

Why does this mnemonic work? Remember that whatever we pick to be dv must be something we can integrate. Since we do not have integration formulas that allow us to integrate simple logarithmic functions and inverse trigonometric functions, it makes sense that they should not be chosen as values for dv. Consequently, they should be at the head of the list as choices for u. Thus, we put LI at the beginning of the mnemonic. (We could just as easily have started with IL, since these two types of functions won't appear together in an integration-by-parts problem.) The exponential and trigonometric functions are at the end of our list because they are fairly easy to integrate and make good choices for dv. Thus, we have TE at the end of our mnemonic. (We could just as easily have used ET at the end, since when these types of functions appear together it usually doesn't really matter which one is u and which one is dv.) Algebraic functions are generally easy both to integrate and to differentiate, and they come in the middle of the mnemonic.

✓ Example 7.1.2: Using Integration by Parts

Evaluate

$$\int \frac{\ln x}{x^3}\,dx.$$

Solution

Begin by rewriting the integral:

$$\int \frac{\ln x}{x^3}\,dx = \int x^{-3}\ln x\,dx.$$

Since this integral contains the algebraic function x^{-3} and the logarithmic function $\ln x$, choose $u = \ln x$, since L comes before A in LIATE. After we have chosen $u = \ln x$, we must choose $dv = x^{-3}\,dx$.

Next, since $u = \ln x$, we have $du = \frac{1}{x}\,dx$. Also, $v = \int x^{-3}\,dx = -\frac{1}{2}x^{-2}$. Summarizing,

- $u = \ln x$
- $du = \frac{1}{x}\,dx$
- $dv = x^{-3}\,dx$
- $v = \int x^{-3}\,dx = -\frac{1}{2}x^{-2}$.

Substituting into the integration-by-parts formula (Equation 7.1.2) gives

$$\int \frac{\ln x}{x^3}\,dx = \int x^{-3}\ln x\,dx = (\ln x)(-\frac{1}{2}x^{-2}) - \int (-\frac{1}{2}x^{-2})(\frac{1}{x}\,dx)$$
$$= -\frac{1}{2}x^{-2}\ln x + \int \frac{1}{2}x^{-3}\,dx$$
$$= -\frac{1}{2}x^{-2}\ln x - \frac{1}{4}x^{-2} + C$$
$$= -\frac{1}{2x^2}\ln x - \frac{1}{4x^2} + C$$

? Exercise 7.1.2

Evaluate

$$\int x\ln x\ dx.$$

Hint

Use $u = \ln x$ and $dv = x\ dx$.

Answer

$$\int x\ln x\ dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$$

In some cases, as in the next two examples, it may be necessary to apply integration by parts more than once.

✓ Example 7.1.3A: Applying Integration by Parts More Than Once

Evaluate

$$\int x^2e^{3x}\,dx.$$

Solution

Using LIATE, choose $u = x^2$ and $dv = e^{3x}\,dx$. Thus, $du = 2x\,dx$ and $v = \int e^{3x}\,dx = \left(\frac{1}{3}\right)e^{3x}$. Therefore,

- $u = x^2$
- $du = 2x\,dx$
- $dv = e^{3x}\,dx$
- $v = \int e^{3x}\,dx = \frac{1}{3}e^{3x}$.

Substituting into Equation 7.1.2 produces

$$\int x^2e^{3x}\,dx = \frac{1}{3}x^2e^{3x} - \int \frac{2}{3}xe^{3x}\,dx. \tag{7.1.3}$$

We still cannot integrate $\int \frac{2}{3}xe^{3x}\,dx$ directly, but the integral now has a lower power on x. We can evaluate this new integral by using integration by parts again. To do this, choose

$$u = x$$

and

$$dv = \frac{2}{3}e^{3x}\,dx.$$

Thus,

$$du = dx$$

and

$$v = \int \left(\frac{2}{3}\right) e^{3x}\, dx = \left(\frac{2}{9}\right) e^{3x}.$$

Now we have

- $u = x$
- $du = dx$
- $dv = \frac{2}{3} e^{3x}\, dx$
- $v = \int \frac{2}{3} e^{3x}\, dx = \frac{2}{9} e^{3x}$.

Substituting back into Equation 7.1.3 yields

$$\int x^2 e^{3x}\, dx = \frac{1}{3} x^2 e^{3x} - \left(\frac{2}{9} x e^{3x} - \int \frac{2}{9} e^{3x}\, dx\right).$$

After evaluating the last integral and simplifying, we obtain

$$\int x^2 e^{3x}\, dx = \frac{1}{3} x^2 e^{3x} - \frac{2}{9} x e^{3x} + \frac{2}{27} e^{3x} + C.$$

Example 7.1.3B: Applying Integration by Parts When LIATE Does not Quite Work

Evaluate

$$\int t^3 e^{t^2} dt.$$

Solution

If we use a strict interpretation of the mnemonic LIATE to make our choice of u, we end up with $u = t^3$ and $dv = e^{t^2} dt$. Unfortunately, this choice won't work because we are unable to evaluate $\int e^{t^2} dt$. However, since we can evaluate $\int t e^{t^2}\, dx$, we can try choosing $u = t^2$ and $dv = t e^{t^2} dt$. With these choices we have

- $u = t^2$
- $du = 2t dt$
- $dv = t e^{t^2} dt$
- $v = \int t e^{t^2} dt = \frac{1}{2} e^{t^2}$.

Thus, we obtain

$$\begin{aligned} \int t^3 e^{t^2} dt &= \frac{1}{2} t^2 e^{t^2} - \int \frac{1}{2} e^{t^2} 2t\, dt \\ &= \frac{1}{2} t^2 e^{t^2} - \frac{1}{2} e^{t^2} + C. \end{aligned}$$

Example 7.1.3C: Applying Integration by Parts More Than Once

Evaluate

$$\int \sin(\ln x)\, dx.$$

Solution

This integral appears to have only one function—namely, $\sin(\ln x)$—however, we can always use the constant function 1 as the other function. In this example, let's choose $u = \sin(\ln x)$ and $dv = 1\, dx$. (The decision to use $u = \sin(\ln x)$ is easy. We can't choose $dv = \sin(\ln x)\, dx$ because if we could integrate it, we wouldn't be using integration by parts in the first place!) Consequently, $du = (1/x)\cos(\ln x)\, dx$ and $v = \int 1\, dx = x$. After applying integration by parts to the integral and simplifying, we have

$$\int \sin(\ln x)\, dx = x\sin(\ln x) - \int \cos(\ln x)\, dx.$$

Unfortunately, this process leaves us with a new integral that is very similar to the original. However, let's see what happens when we apply integration by parts again. This time let's choose $u = \cos(\ln x)$ and $dv = 1\, dx$, making $du = -(1/x)\sin(\ln x)\, dx$ and $v = \int 1\, dx = x$.

Substituting, we have

$$\int \sin(\ln x)\, dx = x\sin(\ln x) - (x\cos(\ln x) - \int -\sin(\ln x)\, dx).$$

After simplifying, we obtain

$$\int \sin(\ln x)\, dx = x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\, dx.$$

The last integral is now the same as the original. It may seem that we have simply gone in a circle, but now we can actually evaluate the integral. To see how to do this more clearly, substitute $I = \int \sin(\ln x)\, dx$. Thus, the equation becomes

$$I = x\sin(\ln x) - x\cos(\ln x) - I.$$

First, add I to both sides of the equation to obtain

$$2I = x\sin(\ln x) - x\cos(\ln x).$$

Next, divide by 2:

$$I = \frac{1}{2}x\sin(\ln x) - \frac{1}{2}x\cos(\ln x).$$

Substituting $I = \int \sin(\ln x)\, dx$ again, we have

$$\int \sin(\ln x)\, dx = \frac{1}{2}x\sin(\ln x) - \frac{1}{2}x\cos(\ln x).$$

From this we see that $(1/2)x\sin(\ln x) - (1/2)x\cos(\ln x)$ is an antiderivative of $\sin(\ln x)\, dx$. For the most general antiderivative, add $+C$:

$$\int \sin(\ln x)\, dx = \frac{1}{2}x\sin(\ln x) - \frac{1}{2}x\cos(\ln x) + C.$$

Analysis

If this method feels a little strange at first, we can check the answer by differentiation:

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{2}x\sin(\ln x) - \frac{1}{2}x\cos(\ln x)\right) &\\ = \frac{1}{2}(\sin(\ln x)) + \cos(\ln x)\cdot\frac{1}{x}\cdot\frac{1}{2}x &\\ - \left(\frac{1}{2}\cos(\ln x) - \sin(\ln x)\cdot\frac{1}{x}\cdot\frac{1}{2}x\right) &\\ = \sin(\ln x). &\end{aligned}$$

? Exercise 7.1.3

Evaluate

$$\int x^2 \sin x\, dx.$$

Hint

This is similar to Examples 7.1.3A - 7.1.3C.

Answer

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C$$

Integration by Parts for Definite Integrals

Now that we have used integration by parts successfully to evaluate *indefinite integrals*, we turn our attention to definite integrals. The integration technique is really the same, only we add a step to evaluate the integral at the upper and lower limits of integration.

Integration by Parts for Definite Integrals

Let $u = f(x)$ and $v = g(x)$ be functions with continuous derivatives on $[a, b]$. Then

$$\int_a^b u \, dv = uv\Big|_a^b - \int_a^b v \, du$$

Example 7.1.4A: Finding the Area of a Region

Find the area of the region bounded above by the graph of $y = \tan^{-1} x$ and below by the x-axis over the interval $[0, 1]$.

Solution

This region is shown in Figure 7.1.1. To find the area, we must evaluate

$$\int_0^1 \tan^{-1} x \; dx.$$

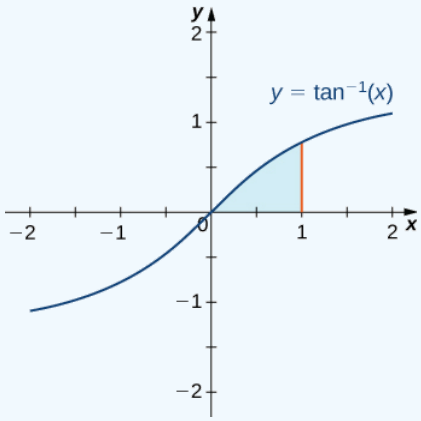

Figure 7.1.1: To find the area of the shaded region, we have to use integration by parts.

For this integral, let's choose $u = tan^{-1}x$ and $dv = dx$, thereby making $du = \dfrac{1}{x^2+1}\, dx$ and $v = x$. After applying the integration-by-parts formula (Equation 7.1.2) we obtain

$$\text{Area} = x \tan^{-1} x\Big|_0^1 - \int_0^1 \frac{x}{x^2+1}\, dx.$$

Use u-substitution to obtain

$$\int_0^1 \frac{x}{x^2+1}\, dx = \frac{1}{2}\ln\left(x^2+1\right)\Big|_0^1.$$

Thus,

$$\text{Area} = x \tan^{-1} x\Big|_0^1 - \frac{1}{2}\ln\left(x^2+1\right)\Big|_0^1 = \left(\frac{\pi}{4} - \frac{1}{2}\ln 2\right) \text{ units}^2.$$

At this point it might not be a bad idea to do a "reality check" on the reasonableness of our solution. Since $\frac{\pi}{4} - \frac{1}{2}\ln 2 \approx 0.4388$ units^2, and from Figure 7.1.1 we expect our area to be slightly less than 0.5 units^2, this solution appears to be reasonable.

Example 7.1.4B: Finding a Volume of Revolution

Find the volume of the solid obtained by revolving the region bounded by the graph of $f(x) = e^{-x}$, the x-axis, the y-axis, and the line $x = 1$ about the y-axis.

Solution

The best option to solving this problem is to use the shell method. Begin by sketching the region to be revolved, along with a typical rectangle (Figure 7.1.2).

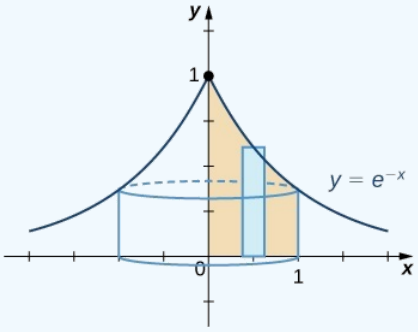

Figure 7.1.2: We can use the shell method to find a volume of revolution.

To find the volume using shells, we must evaluate

$$2\pi \int_0^1 xe^{-x}\,dx. \tag{7.1.4}$$

To do this, let $u = x$ and $dv = e^{-x}$. These choices lead to $du = dx$ and $v = \int e^{-x}\,dx = -e^{-x}$. Using the Shell Method formula, we obtain

$$\begin{aligned}
\text{Volume} &= 2\pi \int_0^1 xe^{-x}\,dx \\
&= 2\pi \left(-xe^{-x}\Big|_0^1 + \int_0^1 e^{-x}\,dx\right) && \text{(Use integration by parts)} \\
&= 2\pi \left(-e^{-1} + 0 - e^{-x}\Big|_0^1\right) \\
&= 2\pi \left(-e^{-1} - e^{-1} + 1\right) \\
&= 2\pi \left(1 - \frac{2}{e}\right) && \text{(Evaluate and simplify)} \\
&\quad \text{units}^3.
\end{aligned}$$

Analysis

Again, it is a good idea to check the reasonableness of our solution. We observe that the solid has a volume slightly less than that of a cylinder of radius 1 and height of $1/e$ added to the volume of a cone of base radius 1 and height of $1 - \frac{1}{e}$. Consequently, the solid should have a volume a bit less than

$$\pi(1)^2 \frac{1}{e} + \left(\frac{\pi}{3}\right)(1)^2 \left(1 - \frac{1}{e}\right) = \frac{2\pi}{3e} + \frac{\pi}{3} \approx 1.8177 \text{ units}^3.$$

Since $2\pi - \frac{4\pi}{e} \approx 1.6603$, we see that our calculated volume is reasonable.

? Exercise 7.1.4

Evaluate

$$\int_0^{\pi/2} x \cos x \, dx.$$

Hint

Use Equation 7.1.2 with $u = x$ and $dv = \cos x \, dx$.

Answer

$$\int_0^{\pi/2} x \cos x \, dx = \frac{\pi}{2} - 1$$

Key Concepts

- The integration-by-parts formula (Equation 7.1.2) allows the exchange of one integral for another, possibly easier, integral.
- Integration by parts applies to both definite and indefinite integrals.

Key Equations

- **Integration by parts formula**

  $$\int u \, dv = uv - \int v \, du$$

- **Integration by parts for definite integrals**

  $$\int_a^b u \, dv = uv\Big|_a^b - \int_a^b v \, du$$

Glossary

integration by parts

a technique of integration that allows the exchange of one integral for another using the formula $\int u \, dv = uv - \int v \, du$

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7.1E: EXERCISES FOR SECTION 7.1

In using the technique of integration by parts, you must carefully choose which expression is u. For each of the following problems, use the guidelines in this section to choose u. Do not evaluate the integrals.

1) $\int x^3 e^{2x}\,dx$

Answer

$u = x^3$

2) $\int x^3 \ln(x)\,dx$

3) $\int y^3 \cos y\,dy$

Answer

$u = y^3$

4) $\int x^2 \arctan x\,dx$

5) $\int e^{3x} \sin(2x)\,dx$

Answer

$u = \sin(2x)$

In exercises 6 - 37, find the integral by using the simplest method. Not all problems require integration by parts.

6) $\int v \sin v\,dv$

7) $\int \ln x\,dx$ (Hint: $\int \ln x\,dx$ is equivalent to $\int 1 \cdot \ln(x)\,dx$.)

Answer

$$\int \ln x\,dx \quad = \quad -x + x\ln x + C$$

8) $\int x \cos x\,dx$

9) $\int \tan^{-1} x\,dx$

Answer

$$\int \tan^{-1} x\,dx \quad = \quad x\tan^{-1} x - \tfrac{1}{2}\ln(1+x^2) + C$$

10) $\int x^2 e^x\,dx$

11) $\int x \sin(2x)\,dx$

Answer

$$\int x \sin(2x)\,dx \quad = \quad -\tfrac{1}{2}x\cos(2x) + \tfrac{1}{4}\sin(2x) + C$$

12) $\int x e^{4x}\,dx$

13) $\int x e^{-x}\,dx$

Answer

$$\int x e^{-x}\,dx \quad = \quad e^{-x}(-1-x) + C$$

14) $\int x \cos 3x\,dx$

https://math.libretexts.org/@go/page/70409

15) $\int x^2 \cos x \, dx$

Answer

$$\int x^2 \cos x \, dx = 2x \cos x + (-2 + x^2) \sin x + C$$

16) $\int x \ln x \, dx$

17) $\int \ln(2x+1) \, dx$

Answer

$$\int \ln(2x+1) \, dx = \tfrac{1}{2}(1+2x)(-1+\ln(1+2x)) + C$$

18) $\int x^2 e^{4x} \, dx$

19) $\int e^x \sin x \, dx$

Answer

$$\int e^x \sin x \, dx = \tfrac{1}{2} e^x (-\cos x + \sin x) + C$$

20) $\int e^x \cos x \, dx$

21) $\int x e^{-x^2} \, dx$

Answer

$$\int x e^{-x^2} \, dx = -\frac{e^{-x^2}}{2} + C$$

22) $\int x^2 e^{-x} \, dx$

23) $\int \sin(\ln(2x)) \, dx$

Answer

$$\int \sin(\ln(2x)) \, dx = -\tfrac{1}{2} x \cos[\ln(2x)] + \tfrac{1}{2} x \sin[\ln(2x)] + C$$

24) $\int \cos(\ln x) \, dx$

25) $\int (\ln x)^2 \, dx$

Answer

$$\int (\ln x)^2 \, dx = 2x - 2x \ln x + x (\ln x)^2 + C$$

26) $\int \ln(x^2) \, dx$

27) $\int x^2 \ln x \, dx$

Answer

$$\int x^2 \ln x \, dx = -\frac{x^3}{9} + \tfrac{1}{3} x^3 \ln x + C$$

28) $\int \sin^{-1} x \, dx$

29) $\int \cos^{-1}(2x) \, dx$

Answer

$$\int \cos^{-1}(2x) \, dx = -\tfrac{1}{2} \sqrt{1 - 4x^2} + x \cos^{-1}(2x) + C$$

30) $\int x \arctan x \, dx$

31) $\int x^2 \sin x \, dx$

Answer

$$\int x^2 \sin x \, dx \quad = \quad -(-2+x^2)\cos x + 2x \sin x + C$$

32) $\int x^3 \cos x \, dx$

33) $\int x^3 \sin x \, dx$

Answer

$$\int x^3 \sin x \, dx \quad = \quad -x(-6+x^2)\cos x + 3(-2+x^2)\sin x + C$$

34) $\int x^3 e^x \, dx$

35) $\int x \sec^{-1} x \, dx$

Answer

$$\int x \sec^{-1} x \, dx \quad = \quad \tfrac{1}{2}x\left(-\sqrt{1-\frac{1}{x^2}} + x \cdot \sec^{-1} x\right) + C$$

36) $\int x \sec^2 x \, dx$

37) $\int x \cosh x \, dx$

Answer

$$\int x \cosh x \, dx \quad = \quad -\cosh x + x \sinh x + C$$

In exercises 38 - 46, compute the definite integrals. Use a graphing utility to confirm your answers.

38) $\int_{1/e}^{1} \ln x \, dx$

39) $\int_0^1 x e^{-2x} \, dx$ (Express the answer in exact form.)

Answer

$$\int_0^1 x e^{-2x} \, dx \quad = \quad \frac{1}{4} - \frac{3}{4e^2}$$

40) $\int_0^1 e^{\sqrt{x}} \, dx \quad (\text{let } u = \sqrt{x})$

41) $\int_1^e \ln(x^2) \, dx$

Answer

$$\int_1^e \ln(x^2) \, dx \quad = \quad 2$$

42) $\int_0^{\pi} x \cos x \, dx$

43) $\int_{-\pi}^{\pi} x \sin x \, dx$ (Express the answer in exact form.)

Answer

$$\int_{-\pi}^{\pi} x \sin x \, dx \quad = \quad 2\pi$$

44) $\int_0^3 \ln(x^2+1) \, dx$ (Express the answer in exact form.)

45) $\int_0^{\pi/2} x^2 \sin x \, dx$ (Express the answer in exact form.)

Answer

$$\int_0^{\pi/2} x^2 \sin x \, dx \quad = \quad -2 + \pi$$

46) $\int_0^1 x 5^x \, dx$ (Express the answer using five significant digits.)

47) Evaluate $\int \cos x \ln(\sin x) \, dx$

Answer

$$\int \cos x \ln(\sin x) \, dx \quad = \quad -\sin(x) + \ln[\sin(x)] \sin x + C$$

In exercises 48 - 50, derive the following formulas using the technique of integration by parts. Assume that n is a positive integer. These formulas are called *reduction formulas* because the exponent in the x term has been reduced by one in each case. The second integral is simpler than the original integral.

48) $\int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx$

49) $\int x^n \cos x \, dx = x^n \sin x - n \int x^{n-1} \sin x \, dx$

Answer

Answers vary

50) $\int x^n \sin x \, dx =$_____

51) Integrate $\int 2x\sqrt{2x-3} \, dx$ using two methods:

a. Using parts, letting $dv = \sqrt{2x-3} \, dx$

b. Substitution, letting $u = 2x - 3$

Answer

a. $\int 2x\sqrt{2x-3} \, dx \quad = \quad \frac{2}{5}(1+x)(-3+2x)^{3/2} + C$

b. $\int 2x\sqrt{2x-3} \, dx \quad = \quad \frac{2}{5}(1+x)(-3+2x)^{3/2} + C$

In exercises 52 - 57, state whether you would use integration by parts to evaluate the integral. If so, identify u and dv. If not, describe the technique used to perform the integration without actually doing the problem.

52) $\int x \ln x \, dx$

53) $\int \frac{\ln^2 x}{x} \, dx$

Answer

Do not use integration by parts. Choose u to be $\ln x$, and the integral is of the form $\int u^2 \, du$.

54) $\int x e^x \, dx$

55) $\int x e^{x^2-3} \, dx$

Answer

Do not use integration by parts. Let $u = x^2 - 3$, and the integral can be put into the form $\int e^u \, du$.

56) $\int x^2 \sin x \, dx$

57) $\int x^2 \sin(3x^3 + 2) \, dx$

Answer

Do not use integration by parts. Choose u to be $u = 3x^3 + 2$ and the integral can be put into the form $\int \sin(u)\, du$.

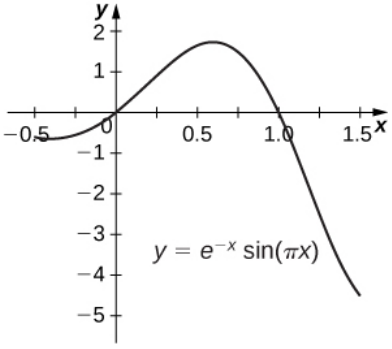
In exercises 58-59, sketch the region bounded above by the curve, the x-axis, and $x = 1$, and find the area of the region. Provide the exact form or round answers to the number of places indicated.

58) $y = 2xe^{-x}$ (Approximate answer to four decimal places.)

59) $y = e^{-x}\sin(\pi x)$ (Approximate answer to five decimal places.)

Answer

The area under graph is 0.39535 units^2.



In exercises 60 - 61, find the volume generated by rotating the region bounded by the given curves about the specified line. Express the answers in exact form or approximate to the number of decimal places indicated.

60) $y = \sin x$, $y = 0$, $x = 2\pi$, $x = 3\pi$; about the y-axis (Express the answer in exact form.)

61) $y = e^{-x}$, $y = 0$, $x = -1$, $x = 0$; about $x = 1$ (Express the answer in exact form.)

Answer

$V = 2\pi e \text{ units}^3$

62) A particle moving along a straight line has a velocity of $v(t) = t^2e^{-t}$ after t sec. How far does it travel in the first 2 sec? (Assume the units are in feet and express the answer in exact form.)

63) Find the area under the graph of $y = \sec^3 x$ from $x = 0$ to $x = 1$. (Round the answer to two significant digits.)

Answer

$A = 2.05 \text{ units}^2$

64) Find the area between $y = (x - 2)e^x$ and the x-axis from $x = 2$ to $x = 5$. (Express the answer in exact form.)

65) Find the area of the region enclosed by the curve $y = x\cos x$ and the x-axis for $\frac{11\pi}{2} \le x \le \frac{13\pi}{2}$. (Express the answer in exact form.)

Answer

$A = 12\pi \text{ units}^2$

66) Find the volume of the solid generated by revolving the region bounded by the curve $y = \ln x$, the x-axis, and the vertical line $x = e^2$ about the x-axis. (Express the answer in exact form.)

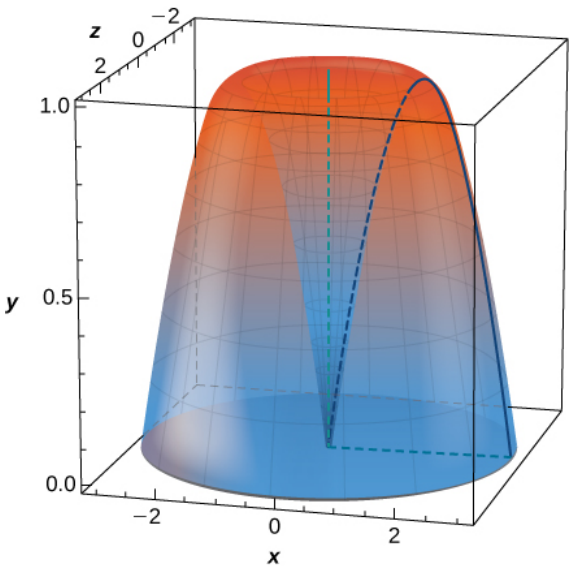
67) Find the volume of the solid generated by revolving the region bounded by the curve $y = 4\cos x$ and the x-axis, $\frac{\pi}{2} \le x \le \frac{3\pi}{2}$, about the x-axis. (Express the answer in exact form.)

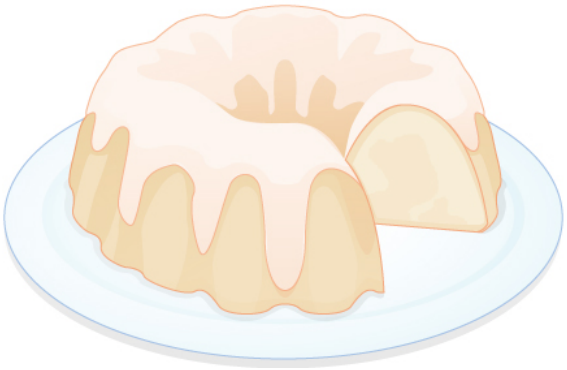
Answer

$V = 8\pi^2 \text{ units}^3$

68) Find the volume of the solid generated by revolving the region in the first quadrant bounded by $y = e^x$ and the x-axis, from $x = 0$ to $x = \ln(7)$, about the y-axis. (Express the answer in exact form.)

69) What is the volume of the Bundt cake that comes from rotating $y = \sin x$ around the y-axis from $x = 0$ to $x = \pi$?





Answer

$V = 2\pi^2$ units^3

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7.2: Trigonometric Integrals

Learning Objectives

- Solve integration problems involving products and powers of $\sin x$ and $\cos x$.
- Solve integration problems involving products and powers of $\tan x$ and $\sec x$.
- Use reduction formulas to solve trigonometric integrals.

In this section we look at how to integrate a variety of products of trigonometric functions. These integrals are called **trigonometric integrals**. They are an important part of the integration technique called *trigonometric substitution*, which is featured in Trigonometric Substitution. This technique allows us to convert algebraic expressions that we may not be able to integrate into expressions involving trigonometric functions, which we may be able to integrate using the techniques described in this section. In addition, these types of integrals appear frequently when we study polar, cylindrical, and spherical coordinate systems later. Let's begin our study with products of $\sin x$ and $\cos x$.

Integrating Products and Powers of sin x and cos x

A key idea behind the strategy used to integrate combinations of products and powers of $\sin x$ and $\cos x$ involves rewriting these expressions as sums and differences of integrals of the form $\int \sin^j x \cos x\, dx$ or $\int \cos^j x \sin x\, dx$. After rewriting these integrals, we evaluate them using u-substitution. Before describing the general process in detail, let's take a look at the following examples.

Example 7.2.1: Integrating $\int \cos^j x \sin x\, dx$

Evaluate $\int \cos^3 x \sin x\, dx$.

Solution

Use u-substitution and let $u = \cos x$. In this case, $du = -\sin x\, dx$.

Thus,

$$\int \cos^3 x \sin x\, dx = -\int u^3\, du = -\frac{1}{4}u^4 + C = -\frac{1}{4}\cos^4 x + C.$$

Exercise 7.2.1

Evaluate $\int \sin^4 x \cos x\, dx$.

Hint

Let $u = \sin x$.

Answer

$$\int \sin^4 x \cos x\, dx = \frac{1}{5}\sin^5 x + C$$

Example 7.2.2: A Preliminary Example: Integrating $\int \cos^j x \sin^k x\, dx$ where k is Odd

Evaluate $\int \cos^2 x \sin^3 x\, dx$.

Solution

To convert this integral to integrals of the form $\int \cos^j x \sin x\, dx$, rewrite $\sin^3 x = \sin^2 x \sin x$ and make the substitution $\sin^2 x = 1 - \cos^2 x$.

Thus,

$$\int \cos^2 x \sin^3 x\, dx = \int \cos^2 x(1-\cos^2 x)\sin x\, dx \qquad \text{Let } u = \cos x;\ \text{then } du = -\sin x\, dx.$$

$$= -\int u^2(1-u^2)\, du$$

$$= \int (u^4 - u^2)\, du$$

$$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C$$

$$= \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C.$$

? Exercise 7.2.2

Evaluate $\int \cos^3 x \sin^2 x\, dx$.

Hint

Write $\cos^3 x = \cos^2 x \cos x = (1-\sin^2 x)\cos x$ and let $u = \sin x$.

Answer

$$\int \cos^3 x \sin^2 x\, dx = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$$

In the next example, we see the strategy that must be applied when there are only even powers of $\sin x$ and $\cos x$. For integrals of this type, the identities

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x) = \frac{1-\cos(2x)}{2}$$

and

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x) = \frac{1+\cos(2x)}{2}$$

are invaluable. These identities are sometimes known as **power-reducing identities** and they may be derived from the double-angle identity $\cos(2x) = \cos^2 x - \sin^2 x$ and the Pythagorean identity $\cos^2 x + \sin^2 x = 1$.

✓ Example 7.2.3: Integrating an Even Power of $\sin x$

Evaluate $\int \sin^2 x\, dx$.

Solution

To evaluate this integral, let's use the trigonometric identity $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$. Thus,

$$\int \sin^2 x\, dx = \int \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right) dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C.$$

? Exercise 7.2.3

Evaluate $\int \cos^2 x\, dx$.

Hint

$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$

Answer

$$\int \cos^2 x\, dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$$

The general process for integrating products of powers of $\sin x$ and $\cos x$ is summarized in the following set of guidelines.

Problem-Solving Strategy: Integrating Products and Powers of $\sin x$ and $cosx$

To integrate $\int \cos^j x \sin^k x\, dx$ use the following strategies:

1. If k is odd, rewrite $\sin^k x = \sin^{k-1} x \sin x$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to rewrite $\sin^{k-1} x$ in terms of $\cos x$. Integrate using the substitution $u = \cos x$. This substitution makes $du = -\sin x\, dx$.

2. If j is odd, rewrite $\cos^j x = \cos^{j-1} x \cos x$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to rewrite $\cos^{j-1} x$ in terms of $\sin x$. Integrate using the substitution $u = \sin x$. This substitution makes $du = \cos x\, dx$. (Note: If both j and k are odd, either strategy 1 or strategy 2 may be used.)

3. If both j and k are even, use $\sin^2 x = \dfrac{1 - \cos(2x)}{2}$ and $\cos^2 x = \dfrac{1 + \cos(2x)}{2}$. After applying these formulas, simplify and reapply strategies 1 through 3 as appropriate.

Example 7.2.4: Integrating $\int \cos^j x \sin^k x\, dx$ where k is Odd

Evaluate $\int \cos^8 x \sin^5 x\, dx$.

Solution

Since the power on $\sin x$ is odd, use strategy 1. Thus,

$$\begin{aligned}
\int \cos^8 x \sin^5 x\, dx &= \int \cos^8 x \sin^4 x \sin x\, dx && \text{Break off } \sin x.\\
&= \int \cos^8 x (\sin^2 x)^2 \sin x\, dx && \text{Rewrite } \sin^4 x = (\sin^2 x)^2.\\
&= \int \cos^8 x (1 - \cos^2 x)^2 \sin x\, dx && \text{Substitute } \sin^2 x = 1 - \cos^2 x.\\
&= \int u^8 (1 - u^2)^2 (-du) && \text{Let } u = \cos x \text{ and } du = -\sin x\, dx.\\
&= \int (-u^8 + 2u^{10} - u^{12}) du && \text{Expand.}\\
&= -\frac{1}{9}u^9 + \frac{2}{11}u^{11} - \frac{1}{13}u^{13} + C && \text{Evaluate the integral.}\\
&= -\frac{1}{9}\cos^9 x + \frac{2}{11}\cos^{11} x - \frac{1}{13}\cos^{13} x + C && \text{Substitute } u = \cos x.
\end{aligned}$$

Example 7.2.5: Integrating $\int \cos^j x \sin^k x\, dx$ where k and j are Even

Evaluate $\int \sin^4 x\, dx$.

Solution: Since the power on $\sin x$ is even $(k = 4)$ and the power on $\cos x$ is even $(j = 0)$, we must use strategy 3. Thus,

$$\begin{aligned}
\int \sin^4 x\, dx &= \int \left(\sin^2 x\right)^2 dx && \text{Rewrite } \sin^4 x = \left(\sin^2 x\right)^2.\\
&= \int \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)^2 dx && \text{Substitute } \sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x).\\
&= \int \left(\frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{1}{4}\cos^2(2x)\right) dx && \text{Expand } \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)^2.\\
&= \int \left(\frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}\cos(4x)\right)\right) && \text{Use Power Reduction Formula on } \cos^2(2x).\\
&= \int \left(\frac{3}{8} - \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)\right) dx && \text{Simplify.}\\
&= \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C && \text{Evaluate the integral.}
\end{aligned}$$

Exercise 7.2.4

Evaluate $\int \cos^3 x\, dx$.

Hint

Use strategy 2. Write $\cos^3 x = \cos^2 x \cos x$ and substitute $\cos^2 x = 1 - \sin^2 x$.

Answer

$$\int \cos^3 x\, dx = \sin x - \frac{1}{3}\sin^3 x + C$$

Exercise 7.2.5

Evaluate $\int \cos^2(3x)\, dx$.

Hint

Use strategy 3. Substitute $\cos^2(3x) = \frac{1}{2} + \frac{1}{2}\cos(6x)$

Answer

$$\int \cos^2(3x)\, dx = \frac{1}{2}x + \frac{1}{12}\sin(6x) + C$$

Product to Sum Formulas

In some areas of physics, such as quantum mechanics, signal processing, and the computation of Fourier series, it is often necessary to integrate products that include $sin(ax), sin(bx), cos(ax),$ and $cos(bx)$. These integrals are evaluated by applying trigonometric identities, as outlined in the following rule.

Rule: Integrating Products of Sines and Cosines of Different Angles

To integrate products involving $\sin(ax)$, $\sin(bx)$, $\cos(ax)$, and $\cos(bx)$, use the substitutions

$$\sin(ax)\sin(bx) = \frac{1}{2}\cos((a-b)x) - \frac{1}{2}\cos((a+b)x)$$

$$\sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x)$$

$$\cos(ax)\cos(bx) = \frac{1}{2}\cos((a-b)x) + \frac{1}{2}\cos((a+b)x)$$

These formulas may be derived from the sum-of-angle formulas for sine and cosine.

Example 7.2.6: Evaluating $\int \sin(ax)\cos(bx)\, dx$

Evaluate $\int \sin(5x)\cos(3x)\, dx$.

Solution: Apply the identity $\sin(5x)\cos(3x) = \frac{1}{2}\sin(2x) + \frac{1}{2}\sin(8x)$. Thus,

$$\int \sin(5x)\cos(3x)\, dx = \int \frac{1}{2}\sin(2x) + \frac{1}{2}\sin(8x)\, dx = -\frac{1}{4}\cos(2x) - \frac{1}{16}\cos(8x) + C.$$

Exercise 7.2.6

Evaluate $\int \cos(6x)\cos(5x)\, dx$.

Hint

Substitute $\cos(6x)\cos(5x) = \frac{1}{2}\cos x + \frac{1}{2}\cos(11x)$.

Answer

$$\int \cos(6x)\cos(5x)\,dx = \frac{1}{2}\sin x + \frac{1}{22}\sin(11x) + C$$

Integrating Products and Powers of tan x and sec x

Before discussing the integration of products and powers of $\tan x$ and $\sec x$, it is useful to recall the integrals involving $\tan x$ and $\sec x$ we have already learned:

1. $\int \sec^2 x\,dx = \tan x + C$
2. $\int \sec x \tan x\,dx = \sec x + C$
3. $\int \tan x\,dx = \ln|\sec x| + C$
4. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$.

For most integrals of products and powers of $\tan x$ and $\sec x$, we rewrite the expression we wish to integrate as the sum or difference of integrals of the form $\int \tan^j x \sec^2 x\,dx$ or $\int \sec^j x \tan x\,dx$. As we see in the following example, we can evaluate these new integrals by using u-substitution.

Example 7.2.7: Evaluating $\int \sec^j x \tan x\,dx$

Evaluate $\int \sec^5 x \tan x\,dx$.

Solution: Start by rewriting $\sec^5 x \tan x$ as $\sec^4 x \sec x \tan x$.

$$\begin{aligned}
\int \sec^5 x \tan x\,dx &= \int \sec^4 x \sec x \tan x\,dx \\
&= \int u^4\,du && \text{Let } u = \sec x;\ \text{then, } du = \sec x \tan x\,dx. \\
&= \tfrac{1}{5}u^5 + C && \text{Evaluate the integral.} \\
&= \tfrac{1}{5}\sec^5 x + C && \text{Substitute } \sec x = u.
\end{aligned}$$

You can read some interesting information at this website to learn about a common integral involving the secant.

Exercise 7.2.7

Evaluate $\int \tan^5 x \sec^2 x\,dx$.

Hint

Let $u = \tan x$ and $du = \sec^2 x$.

Answer

$$\int \tan^5 x \sec^2 x\,dx = \tfrac{1}{6}\tan^6 x + C$$

We now take a look at the various strategies for integrating products and powers of $\sec x$ and $\tan x$.

Problem-Solving Strategy: Integrating $\int \tan^k x \sec^j x\, dx$

To integrate $\int \tan^k x \sec^j x\, dx$, use the following strategies:

1. If j is even and $j \geq 2$, rewrite $\sec^j x = \sec^{j-2} x \sec^2 x$ and use $\sec^2 x = \tan^2 x + 1$ to rewrite $\sec^{j-2} x$ in terms of $\tan x$. Let $u = \tan x$ and $du = \sec^2 x$.
2. If k is odd and $j \geq 1$, rewrite $\tan^k x \sec^j x = \tan^{k-1} x \sec^{j-1} x \sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to rewrite $\tan^{k-1} x$ in terms of $\sec x$. Let $u = \sec x$ and $du = \sec x \tan x\, dx$. (Note: If j is even and k is odd, then either strategy 1 or strategy 2 may be used.)
3. If k is odd where $k \geq 3$ and $j = 0$, rewrite $\tan^k x = \tan^{k-2} x \tan^2 x = \tan^{k-2} x(\sec^2 x - 1) = \tan^{k-2} x \sec^2 x - \tan^{k-2} x$. It may be necessary to repeat this process on the $\tan^{k-2} x$ term.
4. If k is even and j is odd, then use $\tan^2 x = \sec^2 x - 1$ to express $\tan^k x$ in terms of $\sec x$. Use integration by parts to integrate odd powers of $\sec x$.

Example 7.2.8: Integrating $\int \tan^k x \sec^j x\, dx$ when j is Even

Evaluate $\int \tan^6 x \sec^4 x\, dx$.

Solution

Since the power on $\sec x$ is even, rewrite $\sec^4 x = \sec^2 x \sec^2 x$ and use $\sec^2 x = \tan^2 x + 1$ to rewrite the first $\sec^2 x$ in terms of $\tan x$. Thus,

$$\begin{aligned}
\int \tan^6 x \sec^4 x\, dx &= \int \tan^6 x(\tan^2 x + 1)\sec^2 x\, dx \\
&= \int u^6(u^2+1)\, du && \text{Let } u = \tan x \text{ and } du = \sec^2 x. \\
&= \int (u^8 + u^6)\, du && \text{Expand.} \\
&= \frac{1}{9}u^9 + \frac{1}{7}u^7 + C && \text{Evaluate the integral.} \\
&= \frac{1}{9}\tan^9 x + \frac{1}{7}\tan^7 x + C. && \text{Substitute } \tan x = u.
\end{aligned}$$

Example 7.2.9: Integrating $\int \tan^k x \sec^j x\, dx$ when k is Odd

Evaluate $\int \tan^5 x \sec^3 x\, dx$.

Solution

Since the power on $\tan x$ is odd, begin by rewriting $\tan^5 x \sec^3 x = \tan^4 x \sec^2 x \sec x \tan x$. Thus,

$$\begin{aligned}
\int \tan^5 x \sec^3 x\, dx &= \int \tan^4 x \sec^2 x \sec x \tan x\, dx. \\
&= \int (\tan^2 x)^2 \sec^2 x \sec x \tan x\, dx && \text{Write } \tan^4 x = (\tan^2 x)^2. \\
&= \int (\sec^2 x - 1)^2 \sec^2 x \sec x \tan x\, dx && \text{Use } \tan^2 x = \sec^2 x - 1. \\
&= \int (u^2-1)^2 u^2 du && \text{Let } u = \sec x \text{ and } du = \sec x \tan x\, dx \\
&= \int (u^6 - 2u^4 + u^2) du && \text{Expand.} \\
&= \frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3 + C && \text{Integrate.} \\
&= \frac{1}{7}\sec^7 x - \frac{2}{5}\sec^5 x + \frac{1}{3}\sec^3 x + C && \text{Substitute } \sec x = u.
\end{aligned}$$

Example 7.2.10: Integrating $\int \tan^k x\,dx$ where k is Odd and $k \geq 3$

Evaluate $\int \tan^3 x\,dx$.

Solution

Begin by rewriting $\tan^3 x = \tan x \tan^2 x = \tan x(\sec^2 x - 1) = \tan x \sec^2 x - \tan x$. Thus,

$$\begin{aligned}\int \tan^3 x\,dx &= \int (\tan x \sec^2 x - \tan x)\,dx \\ &= \int \tan x \sec^2 x\,dx - \int \tan x\,dx \\ &= \frac{1}{2}\tan^2 x - \ln|\sec x| + C.\end{aligned}$$

For the first integral, use the substitution $u = \tan x$. For the second integral, use the formula.

Example 7.2.11: Integrating $\int \sec^3 x\,dx$

Integrate $\int \sec^3 x\,dx$.

Solution

This integral requires integration by parts. To begin, let $u = \sec x$ and $dv = \sec^2 x$. These choices make $du = \sec x \tan x$ and $v = \tan x$. Thus,

$$\begin{aligned}\int \sec^3 x\,dx &= \sec x \tan x - \int \tan x \sec x \tan x\,dx \\ &= \sec x \tan x - \int \tan^2 x \sec x\,dx && \text{Simplify.} \\ &= \sec x \tan x - \int (\sec^2 x - 1)\sec x\,dx && \text{Substitute } \tan^2 x = \sec^2 x - 1. \\ &= \sec x \tan x + \int \sec x\,dx - \int \sec^3 x\,dx && \text{Rewrite.} \\ &= \sec x \tan x + \ln|\sec x + \tan x| - \int \sec^3 x\,dx. && \text{Evaluate } \int \sec x\,dx.\end{aligned}$$

We now have

$$\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x| - \int \sec^3 x\,dx.$$

Since the integral $\int \sec^3 x\,dx$ has reappeared on the right-hand side, we can solve for $\int \sec^3 x\,dx$ by adding it to both sides. In doing so, we obtain

$$2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x|.$$

Dividing by 2, we arrive at

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$$

Exercise 7.2.8

Evaluate $\int \tan^3 x \sec^7 x\,dx$.

Hint

Use Example 7.2.9 as a guide.

Answer

$$\int \tan^3 x \sec^7 x \, dx = \frac{1}{9}\sec^9 x - \frac{1}{7}\sec^7 x + C$$

Reduction Formulas

Evaluating $\int \sec^n x \, dx$ for values of n where n is odd requires integration by parts. In addition, we must also know the value of $\int \sec^{n-2} x \, dx$ to evaluate $\int \sec^n x \, dx$. The evaluation of $\int \tan^n x \, dx$ also requires being able to integrate $\int \tan^{n-2} x \, dx$. To make the process easier, we can derive and apply the following power reduction formulas. These rules allow us to replace the integral of a power of $\sec x$ or $\tan x$ with the integral of a lower power of $\sec x$ or $\tan x$.

Rule: Reduction Formulas for $\int \sec^n x \, dx$ and $\int \tan^n x \, dx$

$$\int \sec^n x \, dx = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x \, dx$$

$$\int \tan^n x \, dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x \, dx$$

The first power reduction rule may be verified by applying integration by parts. The second may be verified by following the strategy outlined for integrating odd powers of $\tan x$.

Example 7.2.12: Revisiting $\int \sec^3 x \, dx$

Apply a reduction formula to evaluate $\int \sec^3 x \, dx$.

Solution: By applying the first reduction formula, we obtain

$$\begin{aligned}\int \sec^3 x \, dx &= \frac{1}{2}\sec x \tan x + \frac{1}{2}\int \sec x \, dx \\ &= \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.\end{aligned}$$

Example 7.2.13: Using a Reduction Formula

Evaluate $\int \tan^4 x \, dx$.

Solution: Applying the reduction formula for $\int \tan^4 x \, dx$ we have

$$\begin{aligned}\int \tan^4 x \, dx &= \frac{1}{3}\tan^3 x - \int \tan^2 x \, dx \\ &= \frac{1}{3}\tan^3 x - (\tan x - \int \tan^0 x \, dx) && \text{Apply the reduction formula to } \int \tan^2 x \, dx. \\ &= \frac{1}{3}\tan^3 x - \tan x + \int 1 \, dx && \text{Simplify.} \\ &= \frac{1}{3}\tan^3 x - \tan x + x + C && \text{Evaluate } \int 1 \, dx\end{aligned}$$

Exercise 7.2.9

Apply the reduction formula to $\int \sec^5 x \, dx$.

Hint

Use reduction formula 1 and let $n = 5$.

Answer

$$\int \sec^5 x\,dx = \frac{1}{4}\sec^3 x\tan x + \frac{3}{4}\int \sec^3 x$$

Key Concepts

Integrals of trigonometric functions can be evaluated by the use of various strategies. These strategies include

1. Applying trigonometric identities to rewrite the integral so that it may be evaluated by u-substitution
2. Using integration by parts
3. Applying trigonometric identities to rewrite products of sines and cosines with different arguments as the sum of individual sine and cosine functions
4. Applying reduction formulas

Key Equations

To integrate products involving $\sin(ax)$, $\sin(bx)$, $\cos(ax)$, and $\cos(bx)$, use the substitutions.

- **Sine Products**

  $$\sin(ax)\sin(bx) = \tfrac{1}{2}\cos((a-b)x) - \tfrac{1}{2}\cos((a+b)x)$$

- **Sine and Cosine Products**

  $$\sin(ax)\cos(bx) = \tfrac{1}{2}\sin((a-b)x) + \tfrac{1}{2}\sin((a+b)x)$$

- **Cosine Products**

  $$\cos(ax)\cos(bx) = \tfrac{1}{2}\cos((a-b)x) + \tfrac{1}{2}\cos((a+b)x)$$

- **Power Reduction Formula**

  $$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2} x\tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$$

- **Power Reduction Formula**

  $$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx$$

Glossary

power reduction formula
a rule that allows an integral of a power of a trigonometric function to be exchanged for an integral involving a lower power

trigonometric integral
an integral involving powers and products of trigonometric functions

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7.2E: Exercises for Section 7.2

Fill in the blank to make a true statement.

1) $\sin^2 x + \_\_\_\_\_ = 1$

Answer

$\cos^2 x$

2) $\sec^2 x - 1 = \_\_\_\_\_$

Answer

$\tan^2 x$

Use an identity to reduce the power of the trigonometric function to a trigonometric function raised to the first power.

3) $\sin^2 x = \_\_\_\_\_$

Answer

$\dfrac{1-\cos(2x)}{2}$

4) $\cos^2 x = \_\_\_\_\_$

Answer

$\dfrac{1+\cos(2x)}{2}$

Evaluate each of the following integrals by u-substitution.

5) $\int \sin^3 x \cos x \, dx$

Answer

$$\int \sin^3 x \cos x \, dx = \frac{\sin^4 x}{4} + C$$

6) $\int \sqrt{\cos x} \sin x \, dx$

7) $\int \tan^5(2x) \sec^2(2x) \, dx$

Answer

$$\int \tan^5(2x) \sec^2(2x) \, dx = \tfrac{1}{12}\tan^6(2x) + C$$

8) $\int \sin^7(2x) \cos(2x) \, dx$

9) $\int \tan(\frac{x}{2}) \sec^2(\frac{x}{2}) \, dx$

Answer

$$\int \tan(\frac{x}{2}) \sec^2(\frac{x}{2}) \, dx = \tan^2(\frac{x}{2}) + C$$

10) $\int \tan^2 x \sec^2 x \, dx$

Compute the following integrals using the guidelines for integrating powers of trigonometric functions. Use a CAS to check the solutions. (*Note*: Some of the problems may be done using techniques of integration learned previously.)

11) $\int \sin^3 x \, dx$

Answer

$$\int \sin^3 x\,dx \quad = -\cos x + \frac{\cos^3 x}{3} + C$$

12) $\int \cos^3 x\,dx$

13) $\int \sin x \cos x\,dx$

Answer

$$\int \sin x \cos x\,dx \quad = \quad -\tfrac{1}{2}\cos^2 x + C$$

14) $\int \cos^5 x\,dx$

15) $\int \sin^5 x \cos^2 x\,dx$

Answer

$$\int \sin^5 x \cos^2 x\,dx \quad = \quad -\frac{5\cos x}{64} - \tfrac{1}{192}\cos(3x) + \tfrac{3}{320}\cos(5x) - \tfrac{1}{448}\cos(7x) + C$$

16) $\int \sin^3 x \cos^3 x\,dx$

17) $\int \sqrt{\sin x}\cos x\,dx$

Answer

$$\int \sqrt{\sin x}\cos x\,dx \quad = \quad \tfrac{2}{3}(\sin x)^{3/2} + C$$

18) $\int \sqrt{\sin x}\cos^3 x\,dx$

19) $\int \sec x \tan x\,dx$

Answer

$$\int \sec x \tan x\,dx \quad = \quad \sec x + C$$

20) $\int \tan(5x)\,dx$

21) $\int \tan^2 x \sec x\,dx$

Answer

$$\int \tan^2 x \sec x\,dx \quad = \quad \tfrac{1}{2}\sec x \tan x - \tfrac{1}{2}\ln(\sec x + \tan x) + C$$

22) $\int \tan x \sec^3 x\,dx$

23) $\int \sec^4 x\,dx$

Answer

$$\int \sec^4 x\,dx \quad = \quad \frac{2\tan x}{3} + \tfrac{1}{3}\sec^2 x \tan x = \tan x + \frac{\tan^3 x}{3} + C$$

24) $\int \cot x\,dx$

25) $\int \csc x\,dx$

Answer

$$\int \csc x\,dx \quad = \quad -\ln|\cot x + \csc x| + C$$

26) $\int \frac{\tan^3 x}{\sqrt{\sec x}}\,dx$

For exercises 27 - 28, find a general formula for the integrals.

27) $\int \sin^2 ax \cos ax\,dx$

Answer

$$\int \sin^2 ax \cos ax\,dx \quad = \quad \frac{\sin^3(ax)}{3a} + C$$

28) $\int \sin ax \cos ax\,dx.$

Use the double-angle formulas to evaluate the integrals in exercises 29 - 34.

29) $\int_0^{\pi} \sin^2 x\,dx$

Answer

$$\int_0^{\pi} \sin^2 x\,dx \quad = \quad \frac{\pi}{2}$$

30) $\int_0^{\pi} \sin^4 x\,dx$

31) $\int \cos^2 3x\,dx$

Answer

$$\int \cos^2 3x\,dx \quad = \quad \frac{x}{2} + \tfrac{1}{12}\sin(6x) + C$$

32) $\int \sin^2 x \cos^2 x\,dx$

33) $\int \sin^2 x\,dx + \int \cos^2 x\,dx$

Answer

$$\int \sin^2 x\,dx + \int \cos^2 x\,dx \quad = \quad x + C$$

34) $\int \sin^2 x \cos^2(2x)\,dx$

For exercises 35 - 43, evaluate the definite integrals. Express answers in exact form whenever possible.

35) $\int_0^{2\pi} \cos x \sin 2x\,dx$

Answer

$$\int_0^{2\pi} \cos x \sin 2x\,dx \quad = \quad 0$$

36) $\int_0^{\pi} \sin 3x \sin 5x\,dx$

37) $\int_0^{\pi} \cos(99x)\sin(101x)\,dx$

Answer

$$\int_0^{\pi} \cos(99x)\sin(101x)\,dx \quad = \quad 0$$

38) $\int_{-\pi}^{\pi} \cos^2(3x)\,dx$

39) $\int_{0}^{2\pi} \sin x \sin(2x) \sin(3x)\,dx$

Answer

$$\int_{0}^{2\pi} \sin x \sin(2x) \sin(3x)\,dx \quad = \quad 0$$

40) $\int_{0}^{4\pi} \cos(x/2) \sin(x/2)\,dx$

41) $\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}}\,dx$ (Round this answer to three decimal places.)

Answer

$$\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}}\,dx \quad \approx \quad 0.239$$

42) $\int_{-\pi/3}^{\pi/3} \sqrt{\sec^2 x - 1}\,dx$

43) $\int_{0}^{\pi/2} \sqrt{1 - \cos(2x)}\,dx$

Answer

$$\int_{0}^{\pi/2} \sqrt{1 - \cos(2x)}\,dx \quad = \quad \sqrt{2}$$

44) Find the area of the region bounded by the graphs of the equations $y = \sin x$, $y = \sin^3 x$, $x = 0$, and $x = \frac{\pi}{2}$.

45) Find the area of the region bounded by the graphs of the equations $y = \cos^2 x$, $y = \sin^2 x$, $x = -\frac{\pi}{4}$, and $x = \frac{\pi}{4}$.

Answer

$A = 1 \text{ unit}^2$

46) A particle moves in a straight line with the velocity function $v(t) = \sin(\omega t)\cos^2(\omega t)$. Find its position function $x = f(t)$ if $f(0) = 0$.

47) Find the average value of the function $f(x) = \sin^2 x \cos^3 x$ over the interval $[-\pi, \pi]$.

Answer

0

For exercises 48 - 49, solve the differential equations.

48) $\frac{dy}{dx} = \sin^2 x$. The curve passes through point $(0, 0)$.

49) $\frac{dy}{d\theta} = \sin^4(\pi\theta)$

Answer

$$f(x) = \frac{3\theta}{8} - \frac{1}{4\pi}\sin(2\pi\theta) + \frac{1}{32\pi}\sin(4\pi\theta) + C$$

50) Find the length of the curve $y = \ln(\csc x)$, for $\frac{\pi}{4} \le x \le \frac{\pi}{2}$.

51) Find the length of the curve $y = \ln(\sin x)$, for $\frac{\pi}{3} \le x \le \frac{\pi}{2}$.

Answer

$s = \ln(\sqrt{3})$

52) Find the volume generated by revolving the curve $y = \cos(3x)$ about the x-axis, for $0 \le x \le \frac{\pi}{36}$.

For exercises 53 - 54, use this information: The inner product of two functions f and g over $[a, b]$ is defined by $f(x) \cdot g(x) = \langle f, g \rangle = \int_a^b f \cdot g \, dx$. Two distinct functions f and g are said to be orthogonal if $\langle f, g \rangle = 0$.

53) Show that $\sin(2x)$, $\cos(3x)$ are orthogonal over the interval $[-\pi, \pi]$.

Answer

$$\int_{-\pi}^{\pi} \sin(2x)\cos(3x)\,dx = 0$$

54) Evaluate $\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx$.

55) Integrate $y' = \sqrt{\tan x}\sec^4 x$.

Answer

$$y = \int \sqrt{\tan x}\sec^4 x\,dx \quad = \quad \tfrac{2}{3}(\tan x)^{3/2} + \tfrac{2}{7}(\tan x)^{7/2} + C = \tfrac{2}{21}(\tan x)^{3/2}\left[7 + 3\tan^2 x\right] + C$$

For each pair of integrals in exercises 56 - 57, determine which one is more difficult to evaluate. Explain your reasoning.

56) $\int \sin^{456} x \cos x\,dx$ or $\int \sin^2 x \cos^2 x\,dx$

57) $\int \tan^{350} x \sec^2 x\,dx$ or $\int \tan^{350} x \sec x\,dx$

Answer

The second integral is more difficult because the first integral is simply a u-substitution type.

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7.3: Trigonometric Substitution

Learning Objectives

- Solve integration problems involving the square root of a sum or difference of two squares.

In this section, we explore integrals containing expressions of the form $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, and $\sqrt{x^2-a^2}$, where the values of a are positive. We have already encountered and evaluated integrals containing some expressions of this type, but many still remain inaccessible. The technique of *trigonometric substitution* comes in very handy when evaluating these integrals. This technique uses substitution to rewrite these integrals as trigonometric integrals.

Integrals Involving $\sqrt{a^2-x^2}$

Before developing a general strategy for integrals containing $\sqrt{a^2-x^2}$, consider the integral $\int \sqrt{9-x^2}dx$. This integral cannot be evaluated using any of the techniques we have discussed so far. However, if we make the substitution $x=3\sin\theta$, we have $dx=3\cos\theta\,d\theta$. After substituting into the integral, we have

$$\int \sqrt{9-x^2}\,dx = \int \sqrt{9-(3\sin\theta)^2}\cdot 3\cos\theta\,d\theta.$$

After simplifying, we have

$$\int \sqrt{9-x^2}\,dx = \int 9\sqrt{1-\sin^2\theta}\cdot\cos\theta\,d\theta.$$

Letting $1-\sin^2\theta=\cos^2\theta$, we now have

$$\int \sqrt{9-x^2}\,dx = \int 9\sqrt{\cos^2\theta}\cos\theta\,d\theta.$$

Assuming that $\cos\theta\geq 0$, we have

$$\int \sqrt{9-x^2}\,dx = \int 9\cos^2\theta\,d\theta.$$

At this point, we can evaluate the integral using the techniques developed for integrating powers and products of trigonometric functions. Before completing this example, let's take a look at the general theory behind this idea.

To evaluate integrals involving $\sqrt{a^2-x^2}$, we make the substitution $x=a\sin\theta$ and $dx=a\cos\theta$. To see that this actually makes sense, consider the following argument: The domain of $\sqrt{a^2-x^2}$ is $[-a,a]$. Thus,

$$-a\leq x\leq a.$$

Consequently,

$$-1\leq\frac{x}{a}\leq 1.$$

Since the range of $\sin x$ over $[-(\pi/2),\pi/2]$ is $[-1,1]$, there is a unique angle θ satisfying $-(\pi/2)\leq\theta\leq\pi/2$ so that $\sin\theta=x/a$, or equivalently, so that $x=a\sin\theta$. If we substitute $x=a\sin\theta$ into $\sqrt{a^2-x^2}$, we get

$$\begin{aligned}
\sqrt{a^2-x^2} &= \sqrt{a^2-(a\sin\theta)^2} && \text{Let } x=a\sin\theta \text{ where } -\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}.\\
&&& \text{Simplify.}\\
&= \sqrt{a^2-a^2\sin^2\theta} && \text{Factor out } a^2.\\
&= \sqrt{a^2(1-\sin^2\theta)} && \text{Substitute } 1-\sin^2 x=\cos^2 x.\\
&= \sqrt{a^2\cos^2\theta} && \text{Take the square root.}\\
&= |a\cos\theta|\\
&= a\cos\theta
\end{aligned}$$

Since $\cos x \geq 0$ on $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and $a > 0, |a\cos\theta| = a\cos\theta$. We can see, from this discussion, that by making the substitution $x = a\sin\theta$, we are able to convert an integral involving a radical into an integral involving trigonometric functions. After we evaluate the integral, we can convert the solution back to an expression involving x. To see how to do this, let's begin by assuming that $0 < x < a$. In this case, $0 < \theta < \frac{\pi}{2}$. Since $\sin\theta = \frac{x}{a}$, we can draw the reference triangle in Figure 7.3.1 to assist in expressing the values of $\cos\theta$, $\tan\theta$, and the remaining trigonometric functions in terms of x. It can be shown that this triangle actually produces the correct values of the trigonometric functions evaluated at θ for all θ satisfying $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. It is useful to observe that the expression $\sqrt{a^2 - x^2}$ actually appears as the length of one side of the triangle. Last, should θ appear by itself, we use $\theta = \sin^{-1}\left(\frac{x}{a}\right)$.

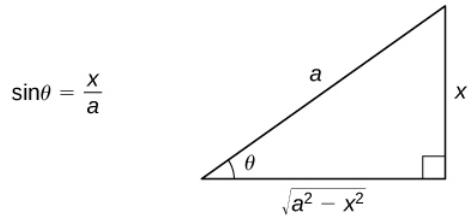

Figure 7.3.1: A reference triangle can help express the trigonometric functions evaluated at θ in terms of x.

The essential part of this discussion is summarized in the following problem-solving strategy.

Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{a^2 - x^2}$

1. It is a good idea to make sure the integral cannot be evaluated easily in another way. For example, although this method can be applied to integrals of the form $\int \frac{1}{\sqrt{a^2 - x^2}}dx$, $\int \frac{x}{\sqrt{a^2 - x^2}}dx$, and $\int x\sqrt{a^2 - x^2}\,dx$, they can each be integrated directly either by formula or by a simple u-substitution.
2. Make the substitution $x = a\sin\theta$ and $dx = a\cos\theta\,d\theta$. Note: This substitution yields $\sqrt{a^2 - x^2} = a\cos\theta$.
3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangle from Figure 1 to rewrite the result in terms of x. You may also need to use some trigonometric identities and the relationship $\theta = \sin^{-1}\left(\frac{x}{a}\right)$.

The following example demonstrates the application of this problem-solving strategy.

Example 7.3.1: Integrating an Expression Involving $\sqrt{a^2 - x^2}$

Evaluate

$$\int \sqrt{9 - x^2}dx.$$

Solution

Begin by making the substitutions $x = 3\sin\theta$ and $dx = 3\cos\theta\,d\theta$. Since $\sin\theta = \frac{x}{3}$, we can construct the reference triangle shown in Figure 2.

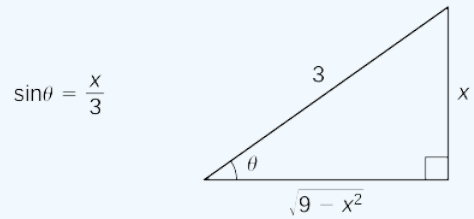

Figure 7.3.2: A reference triangle can be constructed for Example 7.3.1.

Thus,

$$\int \sqrt{9 - x^2}\,dx = \int \sqrt{9 - (3\sin\theta)^2}3\cos\theta\,d\theta \text{ Substitute } x = 3\sin\theta \text{ and } dx = 3\cos\theta\,d\theta.$$

$$= \int \sqrt{9(1 - \sin^2\theta)} \cdot 3\cos\theta\,d\theta \text{ Simplify.}$$

$= \int \sqrt{9\cos^2\theta} \cdot 3\cos\theta\, d\theta$ Substitute $\cos^2\theta = 1 - \sin^2\theta$.

$= \int 3|\cos\theta| 3\cos\theta\, d\theta$ Take the square root.

$= \int 9\cos^2\theta\, d\theta$ Simplify. Since $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \cos\theta \ge 0$ and $|\cos\theta| = \cos\theta$.

$= \int 9\left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right) d\theta$ Use the strategy for integrating an even power of $\cos\theta$.

$= \frac{9}{2}\theta + \frac{9}{4}\sin(2\theta) + C$ Evaluate the integral.

$= \frac{9}{2}\theta + \frac{9}{4}(2\sin\theta\cos\theta) + C$ Substitute $\sin(2\theta) = 2\sin\theta\cos\theta$.

$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{9}{2} \cdot \frac{x}{3} \cdot \frac{\sqrt{9-x^2}}{3} + C$ Substitute $\sin^{-1}\left(\frac{x}{3}\right) = \theta$ and $\sin\theta = \frac{x}{3}$.

Use the reference triangle to see that $\cos\theta = \frac{\sqrt{9-x^2}}{3}$ and make this substitution. Simplify.

$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{x\sqrt{9-x^2}}{2} + C.$ Simplify.

Example 7.3.2: Integrating an Expression Involving $\sqrt{a^2 - x^2}$

Evaluate

$$\int \frac{\sqrt{4-x^2}}{x} dx.$$

Solution

First make the substitutions $x = 2\sin\theta$ and $dx = 2\cos\theta\, d\theta$. Since $\sin\theta = \frac{x}{2}$, we can construct the reference triangle shown in Figure 7.3.3.

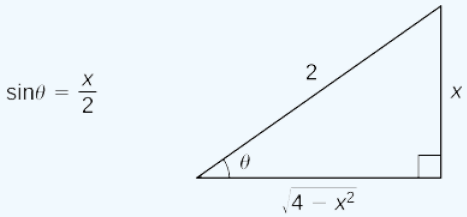

Figure 7.3.3: A reference triangle can be constructed for Example 7.3.2.

Thus,

$\int \frac{\sqrt{4-x^2}}{x} dx = \int \frac{\sqrt{4-(2\sin\theta)^2}}{2\sin\theta} 2\cos\theta\, d\theta$ Substitute $x = 2\sin\theta$ and $dx = 2\cos\theta\, d\theta$.

$= \int \frac{2\cos^2\theta}{\sin\theta} d\theta$ Substitute $\cos^2\theta = 1 - \sin^2\theta$ and simplify.

$= \int \frac{2(1-\sin^2\theta)}{\sin\theta} d\theta$ Substitute $\cos^2\theta = 1 - \sin^2\theta$.

$= \int (2\csc\theta - 2\sin\theta)\, d\theta$ Separate the numerator, simplify, and use $\csc\theta = \frac{1}{\sin\theta}$.

$= 2\ln|\csc\theta - \cot\theta| + 2\cos\theta + C$ Evaluate the integral.

$= 2\ln\left|\frac{2}{x} - \frac{\sqrt{4-x^2}}{x}\right| + \sqrt{4-x^2} + C.$ Use the reference triangle to rewrite the expression in terms of x and simplify.

In the next example, we see that we sometimes have a choice of methods.

Example 7.3.3: Integrating an Expression Involving $\sqrt{a^2 - x^2}$ Two Ways

Evaluate $\int x^3\sqrt{1-x^2}dx$ two ways: first by using the substitution $u = 1 - x^2$ and then by using a trigonometric substitution.

Method 1

Let $u = 1 - x^2$ and hence $x^2 = 1 - u$. Thus, $du = -2x\,dx$. In this case, the integral becomes

$$\begin{aligned}\int x^3\sqrt{1-x^2}\,dx &= -\frac{1}{2}\int x^2\sqrt{1-x^2}(-2x\,dx) \text{ Make the substitution.}\\ &= -\frac{1}{2}\int(1-u)\sqrt{u}\,du \text{ Expand the expression.}\\ &= -\frac{1}{2}\int(u^{1/2}-u^{3/2})\,du \text{ Evaluate the integral.}\\ &= -\frac{1}{2}(\frac{2}{3}u^{3/2}-\frac{2}{5}u^{5/2})+C \text{ Rewrite in terms of x.}\\ &= -\frac{1}{3}(1-x^2)^{3/2}+\frac{1}{5}(1-x^2)^{5/2}+C.\end{aligned}$$

Method 2

Let $x = \sin\theta$. In this case, $dx = \cos\theta\,d\theta$. Using this substitution, we have

$$\begin{aligned}\int x^3\sqrt{1-x^2}dx &= \int \sin^3\theta\cos^2\theta\,d\theta\\ &= \int(1-\cos^2\theta)\cos^2\theta\sin\theta\,d\theta \text{ Let } u=\cos\theta. \text{ Thus,} du=-\sin\theta\,d\theta.\\ &= \int(u^4-u^2)\,du\\ &= \frac{1}{5}u^5-\frac{1}{3}u^3+C \text{ Substitute } \cos\theta = u.\\ &= \frac{1}{5}\cos^5\theta-\frac{1}{3}\cos^3\theta+C \text{ Use a reference triangle to see that } \cos\theta=\sqrt{1-x^2}.\\ &= \frac{1}{5}(1-x^2)^{5/2}-\frac{1}{3}(1-x^2)^{3/2}+C.\end{aligned}$$

Exercise 7.3.1

Rewrite the integral $\displaystyle\int\frac{x^3}{\sqrt{25-x^2}}\,dx$ using the appropriate trigonometric substitution (do not evaluate the integral).

Hint

Substitute $x = 5\sin\theta$ and $dx = 5\cos\theta\,d\theta$.

Answer

$$\int 125\sin^3\theta\,d\theta$$

Integrating Expressions Involving $\sqrt{a^2 + x^2}$

For integrals containing $\sqrt{a^2+x^2}$,let's first consider the domain of this expression. Since $\sqrt{a^2+x^2}$ is defined for all real values of x, we restrict our choice to those trigonometric functions that have a range of all real numbers. Thus, our choice is restricted to selecting either $x = a\tan\theta$ or $x = a\cot\theta$. Either of these substitutions would actually work, but the standard substitution is $x = a\tan\theta$ or, equivalently, $\tan\theta = x/a$. With this substitution, we make the assumption that $-(\pi/2) < \theta < \pi/2$, so that we also have $\theta = \tan^{-1}(x/a)$. The procedure for using this substitution is outlined in the following problem-solving strategy.

Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{a^2 + x^2}$

1. Check to see whether the integral can be evaluated easily by using another method. In some cases, it is more convenient to use an alternative method.
2. Substitute $x = a\tan\theta$ and $dx = a\sec^2\theta\,d\theta$. This substitution yields $\sqrt{a^2+x^2} = \sqrt{a^2+(a\tan\theta)^2} = \sqrt{a^2(1+\tan^2\theta)} = \sqrt{a^2sec^2\theta}$ $= |a\sec\theta| = a\sec\theta$. (Since $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $\sec\theta > 0$ over this interval, $|a\sec\theta| = a\sec\theta$.)

3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangle from Figure 7.3.4 to rewrite the result in terms of x. You may also need to use some trigonometric identities and the relationship $\theta = \tan^{-1}\left(\frac{x}{a}\right)$. (Note: The reference triangle is based on the assumption that $x > 0$; however, the trigonometric ratios produced from the reference triangle are the same as the ratios for which $x \leq 0$.)

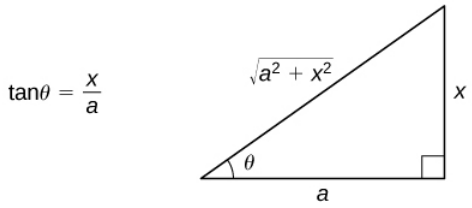

Figure 7.3.4: A reference triangle can be constructed to express the trigonometric functions evaluated at θ in terms of x.

Example 7.3.4: Integrating an Expression Involving $\sqrt{a^2 + x^2}$

Evaluate $\int \frac{dx}{\sqrt{1+x^2}}$ and check the solution by differentiating.

Solution

Begin with the substitution $x = \tan\theta$ and $dx = sec^2\theta\, d\theta$. Since $\tan\theta = x$, draw the reference triangle in Figure 7.3.5.

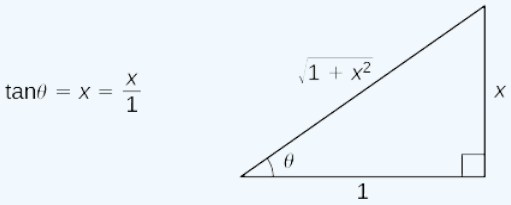

Figure 7.3.5: The reference triangle for Example 7.3.4.

Thus,

$$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{\sec^2\theta}{\sec\theta} d\theta \qquad \text{Substitute } x = \tan\theta \text{ and } dx = \sec^2\theta\, d\theta.$$

$$\text{This substitution makes } \sqrt{1+x^2} = \sec\theta. \text{ Simplify.}$$

$$= \int \sec\theta\, d\theta \qquad \text{Evaluate the integral.}$$

$$= \ln|\sec\theta + \tan\theta| + C \qquad \text{Use the reference triangle to express the result in terms of } x.$$

$$= \ln|\sqrt{1+x^2} + x| + C$$

To check the solution, differentiate:

$$\frac{d}{dx}\left(\ln|\sqrt{1+x^2}+x|\right) = \frac{1}{\sqrt{1+x^2}+x} \cdot \left(\frac{x}{\sqrt{1+x^2}} + 1\right) = \frac{1}{\sqrt{1+x^2}+x} \cdot \frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}} = \frac{1}{\sqrt{1+x^2}}.$$

Since $\sqrt{1+x^2} + x > 0$ for all values of x, we could rewrite $\ln|\sqrt{1+x^2}+x| + C = \ln(\sqrt{1+x^2}+x) + C$, if desired.

Example 7.3.5: Evaluating $\int \frac{dx}{\sqrt{1+x^2}}$ Using a Different Substitution

Use the substitution $x = \sinh\theta$ to evaluate $\int \frac{dx}{\sqrt{1+x^2}}$.

Solution

Because $\sinh\theta$ has a range of all real numbers, and $1 + \sinh^2\theta = \cosh^2\theta$, we may also use the substitution $x = \sinh\theta$ to evaluate this integral. In this case, $dx = \cosh\theta\, d\theta$. Consequently,

$$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{\cosh\theta}{\sqrt{1+\sinh^2\theta}}d\theta \qquad \text{Substitute } x=\sinh\theta \text{ and } dx=\cosh\theta\,d\theta.$$

$$\text{Substitute } 1+\sinh^2\theta=\cosh^2\theta.$$

$$= \int \frac{\cosh\theta}{\sqrt{\cosh^2\theta}}d\theta \qquad \text{Since } \sqrt{\cosh^2\theta}=|\cosh\theta|$$

$$= \int \frac{\cosh\theta}{|\cosh\theta|}d\theta \qquad |\cosh\theta|=\cosh\theta \text{ since } \cosh\theta>0 \text{ for all } \theta.$$

$$= \int \frac{\cosh\theta}{\cosh\theta}d\theta \qquad \text{Simplify.}$$

$$= \int 1\,d\theta \qquad \text{Evaluate the integral.}$$

$$= \theta + C \qquad \text{Since } x=\sinh\theta, \text{ we know } \theta=\sinh^{-1}x.$$

$$= \sinh^{-1}x + C.$$

Analysis

This answer looks quite different from the answer obtained using the substitution $x=\tan\theta$. To see that the solutions are the same, set $y=\sinh^{-1}x$. Thus, $\sinh y = x$. From this equation we obtain:

$$\frac{e^y - e^{-y}}{2} = x.$$

After multiplying both sides by $2e^y$ and rewriting, this equation becomes:

$$e^{2y} - 2xe^y - 1 = 0.$$

Use the quadratic equation to solve for e^y:

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}.$$

Simplifying, we have:

$$e^y = x \pm \sqrt{x^2+1}.$$

Since $x-\sqrt{x^2+1}<0$, it must be the case that $e^y = x+\sqrt{x^2+1}$. Thus,

$$y = \ln(x+\sqrt{x^2+1}).$$

Last, we obtain

$$\sinh^{-1}x = \ln(x+\sqrt{x^2+1}).$$

After we make the final observation that, since $x+\sqrt{x^2+1}>0$,

$$\ln(x+\sqrt{x^2+1}) = \ln|\sqrt{1+x^2}+x|,$$

we see that the two different methods produced equivalent solutions.

Example 7.3.6: Finding an Arc Length

Find the length of the curve $y=x^2$ over the interval $[0,\frac{1}{2}]$.

Solution

Because $\frac{dy}{dx}=2x$, the arc length is given by

$$\int_0^{1/2}\sqrt{1+(2x)^2}dx = \int_0^{1/2}\sqrt{1+4x^2}dx.$$

To evaluate this integral, use the substitution $x = \frac{1}{2}\tan\theta$ and $dx = \frac{1}{2}\sec^2\theta\, d\theta$. We also need to change the limits of integration. If $x = 0$, then $\theta = 0$ and if $x = \frac{1}{2}$, then $\theta = \frac{\pi}{4}$. Thus,

$\int_0^{1/2} \sqrt{1+4x^2}dx = \int_0^{\pi/4} \sqrt{1+\tan^2\theta}\cdot \frac{1}{2}\sec^2\theta\, d\theta$ After substitution, $\sqrt{1+4x^2} = \sec\theta$. (Substitute $1+\tan^2\theta = \sec^2\theta$ and simplify.)

$= \frac{1}{2}\int_0^{\pi/4} \sec^3\theta\, d\theta$ We derived this integral in the previous section.

$= \frac{1}{2}(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta|)\,|_0^{\pi/4}$ Evaluate and simplify.

$= \frac{1}{4}(\sqrt{2}+\ln(\sqrt{2}+1))$.

? Exercise 7.3.2

Rewrite $\int x^3\sqrt{x^2+4}dx$ by using a substitution involving $\tan\theta$.

Hint

Use $x = 2\tan\theta$ and $dx = 2\sec^2\theta\, d\theta$.

Answer

$$\int 32\tan^3\theta\sec^3\theta\, d\theta$$

Integrating Expressions Involving $\sqrt{x^2-a^2}$

The domain of the expression $\sqrt{x^2-a^2}$ is $(-\infty,-a]\cup[a,+\infty)$. Thus, either $x \le -a$ or $x \ge a$. Hence, $\frac{x}{a} \le -1$ or $\frac{x}{a} \ge 1$. Since these intervals correspond to the range of $\sec\theta$ on the set $[0,\frac{\pi}{2})\cup(\frac{\pi}{2},\pi]$, it makes sense to use the substitution $\sec\theta = \frac{x}{a}$ or, equivalently, $x = a\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \le \pi$. The corresponding substitution for dx is $dx = a\sec\theta\tan\theta\, d\theta$. The procedure for using this substitution is outlined in the following problem-solving strategy.

Problem-Solving Strategy: Integrals Involving $\sqrt{x^2-a^2}$

1. Check to see whether the integral cannot be evaluated using another method. If so, we may wish to consider applying an alternative technique.
2. Substitute $x = a\sec\theta$ and $dx = a\sec\theta\tan\theta\, d\theta$. This substitution yields

$$\sqrt{x^2-a^2} = \sqrt{(a\sec\theta)^2-a^2} = \sqrt{a^2(\sec^2\theta-1)} = \sqrt{a^2\tan^2\theta} = |a\tan\theta|.$$

   For $x \ge a, |a\tan\theta| = a\tan\theta$ and for $x \le -a, |a\tan\theta| = -a\tan\theta$.
3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangles from Figure 7.3.6 to rewrite the result in terms of x.
6. You may also need to use some trigonometric identities and the relationship $\theta = \sec^{-1}\left(\frac{x}{a}\right)$. (Note: We need both reference triangles, since the values of some of the trigonometric ratios are different depending on whether $x > a$ or $x < -a$.)

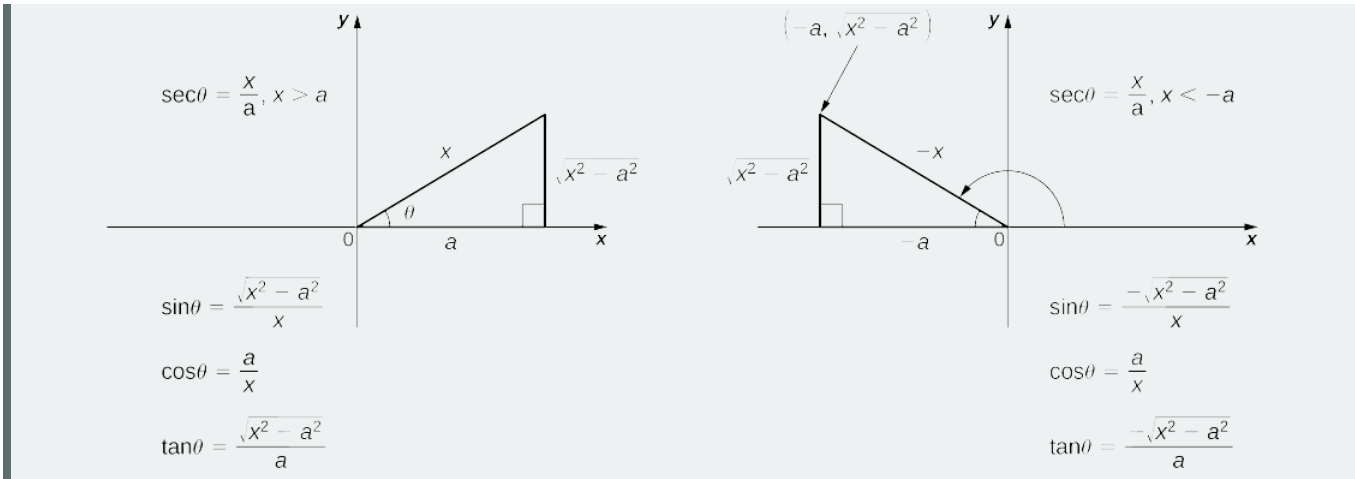

Figure 7.3.6: Use the appropriate reference triangle to express the trigonometric functions evaluated at θ in terms of x.

Example 7.3.7: Finding the Area of a Region

Find the area of the region between the graph of $f(x) = \sqrt{x^2 - 9}$ and the x-axis over the interval $[3, 5]$.

Solution

First, sketch a rough graph of the region described in the problem, as shown in the following figure.

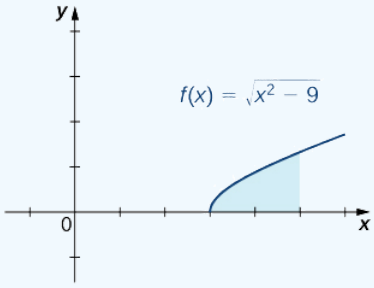

Figure 7.3.7: Calculating the area of the shaded region requires evaluating an integral with a trigonometric substitution.

We can see that the area is $A = \int_3^5 \sqrt{x^2 - 9}dx$. To evaluate this definite integral, substitute $x = 3\sec\theta$ and $dx = 3\sec\theta\tan\theta\, d\theta$. We must also change the limits of integration. If $x = 3$, then $3 = 3\sec\theta$ and hence $\theta = 0$. If $x = 5$, then $\theta = \sec^{-1}(\frac{5}{3})$. After making these substitutions and simplifying, we have

$$\begin{aligned}
\textbf{Area} &= \int_3^5 \sqrt{x^2 - 9}dx \\
&= \int_0^{\sec^{-1}(5/3)} 9\tan^2\theta\sec\theta\, d\theta \text{ Use } \tan^2\theta = \sec^2\theta - 1. \\
&= \int_0^{\sec^{-1}(5/3)} 9(\sec^2\theta - 1)\sec\theta\, d\theta \text{ Expand.} \\
&= \int_0^{\sec^{-1}(5/3)} 9(\sec^3\theta - \sec\theta)\, d\theta \text{ Evaluate the integral.} \\
&= (\frac{9}{2}\ln|\sec\theta + \tan\theta| + \frac{9}{2}\sec\theta\tan\theta) - 9\ln|\sec\theta + \tan\theta|\,|_0^{\sec^{-1}(5/3)} \text{ Simplify.} \\
&= \frac{9}{2}\sec\theta\tan\theta - \frac{9}{2}\ln|\sec\theta + \tan\theta|\,|_0^{\sec^{-1}(5/3)} \text{ Evaluate. Use } \sec(\sec^{-1}\frac{5}{3}) = \frac{5}{3} \text{ and } \tan(\sec^{-1}\frac{5}{3}) = \frac{4}{3}. \\
&= \frac{9}{2}\cdot\frac{5}{3}\cdot\frac{4}{3} - \frac{9}{2}\ln|\frac{5}{3} + \frac{4}{3}| - (\frac{9}{2}\cdot 1\cdot 0 - \frac{9}{2}\ln|1 + 0|) \\
&= 10 - \frac{9}{2}\ln 3
\end{aligned}$$

Exercise 7.3.3

Evaluate

$$\int \frac{dx}{\sqrt{x^2-4}}.$$

Assume that $x > 2$.

Hint

Substitute $x = 2\sec\theta$ and $dx = 2\sec\theta\tan\theta\, d\theta$.

Answer

$$\ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| + C$$

Key Concepts

- For integrals involving $\sqrt{a^2-x^2}$, use the substitution $x = a\sin\theta$ and $dx = a\cos\theta\, d\theta$.
- For integrals involving $\sqrt{a^2+x^2}$, use the substitution $x = a\tan\theta$ and $dx = a\sec^2\theta\, d\theta$.
- For integrals involving $\sqrt{x^2-a^2}$, substitute $x = a\sec\theta$ and $dx = a\sec\theta\tan\theta\, d\theta$.

Glossary

trigonometric substitution

an integration technique that converts an algebraic integral containing expressions of the form $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, or $\sqrt{x^2-a^2}$ into a trigonometric integral

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7.3E: Exercises for Section 7.3

Simplify the expressions in exercises 1 - 5 by writing each one using a single trigonometric function.

1) $4 - 4\sin^2\theta$

2) $9\sec^2\theta - 9$

Answer

$$9\sec^2\theta - 9 \quad = \quad 9\tan^2\theta$$

3) $a^2 + a^2\tan^2\theta$

4) $a^2 + a^2\sinh^2\theta$

Answer

$$a^2 + a^2\sinh^2\theta \quad = \quad a^2\cosh^2\theta$$

5) $16\cosh^2\theta - 16$

Use the technique of completing the square to express each trinomial in exercises 6 - 8 as the square of a binomial.

6) $4x^2 - 4x + 1$

Answer

$4(x - \frac{1}{2})^2$

7) $2x^2 - 8x + 3$

8) $-x^2 - 2x + 4$

Answer

$-(x+1)^2 + 5$

In exercises 9 - 28, integrate using the method of trigonometric substitution. Express the final answer in terms of the original variable.

9) $\int \frac{dx}{\sqrt{4 - x^2}}$

10) $\int \frac{dx}{\sqrt{x^2 - a^2}}$

Answer

$$\int \frac{dx}{\sqrt{x^2 - a^2}} \quad = \quad \ln\left| x + \sqrt{-a^2 + x^2}\right| + C$$

11) $\int \sqrt{4 - x^2}\, dx$

12) $\int \frac{dx}{\sqrt{1 + 9x^2}}$

Answer

$$\int \frac{dx}{\sqrt{1 + 9x^2}} \quad = \quad \tfrac{1}{3}\ln\left| \sqrt{9x^2 + 1} + 3x\right| + C$$

13) $\int \frac{x^2\, dx}{\sqrt{1 - x^2}}$

14) $\int \frac{dx}{x^2\sqrt{1 - x^2}}$

Answer

$$\int \frac{dx}{x^2\sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{x} + C$$

15) $\int \frac{dx}{(1+x^2)^2}$

16) $\int \sqrt{x^2+9}\,dx$

Answer

$$\int \sqrt{x^2+9}\,dx = 9\left[\frac{x\sqrt{x^2+9}}{18} + \tfrac{1}{2}\ln\left|\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right|\right] + C$$

17) $\int \frac{\sqrt{x^2-25}}{x}\,dx$

18) $\int \frac{\theta^3}{\sqrt{9-\theta^2}}\,d\theta$

Answer

$$\int \frac{\theta^3 d\theta}{\sqrt{9-\theta^2}}\,d\theta = -\tfrac{1}{3}\sqrt{9-\theta^2}(18+\theta^2) + C$$

19) $\int \frac{dx}{\sqrt{x^6-x^2}}$

20) $\int \sqrt{x^6-x^8}\,dx$

Answer

$$\int \sqrt{x^6-x^8}\,dx = \frac{(-1+x^2)(2+3x^2)\sqrt{x^6-x^8}}{15x^3} + C$$

21) $\int \frac{dx}{(1+x^2)^{3/2}}$

22) $\int \frac{dx}{(x^2-9)^{3/2}}$

Answer

$$\int \frac{dx}{(x^2-9)^{3/2}} = -\frac{x}{9\sqrt{x^2-9}} + C$$

23) $\int \frac{\sqrt{1+x^2}}{x}\,dx$

24) $\int \frac{x^2}{\sqrt{x^2-1}}\,dx$

Answer

$$\int \frac{x^2}{\sqrt{x^2-1}}\,dx = \tfrac{1}{2}(\ln\,|\,x+\sqrt{x^2-1}\,|\,+x\sqrt{x^2-1}) + C$$

25) $\int \frac{x^2}{x^2+4}\,dx$

26) $\int \frac{dx}{x^2\sqrt{x^2+1}}$

Answer

$$\int \frac{dx}{x^2\sqrt{x^2+1}} = -\frac{\sqrt{1+x^2}}{x} + C$$

27) $\int \frac{x^2}{\sqrt{1+x^2}}\,dx$

28) $\int_{-1}^{1}(1-x^2)^{3/2}\,dx$

Answer

$$\int_{-1}^{1}(1-x^2)^{3/2}\,dx \quad = \quad \tfrac{1}{8}\left(x(5-2x^2)\sqrt{1-x^2}+3\arcsin x\right)+C$$

In exercises 29 - 34, use the substitutions $x=\sinh\theta$, $\cosh\theta$, or $\tanh\theta$. Express the final answers in terms of the variable x.

29) $\int \frac{dx}{\sqrt{x^2-1}}$

30) $\int \frac{dx}{x\sqrt{1-x^2}}$

Answer

$$\int \frac{dx}{x\sqrt{1-x^2}} \quad = \quad \ln x - \ln\left|\,1+\sqrt{1-x^2}\,\right| + C$$

31) $\int \sqrt{x^2-1}\,dx$

32) $\int \frac{\sqrt{x^2-1}}{x^2}\,dx$

Answer

$$\int \frac{\sqrt{x^2-1}}{x^2}\,dx \quad = \quad -\frac{\sqrt{-1+x^2}}{x}+\ln\left|x+\sqrt{-1+x^2}\right|+C$$

33) $\int \frac{dx}{1-x^2}$

34) $\int \frac{\sqrt{1+x^2}}{x^2}\,dx$

Answer

$$\int \frac{\sqrt{1+x^2}}{x^2}\,dx \quad = \quad -\frac{\sqrt{1+x^2}}{x}+\operatorname{arcsinh} x+C$$

Use the technique of completing the square to evaluate the integrals in exercises 35 - 39.

35) $\int \frac{1}{x^2-6x}\,dx$

36) $\int \frac{1}{x^2+2x+1}\,dx$

Answer

$$\int \frac{1}{x^2+2x+1}\,dx \quad = \quad -\frac{1}{1+x}+C$$

37) $\int \frac{1}{\sqrt{-x^2+2x+8}}\,dx$

38) $\int \frac{1}{\sqrt{-x^2+10x}}\,dx$

Answer

$$\int \frac{1}{\sqrt{-x^2+10x}}\,dx \quad = \quad \arcsin\left(\frac{x-5}{5}\right)+C$$

39) $\int \frac{1}{\sqrt{x^2+4x-12}}\,dx$

40) Evaluate the integral without using calculus: $\int_{-3}^{3}\sqrt{9-x^2}\,dx$.

Answer

$\int_{-3}^{3} \sqrt{9-x^2}\,dx = \frac{9\pi}{2}$; area of a semicircle with radius 3

41) Find the area enclosed by the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

42) Evaluate the integral $\int \frac{dx}{\sqrt{1-x^2}}$ using two different substitutions. First, let $x = \cos\theta$ and evaluate using trigonometric substitution. Second, let $x = \sin\theta$ and use trigonometric substitution. Are the answers the same?

Answer

$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin(x) + C$ is the common answer.

43) Evaluate the integral $\int \frac{dx}{x\sqrt{x^2-1}}$ using the substitution $x = \sec\theta$. Next, evaluate the same integral using the substitution $x = \csc\theta$. Show that the results are equivalent.

44) Evaluate the integral $\int \frac{x}{x^2+1}\,dx$ using the form $\int \frac{1}{u}\,du$. Next, evaluate the same integral using $x = \tan\theta$. Are the results the same?

Answer

$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln(1+x^2) + C$ is the result using either method.

45) State the method of integration you would use to evaluate the integral $\int x\sqrt{x^2+1}\,dx$. Why did you choose this method?

46) State the method of integration you would use to evaluate the integral $\int x^2\sqrt{x^2-1}\,dx$. Why did you choose this method?

Answer

Use trigonometric substitution. Let $x = \sec(\theta)$.

47) Evaluate $\int_{-1}^{1} \frac{x}{x^2+1}\,dx$

48) Find the length of the arc of the curve over the specified interval: $y = \ln x, \quad [1, 5]$. Round the answer to three decimal places.

Answer

$s = 4.367$ units

49) Find the surface area of the solid generated by revolving the region bounded by the graphs of $y = x^2$, $y = 0$, $x = 0$, and $x = \sqrt{2}$ about the x-axis. (Round the answer to three decimal places).

50) The region bounded by the graph of $f(x) = \frac{1}{1+x^2}$ and the x-axis between $x = 0$ and $x = 1$ is revolved about the x-axis. Find the volume of the solid that is generated.

Answer

$V = \left(\frac{\pi^2}{8} + \frac{\pi}{4}\right)$ units^3

In exercises 51 - 52, solve the initial-value problem for y as a function of x.

51) $(x^2+36)\frac{dy}{dx} = 1, \quad y(6) = 0$

52) $(64-x^2)\frac{dy}{dx} = 1, \quad y(0) = 3$

Answer

$y = \frac{1}{16}\ln\left|\frac{x+8}{x-8}\right| + 3$

53) Find the area bounded by $y = \dfrac{2}{\sqrt{64 - 4x^2}}$, $x = 0$, $y = 0$, and $x = 2$.

54) An oil storage tank can be described as the volume generated by revolving the area bounded by $y = \dfrac{16}{\sqrt{64 + x^2}}$, $x = 0$, $y = 0$, $x = 2$ about the x-axis. Find the volume of the tank (in cubic meters).

Answer

$V = 24.6 \text{ m}^3$

55) During each cycle, the velocity v (in feet per second) of a robotic welding device is given by $v = 2t - \dfrac{14}{4 + t^2}$, where t is time in seconds. Find the expression for the displacement s (in feet) as a function of t if $s = 0$ when $t = 0$.

56) Find the length of the curve $y = \sqrt{16 - x^2}$ between $x = 0$ and $x = 2$.

Answer

$s = \frac{2\pi}{3}$ units

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7.4: Partial Fractions

Learning Objectives

- Integrate a rational function using the method of partial fractions.
- Recognize simple linear factors in a rational function.
- Recognize repeated linear factors in a rational function.
- Recognize quadratic factors in a rational function.

We have seen some techniques that allow us to integrate specific rational functions. For example, we know that

$$\int \frac{du}{u} = \ln|u| + C$$

and

$$\int \frac{du}{u^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C.$$

However, we do not yet have a technique that allows us to tackle arbitrary quotients of this type. Thus, it is not immediately obvious how to go about evaluating

$$\int \frac{3x}{x^2 - x - 2}\, dx.$$

However, we know from material previously developed that

$$\int \left(\frac{1}{x+1} + \frac{2}{x-2}\right) dx = \ln|x+1| + 2\ln|x-2| + C.$$

In fact, by getting a common denominator, we see that

$$\frac{1}{x+1} + \frac{2}{x-2} = \frac{3x}{x^2 - x - 2}.$$

Consequently,

$$\int \frac{3x}{x^2 - x - 2}\, dx = \int \left(\frac{1}{x+1} + \frac{2}{x-2}\right) dx.$$

In this section, we examine the method of **partial fraction decomposition**, which allows us to decompose **rational functions** into sums of simpler, more easily integrated rational functions. Using this method, we can rewrite an expression such as:

$$\frac{3x}{x^2 - x - 2}$$

as an expression such as

$$\frac{1}{x+1} + \frac{2}{x-2}.$$

The key to the method of partial fraction decomposition is being able to anticipate the form that the decomposition of a rational function will take. As we shall see, this form is both predictable and highly dependent on the factorization of the denominator of the rational function. It is also extremely important to keep in mind that partial fraction decomposition can be applied to a rational function $\frac{P(x)}{Q(x)}$ only if $deg(P(x)) < deg(Q(x))$. In the case when $deg(P(x)) \geq deg(Q(x))$, we must first perform long division to rewrite the quotient $\frac{P(x)}{Q(x)}$ in the form $A(x) + \frac{R(x)}{Q(x)}$, where $deg(R(x)) < deg(Q(x))$. We then do a partial fraction decomposition on $\frac{R(x)}{Q(x)}$. The following example, although not requiring partial fraction decomposition, illustrates our approach to integrals of rational functions of the form $\int \frac{P(x)}{Q(x)}\, dx$, where $deg(P(x)) \geq deg(Q(x))$.

Example 7.4.1: Integrating $\int \frac{P(x)}{Q(x)}\,dx$, where $deg(P(x)) \geq deg(Q(x))$

Evaluate

$$\int \frac{x^2+3x+5}{x+1}\,dx.$$

Solution

Since $deg(x^2+3x+5) \geq deg(x+1)$, we perform long division to obtain

$$\frac{x^2+3x+5}{x+1} = x+2+\frac{3}{x+1}.$$

Thus,

$$\int \frac{x^2+3x+5}{x+1}\,dx = \int \left(x+2+\frac{3}{x+1}\right)\,dx = \frac{1}{2}x^2+2x+3\ln|x+1|+C.$$

Visit this website for a review of long division of polynomials.

Exercise 7.4.1

Evaluate

$$\int \frac{x-3}{x+2}\,dx.$$

Hint

Use long division to obtain $\frac{x-3}{x+2} = 1-\frac{5}{x+2}$.

Answer

$$x-5\ln|x+2|+C$$

To integrate $\int \frac{P(x)}{Q(x)}\,dx$, where $deg(P(x)) < deg(Q(x))$, we must begin by factoring $Q(x)$.

Nonrepeated Linear Factors

If $Q(x)$ can be factored as $(a_1x+b_1)(a_2x+b_2)\ldots(a_nx+b_n)$, where each linear factor is distinct, then it is possible to find constants $A_1, A_2, \ldots A_n$ satisfying

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1}+\frac{A_2}{a_2x+b_2}+\cdots+\frac{A_n}{a_nx+b_n}. \tag{7.4.1}$$

The proof that such constants exist is beyond the scope of this course.

In this next example, we see how to use partial fractions to integrate a rational function of this type.

Example 7.4.2: Partial Fractions with Nonrepeated Linear Factors

Evaluate $\int \frac{3x+2}{x^3-x^2-2x}\,dx$.

Solution

Since $deg(3x+2) < deg(x^3-x^2-2x)$, we begin by factoring the denominator of $\frac{3x+2}{x^3-x^2-2x}$. We can see that $x^3-x^2-2x = x(x-2)(x+1)$. Thus, there are constants A, B, and C satisfying Equation 7.4.1 such that

$$\frac{3x+2}{x(x-2)(x+1)} = \frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+1}.$$

We must now find these constants. To do so, we begin by getting a common denominator on the right. Thus,

$$\frac{3x+2}{x(x-2)(x+1)} = \frac{A(x-2)(x+1)+Bx(x+1)+Cx(x-2)}{x(x-2)(x+1)}.$$

Now, we set the numerators equal to each other, obtaining

$$3x+2 = A(x-2)(x+1)+Bx(x+1)+Cx(x-2). \tag{7.4.2}$$

There are two different strategies for finding the coefficients A, B, and C. We refer to these as the *method of equating coefficients* and the *method of strategic substitution*.

Strategy one: Method of Equating Coefficients

Rewrite Equation 7.4.2 in the form

$$3x+2 = (A+B+C)x^2+(-A+B-2C)x+(-2A).$$

Equating coefficients produces the system of equations

$$\begin{aligned} A+B+C &= 0 \\ -A+B-2C &= 3 \\ -2A &= 2. \end{aligned}$$

To solve this system, we first observe that $-2A = 2 \Rightarrow A = -1$. Substituting this value into the first two equations gives us the system

$$\begin{aligned} B+C &= 1 \\ B-2C &= 2. \end{aligned}$$

Multiplying the second equation by -1 and adding the resulting equation to the first produces

$$-3C = 1,$$

which in turn implies that $C = -\frac{1}{3}$. Substituting this value into the equation $B+C=1$ yields $B = \frac{4}{3}$. Thus, solving these equations yields $A = -1, B = \frac{4}{3}$, and $C = -\frac{1}{3}$.

It is important to note that the system produced by this method is consistent if and only if we have set up the decomposition correctly. If the system is inconsistent, there is an error in our decomposition.

Strategy two: Method of Strategic Substitution

The method of strategic substitution is based on the assumption that we have set up the decomposition correctly. If the decomposition is set up correctly, then there must be values of A, B, and C that satisfy Equation 7.4.2 for all values of x. That is, this equation must be true for any value of x we care to substitute into it. Therefore, by choosing values of x carefully and substituting them into the equation, we may find A, B, and C easily. For example, if we substitute $x = 0$, the equation reduces to $2 = A(-2)(1)$. Solving for A yields $A = -1$. Next, by substituting $x = 2$, the equation reduces to $8 = B(2)(3)$, or equivalently $B = 4/3$. Last, we substitute $x = -1$ into the equation and obtain $-1 = C(-1)(-3)$. Solving, we have $C = -\frac{1}{3}$.

It is important to keep in mind that if we attempt to use this method with a decomposition that has not been set up correctly, we are still able to find values for the constants, but these constants are meaningless. If we do opt to use the method of strategic substitution, then it is a good idea to check the result by recombining the terms algebraically.

Now that we have the values of A, B, and C, we rewrite the original integral:

$$\int \frac{3x+2}{x^3-x^2-2x}\,dx = \int \left(-\frac{1}{x}+\frac{4}{3}\cdot\frac{1}{x-2}-\frac{1}{3}\cdot\frac{1}{x+1}\right)dx.$$

Evaluating the integral gives us

$$\int \frac{3x+2}{x^3-x^2-2x}\,dx = -\ln|x|+\frac{4}{3}\ln|x-2|-\frac{1}{3}\ln|x+1|+C.$$

In the next example, we integrate a rational function in which the degree of the numerator is not less than the degree of the denominator.

Example 7.4.3: Dividing before Applying Partial Fractions

Evaluate $\int \frac{x^2+3x+1}{x^2-4}\,dx$.

Solution

Since $deg(x^2+3x+1) \geq deg(x^2-4)$, we must perform long division of polynomials. This results in

$$\frac{x^2+3x+1}{x^2-4} = 1 + \frac{3x+5}{x^2-4}$$

Next, we perform partial fraction decomposition on $\frac{3x+5}{x^2-4} = \frac{3x+5}{(x+2)(x-2)}$. We have

$$\frac{3x+5}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}.$$

Thus,

$$3x+5 = A(x+2) + B(x-2).$$

Solving for A and B using either method, we obtain $A = 11/4$ and $B = 1/4$.

Rewriting the original integral, we have

$$\int \frac{x^2+3x+1}{x^2-4}\,dx = \int \left(1 + \frac{11}{4}\cdot\frac{1}{x-2} + \frac{1}{4}\cdot\frac{1}{x+2}\right)\,dx.$$

Evaluating the integral produces

$$\int \frac{x^2+3x+1}{x^2-4}\,dx = x + \frac{11}{4}\ln|x-2| + \frac{1}{4}\ln|x+2| + C.$$

As we see in the next example, it may be possible to apply the technique of partial fraction decomposition to a nonrational function. The trick is to convert the nonrational function to a rational function through a substitution.

Example 7.4.4: Applying Partial Fractions after a Substitution

Evaluate $\int \frac{\cos x}{\sin^2 x - \sin x}\,dx$.

Solution

Let's begin by letting $u = \sin x$. Consequently, $du = \cos x\,dx$. After making these substitutions, we have

$$\int \frac{\cos x}{\sin^2 x - \sin x}\,dx = \int \frac{du}{u^2-u} = \int \frac{du}{u(u-1)}.$$

Applying partial fraction decomposition to $\frac{1}{u(u-1)}$ gives $\frac{1}{u(u-1)} = -\frac{1}{u} + \frac{1}{u-1}$.

Thus,

$$\int \frac{\cos x}{\sin^2 x - \sin x}\,dx = -\ln|u| + \ln|u-1| + C = -\ln|\sin x| + \ln|\sin x - 1| + C.$$

Exercise 7.4.2

Evaluate $\int \frac{x+1}{(x+3)(x-2)}\,dx$.

Hint

$$\frac{x+1}{\quad} = \frac{A}{\quad} + \frac{B}{\quad}$$

$$\frac{}{(x+3)(x-2)} = \frac{}{x+3} + \frac{}{x-2}$$

Answer

$$\frac{2}{5}\ln|x+3| + \frac{3}{5}\ln|x-2| + C$$

Repeated Linear Factors

For some applications, we need to integrate rational expressions that have denominators with repeated linear factors—that is, rational functions with at least one factor of the form $(ax+b)^n$, where n is a positive integer greater than or equal to 2. If the denominator contains the repeated linear factor $(ax+b)^n$, then the decomposition must contain

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}. \tag{7.4.3}$$

As we see in our next example, the basic technique used for solving for the coefficients is the same, but it requires more algebra to determine the numerators of the partial fractions.

Example 7.4.5: Partial Fractions with Repeated Linear Factors

Evaluate $\int \frac{x-2}{(2x-1)^2(x-1)}\,dx$.

Solution

We have $deg(x-2) < deg((2x-1)^2(x-1))$, so we can proceed with the decomposition. Since $(2x-1)^2$ is a repeated linear factor, include

$$\frac{A}{2x-1} + \frac{B}{(2x-1)^2}$$

in the decomposition in Equation 7.4.3. Thus,

$$\frac{x-2}{(2x-1)^2(x-1)} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2} + \frac{C}{x-1}.$$

After getting a common denominator and equating the numerators, we have

$$x-2 = A(2x-1)(x-1) + B(x-1) + C(2x-1)^2. \tag{7.4.4}$$

We then use the method of equating coefficients to find the values of A, B, and C.

$$x-2 = (2A+4C)x^2 + (-3A+B-4C)x + (A-B+C).$$

Equating coefficients yields $2A+4C=0$, $-3A+B-4C=1$, and $A-B+C=-2$. Solving this system yields $A=2, B=3$, and $C=-1$.

Alternatively, we can use the method of strategic substitution. In this case, substituting $x=1$ and $x=1/2$ into Equation 7.4.4 easily produces the values $B=3$ and $C=-1$. At this point, it may seem that we have run out of good choices for x, however, since we already have values for B and C, we can substitute in these values and choose any value for x not previously used. The value $x=0$ is a good option. In this case, we obtain the equation $-2 = A(-1)(-1) + 3(-1) + (-1)(-1)^2$ or, equivalently, $A=2$.

Now that we have the values for A, B, and C, we rewrite the original integral and evaluate it:

$$\begin{aligned}\int \frac{x-2}{(2x-1)^2(x-1)}\,dx &= \int \left(\frac{2}{2x-1} + \frac{3}{(2x-1)^2} - \frac{1}{x-1}\right) dx \\ &= \ln|2x-1| - \frac{3}{2(2x-1)} - \ln|x-1| + C.\end{aligned}$$

Exercise 7.4.3

Set up the partial fraction decomposition for

$$\int \frac{x+2}{(x+3)^3(x-4)^2}\,dx.$$

(Do not solve for the coefficients or complete the integration.)

Hint

Use the problem-solving method of Example 7.4.5 for guidance.

Answer

$$\frac{x+2}{(x+3)^3(x-4)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{(x+3)^3} + \frac{D}{(x-4)} + \frac{E}{(x-4)^2}$$

The General Method

Now that we are beginning to get the idea of how the technique of partial fraction decomposition works, let's outline the basic method in the following problem-solving strategy.

Problem-Solving Strategy: Partial Fraction Decomposition

To decompose the rational function $P(x)/Q(x)$, use the following steps:

1. Make sure that $deg(P(x)) < deg(Q(x))$. If not, perform long division of polynomials.
2. Factor $Q(x)$ into the product of linear and irreducible quadratic factors. An irreducible quadratic is a quadratic that has no real zeros.
3. Assuming that $deg(P(x)) < deg(Q(x)$, the factors of $Q(x)$ determine the form of the decomposition of $P(x)/Q(x)$.
   a. If $Q(x)$ can be factored as $(a_1x+b_1)(a_2x+b_2)\ldots(a_nx+b_n)$, where each linear factor is distinct, then it is possible to find constants $A_1, A_2, \ldots A_n$ satisfying
   $$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n}.$$
   b. If $Q(x)$ contains the repeated linear factor $(ax+b)^n$, then the decomposition must contain
   $$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}.$$
   c. For each irreducible quadratic factor ax^2+bx+c that $Q(x)$ contains, the decomposition must include
   $$\frac{Ax+B}{ax^2+bx+c}.$$
   d. For each repeated irreducible quadratic factor $(ax^2+bx+c)^n$, the decomposition must include
   $$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}.$$
   e. After the appropriate decomposition is determined, solve for the constants.
   f. Last, rewrite the integral in its decomposed form and evaluate it using previously developed techniques or integration formulas.

Simple Quadratic Factors

Now let's look at integrating a rational expression in which the denominator contains an irreducible quadratic factor. Recall that the quadratic ax^2+bx+c is irreducible if $ax^2+bx+c=0$ has no real zeros—that is, if $b^2-4ac<0$.

Example 7.4.6: Rational Expressions with an Irreducible Quadratic Factor

Evaluate

$$\int \frac{2x-3}{x^3+x}\,dx.$$

Solution

Since $deg(2x-3) < deg(x^3+x)$, factor the denominator and proceed with partial fraction decomposition. Since $x^3+x = x(x^2+1)$ contains the irreducible quadratic factor x^2+1, include $\frac{Ax+B}{x^2+1}$ as part of the decomposition, along with $\frac{C}{x}$ for the linear term x. Thus, the decomposition has the form

$$\frac{2x-3}{x(x^2+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x}.$$

After getting a common denominator and equating the numerators, we obtain the equation

$$2x-3 = (Ax+B)x + C(x^2+1).$$

Solving for A, B, and C, we get $A = 3, B = 2$, and $C = -3$.

Thus,

$$\frac{2x-3}{x^3+x} = \frac{3x+2}{x^2+1} - \frac{3}{x}.$$

Substituting back into the integral, we obtain

$$\begin{aligned}
\int \frac{2x-3}{x^3+x}\,dx &= \int \left(\frac{3x+2}{x^2+1} - \frac{3}{x}\right) dx \\
&= 3\int \frac{x}{x^2+1}\,dx + 2\int \frac{1}{x^2+1}\,dx - 3\int \frac{1}{x}\,dx \qquad \text{Split up the integral} \\
&= \frac{3}{2}\ln\,|\,x^2+1\,| + 2\tan^{-1}x - 3\ln|x| + C. \qquad \text{Evaluate each integral}
\end{aligned}$$

Note: We may rewrite $\ln\,|\,x^2+1\,| = \ln(x^2+1)$, if we wish to do so, since $x^2+1 > 0$.

Example 7.4.7: Partial Fractions with an Irreducible Quadratic Factor

Evaluate $\int \frac{dx}{x^3-8}$.

Solution: We can start by factoring $x^3-8 = (x-2)(x^2+2x+4)$. We see that the quadratic factor x^2+2x+4 is irreducible since $2^2 - 4(1)(4) = -12 < 0$. Using the decomposition described in the problem-solving strategy, we get

$$\frac{1}{(x-2)(x^2+2x+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+4}.$$

After obtaining a common denominator and equating the numerators, this becomes

$$1 = A(x^2+2x+4) + (Bx+C)(x-2).$$

Applying either method, we get $A = \frac{1}{12}, B = -\frac{1}{12}$, and $C = -\frac{1}{3}$.

Rewriting $\int \frac{dx}{x^3-8}$, we have

$$\int \frac{dx}{x^3-8} = \frac{1}{12}\int \frac{1}{x-2}\,dx - \frac{1}{12}\int \frac{x+4}{x^2+2x+4}\,dx.$$

We can see that

$$\int \frac{1}{x-2}\,dx = \ln|x-2| + C,$$

but

$$\int \frac{x+4}{x^2+2x+4}\,dx$$

requires a bit more effort. Let's begin by **completing the square** on x^2+2x+4 to obtain

$$x^2+2x+4=(x+1)^2+3.$$

By letting $u=x+1$ and consequently $du=dx$, we see that

$$\begin{aligned}
\int \frac{x+4}{x^2+2x+4}\,dx &= \int \frac{x+4}{(x+1)^2+3}\,dx && \text{Complete the square on the denominator}\\
&= \int \frac{u+3}{u^2+3}\,du && \text{Substitute } u=x+1,\ x=u-1, \text{ and } du=dx\\
&= \int \frac{u}{u^2+3}du+\int \frac{3}{u^2+3}du && \text{Split the numerator apart}\\
&= \frac{1}{2}\ln\mid u^2+3\mid && \text{Evaluate each integral}\\
&\quad +\frac{3}{\sqrt{3}}\tan^{-1}\frac{u}{\sqrt{3}}+C\\
&= \frac{1}{2}\ln\mid x^2+2x+4\mid && \text{Rewrite in terms of } x \text{ and simplify}\\
&\quad +\sqrt{3}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right)+C
\end{aligned}$$

Substituting back into the original integral and simplifying gives

$$\int \frac{dx}{x^3-8}=\frac{1}{12}\ln|x-2|-\frac{1}{24}\ln|x^2+2x+4|-\frac{\sqrt{3}}{12}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right)+C.$$

Here again, we can drop the absolute value if we wish to do so, since $x^2+2x+4>0$ for all x.

Example 7.4.8: Finding a Volume

Find the volume of the solid of revolution obtained by revolving the region enclosed by the graph of $f(x)=\dfrac{x^2}{(x^2+1)^2}$ and the x-axis over the interval $[0,1]$ about the y-axis.

Solution

Let's begin by sketching the region to be revolved (see Figure 7.4.1). From the sketch, we see that the shell method is a good choice for solving this problem.

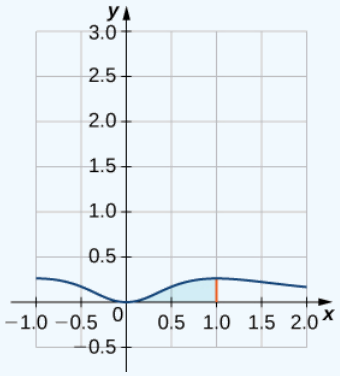

Figure 7.4.1: We can use the shell method to find the volume of revolution obtained by revolving the region shown about the y-axis.

The volume is given by

$$V=2\pi\int^{1} x\cdot\frac{x^2}{(x^2+1)^2}\,dx=2\pi\int^{1}\frac{x^3}{(x^2+1)^2}\,dx.$$

$$\int_0^1 \frac{x^3}{(x^2+1)^2}\,dx \qquad \int_0^1 \frac{x^3}{(x^2+1)^2}\,dx$$

Since $deg((x^2+1)^2) = 4 > 3 = deg(x^3)$, we can proceed with partial fraction decomposition. Note that $(x^2+1)^2$ is a repeated irreducible quadratic. Using the decomposition described in the problem-solving strategy, we get

$$\frac{x^3}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}.$$

Finding a common denominator and equating the numerators gives

$$x^3 = (Ax+B)(x^2+1) + Cx + D.$$

Solving, we obtain $A = 1, B = 0, C = -1$, and $D = 0$. Substituting back into the integral, we have

$$\begin{aligned} V &= 2\pi \int_0^1 \frac{x^3}{(x^2+1)^2}\,dx = 2\pi \int_0^1 \left(\frac{x}{x^2+1} - \frac{x}{(x^2+1)^2}\right) dx \\ &= 2\pi \left(\frac{1}{2}\ln(x^2+1) + \frac{1}{2}\cdot\frac{1}{x^2+1}\right)\Big|_0^1 = \pi\left(\ln 2 - \tfrac{1}{2}\right). \end{aligned}$$

? Exercise 7.4.4

Set up the partial fraction decomposition for

$$\int \frac{x^2+3x+1}{(x+2)(x-3)^2(x^2+4)^2}\,dx.$$

Hint

Use the problem-solving strategy.

Answer

$$\frac{x^2+3x+1}{(x+2)(x-3)^2(x^2+4)^2} = \frac{A}{x+2} + \frac{B}{x-3} + \frac{C}{(x-3)^2} + \frac{Dx+E}{x^2+4} + \frac{Fx+G}{(x^2+4)^2}$$

Key Concepts

- Partial fraction decomposition is a technique used to break down a rational function into a sum of simple rational functions that can be integrated using previously learned techniques.
- When applying partial fraction decomposition, we must make sure that the degree of the numerator is less than the degree of the denominator. If not, we need to perform long division before attempting partial fraction decomposition.
- The form the decomposition takes depends on the type of factors in the denominator. The types of factors include nonrepeated linear factors, repeated linear factors, nonrepeated irreducible quadratic factors, and repeated irreducible quadratic factors.

Glossary

partial fraction decomposition

a technique used to break down a rational function into the sum of simple rational functions

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7.4E: Exercises for Section 7.4

Use partial fraction decomposition (or a simpler technique) to express the rational function as a sum or difference of two or more simpler rational expressions.

1) $\dfrac{1}{(x-3)(x-2)}$

2) $\dfrac{x^2+1}{x(x+1)(x+2)}$

Answer

$$\frac{x^2+1}{x(x+1)(x+2)} = -\frac{2}{x+1}+\frac{5}{2(x+2)}+\frac{1}{2x}$$

3) $\dfrac{1}{x^3-x}$

4) $\dfrac{3x+1}{x^2}$

Answer

$$\frac{3x+1}{x^2} = \frac{1}{x^2}+\frac{3}{x}$$

5) $\dfrac{3x^2}{x^2+1}$ (Hint: Use long division first.)

6) $\dfrac{2x^4}{x^2-2x}$

Answer

$$\frac{2x^4}{x^2-2x} = 2x^2+4x+8+\frac{16}{x-2}$$

7) $\dfrac{1}{(x-1)(x^2+1)}$

8) $\dfrac{1}{x^2(x-1)}$

Answer

$$\frac{1}{x^2(x-1)} = -\frac{1}{x^2}-\frac{1}{x}+\frac{1}{x-1}$$

9) $\dfrac{x}{x^2-4}$

10) $\dfrac{1}{x(x-1)(x-2)(x-3)}$

Answer

$$\frac{1}{x(x-1)(x-2)(x-3)} = -\frac{1}{2(x-2)}+\frac{1}{2(x-1)}-\frac{1}{6x}+\frac{1}{6(x-3)}$$

11) $\dfrac{1}{x^4-1} = \dfrac{1}{(x+1)(x-1)(x^2+1)}$

12) $\dfrac{3x^2}{x^3-1} = \dfrac{3x^2}{(x-1)(x^2+x+1)}$

Answer

$$\frac{3x^2}{x^3-1} = \frac{1}{x-1}+\frac{2x+1}{x^2+x+1}$$

13) $\dfrac{2x}{(x+2)^2}$

14) $\dfrac{3x^4+x^3+20x^2+3x+31}{(x+1)(x^2+4)^2}$

Answer

$$\frac{3x^4+x^3+20x^2+3x+31}{(x+1)(x^2+4)^2} = \frac{2}{x+1}+\frac{x}{x^2+4}-\frac{1}{(x^2+4)^2}$$

In exercises 15 - 25, use the method of partial fractions to evaluate each of the following integrals.

15) $\displaystyle\int \frac{dx}{(x-3)(x-2)}$

16) $\displaystyle\int \frac{3x}{x^2+2x-8}\,dx$

Answer

$$\int \frac{3x}{x^2+2x-8}\,dx = 2\ln|x+4|+\ln|x-2|+C = \ln\left|(x+4)^2(x-2)\right|+C$$

17) $\displaystyle\int \frac{dx}{x^3-x}$

18) $\displaystyle\int \frac{x}{x^2-4}\,dx$

Answer

Note that you don't need Partial Fractions here. We use a simple u-substitution.

$$\int \frac{x}{x^2-4}\,dx = \tfrac{1}{2}\ln|4-x^2|+C$$

19) $\displaystyle\int \frac{dx}{x(x-1)(x-2)(x-3)}$

20) $\displaystyle\int \frac{2x^2+4x+22}{x^2+2x+10}\,dx$

Answer

Note that since the degree of the numerator is equal to the degree of the denominator, we need to start with long division. Then note that we will need to use completing the square to continue since we cannot factor the trinomial in the denominator.

$$\int \frac{2x^2+4x+22}{x^2+2x+10}\,dx = 2\left(x+\tfrac{1}{3}\arctan\left(\frac{1+x}{3}\right)\right)+C$$

21) $\displaystyle\int \frac{dx}{x^2-5x+6}$

22) $\displaystyle\int \frac{2-x}{x^2+x}\,dx$

Answer

$$\int \frac{2-x}{x^2+x}\,dx = 2\ln|x|-3\ln|1+x|+C = \ln\left|\frac{x^2}{(1+x)^3}\right|+C$$

23) $\displaystyle\int \frac{2}{x^2-x-6}\,dx$

24) $\displaystyle\int \frac{dx}{x^3-2x^2-4x+8}$

Answer

$$\int \frac{dx}{x^3-2x^2-4x+8} = \tfrac{1}{16}\left(-\frac{4}{-2+x}-\ln|-2+x|+\ln|2+x|\right)+C = \tfrac{1}{16}\left(-\frac{4}{-2+x}+\ln\left|\frac{x+2}{x-2}\right|\right)+C$$

25) $\displaystyle\int \frac{dx}{x^4-10x^2+9}$

In exercises 26 - 29, evaluate the integrals with irreducible quadratic factors in the denominators.

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26) $\int \frac{2}{(x-4)(x^2+2x+6)}\,dx$

Answer

$$\int \frac{2}{(x-4)(x^2+2x+6)}\,dx \quad = \quad \tfrac{1}{30}(-2\sqrt{5}\arctan\left[\frac{1+x}{\sqrt{5}}\right] + 2\ln|-4+x| - \ln|6+2x+x^2|) + C$$

27) $\int \frac{x^2}{x^3-x^2+4x-4}\,dx$

28) $\int \frac{x^3+6x^2+3x+6}{x^3+2x^2}\,dx$

Answer

Note that we need to use long division first, since the degree of the numerator is greater than the degree of the denominator.

$$\int \frac{x^3+6x^2+3x+6}{x^3+2x^2}\,dx \quad = \quad -\frac{3}{x} + 4\ln|x+2| + x + C$$

29) $\int \frac{x}{(x-1)(x^2+2x+2)^2}\,dx$

In exercises 30 - 32, use the method of partial fractions to evaluate the integrals.

30) $\int \frac{3x+4}{(x^2+4)(3-x)}\,dx$

Answer

$$\int \frac{3x+4}{(x^2+4)(3-x)}\,dx \quad = \quad -\ln|3-x| + \tfrac{1}{2}\ln|x^2+4| + C$$

31) $\int \frac{2}{(x+2)^2(2-x)}\,dx$

32) $\int \frac{3x+4}{x^3-2x-4}\,dx$ (Hint: Use the rational root theorem.)

Answer

$$\int \frac{3x+4}{x^3-2x-4}\,dx \quad = \quad \ln|x-2| - \tfrac{1}{2}\ln|x^2+2x+2| + C$$

In exercises 33 - 46, use substitution to convert the integrals to integrals of rational functions. Then use partial fractions to evaluate the integrals.

33) $\int_0^1 \frac{e^x}{36-e^{2x}}\,dx$ (Give the exact answer and the decimal equivalent. Round to five decimal places.)

34) $\int \frac{e^x}{e^{2x}-e^x}\,dx$

Answer

$$\int \frac{e^x}{e^{2x}-e^x}\,dx \quad = \quad -x + \ln|1-e^x| + C$$

35) $\int \frac{\sin x\,dx}{1-\cos^2 x}$

36) $\int \frac{\sin x}{\cos^2 x + \cos x - 6}\,dx$

Answer

$$\int \frac{\sin x}{\cos^2 x + \cos x - 6}\,dx \quad = \quad \tfrac{1}{5}\ln\left|\frac{\cos x+3}{\cos x-2}\right| + C$$

37) $\int \frac{1-\sqrt{x}}{1+\sqrt{x}}\,dx$

38) $\int \frac{dt}{(e^t-e^{-t})^2}$

Answer

$$\int \frac{dt}{(e^t - e^{-t})^2} = \frac{1}{2 - 2e^{2t}} + C$$

39) $\int \frac{1 + e^x}{1 - e^x}\, dx$

40) $\int \frac{dx}{1 + \sqrt{x+1}}$

Answer

$$\int \frac{dx}{1 + \sqrt{x+1}} = 2\sqrt{1+x} - 2\ln|1 + \sqrt{1+x}| + C$$

41) $\int \frac{dx}{\sqrt{x} + \sqrt[4]{x}}$

42) $\int \frac{\cos x}{\sin x(1 - \sin x)}\, dx$

Answer

$$\int \frac{\cos x}{\sin x(1 - \sin x)}\, dx = \ln\left|\frac{\sin x}{1 - \sin x}\right| + C$$

43) $\int \frac{e^x}{(e^{2x} - 4)^2}\, dx$

44) $\int_1^2 \frac{1}{x^2\sqrt{4 - x^2}}\, dx$

Answer

$$\int_1^2 \frac{1}{x^2\sqrt{4 - x^2}}\, dx = \frac{\sqrt{3}}{4}$$

45) $\int \frac{1}{2 + e^{-x}}\, dx$

46) $\int \frac{1}{1 + e^x}\, dx$

Answer

$$\int \frac{1}{1 + e^x}\, dx = x - \ln(1 + e^x) + C$$

In exercises 47 - 48, use the given substitution to convert the integral to an integral of a rational function, then evaluate.

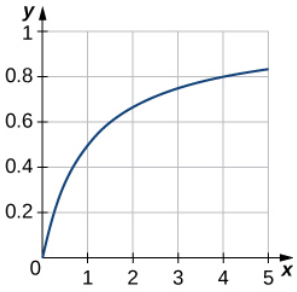
47) $\int \frac{1}{t - \sqrt[3]{t}}\, dt; \quad t = x^3$

48) $\int \frac{1}{\sqrt{x} + \sqrt[3]{x}}\, dx; \quad x = u^6$

Answer

$$\int \frac{1}{\sqrt{x} + \sqrt[3]{x}}\, dx = 6x^{1/6} - 3x^{1/3} + 2\sqrt{x} - 6\ln(1 + x^{1/6}) + C$$

49) Graph the curve $y = \frac{x}{1+x}$ over the interval $[0, 5]$. Then, find the area of the region bounded by the curve, the x-axis, and the line $x = 4$.



50) Find the volume of the solid generated when the region bounded by $y = \dfrac{1}{\sqrt{x(3-x)}}$, $y = 0$, $x = 1$, and $x = 2$ is revolved about the x-axis.

Answer

$V = \frac{4}{3}\pi \text{arctanh}\ \left[\frac{1}{3}\right] = \frac{1}{3}\pi \ln 4 \text{ units}^3$

51) The velocity of a particle moving along a line is a function of time given by $v(t) = \dfrac{88t^2}{t^2+1}$. Find the distance that the particle has traveled after $t = 5$ sec.

In exercises 52 - 54, solve the initial-value problem for x as a function of t.

52) $(t^2 - 7t + 12)\dfrac{dx}{dt} = 1, \quad t > 4,\ x(5) = 0$

Answer

$x = -\ln|t-3| + \ln|t-4| + \ln 2 = \ln\left|\dfrac{2(t-4)}{t-3}\right|$

53) $(t+5)\dfrac{dx}{dt} = x^2 + 1, \quad t > -5,\ x(1) = \tan 1$

54) $(2t^3 - 2t^2 + t - 1)\dfrac{dx}{dt} = 3, \quad x(2) = 0$

Answer

$x = \ln|t-1| - \sqrt{2}\arctan(\sqrt{2}t) - \frac{1}{2}\ln(t^2 + \frac{1}{2}) + \sqrt{2}\arctan(2\sqrt{2}) + \frac{1}{2}\ln 4.5$

55) Find the x-coordinate of the centroid of the area bounded by $y(x^2 - 9) = 1$, $y = 0$, $x = 4$, and $x = 5$. (Round the answer to two decimal places.)

56) Find the volume generated by revolving the area bounded by $y = \dfrac{1}{x^3 + 7x^2 + 6x}$, $x = 1$, $x = 7$, and $y = 0$ about the y-axis.

Answer

$V = \frac{2}{5}\pi \ln \frac{28}{13} \text{ units}^3$

57) Find the area bounded by $y = \dfrac{x - 12}{x^2 - 8x - 20}$, $y = 0$, $x = 2$, and $x = 4$. (Round the answer to the nearest hundredth.)

58) Evaluate the integral $\displaystyle\int \frac{dx}{x^3 + 1}$.

Answer

$$\int \frac{dx}{x^3+1} = \frac{\arctan\left[\frac{-1+2x}{\sqrt{3}}\right]}{\sqrt{3}} + \frac{1}{3}\ln|1+x| - \frac{1}{6}\ln|1 - x + x^2| + C$$

For problems 59 - 62, use the substitutions $\tan(\frac{x}{2}) = t$, $dx = \dfrac{2}{1+t^2}\,dt$, $\sin x = \dfrac{2t}{1+t^2}$, and $\cos x = \dfrac{1-t^2}{1+t^2}$.

59) $\displaystyle\int \frac{dx}{3 - 5\sin x}$

60) Find the area under the curve $y = \dfrac{1}{1 + \sin x}$ between $x = 0$ and $x = \pi$. (Assume the dimensions are in inches.)

Answer

2.0 in.^{2}

61) Given $\tan\left(\frac{x}{2}\right) = t$, derive the formulas $dx = \dfrac{2}{1+t^2}\,dt$, $\sin x = \dfrac{2t}{1+t^2}$, and $\cos x = \dfrac{1-t^2}{1+t^2}$.

62) Evaluate $\displaystyle\int \frac{\sqrt[3]{x-8}}{x}\,dx$.

Answer

$$\int \frac{\sqrt[3]{x-8}}{x}\,dx = 3(-8+x)^{1/3} - 2\sqrt{3}\arctan\left[\frac{-1+(-8+x)^{1/3}}{\sqrt{3}}\right] - 2\ln\left[2+(-8+x)^{1/3}\right] + \ln\left[4-2(-8+x)^{1/3}+(-8+x)^{2/3}\right] + C$$

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7.5: Other Strategies for Integration

Learning Objectives

- Use a table of integrals to solve integration problems.
- Use a computer algebra system (CAS) to solve integration problems.

In addition to the techniques of integration we have already seen, several other tools are widely available to assist with the process of integration. Among these tools are **integration tables**, which are readily available in many books, including the appendices to this one. Also widely available are **computer algebra systems (CAS)**, which are found on calculators and in many campus computer labs, and are free online.

Tables of Integrals

Integration tables, if used in the right manner, can be a handy way either to evaluate or check an integral quickly. Keep in mind that when using a table to check an answer, it is possible for two completely correct solutions to look very different. For example, in Trigonometric Substitution, we found that, by using the substitution $x = \tan\theta$, we can arrive at

$$\int \frac{dx}{\sqrt{1+x^2}} = \ln\left|x+\sqrt{x^2+1}\right| + C.$$

However, using $x = \sinh\theta$, we obtained a different solution—namely,

$$\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x + C.$$

We later showed algebraically that the two solutions are equivalent. That is, we showed that $\sinh^{-1} x = \ln\left|x+\sqrt{x^2+1}\right|$. In this case, the two antiderivatives that we found were actually equal. This need not be the case. However, as long as the difference in the two antiderivatives is a constant, they are equivalent.

Example 7.5.1: Using a Formula from a Table to Evaluate an Integral

Use the table formula

$$\int \frac{\sqrt{a^2-u^2}}{u^2}du = -\frac{\sqrt{a^2-u^2}}{u} - \sin^{-1}\frac{u}{a} + C$$

to evaluate $\displaystyle\int \frac{\sqrt{16-e^{2x}}}{e^x}dx$.

Solution

If we look at integration tables, we see that several formulas contain expressions of the form $\sqrt{a^2-u^2}$. This expression is actually similar to $\sqrt{16-e^{2x}}$, where $a=4$ and $u=e^x$. Keep in mind that we must also have $du=e^x$. Multiplying the numerator and the denominator of the given integral by e^x should help to put this integral in a useful form. Thus, we now have

$$\int \frac{\sqrt{16-e^{2x}}}{e^x}dx = \int \frac{\sqrt{16-e^{2x}}}{e^{2x}}e^x dx.$$

Substituting $u=e^x$ and $du=e^x\,dx$ produces $\displaystyle\int \frac{\sqrt{a^2-u^2}}{u^2}du$. From the integration table (#88 in Appendix A),

$$\int \frac{\sqrt{a^2-u^2}}{u^2}du = -\frac{\sqrt{a^2-u^2}}{u} - \sin^{-1}\frac{u}{a} + C.$$

Thus,

$$\begin{aligned}\int \frac{\sqrt{16-e^{2x}}}{e^x}dx &= \int \frac{\sqrt{16-e^{2x}}}{e^{2x}}e^x dx \text{ Substitute } u=e^x \text{ and } du=e^x dx.\\ &= \int \frac{\sqrt{4^2-u^2}}{u^2}du \text{ Apply the formula using } a=4.\\ &= -\frac{\sqrt{4^2-u^2}}{u} - \sin^{-1}\frac{u}{4} + C \text{ Substitute } u=e^x.\end{aligned}$$

$$= -\frac{\sqrt{16 - e^{2x}}}{e^x} - \sin^{-1}\left(\frac{e^x}{4}\right) + C$$

Computer Algebra Systems

If available, a CAS is a faster alternative to a table for solving an integration problem. Many such systems are widely available and are, in general, quite easy to use.

Example 7.5.2: Using a Computer Algebra System to Evaluate an Integral

Use a computer algebra system to evaluate $\int \frac{dx}{\sqrt{x^2-4}}$. Compare this result with $\ln\left|\frac{\sqrt{x^2-4}}{2} + \frac{x}{2}\right| + C$, a result we might have obtained if we had used trigonometric substitution.

Solution

Using Wolfram Alpha, we obtain

$$\int \frac{dx}{\sqrt{x^2-4}} = \ln\left|\sqrt{x^2-4} + x\right| + C.$$

Notice that

$$\ln\left|\frac{\sqrt{x^2-4}}{2} + \frac{x}{2}\right| + C = \ln\left|\frac{\sqrt{x^2-4}+x}{2}\right| + C = \ln\left|\sqrt{x^2-4}+x\right| - \ln 2 + C.$$

Since these two antiderivatives differ by only a constant, the solutions are equivalent. We could have also demonstrated that each of these antiderivatives is correct by differentiating them.

You can access an integral calculator for more examples.

Example 7.5.3: Using a CAS to Evaluate an Integral

Evaluate $\int \sin^3 x\, dx$ using a CAS. Compare the result to $\frac{1}{3}\cos^3 x - \cos x + C$, the result we might have obtained using the technique for integrating odd powers of $\sin x$ discussed earlier in this chapter.

Solution

Using Wolfram Alpha, we obtain

$$\int \sin^3 x\, dx = \frac{1}{12}(\cos(3x) - 9\cos x) + C.$$

This looks quite different from $\frac{1}{3}\cos^3 x - \cos x + C$. To see that these antiderivatives are equivalent, we can make use of a few trigonometric identities:

$$\begin{aligned}
\frac{1}{12}(\cos(3x) - 9\cos x) &= \frac{1}{12}(\cos(x+2x) - 9\cos x) \\
&= \frac{1}{12}(\cos(x)\cos(2x) - \sin(x)\sin(2x) - 9\cos x) \\
&= \frac{1}{12}(\cos x(2\cos^2 x - 1) - \sin x(2\sin x\cos x) - 9\cos x) \\
&= \frac{1}{12}(2\cos^3 x - \cos x - 2\cos x(1-\cos^2 x) - 9\cos x) \\
&= \frac{1}{12}(4\cos^3 x - 12\cos x) \\
&= \frac{1}{3}\cos^3 x - \cos x.
\end{aligned}$$

Thus, the two antiderivatives are identical.

We may also use a CAS to compare the graphs of the two functions, as shown in the following figure.

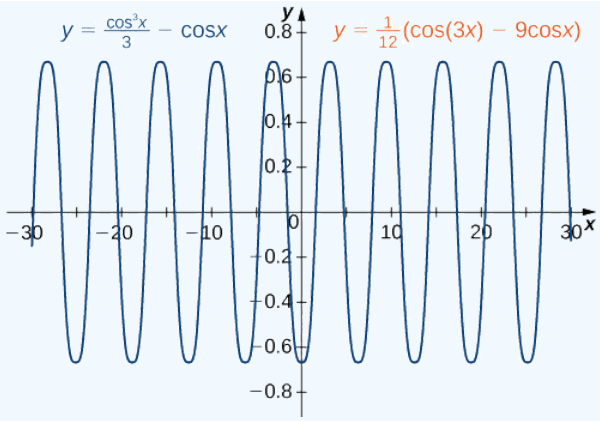

Figure 7.5.1: The graphs of $y = \frac{1}{3}\cos^3 x - \cos x$ and $y = \frac{1}{12}(\cos(3x) - 9\cos x)$ are identical.

? Exercise 7.5.1

Use a CAS to evaluate $\int \frac{dx}{\sqrt{x^2+4}}$.

Hint

Answers may vary.

Answer

Possible solutions include $\sinh^{-1}\left(\frac{x}{2}\right) + C$ and $\ln\left|\sqrt{x^2+4} + x\right| + C$.

Key Concepts

- An integration table may be used to evaluate indefinite integrals.
- A CAS (or computer algebra system) may be used to evaluate indefinite integrals.
- It may require some effort to reconcile equivalent solutions obtained using different methods.

Glossary

computer algebra system (CAS)

technology used to perform many mathematical tasks, including integration

integration table

a table that lists integration formulas

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7.5E: Exercises for Section 7.5

Use a table of integrals to evaluate the following integrals.

1) $\int_0^4 \frac{x}{\sqrt{1+2x}}\,dx$

2) $\int \frac{x+3}{x^2+2x+2}\,dx$

Answer

$$\int \frac{x+3}{x^2+2x+2}\,dx = \tfrac{1}{2}\ln|x^2+2x+2| + 2\arctan(x+1) + C$$

3) $\int x^3\sqrt{1+2x^2}\,dx$

4) $\int \frac{1}{\sqrt{x^2+6x}}\,dx$

Answer

$$\int \frac{1}{\sqrt{x^2+6x}}\,dx = \cosh^{-1}\left(\frac{x+3}{3}\right) + C$$

5) $\int \frac{x}{x+1}\,dx$

6) $\int x\cdot 2^{x^2}\,dx$

Answer

$$\int x\cdot 2^{x^2}\,dx = \frac{2^{x^2-1}}{\ln 2} + C$$

7) $\int \frac{1}{4x^2+25}\,dx$

8) $\int \frac{dy}{\sqrt{4-y^2}}$

Answer

$$\int \frac{dy}{\sqrt{4-y^2}} = \arcsin\left(\frac{y}{2}\right) + C$$

9) $\int \sin^3(2x)\cos(2x)\,dx$

10) $\int \csc(2w)\cot(2w)\,dw$

Answer

$$\int \csc(2w)\cot(2w)\,dw = -\tfrac{1}{2}\csc(2w) + C$$

11) $\int 2^y\,dy$

12) $\int_0^1 \frac{3x}{\sqrt{x^2+8}}\,dx$

Answer

$$\int_0^1 \frac{3x}{\sqrt{x^2+8}}\,dx = 9 - 6\sqrt{2}$$

13) $\int_{-1/4}^{1/4} \sec^2(\pi x)\tan(\pi x)\,dx$

14) $\int_0^{\pi/2} \tan^2\left(\frac{x}{2}\right)\,dx$

Answer

$$\int_0^{\pi/2} \tan^2\left(\frac{x}{2}\right)\,dx = 2 - \frac{\pi}{2}$$

15) $\int \cos^3 x\,dx$

16) $\int \tan^5(3x)\,dx$

Answer

$$\int \tan^5(3x)\,dx = \tfrac{1}{12}\tan^4(3x) - \tfrac{1}{6}\tan^2(3x) + \tfrac{1}{3}\ln|\sec 3x| + C$$

17) $\int \sin^2 y \cos^3 y\,dy$

Use a CAS to evaluate the following integrals. Tables can also be used to verify the answers.

18) [T] $\int \frac{dw}{1+\sec\left(\frac{w}{2}\right)}$

Answer

$$\int \frac{dw}{1+\sec\left(\frac{w}{2}\right)} = 2\cot\left(\tfrac{w}{2}\right) - 2\csc\left(\tfrac{w}{2}\right) + w + C$$

19) [T] $\int \frac{dw}{1-\cos(7w)}$

20) [T] $\int_0^t \frac{dt}{4\cos t + 3\sin t}$

Answer

$$\int_0^t \frac{dt}{4\cos t + 3\sin t} = \tfrac{1}{5}\ln\left|\frac{2(5+4\sin t - 3\cos t)}{4\cos t + 3\sin t}\right|$$

21) [T] $\int \frac{\sqrt{x^2-9}}{3x}\,dx$

22) [T] $\int \frac{dx}{x^{1/2}+x^{1/3}}$

Answer

$$\int \frac{dx}{x^{1/2}+x^{1/3}} = 6x^{1/6} - 3x^{1/3} + 2\sqrt{x} - 6\ln[1+x^{1/6}] + C$$

23) [T] $\int \frac{dx}{x\sqrt{x-1}}$

24) [T] $\int x^3 \sin x\,dx$

Answer

$$\int x^3 \sin x\,dx = -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C$$

25) [T] $\int x\sqrt{x^4-9}\,dx$

26) [T] $\int \frac{x}{1+e^{-x^2}}\,dx$

Answer

$$\int \frac{x}{1+e^{-x^2}}\,dx = \tfrac{1}{2}\left(x^2 + \ln|1+e^{-x^2}|\right) + C$$

27) [T] $\int \frac{\sqrt{3-5x}}{2x}\,dx$

28) [T] $\int \frac{dx}{x\sqrt{x-1}}$

Answer

$$\int \frac{dx}{x\sqrt{x-1}} = 2\arctan\left(\sqrt{x-1}\right) + C$$

29) [T] $\int e^x \cos^{-1}(e^x)\,dx$

Use a calculator or CAS to evaluate the following integrals.

30) [T] $\int_0^{\pi/4} \cos 2x\,dx$

Answer

$$\int_0^{\pi/4} \cos 2x\,dx = 0.5 = \frac{1}{2}$$

31) [T] $\int_0^1 x \cdot e^{-x^2}\,dx$

32) [T] $\int_0^8 \frac{2x}{\sqrt{x^2+36}}\,dx$

Answer

$$\int_0^8 \frac{2x}{\sqrt{x^2+36}}\,dx = 8.0$$

33) [T] $\int_0^{2/\sqrt{3}} \frac{1}{4+9x^2}\,dx$

34) [T] $\int \frac{dx}{x^2+4x+13}$

Answer

$$\int \frac{dx}{x^2+4x+13} = \tfrac{1}{3}\arctan\left(\tfrac{1}{3}(x+2)\right) + C$$

35) [T] $\int \frac{dx}{1+\sin x}$

Use tables to evaluate the integrals. You may need to complete the square or change variables to put the integral into a form given in the table.

36) $\int \frac{dx}{x^2+2x+10}$

Answer

$$\int \frac{dx}{x^2+2x+10} = \tfrac{1}{3}\arctan\left(\frac{x+1}{3}\right) + C$$

37) $\int \frac{dx}{\sqrt{x^2-6x}}$

38) $\int \frac{e^x}{\sqrt{e^{2x}-4}}\,dx$

Answer

$$\int \frac{e^x}{\sqrt{e^{2x}-4}}\,dx = \ln\left(e^x + \sqrt{4+e^{2x}}\right) + C$$

39) $\int \frac{\cos x}{\sin^2 x + 2\sin x}\,dx$

40) $\int \frac{\arctan(x^3)}{x^4}\,dx$

Answer

$$\int \frac{\arctan(x^3)}{x^4}\,dx = \ln x - \tfrac{1}{6}\ln(x^6+1) - \frac{\arctan(x^3)}{3x^3} + C$$

41) $\int \frac{\ln|x|\arcsin(\ln|x|)}{x}\,dx$

Use tables to perform the integration.

42) $\int \frac{dx}{\sqrt{x^2+16}}$

Answer

$$\int \frac{dx}{\sqrt{x^2+16}} = \ln|x| + \sqrt{16+x^2}\,| + C$$

43) $\int \frac{3x}{2x+7}\,dx$

44) $\int \frac{dx}{1-\cos 4x}$

Answer

$$\int \frac{dx}{1-\cos 4x} = -\frac{1}{4}\cot 2x + C$$

45) $\int \frac{dx}{\sqrt{4x+1}}$

46) Find the area bounded by $y(4+25x^2)=5,\ x=0,\ y=0$, and $x=4$. Use a table of integrals or a CAS.

Answer

$\frac{1}{2}\arctan 10$ units^2

47) The region bounded between the curve $y=\frac{1}{\sqrt{1+\cos x}},\ 0.3\le x\le 1.1$, and the x-axis is revolved about the x-axis to generate a solid. Use a table of integrals to find the volume of the solid generated. (Round the answer to two decimal places.)

48) Use substitution and a table of integrals to find the area of the surface generated by revolving the curve $y=e^x,\ 0\le x\le 3$, about the x-axis. (Round the answer to two decimal places.)

Answer

1276.14 units^2

49) [T] Use an integral table and a calculator to find the area of the surface generated by revolving the curve $y=\frac{x^2}{2},\ 0\le x\le 1$, about the x-axis. (Round the answer to two decimal places.)

50) [T] Use a CAS or tables to find the area of the surface generated by revolving the curve $y=\cos x,\ 0\le x\le \frac{\pi}{2}$, about the x-axis. (Round the answer to two decimal places.)

Answer

7.21 units^2

51) Find the length of the curve $y=\frac{x^2}{4}$ over $[0,8]$.

52) Find the length of the curve $y=e^x$ over $[0,\ \ln(2)]$.

Answer

$\left(\sqrt{5}-\sqrt{2}+\ln\left|\frac{2+2\sqrt{2}}{1+\sqrt{5}}\right|\right)$ units

53) Find the area of the surface formed by revolving the graph of $y = 2\sqrt{x}$ over the interval $[0, 9]$ about the x-axis.

54) Find the average value of the function $f(x) = \dfrac{1}{x^2 + 1}$ over the interval $[-3, 3]$.

Answer

$\frac{1}{3}\arctan(3) \approx 0.416$

55) Approximate the arc length of the curve $y = \tan \pi x$ over the interval $\left[0, \frac{1}{4}\right]$. (Round the answer to three decimal places.)

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7.6: Numerical Integration

Learning Objectives

- Approximate the value of a definite integral by using the midpoint and trapezoidal rules.
- Determine the absolute and relative error in using a numerical integration technique.
- Estimate the absolute and relative error using an error-bound formula.
- Recognize when the midpoint and trapezoidal rules over- or underestimate the true value of an integral.
- Use Simpson's rule to approximate the value of a definite integral to a given accuracy.

The antiderivatives of many functions either cannot be expressed or cannot be expressed easily in closed form (that is, in terms of known functions). Consequently, rather than evaluate definite integrals of these functions directly, we resort to various techniques of **numerical integration** to approximate their values. In this section, we explore several of these techniques. In addition, we examine the process of estimating the error in using these techniques.

The Midpoint Rule

Earlier in this text we defined the definite integral of a function over an interval as the limit of **Riemann sums**. In general, any Riemann sum of a function $f(x)$ over an interval $[a,b]$ may be viewed as an estimate of $\int_a^b f(x)\,dx$. Recall that a Riemann sum of a function $f(x)$ over an interval $[a,b]$ is obtained by selecting a partition

$$P = \{x_0, x_1, x_2, \ldots, x_n\}$$

where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$

and a set

$$S = \{x_1^*, x_2^*, \ldots, x_n^*\}$$

where $x_{i-1} \le x_i^* \le x_i \quad \text{for all } i.$

The Riemann sum corresponding to the partition P and the set S is given by $\sum_{i=1}^{n} f(x_i^*)\Delta x_i$, where $\Delta x_i = x_i - x_{i-1}$, the length of the ith subinterval.

The midpoint rule for estimating a definite integral uses a Riemann sum with subintervals of equal width and the midpoints, m_i, of each subinterval in place of x_i^*. Formally, we state a theorem regarding the convergence of the midpoint rule as follows.

The Midpoint Rule

Assume that $f(x)$ is continuous on $[a,b]$. Let n be a positive integer and $\Delta x = \dfrac{b-a}{n}$. If $[a,b]$ is divided into n subintervals, each of length Δx, and m_i is the midpoint of the ith subinterval, set

$$M_n = \sum_{i=1}^{n} f(m_i)\Delta x.$$

Then $\lim_{n\to\infty} M_n = \int_a^b f(x)\,dx.$

As we can see in Figure 7.6.1, if $f(x) \ge 0$ over $[a,b]$, then $\sum_{i=1}^{n} f(m_i)\Delta x$ corresponds to the sum of the areas of rectangles approximating the area between the graph of $f(x)$ and the x-axis over $[a,b]$. The graph shows the rectangles corresponding to M_4 for a nonnegative function over a closed interval $[a,b]$.

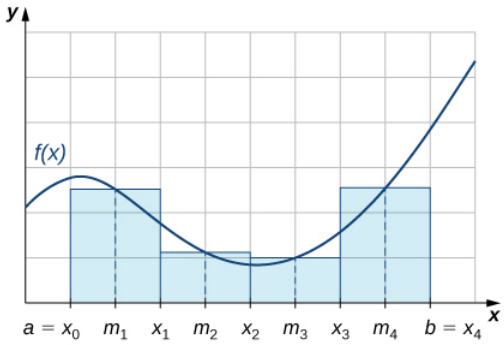

Figure 7.6.1: The midpoint rule approximates the area between the graph of $f(x)$ and the x-axis by summing the areas of rectangles with midpoints that are points on $f(x)$.

Example 7.6.1: Using the Midpoint Rule with M_4

Use the midpoint rule to estimate $\int_0^1 x^2\,dx$ using four subintervals. Compare the result with the actual value of this integral.

Solution: Each subinterval has length $\Delta x = \dfrac{1-0}{4} = \dfrac{1}{4}$. Therefore, the subintervals consist of

$$\left[0, \tfrac{1}{4}\right],\ \left[\tfrac{1}{4}, \tfrac{1}{2}\right],\ \left[\tfrac{1}{2}, \tfrac{3}{4}\right],\ \text{and}\ \left[\tfrac{3}{4}, 1\right].$$

The midpoints of these subintervals are $\left\{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\right\}$. Thus,

$$\begin{aligned} M_4 &= \frac{1}{4}\cdot f\left(\frac{1}{8}\right) + \frac{1}{4}\cdot f\left(\frac{3}{8}\right) + \frac{1}{4}\cdot f\left(\frac{5}{8}\right) + \frac{1}{4}\cdot f\left(\frac{7}{8}\right) \\ &= \frac{1}{4}\cdot\frac{1}{64} + \frac{1}{4}\cdot\frac{9}{64} + \frac{1}{4}\cdot\frac{25}{64} + \frac{1}{4}\cdot\frac{49}{64} \\ &= \frac{21}{64} = 0.328125. \end{aligned}$$

Since

$$\int_0^1 x^2\,dx = \frac{1}{3},$$

the absolute error in this approximation is:

$$\left|\frac{1}{3} - \frac{21}{64}\right| = \frac{1}{192} \approx 0.0052,$$

and we see that the midpoint rule produces an estimate that is somewhat close to the actual value of the definite integral.

Example 7.6.2: Using the Midpoint Rule with M_6

Use M_6 to estimate the length of the curve $y = \frac{1}{2}x^2$ on $[1, 4]$.

Solution: The length of $y = \frac{1}{2}x^2$ on $[1, 4]$ is

$$s = \int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$$

Since $\dfrac{dy}{dx} = x$, this integral becomes $\int_1^4 \sqrt{1+x^2}\,dx$.

If $[1, 4]$ is divided into six subintervals, then each subinterval has length $\Delta x = \dfrac{4-1}{6} = \dfrac{1}{2}$ and the midpoints of the subintervals are $\left\{\frac{5}{4}, \frac{7}{4}, \frac{9}{4}, \frac{11}{4}, \frac{13}{4}, \frac{15}{4}\right\}$. If we set $f(x) = \sqrt{1+x^2}$,

$$\begin{aligned} M_6 &= \tfrac{1}{2}\cdot f\left(\frac{5}{4}\right) + \tfrac{1}{2}\cdot f\left(\frac{7}{4}\right) + \frac{1}{2}\cdot f\left(\frac{9}{4}\right) + \frac{1}{2}\cdot f\left(\frac{11}{4}\right) + \frac{1}{2}\cdot f\left(\frac{13}{4}\right) + \frac{1}{2}\cdot f\left(\frac{15}{4}\right) \\ &\approx \frac{1}{2}(1.6008 + 2.0156 + 2.4622 + 2.9262 + 3.4004 + 3.8810) = 8.1431\ \text{ units}. \end{aligned}$$

? Exercise 7.6.1

Use the midpoint rule with $n = 2$ to estimate $\int_1^2 \frac{1}{x}\,dx$.

Hint

$\Delta x = \frac{1}{2}, \quad m_1 = \frac{5}{4}, \quad \text{and} \quad m_2 = \frac{7}{4}.$

Answer

$\frac{24}{35} \approx 0.685714$

The Trapezoidal Rule

We can also approximate the value of a definite integral by using trapezoids rather than rectangles. In Figure 7.6.2, the area beneath the curve is approximated by trapezoids rather than by rectangles.

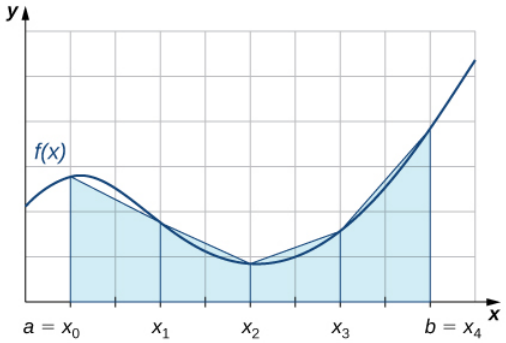

Figure 7.6.2: Trapezoids may be used to approximate the area under a curve, hence approximating the definite integral.

The **trapezoidal rule** for estimating definite integrals uses trapezoids rather than rectangles to approximate the area under a curve. To gain insight into the final form of the rule, consider the trapezoids shown in Figure 7.6.2. We assume that the length of each subinterval is given by Δx. First, recall that the area of a trapezoid with a height of h and bases of length b_1 and b_2 is given by $\text{Area} = \frac{1}{2}h(b_1 + b_2)$. We see that the first trapezoid has a height Δx and parallel bases of length $f(x_0)$ and $f(x_1)$. Thus, the area of the first trapezoid in Figure 7.6.2 is

$$\frac{1}{2}\Delta x\Big(f(x_0) + f(x_1)\Big).$$

The areas of the remaining three trapezoids are

$$\frac{1}{2}\Delta x\Big(f(x_1) + f(x_2)\Big), \ \frac{1}{2}\Delta x\Big(f(x_2) + f(x_3)\Big), \text{ and } \frac{1}{2}\Delta x\Big(f(x_3) + f(x_4)\Big).$$

Consequently,

$$\int_a^b f(x)\,dx$$
$$\approx \frac{1}{2}\Delta x\Big(f(x_0) + f(x_1)\Big) + \frac{1}{2}\Delta x\Big(f(x_1) + f(x_2)\Big) + \frac{1}{2}\Delta x\Big(f(x_2) + f(x_3)\Big) + \frac{1}{2}\Delta x\Big(f(x_3) + f(x_4)\Big).$$

After taking out a common factor of $\frac{1}{2}\Delta x$ and combining like terms, we have

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\Big[f(x_0) + 2\,f(x_1) + 2\,f(x_2) + 2\,f(x_3) + f(x_4)\Big].$$

Generalizing, we formally state the following rule.

The Trapezoidal Rule

Assume that $f(x)$ is continuous over $[a,b]$. Let n be a positive integer and $\Delta x = \dfrac{b-a}{n}$. Let $[a,b]$ be divided into n subintervals, each of length Δx, with endpoints at $P = \{x_0, x_1, x_2 \ldots, x_n\}$.

Set

$$T_n = \frac{\Delta x}{2}\Big[f(x_0) + 2\,f(x_1) + 2\,f(x_2) + \cdots + 2\,f(x_{n-1}) + f(x_n)\Big].$$

Then, $\lim\limits_{n\to+\infty} T_n = \displaystyle\int_a^b f(x)\,dx.$

Before continuing, let's make a few observations about the trapezoidal rule. First of all, it is useful to note that

$$T_n = \frac{1}{2}(L_n + R_n) \text{ where } L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x \text{ and } R_n = \sum_{i=1}^{n} f(x_i)\Delta x.$$

That is, L_n and R_n approximate the integral using the left-hand and right-hand endpoints of each subinterval, respectively. In addition, a careful examination of Figure 7.6.3 leads us to make the following observations about using the trapezoidal rules and midpoint rules to estimate the definite integral of a nonnegative function. The trapezoidal rule tends to overestimate the value of a definite integral systematically over intervals where the function is concave up and to underestimate the value of a definite integral systematically over intervals where the function is concave down. On the other hand, the midpoint rule tends to average out these errors somewhat by partially overestimating and partially underestimating the value of the definite integral over these same types of intervals. This leads us to hypothesize that, in general, the midpoint rule tends to be more accurate than the trapezoidal rule.

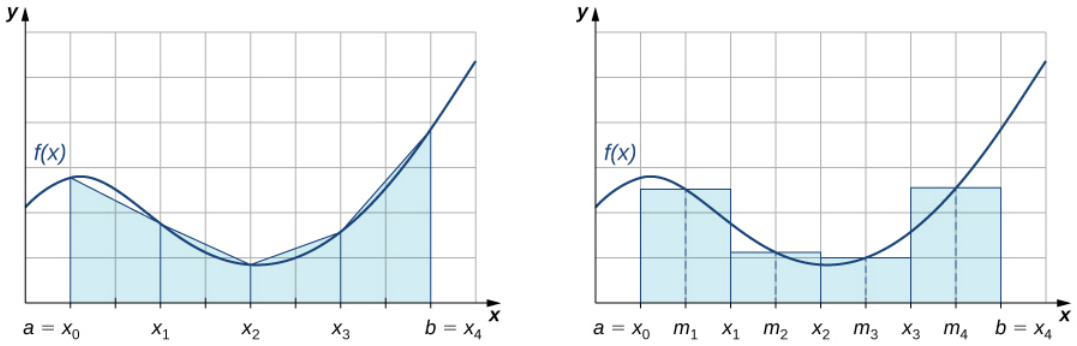

Figure 7.6.3:The trapezoidal rule tends to be less accurate than the midpoint rule.

Example 7.6.3: Using the Trapezoidal Rule

Use the trapezoidal rule to estimate $\displaystyle\int_0^1 x^2\,dx$ using four subintervals.

Solution

The endpoints of the subintervals consist of elements of the set $P = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$ and $\Delta x = \frac{1-0}{4} = \frac{1}{4}$. Thus,

$$\begin{aligned}\int_0^1 x^2 dx &\approx \frac{1}{2}\cdot\frac{1}{4}\Big[f(0) + 2\,f\left(\tfrac{1}{4}\right) + 2\,f\left(\tfrac{1}{2}\right) + 2\,f\left(\tfrac{3}{4}\right) + f(1)\Big] \\ &= \tfrac{1}{8}\Big(0 + 2\cdot\tfrac{1}{16} + 2\cdot\tfrac{1}{4} + 2\cdot\tfrac{9}{16} + 1\Big) \\ &= \frac{11}{32} = 0.34375\end{aligned}$$

Exercise 7.6.2

Use the trapezoidal rule with $n = 2$ to estimate $\displaystyle\int_1^2 \frac{1}{x}\,dx$.

Hint

Set $\Delta x = \frac{1}{2}$. The endpoints of the subintervals are the elements of the set $P = \left\{1, \frac{3}{2}, 2\right\}$.

Answer

$\frac{17}{24} \approx 0.708333$

Absolute and Relative Error

An important aspect of using these numerical approximation rules consists of calculating the error in using them for estimating the value of a definite integral. We first need to define absolute error and relative error.

Definition: absolute and relative error

If B is our estimate of some quantity having an actual value of A, then the **absolute error** is given by $|A - B|$.

The **relative error** is the error as a percentage of the actual value and is given by

$$\left|\frac{A-B}{A}\right| \cdot 100\%.$$

Example 7.6.4: Calculating Error in the Midpoint Rule

Calculate the absolute and relative error in the estimate of $\int_0^1 x^2\,dx$ using the midpoint rule, found in Example 7.6.1.

Solution: The calculated value is $\int_0^1 x^2\,dx = \frac{1}{3}$ and our estimate from the example is $M_4 = \frac{21}{64}$. Thus, the absolute error is given by $\left|\frac{1}{3} - \frac{21}{64}\right| = \frac{1}{192} \approx 0.0052$.

The relative error is

$$\frac{1/192}{1/3} = \frac{1}{64} \approx 0.015625 \approx 1.6\%.$$

Example 7.6.5: Calculating Error in the Trapezoidal Rule

Calculate the absolute and relative error in the estimate of $\int_0^1 x^2\,dx$ using the trapezoidal rule, found in Example 7.6.3.

Solution: The calculated value is $\int_0^1 x^2\,dx = \frac{1}{3}$ and our estimate from the example is $T_4 = \frac{11}{32}$. Thus, the absolute error is given by $\left|\frac{1}{3} - \frac{11}{32}\right| = \frac{1}{96} \approx 0.0104$.

The relative error is given by

$$\frac{1/96}{1/3} = 0.03125 \approx 3.1\%.$$

Exercise 7.6.3

In an earlier checkpoint, we estimated $\int_1^2 \frac{1}{x}\,dx$ to be $\frac{24}{35}$ using M_2. The actual value of this integral is $\ln 2$. Using $\frac{24}{35} \approx 0.6857$ and $\ln 2 \approx 0.6931$, calculate the absolute error and the relative error.

Hint

Use the previous examples as a guide.

Answer

absolute error ≈ 0.0074, and relative error $\approx 1.1\%$

Error Bounds on the Midpoint and Trapezoidal Rules

In the two previous examples, we were able to compare our estimate of an integral with the actual value of the integral; however, we do not typically have this luxury. In general, if we are approximating an integral, we are doing so because we cannot compute the exact value of the integral itself easily. Therefore, it is often helpful to be able to determine an upper bound for the error in an approximation of an integral. The following theorem provides error bounds for the midpoint and trapezoidal rules. The theorem is stated without proof.

Error Bounds for the Midpoint and Trapezoidal Rules

Let $f(x)$ be a continuous function over $[a,b]$, having a second derivative $f''(x)$ over this interval. If M is the maximum value of $|f''(x)|$ over $[a,b]$, then the upper bounds for the error in using M_n and T_n to estimate $\int_a^b f(x)\,dx$ are

$$\text{Error in } M_n \le \frac{M(b-a)^3}{24n^2} \tag{7.6.1}$$

and

$$\text{Error in } T_n \le \frac{M(b-a)^3}{12n^2}$$

.

We can use these bounds to determine the value of n necessary to guarantee that the error in an estimate is less than a specified value.

Example 7.6.6: Determining the Number of Intervals to Use

What value of n should be used to guarantee that an estimate of $\int_0^1 e^{x^2}\,dx$ is accurate to within 0.01 if we use the midpoint rule?

Solution

We begin by determining the value of M, the maximum value of $|f''(x)|$ over $[0,1]$ for $f(x)=e^{x^2}$. Since $f'(x)=2xe^{x^2}$, we have

$$f''(x)=2e^{x^2}+4x^2e^{x^2}.$$

Thus,

$$|f''(x)|=2e^{x^2}(1+2x^2)\le 2\cdot e\cdot 3=6e.$$

From the error-bound Equation 7.6.1, we have

$$\text{Error in } M_n \le \frac{M(b-a)^3}{24n^2} \le \frac{6e(1-0)^3}{24n^2}=\frac{6e}{24n^2}.$$

Now we solve the following inequality for n:

$$\frac{6e}{24n^2}\le 0.01.$$

Thus, $n\ge\sqrt{\frac{600e}{24}}\approx 8.24$. Since n must be an integer satisfying this inequality, a choice of $n=9$ would guarantee that

$$\left|\int_0^1 e^{x^2}\,dx - M_n\right| < 0.01.$$

Analysis

We might have been tempted to round 8.24 down and choose $n=8$, but this would be incorrect because we must have an integer greater than or equal to 8.24. We need to keep in mind that the error estimates provide an upper bound only for the error. The actual estimate may, in fact, be a much better approximation than is indicated by the error bound.

? Exercise 7.6.4

Use Equation 7.6.1 to find an upper bound for the error in using M_4 to estimate $\int_0^1 x^2\,dx$.

Hint

$f''(x) = 2$, so $M = 2$.

Answer

$\frac{1}{192}$

Simpson's Rule

With the midpoint rule, we estimated areas of regions under curves by using rectangles. In a sense, we approximated the curve with piecewise constant functions. With the trapezoidal rule, we approximated the curve by using piecewise linear functions. What if we were, instead, to approximate a curve using piecewise quadratic functions? With **Simpson's rule,** we do just this. We partition the interval into an even number of subintervals, each of equal width. Over the first pair of subintervals we approximate $\int_{x_0}^{x_2} f(x)\,dx$ with $\int_{x_0}^{x_2} p(x)\,dx$, where $p(x) = Ax^2 + Bx + C$ is the quadratic function passing through $(x_0, f(x_0))$, $(x_1, f(x_1))$, and $(x_2, f(x_2))$ (Figure 7.6.4). Over the next pair of subintervals we approximate $\int_{x_2}^{x_4} f(x)\,dx$ with the integral of another quadratic function passing through $(x_2, f(x_2))$, $(x_3, f(x_3))$, and $(x_4, f(x_4))$. This process is continued with each successive pair of subintervals.

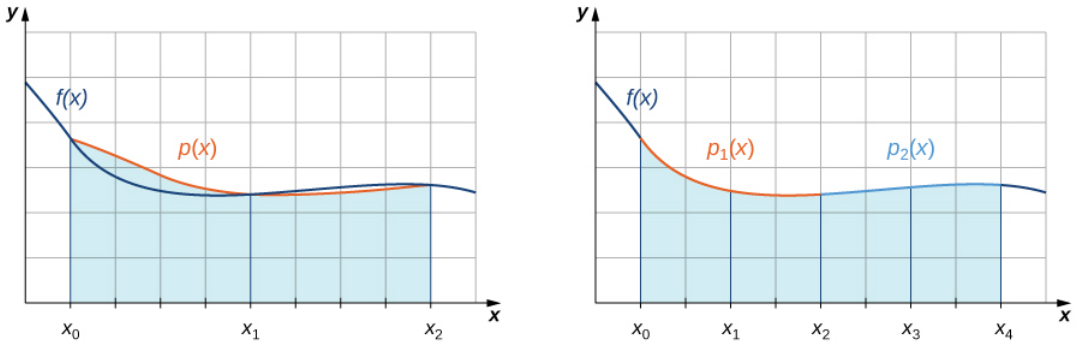

Figure 7.6.4: With Simpson's rule, we approximate a definite integral by integrating a piecewise quadratic function.

To understand the formula that we obtain for Simpson's rule, we begin by deriving a formula for this approximation over the first two subintervals. As we go through the derivation, we need to keep in mind the following relationships:

$$f(x_0) = p(x_0) = Ax_0^2 + Bx_0 + C$$

$$f(x_1) = p(x_1) = Ax_1^2 + Bx_1 + C$$

$$f(x_2) = p(x_2) = Ax_2^2 + Bx_2 + C$$

$x_2 - x_0 = 2\Delta x$, where Δx is the length of a subinterval.

$x_2 + x_0 = 2x_1$, since $x_1 = \dfrac{(x_2 + x_0)}{2}$.

Thus,

$$
\begin{aligned}
\int_{x_0}^{x_2} f(x)\,dx &\approx \int_{x_0}^{x_2} p(x)\,dx \\
&= \int_{x_0}^{x_2} (Ax^2 + Bx + C)\,dx \\
&= \left(\frac{A}{3}x^3 + \frac{B}{2}x^2 + Cx\right)\Big|_{x_0}^{x_2} && \text{Find the antiderivative.} \\
&= \frac{A}{3}(x_2^3 - x_0^3) + \frac{B}{2}(x_2^2 - x_0^2) + C(x_2 - x_0) && \text{Evaluate the antiderivative.} \\
&= \frac{A}{3}(x_2 - x_0)(x_2^2 + x_2x_0 + x_0^2) \\
&\quad + \frac{B}{2}(x_2 - x_0)(x_2 + x_0) + C(x_2 - x_0) \\
&= \frac{x_2 - x_0}{6} && \text{Factor out } \frac{x_2 - x_0}{6}. \\
&\quad \Big(2A(x_2^2 + x_2x_0 + x_0^2) + 3B(x_2 + x_0) + 6C\Big) \\
&= \frac{\Delta x}{3}\Big((Ax_2^2 + Bx_2 + C) + (Ax_0^2 + Bx_0 + C) && \text{Rearrange the terms. Note:} \\
&\quad + A(x_2^2 + 2x_2x_0 + x_0^2) + 2B(x_2 + x_0) + 4C\Big) && \Delta x = \frac{x_2 - x_0}{2} \\
&= \frac{\Delta x}{3}\Big(f(x_2) + f(x_0) + A(x_2 + x_0)^2 && \text{Factor and substitute:} \\
&\quad + 2B(x_2 + x_0) + 4C\Big) \\
& && f(x_2) = Ax_2^2 + Bx_2 + C \text{ and} \\
& && f(x_0) = Ax_0^2 + Bx_0 + C. \\
&= \frac{\Delta x}{3}\Big(f(x_2) + f(x_0) + A(2x_1)^2 + 2B(2x_1) + 4C\Big) && \text{Substitute } x_2 + x_0 = 2x_1. \\
& && \text{Note: } x_1 = \frac{x_2 + x_0}{2} \text{ is} \\
& && \text{the midpoint.} \\
&= \frac{\Delta x}{3}\Big(f(x_2) + 4f(x_1) + f(x_0)\Big). && \text{Expand and substitute} \\
& && f(x_1) = Ax_1^2 + Bx_1 + C.
\end{aligned}
$$

If we approximate $\int_{x_2}^{x_4} f(x)\,dx$ using the same method, we see that we have

$$\int_{x_2}^{x_4} f(x)\,dx \approx \frac{\Delta x}{3}(f(x_4) + 4\,f(x_3) + f(x_2)).$$

Combining these two approximations, we get

$$\int_{x_0}^{x_4} f(x)\,dx \approx \frac{\Delta x}{3}(f(x_0) + 4\,f(x_1) + 2\,f(x_2) + 4\,f(x_3) + f(x_4)).$$

The pattern continues as we add pairs of subintervals to our approximation. The general rule may be stated as follows.

Simpson's Rule

Assume that $f(x)$ is continuous over $[a, b]$. Let n be a positive even integer and $\Delta x = \frac{b-a}{n}$. Let $[a, b]$ be divided into n subintervals, each of length Δx, with endpoints at $P = \{x_0, x_1, x_2, \ldots, x_n\}$. Set

$$S_n = \frac{\Delta x}{3}\Big[f(x_0) + 4\,f(x_1) + 2\,f(x_2) + 4\,f(x_3) + 2\,f(x_4) + \cdots + 2\,f(x_{n-2}) + 4\,f(x_{n-1}) + f(x_n)\Big].$$

Then,

$$\lim_{n\to+\infty} S_n = \int_a^b f(x)\,dx.$$

Just as the trapezoidal rule is the average of the left-hand and right-hand rules for estimating definite integrals, Simpson's rule may be obtained from the midpoint and trapezoidal rules by using a weighted average. It can be shown that $S_{2n} = \left(\frac{2}{3}\right) M_n + \left(\frac{1}{3}\right) T_n$.

It is also possible to put a bound on the error when using Simpson's rule to approximate a definite integral. The bound in the error is given by the following rule:

Rule: Error Bound for Simpson's Rule

Let $f(x)$ be a continuous function over $[a, b]$ having a fourth derivative, $f^{(4)}(x)$, over this interval. If M is the maximum value of $| f^{(4)}(x) |$ over $[a, b]$, then the upper bound for the error in using S_n to estimate $\int_a^b f(x)\, dx$ is given by

$$\text{Error in } S_n \leq \frac{M(b-a)^5}{180n^4}.$$

Example 7.6.7: Applying Simpson's Rule 1

Use S_2 to approximate $\int_0^1 x^3\, dx$. Estimate a bound for the error in S_2.

Solution

Since $[0, 1]$ is divided into two intervals, each subinterval has length $\Delta x = \frac{1-0}{2} = \frac{1}{2}$. The endpoints of these subintervals are $\left\{0, \frac{1}{2}, 1\right\}$. If we set $f(x) = x^3$, then

$$S_2 = \frac{1}{3} \cdot \frac{1}{2}(f(0) + 4\, f(\frac{1}{2}) + f(1)) = \frac{1}{6}(0 + 4 \cdot \frac{1}{8} + 1) = \frac{1}{4}.$$

Since $f^{(4)}(x) = 0$ and consequently $M = 0$, we see that

$$\text{Error in } S_2 \leq \frac{0(1)^5}{180 \cdot 2^4} = 0.$$

This bound indicates that the value obtained through Simpson's rule is exact. A quick check will verify that, in fact, $\int_0^1 x^3\, dx = \frac{1}{4}$.

Example 7.6.8: Applying Simpson's Rule 2

Use S_6 to estimate the length of the curve $y = \frac{1}{2}x^2$ over $[1, 4]$.

Solution

The length of $y = \frac{1}{2}x^2$ over $[1, 4]$ is $\int_1^4 \sqrt{1 + x^2}\, dx$. If we divide $[1, 4]$ into six subintervals, then each subinterval has length $\Delta x = \frac{4-1}{6} = \frac{1}{2}$, and the endpoints of the subintervals are $\left\{1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4\right\}$. Setting $f(x) = \sqrt{1 + x^2}$,

$$S_6 = \frac{1}{3} \cdot \frac{1}{2}(f(1) + 4f(\frac{3}{2}) + 2f(2) + 4f(\frac{5}{2}) + 2f(3) + 4f(\frac{7}{2}) + f(4)).$$

After substituting, we have

$$S_6 = \frac{1}{6}(1.4142 + 4 \cdot 1.80278 + 2 \cdot 2.23607 + 4 \cdot 2.69258 + 2 \cdot 3.16228 + 4 \cdot 3.64005 + 4.12311)$$
$$\approx 8.14594 \text{ units}.$$

Exercise 7.6.5

Use S_2 to estimate $\int_1^2 \frac{1}{x}\, dx$.

Hint

$$S_2 = \frac{1}{3}\Delta x\, (f(x_0) + 4f(x_1) + f(x_2))$$

Answer

$\frac{25}{36} \approx 0.694444$

Key Concepts

- We can use numerical integration to estimate the values of definite integrals when a closed form of the integral is difficult to find or when an approximate value only of the definite integral is needed.
- The most commonly used techniques for numerical integration are the midpoint rule, trapezoidal rule, and Simpson's rule.
- The midpoint rule approximates the definite integral using rectangular regions whereas the trapezoidal rule approximates the definite integral using trapezoidal approximations.
- Simpson's rule approximates the definite integral by first approximating the original function using piecewise quadratic functions.

Key Equations

- **Midpoint rule**

  $M_n = \sum_{i=1}^{n} f(m_i)\Delta x$

- **Trapezoidal rule**

  $T_n = \frac{\Delta x}{2}\left[f(x_0) + 2\,f(x_1) + 2\,f(x_2) + \cdots + 2\,f(x_{n-1}) + f(x_n)\right]$

- **Simpson's rule**

  $S_n = \frac{\Delta x}{3}\left[f(x_0) + 4\,f(x_1) + 2\,f(x_2) + 4\,f(x_3) + 2\,f(x_4) + 4\,f(x_5) + \cdots + 2\,f(x_{n-2}) + 4\,f(x_{n-1}) + f(x_n)\right]$

- **Error bound for midpoint rule**

  Error in $M_n \leq \frac{M(b-a)^3}{24n^2}$, where M is the maximum value of $|f''(x)|$ over $[a, b]$.

- **Error bound for trapezoidal rule**

  Error in $T_n \leq \frac{M(b-a)^3}{12n^2}$, where M is the maximum value of $|f''(x)|$ over $[a, b]$.

- **Error bound for Simpson's rule**

  Error in $S_n \leq \frac{M(b-a)^5}{180n^4}$, where M is the maximum value of $|\,f^{(4)}(x)\,|$ over $[a, b]$.

Glossary

absolute error

if B is an estimate of some quantity having an actual value of A, then the absolute error is given by $|A - B|$

midpoint rule

a rule that uses a Riemann sum of the form $M_n = \sum_{i=1}^{n} f(m_i)\Delta x$, where m_i is the midpoint of the ith subinterval to approximate $\int_a^b f(x)\,dx$

numerical integration

the variety of numerical methods used to estimate the value of a definite integral, including the midpoint rule, trapezoidal rule, and Simpson's rule

relative error

error as a percentage of the actual value, given by

$$\text{relative error} = \left|\frac{A - B}{A}\right| \cdot 100\%$$

Simpson's rule

a rule that approximates $\int_a^b f(x)\,dx$ using the area under a piecewise quadratic function.

The approximation S_n to $\int_a^b f(x)\,dx$ is given by

$$S_n = \frac{\Delta x}{3}\Big(f(x_0) + 4\,f(x_1) + 2\,f(x_2) + 4\,f(x_3) + 2\,f(x_4) + \cdots + 2\,f(x_{n-2}) + 4\,f(x_{n-1}) + f(x_n)\Big).$$

trapezoidal rule

a rule that approximates $\int_a^b f(x)\,dx$ using the area of trapezoids.

The approximation T_n to $\int_a^b f(x)\,dx$ is given by

$$T_n = \frac{\Delta x}{2}\Big(f(x_0) + 2\,f(x_1) + 2\,f(x_2) + \cdots + 2\,f(x_{n-1}) + f(x_n)\Big).$$

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- Edited by Paul Seeburger (Monroe Community College). Notes added to development of area under a parabola and typos fixed in original text.

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7.6E: Exercises for Section 7.6

In exercises 1 - 5, approximate the following integrals using either the midpoint rule, trapezoidal rule, or Simpson's rule as indicated. (Round answers to three decimal places.)

1) $\int_1^2 \frac{dx}{x}$; trapezoidal rule; $n = 5$

Answer

0.696

2) $\int_0^3 \sqrt{4+x^3}\ dx$; trapezoidal rule; $n = 6$

3) $\int_0^3 \sqrt{4+x^3}\ dx$; Simpson's rule; $n = 6$

Answer

9.279

4) $\int_0^{12} x^2\ dx$; midpoint rule; $n = 6$

5) $\int_0^1 \sin^2(\pi x)\ dx$; midpoint rule; $n = 3$

Answer

0.500

6) Use the midpoint rule with eight subdivisions to estimate $\int_2^4 x^2\ dx$.

7) Use the trapezoidal rule with four subdivisions to estimate $\int_2^4 x^2\ dx$.

Answer

$T_4 = 18.75$

8) Find the exact value of $\int_2^4 x^2\ dx$. Find the error of approximation between the exact value and the value calculated using the trapezoidal rule with four subdivisions. Draw a graph to illustrate.

Approximate the integral to four decimal places using the indicated rule.

9) $\int_0^1 \sin^2(\pi x)\ dx$; trapezoidal rule; $n = 6$

Answer

0.5000

10) $\int_0^3 \frac{1}{1+x^3}\ dx$; trapezoidal rule; $n = 6$

11) $\int_0^3 \frac{1}{1+x^3}\ dx$; Simpson's rule; $n = 6$

Answer

1.1614

12) $\int_0^{0.8} e^{-x^2}\ dx$; trapezoidal rule; $n = 4$

13) $\int_0^{0.8} e^{-x^2}\ dx$; Simpson's rule; $n = 4$

Answer

0.6577

14) $\int_0^{0.4} \sin(x^2)\ dx$; trapezoidal rule; $n = 4$

15) $\int_0^{0.4} \sin(x^2)\ dx$; Simpson's rule; $n = 4$

Answer

0.0213

16) $\int_{0.1}^{0.5} \frac{\cos x}{x}\ dx$; trapezoidal rule; $n = 4$

17) $\int_{0.1}^{0.5} \frac{\cos x}{x}\ dx$; Simpson's rule; $n = 4$

Answer

1.5629

18) Evaluate $\int_0^1 \frac{dx}{1+x^2}$ exactly and show that the result is $\pi/4$. Then, find the approximate value of the integral using the trapezoidal rule with $n = 4$ subdivisions. Use the result to approximate the value of π.

19) Approximate $\int_2^4 \frac{1}{\ln x}\ dx$ using the midpoint rule with four subdivisions to four decimal places.

Answer

1.9133

20) Approximate $\int_2^4 \frac{1}{\ln x}\ dx$ using the trapezoidal rule with eight subdivisions to four decimal places.

21) Use the trapezoidal rule with four subdivisions to estimate $\int_0^{0.8} x^3\ dx$ to four decimal places.

Answer

$T(4) = 0.1088$

22) Use the trapezoidal rule with four subdivisions to estimate $\int_0^{0.8} x^3\ dx$. Compare this value with the exact value and find the error estimate.

23) Using Simpson's rule with four subdivisions, find $\int_0^{\pi/2} \cos(x)\ dx$.

Answer

$\int_0^{\pi/2} \cos(x)\ dx \approx \quad 1.0$

24) Show that the exact value of $\int_0^1 xe^{-x}\ dx = 1 - \frac{2}{e}$. Find the absolute error if you approximate the integral using the midpoint rule with 16 subdivisions.

25) Given $\int_0^1 xe^{-x}\ dx = 1 - \frac{2}{e}$, use the trapezoidal rule with 16 subdivisions to approximate the integral and find the absolute error.

Answer

Approximate error is 0.000325.

26) Find an upper bound for the error in estimating $\int_0^3 (5x + 4)\ dx$ using the trapezoidal rule with six steps.

27) Find an upper bound for the error in estimating $\int_4^5 \frac{1}{(x-1)^2}\,dx$ using the trapezoidal rule with seven subdivisions.

Answer

$\frac{1}{7938}$

28) Find an upper bound for the error in estimating $\int_0^3 (6x^2-1)\,dx$ using Simpson's rule with $n=10$ steps.

29) Find an upper bound for the error in estimating $\int_2^5 \frac{1}{x-1}\,dx$ using Simpson's rule with $n=10$ steps.

Answer

$\frac{81}{25{,}000}$

30) Find an upper bound for the error in estimating $\int_0^\pi 2x\cos(x)\,dx$ using Simpson's rule with four steps.

31) Estimate the minimum number of subintervals needed to approximate the integral $\int_1^4 (5x^2+8)\,dx$ with an error magnitude of less than 0.0001 using the trapezoidal rule.

Answer

475

32) Determine a value of n such that the trapezoidal rule will approximate $\int_0^1 \sqrt{1+x^2}\,dx$ with an error of no more than 0.01.

33) Estimate the minimum number of subintervals needed to approximate the integral $\int_2^3 (2x^3+4x)\,dx$ with an error of magnitude less than 0.0001 using the trapezoidal rule.

Answer

174

34) Estimate the minimum number of subintervals needed to approximate the integral $\int_3^4 \frac{1}{(x-1)^2}\,dx$ with an error magnitude of less than 0.0001 using the trapezoidal rule.

35) Use Simpson's rule with four subdivisions to approximate the area under the probability density function $y=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ from $x=0$ to $x=0.4$.

Answer

0.1554

36) Use Simpson's rule with $n=14$ to approximate (to three decimal places) the area of the region bounded by the graphs of $y=0, x=0$, and $x=\pi/2$.

37) The length of one arch of the curve $y=3\sin(2x)$ is given by $L=\int_0^{\pi/2}\sqrt{1+36\cos^2(2x)}\,dx$. Estimate L using the trapezoidal rule with $n=6$.

Answer

6.2807

38) The length of the ellipse $x=a\cos(t), y=b\sin(t), 0\le t\le 2\pi$ is given by $L=4a\int_0^{\pi/2}\sqrt{1-e^2\cos^2(t)}dt$, where e is the eccentricity of the ellipse. Use Simpson's rule with $n=6$ subdivisions to estimate the length of the ellipse when $a=2$ and $e=1/3$.

39) Estimate the area of the surface generated by revolving the curve $y=\cos(2x), 0\le x\le \frac{\pi}{4}$ about the x-axis. Use the trapezoidal rule with six subdivisions.

Answer

4.606

40) Estimate the area of the surface generated by revolving the curve $y = 2x^2, 0 \leq x \leq 3$ about the x-axis. Use Simpson's rule with $n = 6$.

41) The growth rate of a certain tree (in feet) is given by $y = \frac{2}{t+1} + e^{-t^2/2}$, where t is time in years. Estimate the growth of the tree through the end of the second year by using Simpson's rule, using two subintervals. (Round the answer to the nearest hundredth.)

Answer

3.41 ft

42) [T] Use a calculator to approximate $\int_0^1 \sin(\pi x)\, dx$ using the midpoint rule with 25 subdivisions. Compute the relative error of approximation.

43) [T] Given $\int_1^5 (3x^2 - 2x)\, dx = 100$, approximate the value of this integral using the midpoint rule with 16 subdivisions and determine the absolute error.

Answer

$T_{16} = 100.125$; absolute error = 0.125

44) Given that we know the Fundamental Theorem of Calculus, why would we want to develop numerical methods for definite integrals?

45) The table represents the coordinates (x, y) that give the boundary of a lot. The units of measurement are meters. Use the trapezoidal rule to estimate the number of square meters of land that is in this lot.

x	y	x	y
0	125	600	95
100	125	700	88
200	120	800	75
300	112	900	35
400	90	1000	0
500	90		

Answer

about 89,250 m^2

46) Choose the correct answer. When Simpson's rule is used to approximate the definite integral, it is necessary that the number of partitions be____

a. an even number

b. odd number

c. either an even or an odd number

d. a multiple of 4

47) The "Simpson" sum is based on the area under a ____.

Answer

parabola

48) The error formula for Simpson's rule depends on___.

a. $f(x)$

b. $f'(x)$

c. $f^{(4)}(x)$

d. the number of steps

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7.7: Improper Integrals

Learning Objectives

- Evaluate an integral over an infinite interval.
- Evaluate an integral over a closed interval with an infinite discontinuity within the interval.
- Use the comparison theorem to determine whether a definite integral is convergent.

Is the area between the graph of $f(x) = \dfrac{1}{x}$ and the x-axis over the interval $[1, +\infty)$ finite or infinite? If this same region is revolved about the x-axis, is the volume finite or infinite? Surprisingly, the area of the region described is infinite, but the volume of the solid obtained by revolving this region about the x-axis is finite.

In this section, we define integrals over an infinite interval as well as integrals of functions containing a discontinuity on the interval. Integrals of these types are called improper integrals. We examine several techniques for evaluating improper integrals, all of which involve taking limits.

Integrating over an Infinite Interval

How should we go about defining an integral of the type $\displaystyle\int_a^{+\infty} f(x)\,dx$? We can integrate $\displaystyle\int_a^t f(x)\,dx$ for any value of t, so it is reasonable to look at the behavior of this integral as we substitute larger values of t. Figure 7.7.1 shows that $\displaystyle\int_a^t f(x)\,dx$ may be interpreted as area for various values of t. In other words, we may define an improper integral as a limit, taken as one of the limits of integration increases or decreases without bound.

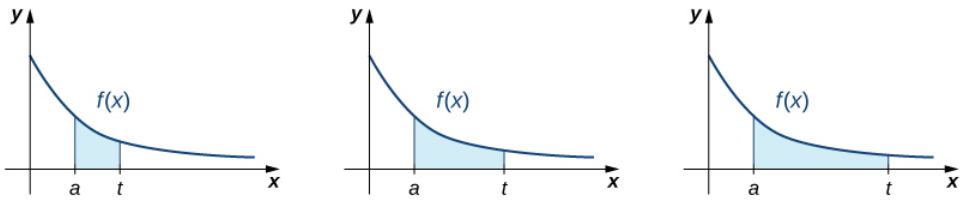

Figure 7.7.1: To integrate a function over an infinite interval, we consider the limit of the integral as the upper limit increases without bound.

Definition: Improper Integral

1. Let $f(x)$ be continuous over an interval of the form $[a, +\infty)$. Then

$$\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty}\int_a^t f(x)\,dx, \tag{7.7.1}$$

   provided this limit exists.

2. Let $f(x)$ be continuous over an interval of the form $(-\infty, b]$. Then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx, \tag{7.7.2}$$

   provided this limit exists.

In each case, if the limit exists, then the *improper integral* is said to converge. If the limit does not exist, then the improper integral is said to diverge.

3. Let $f(x)$ be continuous over $(-\infty, +\infty)$. Then

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^{+\infty} f(x)\,dx, \tag{7.7.3}$$

   provided that $\displaystyle\int_{-\infty}^0 f(x)\,dx$ and $\displaystyle\int_0^{+\infty} f(x)\,dx$ both converge.

   If either of these two integrals diverge, then $\displaystyle\int_{-\infty}^{+\infty} f(x)\,dx$ diverges.

(It can be shown that, in fact, $\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{+\infty} f(x)\,dx$ for any value of a.).

In our first example, we return to the question we posed at the start of this section: Is the area between the graph of $f(x) = \frac{1}{x}$ and the x-axis over the interval $[1, +\infty)$ finite or infinite?

Example 7.7.1: Finding an Area

Determine whether the area between the graph of $f(x) = \frac{1}{x}$ and the x-axis over the interval $[1, +\infty)$ is finite or infinite.

Solution

We first do a quick sketch of the region in question, as shown in Figure 7.7.2.

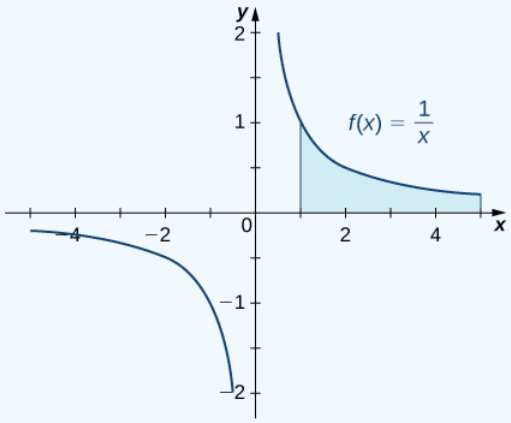

Figure 7.7.2: We can find the area between the curve $f(x) = 1/x$ and the x-axis on an infinite interval.

We can see that the area of this region is given by

$$A = \int_1^{\infty} \frac{1}{x}\,dx.$$

which can be evaluated using Equation 7.7.1:

$$\begin{aligned} A &= \int_1^{\infty} \frac{1}{x}\,dx \\ &= \lim_{t\to+\infty} \int_1^t \frac{1}{x}\,dx && \text{Rewrite the improper integral as a limit} \\ &= \lim_{t\to+\infty} \ln|x|\,|_1^t && \text{Find the antiderivative} \\ &= \lim_{t\to+\infty} (\ln|t| - \ln 1) && \text{Evaluate the antiderivative} \\ &= +\infty. && \text{Evaluate the limit.} \end{aligned}$$

Since the improper integral diverges to $+\infty$, the area of the region is infinite.

Example 7.7.2: Finding a Volume

Find the volume of the solid obtained by revolving the region bounded by the graph of $f(x) = \frac{1}{x}$ and the x-axis over the interval $[1, +\infty)$ about the x-axis.

Solution

The solid is shown in Figure 7.7.3. Using the disk method, we see that the volume V is

$$V = \pi \int_1^{+\infty} \frac{1}{x^2}\,dx.$$

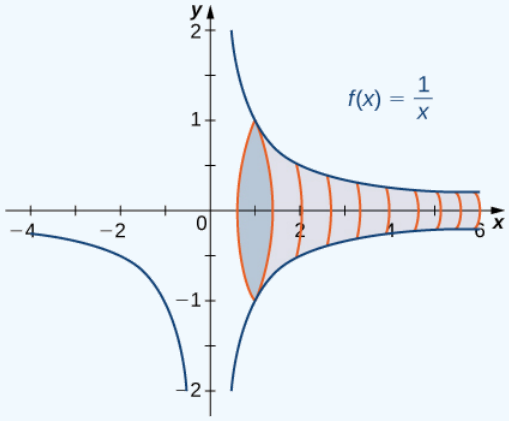

Figure 7.7.3: The solid of revolution can be generated by rotating an infinite area about the x-axis.

Then we have

$$
\begin{aligned}
V &= \pi \int_1^{+\infty} \frac{1}{x^2}\,dx \\
&= \pi \lim_{t\to+\infty} \int_1^t \frac{1}{x^2}\,dx && \text{Rewrite as a limit.} \\
&= \pi \lim_{t\to+\infty} -\frac{1}{x}\,|_1^t && \text{Find the antiderivative.} \\
&= \pi \lim_{t\to+\infty} \left(-\frac{1}{t}+1\right) && \text{Evaluate the antiderivative.} \\
&= \pi \lim_{t\to+\infty} \left(-\frac{1}{\not{t}^{\,0}}+1\right) && \text{Evaluate the antiderivative.} \\
&= \pi
\end{aligned}
$$

The improper integral converges to π. Therefore, the volume of the solid of revolution is π.

In conclusion, although the area of the region between the x-axis and the graph of $f(x) = 1/x$ over the interval $[1, +\infty)$ is infinite, the volume of the solid generated by revolving this region about the x-axis is finite. The solid generated is known as Gabriel's Horn.

Note: Gabriel's horn (also called Torricelli's trumpet) is a geometric figure which has *infinite* surface area, but *finite* volume. The name refers to the tradition identifying the Archangel Gabriel as the angel who blows the horn to announce Judgment Day, associating the divine, or infinite, with the finite. The properties of this figure were first studied by Italian physicist and mathematician Evangelista Torricelli in the 17th century.

Example 7.7.3: Traffic Accidents in a City

Suppose that at a busy intersection, traffic accidents occur at an average rate of one every three months. After residents complained, changes were made to the traffic lights at the intersection. It has now been eight months since the changes were made and there have been no accidents. Were the changes effective or is the 8-month interval without an accident a result of chance?

Figure 7.7.4: Modification of work by David McKelvey, Flickr.

Probability theory tells us that if the average time between events is k, the *probability* that X, the time between events, is between a and b is given by

$$(P(a \le x \le b) = \int_a^b f(x)\,dx$$

where

$$f(x) = \begin{cases} 0, & \text{if } x < 0 \\ ke^{-kx}, & \text{if } x \ge 0 \end{cases}.$$

Thus, if accidents are occurring at a rate of one every 3 months, then the probability that X, the time between accidents, is between a and b is given by

$$P(a \le x \le b) = \int_a^b f(x)\,dx$$

where

$$f(x) = \begin{cases} 0, & \text{if } x < 0 \\ 3e^{-3x}, & \text{if } x \ge 0 \end{cases}.$$

To answer the question, we must compute $P(X \ge 8) = \int_8^{+\infty} 3e^{-3x}\,dx$ and decide whether it is likely that 8 months could have passed without an accident if there had been no improvement in the traffic situation.

Solution

We need to calculate the probability as an improper integral:

$$\begin{aligned} P(X \ge 8) &= \int_8^{+\infty} 3e^{-3x}\,dx \\ &= \lim_{t\to+\infty} \int_8^t 3e^{-3x}\,dx \\ &= \lim_{t\to+\infty} -e^{-3x}\,|_8^t \\ &= \lim_{t\to+\infty} (-e^{-3t} + e^{-24}) \\ &\approx 3.8 \times 10^{-11}. \end{aligned}$$

The value 3.8×10^{-11} represents the probability of no accidents in 8 months under the initial conditions. Since this value is very, very small, it is reasonable to conclude the changes were effective.

Example 7.7.4: Evaluating an Improper Integral over an Infinite Interval

Evaluate $\int_{-\infty}^0 \frac{1}{x^2+4}\,dx$. State whether the improper integral converges or diverges.

Solution

Begin by rewriting $\int_{-\infty}^0 \frac{1}{x^2+4}\,dx$ as a limit using Equation 7.7.2 from the definition. Thus,

$$\begin{aligned} \int_{-\infty}^0 \frac{1}{x^2+4}\,dx &= \lim_{t\to-\infty} \int_t^0 \frac{1}{x^2+4}\,dx && \text{Rewrite as a limit.} \\ &= \lim_{t\to-\infty} \frac{1}{2}\tan^{-1}\frac{x}{2}\,|_t^0 && \text{Find the antiderivative.} \\ &= \lim_{t\to-\infty} \left(\frac{1}{2}\tan^{-1}0 - \frac{1}{2}\tan^{-1}\frac{t}{2}\right) && \text{Evaluate the antiderivative.} \\ &= \frac{\pi}{4}. && \text{Evaluate the limit and simplify.} \end{aligned}$$

The improper integral converges to $\frac{\pi}{4}$.

Example 7.7.5: Evaluating an Improper Integral on $(-\infty, +\infty)$

Evaluate $\int_{-\infty}^{+\infty} xe^x\,dx$. State whether the improper integral converges or diverges.

Solution

Start by splitting up the integral:

$$\int_{-\infty}^{+\infty} xe^x\,dx = \int_{-\infty}^{0} xe^x\,dx + \int_{0}^{+\infty} xe^x\,dx.$$

If either $\int_{-\infty}^{0} xe^x\,dx$ or $\int_{0}^{+\infty} xe^x\,dx$ diverges, then $\int_{-\infty}^{+\infty} xe^x\,dx$ diverges.

Compute each integral separately.

For the first integral,

$$\begin{aligned}
\int_{-\infty}^{0} xe^x\,dx &= \lim_{t\to-\infty}\int_{t}^{0} xe^x\,dx && \text{Rewrite as a limit.}\\
&= \lim_{t\to-\infty}\left(xe^x - e^x\right)\Big|_t^0 && \text{Use integration by parts to find the antiderivative. (Here } u = x \text{ and } dv = e^x\,dx.)\\
&= \lim_{t\to-\infty}\left(-1 - te^t + e^t\right) && \text{Evaluate the antiderivative.}\\
&= -1.
\end{aligned}$$

Evaluate the limit. Note: $\lim_{t\to-\infty} te^t$ is indeterminate of the form $0\cdot\infty$.Thus, $\lim_{t\to-\infty} te^t = \lim_{t\to-\infty}\frac{t}{e^{-t}} = \lim_{t\to-\infty}\frac{-1}{e^{-t}} = \lim_{t\to-\infty} -e^t = 0$ by L'Hôpital's Rule.

The first improper integral converges. For the second integral,

$$\begin{aligned}
\int_{0}^{+\infty} xe^x\,dx &= \lim_{t\to+\infty}\int_{0}^{t} xe^x\,dx && \text{Rewrite as a limit.}\\
&= \lim_{t\to+\infty}\left(xe^x - e^x\right)\Big|_0^t && \text{Find the antiderivative.}\\
&= \lim_{t\to+\infty}\left(te^t - e^t + 1\right) && \text{Evaluate the antiderivative.}\\
&= \lim_{t\to+\infty}\left((t-1)e^t + 1\right) && \text{Rewrite. } (te^t - e^t \text{ is indeterminate.})\\
&= +\infty. && \text{Evaluate the limit.}
\end{aligned}$$

Thus, $\int_{0}^{+\infty} xe^x\,dx$ diverges. Since this integral diverges, $\int_{-\infty}^{+\infty} xe^x\,dx$ diverges as well.

Exercise 7.7.1

Evaluate $\int_{-3}^{+\infty} e^{-x}\,dx$. State whether the improper integral converges or diverges.

Hint

$$\int_{-3}^{+\infty} e^{-x}\,dx = \lim_{t\to+\infty}\int_{-3}^{t} e^{-x}\,dx$$

Answer

It converges to e^3.

Integrating a Discontinuous Integrand

Now let's examine integrals of functions containing an infinite discontinuity in the interval over which the integration occurs. Consider an integral of the form $\int_a^b f(x)\,dx$, where $f(x)$ is continuous over $[a,b)$ and discontinuous at b. Since the function $f(x)$ is continuous over $[a,t]$ for all values of t satisfying $a \le t < b$, the integral $\int_a^t f(x)\,dx$ is defined for all such values of t. Thus, it makes

sense to consider the values of $\int_a^t f(x)\,dx$ as t approaches b for $a \le t < b$. That is, we define $\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$, provided this limit exists. Figure 7.7.5 illustrates $\int_a^t f(x)\,dx$ as areas of regions for values of t approaching b.

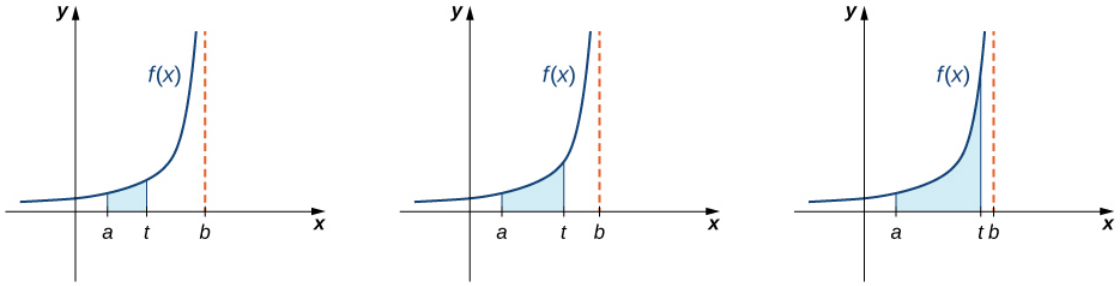

Figure 7.7.5: As t approaches b from the left, the value of the area from a to t approaches the area from a to b.

We use a similar approach to define $\int_a^b f(x)\,dx$, where $f(x)$ is continuous over $(a, b]$ and discontinuous at a. We now proceed with a formal definition.

Definition: Converging and Diverging Improper Integral

1. Let $f(x)$ be continuous over $[a, b)$. Then,

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx. \tag{7.7.4}$$

2. Let $f(x)$ be continuous over $(a, b]$. Then,

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx. \tag{7.7.5}$$

In each case, if the limit exists, then the improper integral is said to converge. If the limit does not exist, then the improper integral is said to diverge.

3. If $f(x)$ is continuous over $[a, b]$ except at a point c in (a, b), then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \tag{7.7.6}$$

provided both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ converge. If either of these integrals diverges, then $\int_a^b f(x)\,dx$ diverges.

The following examples demonstrate the application of this definition.

Example 7.7.6: Integrating a Discontinuous Integrand

Evaluate $\int_0^4 \frac{1}{\sqrt{4-x}}\,dx$, if possible. State whether the integral converges or diverges.

Solution

The function $f(x) = \frac{1}{\sqrt{4-x}}$ is continuous over $[0, 4)$ and discontinuous at 4. Using Equation 7.7.4 from the definition, rewrite $\int_0^4 \frac{1}{\sqrt{4-x}}\,dx$ as a limit:

$$\begin{aligned}
\int_0^4 \frac{1}{\sqrt{4-x}}\,dx &= \lim_{t\to 4^-} \int_0^t \frac{1}{\sqrt{4-x}}\,dx && \text{Rewrite as a limit.}\\
&= \lim_{t\to 4^-} \left(-2\sqrt{4-x}\right)\Big|_0^t && \text{Find the antiderivative.}\\
&= \lim_{t\to 4^-} \left(-2\sqrt{4-t}+4\right) && \text{Evaluate the antiderivative.}\\
&= 4. && \text{Evaluate the limit.}
\end{aligned}$$

The improper integral converges.

Example 7.7.7: Integrating a Discontinuous Integrand

Evaluate $\int_0^2 x \ln x \, dx$. State whether the integral converges or diverges.

Solution

Since $f(x) = x \ln x$ is continuous over $(0, 2]$ and is discontinuous at zero, we can rewrite the integral in limit form using Equation 7.7.5:

$$\begin{aligned}
\int_0^2 x \ln x \, dx &= \lim_{t \to 0^+} \int_t^2 x \ln x \, dx && \text{Rewrite as a limit.} \\
&= \lim_{t \to 0^+} \left(\frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 \right) \Big|_t^2 && \text{Evaluate } \int x \ln x \, dx \text{ using integration by parts with } u = \ln x \text{ and } dv = x. \\
&= \lim_{t \to 0^+} \left(2 \ln 2 - 1 - \frac{1}{2} t^2 \ln t + \frac{1}{4} t^2 \right) && \text{Evaluate the antiderivative.} \\
&= 2 \ln 2 - 1. && \text{Evaluate the limit. Note that } \lim_{t \to 0^+} t^2 \ln t \text{ is indeterminate.} \\
& && \text{To evaluate it, rewrite it as a quotient and apply L'Hôpital's rule.}
\end{aligned}$$

The improper integral converges.

Example 7.7.8: Integrating a Discontinuous Integrand

Evaluate $\int_{-1}^1 \frac{1}{x^3} \, dx$. State whether the improper integral converges or diverges.

Solution

Since $f(x) = 1/x^3$ is discontinuous at zero, using Equation 7.7.6, we can write

$$\int_{-1}^1 \frac{1}{x^3} \, dx = \int_{-1}^0 \frac{1}{x^3} \, dx \quad + \quad \int_0^1 \frac{1}{x^3} \, dx.$$

If either of the two integrals diverges, then the original integral diverges. Begin with $\int_{-1}^0 \frac{1}{x^3} \, dx$:

$$\begin{aligned}
\int_{-1}^0 \frac{1}{x^3} \, dx &= \lim_{t \to 0^-} \int_{-1}^t \frac{1}{x^3} \, dx && \text{Rewrite as a limit.} \\
&= \lim_{t \to 0^-} \left(-\frac{1}{2x^2} \right) \Big|_{-1}^t && \text{Find the antiderivative.} \\
&= \lim_{t \to 0^-} \left(-\frac{1}{2t^2} + \frac{1}{2} \right) && \text{Evaluate the antiderivative.} \\
&= +\infty. && \text{Evaluate the limit.}
\end{aligned}$$

Therefore, $\int_{-1}^0 \frac{1}{x^3} \, dx$ diverges.

Since $\int_{-1}^0 \frac{1}{x^3} \, dx$ diverges, $\int_{-1}^1 \frac{1}{x^3} \, dx$ also diverges.

Exercise 7.7.2

Evaluate $\int_0^2 \frac{1}{x} \, dx$. State whether the integral converges or diverges.

Hint

Write $\int_0^2 \frac{1}{x} \, dx$ in limit form using Equation 7.7.5.

Answer

$+\infty$, It diverges.

A Comparison Theorem

It is not always easy or even possible to evaluate an improper integral directly; however, by comparing it with another carefully chosen integral, it may be possible to determine its convergence or divergence. To see this, consider two continuous functions $f(x)$ and $g(x)$ satisfying $0 \le f(x) \le g(x)$ for $x \ge a$ (Figure 7.7.6). In this case, we may view integrals of these functions over intervals of the form $[a, t]$ as areas, so we have the relationship

$$0 \le \int_a^t f(x)\,dx \le \int_a^t g(x)\,dx$$

for $t \ge a$.

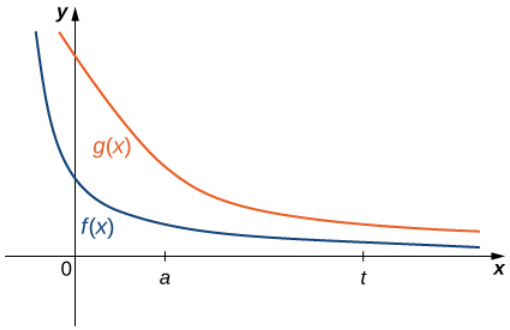

Figure 7.7.6: If $0 \le f(x) \le g(x)$ for $x \ge a$, then for $t \ge a$, $\int_a^t f(x)\,dx \le \int_a^t g(x)\,dx$.

Thus, if

$$\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty} \int_a^t f(x)\,dx = +\infty,$$

then

$$\int_a^{+\infty} g(x)\,dx = \lim_{t\to+\infty} \int_a^t g(x)\,dx = +\infty$$

as well. That is, if the area of the region between the graph of $f(x)$ and the x-axis over $[a, +\infty)$ is infinite, then the area of the region between the graph of $g(x)$ and the x-axis over $[a, +\infty)$ is infinite too.

On the other hand, if

$$\int_a^{+\infty} g(x)\,dx = \lim_{t\to+\infty} \int_a^t g(x)\,dx = L$$

for some real number L, then

$$\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty} \int_a^t f(x)\,dx$$

must converge to some value less than or equal to L, since $\int_a^t f(x)\,dx$ increases as t increases and $\int_a^t f(x)\,dx \le L$ for all $t \ge a$.

If the area of the region between the graph of $g(x)$ and the x-axis over $[a, +\infty)$ is finite, then the area of the region between the graph of $f(x)$ and the x-axis over $[a, +\infty)$ is also finite.

These conclusions are summarized in the following theorem.

A Comparison Theorem

Let $f(x)$ and $g(x)$ be continuous over $[a, +\infty)$. Assume that $0 \le f(x) \le g(x)$ for $x \ge a$.

i. If

$$\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty}\int_a^t f(x)\,dx = +\infty,$$

then

$$\int_a^{+\infty} g(x)\,dx = \lim_{t\to+\infty}\int_a^t g(x)\,dx = +\infty.$$

ii. If

$$\int_a^{+\infty} g(x)\,dx = \lim_{t\to+\infty}\int_a^t g(x)\,dx = L,$$

where L is a real number, then

$$\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty}\int_a^t f(x)\,dx = M$$

for some real number $M \le L$.

Example 7.7.9: Applying the Comparison Theorem

Use a comparison to show that $\int_1^{+\infty} \frac{1}{xe^x}\,dx$.

converges.

Solution

We can see that $0 \le \frac{1}{xe^x} \le \frac{1}{e^x} = e^{-x}$,

so if $\int_1^{+\infty} e^{-x}\,dx$ converges, then so does $\int_1^{+\infty} \frac{1}{xe^x}\,dx$.

To evaluate $\int_1^{+\infty} e^{-x}\,dx$, first rewrite it as a limit:

$$\begin{aligned}\int_1^{+\infty} e^{-x}\,dx &= \lim_{t\to+\infty}\int_1^t e^{-x}\,dx \\ &= \lim_{t\to+\infty}\left(-e^{-x}\right)\Big|_1^t \\ &= \lim_{t\to+\infty}\left(-e^{-t}+e^{-1}\right) \\ &= e^{-1}.\end{aligned}$$

Since $\int_1^{+\infty} e^{-x}\,dx$ converges, so does $\int_1^{+\infty} \frac{1}{xe^x}\,dx$.

Example 7.7.10: Applying the Comparison Theorem

Use the comparison theorem to show that $\int_1^{+\infty} \frac{1}{x^p}\,dx$ diverges for all $p < 1$.

Solution

For $p < 1$, $\frac{1}{x} \le \frac{1}{x^p}$ over $[1, +\infty)$.

In Example 7.7.1, we showed that $\int_1^{+\infty} \frac{1}{x}\,dx = +\infty$.

Therefore, $\int_1^{+\infty} \frac{1}{x^p}\,dx$ diverges for all $p < 1$.

Exercise 7.7.3

Use a comparison to show that $\int_e^{+\infty} \frac{\ln x}{x}\,dx$ diverges.

Hint

$\frac{1}{x} \le \frac{\ln x}{x}$ on $[e, +\infty)$

Answer

Since $\int_e^{+\infty} \frac{1}{x}\,dx = +\infty$, $\int_e^{+\infty} \frac{\ln x}{x}\,dx$ diverges.

Laplace Transforms

Laplace Transforms

In the last few chapters, we have looked at several ways to use integration for solving real-world problems. For this next project, we are going to explore a more advanced application of integration: integral transforms. Specifically, we describe the Laplace transform and some of its properties. The **Laplace transform** is used in engineering and physics to simplify the computations needed to solve some problems. It takes functions expressed in terms of time and *transforms* them to functions expressed in terms of frequency. It turns out that, in many cases, the computations needed to solve problems in the frequency domain are much simpler than those required in the time domain.

The Laplace transform is defined in terms of an integral as

$$\mathscr{L}\big(f(t)\big) = F(s) = \int_0^{\infty} e^{-st} f(t)\,dt.$$

Note that the input to a Laplace transform is a function of time, $f(t)$, and the output is a function of frequency, $F(s)$. Although many real-world examples require the use of complex numbers (involving the imaginary number $i = \sqrt{-1}$), in this project we limit ourselves to functions of real numbers.

Let's start with a simple example. Here we calculate the Laplace transform of $f(t) = t$. We have

$$\mathscr{L}(t) = \int_0^{\infty} te^{-st} dt.$$

This is an improper integral, so we express it in terms of a limit, which gives

$$\mathscr{L}(t) = \int_0^{\infty} te^{-st} dt = \lim_{z\to\infty} \int_0^{z} te^{-st} dt.$$

Now we use integration by parts to evaluate the integral. Note that we are integrating with respect to t, so we treat the variable s as a constant. We have

$$\begin{aligned} u &= t & dv &= e^{-st} dt \\ du &= dt & v &= -\tfrac{1}{s} e^{-st} \end{aligned}$$

Then we obtain

$$\begin{aligned}
\lim_{z\to\infty}\int_0^z te^{-st}dt &= \lim_{z\to\infty}\left[-\frac{t}{s}e^{-st}\Big|_0^z+\frac{1}{s}\int_0^z e^{-st}dt\right]\\
&= \lim_{z\to\infty}\left[\left[-\frac{z}{s}e^{-sz}+\frac{0}{s}e^{-0s}\right]+\frac{1}{s}\int_0^z e^{-st}dt\right]\\
&= \lim_{z\to\infty}\left[\left[-\frac{z}{s}e^{-sz}+0\right]-\frac{1}{s}\left[\frac{e^{-st}}{s}\right]\Big|_0^z\right]\\
&= \lim_{z\to\infty}\left[-\frac{z}{s}e^{-sz}-\frac{1}{s^2}\left[e^{-sz}-1\right]\right]\\
&= \lim_{z\to\infty}\left[-\frac{z}{se^{sz}}\right]-\lim_{z\to\infty}\frac{1}{s^2e^{sz}}+\lim_{z\to\infty}\frac{1}{s^2}\\
&= 0-0+\frac{1}{s^2}\\
&\qquad \text{Note that the first limit requires L'Hôpital's rule. See work below.}\\
&= \frac{1}{s^2}.
\end{aligned}$$

As noted above, $\lim_{z\to\infty}\left[-\frac{z}{se^{sz}}\right]$ is indeterminate of the form $-\frac{\infty}{\infty}$ and requires L'Hôpital's rule to evaluate. Using L'Hôpital's rule,

$$\lim_{z\to\infty}\left[-\frac{z}{se^{sz}}\right]=\lim_{z\to\infty}\left[-\frac{1}{s^2e^{sz}}\right]=0.$$

1. Calculate the Laplace transform of $f(t)=1$.
2. Calculate the Laplace transform of $f(t)=e^{-3t}$.
3. Calculate the Laplace transform of $f(t)=t^2$. (Note, you will have to integrate by parts twice.)

Laplace transforms are often used to solve differential equations. Differential equations are not covered in detail until later in this book; but, for now, let's look at the relationship between the Laplace transform of a function and the Laplace transform of its derivative.

Let's start with the definition of the Laplace transform. We have

$$\mathscr{L}\big(f(t)\big)=\int_0^\infty e^{-st}f(t)\,dt=\lim_{z\to\infty}\int_0^z e^{-st}f(t)\,dt.$$

Use integration by parts to evaluate $\lim_{z\to\infty}\int_0^z e^{-st}f(t)\,dt$. (Let $u=f(t)$ and $dv=e^{-st}dt$.)

After integrating by parts and evaluating the limit, you should see that

$$\mathscr{L}\big(f(t)\big)=\frac{f(0)}{s}+\frac{1}{s}\big[\mathscr{L}\big(f'(t)\big)\big].$$

Then,

$$\mathscr{L}\big(f'(t)\big)=s\mathscr{L}\big(f(t)\big)-f(0).$$

Thus, differentiation in the time domain simplifies to multiplication by s in the frequency domain.

The final thing we look at in this project is how the Laplace transforms of $f(t)$ and its antiderivative are related. Let $g(t)=\int_0^t f(u)\,du$.

Then,

$$\mathscr{L}\big(g(t)\big)=\int_0^\infty e^{-st}g(t)\,dt=\lim_{z\to\infty}\int_0^z e^{-st}g(t)\,dt.$$

Use integration by parts to evaluate $\lim_{z\to\infty}\int_0^z e^{-st}g(t)\,dt$. (Let $u=g(t)$ and $dv=e^{-st}dt$. Note, by the way that we have defined $g(t)$, $du=f(t)\,dt$.)

As you might expect, you should see that

$$\mathscr{L}\big(g(t)\big)=\frac{1}{\cdot}\cdot\mathscr{L}\big(f(t)\big).$$

Integration in the time domain simplifies to division by s in the frequency domain.

Key Concepts

- Integrals of functions over infinite intervals are defined in terms of limits.
- Integrals of functions over an interval for which the function has a discontinuity at an endpoint may be defined in terms of limits.
- The convergence or divergence of an improper integral may be determined by comparing it with the value of an improper integral for which the convergence or divergence is known.

Key Equations

- **Improper integrals**

$$\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty}\int_a^t f(x)\,dx$$
$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$$
$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^{+\infty} f(x)\,dx$$

Glossary

improper integral

an integral over an infinite interval or an integral of a function containing an infinite discontinuity on the interval; an improper integral is defined in terms of a limit. The improper integral converges if this limit is a finite real number; otherwise, the improper integral diverges

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7.7E: Exercises for Section 7.7

In exercises 1 - 8, evaluate the following integrals. If the integral is not convergent, answer "It diverges."

1) $\int_2^4 \frac{dx}{(x-3)^2}$

Answer

It diverges.

2) $\int_0^\infty \frac{1}{4+x^2}\,dx$

3) $\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx$

Answer

Converges to $\frac{\pi}{2}$

4) $\int_1^\infty \frac{1}{x\ln x}\,dx$

5) $\int_1^\infty xe^{-x}\,dx$

Answer

Converges to $\frac{2}{e}$

6) $\int_{-\infty}^\infty \frac{x}{x^2+1}\,dx$

7) Without integrating, determine whether the integral $\int_1^\infty \frac{1}{\sqrt{x^3+1}}\,dx$ converges or diverges by comparing the function $f(x) = \frac{1}{\sqrt{x^3+1}}$ with $g(x) = \frac{1}{\sqrt{x^3}}$.

Answer

It converges.

8) Without integrating, determine whether the integral $\int_1^\infty \frac{1}{\sqrt{x+1}}\,dx$ converges or diverges.

In exercises 9 - 25, determine whether the improper integrals converge or diverge. If possible, determine the value of the integrals that converge.

9) $\int_0^\infty e^{-x}\cos x\,dx$

Answer

Converges to $\frac{1}{2}$.

10) $\int_1^\infty \frac{\ln x}{x}\,dx$

11) $\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$

Answer

Converges to -4.

12) $\int_0^1 \ln x\,dx$

13) $\int_{-\infty}^\infty \frac{1}{x^2+1}\,dx$

Answer

Converges to π.

14) $\int_1^5 \frac{dx}{\sqrt{x-1}}$

15) $\int_{-2}^2 \frac{dx}{(1+x)^2}$

Answer

It diverges.

16) $\int_0^\infty e^{-x}\,dx$

17) $\int_0^\infty \sin x\,dx$

Answer

It diverges.

18) $\int_{-\infty}^\infty \frac{e^x}{1+e^{2x}}\,dx$

19) $\int_0^1 \frac{dx}{\sqrt[3]{x}}$

Answer

Converges to 1.5.

20) $\int_0^2 \frac{dx}{x^3}$

21) $\int_{-1}^2 \frac{dx}{x^3}$

Answer

It diverges.

22) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

23) $\int_0^3 \frac{1}{x-1}\,dx$

Answer

It diverges.

24) $\int_1^\infty \frac{5}{x^3}\,dx$

25) $\int_3^5 \frac{5}{(x-4)^2}\,dx$

Answer

It diverges.

In exercises 26 and 27, determine the convergence of each of the following integrals by comparison with the given integral. If the integral converges, find the number to which it converges.

26) $\int_1^\infty \frac{dx}{x^2+4x}$; compare with $\int_1^\infty \frac{dx}{x^2}$.

27) $\int_1^\infty \frac{dx}{\sqrt{x}+1}$; compare with $\int_1^\infty \frac{dx}{2\sqrt{x}}$.

Answer

Both integrals diverge.

In exercises 28 - 38, evaluate the integrals. If the integral diverges, answer "It diverges."

28) $\int_1^{\infty} \frac{dx}{x^e}$

29) $\int_0^1 \frac{dx}{x^{\pi}}$

Answer

It diverges.

30) $\int_0^1 \frac{dx}{\sqrt{1-x}}$

31) $\int_0^1 \frac{dx}{1-x}$

Answer

It diverges.

32) $\int_{-\infty}^0 \frac{dx}{x^2+1}$

33) $\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}}$

Answer

Converges to π.

34) $\int_0^1 \frac{\ln x}{x}\, dx$

35) $\int_0^e \ln(x)\, dx$

Answer

Converges to 0.

36) $\int_0^{\infty} xe^{-x}\, dx$

37) $\int_{-\infty}^{\infty} \frac{x}{(x^2+1)^2}\, dx$

Answer

Converges to 0.

38) $\int_0^{\infty} e^{-x}\, dx$

In exercises 39 - 44, evaluate the improper integrals. Each of these integrals has an infinite discontinuity either at an endpoint or at an interior point of the interval.

39) $\int_0^9 \frac{dx}{\sqrt{9-x}}$

Answer

Converges to 6.

40) $\int_{-27}^1 \frac{dx}{x^{2/3}}$

41) $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

Answer

Converges to $\frac{\pi}{2}$.

42) $\int_6^{24} \frac{dt}{t\sqrt{t^2-36}}$

43) $\int_0^4 x\ln(4x)\,dx$

Answer

Converges to $8\ln(16)-4$.

44) $\int_0^3 \frac{x}{\sqrt{9-x^2}}\,dx$

45) Evaluate $\int_{.5}^t \frac{dx}{\sqrt{1-x^2}}$. (Be careful!) (Express your answer using three decimal places.)

Answer

Converges to about 1.047.

46) Evaluate $\int_1^4 \frac{dx}{\sqrt{x^2-1}}$. (Express the answer in exact form.)

47) Evaluate $\int_2^\infty \frac{dx}{(x^2-1)^{3/2}}$.

Answer

Converges to $-1+\frac{2}{\sqrt{3}}$.

48) Find the area of the region in the first quadrant between the curve $y=e^{-6x}$ and the x-axis.

49) Find the area of the region bounded by the curve $y=\frac{7}{x^2}$, the x-axis, and on the left by $x=1$.

Answer

$A=7.0$ units.^{2}

50) Find the area under the curve $y=\frac{1}{(x+1)^{3/2}}$, bounded on the left by $x=3$.

51) Find the area under $y=\frac{5}{1+x^2}$ in the first quadrant.

Answer

$A=\frac{5\pi}{2}$ units.^{2}

52) Find the volume of the solid generated by revolving about the x-axis the region under the curve $y=\frac{3}{x}$ from $x=1$ to $x=\infty$.

53) Find the volume of the solid generated by revolving about the y-axis the region under the curve $y=6e^{-2x}$ in the first quadrant.

Answer

$V=3\pi$ units^3

54) Find the volume of the solid generated by revolving about the x-axis the area under the curve $y=3e^{-x}$ in the first quadrant.

The **Laplace transform** of a continuous function over the interval $[0,\infty)$ is defined by $F(s)=\int_0^\infty e^{-sx}f(x)\,dx$ (see the Student Project). This definition is used to solve some important initial-value problems in differential equations, as discussed later. The

domain of F is the set of all real numbers s such that the improper integral converges. Find the Laplace transform F of each of the following functions and give the domain of F.

55) $f(x) = 1$

Answer

$\frac{1}{s}, \quad s > 0$

56) $f(x) = x$

57) $f(x) = \cos(2x)$

Answer

$\frac{s}{s^2 + 4}, \quad s > 0$

58) $f(x) = e^{ax}$

59) Use the formula for arc length to show that the circumference of the circle $x^2 + y^2 = 1$ is 2π.

Answer

Answers will vary.

A function is a **probability density function** if it satisfies the following definition: $\int_{-\infty}^{\infty} f(t)\, dt = 1$. The probability that a random variable x lies between a and b is given by $P(a \leq x \leq b) = \int_a^b f(t)\, dt$.

60) Show that $f(x) = \begin{cases} 0, & \text{if } x < 0 \\ 7e^{-7x}, & \text{if } x \geq 0 \end{cases}$ is a probability density function.

61) Find the probability that x is between 0 and 0.3. (Use the function defined in the preceding problem.) Use four-place decimal accuracy.

Answer

0.8775

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7.R: Chapter 7 Review Exercises

In exercises 1 - 4, determine whether the statement is true or false. Justify your answer with a proof or a counterexample.

1) $\int e^x \sin(x)\, dx$ cannot be integrated by parts.

2) $\int \frac{1}{x^4+1}\, dx$ cannot be integrated using partial fractions.

Answer

False

3) In numerical integration, increasing the number of points decreases the error.

4) Integration by parts can always yield the integral.

Answer

False

In exercises 5 - 10, evaluate the integral using the specified method.

5) $\int x^2 \sin(4x)\, dx$, using integration by parts

6) $\int \frac{1}{x^2\sqrt{x^2+16}}\, dx$, using trigonometric substitution

Answer

$$\int \frac{1}{x^2\sqrt{x^2+16}}\, dx = -\frac{\sqrt{x^2+16}}{16x} + C$$

7) $\int \sqrt{x} \ln x\, dx$, using integration by parts

8) $\int \frac{3x}{x^3+2x^2-5x-6}\, dx$, using partial fractions

Answer

$$\int \frac{3x}{x^3+2x^2-5x-6}\, dx = \frac{1}{10}\left(4\ln|2-x| + 5\ln|x+1| - 9\ln|x+3|\right) + C$$

9) $\int \frac{x^5}{(4x^2+4)^{5/2}}\, dx$, using trigonometric substitution

10) $\int \frac{\sqrt{4-\sin^2(x)}}{\sin^2(x)}\cos(x)\, dx$, using a table of integrals or a CAS

Answer

$$\int \frac{\sqrt{4-\sin^2(x)}}{\sin^2(x)}\cos(x)\, dx = -\frac{\sqrt{4-\sin^2(x)}}{\sin(x)} - \frac{x}{2} + C$$

In exercises 11 - 15, integrate using whatever method you choose.

11) $\int \sin^2 x \cos^2 x\, dx$

12) $\int x^3\sqrt{x^2+2}\, dx$

Answer

$$\int x^3\sqrt{x^2+2}\, dx = \frac{1}{15}(x^2+2)^{3/2}(3x^2-4) + C$$

13) $\int \frac{3x^2+1}{x^4-2x^3-x^2+2x}\, dx$

14) $\int \frac{1}{x^4+4}\,dx$

Answer

$$\int \frac{1}{x^4+4}\,dx = \frac{1}{16}\ln\left(\frac{x^2+2x+2}{x^2-2x+2}\right) - \frac{1}{8}\tan^{-1}(1-x) + \frac{1}{8}\tan^{-1}(x+1) + C$$

15) $\int \frac{\sqrt{3+16x^4}}{x^4}\,dx$

In exercises 16 - 18, approximate the integrals using the midpoint rule, trapezoidal rule, and Simpson's rule using four subintervals, rounding to three decimals.

16) [T] $\int_1^2 \sqrt{x^5+2}\,dx$

Answer

$M_4 = 3.312,$
$T_4 = 3.354,$
$S_4 = 3.326$

17) [T] $\int_0^{\sqrt{\pi}} e^{-\sin(x^2)}\,dx$

18) [T] $\int_1^4 \frac{\ln(1/x)}{x}\,dx$

Answer

$M_4 = -0.982,$
$T_4 = -0.917,$
$S_4 = -0.952$

In exercises 19 - 20, evaluate the integrals, if possible.

19) $\int_1^\infty \frac{1}{x^n}\,dx$, for what values of n does this integral converge or diverge?

20) $\int_1^\infty \frac{e^{-x}}{x}\,dx$

Answer

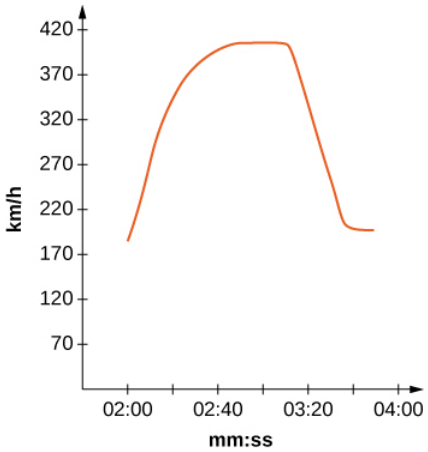
approximately 0.2194

In exercises 21 - 22, consider the gamma function given by $\Gamma(a) = \int_0^\infty e^{-y}y^{a-1}\,dy.$

21) Show that $\Gamma(a) = (a-1)\Gamma(a-1).$

22) Extend to show that $\Gamma(a) = (a-1)!$, assuming a is a positive integer.

The fastest car in the world, the Bugati Veyron, can reach a top speed of 408 km/h. The graph represents its velocity.



23) [T] Use the graph to estimate the velocity every 20 sec and fit to a graph of the form $v(t) = ae^{bx}\sin(cx) + d$. (Hint: Consider the time units.)

24) [T] Using your function from the previous problem, find exactly how far the Bugati Veyron traveled in the 1 min 40 sec included in the graph.

Answer

Answers may vary. Ex: 9.405 km

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CHAPTER OVERVIEW

8: Introduction to Differential Equations

Many real-world phenomena can be modeled mathematically by using differential equations. Population growth, radioactive decay, predator-prey models, and spring-mass systems are four examples of such phenomena. In this chapter we study some of these applications. A goal of this chapter is to develop solution techniques for different types of differential equations. As the equations become more complicated, the solution techniques also become more complicated, and in fact an entire course could be dedicated to the study of these equations. In this chapter we study several types of differential equations and their corresponding methods of solution.

8.0: Prelude to Differential Equations
8.1: Basics of Differential Equations
8.1E: Exercises for Section 8.1
8.2: Direction Fields and Numerical Methods
8.2E: Exercises for Section 8.2
8.3: Separable Equations
8.3E: Exercises for Section 8.3
8.4: The Logistic Equation
8.4E: Exercises for Section 8.4
8.5: First-order Linear Equations
8.5E: Exercises for Section 8.5
8.R: Chapter 8 Review Exercises

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8.0: Prelude to Differential Equations

Many real-world phenomena can be modeled mathematically by using differential equations. Population growth, radioactive decay, predator-prey models, and spring-mass systems are four examples of such phenomena. In this chapter we study some of these applications. Suppose we wish to study a population of deer over time and determine the total number of animals in a given area. We can first observe the population over a period of time, estimate the total number of deer, and then use various assumptions to derive a mathematical model for different scenarios. Some factors that are often considered are environmental impact, threshold population values, and predators. In this chapter we see how differential equations can be used to predict populations over time.

Figure 8.0.1: The white-tailed deer (Odocoileus virginianus) of the eastern United States. Differential equations can be used to study animal populations. (credit: modification of work by Rachel Kramer, Flickr)

Another goal of this chapter is to develop solution techniques for different types of differential equations. As the equations become more complicated, the solution techniques also become more complicated, and in fact an entire course could be dedicated to the study of these equations. In this chapter we study several types of differential equations and their corresponding methods of solution.

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8.1: Basics of Differential Equations

Learning Objectives

- Identify the order of a differential equation.
- Explain what is meant by a solution to a differential equation.
- Distinguish between the general solution and a particular solution of a differential equation.
- Identify an initial-value problem.
- Identify whether a given function is a solution to a differential equation or an initial-value problem.

Calculus is the mathematics of change, and rates of change are expressed by derivatives. Thus, one of the most common ways to use calculus is to set up an equation containing an unknown function $y = f(x)$ and its derivative, known as a differential equation. Solving such equations often provides information about how quantities change and frequently provides insight into how and why the changes occur.

Techniques for solving differential equations can take many different forms, including direct solution, use of graphs, or computer calculations. We introduce the main ideas in this chapter and describe them in a little more detail later in the course. In this section we study what differential equations are, how to verify their solutions, some methods that are used for solving them, and some examples of common and useful equations.

General Differential Equations

Consider the equation $y' = 3x^2$, which is an example of a differential equation because it includes a derivative. There is a relationship between the variables x and y : y is an unknown function of x. Furthermore, the left-hand side of the equation is the derivative of y. Therefore we can interpret this equation as follows: Start with some function $y = f(x)$ and take its derivative. The answer must be equal to $3x^2$. What function has a derivative that is equal to $3x^2$? One such function is $y = x^3$, so this function is considered a *solution* to a differential equation.

Definition: differential equation

A *differential equation* is an equation involving an unknown function $y = f(x)$ and one or more of its derivatives. A solution to a differential equation is a function $y = f(x)$ that satisfies the differential equation when f and its derivatives are substituted into the equation.

Go to this website to explore more on this topic.

Some examples of differential equations and their solutions appear in Table 8.1.1.

Table 8.1.1: Examples of Differential Equations and Their Solutions

Equation	Solution
$y' = 2x$	$y = x^2$
$y' + 3y = 6x + 11$	$y = e^{-3x} + 2x + 3$
$y'' - 3y' + 2y = 24e^{-2x}$	$y = 3e^x - 4e^{2x} + 2e^{-2x}$

Note that a solution to a differential equation is not necessarily unique, primarily because the derivative of a constant is zero. For example, $y = x^2 + 4$ is also a solution to the first differential equation in Table 8.1.1. We will return to this idea a little bit later in this section. For now, let's focus on what it means for a function to be a solution to a differential equation.

Example 8.1.1: Verifying Solutions of Differential Equations

Verify that the function $y = e^{-3x} + 2x + 3$ is a solution to the differential equation $y' + 3y = 6x + 11$.

Solution

To verify the solution, we first calculate y' using the chain rule for derivatives. This gives $y' = -3e^{-3x} + 2$. Next we substitute y and y' into the left-hand side of the differential equation:

$$(-3e^{-2x} + 2) + 3(e^{-2x} + 2x + 3).$$

The resulting expression can be simplified by first distributing to eliminate the parentheses, giving

$$-3e^{-2x} + 2 + 3e^{-2x} + 6x + 9.$$

Combining like terms leads to the expression $6x + 11$, which is equal to the right-hand side of the differential equation. This result verifies that $y = e^{-3x} + 2x + 3$ is a solution of the differential equation.

Exercise 8.1.1

Verify that $y = 2e^{3x} - 2x - 2$ is a solution to the differential equation $y' - 3y = 6x + 4$.

Hint

First calculate y' then substitute both y' and y into the left-hand side.

It is convenient to define characteristics of differential equations that make it easier to talk about them and categorize them. The most basic characteristic of a differential equation is its order.

Definition: order of a differential equation

The *order of a differential equation* is the highest order of any derivative of the unknown function that appears in the equation.

Example 8.1.2: Identifying the Order of a Differential Equation

The highest derivative in the equation is y',

What is the order of each of the following differential equations?

a. $y' - 4y = x^2 - 3x + 4$
b. $x^2y''' - 3xy'' + xy' - 3y = \sin x$
c. $\frac{4}{x}y^{(4)} - \frac{6}{x^2}y'' + \frac{12}{x^4}y = x^3 - 3x^2 + 4x - 12$

Solution

a. The highest derivative in the equation is y',so the order is 1.
b. The highest derivative in the equation is y''', so the order is 3.
c. The highest derivative in the equation is $y^{(4)}$, so the order is 4.

Exercise 8.1.2

What is the order of the following differential equation?

$$(x^4 - 3x)y^{(5)} - (3x^2 + 1)y' + 3y = \sin x \cos x$$

Hint

What is the highest derivative in the equation?

Answer

5

General and Particular Solutions

We already noted that the differential equation $y' = 2x$ has at least two solutions: $y = x^2$ and $y = x^2 + 4$. The only difference between these two solutions is the last term, which is a constant. What if the last term is a different constant? Will this expression still be a solution to the differential equation? In fact, any function of the form $y = x^2 + C$, where C represents any constant, is a solution as

well. The reason is that the derivative of $x^2 + C$ is $2x$, regardless of the value of C. It can be shown that any solution of this differential equation must be of the form $y = x^2 + C$. This is an example of a **general solution** to a differential equation. A graph of some of these solutions is given in Figure 8.1.1. (Note: in this graph we used even integer values for C ranging between -4 and 4. In fact, there is no restriction on the value of C; it can be an integer or not.)

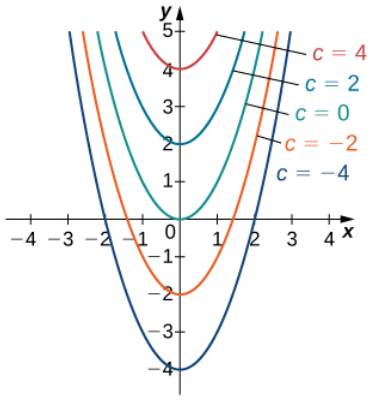

Figure 8.1.1: Family of solutions to the differential equation $y' = 2x$.

In this example, we are free to choose any solution we wish; for example, $y = x^2 - 3$ is a member of the family of solutions to this differential equation. This is called a particular solution to the differential equation. A particular solution can often be uniquely identified if we are given additional information about the problem.

Example 8.1.3: Finding a Particular Solution

Find the particular solution to the differential equation $y' = 2x$ passing through the point $(2, 7)$.

Solution

Any function of the form $y = x^2 + C$ is a solution to this differential equation. To determine the value of C, we substitute the values $x = 2$ and $y = 7$ into this equation and solve for C:

$$\begin{aligned} y &= x^2 + C \\ 7 &= 2^2 + C \\ &= 4 + C \\ C &= 3. \end{aligned}$$

Therefore the particular solution passing through the point $(2, 7)$ is $y = x^2 + 3$.

Exercise 8.1.3

Find the particular solution to the differential equation

$$y' = 4x + 3$$

passing through the point $(1, 7)$, given that $y = 2x^2 + 3x + C$ is a general solution to the differential equation.

Hint

First substitute $x = 1$ and $y = 7$ into the equation, then solve for C.

Answer

$$y = 2x^2 + 3x + 2$$

Initial-Value Problems

Usually a given differential equation has an infinite number of solutions, so it is natural to ask which one we want to use. To choose one solution, more information is needed. Some specific information that can be useful is an *initial value*, which is an ordered pair

that is used to find a particular solution.

A differential equation together with one or more initial values is called an *initial-value problem*. The general rule is that the number of initial values needed for an initial-value problem is equal to the order of the differential equation. For example, if we have the differential equation $y' = 2x$, then $y(3) = 7$ is an initial value, and when taken together, these equations form an initial-value problem. The differential equation $y'' - 3y' + 2y = 4e^x$ is second order, so we need two initial values. With initial-value problems of order greater than one, the same value should be used for the independent variable. An example of initial values for this second-order equation would be $y(0) = 2$ and $y'(0) = -1$. These two initial values together with the differential equation form an initial-value problem. These problems are so named because often the independent variable in the unknown function is t, which represents time. Thus, a value of $t = 0$ represents the beginning of the problem.

Example 8.1.4: Verifying a Solution to an Initial-Value Problem

Verify that the function $y = 2e^{-2t} + e^t$ is a solution to the initial-value problem

$$y' - y = -6e^{-2t}, \quad y(0) = 3.$$

Solution

For a function to satisfy an initial-value problem, it must satisfy both the differential equation and the initial condition. To show that y satisfies the differential equation, we start by calculating y'. This gives $y' = -4e^{-2t} + e^t$. Next we substitute both y and y' into the left-hand side of the differential equation and simplify:

$$\begin{aligned} y' - y &= (-4e^{-2t} + e^t) - (2e^{-2t} + e^t) \\ &= -4e^{-2t} + e^t - 2e^{-2t} - e^t = -6e^{-2t}. \end{aligned}$$

This is equal to the right-hand side of the differential equation, so $y = 2e^{-2t} + e^t$ solves the differential equation. Next we calculate $y(0)$:

$$y(0) = 2e^{-2(0)} + e^0 = 2 + 1 = 3.$$

This result verifies the initial value. Therefore the given function satisfies the initial-value problem.

Exercise 8.1.4

Verify that $y = 3e^{2t} + 4\sin t$ is a solution to the initial-value problem

$$y' - 2y = 4\cos t - 8\sin t, y(0) = 3.$$

Hint

First verify that y solves the differential equation. Then check the initial value.

In Example 8.1.4, the initial-value problem consisted of two parts. The first part was the differential equation $y' - y = -6e^{-2t}$, and the second part was the initial value $y(0) = 3$. These two equations together formed the initial-value problem.

The same is true in general. An initial-value problem will consists of two parts: the differential equation and the initial condition. The differential equation has a family of solutions, and the initial condition determines the value of C. The family of solutions to the differential equation in Example 8.1.4 is given by $y = 2e^{-2t} + Ce^t$. This family of solutions is shown in Figure 8.1.2, with the particular solution $y = 2e^{-2t} + e^t$ labeled.

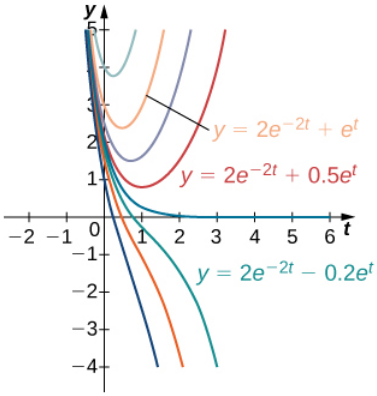

Figure 8.1.2: A family of solutions to the differential equation $y' - y = -6e^{-2t}$. The particular solution $y = 2e^{-2t} + e^t$ is labeled.

Example 8.1.5: Solving an Initial-value Problem

Solve the following initial-value problem:

$$y' = 3e^x + x^2 - 4, y(0) = 5.$$

Solution

The first step in solving this initial-value problem is to find a general family of solutions. To do this, we find an antiderivative of both sides of the differential equation

$$\int y'\,dx = \int (3e^x + x^2 - 4)\,dx,$$

namely,

$$y + C_1 = 3e^x + \tfrac{1}{3}x^3 - 4x + C_2.$$

We are able to integrate both sides because the y term appears by itself. Notice that there are two integration constants: C_1 and C_2. Solving this equation for y gives

$$y = 3e^x + \tfrac{1}{3}x^3 - 4x + C_2 - C_1.$$

Because C_1 and C_2 are both constants, $C_2 - C_1$ is also a constant. We can therefore define $C = C_2 - C_1$, which leads to the equation

$$y = 3e^x + \tfrac{1}{3}x^3 - 4x + C.$$

Next we determine the value of C. To do this, we substitute $x = 0$ and $y = 5$ into this equation and solve for C:

$$\begin{aligned} 5 &= 3e^0 + \frac{1}{3}0^3 - 4(0) + C \\ 5 &= 3 + C \\ C &= 2 \end{aligned}$$

Now we substitute the value $C = 2$ into the general equation. The solution to the initial-value problem is $y = 3e^x + \frac{1}{3}x^3 - 4x + 2$.

Analysis

The difference between a general solution and a particular solution is that a general solution involves a family of functions, either explicitly or implicitly defined, of the independent variable. The initial value or values determine which particular solution in the family of solutions satisfies the desired conditions.

Exercise 8.1.5

Solve the initial-value problem

$$y' = x^2 - 4x + 3 - 6e^x, y(0) = 8.$$

Hint

First take the antiderivative of both sides of the differential equation. Then substitute $x = 0$ and $y = 8$ into the resulting equation and solve for C.

Answer

$y = \frac{1}{3}x^3 - 2x^2 + 3x - 6e^x + 14$

Applications

In physics and engineering applications, we often consider the forces acting upon an object, and use this information to understand the resulting motion that may occur. For example, if we start with an object at Earth's surface, the primary force acting upon that object is gravity. Physicists and engineers can use this information, along with Newton's **second law of motion** (in equation form $F = ma$, where F represents force, m represents mass, and a represents acceleration), to derive an equation that can be solved.

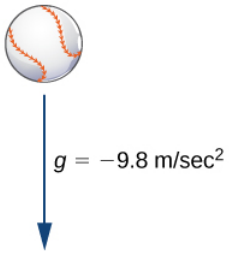

Figure 8.1.3: For a baseball falling in air, the only force acting on it is gravity (neglecting air resistance).

In Figure 8.1.3 we assume that the only force acting on a baseball is the force of gravity. This assumption ignores air resistance. (The force due to air resistance is considered in a later discussion.) The acceleration due to gravity at Earth's surface, g, is approximately $9.8\,\text{m/s}^2$. We introduce a frame of reference, where Earth's surface is at a height of 0 meters. Let $v(t)$ represent the velocity of the object in meters per second. If $v(t) > 0$, the ball is rising, and if $v(t) < 0$, the ball is falling (Figure).

Figure 8.1.4: Possible velocities for the rising/falling baseball.

Our goal is to solve for the velocity $v(t)$ at any time t. To do this, we set up an initial-value problem. Suppose the mass of the ball is m, where m is measured in kilograms. We use Newton's second law, which states that the force acting on an object is equal to its mass times its acceleration $(F = ma)$. Acceleration is the derivative of velocity, so $a(t) = v'(t)$. Therefore the force acting on the baseball is given by $F = mv'(t)$. However, this force must be equal to the force of gravity acting on the object, which (again using Newton's second law) is given by $F_g = -mg$, since this force acts in a downward direction. Therefore we obtain the equation $F = F_g$, which becomes $mv'(t) = -mg$. Dividing both sides of the equation by m gives the equation

$$v'(t) = -g.$$

Notice that this differential equation remains the same regardless of the mass of the object.

We now need an initial value. Because we are solving for velocity, it makes sense in the context of the problem to assume that we know the **initial velocity**, or the velocity at time $t = 0$. This is denoted by $v(0) = v_0$.

Example 8.1.6: Velocity of a Moving Baseball

A baseball is thrown upward from a height of 3 meters above Earth's surface with an initial velocity of 10 m/s, and the only force acting on it is gravity. The ball has a mass of 0.15 kg at Earth's surface.

a. Find the velocity $v(t)$ of the basevall at time t.
b. What is its velocity after 2 seconds?

Solution

a. From the preceding discussion, the differential equation that applies in this situation is

$$v'(t) = -g,$$

where $g = 9.8\ \text{m/s}^2$. The initial condition is $v(0) = v_0$, where $v_0 = 10$ m/s. Therefore the initial-value problem is $v'(t) = -9.8\ \text{m/s}^2$, $v(0) = 10$ m/s.

The first step in solving this initial-value problem is to take the antiderivative of both sides of the differential equation. This gives

$$\int v'(t)\,dt = \int -9.8\,dt$$

$$v(t) = -9.8t + C.$$

The next step is to solve for C. To do this, substitute $t = 0$ and $v(0) = 10$:

$$\begin{aligned} v(t) &= -9.8t + C \\ v(0) &= -9.8(0) + C \\ 10 &= C. \end{aligned}$$

Therefore $C = 10$ and the velocity function is given by $v(t) = -9.8t + 10$.

b. To find the velocity after 2 seconds, substitute $t = 2$ into $v(t)$.

$$\begin{aligned} v(t) &= -9.8t + 10 \\ v(2) &= -9.8(2) + 10 \\ v(2) &= -9.6 \end{aligned}$$

The units of velocity are meters per second. Since the answer is negative, the object is falling at a speed of 9.6 m/s.

Exercise 8.1.6

Suppose a rock falls from rest from a height of 100 meters and the only force acting on it is gravity. Find an equation for the velocity $v(t)$ as a function of time, measured in meters per second.

Hint

What is the initial velocity of the rock? Use this with the differential equation in Example 8.1.6 to form an initial-value problem, then solve for $v(t)$.

Answer

$v(t) = -9.8t$

A natural question to ask after solving this type of problem is how high the object will be above Earth's surface at a given point in time. Let $s(t)$ denote the height above Earth's surface of the object, measured in meters. Because velocity is the derivative of position (in this case height), this assumption gives the equation $s'(t) = v(t)$. An initial value is necessary; in this case the initial height of the object works well. Let the initial height be given by the equation $s(0) = s_0$. Together these assumptions give the initial-value problem

$$s'(t) = v(t), s(0) = s_0.$$

If the velocity function is known, then it is possible to solve for the position function as well.

Example 8.1.7: Height of a Moving Baseball

A baseball is thrown upward from a height of 3 meters above Earth's surface with an initial velocity of $10m/s$, and the only force acting on it is gravity. The ball has a mass of 0.15 kilogram at Earth's surface.

a. Find the position $s(t)$ of the baseball at time t.
b. What is its height after 2 seconds?

Solution

We already know the velocity function for this problem is $v(t) = -9.8t + 10$. The initial height of the baseball is 3 meters, so $s_0 = 3$. Therefore the initial-value problem for this example is

To solve the initial-value problem, we first find the antiderivatives:

$$\int s'(t)\,dt = \int (-9.8t + 10)\,dt$$

$$s(t) = -4.9t^2 + 10t + C.$$

Next we substitute $t = 0$ and solve for C:

$$s(t) = -4.9t^2 + 10t + C$$

$$s(0) = -4.9(0)^2 + 10(0) + C$$

$$3 = C.$$

Therefore the position function is $s(t) = -4.9t^2 + 10t + 3$.

b. The height of the baseball after 2 sec is given by $s(2)$:

$$s(2) = -4.9(2)^2 + 10(2) + 3 = -4.9(4) + 23 = 3.4.$$

Therefore the baseball is 3.4 meters above Earth's surface after 2 seconds. It is worth noting that the mass of the ball cancelled out completely in the process of solving the problem.

Key Concepts

- A differential equation is an equation involving a function $y = f(x)$ and one or more of its derivatives. A solution is a function $y = f(x)$ that satisfies the differential equation when f and its derivatives are substituted into the equation.
- The order of a differential equation is the highest order of any derivative of the unknown function that appears in the equation.
- A differential equation coupled with an initial value is called an initial-value problem. To solve an initial-value problem, first find the general solution to the differential equation, then determine the value of the constant. Initial-value problems have many applications in science and engineering.

Glossary

differential equation
an equation involving a function $y = y(x)$ and one or more of its derivatives

general solution (or family of solutions)
the entire set of solutions to a given differential equation

initial value(s)
a value or set of values that a solution of a differential equation satisfies for a fixed value of the independent variable

initial velocity
the velocity at time $t = 0$

initial-value problem

a differential equation together with an initial value or values

order of a differential equation

the highest order of any derivative of the unknown function that appears in the equation

particular solution

member of a family of solutions to a differential equation that satisfies a particular initial condition

solution to a differential equation

a function $y = f(x)$ that satisfies a given differential equation

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8.1E: Exercises for Section 8.1

In exercises 1 - 7, determine the order of each differential equation.

1) $y'+y=3y^2$

Answer

1st-order

2) $(y')^2=y'+2y$

3) $y'''+y''y'=3x^2$

Answer

3rd-order

4) $y'=y''+3t^2$

5) $\frac{dy}{dt}=t$

Answer

1st-order

6) $\frac{dy}{dx}+\frac{d^2y}{dx^2}=3x^4$

7) $\left(\frac{dy}{dt}\right)^2+8\frac{dy}{dt}+3y=4t$

Answer

1st-order

In exercises 8 - 17, verify that the given function is a solution to the given differential equation.

8) $y=\frac{x^3}{3}$ solves $y'=x^2$

9) $y=2e^{-x}+x-1$ solves $y'=x-y$

10) $y=e^{3x}-\frac{e^x}{2}$ solves $y'=3y+e^x$

11) $y=\frac{1}{1-x}$ solves $y'=y^2$

12) $y=\frac{e^{x^2}}{2}$ solves $y'=2xy$

13) $y=4+\ln x$ solves $xy'=1$

14) $y=3-x+x\ln x$ solves $y'=\ln x$

15) $y=2e^x-x-1$ solves $y'=y+x$

16) $y=e^x+\frac{\sin x}{2}-\frac{\cos x}{2}$ solves $y'=\cos x+y$

17) $y=\pi e^{-\cos x}$ solves $y'=y\sin x$

In exercises 18 - 27, verify the given general solution and find the particular solution.

18) Find the particular solution to the differential equation $y'=4x^2$ that passes through $(-3,-30)$, given that $y=C+\frac{4x^3}{3}$ is a general solution.

19) Find the particular solution to the differential equation $y'=3x^3$ that passes through $(1,4.75)$, given that $y=C+\frac{3x^4}{4}$ is a general solution.

Answer

$y = 4 + \frac{3x^4}{4}$

20) Find the particular solution to the differential equation $y' = 3x^2y$ that passes through $(0, 12)$, given that $y = Ce^{x^3}$ is a general solution.

21) Find the particular solution to the differential equation $y' = 2xy$ that passes through $(0, \frac{1}{2})$, given that $y = Ce^{x^2}$ is a general solution.

Answer

$y = \frac{1}{2}e^{x^2}$

22) Find the particular solution to the differential equation $y' = \left(2xy\right)^2$ that passes through $(1, -\frac{1}{2})$, given that $y = -\frac{3}{C + 4x^3}$ is a general solution.

23) Find the particular solution to the differential equation $y'x^2 = y$ that passes through $(1, \frac{2}{e})$, given that $y = Ce^{-1/x}$ is a general solution.

Answer

$y = 2e^{-1/x}$

24) Find the particular solution to the differential equation $8\frac{dx}{dt} = -2\cos(2t) - \cos(4t)$ that passes through (π, π), given that $x = C - \frac{1}{8}\sin(2t) - \frac{1}{32}\sin(4t)$ is a general solution.

25) Find the particular solution to the differential equation $\frac{du}{dt} = \tan u$ that passes through $(1, \frac{\pi}{2})$, given that $u = \sin^{-1}\left(e^{C+t}\right)$ is a general solution.

Answer

$u = \sin^{-1}\left(e^{-1+t}\right)$

26) Find the particular solution to the differential equation $\frac{dy}{dt} = e^{t+y}$ that passes through $(1, 0)$, given that $y = -\ln(C - e^t)$ is a general solution.

27) Find the particular solution to the differential equation $y'(1 - x^2) = 1 + y$ that passes through $(0, -2)$, given that $y = C\frac{\sqrt{x+1}}{\sqrt{1-x}} - 1$ is a general solution.

Answer

$y = -\frac{\sqrt{x+1}}{\sqrt{1-x}} - 1$

In exercises 28 - 37, find the general solution to the differential equation.

28) $y' = 3x + e^x$

29) $y' = \ln x + \tan x$

Answer

$y = C - x + x\ln x - \ln(\cos x)$

30) $y' = \sin x e^{\cos x}$

31) $y' = 4^x$

Answer

$y = C + \frac{4^x}{\ln 4}$

32) $y' = \sin^{-1}(2x)$

33) $y' = 2t\sqrt{t^2+16}$

Answer

$y = \frac{2}{3}\sqrt{t^2+16}\left(t^2+16\right) + C$

34) $x' = \coth t + \ln t + 3t^2$

35) $x' = t\sqrt{4+t}$

Answer

$x = \frac{2}{15}\sqrt{4+t}\left(3t^2+4t-32\right) + C$

36) $y' = y$

37) $y' = \frac{y}{x}$

Answer

$y = Cx$

In exercises 38 - 42, solve the initial-value problems starting from $y(t=0)=1$ and $y(t=0)=-1$. Draw both solutions on the same graph.

38) $\frac{dy}{dt} = 2t$

39) $\frac{dy}{dt} = -t$

Answer

$y = 1 - \frac{t^2}{2}$, and $y = -\frac{t^2}{2} - 1$

40) $\frac{dy}{dt} = 2y$

41) $\frac{dy}{dt} = -y$

Answer

$y = e^{-t}$ and $y = -e^{-t}$

42) $\frac{dy}{dt} = 2$

In exercises 43 - 47, solve the initial-value problems starting from $y_0 = 10$. At what time does y increase to 100 or drop to 1?

43) $\frac{dy}{dt} = 4t$

Answer

$y = 2(t^2+5)$, When $t = 3\sqrt{5}$, y will increase to 100.

44) $\frac{dy}{dt} = 4y$

45) $\frac{dy}{dt} = -2y$

Answer

$y = 10e^{-2t}$, When $t = -\frac{1}{2}\ln\left(\frac{1}{10}\right)$, y will decrease to 1.

46) $\frac{dy}{dt} = e^{4t}$

47) $\frac{dy}{dt} = e^{-4t}$

Answer

$y = \frac{1}{4}(41 - e^{-4t})$, Neither condition will ever happen.

Recall that a family of solutions includes solutions to a differential equation that differ by a constant. For exercises 48 - 52, use your calculator to graph a family of solutions to the given differential equation. Use initial conditions from $y(t = 0) = -10$ to $y(t = 0) = 10$ increasing by 2. Is there some critical point where the behavior of the solution begins to change?

48) [T] $y' = y(x)$

49) [T] $xy' = y$

Answer

Solution changes from increasing to decreasing at $y(0) = 0$.

50) [T] $y' = t^3$

51) [T] $y' = x + y$ (Hint: $y = Ce^x - x - 1$ is the general solution)

Answer

Solution changes from increasing to decreasing at $y(0) = 0$.

52) [T] $y' = x \ln x + \sin x$

53) Find the general solution to describe the velocity of a ball of mass 1 lb that is thrown upward at a rate of a ft/sec.

Answer

$v(t) = -32t + a$

54) In the preceding problem, if the initial velocity of the ball thrown into the air is $a = 25$ ft/s, write the particular solution to the velocity of the ball. Solve to find the time when the ball hits the ground.

55) You throw two objects with differing masses m_1 and m_2 upward into the air with the same initial velocity of a ft/s. What is the difference in their velocity after 1 second?

Answer

0 ft/s

56) [T] You throw a ball of mass 1 kilogram upward with a velocity of $a = 25$ m/s on Mars, where the force of gravity is $g = -3.711$ m/s^2. Use your calculator to approximate how much longer the ball is in the air on Mars.

57) [T] For the previous problem, use your calculator to approximate how much higher the ball went on Mars.

Answer

4.86 meters

58) [T] A car on the freeway accelerates according to $a = 15\cos(\pi t)$, where t is measured in hours. Set up and solve the differential equation to determine the velocity of the car if it has an initial speed of 51 mph. After 40 minutes of driving, what is the driver's velocity?

59) [T] For the car in the preceding problem, find the expression for the distance the car has traveled in time t, assuming an initial distance of 0. How long does it take the car to travel 100 miles? Round your answer to hours and minutes.

Answer

$x = 50t - \frac{15}{\pi^2}\cos(\pi t) + \frac{3}{\pi^2}$, 2 hours 1 minute

60) [T] For the previous problem, find the total distance traveled in the first hour.

61) Substitute $y = Be^{3t}$ into $y' - y = 8e^{3t}$ to find a particular solution.

Answer

$y = 4e^{3t}$

62) Substitute $y = a\cos(2t) + b\sin(2t)$ into $y' + y = 4\sin(2t)$ to find a particular solution.

63) Substitute $y = a + bt + ct^2$ into $y' + y = 1 + t^2$ to find a particular solution.

Answer

$y = 1 - 2t + t^2$

64) Substitute $y = ae^t \cos t + be^t \sin t$ into $y' = 2e^t \cos t$ to find a particular solution.

65) Solve $y' = e^{kt}$ with the initial condition $y(0) = 0$ and solve $y' = 1$ with the same initial condition. As k approaches 0, what do you notice?

Answer

$y = \frac{1}{k}(e^{kt} - 1)$ and $y = t$

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8.2: Direction Fields and Numerical Methods

Learning Objectives

- Draw the direction field for a given first-order differential equation.
- Use a direction field to draw a solution curve of a first-order differential equation.
- Use Euler's Method to approximate the solution to a first-order differential equation.

For the rest of this chapter we will focus on various methods for solving differential equations and analyzing the behavior of the solutions. In some cases it is possible to predict properties of a solution to a differential equation without knowing the actual solution. We will also study numerical methods for solving differential equations, which can be programmed by using various computer languages or even by using a spreadsheet program, such as Microsoft Excel.

Creating Direction Fields

Direction fields (also called slope fields) are useful for investigating first-order differential equations. In particular, we consider a first-order differential equation of the form

$$y' = f(x, y).$$

An applied example of this type of differential equation appears in Newton's law of cooling, which we will solve explicitly later in this chapter. First, though, let us create a direction field for the differential equation

$$T'(t) = -0.4(T - 72).$$

Here $T(t)$ represents the temperature (in degrees Fahrenheit) of an object at time t, and the ambient temperature is $72°F$. Figure 8.2.1 shows the direction field for this equation.

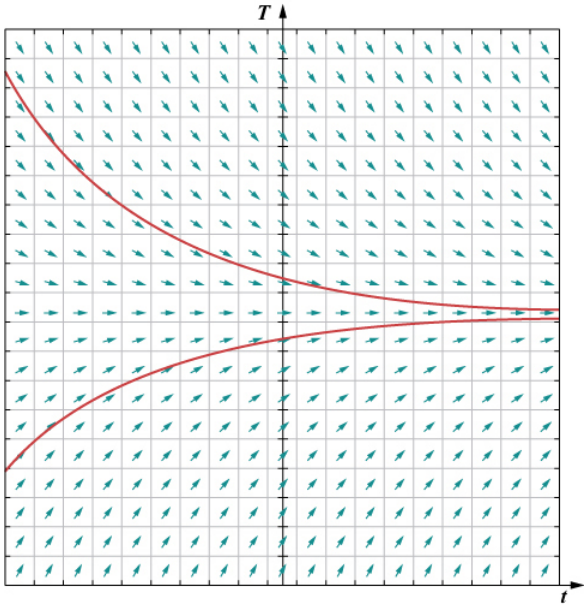


Figure 8.2.1: Direction field for the differential equation $T'(t) = -0.4(T - 72)$. Two solutions are plotted: one with initial temperature less than $72°F$ and the other with initial temperature greater than $72°F$.

The idea behind a direction field is the fact that the derivative of a function evaluated at a given point is the slope of the tangent line to the graph of that function at the same point. Other examples of differential equations for which we can create a direction field include

$$y' = 3x + 2y - 4$$

$$y' = x^2 - y^2$$

$$y' = \frac{2x + 4}{u - 2}.$$

To create a direction field, we start with the first equation: $y' = 3x + 2y - 4$. We let (x_0, y_0) be any ordered pair, and we substitute these numbers into the right-hand side of the differential equation. For example, if we choose $x = 1$ and $y = 2$, substituting into the right-hand side of the differential equation yields

$$y' = 3x + 2y - 4 = 3(1) + 2(2) - 4 = 3.$$

This tells us that if a solution to the differential equation $y' = 3x + 2y - 4$ passes through the point $(1, 2)$, then the slope of the solution at that point must equal 3. To start creating the direction field, we put a short line segment at the point $(1, 2)$ having slope 3. We can do this for any point in the domain of the function $f(x, y) = 3x + 2y - 4$, which consists of all ordered pairs (x, y) in R^2. Therefore any point in the Cartesian plane has a slope associated with it, assuming that a solution to the differential equation passes through that point. The direction field for the differential equation $y' = 3x + 2y - 4$ is shown in Figure 8.2.2.

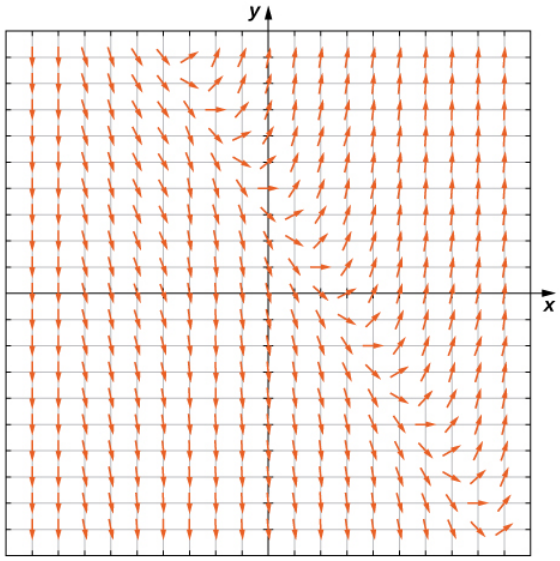


Figure 8.2.2: Direction field for the differential equation $y' = 3x + 2y - 4$.

We can generate a direction field of this type for any differential equation of the form $y' = f(x, y)$.

Definition: Direction Field (Slope Field)

A direction field (slope field) is a mathematical object used to graphically represent solutions to a first-order differential equation. At each point in a direction field, a line segment appears whose slope is equal to the slope of a solution to the differential equation passing through that point.

Using Direction Fields

We can use a direction field to predict the behavior of solutions to a differential equation without knowing the actual solution. For example, the direction field in Figure 8.2.3 serves as a guide to the behavior of solutions to the differential equation $y' = 3x + 2y - 4$.

To use a direction field, we start by choosing any point in the field. The line segment at that point serves as a signpost telling us what direction to go from there. For example, if a solution to the differential equation passes through the point $(0, 1)$, then the slope of the solution passing through that point is given by $y' = 3(0) + 2(1) - 4 = -2$. Now let x increase slightly, say to $x = 0.1$. Using the method of linear approximations gives a formula for the approximate value of y for $x = 0.1$. In particular,

$$L(x) = y_0 + f'(x_0)(x - x_0) = 1 - 2(x_0 - 0) = 1 - 2x_0.$$

Substituting $x_0 = 0.1$ into $L(x)$ gives an approximate y value of 0.8.

At this point the slope of the solution changes (again according to the differential equation). We can keep progressing, recalculating the slope of the solution as we take small steps to the right, and watching the behavior of the solution. Figure 8.2.3 shows a graph of the solution passing through the point $(0, 1)$.

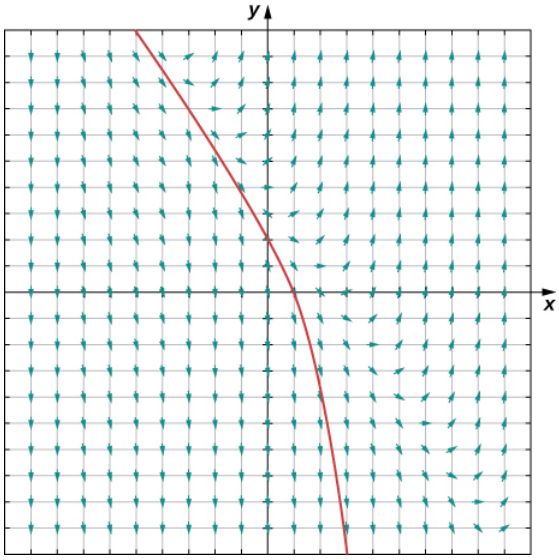


Figure 8.2.3: Direction field for the differential equation $y' = 3x + 2y - 4$ with the solution passing through the point $(0, 1)$.

The curve is the graph of the solution to the initial-value problem

$$y' = 3x + 2y - 4,\ y(0) = 1.$$

This curve is called a solution curve passing through the point $(0, 1)$. The exact solution to this initial-value problem is

$$y = -\frac{3}{2}x + \frac{5}{4} - \frac{1}{4}e^{2x},$$

and the graph of this solution is identical to the curve in Figure 8.2.3.

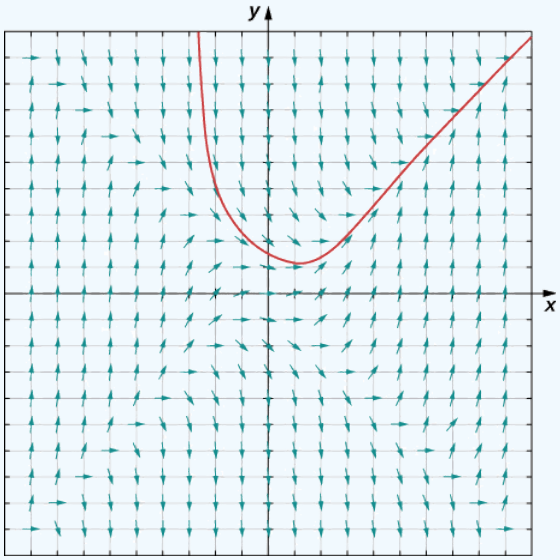
? Exercise 8.2.1

Create a direction field for the differential equation $y' = x^2 - y^2$ and sketch a solution curve passing through the point $(-1, 2)$.

Hint

Use x and y values ranging from -5 to 5. For each coordinate pair, calculate y' using the right-hand side of the differential equation.

Answer



Now consider the direction field for the differential equation $y' = (x-3)(y^2-4)$, shown in Figure 8.2.4. This direction field has several interesting properties. First of all, at $y = -2$ and $y = 2$, horizontal dashes appear all the way across the graph. This means that if $y = -2$, then $y' = 0$. Substituting this expression into the right-hand side of the differential equation gives

$$\begin{aligned}(x-3)(y^2-4) &= (x-3)((-2)^2-4)\\ &= (x-3)(0)\\ &= 0\\ &= y'.\end{aligned}$$

Therefore $y = -2$ is a solution to the differential equation. Similarly, $y = 2$ is a solution to the differential equation. These are the only constant-valued solutions to the differential equation, as we can see from the following argument. Suppose $y = k$ is a constant solution to the differential equation. Then $y' = 0$. Substituting this expression into the differential equation yields $0 = (x-3)(k^2-4)$. This equation must be true for all values of x, so the second factor must equal zero. This result yields the equation $k^2 - 4 = 0$. The solutions to this equation are $k = -2$ and $k = 2$, which are the constant solutions already mentioned. These are called the equilibrium solutions to the differential equation.

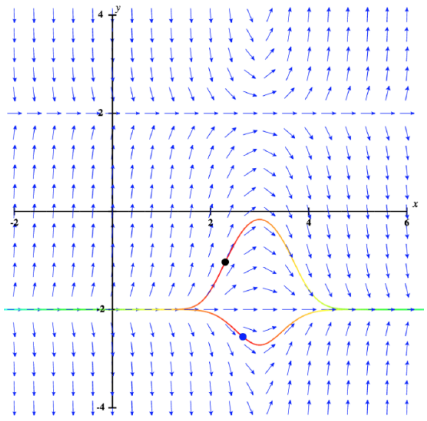


Figure 8.2.4: Direction field for the differential equation $y' = (x-3)(y^2-4)$ showing two solutions. These solutions are very close together, but one is barely above the equilibrium solution $y = -2$ and the other is barely below the same equilibrium solution.

Definition: Equilibrium Solutions

Consider the differential equation $y' = f(x, y)$. An *equilibrium solution* is any solution to the differential equation of the form $y = c$, where c is a constant.

To determine the equilibrium solutions to the differential equation $y' = f(x, y)$, set the right-hand side equal to zero. An equilibrium solution of the differential equation is any function of the form $y = k$ such that $f(x, k) = 0$ for all values of x in the domain of f.

An important characteristic of equilibrium solutions concerns whether or not they approach the line $y = k$ as an asymptote for large values of x.

Definition: asymptotically Stable, Unstable and Semi-Stable Solutions

Consider the differential equation $y' = f(x, y)$, and assume that all solutions to this differential equation are defined for $x \geq x_0$. Let $y = k$ be an equilibrium solution to the differential equation.

1. $y = k$ is an *asymptotically stable solution* to the differential equation if there exists $\varepsilon > 0$ such that for any value $c \in (k-\varepsilon,\ k+\varepsilon)$ the solution to the initial-value problem $y' = f(x, y),\ y(x_0) = c$ approaches k as x approaches infinity.
2. $y = k$ is an *asymptotically unstable solution* to the differential equation if there exists $\varepsilon > 0$ such that for any value $c \in (k-\varepsilon,\ k+\varepsilon)$ the solution to the initial-value problem $y' = f(x, y),\ y(x_0) = c$ never approaches k as x approaches infinity.
3. $y = k$ is an *asymptotically semi-stable solution* to the differential equation if it is neither asymptotically stable nor asymptotically unstable.

Now we return to the differential equation $y' = (x-3)(y^2-4)$, with the initial condition $y(0) = 0.5$. The direction field for this initial-value problem, along with the corresponding solution, is shown in Figure 8.2.5.

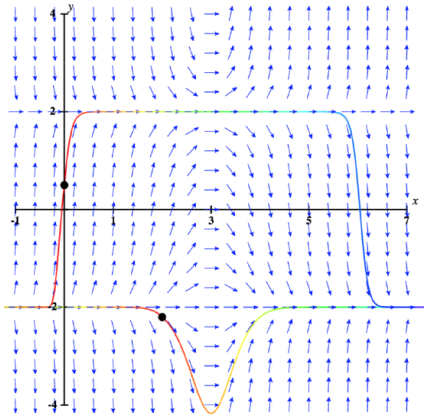


Figure 8.2.5: Direction field for the initial-value problem $y' = (x-3)(y^2-4),\ y(0) = 0.5$. A second solution curve is graphed with initial condition $(2, -2.2)$.

The values of the solution to this initial-value problem stay between $y = -2$ and $y = 2$, which are the equilibrium solutions to the differential equation. Furthermore, as x approaches infinity, although they initially appear to approach the line y = 2, the y-coordinates clearly approach -2. The behavior of solutions is similar if the initial value is below -2, for example, $y(2) = -2.2$. In this case, the solutions increase and approach $y = -2$ as x approaches infinity. Therefore $y = -2$ is an asymptotically stable solution to this differential equation.

What happens when the initial value is above $y = 2$? This scenario is illustrated in Figure 8.2.6, with the initial value $y(0) = 3$.

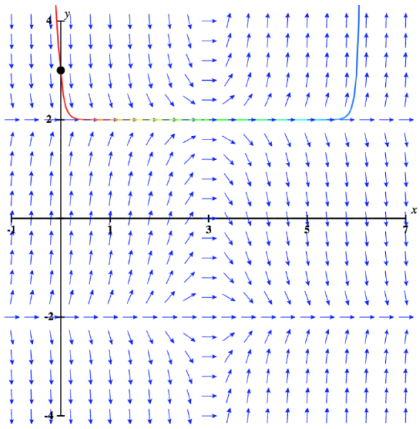


Figure 8.2.6: Direction field for the initial-value problem $y' = (x-3)(y^2-4),\ y(0) = 3$.

The solution increases rapidly toward positive infinity as x approaches infinity. Furthermore, if the initial value is slightly below 2, then the solution approaches -2, which is the other equilibrium solution. Therefore in neither case does the solution approach $y = 2$, so $y = 2$ is called an asymptotically unstable, or unstable, equilibrium solution.

Example 8.2.1: Stability of an Equilibrium Solution

Create a direction field for the differential equation $y' = (y-3)^2(y^2+y-2)$ and identify any equilibrium solutions. Classify each of the equilibrium solutions as stable, unstable, or semi-stable.

Solution

The direction field is shown in Figure 8.2.7.

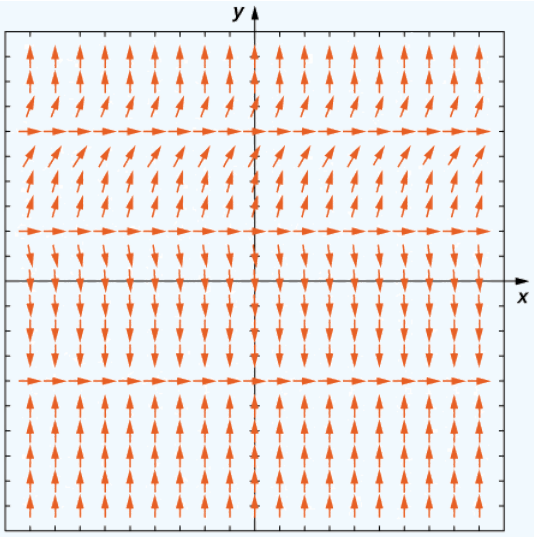


Figure 8.2.7: Direction field for the differential equation $y' = (y-3)^2(y^2+y-2)$.

The equilibrium solutions are $y = -2$, $y = 1$, and $y = 3$. To classify each of the solutions, look at an arrow directly above or below each of these values. For example, at $y = -2$ the arrows directly below this solution point up, and the arrows directly above the solution point down. Therefore all initial conditions close to $y = -2$ approach $y = -2$, and the solution is stable. For the solution $y = 1$, all initial conditions above and below $y = 1$ are repelled (pushed away) from $y = 1$, so this solution is unstable. The solution $y = 3$ is semi-stable, because for initial conditions slightly greater than 3, the solution approaches infinity, and for initial conditions slightly less than 3, the solution approaches $y = 1$.

Analysis

It is possible to find the equilibrium solutions to the differential equation by setting the right-hand side equal to zero and solving for y. This approach gives the same equilibrium solutions as those we saw in the direction field.

? Exercise 8.2.2

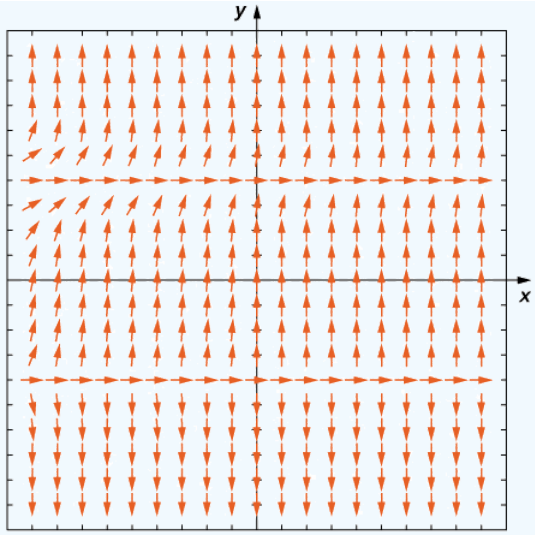
Create a direction field for the differential equation $y' = (x+5)(y+2)(y^2-4y+4)$ and identify any equilibrium solutions. Classify each of the equilibrium solutions as stable, unstable, or semi-stable.

Hint

First create the direction field and look for horizontal dashes that go all the way across. Then examine the slope lines directly above and below the equilibrium solutions.

Answer

The equilibrium solutions are $y = -2$ and $y = 2$. For this equation, $y = -2$ is an unstable equilibrium solution, and $y = 2$ is a semi-stable equilibrium solution.



Euler's Method

Consider the initial-value problem

$$y' = 2x - 3,\ y(0) = 3.$$

Integrating both sides of the differential equation gives $y = x^2 - 3x + C$, and solving for C yields the particular solution $y = x^2 - 3x + 3$. The solution for this initial-value problem appears as the parabola in Figure 8.2.8.

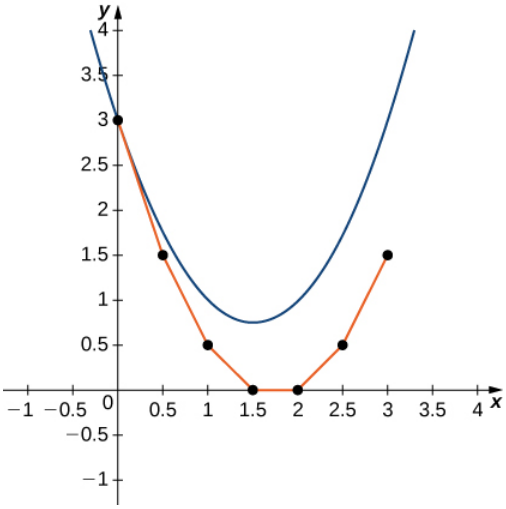


Figure 8.2.8: Euler's Method for the initial-value problem $y' = 2x - 3, y(0) = 3$.

The red graph consists of line segments that approximate the solution to the initial-value problem. The graph starts at the same initial value of $(0, 3)$. Then the slope of the solution at any point is determined by the right-hand side of the differential equation, and the length of the line segment is determined by increasing the x value by 0.5 each time (the step size). This approach is the basis of Euler's Method.

Before we state Euler's Method as a theorem, let's consider another initial-value problem:

$$y' = x^2 - y^2,\ y(-1) = 2.$$

The idea behind direction fields can also be applied to this problem to study the behavior of its solution. For example, at the point $(-1, 2)$, the slope of the solution is given by $y' = (-1)^2 - 2^2 = -3$, so the slope of the tangent line to the solution at that point is also equal to -3. Now we define $x_0 = -1$ and $y_0 = 2$. Since the slope of the solution at this point is equal to -3, we can use the method of linear approximation to approximate y near $(-1, 2)$.

$$L(x) = y_0 + f'(x_0)(x - x_0)$$

$$L(x) = y_0 + f'(x_0)(x - x_0).$$

Here $x_0 = -1, y_0 = 2$, and $f'(x_0) = -3$, so the linear approximation becomes

$$L(x) = 2 - 3(x - (-1)) = 2 - 3x - 3 = -3x - 1.$$

Now we choose a **step size**. The step size is a small value, typically 0.1 or less, that serves as an increment for x; it is represented by the variable h. In our example, let $h = 0.1$. Incrementing x_0 by h gives our next x value:

$$x_1 = x_0 + h = -1 + 0.1 = -0.9.$$

We can substitute $x_1 = -0.9$ into the linear approximation to calculate y_1.

$$y_1 = L(x_1) = -3(-0.9) - 1 = 1.7.$$

Therefore the approximate y value for the solution when $x = -0.9$ is $y = 1.7$. We can then repeat the process, using $x_1 = -0.9$ and $y_1 = 1.7$ to calculate x_2 and y_2. The new slope is given by $y' = (-0.9)^2 - (1.7)^2 = -2.08$. First, $x_2 = x_1 + h = -0.9 + 0.1 = -0.8$. Using linear approximation gives

$$\begin{aligned} L(x) &= y_1 + f'(x_1)(x - x_1) \\ &= 1.7 - 2.08(x - (-0.9)) \\ &= 1.7 - 2.08x - 1.872 \\ &= -2.08x - 0.172. \end{aligned}$$

Finally, we substitute $x_2 = -0.8$ into the linear approximation to calculate y_2.

$$\begin{aligned} y_2 &= L(x_2) \\ &= -2.08x_2 - 0.172 \\ &= -2.08(-0.8) - 0.172 \\ &= 1.492. \end{aligned}$$

Therefore the approximate value of the solution to the differential equation is $y = 1.492$ when $x = -0.8$.

What we have just shown is the idea behind **Euler's Method**. Repeating these steps gives a list of values for the solution. These values are shown in the table, rounded off to four decimal places.

Table 8.2.1: Using Euler's Method to Approximate Solutions to a Differential Equation

n	0	1	2	3	4	5
x_n	−1	−0.9	−0.8	−0.7	−0.6	−0.5
y_n	2	1.7	1.492	1.3334	1.2046	1.0955
n	6	7	8	9	10	
x_n	−0.4	−0.3	−0.2	−0.1	0	
y_n	1.0004	1.9164	1.8414	1.7746	1.7156	

Euler's method

Consider the initial-value problem

$$y' = f(x, y),\ y(x_0) = y_0.$$

To approximate a solution to this problem using Euler's method, define

$$x_n = x_0 + nh$$

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1}).$$

Here $h > 0$ represents the step size and n is an integer, starting with 1. The number of steps taken is counted by the variable n.

Typically h is a small value, say 0.1 or 0.05. The smaller the value of h, the more calculations are needed. The higher the value of h, the fewer calculations are needed. However, the tradeoff results in a lower degree of accuracy for larger step size, as illustrated in

Figure 8.2.9.

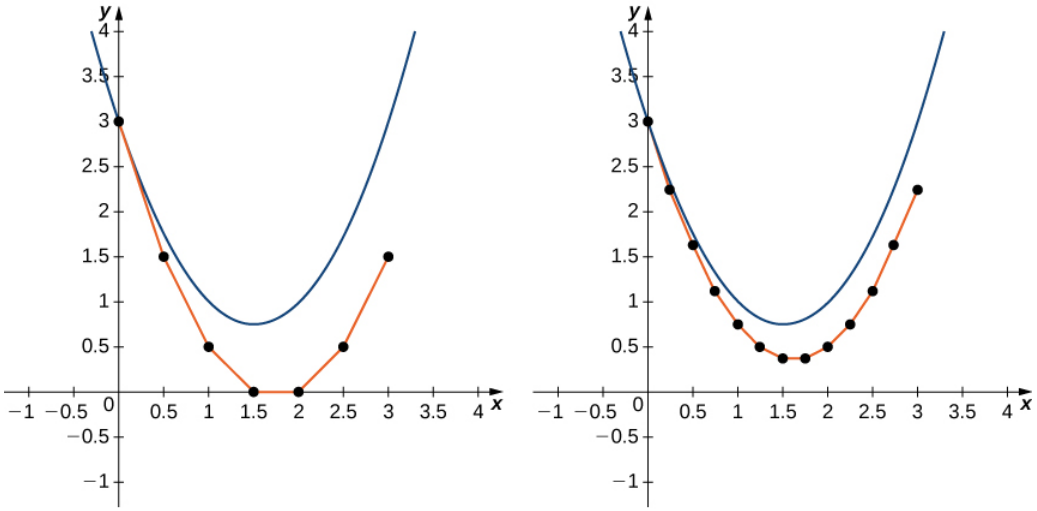


Figure 8.2.9: Euler's method for the initial-value problem $y' = 2x - 3,\ y(0) = 3$ with (a) a step size of $h = 0.5$; and (b) a step size of $h = 0.25$.

Example 8.2.2: Using Euler's Method

Consider the initial-value problem

$$y' = 3x^2 - y^2 + 1,\ y(0) = 2.$$

Use Euler's method with a step size of 0.1 to generate a table of values for the solution for values of x between 0 and 1.

Solution

We are given $h = 0.1$ and $f(x, y) = 3x^2 - y^2 + 1$. Furthermore, the initial condition $y(0) = 2$ gives $x_0 = 0$ and $y_0 = 2$. Using Equation with $n = 0$, we can generate this table.

Table 8.2.2: Using Euler's Method to Approximate Solutions to a Differential Equation

n	x_n	$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$
0	0	2
1	0.1	$y_1 = y_0 + hf(x_0, y_0) = 1.7$
2	0.2	$y_2 = y_1 + hf(x_1, y_1) = 1.514$
3	0.3	$y_3 = y_2 + hf(x_2, y_2) = 1.3968$
4	0.4	$y_4 = y_3 + hf(x_3, y_3) = 1.3287$
5	0.5	$y_5 = y_4 + hf(x_4, y_4) = 1.3001$
6	0.6	$y_6 = y_5 + hf(x_5, y_5) = 1.3061$
7	0.7	$y_7 = y_6 + hf(x_6, y_6) = 1.3435$
8	0.8	$y_8 = y_7 + hf(x_7, y_7) = 1.4100$
9	0.9	$y_9 = y_8 + hf(x_8, y_8) = 1.5032$
10	1.0	$y_{10} = y_9 + hf(x_9, y_9) = 1.6202$

With ten calculations, we are able to approximate the values of the solution to the initial-value problem for values of x between 0 and 1.

Go to this website for a chance to visually explore Euler's method.

Exercise 8.2.3

Consider the initial-value problem

$$y' = x^3 + y^2,\ y(1) = -2.$$

Using a step size of 0.1, generate a table with approximate values for the solution to the initial-value problem for values of x between 1 and 2.

Hint

Start by identifying the value of h, then figure out what $f(x, y)$ is. Then use the formula for Euler's Method to calculate $y_1, y_2,$ and so on.

Answer

Table 8.2.3: Using Euler's Method to approximate solutions to the differential equation in Exercise 8.2.3.

n	\(x_n)	$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$
0	1	-2
1	1.1	$y_1 = y_0 + hf(x_0, y_0) = -1.5$
2	1.2	$y_2 = y_1 + hf(x_1, y_1) = -1.1419$
3	1.3	$y_3 = y_2 + hf(x_2, y_2) = -0.8387$
4	1.4	$y_4 = y_3 + hf(x_3, y_3) = -0.5487$
5	1.5	$y_5 = y_4 + hf(x_4, y_4) = -0.2442$
6	1.6	$y_6 = y_5 + hf(x_5, y_5) = 0.0993$
7	1.7	$y_7 = y_6 + hf(x_6, y_6) = 0.5099$
8	1.8	$y_8 = y_7 + hf(x_7, y_7) = 1.0272$
9	1.9	$y_9 = y_8 + hf(x_8, y_8) = 1.7159$
10	2	$y_{10} = y_9 + hf(x_9, y_9) = 2.6962$

Key Concepts

- A direction field is a mathematical object used to graphically represent solutions to a first-order differential equation.
- Euler's Method is a numerical technique that can be used to approximate solutions to a differential equation.

Key Equations

- **Euler's Method**

  $x_n = x_0 + nh$

  $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$, where h is the step size

Glossary

asymptotically semi-stable solution

$y = k$ if it is neither asymptotically stable nor asymptotically unstable

asymptotically stable solution

$y = k$ if there exists $\varepsilon > 0$ such that for any value $c \in (k - \varepsilon,\ k + \varepsilon)$ the solution to the initial-value problem $y' = f(x, y),\ y(x_0) = c$ approaches k as x approaches infinity

asymptotically unstable solution

$y = k$ if there exists $\varepsilon > 0$ such that for any value $c \in (k - \varepsilon,\ k + \varepsilon)$ the solution to the initial-value problem $y' = f(x, y),\ y(x_0) = c$ never approaches k as $x$approaches infinity

direction field (slope field)

a mathematical object used to graphically represent solutions to a first-order differential equation; at each point in a direction field, a line segment appears whose slope is equal to the slope of a solution to the differential equation passing through that point

equilibrium solution

any solution to the differential equation of the form $y = c$, where c is a constant

Euler's Method

a numerical technique used to approximate solutions to an initial-value problem

solution curve

a curve graphed in a direction field that corresponds to the solution to the initial-value problem passing through a given point in the direction field

step size

the increment hh that is added to the xx value at each step in Euler's Method

- Gilbert Strang (MIT) and Edwin "Jed" Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.
- Paul Seeburger (Monroe Community College) edited this section to adjust the explanation of equilibrium points in the example shown in Figures 8.2.4 - 8.2.6. He also created these figures.

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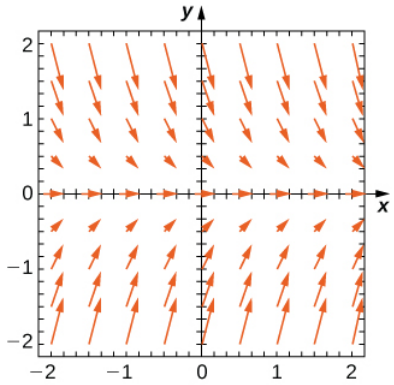
8.2E: Exercises for Section 8.2

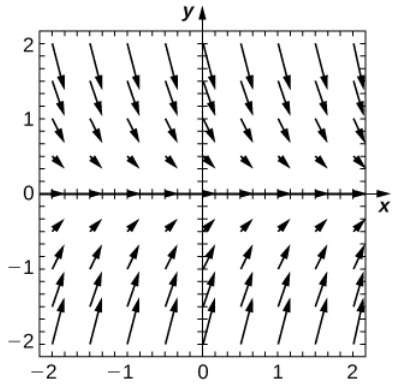
For exercises 1 - 3, use the direction field below from the differential equation $y' = -2y$. Sketch the graph of the solution for the given initial conditions.



1) $y(0) = 1$

2) $y(0) = 0$

Answer



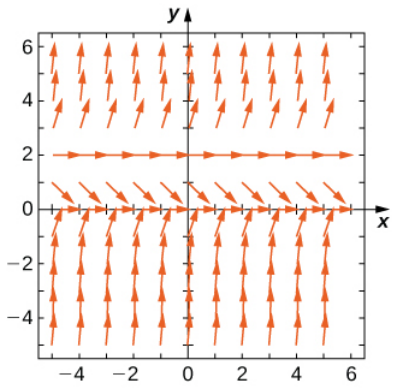
3) $y(0) = -1$

4) Are there any equilibria among the solutions of the differential equation from exercises 1 - 3? List any equilibria along with their stabilities.

Answer

$y = 0$ is a stable equilibrium

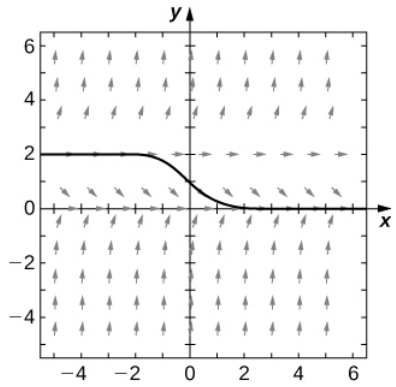
For exercises 5 - 7, use the direction field below from the differential equation $y' = y^2 - 2y$. Sketch the graph of the solution for the given initial conditions.



5) $y(0) = 3$

6) $y(0) = 1$

Answer



7) $y(0) = -1$

8) Are there any equilibria among the solutions of the differential equation from exercises 5 - 7? List any equilibria along with their stabilities.

Answer

$y = 0$ is a stable equilibrium and $y = 2$ is unstable

In exercises 9 - 13, draw the direction field for the following differential equations, then solve the differential equation. Draw your solution on top of the direction field. Does your solution follow along the arrows on your direction field?

9) $y' = t^3$

10) $y' = e^t$

Answer

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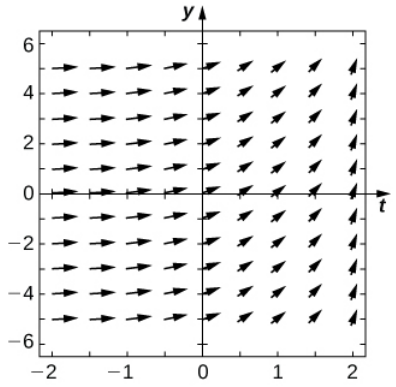
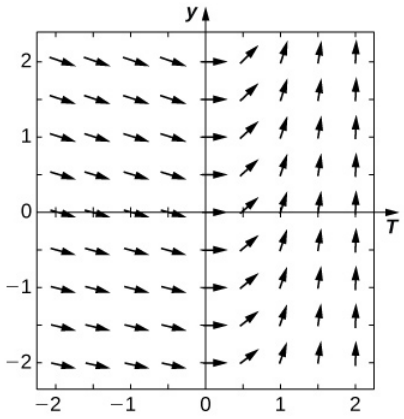


11) $\frac{dy}{dx} = x^2 \cos x$

12) $\frac{dy}{dt} = te^t$

Answer



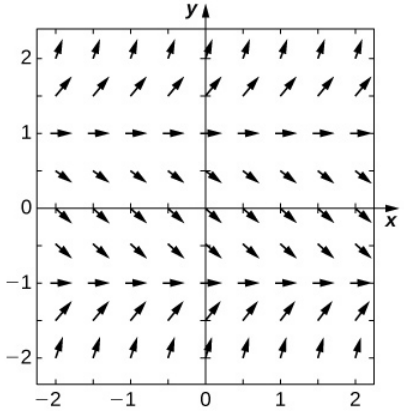
13) $\frac{dx}{dt} = \cosh(t)$

In exercises 14 - 18, draw the directional field for the following differential equations. What can you say about the behavior of the solution? Are there equilibria? What stability do these equilibria have?

14) $y' = y^2 - 1$

Answer

There appear to be equlibria at $y = -1$ (stable) and $y = 1$ (unstable).

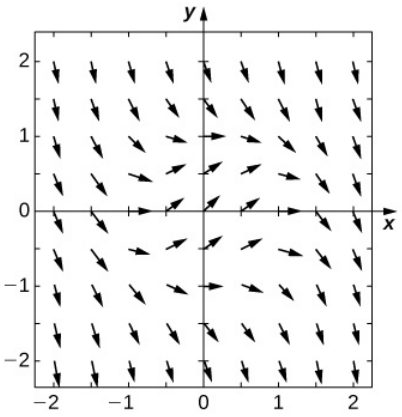


15) $y' = y - x$

16) $y' = 1 - y^2 - x^2$

Answer

There do not appear to be any equilibria.

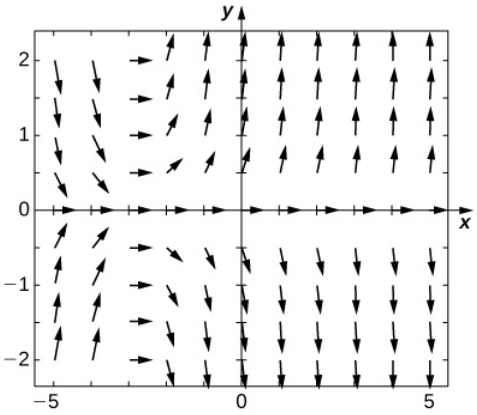


17) $y' = t^2 \sin y$

18) $y' = 3y + xy$

Answer

There appears to be an unstable equilibrium at $y = 0$.



Match the direction field with the given differential equations. Explain your selections.

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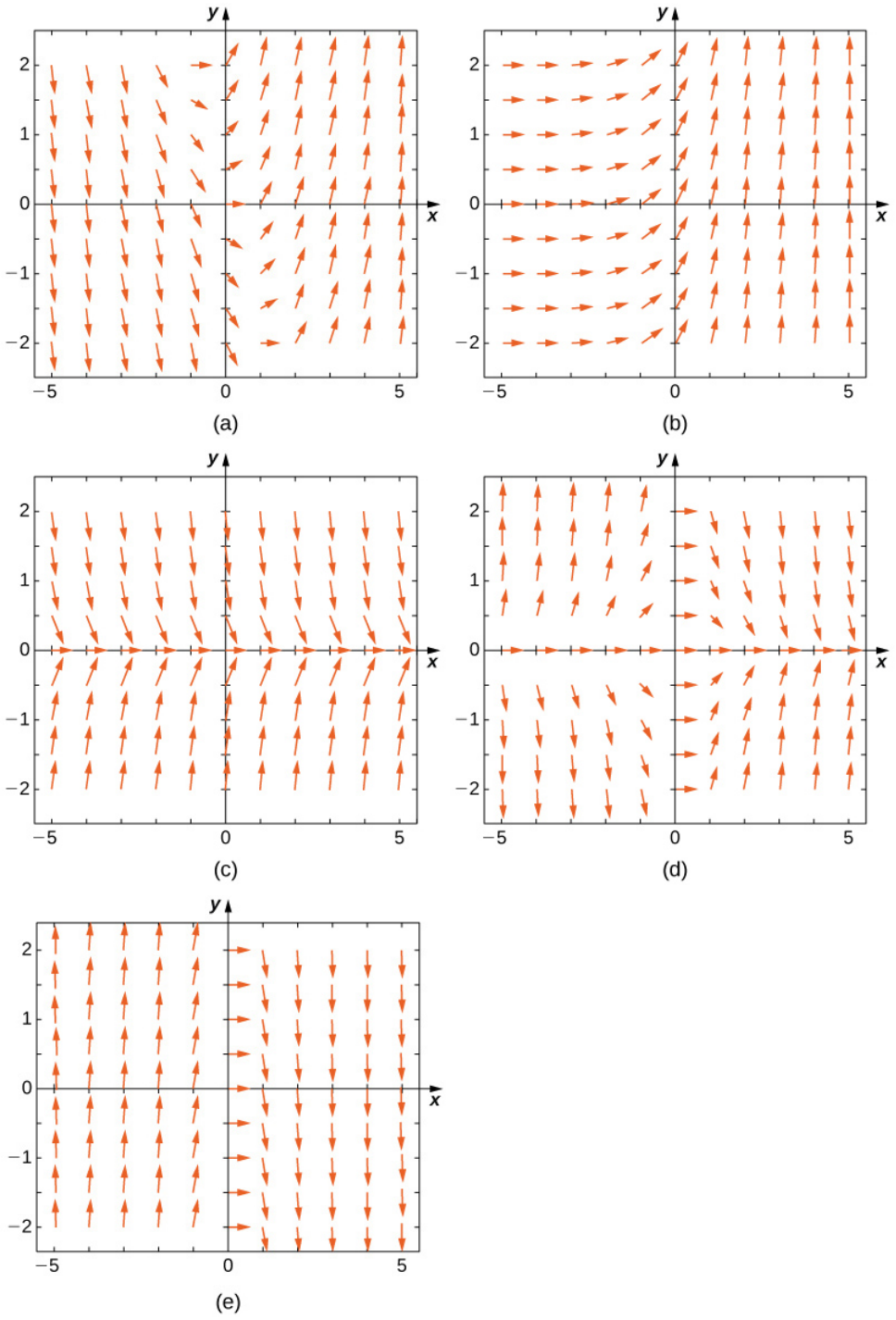


19) $y' = -3y$

20) $y' = -3t$

Answer

E

21) $y' = e^t$

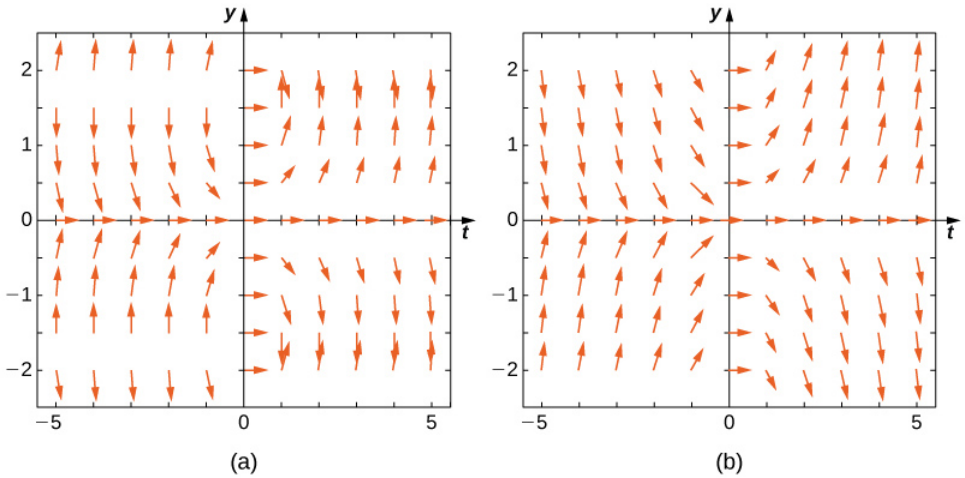
22) $y' = \frac{1}{2}y + t$

Answer

A

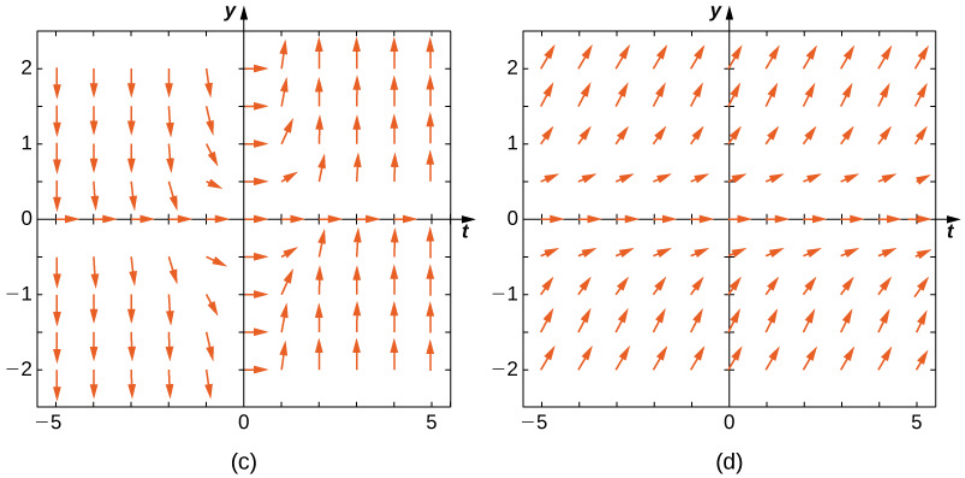
23) $y' = -ty$

Match the direction field with the given differential equations. Explain your selections.



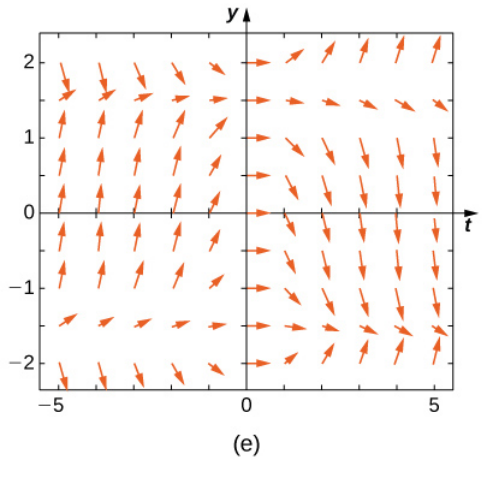
(a)

(b)



(c)

(d)



(e)

24) $y' = t \sin y$

Answer

B

25) $y' = -t \cos y$

26) $y' = t \tan y$

Answer

A

27) $y' = \sin^2 y$

28) $y' = y^2 t^3$

Answer

C

Estimate the following solutions using Euler's method with $n = 5$ steps over the interval $t = [0, 1]$. If you are able to solve the initial-value problem exactly, compare your solution with the exact solution. If you are unable to solve the initial-value problem, the exact solution will be provided for you to compare with Euler's method. How accurate is Euler's method?

29) $y' = -3y, \quad y(0) = 1$

30) $y' = t^2, \quad y(0) = 1$

Answer

1.24, exact: $\frac{4}{3} \approx 1.33$

Solution:

31) $y' = 3t - y, \quad y(0) = 1$. Exact solution is $y = 3t + 4e^{-t} - 3$

32) $y' = y + t^2, \quad y(0) = 3$. Exact solution is $y = 5e^t - 2 - t^2 - 2t$

Answer

7.739364, exact: $5(e - 1)$

33) $y' = 2t, \quad y(0) = 0$

34) [T] $y' = e^{x+y}, y(0) = -1$. Exact solution is $y = -\ln(e + 1 - e^x)$

Answer

-0.2535, exact: 0

35) $y' = y^2 \ln(x + 1), \quad y(0) = 1$. Exact solution is $y = -\dfrac{1}{(x + 1)(\ln(x + 1) - 1)}$

36) $y' = 2^x, \quad y(0) = 0$. Exact solution is $y = \dfrac{2^x - 1}{\ln 2}$

Answer

1.345, exact: $\frac{1}{\ln(2)}$

37) $y' = y, \quad y(0) = -1$. Exact solution is $y = -e^x$.

38) $y' = -5t, \quad y(0) = -2$. Exact solution is $y = -\frac{5}{2}t^2 - 2$

Answer

-4, exact: $-9/2 = -4.5$

Differential equations can be used to model **disease epidemics.** In the next set of problems, we examine the change of size of two sub-populations of people living in a city: individuals who are infected and individuals who are susceptible to infection. S represents the size of the susceptible population, and I represents the size of the infected population. We assume that if a susceptible person interacts with an infected person, there is a probability c that the susceptible person will become infected. Each infected person recovers from the infection at a rate r and becomes susceptible again. We consider the case of influenza, where we

assume that no one dies from the disease, so we assume that the total population size of the two sub-populations is a constant number, N. The differential equations that model these population sizes are

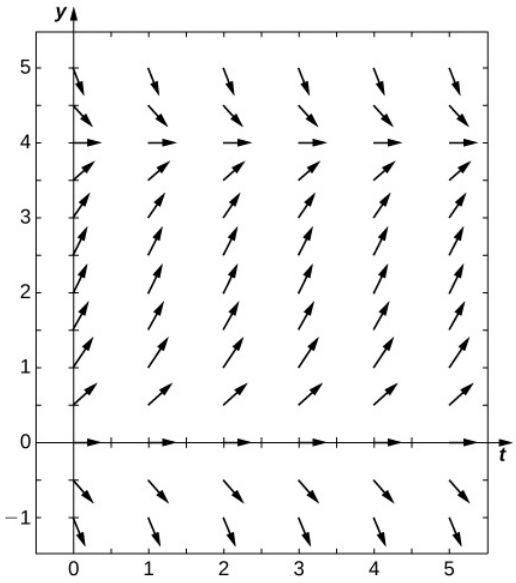
$$S' = rI - cSI \text{ and } I' = cSI - rI.$$

Here c represents the contact rate and r is the recovery rate.

39) Show that, by our assumption that the total population size is constant $(S + I = N)$, you can reduce the system to a single differential equation in $I : I' = c(N - I)I - rI$.

40) Assuming the parameters are $c = 0.5, N = 5$, and $r = 0.5$, draw the resulting directional field.

Answer



41) [T] Use computational software or a calculator to compute the solution to the initial-value problem $y' = ty, y(0) = 2$ using Euler's Method with the given step size h. Find the solution at $t = 1$. For a hint, here is "pseudo-code" for how to write a computer program to perform Euler's Method for $y' = f(t, y), y(0) = 2$:

Create function $f(t, y)$

Define parameters $y(1) = y_0, t(0) = 0$, step size h, and total number of steps, N

Write a for-loop:

for $k = 1$ to N

$fn = f(t(k), y(k))$

$y(k + 1) = y(k) + h * fn$

$t(k + 1) = t(k) + h$

42) Solve the initial-value problem for the exact solution.

Answer

$y' = 2e^{t^2/2}$

43) Draw the directional field

44) $h = 1$

Answer

2

45) [T] $h = 10$

46) [T] $h = 100$

Answer

3.2756

47) [T] $h = 1000$

48) [T] Evaluate the exact solution at $t = 1$. Make a table of errors for the relative error between the Euler's method solution and the exact solution. How much does the error change? Can you explain?

Answer

Exact solution: y $= 2\sqrt{e}$.

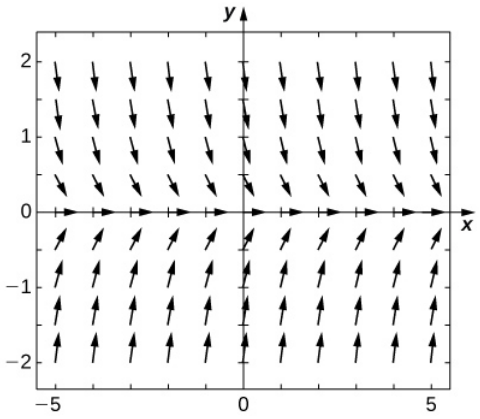
Step Size	Error
$h = 1$	0.3935
$h = 10$	0.06163
$h = 100$	0.006612
$h = 10000$	0.0006661

For exercises 49 - 53, consider the initial-value problem $y' = -2y$, with $y(0) = 2$.

49) Show that $y = 2e^{-2x}$ solves this initial-value problem.

50) Draw the directional field of this differential equation.

Answer



51) [T] By hand or by calculator or computer, approximate the solution using Euler's Method at $t = 10$ using $h = 5$.

52) [T] By calculator or computer, approximate the solution using Euler's Method at $t = 10$ using $h = 100$.

Answer

$4.0741e^{-10}$

53) [T] Plot exact answer and each Euler approximation (for $h = 5$ and $h = 100$) at each h on the directional field. What do you notice?

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8.3: Separable Equations

Learning Objectives

- Use separation of variables to solve a differential equation.
- Solve applications using separation of variables.

We now examine a solution technique for finding exact solutions to a class of differential equations known as separable differential equations. These equations are common in a wide variety of disciplines, including physics, chemistry, and engineering. We illustrate a few applications at the end of the section.

Separation of Variables

We start with a definition and some examples.

Definition: Separable Differential Equations

A *separable differential equation* is any equation that can be written in the form

$$y' = f(x)g(y). \tag{8.3.1}$$

The term 'separable' refers to the fact that the right-hand side of Equation 8.3.1 can be separated into a function of x times a function of y. Examples of separable differential equations include

$$y' = (x^2 - 4)(3y + 2) \tag{8.3.2}$$

$$y' = 6x^2 + 4x \tag{8.3.3}$$

$$y' = \sec y + \tan y \tag{8.3.4}$$

$$y' = xy + 3x - 2y - 6. \tag{8.3.5}$$

Equation 8.3.3 is separable with $f(x) = 6x^2 + 4x$ and $g(y) = 1$, Equation 8.3.4 is separable with $f(x) = 1$ and $g(y) = \sec y + \tan y$, and the right-hand side of Equation 8.3.5 can be factored as $(x + 3)(y - 2)$, so it is separable as well. Equation 8.3.4 is also called an *autonomous differential equation* because the right-hand side of the equation is a function of y alone. If a differential equation is separable, then it is possible to solve the equation using the method of *separation of variables*.

Problem-Solving Strategy: Separation of Variables

1. Check for any values of y that make $g(y) = 0$. These correspond to constant solutions.
2. Rewrite the differential equation in the form

$$\frac{dy}{g(y)} = f(x)dx.$$

3. Integrate both sides of the equation.
4. Solve the resulting equation for y if possible.
5. If an initial condition exists, substitute the appropriate values for x and y into the equation and solve for the constant.

Note that Step 4 states "Solve the resulting equation for y if possible." It is not always possible to obtain y as an explicit function of x. Quite often we have to be satisfied with finding y as an implicit function of x.

Example 8.3.1: Using Separation of Variables

Find a general solution to the differential equation $y' = (x^2 - 4)(3y + 2)$ using the method of separation of variables.

Solution

Follow the five-step method of separation of variables.

1. In this example, $f(x) = x^2 - 4$ and $g(y) = 3y + 2$. Setting $g(y) = 0$ gives $y = -\frac{2}{3}$ as a constant solution.

2. Rewrite the differential equation in the form

$$\frac{dy}{3y+2} = (x^2-4)\,dx.$$

3. Integrate both sides of the equation:

$$\int \frac{dy}{3y+2} = \int (x^2-4)\,dx.$$

Let $u = 3y+2$. Then $du = 3\dfrac{dy}{dx}\,dx$, so the equation becomes

$$\frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}x^3 - 4x + C$$

$$\frac{1}{3}\ln|u| = \frac{1}{3}x^3 - 4x + C$$

$$\frac{1}{3}\ln|3y+2| = \frac{1}{3}x^3 - 4x + C.$$

4. To solve this equation for y, first multiply both sides of the equation by 3.

$$\ln|3y+2| = x^3 - 12x + 3C$$

Now we use some logic in dealing with the constant C. Since C represents an arbitrary constant, $3C$ also represents an arbitrary constant. If we call the second arbitrary constant C_1, where $C_1 = 3C$, the equation becomes

$$\ln|3y+2| = x^3 - 12x + C_1.$$

Now exponentiate both sides of the equation (i.e., make each side of the equation the exponent for the base e).

$$e^{\ln|3y+2|} = e^{x^3-12x+C_1}$$
$$|3y+2| = e^{C_1}e^{x^3-12x}$$

Again define a new constant $C_2 = e^{C_1}$ (note that $C_2 > 0$):

$$|3y+2| = C_2e^{x^3-12x}.$$

Because of the absolute value on the left side of the equation, this corresponds to two separate equations:

$$3y+2 = C_2e^{x^3-12x}$$

and

$$3y+2 = -C_2e^{x^3-12x}.$$

The solution to either equation can be written in the form

$$y = \frac{-2 \pm C_2e^{x^3-12x}}{3}.$$

Since $C_2 > 0$, it does not matter whether we use plus or minus, so the constant can actually have either sign. Furthermore, the subscript on the constant C is entirely arbitrary, and can be dropped. Therefore the solution can be written as

$$y = \frac{-2 + Ce^{x^3-12x}}{3}, \text{ where } C = \pm C_2 \text{ or } C = 0.$$

Note that in writing a single general solution in this way, we are also allowing C to equal 0. This gives us the singular solution, $y = -\dfrac{2}{3}$, for the given differential equation. Check that this is indeed a solution of this differential equation!

5. No initial condition is imposed, so we are finished.

Exercise 8.3.1

Use the method of separation of variables to find a general solution to the differential equation

$$y' = 2xy + 3y - 4x - 6.$$

Hint

First factor the right-hand side of the equation by grouping, then use the five-step strategy of separation of variables.

Answer

$$y = 2 + Ce^{x^2+3x}$$

Example 8.3.2: Solving an Initial-Value Problem

Using the method of separation of variables, solve the initial-value problem

$$y' = (2x + 3)(y^2 - 4), \quad y(0) = -1.$$

Solution

Follow the five-step method of separation of variables.

1. In this example, $f(x) = 2x + 3$ and $g(y) = y^2 - 4$. Setting $g(y) = 0$ gives $y = \pm 2$ as constant solutions.

2. Divide both sides of the equation by $y^2 - 4$ and multiply by dx. This gives the equation

$$\frac{dy}{y^2 - 4} = (2x + 3)\, dx.$$

3. Next integrate both sides:

$$\int \frac{1}{y^2 - 4} dy = \int (2x + 3)\, dx. \tag{8.3.6}$$

To evaluate the left-hand side, use the method of partial fraction decomposition. This leads to the identity

$$\frac{1}{y^2 - 4} = \frac{1}{4}\left(\frac{1}{y - 2} - \frac{1}{y + 2}\right).$$

Then Equation 8.3.6 becomes

$$\frac{1}{4}\int \left(\frac{1}{y - 2} - \frac{1}{y + 2}\right) dy = \int (2x + 3)\, dx$$

$$\frac{1}{4}\left(\ln|y - 2| - \ln|y + 2|\right) = x^2 + 3x + C.$$

Multiplying both sides of this equation by 4 and replacing $4C$ with C_1 gives

$$\ln|y - 2| - \ln|y + 2| = 4x^2 + 12x + C_1$$

$$\ln\left|\frac{y - 2}{y + 2}\right| = 4x^2 + 12x + C_1.$$

4. It is possible to solve this equation for y. First exponentiate both sides of the equation and define $C_2 = e^{C_1}$:

$$\left|\frac{y - 2}{y + 2}\right| = C_2 e^{4x^2+12x}.$$

Next we can remove the absolute value and let a new constant C_3 be positive, negative, or zero, i.e., $C_3 = \pm C_2$ or $C_3 = 0$.

Then multiply both sides by $y + 2$.

$$y - 2 = C_3(y + 2)e^{4x^2+12x}$$

$$y - 2 = C_3 y e^{4x^2+12x} + 2C_3 e^{4x^2+12x}.$$

Now collect all terms involving y on one side of the equation, and solve for y:

$$y - C_3 y e^{4x^2+12x} = 2 + 2C_3 e^{4x^2+12x}$$

$$y\left(1 - C_3 e^{4x^2+12x}\right) = 2 + 2C_3 e^{4x^2+12x}$$

$$y = \frac{2 + 2C_3 e^{4x^2+12x}}{1 - C_3 e^{4x^2+12x}}.$$

5. To determine the value of C_3, substitute $x = 0$ and $y = -1$ into the general solution. Alternatively, we can put the same values into an earlier equation, namely the equation $\frac{y-2}{y+2} = C_3 e^{4x^2+12}$. This is much easier to solve for C_3:

$$\frac{y-2}{y+2} = C_3 e^{4x^2+12x}$$

$$\frac{-1-2}{-1+2} = C_3 e^{4(0)^2+12(0)}$$

$$C_3 = -3.$$

Therefore the solution to the initial-value problem is

$$y = \frac{2 - 6e^{4x^2+12x}}{1 + 3e^{4x^2+12x}}.$$

A graph of this solution appears in Figure 8.3.1.

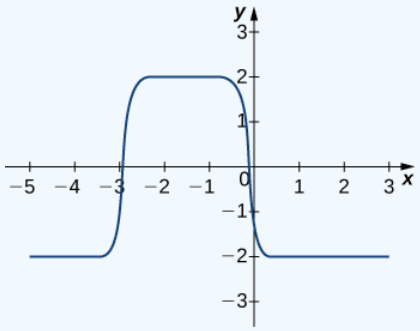

Figure 8.3.1: Graph of the solution to the initial-value problem $y' = (2x+3)(y^2-4), y(0) = -1$.

? Exercise 8.3.2

Find the solution to the initial-value problem

$$6y' = (2x+1)(y^2 - 2y - 8)$$

with $y(0) = -3$ using the method of separation of variables.

Hint

Follow the steps for separation of variables to solve the initial-value problem.

Answer

$$y = \frac{4 + 14e^{x^2+x}}{1 - 7e^{x^2+x}}$$

Applications of Separation of Variables

Many interesting problems can be described by separable equations. We illustrate two types of problems: solution concentrations and Newton's law of cooling.

Solution concentrations

Consider a tank being filled with a salt solution. We would like to determine the amount of salt present in the tank as a function of time. We can apply the process of separation of variables to solve this problem and similar problems involving *solution concentrations*.

Example 8.3.3: Determining Salt Concentration over Time

A tank containing 100 L of a brine solution initially has 4 kg of salt dissolved in the solution. At time $t = 0$, another brine solution flows into the tank at a rate of 2 L/min. This brine solution contains a concentration of 0.5 kg/L of salt. At the same time, a stopcock is opened at the bottom of the tank, allowing the combined solution to flow out at a rate of 2 L/min, so that the level of liquid in the tank remains constant (Figure 8.3.2). Find the amount of salt in the tank as a function of time (measured in minutes), and find the limiting amount of salt in the tank, assuming that the solution in the tank is well mixed at all times.

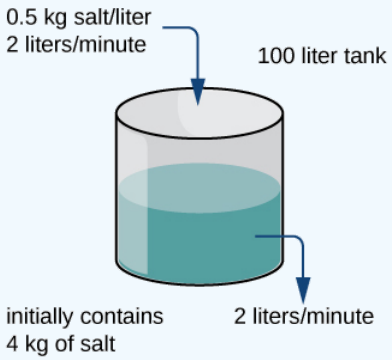

Figure 8.3.2: A brine tank with an initial amount of salt solution accepts an input flow and delivers an output flow. How does the amount of salt change with time?

Solution

First we define a function $u(t)$ that represents the amount of salt in kilograms in the tank as a function of time. Then $\frac{du}{dt}$ represents the rate at which the amount of salt in the tank changes as a function of time. Also, $u(0)$ represents the amount of salt in the tank at time $t = 0$, which is 4 kilograms.

The general setup for the differential equation we will solve is of the form

$$\frac{du}{dt} = \text{INFLOW RATE} - \text{OUTFLOW RATE}.$$

INFLOW RATE represents the rate at which salt enters the tank, and OUTFLOW RATE represents the rate at which salt leaves the tank. Because solution enters the tank at a rate of 2 L/min, and each liter of solution contains 0.5 kilogram of salt, every minute $2(0.5) = 1$ kilogram of salt enters the tank. Therefore INFLOW RATE = 1.

To calculate the rate at which salt leaves the tank, we need the concentration of salt in the tank at any point in time. Since the actual amount of salt varies over time, so does the concentration of salt. However, the volume of the solution remains fixed at 100 liters. The number of kilograms of salt in the tank at time t is equal to $u(t)$. Thus, the concentration of salt is $\frac{u(t)}{100}$ kg/L, and the solution leaves the tank at a rate of 2 L/min. Therefore salt leaves the tank at a rate of $\frac{u(t)}{100} \cdot 2 = \frac{u(t)}{50}$ kg/min, and OUTFLOW RATE is equal to $\frac{u(t)}{50}$. Therefore the differential equation becomes $\frac{du}{dt} = 1 - \frac{u}{50}$, and the initial condition is $u(0) = 4$. The initial-value problem to be solved is

$$\frac{du}{dt} = 1 - \frac{u}{50}, u(0) = 4.$$

The differential equation is a separable equation, so we can apply the five-step strategy for solution.

Step 1. Setting $1 - \frac{u}{50} = 0$ gives $u = 50$ as a constant solution. Since the initial amount of salt in the tank is 4 kilograms, this solution does not apply.

Step 2. Rewrite the equation as

$$\frac{du}{\quad} \quad \frac{50 - u}{\quad}$$

$$\frac{du}{dt} = \frac{50 - u}{50}.$$

Then multiply both sides by dt and divide both sides by $50 - u$:

$$\frac{du}{50 - u} = \frac{dt}{50}.$$

Step 3. Integrate both sides:

$$\int \frac{du}{50 - u} = \int \frac{dt}{50}$$
$$-\ln|50 - u| = \frac{t}{50} + C.$$

Step 4. Solve for $u(t)$:

$$\ln|50 - u| = -\frac{t}{50} - C$$

$$e^{\ln|50-u|} = e^{-(t/50)-C}$$

$$|50 - u| = C_1 e^{-t/50}, \text{ where } C_1 = e^{-C}.$$

Eliminate the absolute value by allowing the constant to be positive, negative, or zero, i.e., $C_1 = \pm e^{-C}$ or $C_1 = 0$:

$$50 - u = C_1 e^{-t/50}.$$

Finally, solve for $u(t)$:

$$u(t) = 50 - C_1 e^{-t/50}.$$

Step 5. Solve for C_1:

$$u(0) = 50 - C_1 e^{-0/50}$$
$$4 = 50 - C_1$$
$$C_1 = 46.$$

The solution to the initial value problem is $u(t) = 50 - 46e^{-t/50}$. To find the limiting amount of salt in the tank, take the limit as t approaches infinity:

$$\lim_{t\to\infty} u(t) = 50 - 46e^{-t/50}$$
$$= 50 - 46(0) = 50.$$

Note that this was the constant solution to the differential equation. If the initial amount of salt in the tank is 50 kilograms, then it remains constant. If it starts at less than 50 kilograms, then it approaches 50 kilograms over time.

? Exercise 8.3.3

A tank contains 3 kilograms of salt dissolved in 75 liters of water. A salt solution of 0.4 kg salt/L is pumped into the tank at a rate of 6 L/min and is drained at the same rate. Solve for the salt concentration at time t. Assume the tank is well mixed at all times.

Hint

Follow the steps in Example 8.3.3 and determine an expression for INFLOW and OUTFLOW. Formulate an initial-value problem, and then solve it.

Initial value problem:

$$\frac{du}{dt} = 2.4 - \frac{2u}{25},\ u(0) = 3$$

Answer

$$u(t) = 30 - 27e^{-t/50}$$

Newton's Law of Cooling

Newton's law of cooling states that the rate of change of an object's temperature is proportional to the difference between its own temperature and the ambient temperature (i.e., the temperature of its surroundings). If we let $T(t)$ represent the temperature of an object as a function of time, then $\frac{dT}{dt}$ represents the rate at which that temperature changes. The temperature of the object's surroundings can be represented by T_s. Then Newton's law of cooling can be written in the form

$$\frac{dT}{dt} = k(T(t) - T_s)$$

or simply

$$\frac{dT}{dt} = k(T - T_s).$$

The temperature of the object at the beginning of any experiment is the initial value for the initial-value problem. We call this temperature T_0. Therefore the initial-value problem that needs to be solved takes the form

$$\frac{dT}{dt} = k(T - T_s) \tag{8.3.7}$$

with $T(0) = T_0$, where k is a constant that needs to be either given or determined in the context of the problem. We use these equations in Example 8.3.4.

Example 8.3.4: Waiting for a Pizza to Cool

A pizza is removed from the oven after baking thoroughly, and the temperature of the oven is $350°F$. The temperature of the kitchen is $75°F$, and after 5 minutes the temperature of the pizza is $340°F$. We would like to wait until the temperature of the pizza reaches $300°F$ before cutting and serving it (Figure 8.3.3). How much longer will we have to wait?

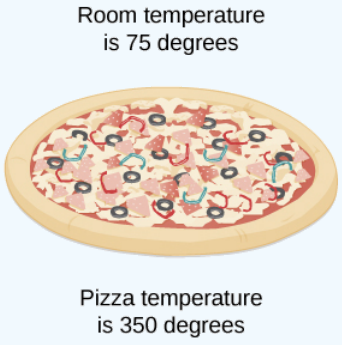

Figure 8.3.3: From Newton's law of cooling, if the pizza cools $10°F$ in 5 minutes, how long before it cools to $300°F$?

Solution

The ambient temperature (surrounding temperature) is $75°F$, so $T_s = 75$. The temperature of the pizza when it comes out of the oven is $350°F$, which is the initial temperature (i.e., initial value), so $T_0 = 350$. Therefore Equation 8.3.7 becomes

$$\frac{dT}{dt} = k(T - 75)$$

with $T(0) = 350$.

To solve the differential equation, we use the five-step technique for solving separable equations.

1. Setting the right-hand side equal to zero gives $T = 75$ as a constant solution. Since the pizza starts at $350°F$, this is not the solution we are seeking.

2. Rewrite the differential equation by multiplying both sides by dt and dividing both sides by $T - 75$:

$$\frac{dT}{T - 75} = k\,dt.$$

3. Integrate both sides:

$$\int \frac{dT}{T-75} = \int k\,dt$$
$$\ln|T-75| = kt + C.$$

4. Solve for T by first exponentiating both sides:

$$\begin{aligned} e^{\ln|T-75|} &= e^{kt+C} \\ |T-75| &= C_1 e^{kt}, && \text{where } C_1 = e^C. \\ T-75 &= \pm C_1 e^{kt} \\ T-75 &= Ce^{kt}, && \text{where } C = \pm C_1 \text{ or } C = 0. \\ T(t) &= 75 + Ce^{kt}. \end{aligned}$$

5. Solve for C by using the initial condition $T(0) = 350$:

$$\begin{aligned} T(t) &= 75 + Ce^{kt} \\ T(0) &= 75 + Ce^{k(0)} \\ 350 &= 75 + C \\ C &= 275. \end{aligned}$$

Therefore the solution to the initial-value problem is

$$T(t) = 75 + 275e^{kt}.$$

To determine the value of k, we need to use the fact that after 5 minutes the temperature of the pizza is $340°F$. Therefore $T(5) = 340$. Substituting this information into the solution to the initial-value problem, we have

$$T(t) = 75 + 275e^{kt}$$

$$T(5) = 340 = 75 + 275e^{5k}$$

$$265 = 275e^{5k}$$

$$e^{5k} = \frac{53}{55}$$

$$\ln e^{5k} = \ln(\frac{53}{55})$$

$$5k = \ln(\frac{53}{55})$$

$$k = \frac{1}{5}\ln(\frac{53}{55}) \approx -0.007408.$$

So now we have $T(t) = 75 + 275e^{-0.007048t}$. When is the temperature $300°F$? Solving for t, we find

$$T(t) = 75 + 275e^{-0.007048t}$$

$$300 = 75 + 275e^{-0.007048t}$$

$$225 = 275e^{-0.007048t}$$

$$e^{-0.007048t} = \frac{9}{11}$$

$$\ln e^{-0.007048t} = \ln\frac{9}{11}$$

$$-0.007048t = \ln\frac{9}{11}$$

$$t = -\frac{1}{0.007048}\ln\frac{9}{11} \approx 28.5.$$

Therefore we need to wait an additional 23.5 minutes (after the temperature of the pizza reached $340°F$). That should be just enough time to finish this calculation.

Exercise 8.3.4

A cake is removed from the oven after baking thoroughly, and the temperature of the oven is $450°F$. The temperature of the kitchen is $70°F$, and after 10 minutes the temperature of the cake is $430°F$.

a. Write the appropriate initial-value problem to describe this situation.
b. Solve the initial-value problem for $T(t)$.
c. How long will it take until the temperature of the cake is within $5°F$ of room temperature?

Hint

Determine the values of T_s and T_0 then use Equation 8.3.7.

Answer a

Initial-value problem

$$\frac{dT}{dt} = k(T - 70), \quad T(0) = 450$$

Answer b

$$T(t) = 70 + 380e^{kt}$$

Answer c

Approximately 114 minutes.

Key Concepts

- A separable differential equation is any equation that can be written in the form $y' = f(x)g(y)$.
- The method of separation of variables is used to find the general solution to a separable differential equation.

Key Equations

- **Separable differential equation**

  $y' = f(x)g(y)$

- **Solution concentration**

  $\frac{du}{dt} = \text{INFLOW RATE} - \text{OUTFLOW RATE}$

- **Newton's law of cooling**

  $\frac{dT}{dt} = k(T - T_s)$

Glossary

autonomous differential equation
an equation in which the right-hand side is a function of y alone

separable differential equation
any equation that can be written in the form $y' = f(x)g(y)$

separation of variables
a method used to solve a separable differential equation

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8.3E: Exercises for Section 8.3

In exercises 1 - 4, solve the following initial-value problems with the initial condition $y_0 = 0$ and graph the solution.

1) $\frac{dy}{dt} = y + 1$

Answer

$y = e^t - 1$

2) $\frac{dy}{dt} = y - 1$

3) $\frac{dy}{dt} = -y + 1$

Answer

$y = 1 - e^{-t}$

4) $\frac{dy}{dt} = -y - 1$

In exercises 5 - 14, find the general solution to the differential equation.

5) $x^2 y' = (x + 1)y$

Answer

$y = Cxe^{-1/x}$

6) $y' = \tan(y)x$

7) $y' = 2xy^2$

Answer

$y = \frac{1}{C - x^2}$

8) $\frac{dy}{dt} = y\cos(3t + 2)$

9) $2x\frac{dy}{dx} = y^2$

Answer

$y = -\frac{2}{C + \ln|x|}$

10) $y' = e^y x^2$

11) $(1 + x)y' = (x + 2)(y - 1)$

Answer

$y = Ce^x(x + 1) + 1$

12) $\frac{dx}{dt} = 3t^2(x^2 + 4)$

13) $t\frac{dy}{dt} = \sqrt{1 - y^2}$

Answer

$y = \sin(\ln|t| + C)$

14) $y' = e^x e^y$

In exercises 15 - 24, find the solution to the initial-value problem.

15) $y' = e^{y-x}, \quad y(0) = 0$

Answer

$y = -\ln(e^{-x})$ which simplifies to $y = x$

16) $y' = y^2(x+1), \quad y(0) = 2$

17) $\dfrac{dy}{dx} = y^3 x e^{x^2}, \quad y(0) = 1$

Answer

$$y = \frac{1}{\sqrt{2 - e^{x^2}}}$$

18) $\dfrac{dy}{dt} = y^2 e^x \sin(3x), \quad y(0) = 1$

19) $y' = \dfrac{x}{\operatorname{sech}^2 y}, \quad y(0) = 0$

Answer

$$y = \tanh^{-1}\left(\frac{x^2}{2}\right)$$

20) $y' = 2xy(1+2y), \quad y(0) = -1$

21) $\dfrac{dx}{dt} = \ln(t)\sqrt{1-x^2}, \quad x(1) = 0$

Answer

$x = \sin(1 - t + t\ln t)$

22) $y' = 3x^2(y^2+4), \quad y(0) = 0$

23) $y' = e^y 5^x, \quad y(0) = \ln(\ln(5))$

Answer

$y = \ln(\ln(5)) - \ln(2 - 5^x)$

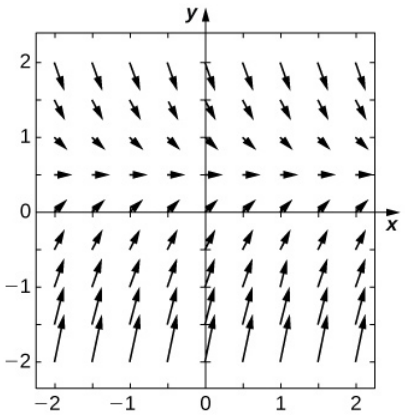
24) $y' = -2x\tan(y), \quad y(0) = \dfrac{\pi}{2}$

For problems 25 - 29, use a software program or your calculator to generate the directional fields. Solve explicitly and draw solution curves for several initial conditions. Are there some critical initial conditions that change the behavior of the solution?

25) [T] $y' = 1 - 2y$

Answer

$$y = Ce^{-2}x + \frac{1}{2}$$

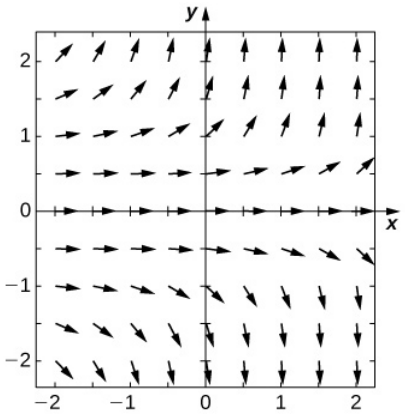


26) [T] $y' = y^2x^3$

27) [T] $y' = y^3e^x$

Answer

$$y = \frac{1}{\sqrt{2}\sqrt{C - e^x}}$$

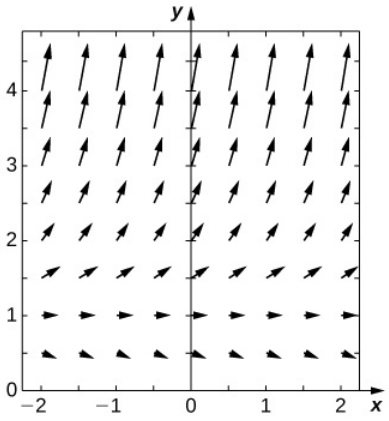


28) [T] $y' = e^y$

29) [T] $y' = y\ln(x)$

Answer

$$y = Ce^{-x}x^x$$



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30) Most **drugs in the bloodstream** decay according to the equation $y' = cy$, where y is the concentration of the drug in the bloodstream. If the half-life of a drug is 2 hours, what fraction of the initial dose remains after 6 hours?

31) A drug is administered intravenously to a patient at a rate r mg/h and is cleared from the body at a rate proportional to the amount of drug still present in the body, d. Set up and solve the differential equation, assuming there is no drug initially present in the body.

Answer

$y = \frac{r}{d}(1 - e^{-dt})$

32) [T] How often should a drug be taken if its dose is 3 mg, it is cleared at a rate $c = 0.1$ mg/h, and 1 mg is required to be in the bloodstream at all times?

33) A tank contains 1 kilogram of salt dissolved in 100 liters of water. A salt solution of 0.1 kg salt/L is pumped into the tank at a rate of 2 L/min and is drained at the same rate. Solve for the salt concentration at time t. Assume the tank is well mixed.

Answer

Since the amount of salt at time t is given by the equation $A(t) = 10 - 9e^{-t/50}$, and the tank volume stays constant at 100 liters, the concentration at time t would be $c(t) = \frac{1}{100}\left(10 - 9e^{-t/50}\right)$.

34) A tank containing 10 kilograms of salt dissolved in 1000 liters of water has two salt solutions pumped in. The first solution of 0.2 kg salt/L is pumped in at a rate of 20 L/min and the second solution of 0.05 kg salt/L is pumped in at a rate of 5 L/min. The tank drains at 25 L/min. Assume the tank is well mixed. Solve for the salt concentration at time t.

35) [T] For the preceding problem, find how much salt is in the tank 1 hour after the process begins.

Answer

134.3 kilograms

36) Torricelli's law states that for a water tank with a hole in the bottom that has a cross-section of A and with a height of water h above the bottom of the tank, the rate of change of volume of water flowing from the tank is proportional to the square root of the height of water, according to $\frac{dV}{dt} = -A\sqrt{2gh}$, where g is the acceleration due to gravity. Note that $\frac{dV}{dt} = A\frac{dh}{dt}$. Solve the resulting initial-value problem for the height of the water, assuming a tank with a hole of radius 2 ft. The initial height of water is 100 ft.

37) For the preceding problem, determine how long it takes the tank to drain.

Answer

720 seconds

For problems 38 - 44, use Newton's law of cooling.

38) The liquid base of an ice cream has an initial temperature of $200°F$ before it is placed in a freezer with a constant temperature of $0°F$. After 1 hour, the temperature of the ice-cream base has decreased to $140°F$. Formulate and solve the initial-value problem to determine the temperature of the ice cream.

39) [T] The liquid base of an ice cream has an initial temperature of $210°F$ before it is placed in a freezer with a constant temperature of $20°F$. After 2 hours, the temperature of the ice-cream base has decreased to $170°F$. At what time will the ice cream be ready to eat? (Assume $30°F$ is the optimal eating temperature.)

Answer

24 hours 55 minutes

40) [T] You are organizing an ice cream social. The outside temperature is $80°F$ and the ice cream is at $10°F$. After 10 minutes, the ice cream temperature has risen by $10°F$. How much longer can you wait before the ice cream melts at $40°F$?

41) You have a cup of coffee at temperature $70°C$ and the ambient temperature in the room is $20°C$. Assuming a cooling rate k of 0.125, write and solve the differential equation to describe the temperature of the coffee with respect to time.

Answer

$T(t) = 20 + 50e^{-0.125t}$

42) [T] You have a cup of coffee at temperature $70°C$ that you put outside, where the ambient temperature is $0°C$. After 5 minutes, how much colder is the coffee?

43) You have a cup of coffee at temperature $70°C$ and you immediately pour in 1 part milk to 5 parts coffee. The milk is initially at temperature $1°C$. Write and solve the differential equation that governs the temperature of this coffee.

Answer

$T(t) = 20 + 38.5e^{-0.125t}$

44) You have a cup of coffee at temperature $70°C$, which you let cool 10 minutes before you pour in the same amount of milk at $1°C$ as in the preceding problem. How does the temperature compare to the previous cup after 10 minutes?

45) Solve the generic problem $y' = ay + b$ with initial condition $y(0) = c$.

Answer

$y = (c + ba)e^{ax} - \frac{b}{a}$

46) Prove the basic continual compounded interest equation. Assuming an initial deposit of P_0 and an interest rate of r, set up and solve an equation for continually compounded interest.

47) Assume an initial nutrient amount of I kilograms in a tank with L liters. Assume a concentration of c kg/L being pumped in at a rate of r L/min. The tank is well mixed and is drained at a rate of r L/min. Find the equation describing the amount of nutrient in the tank.

Answer

$y(t) = cL + (I - cL)e^{-rt/L}$

48) Leaves accumulate on the forest floor at a rate of 2 g/cm^2/yr and also decompose at a rate of 90% per year. Write a differential equation governing the number of grams of leaf litter per square centimeter of forest floor, assuming at time 0 there is no leaf litter on the ground. Does this amount approach a steady value? What is that value?

49) Leaves accumulate on the forest floor at a rate of 4 g/cm^2/yr. These leaves decompose at a rate of 10% per year. Write a differential equation governing the number of grams of leaf litter per square centimeter of forest floor. Does this amount approach a steady value? What is that value?

Answer

Differential Equation: $\frac{dy}{dt} = 4 - 0.1y$
Solution, the model for this situation: $y = 40(1 - e^{-0.1t})$,
Amount approaches a steady value of 40 g/cm^2

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8.4: The Logistic Equation

Learning Objectives

- Describe the concept of environmental carrying capacity in the logistic model of population growth.
- Draw a direction field for a logistic equation and interpret the solution curves.
- Solve a logistic equation and interpret the results.

Differential equations can be used to represent the size of a population as it varies over time. We saw this in an earlier chapter in the section on exponential growth and decay, which is the simplest model. A more realistic model includes other factors that affect the growth of the population. In this section, we study the logistic differential equation and see how it applies to the study of population dynamics in the context of biology.

Population Growth and Carrying Capacity

To model population growth using a differential equation, we first need to introduce some variables and relevant terms. The variable t. will represent time. The units of time can be hours, days, weeks, months, or even years. Any given problem must specify the units used in that particular problem. The variable P will represent population. Since the population varies over time, it is understood to be a function of time. Therefore we use the notation $P(t)$ for the population as a function of time. If $P(t)$ is a differentiable function, then the first derivative $\frac{dP}{dt}$ represents the instantaneous rate of change of the population as a function of time.

In Exponential Growth and Decay, we studied the exponential growth and decay of populations and radioactive substances. An example of an exponential growth function is $P(t) = P_0 e^{rt}$. In this function, $P(t)$ represents the population at time t, P_0 represents the **initial population** (population at time $t = 0$), and the constant $r > 0$ is called the **growth rate**. Figure 8.4.1 shows a graph of $P(t) = 100e^{0.03t}$. Here $P_0 = 100$ and $r = 0.03$.

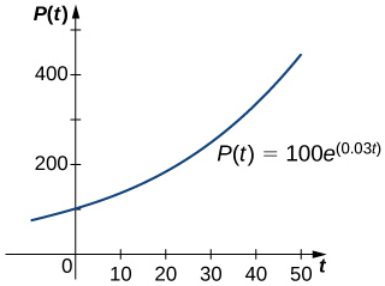


Figure 8.4.1: An exponential growth model of population.

We can verify that the function $P(t) = P_0 e^{rt}$ satisfies the initial-value problem

$$\frac{dP}{dt} = rP$$

with $P(0) = P_0$.

This differential equation has an interesting interpretation. The left-hand side represents the rate at which the population increases (or decreases). The right-hand side is equal to a positive constant multiplied by the current population. Therefore the differential equation states that the rate at which the population increases is proportional to the population at that point in time. Furthermore, it states that the constant of proportionality never changes.

One problem with this function is its prediction that as time goes on, the population grows without bound. This is unrealistic in a real-world setting. Various factors limit the rate of growth of a particular population, including birth rate, death rate, food supply, predators, and so on. The growth constant r usually takes into consideration the birth and death rates but none of the other factors, and it can be interpreted as a net (birth minus death) percent growth rate per unit time. A natural question to ask is whether the population growth rate stays constant, or whether it changes over time. Biologists have found that in many biological systems, the population grows until a certain steady-state population is reached. This possibility is not taken into account with exponential

growth. However, the concept of carrying capacity allows for the possibility that in a given area, only a certain number of a given organism or animal can thrive without running into resource issues.

Definition: Carrying Capacity

The *carrying capacity* of an organism in a given environment is defined to be the maximum population of that organism that the environment can sustain indefinitely.

We use the variable K to denote the carrying capacity. The growth rate is represented by the variable r. Using these variables, we can define the logistic differential equation.

Definition: Logistic Differential Equation

Let K represent the carrying capacity for a particular organism in a given environment, and let r be a real number that represents the growth rate. The function $P(t)$ represents the population of this organism as a function of time t, and the constant P_0 represents the initial population (population of the organism at time $t = 0$). Then the *logistic differential equation* is

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right). \tag{8.4.1}$$

The logistic equation was first published by Pierre Verhulst in 1845. This differential equation can be coupled with the initial condition $P(0) = P_0$ to form an initial-value problem for $P(t)$.

Suppose that the initial population is small relative to the carrying capacity. Then $\frac{P}{K}$ is small, possibly close to zero. Thus, the quantity in parentheses on the right-hand side of Equation 8.4.1 is close to 1, and the right-hand side of this equation is close to rP. If $r > 0$, then the population grows rapidly, resembling exponential growth.

However, as the population grows, the ratio $\frac{P}{K}$ also grows, because K is constant. If the population remains below the carrying capacity, then $\frac{P}{K}$ is less than 1, so $1 - \frac{P}{K} > 0$. Therefore the right-hand side of Equation 8.4.1 is still positive, but the quantity in parentheses gets smaller, and the growth rate decreases as a result. If $P = K$ then the right-hand side is equal to zero, and the population does not change.

Now suppose that the population starts at a value higher than the carrying capacity. Then $\frac{P}{K} > 1$, and $1 - \frac{P}{K} < 0$. Then the right-hand side of Equation 8.4.1 is negative, and the population decreases. As long as $P > K$, the population decreases. It never actually reaches K because $\frac{dP}{dt}$ will get smaller and smaller, but the population approaches the carrying capacity as t approaches infinity. This analysis can be represented visually by way of a phase line. A **phase line** describes the general behavior of a solution to an autonomous differential equation, depending on the initial condition. For the case of a carrying capacity in the logistic equation, the phase line is as shown in Figure $8.4.2$.

Figure $8.4.2$: A phase line for the differential equation $\frac{dP}{dt} = rP(1 - \frac{P}{K})$.

This phase line shows that when P is less than zero or greater than K, the population decreases over time. When P is between 0 and K, the population increases over time.

Example 8.4.1: Examining the Carrying Capacity of a Deer Population

Let's consider the population of white-tailed deer (*Odocoileus virginianus*) in the state of Kentucky. The Kentucky Department of Fish and Wildlife Resources (KDFWR) sets guidelines for hunting and fishing in the state. Before the hunting season of 2004, it estimated a population of 900,000 deer. Johnson notes: "A deer population that has plenty to eat and is not hunted by humans or other predators will double every three years." (George Johnson, "The Problem of Exploding Deer Populations Has No Attractive Solutions," January 12,2001, accessed April 9, 2015)

Figure 8.4.3: (credit: modification of work by Rachel Kramer, Flickr)

This observation corresponds to a rate of increase $r = \frac{\ln(2)}{3} = 0.2311$, so the approximate growth rate is 23.11% per year. (This assumes that the population grows exponentially, which is reasonable—at least in the short term—with plentiful food supply and no predators.) The KDFWR also reports deer population densities for 32 counties in Kentucky, the average of which is approximately 27 deer per square mile. Suppose this is the deer density for the whole state (39,732 square miles). The carrying capacity K is 39,732 square miles times 27 deer per square mile, or 1,072,764 deer.

a. For this application, we have $P_0 = 900,000$, $K = 1,072,764$, and $r = 0.2311$. Substitute these values into Equation 8.4.1 and form the initial-value problem.
b. Solve the initial-value problem from part a.
c. According to this model, what will be the population in 3 years? Recall that the doubling time predicted by Johnson for the deer population was 3 years. How do these values compare?

Suppose the population managed to reach **1,200,000** What does the logistic equation predict will happen to the population in this scenario?

Solution

a. The initial value problem is

$$\frac{dP}{dt} = 0.2311P\left(1 - \frac{P}{1,072,764}\right), \; P(0) = 900,000.$$

b. The logistic equation is an autonomous differential equation, so we can use the method of separation of variables.

Step 1: Setting the right-hand side equal to zero gives $P = 0$ and $P = 1,072,764$. This means that if the population starts at zero it will never change, and if it starts at the carrying capacity, it will never change.

Step 2: Rewrite the differential equation and multiply both sides by:

$$\frac{dP}{dt} = 0.2311P\left(\frac{1,072,764 - P}{1,072,764}\right)$$

$$dP = 0.2311P\left(\frac{1,072,764 - P}{1,072,764}\right)dt$$

$$\frac{dP}{P(1,072,764 - P)} = \frac{0.2311}{1,072,764}dt.$$

Step 3: Integrate both sides of the equation using partial fraction decomposition:

$$\int \frac{dP}{P(1,072,764-P)} = \int \frac{0.2311}{1,072,764}dt$$
$$\frac{1}{1,072,764}\int \left(\frac{1}{P}+\frac{1}{1,072,764-P}\right)dP = \frac{0.2311t}{1,072,764}+C$$
$$\frac{1}{1,072,764}\left(\ln|P|-\ln|1,072,764-P|\right) = \frac{0.2311t}{1,072,764}+C.$$

Step 4: Multiply both sides by 1,072,764 and use the quotient rule for logarithms:

$$\ln\left|\frac{P}{1,072,764-P}\right| = 0.2311t + C_1.$$

Here $C_1 = 1,072,764C$. Next exponentiate both sides and eliminate the absolute value:

$$e^{\ln\left|\frac{P}{1,072,764-P}\right|} = e^{0.2311t+C_1}$$
$$\left|\frac{P}{1,072,764-P}\right| = C_2e^{0.2311t}$$
$$\frac{P}{1,072,764-P} = C_2e^{0.2311t}.$$

Here $C_2 = e^{C_1}$ but after eliminating the absolute value, it can be negative as well. Now solve for:

$$P = C_2e^{0.2311t}(1,072,764-P)$$
$$P = 1,072,764C_2e^{0.2311t} - C_2Pe^{0.2311t}$$
$$P + C_2Pe^{0.2311t} = 1,072,764C_2e^{0.2311t}$$
$$P(1+C_2e^{0.2311t} = 1,072,764C_2e^{0.2311t}$$
$$P(t) = \frac{1,072,764C_2e^{0.2311t}}{1+C_2e^{0.2311t}}.$$

Step 5: To determine the value of C_2, it is actually easier to go back a couple of steps to where C_2 was defined. In particular, use the equation

$$\frac{P}{1,072,764-P} = C_2e^{0.2311t}.$$

The initial condition is $P(0) = 900,000$. Replace P with $900,000$ and t with zero:

$$\frac{P}{1,072,764-P} = C_2e^{0.2311t}$$
$$\frac{900,000}{1,072,764-900,000} = C_2e^{0.2311(0)}$$
$$\frac{900,000}{172,764} = C_2$$
$$C_2 = \frac{25,000}{4,799}$$
$$\approx 5.209.$$

Therefore

$$P(t) = \frac{1,072,764\left(\frac{25000}{4799}\right)e^{0.2311t}}{1+\left(\frac{25000}{4799}\right)e^{0.2311t}}$$
$$= \frac{1,072,764(25000)e^{0.2311t}}{4799+25000e^{0.2311t}.}$$

Dividing the numerator and denominator by **25,000** gives

$$P(t) = \frac{1,072,764e^{0.2311t}}{0.19196 + e^{0.2311t}}.$$

Figure 8.4.4 is a graph of this equation.

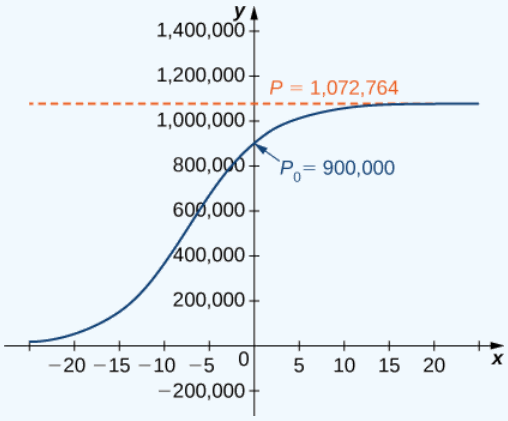


Figure 8.4.4: Logistic curve for the deer population with an initial population of **900,000** deer.

c. Using this model we can predict the population in **3** years.

$$P(3) = \frac{1,072,764e^{0.2311(3)}}{0.19196 + e^{0.2311(3)}} \approx 978,830\, deer$$

This is far short of twice the initial population of $900,000$. Remember that the doubling time is based on the assumption that the growth rate never changes, but the logistic model takes this possibility into account.

d. If the population reached **1,200,000** deer, then the new initial-value problem would be

$$\frac{dP}{dt} = 0.2311P\left(1 - \frac{P}{1,072,764}\right),\ P(0) = 1,200,000.$$

The general solution to the differential equation would remain the same.

$$P(t) = \frac{1,072,764C_2e^{0.2311t}}{1 + C_2e^{0.2311t}}$$

To determine the value of the constant, return to the equation

$$\frac{P}{1,072,764 - P} = C_2e^{0.2311t}.$$

Substituting the values $t = 0$ and $P = 1,200,000$, you get

$$C_2e^{0.2311(0)} = \frac{1,200,000}{1,072,764 - 1,200,000}$$

$$C_2 = -\frac{100,000}{10,603} \approx -9.431.$$

Therefore

$$P(t) = \frac{1,072,764C_2e^{0.2311t}}{1 + C_2e^{0.2311t}}$$

$$= \frac{1,072,764\left(-\frac{100,000}{10,603}\right)e^{0.2311t}}{1 + \left(-\frac{100,000}{10,603}\right)e^{0.2311t}}$$

$$= -\frac{107,276,400,000e^{0.2311t}}{100,000e^{0.2311t} - 10,603}$$

$$\approx \frac{10,117,551e^{0.2311t}}{9.43129e^{0.2311t} - 1}$$

This equation is graphed in Figure 8.4.5.

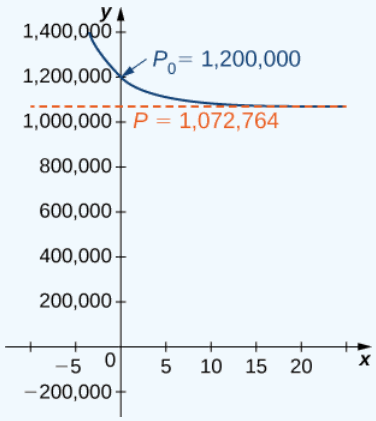


Figure 8.4.5: Logistic curve for the deer population with an initial population of **1,200,000** deer.

Solving the Logistic Differential Equation

The logistic differential equation is an autonomous differential equation, so we can use separation of variables to find the general solution, as we just did in Example 8.4.1.

Step 1: Setting the right-hand side equal to zero leads to $P = 0$ and $P = K$ as constant solutions. The first solution indicates that when there are no organisms present, the population will never grow. The second solution indicates that when the population starts at the carrying capacity, it will never change.

Step 2: Rewrite the differential equation in the form

$$\frac{dP}{dt} = \frac{rP(K-P)}{K}.$$

Then multiply both sides by dt and divide both sides by $P(K-P)$. This leads to

$$\frac{dP}{P(K-P)} = \frac{r}{K}dt.$$

Multiply both sides of the equation by K and integrate:

$$\int \frac{K}{P(K-P)}dP = \int r dt. \tag{8.4.2}$$

The left-hand side of this equation can be integrated using partial fraction decomposition. We leave it to you to verify that

$$\frac{K}{P(K-P)} = \frac{1}{P} + \frac{1}{K-P}.$$

Then the Equation 8.4.2 becomes

$$\int \frac{1}{P} + \frac{1}{K-P}dP = \int r dt$$

$$\ln|P| - \ln|K-P| = rt + C$$

$$\ln\left|\frac{P}{K-P}\right| = rt + C.$$

Now exponentiate both sides of the equation to eliminate the natural logarithm:

$$e^{\ln\left|\frac{P}{K-P}\right|} = e^{rt+C}$$

$$\left|\frac{P}{K-P}\right| = e^C e^{rt}.$$

We define $C_1 = e^c$ so that the equation becomes

$$\frac{P}{K-P} = C_1 e^{rt}. \tag{8.4.3}$$

To solve this equation for $P(t)$, first multiply both sides by $K-P$ and collect the terms containing P on the left-hand side of the equation:

$$\begin{aligned} P &= C_1 e^{rt}(K-P) \\ &= C_1 K e^{rt} - C_1 P e^{rt} \\ P + C_1 P e^{rt} &= C_1 K e^{rt}. \end{aligned}$$

Next, factor P from the left-hand side and divide both sides by the other factor:

$$\begin{aligned} P(1 + C_1 e^{rt}) &= C_1 K e^{rt} \\ P(t) &= \frac{C_1 K e^{rt}}{1 + C_1 e^{rt}}. \end{aligned}$$

The last step is to determine the value of C_1. The easiest way to do this is to substitute $t = 0$ and P_0 in place of P in Equation and solve for C_1:

$$\begin{aligned} \frac{P}{K-P} &= C_1 e^{rt} \\ \frac{P_0}{K-P_0} &= C_1 e^{r(0)} \\ C_1 &= \frac{P_0}{K-P_0}. \end{aligned}$$

Finally, substitute the expression for C_1 into Equation 8.4.3:

$$P(t) = \frac{C_1 K e^{rt}}{1 + C_1 e^{rt}} = \frac{\frac{P_0}{K-P_0} K e^{rt}}{1 + \frac{P_0}{K-P_0} e^{rt}}$$

Now multiply the numerator and denominator of the right-hand side by $(K-P_0)$ and simplify:

$$\begin{aligned} P(t) &= \frac{\frac{P_0}{K-P_0} K e^{rt}}{1 + \frac{P_0}{K-P_0} e^{rt}} \\ &= \frac{\frac{P_0}{K-P_0} K e^{rt}}{1 + \frac{P_0}{K-P_0} e^{rt}} \cdot \frac{K-P_0}{K-P_0} \\ &= \frac{P_0 K e^{rt}}{(K-P_0) + P_0 e^{rt}}. \end{aligned}$$

We state this result as a theorem.

Solution of the Logistic Differential Equation

Consider the logistic differential equation subject to an initial population of P_0 with carrying capacity K and growth rate r. The solution to the corresponding initial-value problem is given by

$$P(t) = \frac{P_0 K e^{rt}}{(K-P_0) + P_0 e^{rt}}$$

.

Now that we have the solution to the initial-value problem, we can choose values for P_0, r, and K and study the solution curve. For example, in Example we used the values $r = 0.2311, K = 1,072,764$, and an initial population of $900,000$ deer. This leads to the

solution

$$P(t) = \frac{P_0 K e^{rt}}{(K - P_0) + P_0 e^{rt}}$$
$$= \frac{900,000(1,072,764)e^{0.2311t}}{(1,072,764 - 900,000) + 900,000e^{0.2311t}}$$
$$= \frac{900,000(1,072,764)e^{0.2311t}}{172,764 + 900,000e^{0.2311t}}.$$

Dividing top and bottom by $900,000$ gives

$$P(t) = \frac{1,072,764e^{0.2311t}}{0.19196 + e^{0.2311t}}.$$

This is the same as the original solution. The graph of this solution is shown again in blue in Figure $8.4.6$, superimposed over the graph of the exponential growth model with initial population $900,000$ and growth rate 0.2311 (appearing in green). The red dashed line represents the carrying capacity, and is a horizontal asymptote for the solution to the logistic equation.

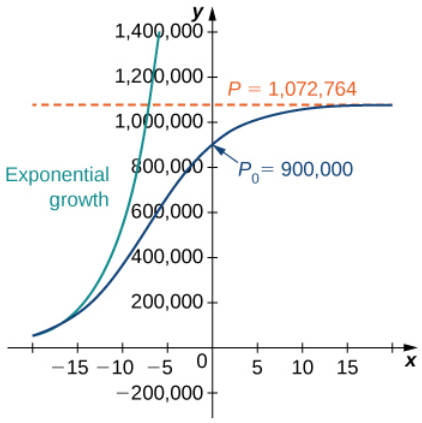


Figure $8.4.6$: A comparison of exponential versus logistic growth for the same initial population of $900,000$ organisms and growth rate of 23.11

Working under the assumption that the population grows according to the logistic differential equation, this graph predicts that approximately 20 years earlier (1984), the growth of the population was very close to exponential. The net growth rate at that time would have been around 23.1 per year. As time goes on, the two graphs separate. This happens because the population increases, and the logistic differential equation states that the growth rate decreases as the population increases. At the time the population was measured (2004), it was close to carrying capacity, and the population was starting to level off.

The solution to the logistic differential equation has a point of inflection. To find this point, set the second derivative equal to zero:

$$P(t) = \frac{P_0 K e^{rt}}{(K - P_0) + P_0 e^{rt}}$$
$$P'(t) = \frac{rP_0 K(K - P_0)e^{rt}}{((K - P_0) + P_0 e^{rt})^2}$$
$$P''(t) = \frac{r^2 P_0 K(K - P_0)^2 e^{rt} - r^2 P_0^2 K(K - P_0)e^{2rt}}{((K - P_0) + P_0 e^{rt})^3}$$
$$= \frac{r^2 P_0 K(K - P_0)e^{rt}((K - P_0) - P_0 e^{rt})}{((K - P_0) + P_0 e^{rt})^3}.$$

Setting the numerator equal to zero,

$$r^2 P_0 K(K - P_0)e^{rt}((K - P_0) - P_0 e^{rt}) = 0.$$

As long as $P_0 \neq K$, the entire quantity before and including $e^{rt}$is nonzero, so we can divide it out:

$$(K - P_0) - P_0 e^{rt} = 0.$$

Solving for t,

$$P_0 e^{rt} = K - P_0$$

$$e^{rt} = \frac{K - P_0}{P_0}$$

$$\ln e^{rt} = \ln \frac{K - P_0}{P_0}$$

$$rt = \ln \frac{K - P_0}{P_0}$$

$$t = \frac{1}{r} \ln \frac{K - P_0}{P_0}.$$

Notice that if $P_0 > K$, then this quantity is undefined, and the graph does not have a point of inflection. In the logistic graph, the point of inflection can be seen as the point where the graph changes from concave up to concave down. This is where the "leveling off" starts to occur, because the net growth rate becomes slower as the population starts to approach the carrying capacity.

Exercise 8.4.1

A population of rabbits in a meadow is observed to be 200 rabbits at time $t = 0$. After a month, the rabbit population is observed to have increased by 4. Using an initial population of 200 and a growth rate of 0.04, with a carrying capacity of 750 rabbits,

a. Write the logistic differential equation and initial condition for this model.
b. Draw a slope field for this logistic differential equation, and sketch the solution corresponding to an initial population of 200 rabbits.
c. Solve the initial-value problem for $P(t)$.
d. Use the solution to predict the population after 1 year.

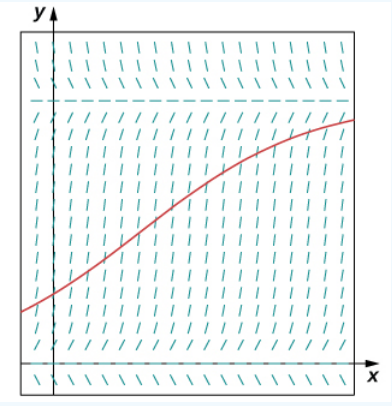
Hint

First determine the values of $r, K,$ and P_0. Then create the initial-value problem, draw the direction field, and solve the problem.

Answer

a. $\frac{dP}{dt} = 0.04(1 - \frac{P}{750}), P(0) = 200$

b.



c. $P(t) = \frac{3000e^{.04t}}{11 + 4e^{.04t}}$

d. After 12 months, the population will be $P(12) \approx 278$ rabbits.

Student Project: Logistic Equation with a Threshold Population

An improvement to the logistic model includes a **threshold population.** The threshold population is defined to be the minimum population that is necessary for the species to survive. We use the variable T to represent the threshold population. A differential equation that incorporates both the threshold population T and carrying capacity K is

$$\frac{dP}{dt} = -rP\left(1 - \frac{P}{K}\right)\left(1 - \frac{P}{T}\right)$$

where r represents the growth rate, as before.

1. The threshold population is useful to biologists and can be utilized to determine whether a given species should be placed on the endangered list. A group of Australian researchers say they have determined the threshold population for any species to survive: 5000 adults. (Catherine Clabby, "A Magic Number," American Scientist 98(1): 24, doi:10.1511/2010.82.24. accessed April 9, 2015, www.americanscientist.org/iss...a-magic-number). Therefore we use $T = 5000$ as the threshold population in this project. Suppose that the environmental carrying capacity in Montana for elk is $25,000$. Set up Equation using the carrying capacity of $25,000$ and threshold population of 5000. Assume an annual net growth rate of 18%.
2. Draw the direction field for the differential equation from step 1, along with several solutions for different initial populations. What are the constant solutions of the differential equation? What do these solutions correspond to in the original population model (i.e., in a biological context)?
3. What is the limiting population for each initial population you chose in step 2? (Hint: use the slope field to see what happens for various initial populations, i.e., look for the horizontal asymptotes of your solutions.)
4. This equation can be solved using the method of separation of variables. However, it is very difficult to get the solution as an explicit function of t. Using an initial population of $18,000$ elk, solve the initial-value problem and express the solution as an implicit function of t, or solve the general initial-value problem, finding a solution in terms of $r, K, T,$ and P_0.

Key Concepts

- When studying population functions, different assumptions—such as exponential growth, logistic growth, or threshold population—lead to different rates of growth.
- The logistic differential equation incorporates the concept of a carrying capacity. This value is a limiting value on the population for any given environment.
- The logistic differential equation can be solved for any positive growth rate, initial population, and carrying capacity.

Key Equations

- **Logistic differential equation and initial-value problem**

  $$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right), \quad P(0) = P_0$$

- **Solution to the logistic differential equation/initial-value problem**

  $$P(t) = \frac{P_0 K e^{rt}}{(K - P_0) + P_0 e^{rt}}$$

- **Threshold population model**

  $$\frac{dP}{dt} = -rP\left(1 - \frac{P}{K}\right)\left(1 - \frac{P}{T}\right)$$

Glossary

carrying capacity

the maximum population of an organism that the environment can sustain indefinitely

growth rate

the constant $r > 0$ in the exponential growth function $P(t) = P_0 e^{rt}$

initial population

the population at time $t = 0$

logistic differential equation

a differential equation that incorporates the carrying capacity K and growth rate rr into a population model

phase line

a visual representation of the behavior of solutions to an autonomous differential equation subject to various initial conditions

threshold population

the minimum population that is necessary for a species to survive

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8.4E: Exercises for Section 8.4

Basic Logistic Model

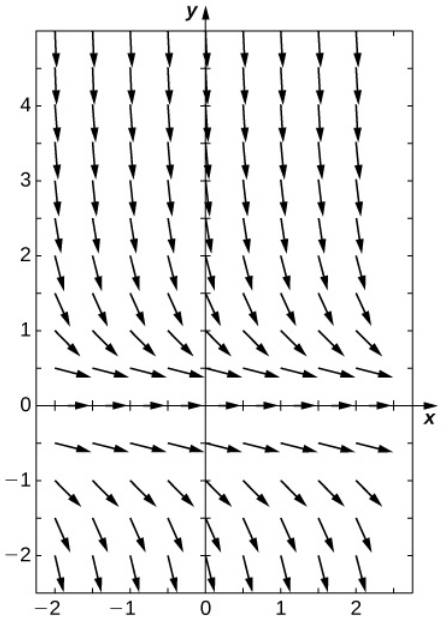
For problems 1 - 11, consider the logistic equation in the form $P' = CP - P^2$. Draw the directional field and find the stability of the equilibria.

1) $C = 3$

2) $C = 0$

Answer

$P = 0$ semi-stable



3) $C = -3$

4) Solve the logistic equation for $C = 10$ and an initial condition of $P(0) = 2$.

Answer

$$P = \frac{10e^{10x}}{e^{10x} + 4}$$

5) Solve the logistic equation for $C = -10$ and an initial condition of $P(0) = 2$.

6) A population of deer inside a park has a carrying capacity of 200 and a growth rate of 2. If the initial population is 50 deer, what is the population of deer at any given time?

Answer

$$P(t) = \frac{10000e^{0.02t}}{150 + 50e^{0.02t}}$$

7) A population of frogs in a pond has a growth rate of 5 If the initial population is 1000 frogs and the carrying capacity is 6000, what is the population of frogs at any given time?

8) [T] Bacteria grow at a rate of 20 per hour in a petri dish. If there is initially one bacterium and a carrying capacity of 1 million cells, how long does it take to reach $500,000$ cells?

Answer

69 hours 5 minutes

9) [T] Rabbits in a park have an initial population of 10 and grow at a rate of 4 per year. If the carrying capacity is 500, at what time does the population reach 100 rabbits?

10) [T] Two monkeys are placed on an island. After 5 years, there are 8 monkeys, and the estimated carrying capacity is 25 monkeys. When does the population of monkeys reach 16 monkeys?

Answer

8 years 11 months

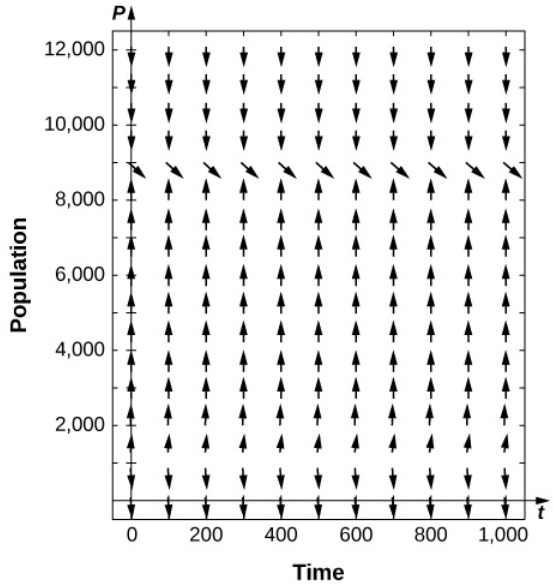
11) [T] A butterfly sanctuary is built that can hold 2000 butterflies, and 400 butterflies are initially moved in. If after 2 months there are now 800 butterflies, when does the population get to 1500 butterflies?

Logistic Population Model with Depletion

The following problems consider the logistic equation with an added term for depletion, either through death or emigration.

12) [T] The population of trout in a pond is given by $P' = 0.4P\left(1 - \dfrac{P}{10000}\right) - 400$, where 400 trout are caught per year. Use your calculator or computer software to draw a directional field and draw a few sample solutions. What do you expect for the behavior?

Answer

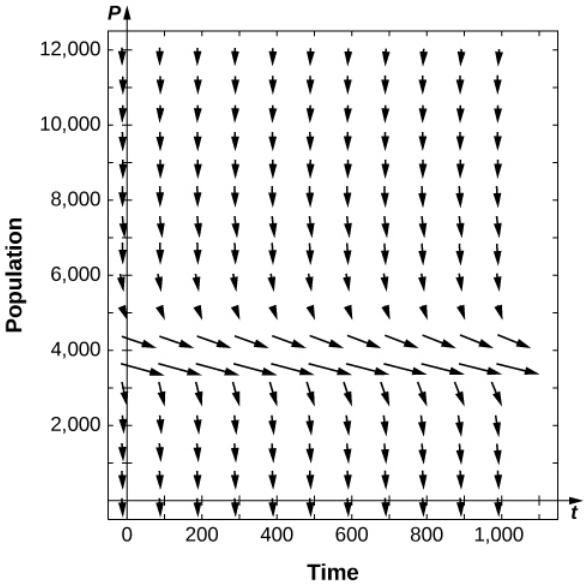


13) In the preceding problem, what are the stabilities of the equilibria $0 < P_1 < P_2$?

14) [T] For the preceding problem, use software to generate a directional field for the value $f = 400$. What are the stabilities of the equilibria?

Answer

P_1 semi-stable

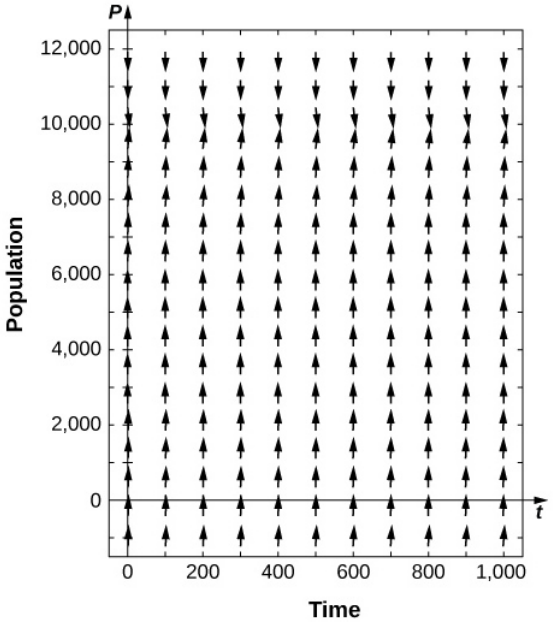


15) [T] For the preceding problems, use software to generate a directional field for the value $f = 600$. What are the stabilities of the equilibria?

16) [T] For the preceding problems, consider the case where a certain number of fish are added to the pond, or $f = -200$. What are the nonnegative equilibria and their stabilities?

Answer

$P_2 > 0$ stable



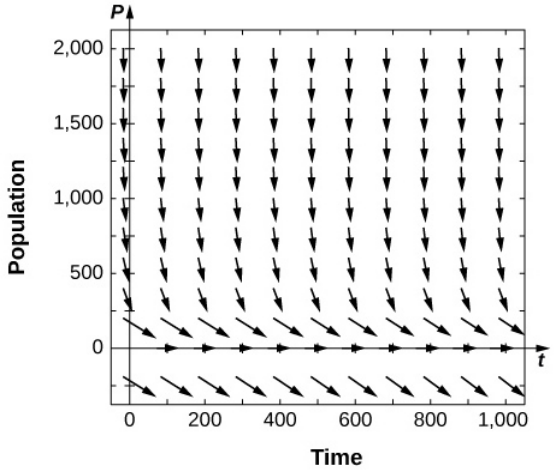
It is more likely that the amount of fishing is governed by the current number of fish present, so instead of a constant number of fish being caught, the rate is proportional to the current number of fish present, with proportionality constant k, as $P' = 0.4P\left(1 - \frac{P}{10000}\right) - kP$.

17) [T] For the previous fishing problem, draw a directional field assuming $k = 0.1$. Draw some solutions that exhibit this behavior. What are the equilibria and what are their stabilities?

18) [T] Use software or a calculator to draw directional fields for $k = 0.4$. What are the nonnegative equilibria and their stabilities?

Answer

$P_1 = 0$ is semi-stable



19) [T] Use software or a calculator to draw directional fields for $k = 0.6$. What are the equilibria and their stabilities?

20) Solve this equation, assuming a value of $k = 0.05$ and an initial condition of 2000 fish.

Answer

$$y = \frac{-20}{4 \times 10^{-6} - 0.002e^{0.01t}}$$

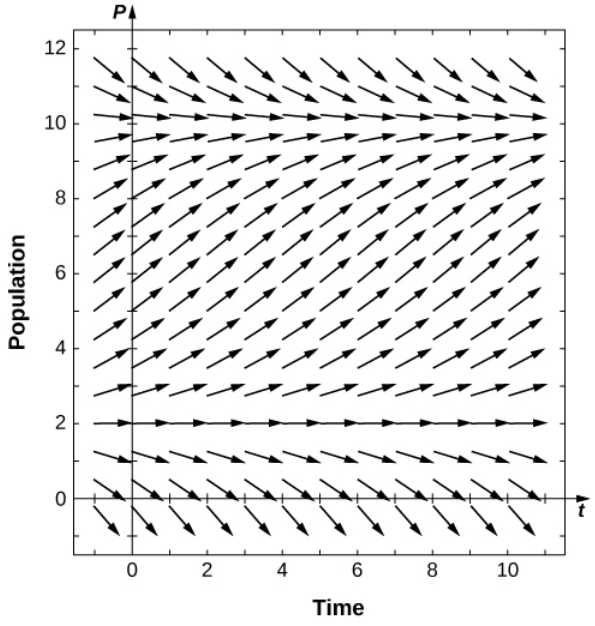
21) Solve this equation, assuming a value of $k = 0.05$ and an initial condition of 5000 fish.

Minimal Sustainable Population Thresholds

The following problems add in a minimal threshold value for the species to survive, T, which changes the differential equation to $P'(t) = rP\left(1 - \frac{P}{K}\right)\left(1 - \frac{T}{P}\right)$.

22) Draw the directional field of the threshold logistic equation, assuming $K = 10, r = 0.1, T = 2$. When does the population survive? When does it go extinct?

Answer



https://math.libretexts.org/@go/page/71709

23) For the preceding problem, solve the logistic threshold equation, assuming the initial condition $P(0) = P_0$.

24) Bengal tigers in a conservation park have a carrying capacity of 100 and need a minimum of 10 to survive. If they grow in population at a rate of 1 per year, with an initial population of 15 tigers, solve for the number of tigers present.

Answer

$$P(t) = \frac{850 + 500e^{0.009t}}{85 + 5e^{0.009t}}$$

25) A forest containing ring-tailed lemurs in Madagascar has the potential to support 5000 individuals, and the lemur population grows at a rate of 5 per year. A minimum of 500 individuals is needed for the lemurs to survive. Given an initial population of 600 lemurs, solve for the population of lemurs.

26) The population of mountain lions in Northern Arizona has an estimated carrying capacity of 250 and grows at a rate of 0.25 per year and there must be 25 for the population to survive. With an initial population of 30 mountain lions, how many years will it take to get the mountain lions off the endangered species list (at least 100)?

Answer

13 years months

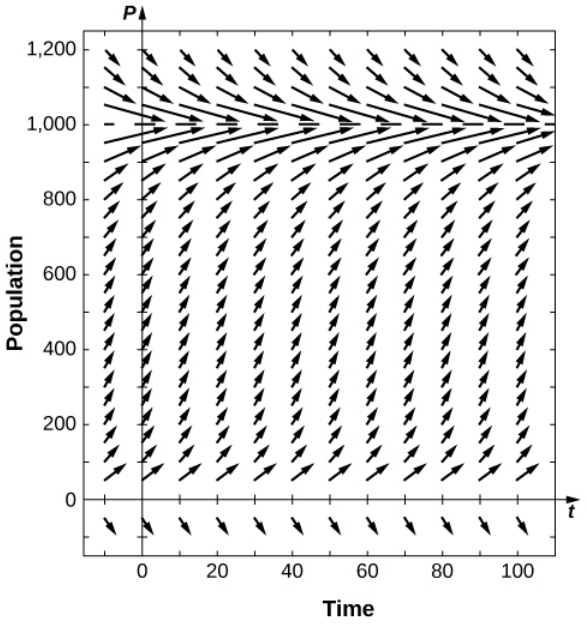
The Gompertz Equation

The following questions consider the Gompertz equation, a modification for logistic growth, which is often used for modeling cancer growth, specifically the number of tumor cells.

27) The Gompertz equation is given by $P(t)' = \alpha \ln\left(\frac{K}{P(t)}\right) P(t)$. Draw the directional fields for this equation assuming all parameters are positive, and given that $K = 1$.

28) Assume that for a population, $K = 1000$ and $\alpha = 0.05$. Draw the directional field associated with this differential equation and draw a few solutions. What is the behavior of the population?

Answer



29) Solve the Gompertz equation for generic α and K and $P(0) = P_0$.

30) [T] The Gompertz equation has been used to model tumor growth in the human body. Starting from one tumor cell on day 1 and assuming $\alpha = 0.1$ and a carrying capacity of 10 million cells, how long does it take to reach "detection" stage at 5 million cells?

Answer

31.465 days

31) [T] It is estimated that the world human population reached 3 billion people in 1959 and 6 billion in 1999. Assuming a carrying capacity of 16 billion humans, write and solve the differential equation for logistic growth, and determine what year the population reached 7 billion.

32) [T] It is estimated that the world human population reached 3 billion people in 1959 and 6 billion in 1999. Assuming a carrying capacity of 16 billion humans, write and solve the differential equation for Gompertz growth, and determine what year the population reached 7 billion. Was logistic growth or Gompertz growth more accurate, considering world population reached 7 billion on October $31, 2011$?

Answer

September 2008

33) Show that the population grows fastest when it reaches half the carrying capacity for the logistic equation $P' = rP\left(1 - \frac{P}{K}\right)$.

34) When does population increase the fastest in the threshold logistic equation $P'(t) = rP\left(1 - \frac{P}{K}\right)\left(1 - \frac{T}{P}\right)$?

Answer

$\frac{K+T}{2}$

35) When does population increase the fastest for the Gompertz equation $P(t)' = \alpha \ln\left(\frac{K}{P(t)}\right) P(t)$?

Below is a table of the populations of whooping cranes in the wild from 1940 to 2000. The population rebounded from near extinction after conservation efforts began. The following problems consider applying population models to fit the data. Assume a carrying capacity of $10,000$ cranes. Fit the data assuming years since 1940 (so your initial population at time 0 would be 22 cranes).

Year (years since conservation began)	Whooping Crane Population
1940(0)	22
1950(10)	31
1960(20)	36
1970(30)	57
1980(40)	91
1990(50)	159
2000(60)	256

Source: www.savingcranes.org/images/...wc_numbers.pdf

36) Find the equation and parameter r that best fit the data for the logistic equation.

Answer

$r = 0.0405$

37) Find the equation and parameters r and T that best fit the data for the threshold logistic equation.

38) Find the equation and parameter α that best fit the data for the Gompertz equation.

Answer

$\alpha = 0.0081$

39) Graph all three solutions and the data on the same graph. Which model appears to be most accurate?

40) Using the three equations found in the previous problems, estimate the population in 2010 (year 70 after conservation). The real population measured at that time was 437. Which model is most accurate?

Answer

Logistic: 361, Threshold: 436, Gompertz: 309.

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8.5: First-order Linear Equations

Learning Objectives

- Write a first-order linear differential equation in standard form.
- Find an integrating factor and use it to solve a first-order linear differential equation.
- Solve applied problems involving first-order linear differential equations.

Earlier, we studied an application of a first-order differential equation that involved solving for the velocity of an object. In particular, if a ball is thrown upward with an initial velocity of v_0 ft/s, then an initial-value problem that describes the velocity of the ball after t seconds is given by

$$\frac{dv}{dt} = -32$$

with $v(0) = v_0$.

This model assumes that the only force acting on the ball is gravity. Now we add to the problem by allowing for the possibility of **air resistance** acting on the ball.

Air resistance always acts in the direction opposite to motion. Therefore if an object is rising, air resistance acts in a downward direction. If the object is falling, air resistance acts in an upward direction (Figure 8.5.1). There is no exact relationship between the velocity of an object and the air resistance acting on it. For very small objects, air resistance is proportional to velocity; that is, the force due to air resistance is numerically equal to some constant k times v. For larger (e.g., baseball-sized) objects, depending on the shape, air resistance can be approximately proportional to the square of the velocity. In fact, air resistance may be proportional to $v^{1.5}$, or $v^{0.9}$, or some other power of v.

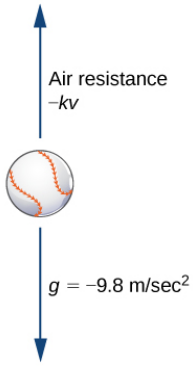

Figure 8.5.1: Forces acting on a moving baseball: gravity acts in a downward direction and air resistance acts in a direction opposite to the direction of motion.

We will work with the linear approximation for air resistance. If we assume $k > 0$, then the expression for the force F_A due to air resistance is given by $F_A = -kv$. Therefore the sum of the forces acting on the object is equal to the sum of the gravitational force and the force due to air resistance. This, in turn, is equal to the mass of the object multiplied by its acceleration at time t(Newton's second law). This gives us the differential equation

$$m\frac{dv}{dt} = -kv - mg.$$

Finally, we impose an initial condition $v(0) = v_0$, where v_0 is the initial velocity measured in meters per second. This makes $g = 9.8m/s^2$. The initial-value problem becomes

$$m\frac{dv}{dt} = -kv - mg$$

with $v(0) = v_0$.

The differential equation in this initial-value problem is an example of a first-order linear differential equation. (Recall that a differential equation is first-order if the highest-order derivative that appears in the equation is 1.) In this section, we study first-

order linear equations and examine a method for finding a general solution to these types of equations, as well as solving initial-value problems involving them.

Definition: Linear first-order differential equation

A first-order differential equation is *linear* if it can be written in the form

$$a(x)y'+b(x)y=c(x), \tag{8.5.1}$$

where $a(x), b(x),$ and $c(x)$ are arbitrary functions of x.

Remember that the unknown function y depends on the variable x; that is, x is the independent variable and y is the dependent variable. Some examples of first-order linear differential equations are

$$\begin{aligned}(3x^2-4)y'+(x-3)y&=\sin x\\ (\sin x)y'-(\cos x)y&=\cot x\\ 4xy'+(3\ln x)y&=x^3-4x.\end{aligned}$$

Examples of first-order nonlinear differential equations include

$$\begin{aligned}(y')^4-(y')^3&=(3x-2)(y+4)\\ 4y'+3y^3&=4x-5\\ (y')^2&=\sin y+\cos x.\end{aligned}$$

These equations are nonlinear because of terms like $(y')^4, y^3,$ etc. Due to these terms, it is impossible to put these equations into the same form as Equation 8.5.1.

Standard Form

Consider the differential equation

$$(3x^2-4)y'+(x-3)y=\sin x.$$

Our main goal in this section is to derive a solution method for equations of this form. It is useful to have the coefficient of y' be equal to 1. To make this happen, we divide both sides by $3x^2-4$.

$$y'+\left(\frac{x-3}{3x^2-4}\right)y=\frac{\sin x}{3x^2-4}$$

This is called the **standard form** of the differential equation. We will use it later when finding the solution to a general first-order linear differential equation. Returning to Equation 8.5.1, we can divide both sides of the equation by $a(x)$. This leads to the equation

$$y'+\frac{b(x)}{a(x)}y=\frac{c(x)}{a(x)}. \tag{8.5.2}$$

Now define

$$p(x)=\frac{b(x)}{a(x)}$$

and

$$q(x)=\frac{c(x)}{a(x)}$$

Then Equation 8.5.2 becomes

$$y'+p(x)y=q(x).$$

We can write any first-order linear differential equation in this form, and this is referred to as the standard form for a first-order linear differential equation.

Example 8.5.1: Writing First-Order Linear Equations in Standard Form

Put each of the following first-order linear differential equations into standard form. Identify $p(x)$ and $q(x)$ for each equation.

a. $y' = 3x - 4y$
b. $\dfrac{3xy'}{4y-3} = 2$ (here $x > 0$)
c. $y = 3y' - 4x^2 + 5$

Solution

a. Add $4y$ to both sides:

$$y' + 4y = 3x.$$

In this equation, $p(x) = 4$ and $q(x) = 3x$.

b. Multiply both sides by $4y - 3$, then subtract $8y$ from each side:

$$\begin{aligned} \frac{3xy'}{4y-3} &= 2 \\ 3xy' &= 2(4y-3) \\ 3xy' &= 8y - 6 \\ 3xy' - 8y &= -6. \end{aligned}$$

Finally, divide both sides by $3x$ to make the coefficient of y' equal to 1:

$$y' - \frac{8}{3x}y = -\frac{2}{3x}.$$

This is allowable because in the original statement of this problem we assumed that $x > 0$. (If $x = 0$ then the original equation becomes $0 = 2$, which is clearly a false statement.)

In this equation, $p(x) = -\dfrac{8}{3x}$ and $q(x) = -\dfrac{2}{3x}$.

c. Subtract y from each side and add $4x^2 - 5$:

$$3y' - y = 4x^2 - 5.$$

Next divide both sides by 3:

$$y' - \frac{1}{3}y = \frac{4}{3}x^2 - \frac{5}{3}.$$

In this equation, $p(x) = -\dfrac{1}{3}$ and $q(x) = \dfrac{4}{3}x^2 - \dfrac{5}{3}$.

Exercise 8.5.1

Put the equation $\dfrac{(x+3)y'}{2x - 3y - 4} = 5$ into standard form and identify $p(x)$ and $q(x)$.

Hint

Multiply both sides by the common denominator, then collect all terms involving y on one side.

Answer

$$y' + \frac{15}{x+3}y = \frac{10x - 20}{x+3}$$

$$p(x) = \frac{15}{x+3}$$

and

$$q(x) = \frac{10x - 20}{x+3}$$

Integrating Factors

We now develop a solution technique for any first-order linear differential equation. We start with the standard form of a first-order linear differential equation:

$$y' + p(x)y = q(x). \tag{8.5.3}$$

The first term on the left-hand side of Equation 8.5.3 is the derivative of the unknown function, and the second term is the product of a known function with the unknown function. This is somewhat reminiscent of the power rule. If we multiply Equation 8.5.3 by a yet-to-be-determined function $\mu(x)$, then the equation becomes

$$\mu(x)y' + \mu(x)p(x)y = \mu(x)q(x). \tag{8.5.4}$$

The left-hand side Equation 8.5.4 can be matched perfectly to the product rule:

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

Matching term by term gives $y = f(x), g(x) = \mu(x)$, and $g'(x) = \mu(x)p(x)$. Taking the derivative of $g(x) = \mu(x)$ and setting it equal to the right-hand side of $g'(x) = \mu(x)p(x)$ leads to

$$\mu'(x) = \mu(x)p(x).$$

This is a first-order, separable differential equation for $\mu(x)$. We know $p(x)$ because it appears in the differential equation we are solving. Separating variables and integrating yields

$$\begin{aligned}
\frac{\mu'(x)}{\mu(x)} &= p(x) \\
\int \frac{\mu'(x)}{\mu(x)}dx &= \int p(x)dx \\
\ln|\mu(x)| &= \int p(x)dx + C \\
e^{\ln|\mu(x)|} &= e^{\int p(x)dx + C} \\
|\mu(x)| &= C_1 e^{\int p(x)dx} \\
\mu(x) &= C_2 e^{\int p(x)dx}.
\end{aligned}$$

Here C_2 can be an arbitrary (positive or negative) constant. This leads to a general method for solving a first-order linear differential equation. We first multiply both sides of Equation 8.5.3 by this integrating factor by **integrating factor** $\mu(x)$. This gives

$$\mu(x)y' + \mu(x)p(x)y = \mu(x)q(x). \tag{8.5.5}$$

Note that, since the non-zero constant C_2 can be divided out of this equation (as it will appear in every term on both sides), we do not need to use it in this process. That is, we can simply use

$$\mu(x) = e^{\int p(x)dx}.$$

The left-hand side of Equation 8.5.5 can be rewritten as $\frac{d}{dx}(\mu(x)y)$.

$$\frac{d}{dx}(\mu(x)y) = \mu(x)q(x). \tag{8.5.6}$$

Next integrate both sides of Equation 8.5.6 with respect to x.

$$\begin{aligned}
\int \frac{d}{dx}(\mu(x)y)dx &= \int \mu(x)q(x)dx \\
\mu(x)y &= \int \mu(x)q(x)dx
\end{aligned}$$

Divide both sides of Equation 8.5.6 by $\mu(x)$:

$$y = \frac{1}{\quad}\left[\int \mu(x)q(x)dx + C\right]. \tag{8.5.7}$$

$$\mu(x)\left[\int \text{[illegible]}\right]$$

Since $\mu(x)$ was previously calculated, we are now finished. An important note about the integrating constant C: It may seem that we are inconsistent in the usage of the integrating constant. However, the integral involving $p(x)$ is necessary in order to find an integrating factor for Equation 8.5.3. Only one integrating factor is needed in order to solve the equation; therefore, it is safe to assign a value for C for this integral. We chose $C = 0$. When calculating the integral inside the brackets in Equation 8.5.7, it is necessary to keep our options open for the value of the integrating constant, because our goal is to find a general family of solutions to Equation 8.5.3. This integrating factor guarantees just that.

Problem-Solving Strategy: Solving a First-order Linear Differential Equation

1. Put the equation into standard form and identify $p(x)$ and $q(x)$.
2. Calculate the integrating factor

$$\mu(x) = e^{\int p(x)dx}. \tag{8.5.8}$$

3. Multiply both sides of the differential equation by $\mu(x)$.
4. Integrate both sides of the equation obtained in step 3, and divide both sides by $\mu(x)$.
5. If there is an initial condition, determine the value of C.

Example 8.5.2: Solving a First-order Linear Equation

Find a general solution for the differential equation $xy' + 3y = 4x^2 - 3x$. Assume $x > 0$.

Solution

1. To put this differential equation into standard form, divide both sides by x:

$$y' + \frac{3}{x}y = 4x - 3.$$

Therefore $p(x) = \dfrac{3}{x}$ and $q(x) = 4x - 3$.

2. The integrating factor is $\mu(x) = e^{\int (3/x)dx} = e^{3\ln x} = x^3$.

3. Multiplying both sides of the differential equation by $\mu(x)$ gives us

$$\begin{aligned} x^3y' + x^3\left(\frac{3}{x}\right)y &= x^3(4x-3) \\ x^3y' + 3x^2y &= 4x^4 - 3x^3 \\ \frac{d}{dx}(x^3y) &= 4x^4 - 3x^3. \end{aligned}$$

4. Integrate both sides of the equation.

$$\begin{aligned} \int \frac{d}{dx}(x^3y)dx &= \int 4x^4 - 3x^3 dx \\ x^3y &= \frac{4x^5}{5} - \frac{3x^4}{4} + C \\ y &= \frac{4x^2}{5} - \frac{3x}{4} + Cx^{-3}. \end{aligned}$$

5. There is no initial value, so the problem is complete.

Analysis

You may have noticed the condition that was imposed on the differential equation; namely, $x > 0$. For any nonzero value of C, the general solution is not defined at $x = 0$. Furthermore, when $x < 0$, the integrating factor changes. The integrating factor is given by Equation 8.5.8 as $\mu(x) = e^{\int p(x)dx}$. For this $p(x)$ we get

$$e^{\int p(x)dx} = e^{\int (3/x)dx}$$
$$= e^{3\ln|x|}$$
$$= |x|^3$$

since $x < 0$. The behavior of the general solution changes at $x = 0$ largely due to the fact that $p(x)$ is not defined there.

? Exercise 8.5.2

Find the general solution to the differential equation $(x-2)y' + y = 3x^2 + 2x$. Assume $x > 2$.

Hint

Use the method outlined in the problem-solving strategy for first-order linear differential equations.

Answer

$$y = \frac{x^3 + x^2 + C}{x - 2}$$

Now we use the same strategy to find the solution to an initial-value problem.

✓ Example 8.5.3: A First-order Linear Initial-Value Problem

Solve the initial-value problem

$$y' + 3y = 2x - 1, y(0) = 3.$$

Solution

1. This differential equation is already in standard form with $p(x) = 3$ and $q(x) = 2x - 1$.

2. The integrating factor is $\mu(x) = e^{\int 3dx} = e^{3x}$.

3. Multiplying both sides of the differential equation by $\mu(x)$ gives

$$e^{3x}y' + 3e^{3x}y = (2x-1)e^{3x}$$
$$\frac{d}{dx}[ye^{3x}] = (2x-1)e^{3x}.$$

Integrate both sides of the equation:

$$\int \frac{d}{dx}[ye^{3x}]dx = \int (2x-1)e^{3x}dx$$
$$ye^{3x} = \frac{e^{3x}}{3}(2x-1) - \int \frac{2}{3}e^{3x}dx$$
$$ye^{3x} = \frac{e^{3x}(2x-1)}{3} - \frac{2e^{3x}}{9} + C$$
$$y = \frac{2x-1}{3} - \frac{2}{9} + Ce^{-3x}$$
$$y = \frac{2x}{3} - \frac{5}{9} + Ce^{-3x}.$$

4. Now substitute $x = 0$ and $y = 3$ into the general solution and solve for C:

$$y = \frac{2}{3}x - \frac{5}{9} + Ce^{-3x}$$
$$3 = \frac{2}{3}(0) - \frac{5}{9} + Ce^{-3(0)}$$
$$3 = -\frac{5}{9} + C$$
$$C = \frac{32}{9}.$$

Therefore the solution to the initial-value problem is

$$y = \frac{2}{3}x - \frac{5}{9} + \frac{32}{9}e^{-3x}.$$

Example 8.5.4:

Solve the initial-value problem

$$y' - 2y = 4x + 3y(0) = -2.$$

Solution

$$y = -2x - 4 + 2e^{2x}$$

Applications of First-order Linear Differential Equations

We look at two different applications of first-order linear differential equations. The first involves air resistance as it relates to objects that are rising or falling; the second involves an electrical circuit. Other applications are numerous, but most are solved in a similar fashion.

Free fall with air resistance

We discussed air resistance at the beginning of this section. The next example shows how to apply this concept for a ball in vertical motion. Other factors can affect the force of air resistance, such as the size and shape of the object, but we ignore them here.

Example 8.5.5: A Ball with Air Resistance

A racquetball is hit straight upward with an initial velocity of 2 m/s. The mass of a racquetball is approximately 0.0427 kg. Air resistance acts on the ball with a force numerically equal to $0.5v$, where v represents the velocity of the ball at time t.

a. Find the velocity of the ball as a function of time.
b. How long does it take for the ball to reach its maximum height?
c. If the ball is hit from an initial height of 1 meter, how high will it reach?

Solution

a. The mass $m = 0.0427\ kg, k = 0.5$, and $g = 9.8\ m/s^2$. The initial velocity is $v_0 = 2\ m/s$. Therefore the initial-value problem is

$$0.0427\frac{dv}{dt} = -0.5v - 0.0427(9.8), v_0 = 2.$$

Dividing the differential equation by 0.0427 gives

$$\frac{dv}{dt} = -11.7096v - 9.8, v_0 = 2.$$

The differential equation is linear. Using the problem-solving strategy for linear differential equations:

Step 1. Rewrite the differential equation as $\frac{dv}{dt} + 11.7096v = -9.8$. This gives $p(t) = 11.7096$ and $q(t) = -9.8$

Step 2. The integrating factor is $\mu(t) = e^{\int 11.7096dt} = e^{11.7096t}$.

Step 3. Multiply the differential equation by $\mu(t)$:

$$e^{11.7096t}\frac{dv}{dt} + 11.7096ve^{11.7096t} = -9.8e^{11.7096t}$$

$$\frac{d}{dt}[ve^{11.7096t}] = -9.8e^{11.7096t}.$$

Step 4. Integrate both sides:

$$\int \frac{d}{dt}[ve^{11.7096t}]dt = \int -9.8e^{11.7096t}dt$$
$$ve^{11.7096t} = \frac{-9.8}{11.7096}e^{11.7096t} + C$$
$$v(t) = -0.8369 + Ce^{-11.7096t}.$$

Step 5. Solve for C using the initial condition $v_0 = v(0) = 2$:

$$v(t) = -0.8369 + Ce^{-11.7096t}$$
$$v(0) = -0.8369 + Ce^{-11.7096(0)}$$
$$2 = -0.8369 + C$$
$$C = 2.8369.$$

Therefore the solution to the initial-value problem is

$$v(t) = 2.8369e^{-11.7096t} - 0.8369.$$

b. The ball reaches its maximum height when the velocity is equal to zero. The reason is that when the velocity is positive, it is rising, and when it is negative, it is falling. Therefore when it is zero, it is neither rising nor falling, and is at its maximum height:

$$2.8369e^{-11.7096t} - 0.8369 = 0$$
$$2.8369e^{-11.7096t} = 0.8369$$
$$e^{-11.7096t} = \frac{0.8369}{2.8369} \approx 0.295$$
$$\ln e^{-11.7096t} = \ln 0.295 \approx -1.221$$
$$-11.7096t = -1.221$$
$$t \approx 0.104.$$

Therefore it takes approximately 0.104 second to reach maximum height.

c. To find the height of the ball as a function of time, use the fact that the derivative of position is velocity, i.e., if $h(t)$ represents the height at time t, then $h'(t) = v(t)$. Because we know $v(t)$ and the initial height, we can form an initial-value problem:

$$h'(t) = 2.8369e^{-11.7096t} - 0.8369,\ h(0) = 1.$$

Integrating both sides of the differential equation with respect to t gives

$$\int h'(t)dt = \int 2.8369e^{-11.7096t} - 0.8369\ dt$$
$$h(t) = -\frac{2.8369}{11.7096}e^{-11.7096t} - 0.8369t + C$$
$$h(t) = -0.2423e^{-11.7096t} - 0.8369t + C.$$

Solve for C by using the initial condition:

$$h(t) = -0.2423e^{-11.7096t} - 0.8369t + C$$
$$h(0) = -0.2423e^{-11.7096(0)} - 0.8369(0) + C$$
$$1 = -0.2423 + C$$
$$C = 1.2423.$$

Therefore

$$h(t) = -0.2423e^{-11.7096t} - 0.8369t + 1.2423.$$

After 0.104 second, the height is given by

$$h(0.2) = -0.2423e^{-11.7096t} - 0.8369t + 1.2423$$
$$\approx 1.0836 \text{ meter.}$$

? Exercise 8.5.3

The weight of a penny is 2.5 grams (United States Mint, "Coin Specifications," accessed April 9, 2015, www.usmint.gov/about_the_mint...specifications), and the upper observation deck of the Empire State Building is 369 meters above the street. Since the penny is a small and relatively smooth object, air resistance acting on the penny is actually quite small. We assume the air resistance is numerically equal to $0.0025v$. Furthermore, the penny is dropped with no initial velocity imparted to it.

a. Set up an initial-value problem that represents the falling penny.
b. Solve the problem for $v(t)$.
c. What is the terminal velocity of the penny (i.e., calculate the limit of the velocity as t approaches infinity)?

Hint

Set up the differential equation the same way as Example ? 8.5.5. Remember to convert from grams to kilograms.

Answer

a. $\frac{dv}{dt} = -v - 9.8\ v(0) = 0$
b. $v(t) = 9.8(e^{-t} - 1)$
c. $\lim_{t\to\infty} v(t) = \lim_{t\to\infty}(9.8(e^{-t} - 1)) = -9.8 \text{ m/s} \approx -21.922 \text{ mph}$

Electrical Circuits

A source of electromotive force (e.g., a battery or generator) produces a flow of current in a closed circuit, and this current produces a voltage drop across each resistor, inductor, and capacitor in the circuit. Kirchhoff's Loop Rule states that the sum of the voltage drops across resistors, inductors, and capacitors is equal to the total electromotive force in a closed circuit. We have the following three results:

1. The voltage drop across a resistor is given by
$$E_R = Ri,$$
where R is a constant of proportionality called the resistance, and i is the current.

2. The voltage drop across an inductor is given by
$$E_L = Li',$$
where L is a constant of proportionality called the inductance, and i again denotes the current.

3. The voltage drop across a capacitor is given by
$$E_C = \frac{1}{C}q,$$
where C is a constant of proportionality called the capacitance, and q is the instantaneous charge on the capacitor. The relationship between i and q is $i = q'$.

We use units of volts (V) to measure voltage E, amperes (A) to measure current i, coulombs (C) to measure charge q, ohms (Ω) to measure resistance R, henrys (H) to measure inductance L, and farads (F) to measure capacitance C. Consider the circuit in Figure 8.5.2.

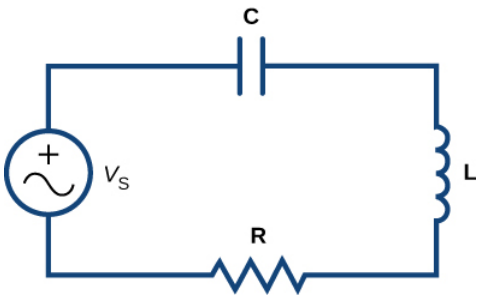

Figure 8.5.2: A typical electric circuit, containing a voltage generator (V_S), capacitor (C), inductor (L), and resistor (R).

Applying Kirchhoff's Loop Rule to this circuit, we let E denote the electromotive force supplied by the voltage generator. Then

$$E_L + E_R + E_C = E.$$

Substituting the expressions for $E_L, E_R,$ and E_C into this equation, we obtain

$$Li' + Ri + \frac{1}{C}q = E. \tag{8.5.9}$$

If there is no capacitor in the circuit, then the equation becomes

$$Li' + Ri = E. \tag{8.5.10}$$

This is a first-order differential equation in i. The circuit is referred to as an **LR circuit**.

Next, suppose there is no inductor in the circuit, but there is a capacitor and a resistor, so $L = 0$, $R \neq 0$, and $C \neq 0$. Then, since $q' = i$, Equation 8.5.9 can be rewritten as

$$Rq' + \frac{1}{C}q = E, \tag{8.5.11}$$

which is a first-order linear differential equation in q. This is referred to as an **RC circuit**. In either case, we can set up and solve an initial-value problem.

Electric Circuit

A circuit has in series an electromotive force given by $E = 50 \sin 20t V$, a resistor of 5Ω, and an inductor of $0.4H$. If the initial current is 0, find the current at time $t > 0$.

Solution

We have a resistor and an inductor in the circuit, so we have an LR circuit and can use Equation 8.5.10. The voltage drop across the resistor is given by $E_R = Ri = 5i$. The voltage drop across the inductor is given by $E_L = Li' = 0.4i'$. The electromotive force becomes the right-hand side of Equation 8.5.10. Therefore Equation 8.5.10 becomes

$$0.4i' + 5i = 50 \sin 20t.$$

Dividing both sides by 0.4 gives the equation

$$i' + 12.5i = 125 \sin 20t.$$

Since the initial current is 0, this result gives an initial condition of $i(0) = 0$. We can solve this initial-value problem using the five-step strategy for solving first-order differential equations.

Step 1. Rewrite the differential equation as $i' + 12.5i = 125 \sin 20t$. This gives $p(t) = 12.5$ and $q(t) = 125 \sin 20t$.

Step 2. The integrating factor is $\mu(t) = e^{\int 12.5dt} = e^{12.5t}$.

Step 3. Multiply the differential equation by $\mu(t)$:

$$e^{12.5t}i' + 12.5e^{12.5t}i = 125e^{12.5t} \sin 20t$$

$$\frac{d}{dt}[ie^{12.5t}] = 125e^{12.5t} \sin 20t.$$

Step 4. Integrate both sides:

$$\int \frac{d}{dt}[ie^{12.5t}]dt = \int 125e^{12.5t}\sin 20t\ dt$$

$$ie^{12.5t} = \left(\frac{250\sin 20t - 400\cos 20t}{89}\right)e^{12.5t} + C$$

$$i(t) = \frac{250\sin 20t - 400\cos 20t}{89} + Ce^{-12.5t}.$$

Step 5. Solve for C using the initial condition $i(0) = 0$:

$$i(t) = \frac{250\sin 20t - 400\cos 20t}{89} + Ce^{-12.5t}$$

$$i(0) = \frac{250\sin 20(0) - 400\cos 20(0)}{89} + Ce^{-12.5(0)}$$

$$0 = -\frac{400}{89} + C$$

$$C = \frac{400}{89}.$$

Therefore the solution to the initial-value problem is

$$i(t) = \frac{250\sin 20t - 400\cos 20t + 400e^{-12.5t}}{89} = \frac{250\sin 20t - 400\cos 20t}{89} + \frac{400e^{-12.5t}}{89}.$$

The first term can be rewritten as a single cosine function. First, multiply and divide by $\sqrt{250^2 + 400^2} = 50\sqrt{89}$:

$$\frac{250\sin 20t - 400\cos 20t}{89} = \frac{50\sqrt{89}}{89}\left(\frac{250\sin 20t - 400\cos 20t}{50\sqrt{89}}\right) = -\frac{50\sqrt{89}}{89}\left(\frac{8\cos 20t}{\sqrt{89}} - \frac{5\sin 20t}{\sqrt{89}}\right).$$

Next, define φ to be an acute angle such that $\cos\varphi = \frac{8}{\sqrt{89}}$. Then $\sin\varphi = \frac{5}{\sqrt{89}}$ and

$$-\frac{50\sqrt{89}}{89}\left(\frac{8\cos 20t}{\sqrt{89}} - \frac{5\sin 20t}{\sqrt{89}}\right) = -\frac{50\sqrt{89}}{89}(\cos\varphi\cos 20t - \sin\varphi\sin 20t) = -\frac{50\sqrt{89}}{89}\cos(20t + \varphi).$$

Therefore the solution can be written as

$$i(t) = -\frac{50\sqrt{89}}{89}cos(20t + \varphi) + \frac{400e^{-12.5t}}{89}.$$

The second term is called the attenuation term, because it disappears rapidly as t grows larger. The phase shift is given by φ, and the amplitude of the steady-state current is given by $\frac{50\sqrt{89}}{89}$. The graph of this solution appears in Figure 8.5.3:

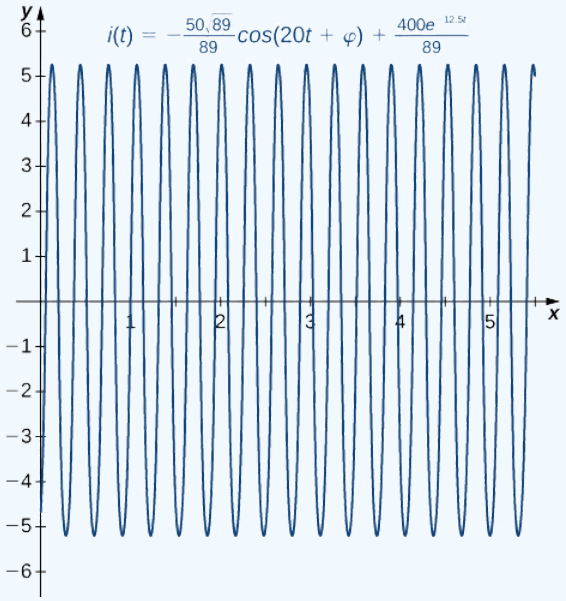

Figure 8.5.3.

Exercise 8.5.4

A circuit has in series an electromotive force given by $E = 20 \sin 5t$ V, a capacitor with capacitance $0.02F$, and a resistor of 8Ω. If the initial charge is $4C$, find the charge at time $t > 0$.

Hint

Use Equation 8.5.11 for an RC circuit to set up an initial-value problem.

Answer

Initial-value problem:

$$8q' + \frac{1}{0.02}q = 20 \sin 5t, q(0) = 4$$

$$q(t) = \frac{10 \sin 5t - 8 \cos 5t + 172e^{-6.25t}}{41}$$

Key Concepts

- Any first-order linear differential equation can be written in the form $y' + p(x)y = q(x)$.
- We can use a five-step problem-solving strategy for solving a first-order linear differential equation that may or may not include an initial value.
- Applications of first-order linear differential equations include determining motion of a rising or falling object with air resistance and finding current in an electrical circuit.

Key Equations

- **standard form**

  $y' + p(x)y = q(x)$

- **integrating factor**

  $\mu(x) = e^{\int p(x)dx}$

Glossary

integrating factor
any function $f(x)$ that is multiplied on both sides of a differential equation to make the side involving the unknown function equal to the derivative of a product of two functions

linear
description of a first-order differential equation that can be written in the form $a(x)y' + b(x)y = c(x)$

standard form
the form of a first-order linear differential equation obtained by writing the differential equation in the form $y' + p(x)y = q(x)$

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8.5E: Exercises for Section 8.5

In exercises 1 - 5, state if each of the following differential equations is linear? Explain your reasoning.

1) $\frac{dy}{dx} = x^2y + \sin x$

2) $\frac{dy}{dt} = ty$

Answer

Yes

3) $\frac{dy}{dt} + y^2 = x$

4) $y' = x^3 + e^x$

Answer

Yes

5) $y' = y + e^y$

In exercises 6 - 10, write the following first-order differential equations in standard form.

6) $y' = x^3y + \sin x$

Answer

$y' - x^3y = \sin x$

7) $y' + 3y - \ln x = 0$

8) $-xy' = (3x + 2)y + xe^x$

Answer

$y' + \frac{(3x+2)}{x}y = -e^x$

9) $\frac{dy}{dt} = 4y + ty + \tan t$

10) $\frac{dy}{dt} = yx(x + 1)$

Answer

$\frac{dy}{dt} - yx(x + 1) = 0$

In exercises 11 - 15, state the integrating factors for each of the following differential equations.

11) $y' = xy + 3$

12) $y' + e^xy = \sin x$

Answer

e^x

13) $y' = x\ln(x)y + 3x$

14) $\frac{dy}{dx} = \tanh(x)y + 1$

Answer

$-\ln(\cosh x)$

15) $\frac{dy}{dt} + 3ty = e^ty$

In exercises 16 - 25, solve each differential equation by using integrating factors.

16) $y' = 3y + 2$

Answer

$y = Ce^{3x} - \frac{2}{3}$

17) $y' = 2y - x^2$

18) $xy' = 3y - 6x^2$

Answer

$y = Cx^3 + 6x^2$

19) $(x + 2)y' = 3x + y$

20) $y' = 3x + xy$

Answer

$y = Ce^{x^2/2} - 3$

21) $xy' = x + y$

22) $\sin(x)y' = y + 2x$

Answer

$y = C\tan\left(\frac{x}{2}\right) - 2x + 4\tan\left(\frac{x}{2}\right)\ln\left(\sin(\frac{x}{2})\right)$

23) $y' = y + e^x$

24) $xy' = 3y + x^2$

Answer

$y = Cx^3 - x^2$

25) $y' + \ln x = \frac{y}{x}$

In exercises 26 - 33, solve the given differential equation. Use your calculator to draw a family of solutions. Are there certain initial conditions that change the behavior of the solution?

26) [T] $(x + 2)y' = 2y - 1$

Answer

$y = C(x + 2)^2 + \frac{1}{2}$

27) [T] $y' = 3e^{t/3} - 2y$

28) [T] $xy' + \frac{y}{2} = \sin(3t)$

Answer

$y = \frac{C}{\sqrt{x}} + 2\sin(3t)$

29) [T] $xy' = 2\frac{\cos x}{x} - 3y$

30) [T] $(x + 1)y' = 3y + x^2 + 2x + 1$

Answer

$y = C(x + 1)^3 - x^2 - 2x - 1$

31) [T] $\sin(x)y' + \cos(x)y = 2x$

32) [T] $\sqrt{x^2+1}y' = y+2$

Answer

$y = Ce^{\sinh^{-1}x} - 2$

33) [T] $x^3y' + 2x^2y = x+1$

In exercises 34 - 43, solve each initial-value problem by using integrating factors.

34) $y' + y = x, \quad y(0) = 3$

Answer

$y = x + 4e^x - 1$

35) $y' = y + 2x^2, \quad y(0) = 0$

36) $xy' = y - 3x^3, \quad y(1) = 0$

Answer

$y = -\dfrac{3x}{2}(x^2 - 1)$

37) $x^2y' = xy - \ln x, \quad y(1) = 1$

38) $(1+x^2)y' = y - 1, \quad y(0) = 0$

Answer

$y = 1 - e^{\tan^{-1}x}$

39) $xy' = y + 2x\ln x, \quad y(1) = 5$

40) $(2+x)y' = y + 2 + x, \quad y(0) = 0$

Answer

$y = (x+2)\ln\left(\dfrac{x+2}{2}\right)$

41) $y' = xy + 2xe^x, \quad y(0) = 2$

42) $\sqrt{x}y' = y + 2x, \quad y(0) = 1$

Answer

$y = 2e^{2\sqrt{x}} - 2x - 2\sqrt{x} - 1$

43) $y' = 2y + xe^x, \quad y(0) = -1$

44) A falling object of mass m can reach terminal velocity when the drag force is proportional to its velocity, with proportionality constant k. Set up the differential equation and solve for the velocity given an initial velocity of 0.

Answer

$v(t) = \dfrac{gm}{k}\left(1 - e^{-kt/m}\right)$

45) Using your expression from the preceding problem, what is the terminal velocity? (Hint: Examine the limiting behavior; does the velocity approach a value?)

46) [T] Using your equation for terminal velocity, solve for the distance fallen. How long does it take to fall 5000 meters if the mass is 100 kilograms, the acceleration due to gravity is 9.8 m/s^2 and the proportionality constant is 4?

Answer

40.451 seconds

47) A more accurate way to describe terminal velocity is that the drag force is proportional to the square of velocity, with a proportionality constant k. Set up the differential equation and solve for the velocity.

48) Using your expression from the preceding problem, what is the terminal velocity? (Hint: Examine the limiting behavior: Does the velocity approach a value?)

Answer

$$\sqrt{\frac{gm}{k}}$$

49) [T] Using your equation for terminal velocity, solve for the distance fallen. How long does it take to fall 5000 meters if the mass is 100 kilograms, the acceleration due to gravity is 9.8 m/s^2 and the proportionality constant is 4? Does it take more or less time than your initial estimate?

In exercises 50 - 54, determine how parameter a affects the solution.

50) Solve the generic equation $y' = ax + y$. How does varying a change the behavior?

Answer

$$y = Ce^x - a(x+1)$$

51) Solve the generic equation $y' = ax + y$. How does varying a change the behavior?

52) Solve the generic equation $y' = ax + xy$. How does varying a change the behavior?

Answer

$$y = Ce^{x^2/2} - a$$

53) Solve the generic equation $y' = x + axy$. How does varying a change the behavior?

54) Solve $y' - y = e^{kt}$ with the initial condition $y(0) = 0$. As k approaches 1, what happens to your formula?

Answer

$$y = \frac{e^{kt} - e^t}{k-1}$$

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8.R: Chapter 8 Review Exercises

True or False? Justify your answer with a proof or a counterexample.

1) The differential equation $y' = 3x^2y - \cos(x)y''$ is linear.

2) The differential equation $y' = x - y$ is separable.

Answer

F

3) You can explicitly solve all first-order differential equations by separation or by the method of integrating factors.

4) You can determine the behavior of all first-order differential equations using directional fields or Euler's method.

Answer

T

For the following problems, find the general solution to the differential equations.

5) $y' = x^2 + 3e^x - 2x$

6) $y' = 2^x + \cos^{-1} x$

Answer

$y(x) = \frac{2^x}{ln(2)} + xcos^{-1}x - \sqrt{1-x^2} + C$

7) $y' = y(x^2 + 1)$

8) $y' = e^{-y} \sin x$

Answer

$y(x) = \ln(C - \cos x)$

9) $y' = 3x - 2y$

10) $y' = y \ln y$

Answer

$y(x) = e^{e^{C+x}}$

For the following problems, find the solution to the initial value problem.

11) $y' = 8x - \ln x - 3x^4, \quad y(1) = 5$

12) $y' = 3x - \cos x + 2, \quad y(0) = 4$

Answer

$y(x) = 4 + \frac{3}{2}x^2 + 2x - \sin x$

13) $xy' = y(x - 2), \quad y(1) = 3$

14) $y' = 3y^2(x + \cos x), \quad y(0) = -2$

Answer

$$y(x) = -\frac{2}{1 + 3(x^2 + 2\sin x)}$$

15) $(x - 1)y' = y - 2, \quad y(0) = 0$

16) $y' = 3y - x + 6x^2, \quad y(0) = -1$

Answer

$y(x) = -2x^2 - 2x - \frac{1}{3} - \frac{2}{3}e^{3x}$

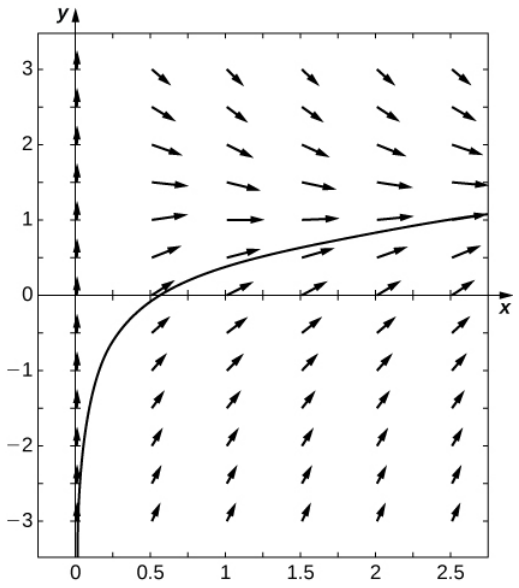
For the following problems, draw the directional field associated with the differential equation, then solve the differential equation. Draw a sample solution on the directional field.

17) $y' = 2y - y^2$

18) $y' = \dfrac{1}{x} + \ln x - y$, for $x > 0$

Answer

$y(x) = Ce^{-x} + \ln x$



For the following problems, use Euler's Method with $n = 5$ steps over the interval $t = [0, 1]$. Then solve the initial-value problem exactly. How close is your Euler's Method estimate?

19) $y' = -4yx, \quad y(0) = 1$

20) $y' = 3^x - 2y, \quad y(0) = 0$

Answer

Euler: 0.6939,

Exact solution: $y(x) = \dfrac{3^x - e^{-2x}}{2 + \ln(3)}$

For the following problems, set up and solve the differential equations.

21) A car drives along a freeway, accelerating according to $a = 5\sin(\pi t)$, where t represents time in minutes. Find the velocity at any time t, assuming the car starts with an initial speed of 60 mph.

22) You throw a ball of mass 2 kilograms into the air with an upward velocity of 8 m/s. Find exactly the time the ball will remain in the air, assuming that gravity is given by $g = 9.8\,\text{m/s}^2$.

Answer

$\frac{40}{49}$ second

23) You drop a ball with a mass of 5 kilograms out an airplane window at a height of 5000 m. How long does it take for the ball to reach the ground?

24) You drop the same ball of mass 5 kilograms out of the same airplane window at the same height, except this time you assume a drag force proportional to the ball's velocity, using a proportionality constant of 3 and the ball reaches terminal velocity. Solve for

the distance fallen as a function of time. How long does it take the ball to reach the ground?

Answer

$x(t) = 5000 + \frac{245}{9} - \frac{49}{3}t - \frac{245}{9}e^{-5/3t}, \quad t = 307.8$ seconds

25) A drug is administered to a patient every 24 hours and is cleared at a rate proportional to the amount of drug left in the body, with proportionality constant 0.2. If the patient needs a baseline level of 5 mg to be in the bloodstream at all times, how large should the dose be?

26) A 1000-liter tank contains pure water and a solution of 0.2 kg salt/L is pumped into the tank at a rate of 1 L/min and is drained at the same rate. Solve for total amount of salt in the tank at time t.

Answer

$T(t) = 200\left(1 - e^{-t/1000}\right)$

27) You boil water to make tea. When you pour the water into your teapot, the temperature is $100°C$. After 5 minutes in your $15°C$ room, the temperature of the tea is $85°C$. Solve the equation to determine the temperatures of the tea at time t. How long must you wait until the tea is at a drinkable temperature ($72°C$)?

28) The human population (in thousands) of Nevada in 1950 was roughly 160. If the carrying capacity is estimated at 10 million individuals, and assuming a growth rate of 2% per year, develop a logistic growth model and solve for the population in Nevada at any time (use 1950 as time = 0). What population does your model predict for 2000? How close is your prediction to the true value of $1,998,257$?

Answer

$P(t) = \dfrac{1600000e^{0.02t}}{9840 + 160e^{0.02t}}$

29) Repeat the previous problem but use Gompertz growth model. Which is more accurate?

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CHAPTER OVERVIEW

9: Sequences and Series

The topic of infinite series may seem unrelated to differential and integral calculus. In fact, an infinite series whose terms involve powers of a variable is a powerful tool that we can use to express functions as “infinite polynomials.” We can use infinite series to evaluate complicated functions, approximate definite integrals, and create new functions. In addition, infinite series are used to solve differential equations that model physical behavior, from tiny electronic circuits to Earth-orbiting satellites.

9.0: Prelude to Sequence and Series
9.1: Sequences
9.1E: Exercises for Section 9.1
9.2: Infinite Series
9.2E: Exercises for Section 9.2
9.3: The Divergence and Integral Tests
9.3E: Exercises for Section 9.3
9.4: Comparison Tests
9.4E: Exercises for Section 9.4
9.5: Alternating Series
9.5E: Exercises for Section 9.5
9.6: Ratio and Root Tests
9.6E: Exercises for Section 9.6
9.R: Chapter 9 Review Exercises

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9.0: Prelude to Sequence and Series

The Koch snowflake is constructed from an infinite number of nonoverlapping equilateral triangles. Consequently, we can express its area as a sum of infinitely many terms. How do we add an infinite number of terms? Can a sum of an infinite number of terms be finite? To answer these questions, we need to introduce the concept of an infinite series, a sum with infinitely many terms. Having defined the necessary tools, we will be able to calculate the area of the Koch snowflake.

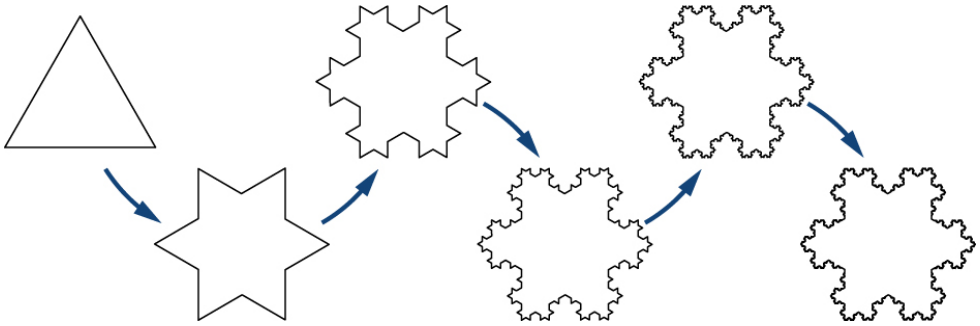

Figure 9.0.1: The Koch snowflake is constructed by using an iterative process. Starting with an equilateral triangle, at each step of the process the middle third of each line segment is removed and replaced with an equilateral triangle pointing outward.

The topic of infinite series may seem unrelated to differential and integral calculus. In fact, an infinite series whose terms involve powers of a variable is a powerful tool that we can use to express functions as “infinite polynomials.” We can use infinite series to evaluate complicated functions, approximate definite integrals, and create new functions. In addition, infinite series are used to solve differential equations that model physical behavior, from tiny electronic circuits to Earth-orbiting satellites.

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9.1: Sequences

Learning Objectives

- Find the formula for the general term of a sequence.
- Calculate the limit of a sequence if it exists.
- Determine the convergence or divergence of a given sequence.

In this section, we introduce sequences and define what it means for a sequence to converge or diverge. We show how to find limits of sequences that converge, often by using the properties of limits for functions discussed earlier. We close this section with the Monotone Convergence Theorem, a tool we can use to prove that certain types of sequences converge.

Terminology of Sequences

To work with this new topic, we need some new terms and definitions. First, an infinite sequence is an ordered list of numbers of the form

$$a_1, a_2, a_3, \ldots, a_n, \ldots.$$

Each of the numbers in the sequence is called a term. The symbol n is called the index variable for the sequence. We use the notation

$$\{a_n\}_{n=1}^{\infty},$$

or simply $\{a_n\}$, to denote this sequence. A similar notation is used for sets, but a sequence is an ordered list, whereas a set is not ordered. Because a particular number a_n exists for each positive integer n, we can also define a sequence as a function whose domain is the set of positive integers.

Let's consider the infinite, ordered list

$$2, 4, 8, 16, 32, \ldots.$$

This is a sequence in which the first, second, and third terms are given by $a_1 = 2, a_2 = 4$, and $a_3 = 8$. You can probably see that the terms in this sequence have the following pattern:

$$a_1 = 2^1,\ a_2 = 2^2,\ a_3 = 2^3,\ a_4 = 2^4 \text{ and } a_5 = 2^5.$$

Assuming this pattern continues, we can write the nth term in the sequence by the explicit formula $a_n = 2^n$. Using this notation, we can write this sequence as

$$\{2^n\}_{n=1}^{\infty}$$

or

$$\{2^n\}.$$

Alternatively, we can describe this sequence in a different way. Since each term is twice the previous term, this sequence can be defined recursively by expressing the nth term a_n in terms of the previous term a_{n-1}. In particular, we can define this sequence as the sequence $\{a_n\}$ where $a_1 = 2$ and for all $n \geq 2$, each term an is defined by the **recurrence relation**

$$a_n = 2a_{n-1}.$$

Definition: Infinite Sequence

An **infinite sequence** $\{a_n\}$ is an ordered list of numbers of the form

$$a_1,\ a_2,\ \ldots,\ a_n,\ \ldots.$$

The subscript n is called the **index variable** of the sequence. Each number a_n is a term of the sequence. Sometimes sequences are defined by **explicit formulas,** in which case $a_n = f(n)$ for some function $f(n)$ defined over the positive integers. In other cases, sequences are defined by using a **recurrence relation**. In a recurrence relation, one term (or more) of the sequence is given explicitly, and subsequent terms are defined in terms of earlier terms in the sequence.

Note that the index does not have to start at $n = 1$ but could start with other integers. For example, a sequence given by the explicit formula $a_n = f(n)$ could start at $n = 0$, in which case the sequence would be

$$a_0, a_1, a_2, \ldots .$$

Similarly, for a sequence defined by a recurrence relation, the term a_0 may be given explicitly, and the terms a_n for $n \geq 1$ may be defined in terms of a_{n-1}. Since a sequence $\{a_n\}$ has exactly one value for each positive integer n, it can be described as a function whose domain is the set of positive integers. As a result, it makes sense to discuss the graph of a sequence. The graph of a sequence $\{a_n\}$ consists of all points (n, a_n) for all positive integers n. Figure 9.1.1 shows the graph of 2^n.

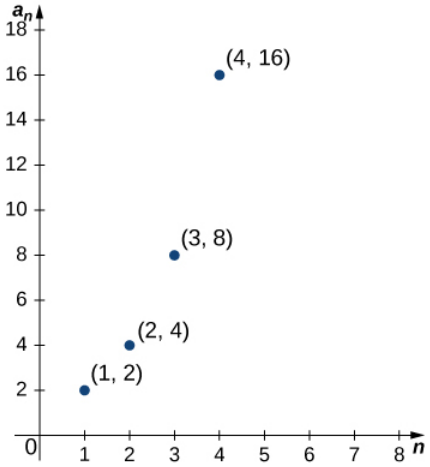

Figure 9.1.1: The plotted points are a graph of the sequence $\{2^n\}$.

Two types of sequences occur often and are given special names: arithmetic sequences and geometric sequences. In an arithmetic sequence, the difference between every pair of consecutive terms is the same. For example, consider the sequence

$$3, 7, 11, 15, 19, \ldots$$

You can see that the difference between every consecutive pair of terms is 4. Assuming that this pattern continues, this sequence is an arithmetic sequence. It can be described by using the recurrence relation

$$\begin{cases} a_1 = 3 \\ a_n = a_{n-1} + 4, \text{ for } n \geq 2 \end{cases}.$$

Note that

$$a_2 = 3 + 4$$

$$a_3 = 3 + 4 + 4 = 3 + 2 \cdot 4$$

$$a_4 = 3 + 4 + 4 + 4 = 3 + 3 \cdot 4.$$

Thus the sequence can also be described using the explicit formula

$$a_n = 3 + 4(n - 1) = 4n - 1.$$

In general, an arithmetic sequence is any sequence of the form $a_n = cn + b$.

In a geometric sequence, the ratio of every pair of consecutive terms is the same. For example, consider the sequence

$$2, -\frac{2}{3}, \frac{2}{9}, -\frac{2}{27}, \frac{2}{81}, \ldots .$$

We see that the ratio of any term to the preceding term is $-\frac{1}{3}$. Assuming this pattern continues, this sequence is a geometric sequence. It can be defined recursively as

$$a_1 = 2$$

$$a_n = -\frac{1}{3} \cdot a_{n-1}, \text{ for } n \geq 2.$$

Alternatively, since

$$a_2 = -\frac{1}{3} \cdot 2$$
$$a_3 = \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)(2) = \left(-\frac{1}{3}\right)^2 \cdot 2$$
$$a_4 = \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)(2) = \left(-\frac{1}{3}\right)^3 \cdot 2,$$

we see that the sequence can be described by using the explicit formula

$$a_n = 2\left(-\frac{1}{3}\right)^{n-1}.$$

The sequence $\{2^n\}$ that we discussed earlier is a geometric sequence, where the ratio of any term to the previous term is 2. In general, a geometric sequence is any sequence of the form $a_n = cr^n$.

Example 9.1.1: Finding Explicit Formulas

For each of the following sequences, find an explicit formula for the nth term of the sequence.

a. $-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \ldots$
b. $\frac{3}{4}, \frac{9}{7}, \frac{27}{10}, \frac{81}{13}, \frac{243}{16}, \ldots.$

Solution

a. First, note that the sequence is alternating from negative to positive. The odd terms in the sequence are negative, and the even terms are positive. Therefore, the nth term includes a factor of $(-1)^n$. Next, consider the sequence of numerators $1, 2, 3, \ldots$ and the sequence of denominators $2, 3, 4, \ldots$. We can see that both of these sequences are arithmetic sequences. The nth term in the sequence of numerators is n, and the nth term in the sequence of denominators is $n + 1$. Therefore, the sequence can be described by the explicit formula

$$a_n = \frac{(-1)^n n}{n+1}.$$

b. The sequence of numerators $3, 9, 27, 81, 243, \ldots$ is a geometric sequence. The numerator of the nth term is 3^n The sequence of denominators $4, 7, 10, 13, 16, \ldots$ is an arithmetic sequence. The denominator of the nth term is $4 + 3(n-1) = 3n + 1$. Therefore, we can describe the sequence by the explicit formula $a_n = \frac{3^n}{3n+1.}$

Exercise 9.1.1

Find an explicit formula for the nth term of the sequence $\left\{\frac{1}{5}, -\frac{1}{7}, \frac{1}{9}, -\frac{1}{11}, \ldots\right\}$.

Hint

The denominators form an arithmetic sequence.

Answer

$$a_n = \frac{(-1)^{n+1}}{3+2n}$$

Example 9.1.2: Defined by Recurrence Relations

For each of the following recursively defined sequences, find an explicit formula for the sequence.

a. $a_1 = 2$, $a_n = -3a_{n-1}$ for $n \geq 2$
b. $a_1 = \left(\frac{1}{2}\right)$, $a_n = a_{n-1} + \left(\frac{1}{2}\right)^n$ for $n \geq 2$

Solution

a. Writing out the first few terms, we have

$$a_1 = 2$$
$$a_2 = -3a_1 = -3(2)$$
$$a_3 = -3a_2 = (-3)^2 2$$
$$a_4 = -3a_3 = (-3)^3 2.$$

In general,

$$a_n = 2(-3)^{n-1}.$$

b. Write out the first few terms:

$$a_1 = \frac{1}{2}$$
$$a_2 = a_1 + \left(\frac{1}{2}\right)^2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$
$$a_3 = a_2 + \left(\frac{1}{2}\right)^3 = \frac{3}{4} + \frac{1}{8} = \frac{7}{8}$$
$$a_4 = a_3 + \left(\frac{1}{2}\right)^4 = \frac{7}{8} + \frac{1}{16} = \frac{15}{16}.$$

From this pattern, we derive the explicit formula

$$a_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}.$$

? Exercise 9.1.2

Find an explicit formula for the sequence defined recursively such that $a_1 = -4$ and $a_n = a_{n-1} + 6$.

Hint

This is an arithmetic sequence.

Answer

$a_n = 6n - 10$

Limit of a Sequence

A fundamental question that arises regarding infinite sequences is the behavior of the terms as n gets larger. Since a sequence is a function defined on the positive integers, it makes sense to discuss the limit of the terms as $n \to \infty$. For example, consider the following four sequences and their different behaviors as $n \to \infty$ (Figure 9.1.2):

a. $\{1 + 3n\} = \{4, 7, 10, 13, ...\}$. The terms $1 + 3n$ become arbitrarily large as $n \to \infty$. In this case, we say that $1 + 3n \to \infty$ as $n \to \infty$.
b. $\left\{1 - \left(\frac{1}{2}\right)^n\right\} = \left\{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16} ...\right\}$. The terms $1 - \left(\frac{1}{2}\right)^n \to 1$ as $n \to \infty$.
c. $\{(-1)^n\} = \{-1, 1, -1, 1, ...\}$. The terms alternate but do not approach one single value as $n \to \infty$.
d. $\left\{\frac{(-1)^n}{n}\right\} = \left\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, ...\right\}$. The terms alternate for this sequence as well, but $\frac{(-1)^n}{n} \to 0$ as $n \to \infty$.

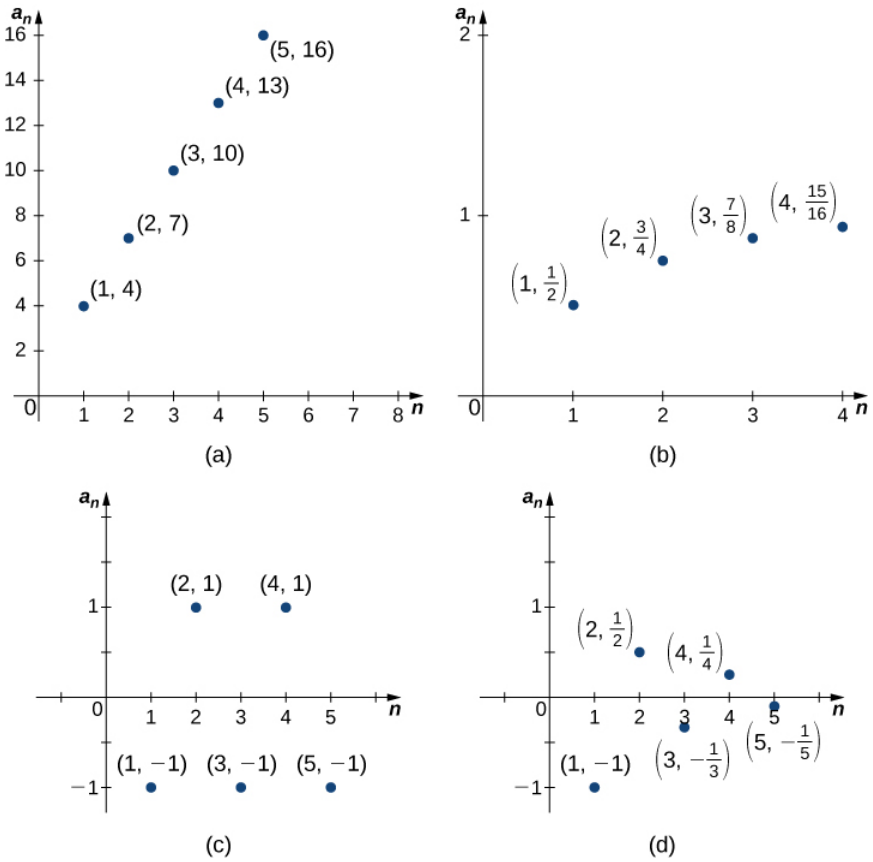

Figure 9.1.2: (a) The terms in the sequence become arbitrarily large as $n \to \infty$. (b) The terms in the sequence approach 1 as $n \to \infty$. (c) The terms in the sequence alternate between 1 and -1 as $n \to \infty$. (d) The terms in the sequence alternate between positive and negative values but approach 0 as $n \to \infty$.

From these examples, we see several possibilities for the behavior of the terms of a sequence as $n \to \infty$. In two of the sequences, the terms approach a finite number as $n \to \infty$. In the other two sequences, the terms do not. If the terms of a sequence approach a finite number L as $n \to \infty$, we say that the sequence is a convergent sequence and the real number L is the limit of the sequence. We can give an informal definition here.

Definition: Convergent and Divergent Sequences

Given a sequence $\{a_n\}$, if the terms a_n become arbitrarily close to a finite number L as n becomes sufficiently large, we say $\{a_n\}$ is a **convergent sequence** and L is the **limit of the sequence.** In this case, we write

$$\lim_{n\to\infty} a_n = L.$$

If a sequence $\{a_n\}$ is not convergent, we say it is a **divergent sequence.**

From Figure 9.1.2*b*, we see that the terms in the sequence $\left\{1 - \left(\frac{1}{2}\right)^n\right\}$ are becoming arbitrarily close to 1 as n becomes very large. We conclude that $\left\{1 - \left(\frac{1}{2}\right)^n\right\}$ is a convergent sequence and its limit is 1. In contrast, from Figure 9.1.2*a*, we see that the terms in the sequence $\{1 + 3n\}$ are not approaching a finite number as n becomes larger. We say that $\{1 + 3n\}$ is a divergent sequence.

In the informal definition for the limit of a sequence, we used the terms “arbitrarily close” and “sufficiently large.” Although these phrases help illustrate the meaning of a converging sequence, they are somewhat vague. To be more precise, we now present the more formal definition of limit for a sequence and show these ideas graphically in Figure 9.1.3.

Definition: Convergence

A sequence $\{a_n\}$ converges to a real number L if for all $\varepsilon > 0$, there exists an integer N such that for all $n \geq N$ $|a_n - L| < \varepsilon$. The number L is the limit of the sequence and we write

$$\lim_{n\to\infty} a_n = L \text{ or } a_n \to L.$$

In this case, we say the sequence $\{a_n\}$ is a convergent sequence. If a sequence does not converge, it is a divergent sequence, and we say the limit does not exist.

We remark that the convergence or divergence of a sequence $\{a_n\}$ depends only on what happens to the terms a_n as $n \to \infty$. Therefore, if a finite number of terms $b_1, b_2, \ldots, b_N$ are placed before a_1 to create a new sequence

$$b_1,\ b_2,\ \ldots,\ b_N,\ a_1,\ a_2,\ \ldots,$$

this new sequence will converge if $\{a_n\}$ converges and diverge if $\{a_n\}$ diverges. Further, if the sequence $\{a_n\}$ converges to L, this new sequence will also converge to L.

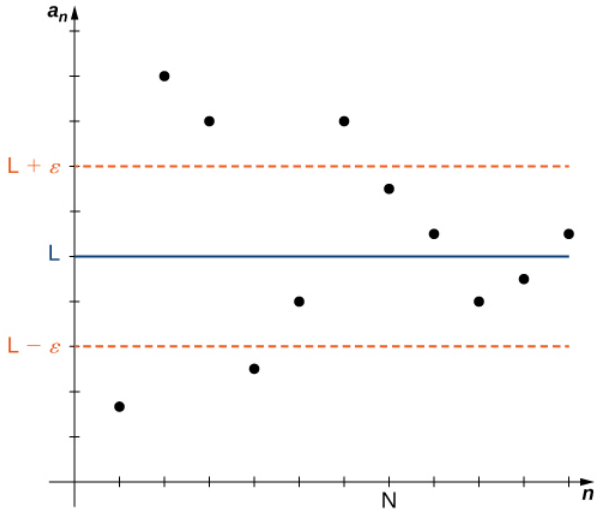

Figure 9.1.3: As n increases, the terms a_n become closer to L. For values of $n \geq N$, the distance between each point (n, a_n) and the line $y = L$ is less than ε.

As defined above, if a sequence does not converge, it is said to be a divergent sequence. For example, the sequences $\{1 + 3n\}$ and $\{(-1)^n\}$ shown in Figure 9.1.2 diverge. However, different sequences can diverge in different ways. The sequence $\{(-1)^n\}$ diverges because the terms alternate between 1 and -1, but do not approach one value as $n \to \infty$. On the other hand, the sequence $\{1 + 3n\}$ diverges because the terms $1 + 3n \to \infty$ as $n \to \infty$. We say the sequence $\{1 + 3n\}$ diverges to infinity and write $\lim_{n\to\infty}(1 + 3n) = \infty$. It is important to recognize that this notation does not imply the limit of the sequence $\{1 + 3n\}$ exists. The sequence is, in fact, divergent. Writing that the limit is infinity is intended only to provide more information about why the sequence is divergent. A sequence can also diverge to negative infinity. For example, the sequence $\{-5n + 2\}$ diverges to negative infinity because $-5n + 2 \to -\infty$ as $n \to -\infty$. We write this as $\lim_{n\to\infty}(-5n + 2) = \to -\infty$.

Because a sequence is a function whose domain is the set of positive integers, we can use properties of limits of functions to determine whether a sequence converges. For example, consider a sequence $\{a_n\}$ and a related function f defined on all positive real numbers such that $f(n) = a_n$ for all integers $n \geq 1$. Since the domain of the sequence is a subset of the domain of f, if $\lim_{x\to\infty} f(x)$ exists, then the sequence converges and has the same limit. For example, consider the sequence $\left\{\frac{1}{n}\right\}$ and the related function $f(x) = \frac{1}{x}$. Since the function f defined on all real numbers $x > 0$ satisfies $f(x) = \frac{1}{x} \to 0$ as $x \to \infty$, the sequence $\left\{\frac{1}{n}\right\}$ must satisfy $\frac{1}{n} \to 0$ as $n \to \infty$.

Limit of a Sequence Defined by a Function

Consider a sequence $\{a_n\}$ such that $a_n = f(n)$ for all $n \geq 1$. If there exists a real number L such that

$$\lim_{x\to\infty} f(x) = L,$$

then $\{a_n\}$ converges and

$$\lim_{n\to\infty} a_n = L.$$

We can use this theorem to evaluate $\lim_{n\to\infty} r^n$ for $0 \leq r \leq 1$. For example, consider the sequence $\{(1/2)^n\}$ and the related exponential function $f(x) = (1/2)^x$. Since $\lim_{x\to\infty} (1/2)^x = 0$, we conclude that the sequence $\{(1/2)^n\}$ converges and its limit is 0. Similarly, for any real number r such that $0 \leq r < 1$, $\lim_{x\to\infty} r^x = 0$, and therefore the sequence $\{r^n\}$ converges. On the other hand, if $r = 1$, then $\lim_{x\to\infty} r^x = 1$, and therefore the limit of the sequence $\{1^n\}$ is 1. If $r > 1$, $\lim_{x\to\infty} r^x = \infty$, and therefore we cannot apply this theorem. However, in this case, just as the function r^x grows without bound as $n \to \infty$, the terms r^n in the sequence become arbitrarily large as $n \to \infty$, and we conclude that the sequence $\{r^n\}$ diverges to infinity if $r > 1$.

We summarize these results regarding the geometric sequence r^n:

$$r^n \to 0 \text{ if } 0 < r < 1$$

$$r^n \to 1 \text{ if } r = 1$$

$$r^n \to \infty \text{ if } r > 1.$$

Later in this section we consider the case when $r < 0$.

We now consider slightly more complicated sequences. For example, consider the sequence $\{(2/3)^n + (1/4)^n\}$. The terms in this sequence are more complicated than other sequences we have discussed, but luckily the limit of this sequence is determined by the limits of the two sequences $\{(2/3)^n\}$ and $\{(1/4)^n\}$. As we describe in the following algebraic limit laws, since $\{(2/3)^n\}$ and $\{1/4)^n\}$ both converge to 0, the sequence $\{(2/3)^n + (1/4)^n\}$ converges to $0 + 0 = 0$. Just as we were able to evaluate a limit involving an algebraic combination of functions f and g by looking at the limits of f and g (see Introduction to Limits), we are able to evaluate the limit of a sequence whose terms are algebraic combinations of a_n and b_n by evaluating the limits of $\{a_n\}$ and $\{b_n\}$.

Algebraic Limit Laws

Given sequences $\{a_n\}$ and $\{b_n\}$ and any real number c, if there exist constants A and B such that $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then

i. $\lim_{n\to\infty} c = c$
ii. $\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n = cA$
iii. $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = A \pm B$
iv. $\lim_{n\to\infty} (a_n \cdot b_n) = \left(\lim_{n\to\infty} a_n\right) \cdot \left(\lim_{n\to\infty} b_n\right) = A \cdot B$
v. $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{A}{B}$, provided $B \neq 0$ and each $b_n \neq 0$.

Proof

We prove part iii.

Let $\epsilon > 0$. Since $\lim_{n\to\infty} a_n = A$, there exists a constant positive integer N_1 such that for all $n \geq N_1$. Since $\lim_{n\to\infty} b_n = B$, there exists a constant N_2 such that $|b_n - B| < \varepsilon/2$ for all $n \geq N_2$. Let N be the largest of N_1 and N_2. Therefore, for all $n \geq N$, $|(a_n + b_n) - (A + B)| \leq |a_n - A| + |b_n - B| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

□

The algebraic limit laws allow us to evaluate limits for many sequences. For example, consider the sequence $a_n = \frac{1}{n^2}$. As shown earlier, $\lim_{n\to\infty} \frac{1}{n} = 0$. Similarly, for any positive integer k, we can conclude that

$$\lim_{n\to\infty} \frac{1}{n^k} = 0.$$

In the next example, we make use of this fact along with the limit laws to evaluate limits for other sequences.

Example 9.1.3: Determining Convergence and Finding Limits

For each of the following sequences, determine whether or not the sequence converges. If it converges, find its limit.

a. $\left\{5 - \frac{3}{n^2}\right\}$
b. $\left\{\frac{3n^4 - 7n^2 + 5}{6 - 4n^4}\right\}$
c. $\left\{\frac{2^n}{n^2}\right\}$
d. $\left\{\left(1 + \frac{4}{n}\right)^n\right\}$

Solution

a. We know that $\lim_{n\to\infty} \frac{1}{n} = 0$. Using this fact, we conclude that

$$\lim_{n\to\infty} \frac{1}{n^2} = \lim_{n\to\infty} \frac{1}{n} . \lim_{n\to\infty} \frac{1}{n} = 0.$$

Therefore,

$$\lim_{n\to\infty} \left(5 - \frac{3}{n^2}\right) = \lim_{n\to\infty} 5 - 3 \lim_{n\to\infty} \frac{1}{n^2} = 5 - 3.0 = 5.$$

The sequence converges and its limit is 5.

b. By factoring n^4 out of the numerator and denominator and using the limit laws above, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{3n^4 - 7n^2 + 5}{6 - 4n^4} &= \lim_{n\to\infty} \frac{3 - \frac{7}{n^2} + \frac{5}{n^4}}{\frac{6}{n^4} - 4} \\
&= \frac{\lim_{n\to\infty} (3 - \frac{7}{n^2} + \frac{5}{n^4})}{\lim_{n\to\infty} (\frac{6}{n^4} - 4)} \\
&= \frac{\lim_{n\to\infty} (3) - \lim_{n\to\infty} \frac{7}{n^2} + \lim_{n\to\infty} \frac{5}{n^4}}{\lim_{n\to\infty} \frac{6}{n^4} - \lim_{n\to\infty} (4)} \\
&= \frac{\lim_{n\to\infty} (3) - 7 \cdot \lim_{n\to\infty} \frac{1}{n^2} + 5 \cdot \lim_{n\to\infty} \frac{1}{n^4}}{6 \cdot \lim_{n\to\infty} \frac{1}{n^4} - \lim_{n\to\infty} (4)} \\
&= \frac{3 - 7 \cdot 0 + 5 \cdot 0}{6 \cdot 0 - 4} = -\frac{3}{4}.
\end{aligned}$$

The sequence converges and its limit is $-3/4$.

c. Consider the related function $f(x) = 2^x/x^2$ defined on all real numbers $x > 0$. Since $2^x \to \infty$ and $x^2 \to \infty$ as $x \to \infty$, apply L'Hôpital's rule and write

$$\begin{aligned}
\lim_{x\to\infty} \frac{2^x}{x^2} &= \lim_{x\to\infty} \frac{2^x \ln 2}{2x} && \text{Take the derivatives of the numerator and denominator.} \\
&= \lim_{x\to\infty} \frac{2^x (\ln 2)^2}{2} && \text{Take the derivatives again.} \\
&= \infty.
\end{aligned}$$

We conclude that the sequence diverges.

d. Consider the function $f(x) = \left(1 + \frac{4}{x}\right)^x$ defined on all real numbers $x > 0$. This function has the indeterminate form 1^∞ as $x \to \infty$. Let

$$y = \lim_{x\to\infty} \left(1 + \frac{4}{x}\right)^x.$$

Now taking the natural logarithm of both sides of the equation, we obtain

$$\ln(y) = \ln\left[\lim_{x\to\infty} \left(1 + \frac{4}{x}\right)^x\right].$$

Since the function $f(x) = \ln(x)$ is continuous on its domain, we can interchange the limit and the natural logarithm. Therefore,

$$\ln(y) = \lim_{x\to\infty} \left[\ln\left(1 + \frac{4}{x}\right)^x\right].$$

Using properties of logarithms, we write

$$\lim_{x\to\infty} \left[\ln\left(1 + \frac{4}{x}\right)^x\right] = \lim_{x\to\infty} x \ln\left(1 + \frac{4}{x}\right).$$

Since the right-hand side of this equation has the indeterminate form $\infty \cdot 0$, rewrite it as a fraction to apply L'Hôpital's rule. Write

$$\lim_{x\to\infty} x \ln\left(1 + \frac{4}{x}\right) = \lim_{x\to\infty} \frac{\ln(1 + 4/x)}{1/x}.$$

Since the right-hand side is now in the indeterminate form 0/0, we are able to apply L'Hôpital's rule. We conclude that

$$\lim_{x\to\infty} \frac{\ln(1 + 4/x)}{1/x} = \lim_{x\to\infty} \frac{4}{1 + 4/x} = 4.$$

Therefore, $\ln(y) = 4$ and $y = e^4$. Therefore, since $\lim_{x\to\infty} \left(1 + \frac{4}{x}\right)^x = e^4$, we can conclude that the sequence $\left\{\left(1 + \frac{4}{n}\right)^n\right\}$ converges to e^4.

? Exercise 9.1.3

Consider the sequence $\{(5n^2 + 1)/e^n\}$. Determine whether or not the sequence converges. If it converges, find its limit.

Hint

Use L'Hôpital's rule.

Answer

The sequence converges, and its limit is 0

Recall that if f is a continuous function at a value L, then $f(x) \to f(L)$ as $x \to L$. This idea applies to sequences as well. Suppose a sequence $a_n \to L$, and a function f is continuous at L. Then $f(a_n) \to f(L)$. This property often enables us to find limits for complicated sequences. For example, consider the sequence $\sqrt{5 - \frac{3}{n^2}}$. From Example a. we know the sequence $5 - \frac{3}{n^2} \to 5$. Since $\sqrt{x}$ is a continuous function at $x = 5$,

$$\lim_{n\to\infty} \sqrt{5 - \frac{3}{n^2}} = \sqrt{\lim_{n\to\infty} (5 - \frac{3}{n^2})} = \sqrt{5}.$$

Continuous Functions Defined on Convergent Sequences

Consider a sequence $\{a_n\}$ and suppose there exists a real number L such that the sequence $\{a_n\}$ converges to L. Suppose f is a continuous function at L. Then there exists an integer N such that f is defined at all values an for $n \geq N$, and the sequence $\{f(a_n)\}$ converges to $f(L)$ (Figure 9.1.4).

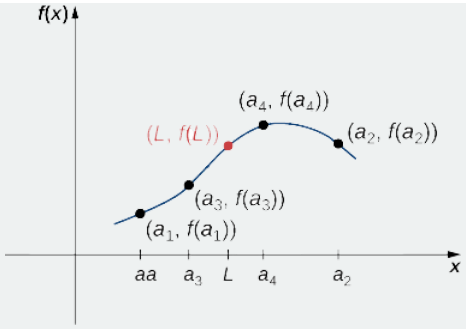

Figure 9.1.4: Because f is a continuous function as the inputs $a_1, a_2, a_3, \ldots$ approach L, the outputs $f(a_1), f(a_2), f(a_3), \ldots$ approach $f(L)$.

Proof

Let $\epsilon > 0$. Since f is continuous at L, there exists $\delta > 0$ such that $|f(x) - f(L)| < \varepsilon$ if $|x - L| < \delta$. Since the sequence $\{a_n\}$ converges to L, there exists N such that $|a_n - L| < \delta$ for all $n \geq N$. Therefore, for all $n \geq N$, $|a_n - L| < \delta$, which implies $|f(a_n) - f(L)| < \varepsilon$. We conclude that the sequence $\{f(a_n)\}$ converges to $f(L)$.

□

Example 9.1.4: Limits Involving Continuous Functions Defined on Convergent Sequences

Determine whether the sequence $\{\cos(3/n^2)\}$ converges. If it converges, find its limit.

Solution:

Since the sequence $\{3/n^2\}$ converges to 0 and $\cos x$ is continuous at $x = 0$, we can conclude that the sequence $\{\cos(3/n^2)\}$ converges and

$$\lim_{n\to\infty} \cos\left(\frac{3}{n^2}\right) = \cos 0 = 1.$$

Exercise 9.1.4

Determine if the sequence $\left\{\sqrt{\frac{2n+1}{3n+5}}\right\}$ converges. If it converges, find its limit.

Hint

Consider the sequence $\left\{\frac{2n+1}{3n+5}\right\}$.

Answer

The sequence converges, and its limit is $\sqrt{2/3}$.

Another theorem involving limits of sequences is an extension of the Squeeze Theorem for limits discussed in Introduction to Limits.

Squeeze Theorem for Sequences

Consider sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$. Suppose there exists an integer N such that

$$a_n \leq b_n \leq c_n \text{ for all } n \geq N.$$

If there exists a real number L such that

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n,$$

then $\{b_n\}$ converges and $\lim_{n\to\infty} b_n = L$ (Figure 9.1.5).

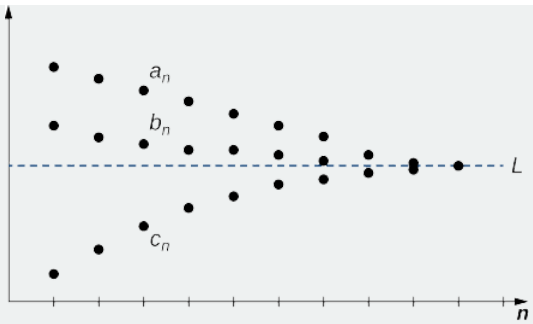

Figure 9.1.5: Each term bn satisfies $a_n \le b_n \le c_n$ and the sequences $\{a_n\}$ and $\{c_n\}$ converge to the same limit, so the sequence $\{b_n\}$ must converge to the same limit as well.

Proof

Let $\varepsilon > 0$. Since the sequence $\{a_n\}$ converges to L, there exists an integer N_1 such that $|a_n - L| < \varepsilon$ for all $n \ge N_1$. Similarly, since $\{c_n\}$ converges to L, there exists an integer N_2 such that $|c_n - L| < \varepsilon$ for all $n \ge N_2$. By assumption, there exists an integer N such that $a_n \le b_n \le c_n$ for all $n \ge N$. Let M be the largest of N_1, N_2, and N. We must show that $|b_n - L| < \varepsilon$ for all $n \ge M$. For all $n \ge M$,

$$-\varepsilon < -|a_n - L| \le a_n - L \le b_n - L \le c_n - L \le |c_n - L| < \varepsilon$$

Therefore, $-\varepsilon < b_n - L < \varepsilon$, and we conclude that $|b_n - L| < \varepsilon$ for all $n \ge M$, and we conclude that the sequence b_n converges to L.

□

Example 9.1.5: Using the Squeeze Theorem

Use the Squeeze Theorem to find the limit of each of the following sequences.

a. $\left\{\frac{\cos n}{n^2}\right\}$

b. $\left\{\left(-\frac{1}{2}\right)^n\right\}$

Solution

a. Since $-1 \le \cos n \le 1$ for all integers n, we have

$$-\frac{1}{n^2} \le \frac{\cos n}{n^2} \le \frac{1}{n^2}.$$

Since $-1/n^2 \to 0$ and $1/n^2 \to 0$, we conclude that $\cos n/n^2 \to 0$ as well.

b. Since

$$-\frac{1}{2^n} \le \left(-\frac{1}{2}\right)^n \le \frac{1}{2^n}$$

for all positive integers n, $-1/2^n \to 0$ and $1/2^n \to 0$, we can conclude that $(-1/2)^n \to 0$.

Exercise 9.1.5

Find $\lim_{n\to\infty} \frac{2n - \sin n}{n}$.

Hint

Use the fact that $-1 \le \sin n \le 1$.

Answer

2

Using the idea from Example $9.1.5$ b we conclude that $r^n \to 0$ for any real number r such that $-1 < r < 0$. If $r < -1$, the sequence r^n diverges because the terms oscillate and become arbitrarily large in magnitude. If $r = -1$, the sequence $r^n = (-1)^n$ diverges, as discussed earlier. Here is a summary of the properties for geometric sequences.

$$r^n \to 0 \text{ if } |r| < 1$$

$$r^n \to 1 \text{ if } r = 1$$

$$r^n \to \infty \text{ if } r > 1$$

$$\{r^n\} \text{ diverges if } r \leq -1$$

Bounded Sequences

We now turn our attention to one of the most important theorems involving sequences: the Monotone Convergence Theorem. Before stating the theorem, we need to introduce some terminology and motivation. We begin by defining what it means for a sequence to be bounded.

Definition: Bound Sequences

A sequence $\{a_n\}$ is **bounded above** if there exists a real number M such that

$$a_n \leq M$$

for all positive integers n.

A sequence $\{a_n\}$ is **bounded below** if there exists a real number m such that

$$m \leq a_n$$

for all positive integers n.

A sequence $\{a_n\}$ is a **bounded sequence** if it is bounded above and bounded below.

If a sequence is not bounded, it is an **unbounded sequence.**

For example, the sequence $\{1/n\}$ is bounded above because $1/n \leq 1$ for all positive integers n. It is also bounded below because $1/n \geq 0$ for all positive integers n. Therefore, $\{1/n\}$ is a bounded sequence. On the other hand, consider the sequence $\{2^n\}$. Because $2^n \geq 2$ for all $n \geq 1$, the sequence is bounded below. However, the sequence is not bounded above. Therefore, $\{2^n\}$ is an unbounded sequence.

We now discuss the relationship between boundedness and convergence. Suppose a sequence $\{a_n\}$ is unbounded. Then it is not bounded above, or not bounded below, or both. In either case, there are terms an that are arbitrarily large in magnitude as n gets larger. As a result, the sequence $\{a_n\}$ cannot converge. Therefore, being bounded is a necessary condition for a sequence to converge.

Convergent Sequences Are Bounded

If a sequence $\{a_n\}$ converges, then it is bounded.

Note that a sequence being bounded is not a sufficient condition for a sequence to converge. For example, the sequence $\{(-1)^n\}$ is bounded, but the sequence diverges because the sequence oscillates between 1 and -1 and never approaches a finite number. We now discuss a sufficient (but not necessary) condition for a bounded sequence to converge.

Consider a bounded sequence $\{a_n\}$. Suppose the sequence $\{a_n\}$ is increasing. That is, $a_1 \leq a_2 \leq a_3 \ldots$. Since the sequence is increasing, the terms are not oscillating. Therefore, there are two possibilities. The sequence could diverge to infinity, or it could converge. However, since the sequence is bounded, it is bounded above and the sequence cannot diverge to infinity. We conclude that $\{a_n\}$ converges. For example, consider the sequence

$$\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}.$$

Since this sequence is increasing and bounded above, it converges. Next, consider the sequence

$$\left\{2,\ 0,\ 3,\ 0,\ 4,\ 0,\ 1,\ -\frac{}{2},\ -\frac{}{3},\ -\frac{}{4},\ \ldots\right\}.$$

Even though the sequence is not increasing for all values of n, we see that $-1/2 < -1/3 < -1/4 < \cdots$. Therefore, starting with the eighth term, $a_8 = -1/2$, the sequence is increasing. In this case, we say the sequence is eventually increasing. Since the sequence is bounded above, it converges. It is also true that if a sequence is decreasing (or eventually decreasing) and bounded below, it also converges.

Definition

A sequence $\{a_n\}$ is increasing for all $n \geq n_0$ if

$$a_n \leq a_{n+1} \text{ for all } n \geq n_0.$$

A sequence $\{a_n\}$ is decreasing for all $n \geq n_0$ if

$$a_n \geq a_{n+1} \text{ for all } n \geq n_0.$$

A sequence $\{a_n\}$ is a *monotone sequence* for all $n \geq n_0$ if it is increasing for all $n \geq n_0$ or decreasing for all $n \geq n_0$.

We now have the necessary definitions to state the Monotone Convergence Theorem, which gives a sufficient condition for convergence of a sequence.

Definition: Monotone Convergence Theorem

If $\{a_n\}$ is a bounded sequence and there exists a positive integer n_0 such that $\{a_n\}$ is monotone for all $n \geq n_0$, then $\{a_n\}$ converges.

The proof of this theorem is beyond the scope of this text. Instead, we provide a graph to show intuitively why this theorem makes sense (Figure 9.1.6).

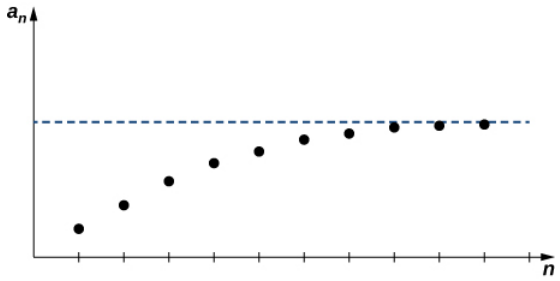

Figure 9.1.6: Since the sequence $\{a_n\}$ is increasing and bounded above, it must converge.

In the following example, we show how the Monotone Convergence Theorem can be used to prove convergence of a sequence.

Example 9.1.6: Using the Monotone Convergence Theorem

For each of the following sequences, use the Monotone Convergence Theorem to show the sequence converges and find its limit.

a. $\left\{\frac{4^n}{n!}\right\}$

b. $\{a_n\}$ defined recursively such that

$$a_1 = 2 \text{ and } a_{n+1} = \frac{a_n}{2} + \frac{1}{2a_n} \text{ for all } n \geq 2.$$

Solution

a. Writing out the first few terms, we see that

$$\left\{\frac{4^n}{n!}\right\} = \left\{4,\ 8,\ \frac{32}{3},\ \frac{32}{3},\ \frac{128}{15},\ \ldots\right\}.$$

At first, the terms increase. However, after the third term, the terms decrease. In fact, the terms decrease for all $n \geq 3$. We can show this as follows.

$$a_{n+1} = \frac{4^{n+1}}{(n+1)!} = \frac{4}{n+1} \cdot \frac{4^n}{n!} = \frac{4}{n+1} \cdot a_n \leq a_n \text{ if } n \geq 3.$$

Therefore, the sequence is decreasing for all $n \geq 3$. Further, the sequence is bounded below by 0 because $4n/n! \geq 0$ for all positive integers n. Therefore, by the Monotone Convergence Theorem, the sequence converges.

To find the limit, we use the fact that the sequence converges and let $L = \lim_{n\to\infty} a_n$. Now note this important observation. Consider $\lim_{n\to\infty} a_{n+1}$. Since

$$\{a_{n+1}\} = \{a_2, a_3, a_4, ...\},$$

the only difference between the sequences $\{a_{n+1}\}$ and $\{a_n\}$ is that $\{a_{n+1}\}$ omits the first term. Since a finite number of terms does not affect the convergence of a sequence,

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L.$$

Combining this fact with the equation

$$a_{n+1} = \frac{4}{n+1} a_n$$

and taking the limit of both sides of the equation

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{4}{n+1} a_n,$$

we can conclude that

$$L = 0 \cdot L = 0.$$

b. Writing out the first several terms,

$$\left\{2, \frac{5}{4}, \frac{41}{40}, \frac{3281}{3280}, ...\right\}.$$

we can conjecture that the sequence is decreasing and bounded below by 1. To show that the sequence is bounded below by 1, we can show that

$$\frac{a_n}{2} + \frac{1}{2a_n} \geq 1.$$

To show this, first rewrite

$$\frac{a_n}{2} + \frac{1}{2a_n} = \frac{a_n^2 + 1}{2a_n}.$$

Since $a_1 > 0$ and a_2 is defined as a sum of positive terms, $a_2 > 0$. Similarly, all terms $a_n > 0$. Therefore,

$$\frac{a_n^2 + 1}{2a_n} \geq 1$$

if and only if

$$a_n^2 + 1 \geq 2a_n.$$

Rewriting the inequality $a_n^2 + 1 \geq 2a_n$ as $a_n^2 - 2a_n + 1 \geq 0$, and using the fact that

$$a_n^2 - 2a_n + 1 = (a_n - 1)^2 \geq 0$$

because the square of any real number is nonnegative, we can conclude that

$$\frac{a^n}{2} + \frac{1}{2a_n} \geq 1.$$

To show that the sequence is decreasing, we must show that $a_{n+1} \leq a_n$ for all $n \geq 1$. Since $1 \leq a_n^2$, it follows that

$$a_n^2 + 1 \leq 2a_n^2.$$

Dividing both sides by $2a_n$, we obtain

$$\frac{a_n}{2} + \frac{1}{2a_n} \leq a_n.$$

Using the definition of a_{n+1}, we conclude that

$$a_{n+1} = \frac{a_n}{2} + \frac{1}{2a_n} \leq a_n.$$

Since $\{a_n\}$ is bounded below and decreasing, by the Monotone Convergence Theorem, it converges.

To find the limit, let $L = \lim_{n\to\infty} a_n$. Then using the recurrence relation and the fact that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$, we have

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \left(\frac{a_n}{2} + \frac{1}{2a_n}\right),$$

and therefore

$$L = \frac{L}{2} + \frac{1}{2L}.$$

Multiplying both sides of this equation by $2L$, we arrive at the equation

$$2L^2 = L^2 + 1.$$

Solving this equation for L, we conclude that $L^2 = 1$, which implies $L = \pm 1$. Since all the terms are positive, the limit $L = 1$.

Exercise 9.1.6

Consider the sequence $\{a_n\}$ defined recursively such that $a_1 = 1$, $a_n = \frac{a_{n-1}}{2}$. Use the Monotone Convergence Theorem to show that this sequence converges and find its limit.

Hint

Show the sequence is decreasing and bounded below.

Answer

0.

Definition: Fibonacci Numbers

The *Fibonacci numbers* are defined recursively by the sequence $\{F_n\}$ where $F_0 = 0$, $F_1 = 1$ and for $n \geq 2$,

$$F_n = F_{n-1} + F_{n-2}.$$

Here we look at properties of the Fibonacci numbers.

1. Write out the first twenty Fibonacci numbers.

2. Find a closed formula for the Fibonacci sequence by using the following steps.

   a. Consider the recursively defined sequence x_n where $x_0 = c$ and $x_{n+1} = ax_n$. Show that this sequence can be described by the closed formula $x_n = ca^n$ for all $n \geq 0$.

   b. Using the result from part a. as motivation, look for a solution of the equation

   $F_n = F_{n-1} + F_{n-2}$

   of the form $F_n = c\lambda^n$. Determine what two values for λ will allow F_n to satisfy this equation.

   c. Consider the two solutions from part b.: λ_1 and λ_2. Let $F_n = c_1\lambda_1^n + c_2\lambda_2^n$. Use the initial conditions F_0 and F_1 to determine the values for the constants c_1 and c_2 and write the closed formula F_n.

3. Use the answer in 2 c. to show that

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}.$$

The number $\phi = (1+\sqrt{5})/2$ is known as the **golden ratio** (Figure 9.1.7 and Figure 9.1.8).

Figure 9.1.7: The seeds in a sunflower exhibit spiral patterns curving to the left and to the right. The number of spirals in each direction is always a Fibonacci number—always. (credit: modification of work by Esdras Calderan, Wikimedia Commons)

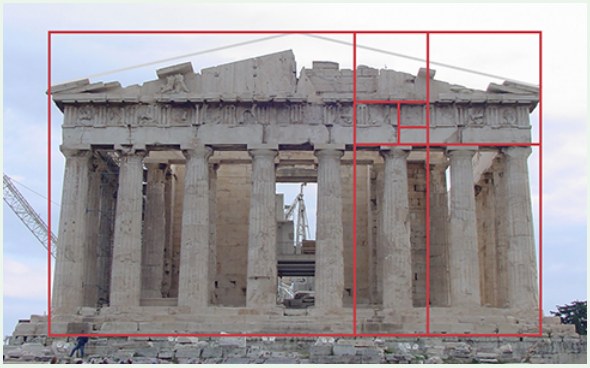

Figure 9.1.8: The proportion of the golden ratio appears in many famous examples of art and architecture. The ancient Greek temple known as the Parthenon was designed with these proportions, and the ratio appears again in many of the smaller details. (credit: modification of work by TravelingOtter, Flickr).

Key Concepts

- To determine the convergence of a sequence given by an explicit formula $a_n = f(n)$, we use the properties of limits for functions.
- If $\{a_n\}$ and $\{b_n\}$ are convergent sequences that converge to A and B, respectively, and c is any real number, then the sequence $\{ca_n\}$converges to $c \cdot A$, the sequences $\{a_n \pm b_n\}$ converge to $A \pm B$, the sequence $\{a_n \cdot b_n\}$ converges to $A \cdot B$, and the sequence $\{a_n/b_n\}$ converges to A/B, provided $B \neq 0$.
- If a sequence is bounded and monotone, then it converges, but not all convergent sequences are monotone.
- If a sequence is unbounded, it diverges, but not all divergent sequences are unbounded.
- The geometric sequence $\{r^n\}$ converges if and only if $|r| < 1$ or $r = 1$.

Glossary

arithmetic sequence

a sequence in which the difference between every pair of consecutive terms is the same is called an arithmetic sequence

bounded above

a sequence $\{a_n\}$ is bounded above if there exists a constant M such that $a_n \leq M$ for all positive integers n

bounded below

a sequence $\{a_n\}$ is bounded below if there exists a constant M such that $M \leq a_n$ for all positive integers n

bounded sequence

a sequence $\{a_n\}$ is bounded if there exists a constant M such that $|a_n| \leq M$ for all positive integers n

convergent sequence

a convergent sequence is a sequence $\{a_n\}$ for which there exists a real number L such that a_n is arbitrarily close to L as long as n is sufficiently large

divergent sequence

a sequence that is not convergent is divergent

explicit formula

a sequence may be defined by an explicit formula such that $a_n = f(n)$

geometric sequence

a sequence $\{a_n\}$ in which the ratio a_{n+1}/a_n is the same for all positive integers n is called a geometric sequence

index variable

the subscript used to define the terms in a sequence is called the index

limit of a sequence

the real number L to which a sequence converges is called the limit of the sequence

monotone sequence

an increasing or decreasing sequence

recurrence relation

a recurrence relation is a relationship in which a term a_n in a sequence is defined in terms of earlier terms in the sequence

sequence

an ordered list of numbers of the form $a_1, a_2, a_3, \ldots$ is a sequence

term

the number a_n in the sequence $\{a_n\}$ is called the nth term of the sequence

unbounded sequence

a sequence that is not bounded is called unbounded

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9.1E: Exercises for Section 9.1

In exercises 1 - 4, find the first six terms of each sequence, starting with $n = 1$.

1) $a_n = 1 + (-1)^n$ for $n \geq 1$

Answer

$a_n = 0$ if n is odd and $a_n = 2$ if n is even

2) $a_n = n^2 - 1$ for $n \geq 1$

3) $a_1 = 1$ and $a_n = a_{n-1} + n$ for $n \geq 2$

Answer

$a_n = 1, 3, 6, 10, 15, 21, \ldots$

4) $a_1 = 1, a_2 = 1$ and $a_n + 2 = a_n + a_{n+1}$ for $n \geq 1$

5) Find an explicit formula for a_n where $a_1 = 1$ and $a_n = a_{n-1} + n$ for $n \geq 2$.

Answer

$$a_n = \frac{n(n+1)}{2}$$

6) Find a formula a_n for the nth term of the arithmetic sequence whose first term is $a_1 = 1$ such that $a_{n+1} - a_n = 17$ for $n \geq 1$.

7) Find a formula a_n for the nth term of the arithmetic sequence whose first term is $a_1 = -3$ such that $a_{n+1} - a_n = 4$ for $n \geq 1$.

Answer

$a_n = 4n - 7$

8) Find a formula a_n for the nth term of the geometric sequence whose first term is $a_1 = 1$ such that $\frac{a_{n+1}}{a_n} = 10$ for $n \geq 1$.

9) Find a formula a_n for the nth term of the geometric sequence whose first term is $a_1 = 3$ such that $\frac{a_{n+1}}{a_n} = 1/10$ for $n \geq 1$.

Answer

$a_n = 3.10^{1-n} = 30.10^{-n}$

10) Find an explicit formula for the nth term of the sequence whose first several terms are $0, 3, 8, 15, 24, 35, 48, 63, 80, 99, \ldots$. (Hint: First add one to each term.)

11) Find an explicit formula for the nth term of the sequence satisfying $a_1 = 0$ and $a_n = 2a_{n-1} + 1$ for $n \geq 2$.

Answer

$a_n = 2^{n-1} - 1$

In exercises 12 and 13, find a formula for the general term a_n of each of the following sequences.

12) $1, 0, -1, 0, 1, 0, -1, 0, \ldots$ (Hint: Find where $\sin x$ takes these values)

13) $1, -1/3, 1/5, -1/7, \ldots$

Answer

$$a_n = \frac{(-1)^{n-1}}{2n-1}$$

In exercises 14-18, find a function $f(n)$ that identifies the nth term a_n of the following recursively defined sequences, as $a_n = f(n)$.

14) $a_1 = 1$ and $a_{n+1} = -a_n$ for $n \geq 1$

15) $a_1 = 2$ and $a_{n+1} = 2a_n$ for $n \geq 1$

Answer

$f(n) = 2^n$

16) $a_1 = 1$ and $a_{n+1} = (n+1)a_n$ for $n \geq 1$

17) $a_1 = 2$ and $a_{n+1} = (n+1)a_n/2$ for $n \geq 1$

Answer

$f(n) = \dfrac{n!}{2^{n-2}}$

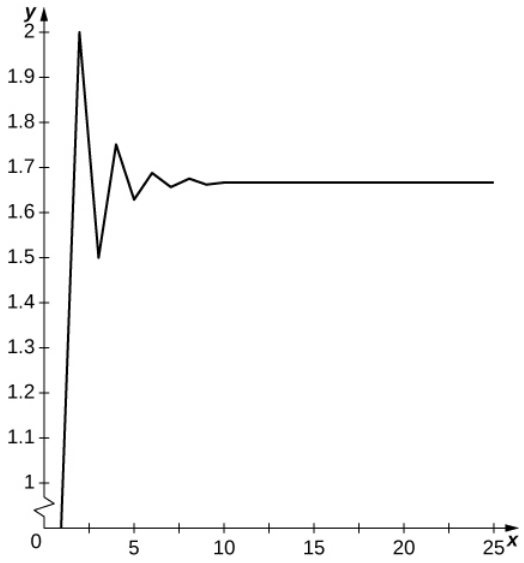
18) $a_1 = 1$ and $a_{n+1} = a_n/2^n$ for $n \geq 1$

In exercises 19 - 22, plot the first N terms of the given sequence. State whether the graphical evidence suggests that the sequence converges or diverges.

19) [T] $a_1 = 1, a_2 = 2$, and for $n \geq 2, a_n = \frac{1}{2}(a_{n-1} + a_{n-2})$; $N = 30$

Answer

Terms oscillate above and below $5/3$ and appear to converge to $5/3$.

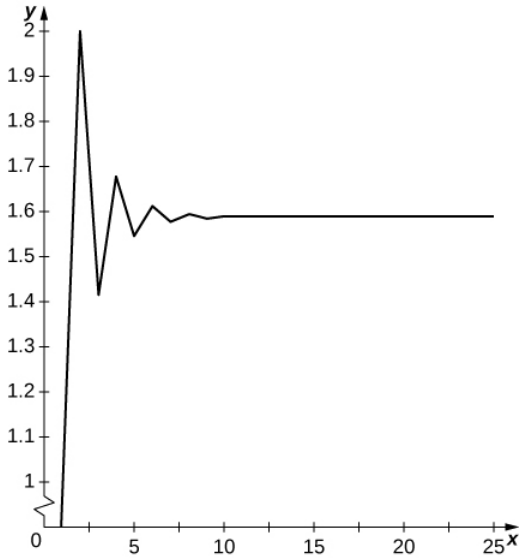


20) [T] $a_1 = 1, a_2 = 2, a_3 = 3$ and for $n \geq 4, a_n = \frac{1}{3}(a_{n-1} + a_{n-2} + a_{n-3}), N = 30$

21) [T] $a_1 = 1, a_2 = 2$, and for $n \geq 3, a_n = \sqrt{a_{n-1}a_{n-2}}; N = 30$

Answer

Terms oscillate above and below $y \approx 1.57..$ and appear to converge to a limit.



22) [T] $a_1 = 1, a_2 = 2, a_3 = 3$, and for $n \geq 4, a_n = \sqrt{a_{n-1}a_{n-2}a_{n-3}}; N = 30$

In exercises 23 - 16, suppose that $\lim_{n\to\infty} a_n = 1$, $\lim_{n\to\infty} b_n = -1$, and $0 < -b_n < a_n$ for all n.

Using this information, evaluate each of the following limits, state that the limit does not exist, or state that there is not enough information to determine whether the limit exists.

23) $\lim_{n\to\infty} 3a_n - 4b_n$

Answer

$\lim_{n\to\infty} 3a_n - 4b_n \quad = \quad 7$

24) $\lim_{n\to\infty} \frac{1}{2}b_n - \frac{1}{2}a_n$

25) $\lim_{n\to\infty} \frac{a_n + b_n}{a_n - b_n}$

Answer

$\lim_{n\to\infty} \frac{a_n + b_n}{a_n - b_n} \quad = \quad 0$

26) $\lim_{n\to\infty} \frac{a_n - b_n}{a_n + b_n}$

In exercises 27 - 30, find the limit of each of the following sequences, using L'Hôpital's rule when appropriate.

27) $\frac{n^2}{2^n}$

Answer

$\lim_{n\to\infty} \frac{n^2}{2^n} \quad = \quad 0$

28) $\frac{(n-1)^2}{(n+1)^2}$

29) $\frac{\sqrt{n}}{\sqrt{n+1}}$

Answer

$\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} \quad = \quad 1$

30) $n^{1/n}$ (Hint: $n^{1/n} = e^{\frac{1}{n}\ln n}$)

In exercises 31 - 37, state whether each sequence is bounded and whether it is eventually monotone, increasing, or decreasing.

31) $n/2^n, n \geq 2$

Answer

bounded, decreasing for $n \geq 1$

32) $\ln\left(1+\frac{1}{n}\right)$

33) $\sin n$

Answer

bounded, not monotone

34) $\cos(n^2)$

35) $n^{1/n}, \quad n \geq 3$

Answer

bounded, decreasing

36) $n^{-1/n}, \quad n \geq 3$

37) $\tan n$

Answer

not monotone, not bounded

In exercises 38 - 39, determine whether the given sequence has a limit. If it does, find the limit.

38) $a_1 = \sqrt{2}, a_2 = \sqrt{2\sqrt{2}}. a_3 = \sqrt{2\sqrt{2\sqrt{2}}}$ etc.

39) $a_1 = 3, a_n = \sqrt{2a_{n-1}}, n = 2, 3, \ldots.$

Answer

a_n is decreasing and bounded below by 2. The limit a must satisfy $a = \sqrt{2a}$ so $a = 2$, independent of the initial value.

Use the Squeeze Theorem to find the limit of each sequence in exercises 40 - 43.

40) $n \sin(1/n)$

41) $\frac{\cos(1/n) - 1}{1/n}$

Answer

0

42) $a_n = \frac{n!}{n^n}$

43) $a_n = \sin n \sin(1/n)$

Answer

0 since $|\sin x| \leq |x|$ and $|\sin x| \leq 1$ so $-\frac{1}{n} \leq a_n \leq \frac{1}{n}$).

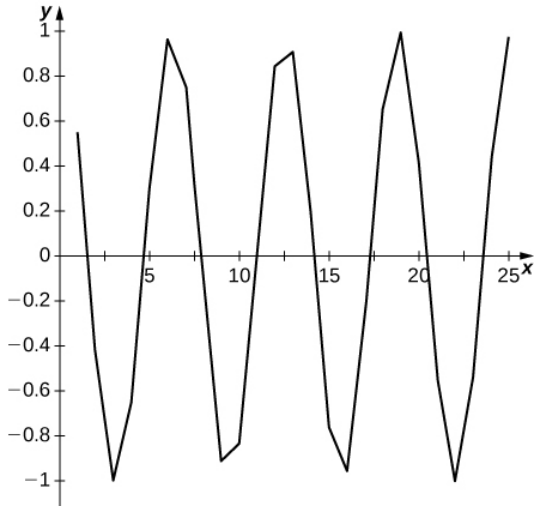
For the sequences in exercises 44 and 45, plot the first 25 terms of the sequence and state whether the graphical evidence suggests that the sequence converges or diverges.

44) [T] $a_n = \sin n$

45) [T] $a_n = \cos n$

Answer

Graph oscillates and suggests no limit.



In exercises 46 - 52, determine the limit of the sequence or show that the sequence diverges. If it converges, find its limit.

46) $a_n = \tan^{-1}(n^2)$

47) $a_n = (2n)^{1/n} - n^{1/n}$

Answer

$n^{1/n} \to 1$ and $2^{1/n} \to 1$, so $a_n \to 0$

48) $a_n = \dfrac{\ln(n^2)}{\ln(2n)}$

49) $a_n = \left(1 - \frac{2}{n}\right)^n$

Answer

Since $(1 + 1/n)^n \to e$, one has $(1 - 2/n)^n \approx (1 + k)^{-2k} \to e^{-2}$ as $k \to \infty$.

50) $a_n = \ln\left(\dfrac{n+2}{n^2-3}\right)$

51) $a_n = \dfrac{2^n + 3^n}{4^n}$

Answer

$2^n + 3^n \le 2 \cdot 3^n$ and $3^n/4^n \to 0$ as $n \to \infty$, so $a_n \to 0$ as $n \to \infty$.

52) $a_n = \dfrac{(1000)^n}{n!}$

53) $a_n = \dfrac{(n!)^2}{(2n)!}$

Answer

$\dfrac{a_{n+1}}{a_n} = n!/(n+1)(n+2)\cdots(2n) = \dfrac{1 \cdot 2 \cdot 3 \cdots n}{(n+1)(n+2)\cdots(2n)} < 1/2^n$. In particular, $a_{n+1}/a_n \le 1/2$, so $a_n \to 0$ as $n \to \infty$.

Newton's method seeks to approximate a solution $f(x) = 0$ that starts with an initial approximation x_0 and successively defines a sequence $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$. For the given choice of f and x_0, write out the formula for x_{n+1}. If the sequence appears to converge, give an exact formula for the solution x, then identify the limit x accurate to four decimal places and the smallest n such that x_n agrees with x up to four decimal places.

54) [T] $f(x) = x^2 - 2, \quad x_0 = 1$

55) [T] $f(x) = (x-1)^2 - 2, \quad x_0 = 2$

Answer

$x_{n+1} = x_n - ((x_n-1)^2 - 2)/2(x_n - 1);\ x = 1 + \sqrt{2},\ x \approx 2.4142,\ n = 5$

56) [T] $f(x) = e^x - 2, \quad x_0 = 1$

57) [T] $f(x) = \ln x - 1, \quad x_0 = 2$

Answer

$x_{n+1} = x_n - x_n(\ln(x_n) - 1);\ x = e,\ x \approx 2.7183,\ n = 5$

58) [T] Suppose you start with one liter of vinegar and repeatedly remove 0.1 L, replace with water, mix, and repeat.

a. Find a formula for the concentration after n steps.

b. After how many steps does the mixture contain less than 10% vinegar?

59) [T] A lake initially contains 2000 fish. Suppose that in the absence of predators or other causes of removal, the fish population increases by 6% each month. However, factoring in all causes, 150 fish are lost each month.

a. Explain why the fish population after n months is modeled by $P_n = 1.06P_{n-1} - 150$ with $P_0 = 2000$.

b. How many fish will be in the pond after one year?

Answer

a. Without losses, the population would obey $P_n = 1.06P_{n-1}$. The subtraction of 150 accounts for fish losses.
b. After 12 months, we have $P_{12} \approx 1494$.

60) [T] A bank account earns 5% interest compounded monthly. Suppose that $\$1000$ is initially deposited into the account, but that $\$10$ is withdrawn each month.

a. Show that the amount in the account after n months is $A_n = (1 + .05/12)A_{n-1} - 10;\ A_0 = 1000.$

b. How much money will be in the account after 1 year?

c. Is the amount increasing or decreasing?

d. Suppose that instead of $\$10$, a fixed amount d dollars is withdrawn each month. Find a value of d such that the amount in the account after each month remains $\$1000$.

e. What happens if d is greater than this amount?

61) [T] A student takes out a college loan of $\$10,000$ at an annual percentage rate of 6%, compounded monthly.

a. If the student makes payments of $\$100$ per month, how much does the student owe after 12 months?

b. After how many months will the loan be paid off?

Answer

a. The student owes $\$9383$ after 12 months.
b. The loan will be paid in full after 139 months or eleven and a half years.

62) [T] Consider a series combining geometric growth and arithmetic decrease. Let $a_1 = 1$. Fix $a > 1$ and $0 < b < a$. Set $a_{n+1} = a.a_n - b$. Find a formula for a_{n+1} in terms of a_n, a, and b and a relationship between a and b such that a_n converges.

63) [T] The binary representation $x = 0.b_1b_2b_3...$ of a number x between 0 and 1 can be defined as follows. Let $b_1 = 0$ if $x < 1/2$ and $b_1 = 1$ if $1/2 \leq x < 1$. Let $x_1 = 2x - b_1$. Let $b_2 = 0$ if $x_1 < 1/2$ and $b_2 = 1$ if $1/2 \leq x < 1$. Let $x_2 = 2x_1 - b_2$ and in general, $x_n = 2x_{n-1} - b_n$ and $b_n{-}1 = 0$ if $x_n < 1/2$ and $b_{n-1} = 1$ if $1/2 \leq x_n < 1$. Find the binary expansion of $1/3$.

Answer

$b_1 = 0, x_1 = 2/3, b_2 = 1, x_2 = 4/3 - 1 = 1/3$, so the pattern repeats, and $1/3 = 0.010101\ldots$.

64) [T] To find an approximation for π, set $a_0 = \sqrt{2+1}, a_1 = \sqrt{2+a_0}$, and, in general, $a_{n+1} = \sqrt{2+a_n}$. Finally, set $p_n = 3.2^n\sqrt{2-a_n}$. Find the first ten terms of p_n and compare the values to π.

For the following two exercises, assume that you have access to a computer program or Internet source that can generate a list of zeros and ones of any desired length. Pseudo-random number generators (PRNGs) play an important role in simulating random noise in physical systems by creating sequences of zeros and ones that appear like the result of flipping a coin repeatedly. One of the simplest types of PRNGs recursively defines a random-looking sequence of N integers $a_1, a_2, \ldots, a_N$ by fixing two special integers (K and M and letting a_{n+1} be the remainder after dividing $K.a_n$ into M, then creates a bit sequence of zeros and ones whose n^{th} term b_n is equal to one if a_n is odd and equal to zero if a_n is even. If the bits b_n are pseudo-random, then the behavior of their average $(b_1 + b_2 + \cdots + b_N)/N$ should be similar to behavior of averages of truly randomly generated bits.

65) [T] Starting with $K = 16,807$ and $M = 2,147,483,647$, using ten different starting values of a_1, compute sequences of bits b_n up to $n = 1000$, and compare their averages to ten such sequences generated by a random bit generator.

Answer

For the starting values $a_1 = 1, a_2 = 2, \ldots, a_1 = 10$, the corresponding bit averages calculated by the method indicated are $0.5220, 0.5000, 0.4960, 0.4870, 0.4860, 0.4680, 0.5130, 0.5210, 0.5040$, and 0.4840. Here is an example of ten corresponding averages of strings of 1000 bits generated by a random number generator:
$0.4880, 0.4870, 0.5150, 0.5490, 0.5130, 0.5180, 0.4860, 0.5030, 0.5050, 0.4980$. There is no real pattern in either type of average. The random-number-generated averages range between 0.4860 and 0.5490, a range of 0.0630, whereas the calculated PRNG bit averages range between 0.4680 and 0.5220, a range of 0.0540.

66) [T] Find the first 1000 digits of π using either a computer program or Internet resource. Create a bit sequence b_n by letting $b_n = 1$ if the nth digit of π is odd and $b_n = 0$ if the nth digit of π is even. Compute the average value of b_n and the average value of $d_n = |b_{n+1} - b_n|, n = 1, \ldots, 999$. Does the sequence b_n appear random? Do the differences between successive elements of b_n appear random?

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9.2: Infinite Series

Learning Objectives

- Explain the meaning of the sum of an infinite series.
- Calculate the sum of a geometric series.
- Evaluate a telescoping series.

We have seen that a sequence is an ordered set of terms. If you add these terms together, you get a series. In this section we define an infinite series and show how series are related to sequences. We also define what it means for a series to converge or diverge. We introduce one of the most important types of series: the geometric series. We will use geometric series in the next chapter to write certain functions as polynomials with an infinite number of terms. This process is important because it allows us to evaluate, differentiate, and integrate complicated functions by using polynomials that are easier to handle. We also discuss the harmonic series, arguably the most interesting divergent series because it just fails to converge.

Sums and Series

An infinite series is a sum of infinitely many terms and is written in the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots.$$

But what does this mean? We cannot add an infinite number of terms in the same way we can add a finite number of terms. Instead, the value of an infinite series is defined in terms of the limit of partial sums. A partial sum of an infinite series is a finite sum of the form

$$\sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k.$$

To see how we use partial sums to evaluate infinite series, consider the following example. Suppose oil is seeping into a lake such that 1000 gallons enters the lake the first week. During the second week, an additional 500 gallons of oil enters the lake. The third week, 250 more gallons enters the lake. Assume this pattern continues such that each week half as much oil enters the lake as did the previous week. If this continues forever, what can we say about the amount of oil in the lake? Will the amount of oil continue to get arbitrarily large, or is it possible that it approaches some finite amount? To answer this question, we look at the amount of oil in the lake after k weeks. Letting S_k denote the amount of oil in the lake (measured in thousands of gallons) after k weeks, we see that

$$S_1 = 1$$

$$S_2 = 1 + 0.5 = 1 + \tfrac{1}{2}$$

$$S_3 = 1 + 0.5 + 0.25 = 1 + \tfrac{1}{2} + \tfrac{1}{4}$$

$$S_4 = 1 + 0.5 + 0.25 + 0.125 = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8}$$

$$S_5 = 1 + 0.5 + 0.25 + 0.125 + 0.0625 = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16}.$$

Looking at this pattern, we see that the amount of oil in the lake (in thousands of gallons) after k weeks is

$$S_k = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^{k-1}} = \sum_{n=1}^{k} \left(\frac{1}{2}\right)^{n-1}.$$

We are interested in what happens as $k \to \infty$. Symbolically, the amount of oil in the lake as $k \to \infty$ is given by the infinite series

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots.$$

At the same time, as $k \to \infty$, the amount of oil in the lake can be calculated by evaluating $\lim_{k\to\infty} S_k$. Therefore, the behavior of the infinite series can be determined by looking at the behavior of the sequence of partial sums S_k. If the sequence of partial sums S_k converges, we say that the infinite series converges, and its sum is given by $\lim_{k\to\infty} S_k$. If the sequence S_k diverges, we say the infinite series diverges. We now turn our attention to determining the limit of this sequence S_k.

First, simplifying some of these partial sums, we see that

$$S_1 = 1$$
$$S_2 = 1 + \tfrac{1}{2} = \tfrac{3}{2}$$
$$S_3 = 1 + \tfrac{1}{2} + \tfrac{1}{4} = \tfrac{7}{4}$$
$$S_4 = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} = \tfrac{15}{8}$$
$$S_5 = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} = \tfrac{31}{16}.$$

Plotting some of these values in Figure 9.2.1, it appears that the sequence S_k could be approaching 2.

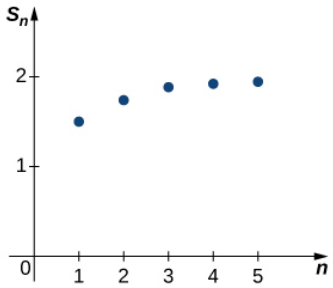

Figure 9.2.1: The graph shows the sequence of partial sums S_k. It appears that the sequence is approaching the value 2.

Let's look for more convincing evidence. In the following table, we list the values of S_k for several values of k.

k	5	10	15	20
S_k	1.9375	1.998	1.999939	1.999998

These data supply more evidence suggesting that the sequence S_k converges to 2. Later we will provide an analytic argument that can be used to prove that $\lim_{k\to\infty} S_k = 2$. For now, we rely on the numerical and graphical data to convince ourselves that the sequence of partial sums does actually converge to 2. Since this sequence of partial sums converges to 2, we say the infinite series converges to 2 and write

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 2.$$

Returning to the question about the oil in the lake, since this infinite series converges to 2, we conclude that the amount of oil in the lake will get arbitrarily close to 2000 gallons as the amount of time gets sufficiently large.

This series is an example of a geometric series. We discuss geometric series in more detail later in this section. First, we summarize what it means for an infinite series to converge.

Definition

An **infinite series** is an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots.$$

For each positive integer k, the sum

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k$$

is called the kth **partial sum** of the infinite series. The partial sums form a sequence S_k. If the sequence of partial sums converges to a real number S, the infinite series converges. If we can describe the **convergence of a series** to S, we call S the sum of the series, and we write

$$\sum_{n=1}^{\infty} a_n = S.$$

If the sequence of partial sums diverges, we have the **divergence of a series.**

Note that the index for a series need not begin with $n = 1$ but can begin with any value. For example, the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$$

can also be written as

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} \text{ or } \sum_{n=5}^{\infty} \left(\frac{1}{2}\right)^{n-5}.$$

Often it is convenient for the index to begin at 1, so if for some reason it begins at a different value, we can re-index by making a change of variables. For example, consider the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2}.$$

By introducing the variable $m = n - 1$, so that $n = m + 1$, we can rewrite the series as

$$\sum_{m=1}^{\infty} \frac{1}{(m+1)^2}.$$

Example 9.2.1: Evaluating Limits of Sequences of Partial Sums

For each of the following series, use the sequence of partial sums to determine whether the series converges or diverges.

a. $\sum_{n=1}^{\infty} \frac{n}{n+1}$

b. $\sum_{n=1}^{\infty} (-1)^n$

c. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Solution

a. The sequence of partial sums S_k satisfies

$$\begin{aligned} S_1 &= \frac{1}{2} \\ S_2 &= \frac{1}{2} + \frac{2}{3} \\ S_3 &= \frac{1}{2} + \frac{2}{3} + \frac{3}{4} \\ S_4 &= \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5}. \end{aligned}$$

Notice that each term added is greater than $1/2$. As a result, we see that

$$\begin{aligned} S_1 &= \frac{1}{2} \\ S_2 &= \frac{1}{2} + \frac{2}{3} > \frac{1}{2} + \frac{1}{2} = 2\left(\frac{1}{2}\right) \\ S_3 &= \frac{1}{2} + \frac{2}{3} + \frac{3}{4} > \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 3\left(\frac{1}{2}\right) \\ S_4 &= \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} > \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 4\left(\frac{1}{2}\right). \end{aligned}$$

From this pattern we can see that $S_k > k\left(\frac{1}{2}\right)$ for every integer k. Therefore, S_k is unbounded and consequently, diverges. Therefore, the infinite series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges.

b. The sequence of partial sums S_k satisfies

$$S_1 = -1$$
$$S_2 = -1 + 1 = 0$$
$$S_3 = -1 + 1 - 1 = -1$$
$$S_4 = -1 + 1 - 1 + 1 = 0.$$

From this pattern we can see the sequence of partial sums is

$$S_k = -1, 0, -1, 0, \ldots.$$

Since this sequence diverges, the infinite series $\sum_{n=1}^{\infty} (-1)^n$ diverges.

c. The sequence of partial sums S_k satisfies

$$S_1 = \frac{1}{1 \cdot 2} = \frac{1}{2}$$
$$S_2 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$
$$S_3 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4}$$
$$S_4 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} = \frac{4}{5}$$
$$S_5 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \frac{1}{5 \cdot 6} = \frac{5}{6}.$$

From this pattern, we can see that the kth partial sum is given by the explicit formula

$$S_k = \frac{k}{k+1}$$

.

Since $k/(k+1) \to 1$, we conclude that the sequence of partial sums converges, and therefore the infinite series converges to 1. We have

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

? Exercise 9.2.1

Determine whether the series $\sum_{n=1}^{\infty} \frac{n+1}{n}$ converges or diverges.

Hint

Look at the sequence of partial sums.

Answer

The series diverges because the kth partial sum $S_k > k$.

The Harmonic Series

A useful series to know about is the **harmonic series.** The harmonic series is defined as

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots.$$

This series is interesting because it diverges, but it diverges very slowly. By this we mean that the terms in the sequence of partial sums S_k approach infinity, but do so very slowly. We will show that the series diverges, but first we illustrate the slow growth of the terms in the sequence S_k in the following table.

k	10	100	1000	10,00	100,000	1,000,000
S_k	2.92897	5.18738	7.48547	9.78761	12.09015	14.39273

Even after $1,000,000$ terms, the partial sum is still relatively small. From this table, it is not clear that this series actually diverges. However, we can show analytically that the sequence of partial sums diverges, and therefore the series diverges.

To show that the sequence of partial sums diverges, we show that the sequence of partial sums is unbounded. We begin by writing the first several partial sums:

$$S_1 = 1$$
$$S_2 = 1 + \frac{1}{2}$$
$$S_3 = 1 + \frac{1}{2} + \frac{1}{3}$$
$$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}.$$

Notice that for the last two terms in S_4,

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4}$$

Therefore, we conclude that

$$S_4 > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{1}{2} + \frac{1}{2} = 1 + 2\left(\frac{1}{2}\right).$$

Using the same idea for S_8, we see that

$$S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + 3\left(\frac{1}{2}\right).$$

From this pattern, we see that $S_1 = 1, S_2 = 1 + 1/2, S_4 > 1 + 2(1/2)$, and $S_8 > 1 + 3(1/2)$. More generally, it can be shown that $S_{2^j} > 1 + j(1/2)$ for all $j > 1$. Since $1 + j(1/2) \to \infty$, we conclude that the sequence S_k is unbounded and therefore diverges. In the previous section, we stated that convergent sequences are bounded. Consequently, since S_k is unbounded, it diverges. Thus, the harmonic series diverges.

Algebraic Properties of Convergent Series

Since the sum of a convergent infinite series is defined as a limit of a sequence, the algebraic properties for series listed below follow directly from the algebraic properties for sequences.

Note 9.2.1: Algebraic Properties of Convergent Series

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series. Then the following algebraic properties hold.

i. The series $\sum_{n=1}^{\infty}(a_n + b_n)$ converges, and $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$. (Sum Rule)

ii. The series $\sum_{n=1}^{\infty}(a_n - b_n)$ converges, and $\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$. (Difference Rule)

iii. For any real number c, the series $\sum_{n=1}^{\infty} ca_n$ converges, and $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$. (Constant Multiple Rule)

Example 9.2.2: Using Algebraic Properties of Convergent Series

Evaluate $\sum_{n=1}^{\infty}\left[\frac{3}{n(n+1)} + \left(\frac{1}{2}\right)^{n-2}\right]$.

Solution

We showed earlier that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

and

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 2.$$

Since both of those series converge, we can apply the properties of Note 9.2.1 to evaluate

$$\sum_{n=1}^{\infty} \left[\frac{3}{n(n+1)} + \left(\frac{1}{2}\right)^{n-2}\right].$$

Using the sum rule, write

$$\sum_{n=1}^{\infty} \left[\frac{3}{n(n+1)} + \left(\frac{1}{2}\right)^{n-2}\right] = \sum_{n=1}^{\infty} \frac{3}{n(n+1)} + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-2}.$$

Then, using the constant multiple rule and the sums above, we can conclude that

$$\sum_{n=1}^{\infty} \frac{3}{n(n+1)} + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-2} = 3\sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \left(\frac{1}{2}\right)^{-1} \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 3(1) + \left(\frac{1}{2}\right)^{-1}(2) = 3 + 2(2) = 7.$$

? Exercise 9.2.2

Evaluate $\sum_{n=1}^{\infty} \frac{5}{2^{n-1}}$.

Hint

Rewrite as $\sum_{n=1}^{\infty} 5\left(\frac{1}{2}\right)^{n-1}$.

Answer

10

Geometric Series

A **geometric series** is any series that we can write in the form

$$a + ar + ar^2 + ar^3 + \cdots = \sum_{n=1}^{\infty} ar^{n-1}.$$

Because the ratio of each term in this series to the previous term is r, the number r is called the ratio. We refer to a as the initial term because it is the first term in the series. For example, the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

is a geometric series with initial term $a = 1$ and ratio $r = 1/2$.

In general, when does a geometric series converge? Consider the geometric series

$$\sum_{n=1}^{\infty} ar^{n-1}$$

when $a > 0$. Its sequence of partial sums S_k is given by

$$S_k = \sum_{n=1}^{k} ar^{n-1} = a + ar + ar^2 + \cdots + ar^{k-1}.$$

Consider the case when $r = 1$. In that case,

$$S_k = a + a(1) + a(1)^2 + \cdots + a(1)^{n-1} = ak.$$

Since $a > 0$, we know $ak \to \infty$ as $k \to \infty$. Therefore, the sequence of partial sums is unbounded and thus diverges. Consequently, the infinite series diverges for $r = 1$. For $r \neq 1$, to find the limit of S_k, multiply Equation by $1 - r$. Doing so, we see that

$$\begin{aligned}(1-r)S_k &= a(1-r)(1 + r + r^2 + r^3 + \cdots + r^{k-1}) \\ &= a[(1 + r + r^2 + r^3 + \cdots + r^{k-1}) - (r + r^2 + r^3 + \cdots + r^k)] = a(1 - r^k).\end{aligned}$$

All the other terms cancel out.

Therefore,

$$S_k = \frac{a(1-r^k)}{1-r} \text{ for } r \neq 1.$$

From our discussion in the previous section, we know that the geometric sequence $r^k \to 0$ if $|r| < 1$ and that r^k diverges if $|r| > 1$ or $r = \pm 1$. Therefore, for $|r| < 1, S_k \to \frac{a}{1-r}$ and we have

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \text{ if } |r| < 1.$$

If $|r| \geq 1$, S_k diverges, and therefore

$$\sum_{n=1}^{\infty} ar^{n-1} \text{ diverges if } |r| \geq 1.$$

Definitions: Diverging and Converging Series

A geometric series is a series of the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots. \tag{9.2.1}$$

If $|r| < 1$, the series **converges**, and

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \text{ for } |r| < 1.$$

If $|r| \geq 1$, the series **diverges**.

Geometric series sometimes appear in slightly different forms. For example, sometimes the index begins at a value other than $n = 1$ or the exponent involves a linear expression for n other than $n - 1$. As long as we can rewrite the series in the form given by Equation 9.2.1, it is a geometric series. For example, consider the series

$$\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n+2}.$$

To see that this is a geometric series, we write out the first several terms:

$$\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n+2} = \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^4 + \cdots = \frac{4}{9} + \frac{4}{9} \cdot \left(\frac{2}{3}\right) + \frac{4}{9} \cdot \left(\frac{2}{3}\right)^2 + \cdots.$$

We see that the initial term is $a = 4/9$ and the ratio is $r = 2/3$. Therefore, the series can be written as

$$\sum_{n=1}^{\infty} \frac{4}{9} \cdot \left(\frac{2}{3}\right)^{n-1}.$$

Since $r = 2/3 < 1$, this series converges, and its sum is given by

$$\sum_{n=1}^{\infty} \frac{4}{9} \cdot \left(\frac{2}{3}\right)^{n-1} = \frac{4/9}{1 - 2/3} = \frac{4}{3}.$$

Example 9.2.3: Determining Convergence or Divergence of a Geometric Series

Determine whether each of the following geometric series converges or diverges, and if it converges, find its sum.

a. $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{4^{n-1}}$

b. $\sum_{n=1}^{\infty} e^{2n}$

Solution

a. Writing out the first several terms in the series, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{4^{n-1}} &= \frac{(-3)^2}{4^0} + \frac{(-3)^3}{4} + \frac{(-3)^4}{4^2} + \cdots \\ &= (-3)^2 + (-3)^2 \cdot \left(\frac{-3}{4}\right) + (-3)^2 \cdot \left(\frac{-3}{4}\right)^2 + \cdots \\ &= 9 + 9 \cdot \left(\frac{-3}{4}\right) + 9 \cdot \left(\frac{-3}{4}\right)^2 + \cdots .\end{aligned}$$

The initial term $a = -3$ and the ratio $r = -3/4$. Since $|r| = 3/4 < 1$, the series converges to

$$\frac{9}{1-(-3/4)} = \frac{9}{7/4} = \frac{36}{7}.$$

b. Writing this series as

$$e^2 \sum_{n=1}^{\infty} (e^2)^{n-1}$$

we can see that this is a geometric series where $r = e^2 > 1$. Therefore, the series diverges.

Exercise 9.2.3

Determine whether the series $\sum_{n=1}^{\infty} \left(\frac{-2}{5}\right)^{n-1}$ converges or diverges. If it converges, find its sum.

Hint

$r = -2/5$

Answer

$5/7$

We now turn our attention to a nice application of geometric series. We show how they can be used to write repeating decimals as fractions of integers.

Example 9.2.4: Writing Repeating Decimals as Fractions of Integers

Use a geometric series to write $3.2\overline{6}$ as a fraction of integers.

Solution

Since $3.2\overline{6}— = 3.262626\ldots$, first we write

$$\begin{aligned}3.262626\ldots &= 3 + \frac{26}{100} + \frac{26}{10,000} + \frac{26}{100,000} + \cdots \\ &= 3 + \frac{26}{10^2} + \frac{26}{10^4} + \frac{26}{10^6} + \cdots .\end{aligned}$$

Ignoring the term 3, the rest of this expression is a geometric series with initial term $a = 26/10^2$ and ratio $r = 1/10^2$. Therefore, the sum of this series is

$$\frac{26/10^2}{1-(1/10^2)} = \frac{26/10^2}{99/10^2} = \frac{26}{99}.$$

Thus,

$$3.262626\ldots = 3 + \tfrac{26}{99} = \tfrac{323}{99}.$$

Exercise 9.2.4

Write $5.2\overline{7}$ as a fraction of integers.

Hint

By expressing this number as a series, find a geometric series with initial term $a = 7/100$ and ratio $r = 1/10$.

Answer

$475/90$

Example 9.2.5: Finding the Area of the Koch Snowflake

Define a sequence of figures $\{F_n\}$ recursively as follows (Figure 9.2.2). Let F_0 be an equilateral triangle with sides of length 1. For $n \geq 1$, let F_n be the curve created by removing the middle third of each side of F_{n-1} and replacing it with an equilateral triangle pointing outward. The limiting figure as $n \to \infty$ is known as **Koch's snowflake.**

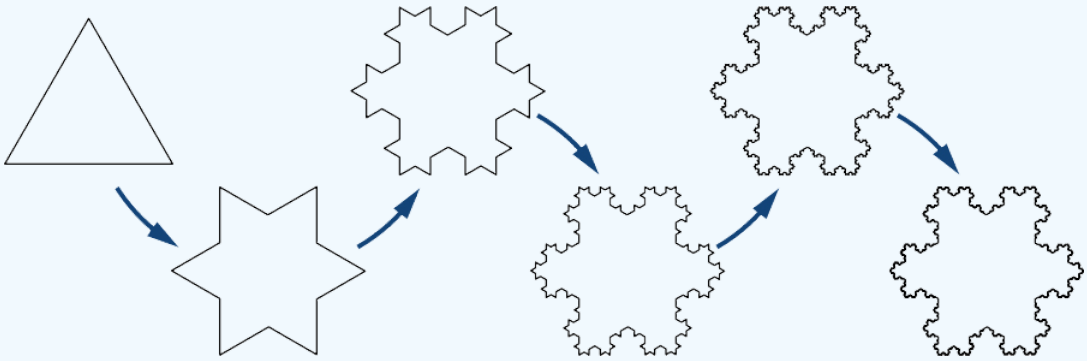

Figure 9.2.2: The first four figures, F_0, F_1, F_2, and F_3, in the construction of the Koch snowflake.

a. Find the length L_n of the perimeter of F_n. Evaluate $\lim_{n\to\infty} L_n$ to find the length of the perimeter of Koch's snowflake.

b. Find the area A_n of figure F_n. Evaluate $\lim_{n\to\infty} A_n$ to find the area of Koch's snowflake.

Solution

a. Let N_n denote the number of sides of figure F_n. Since F_0 is a triangle, $N_0 = 3$. Let l_n denote the length of each side of F_n. Since F_0 is an equilateral triangle with sides of length $l_0 = 1$, we now need to determine N_1 and l_1. Since F_1 is created by removing the middle third of each side and replacing that line segment with two line segments, for each side of F_0, we get four sides in F_1. Therefore, the number of sides for F_1 is

$$N_1 = 4 \cdot 3.$$

Since the length of each of these new line segments is $1/3$ the length of the line segments in F_0, the length of the line segments for F_1 is given by

$$l_1 = \tfrac{1}{3} \cdot 1 = \tfrac{1}{3}.$$

Similarly, for F_2, since the middle third of each side of F_1 is removed and replaced with two line segments, the number of sides in F_2 is given by

$$N_2 = 4N_1 = 4(4 \cdot 3) = 4^2 \cdot 3.$$

Since the length of each of these sides is $1/3$ the length of the sides of F_1, the length of each side of figure F_2 is given by

$$l_2 = \tfrac{1}{3} \cdot l_1 = \tfrac{1}{3} \cdot \tfrac{1}{3} = \left(\tfrac{1}{3}\right)^2.$$

More generally, since F_n is created by removing the middle third of each side of F_{n-1} and replacing that line segment with two line segments of length $\frac{1}{3}l_{n-1}$ in the shape of an equilateral triangle, we know that $N_n = 4N_{n-1}$ and $l_n = \frac{l_{n-1}}{3}$. Therefore, the number of sides of figure F_n is

$$N_n = 4^n \cdot 3$$

and the length of each side is

$$l_n = \left(\frac{1}{3}\right)^n.$$

Therefore, to calculate the perimeter of F_n, we multiply the number of sides N_n and the length of each side l_n. We conclude that the perimeter of F_n is given by

$$L_n = N_n \cdot l_n = 3 \cdot \left(\frac{4}{3}\right)^n$$

Therefore, the length of the perimeter of Koch's snowflake is

$$L = \lim_{n\to\infty} L_n = \infty.$$

b. Let T_n denote the area of each new triangle created when forming F_n. For $n = 0, T_0$ is the area of the original equilateral triangle. Therefore, $T_0 = A_0 = \sqrt{3}/4$. For $n \geq 1$, since the lengths of the sides of the new triangle are $1/3$ the length of the sides of F_{n-1}, we have

$$T_n = \left(\frac{1}{3}\right)^2 \cdot T_{n-1} = \frac{1}{9} \cdot T_{n-1}.$$

Therefore, $T_n = \left(\frac{1}{9}\right)^n \cdot \frac{\sqrt{3}}{4}$. Since a new triangle is formed on each side of F_{n-1},

$$A_n = A_{n-1} + N_{n-1} \cdot T_n = A_{n-1} + (3 \cdot 4_{n-1}) \cdot \left(\frac{1}{9}\right)^n \cdot \frac{\sqrt{3}}{4} = A_{n-1} + \frac{3}{4} \cdot \left(\frac{4}{9}\right)^n \cdot \frac{\sqrt{3}}{4}.$$

Writing out the first few terms A_0, A_1, A_2, we see that

$A_0 = \frac{\sqrt{3}}{4}$

$A_1 = A_0 + \frac{3}{4} \cdot \left(\frac{4}{9}\right) \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4} + \frac{3}{4} \cdot \left(\frac{4}{9}\right) \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4}\left[1 + \frac{3}{4} \cdot \left(\frac{4}{9}\right)\right]$

$A_2 = A_1 + \frac{3}{4} \cdot \left(\frac{4}{9}\right)^2 \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4}\left[1 + \frac{3}{4} \cdot \left(\frac{4}{9}\right)\right] + \frac{3}{4} \cdot \left(\frac{4}{9}\right)^2 \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4}[1 + \frac{3}{4} \cdot \left(\frac{4}{9}\right) + \frac{3}{4} \cdot \left(\frac{4}{9}\right)^2]$.

More generally,

$$A_n = \frac{\sqrt{3}}{4}\left[1 + \frac{3}{4}\left(\frac{4}{9} + \left(\frac{4}{9}\right)^2 + \cdots + \left(\frac{4}{9}\right)^n\right)\right].$$

Factoring $4/9$ out of each term inside the inner parentheses, we rewrite our expression as

$$A_n = \frac{\sqrt{3}}{4}\left[1 + \frac{1}{3}\left(1 + \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \cdots + \left(\frac{4}{9}\right)^{n-1}\right)\right].$$

The expression $1 + \left(\frac{4}{9}\right) + \left(\frac{4}{9}\right)^2 + \cdots + \left(\frac{4}{9}\right)^{n-1}$ is a geometric sum. As shown earlier, this sum satisfies

$$1 + \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \cdots + \left(\frac{4}{9}\right)^{n-1} = \frac{1 - (4/9)^n}{1 - (4/9)}.$$

Substituting this expression into the expression above and simplifying, we conclude that

$$A_n = \frac{\sqrt{3}}{4}\left[1 + \frac{1}{3}\left(\frac{1 - (4/9)^n}{1 - (4/9)}\right)\right] = \frac{\sqrt{3}}{4}\left[\frac{8}{5} - \frac{3}{5}\left(\frac{4}{9}\right)^n\right].$$

Therefore, the area of Koch's snowflake is

$$A = \lim_{n\to\infty} A_n = \frac{2\sqrt{3}}{5}.$$

Analysis

The Koch snowflake is interesting because it has finite area, yet infinite perimeter. Although at first this may seem impossible, recall that you have seen similar examples earlier in the text. For example, consider the region bounded by the curve $y = 1/x^2$

and the x-axis on the interval $[1, \infty)$. Since the improper integral

$$\int_1^\infty \frac{1}{x^2}\,dx$$

converges, the area of this region is finite, even though the perimeter is infinite.

Telescoping Series

Consider the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. We discussed this series in Example 9.2.1c, showing that the series converges by writing out the first several partial sums S_1, S_2, ..., S_6 and noticing that they are all of the form $S_k = \frac{k}{k+1}$. Here we use a different technique to show that this series converges. By using partial fractions, we can write

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Therefore, the series can be written as

$$\sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n+1}\right] = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots.$$

Writing out the first several terms in the sequence of partial sums S_k, we see that

$$S_1 = 1 - \tfrac{1}{2}$$
$$S_2 = \left(1 - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) = 1 - \tfrac{1}{3}$$
$$S_3 = \left(1 - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) + \left(\tfrac{1}{3} - \tfrac{1}{4}\right) = 1 - \tfrac{1}{4}.$$

In general,

$$S_k = \left(1 - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) + \left(\tfrac{1}{3} - \tfrac{1}{4}\right) + \cdots + \left(\tfrac{1}{k} - \tfrac{1}{k+1}\right) = 1 - \frac{1}{k+1}.$$

We notice that the middle terms cancel each other out, leaving only the first and last terms. In a sense, the series collapses like a spyglass with tubes that disappear into each other to shorten the telescope. For this reason, we call a series that has this property a telescoping series. For this series, since $S_k = 1 - 1/(k+1)$ and $1/(k+1) \to 0$ as $k \to \infty$, the sequence of partial sums converges to 1, and therefore the series converges to 1.

Definition

A **telescoping series** is a series in which most of the terms cancel in each of the partial sums, leaving only some of the first terms and some of the last terms.

For example, any series of the form

$$\sum_{n=1}^{\infty} [b_n - b_{n+1}] = (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + \cdots$$

is a telescoping series. We can see this by writing out some of the partial sums. In particular, we see that

$$S_1 = b_1 - b_2$$
$$S_2 = (b_1 - b_2) + (b_2 - b_3) = b_1 - b_3$$
$$S_3 = (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) = b_1 - b_4.$$

In general, the kth partial sum of this series is

$$S_k = b_1 - b_{k+1}.$$

Since the kth partial sum can be simplified to the difference of these two terms, the sequence of partial sums S_k will converge if and only if the sequence b_{k+1} converges. Moreover, if the sequence b_{k+1} converges to some finite number B, then the sequence of partial sums converges to $b_1 - B$, and therefore

$$\sum_{n=1}^{\infty}[b_n - b_{n+1}] = b_1 - B.$$

In the next example, we show how to use these ideas to analyze a telescoping series of this form.

Example 9.2.6: Evaluating a Telescoping Series

Determine whether the telescoping series

$$\sum_{n=1}^{\infty}\left[\cos\left(\frac{1}{n}\right) - \cos\left(\frac{1}{n+1}\right)\right]$$

converges or diverges. If it converges, find its sum.

Solution

By writing out terms in the sequence of partial sums, we can see that

$$S_1 = \cos(1) - \cos(\tfrac{1}{2})$$

$$S_2 = (\cos(1) - \cos(\tfrac{1}{2})) + (\cos(\tfrac{1}{2}) - \cos(\tfrac{1}{3})) = \cos(1) - \cos(\tfrac{1}{3})$$

$$S_3 = (\cos(1) - \cos(\tfrac{1}{2})) + (\cos(\tfrac{1}{2}) - \cos(\tfrac{1}{3})) + (\cos(\tfrac{1}{3}) - \cos(\tfrac{1}{4}))$$

$$= \cos(1) - \cos(\tfrac{1}{4}).$$

In general,

$$S_k = \cos(1) - \cos\left(\tfrac{1}{k+1}\right).$$

Since $1/(k+1) \to 0$ as $k \to \infty$ and $\cos x$ is a continuous function, $\cos(1/(k+1)) \to \cos(0) = 1$. Therefore, we conclude that $S_k \to \cos(1) - 1$. The telescoping series converges and the sum is given by

$$\sum_{n=1}^{\infty}\left[\cos\left(\frac{1}{n}\right) - \cos\left(\frac{1}{n+1}\right)\right] = \cos(1) - 1.$$

Exercise 9.2.5

Determine whether $\sum_{n=1}^{\infty}[e^{1/n} - e^{1/(n+1)}]$ converges or diverges. If it converges, find its sum.

Hint

Write out the sequence of partial sums to see which terms cancel.

Answer

$e - 1$

Euler's Constant

We have shown that the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges. Here we investigate the behavior of the partial sums S_k as $k \to \infty$. In particular, we show that they behave like the natural logarithm function by showing that there exists a constant γ such that

$$\sum_{n=1}^{k}\left(\frac{1}{n} - \ln k\right) \to \gamma \text{ as } k \to \infty.$$

This constant γ is known as **Euler's constant.**

1. Let $T_k = \sum_{n=1}^{k}\left(\frac{1}{n} - \ln k\right)$. Evaluate T_k for various values of k.
2. For T_k as defined in part 1. show that the sequence T_k converges by using the following steps.
   a. Show that the sequence T_k is monotone decreasing. (Hint: Show that $\ln(1 + 1/k > 1/(k+1))$
   b. Show that the sequence T_k is bounded below by zero. (Hint: Express $\ln k$ as a definite integral.)

c. Use the Monotone Convergence Theorem to conclude that the sequence T_k converges. The limit γ is Euler's constant.

3. Now estimate how far T_k is from γ for a given integer k. Prove that for $k \geq 1, 0 < T_k - \gamma \leq 1/k$ by using the following steps.

a. Show that $\ln(k+1) - \ln k < 1/k$.

b. Use the result from part a. to show that for any integer k,

$$T_k - T_{k+1} < \frac{1}{k} - \frac{1}{k+1}.$$

c. For any integers k and j such that $j > k$, express $T_k - T_j$ as a telescoping sum by writing

$$T_k - T_j = (T_k - T_{k+1}) + (T_{k+1} - T_{k+2}) + (T_{k+2} - T_{k+3}) + \cdots + (T_{j-1} - T_j).$$

Use the result from part b. combined with this telescoping sum to conclude that

$$T_k - T_j < \frac{1}{k} - \frac{1}{j}.$$

a. Apply the limit to both sides of the inequality in part c. to conclude that

$$T_k - \gamma \leq \frac{1}{k}.$$

e. Estimate γ to an accuracy of within **0.001**.

Key Concepts

- Given the infinite series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

and the corresponding sequence of partial sums S_k where

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k,$$

the series converges if and only if the sequence S_k converges.

- The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geq 1$. For $|r| < 1$,

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

- The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

diverges.

- A series of the form $\sum_{n=1}^{\infty} [b_n - b_{n+1}] = [b_1 - b_2] + [b_2 - b_3] + [b_3 - b_4] + \cdots + [b_n - b_{n+1}] + \cdots$ is a telescoping series. The kth partial sum of this series is given by $S_k = b_1 - b_{k+1}$. The series will converge if and only if $\lim_{k\to\infty} b_{k+1}$ exists. In that case,

$$\sum_{n=1}^{\infty} [b_n - b_{n+1}] = b_1 - \lim_{k\to\infty} (b_{k+1}).$$

Key Equations

- **Harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

- **Sum of a geometric series**

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \text{ for } |r| < 1$$

Glossary

convergence of a series

a series converges if the sequence of partial sums for that series converges

divergence of a series

a series diverges if the sequence of partial sums for that series diverges

geometric series

a geometric series is a series that can be written in the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$$

harmonic series

the harmonic series takes the form

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

infinite series

an infinite series is an expression of the form

$$a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$$

partial sum

the kth partial sum of the infinite series $\sum_{n=1}^{\infty} a_n$ is the finite sum

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k$$

telescoping series

a telescoping series is one in which most of the terms cancel in each of the partial sums

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9.2E: Exercises for Section 9.2

In exercises 1 - 4, use sigma notation to write each expressions as an infinite series.

1) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$

Answer

$$\sum_{n=1}^{\infty}\frac{1}{n}$$

2) $1-1+1-1+\cdots$

3) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...$

Answer

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$$

4) $\sin 1+\sin\frac{1}{2}+\sin\frac{1}{3}+\sin\frac{1}{4}+\cdots$

In exercises 5 - 8, compute the first four partial sums $S_1,\ldots,S_4$ for the series having n^{th} term a_n starting with $n=1$ as follows.

5) $a_n=n$

Answer

$1,3,6,10$

6) $a_n=1/n$

7) $a_n=\sin\frac{n\pi}{2}$

Answer

$1,1,0,0$

8) $a_n=(-1)^n$

In exercises 9 - 12, compute the general term a_n of the series with the given partial sum S_n. If the sequence of partial sums converges, find its limit S.

9) $S_n=1-\frac{1}{n},\quad n\geq 2$

Answer

$a_n=S_n-S_{n-1}=\dfrac{1}{n-1}-\dfrac{1}{n}$. Since $S=\lim\limits_{n\to\infty}S_n=\lim\limits_{n\to\infty}\left(1-\dfrac{1}{n}\right)=1$, the series converges to $S=1$.

10) $S_n=\dfrac{n(n+1)}{2},\quad n\geq 1$

11) $S_n=\sqrt{n},\quad n\geq 2$

Answer

$a_n=S_n-S_{n-1}=\sqrt{n}-\sqrt{n-1}=\dfrac{1}{\sqrt{n-1}+\sqrt{n}}$.

The series diverges because the partial sums are unbounded.

That is, $\lim\limits_{n\to\infty}S_n=\lim\limits_{n\to\infty}\sqrt{n}=\infty$.

12) $S_n=2-\dfrac{n+2}{2^n},\quad n\geq 1$

For each series in exercises 13 - 16, use the sequence of partial sums to determine whether the series converges or diverges.

13) $\sum_{n=1}^{\infty} \frac{n}{n+2}$

Answer

$S_1 = 1/3,$
$S_2 = 1/3 + 2/4 > 1/3 + 1/3 = 2/3,$
$S_3 = 1/3 + 2/4 + 3/5 > 3 \cdot (1/3) = 1.$
In general $S_k > k/3$, so the series diverges.
Note that the nth Term Test for Divergence could also be used to prove that this series diverges.

14) $\sum_{n=1}^{\infty}(1-(-1)^n))$

15) $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$ (
Hint: Use a partial fraction decomposition like that for $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.)

Answer

$S_1 = 1/(2 \cdot 3) = 1/6 = 2/3 - 1/2,$

$S_2 = 1/(2 \cdot 3) + 1/(3 \cdot 4) = 2/12 + 1/12 = 1/4 = 3/4 - 1/2,$

$S_3 = 1/(2 \cdot 3) + 1/(3 \cdot 4) + 1/(4 \cdot 5) = 10/60 + 5/60 + 3/60 = 3/10 = 4/5 - 1/2,$

$S_4 = 1/(2 \cdot 3) + 1/(3 \cdot 4) + 1/(4 \cdot 5) + 1/(5 \cdot 6) = 10/60 + 5/60 + 3/60 + 2/60 = 1/3 = 5/6 - 1/2.$

The pattern is $S_k = \frac{k+1}{k+2} - \frac{1}{2}$.
Then $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{k+1}{k+2} - \frac{1}{2}\right) = \frac{1}{2}$, so the series converges to $1/2$.

16) $\sum_{n=1}^{\infty} \frac{1}{2n+1}$ (
Hint: Follow the reasoning for $\sum_{n=1}^{\infty} \frac{1}{n}$.)

Suppose that $\sum_{n=1}^{\infty} a_n = 1$, that $\sum_{n=1}^{\infty} b_n = -1$, that $a_1 = 2$, and $b_1 = -3$. Use this information to find the sum of the indicated series in exercises 17 - 20.

17) $\sum_{n=1}^{\infty}(a_n + b_n)$

Answer

$$\sum_{n=1}^{\infty}(a_n + b_n) \quad = \quad \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n \quad = \quad 1 + (-1) \quad = \quad 0$$

18) $\sum_{n=1}^{\infty}(a_n - 2b_n)$

19) $\sum_{n=2}^{\infty}(a_n - b_n)$

Answer

$$\sum_{n=2}^{\infty}(a_n - b_n) \quad = \quad \sum_{n=2}^{\infty} a_n - \sum_{n=2}^{\infty} b_n \quad = \quad \left(\sum_{n=1}^{\infty} a_n - a_1\right) - \left(\sum_{n=1}^{\infty} b_n - b_1\right) \quad = \quad (1-2) - (-1-(-3)) = -1 - 2 \quad = \quad -3$$

20) $\sum_{n=1}^{\infty}(3a_{n+1} - 4b_{n+1})$

In exercises 21 - 26, state whether the given series converges or diverges and explain why.

21) $\sum_{n=1}^{\infty} \frac{1}{n+1000}$ (Hint: Rewrite using a change of index.)

Answer

The series diverges, $\sum_{n=1001}^{\infty} \frac{1}{n}$

22) $\sum_{n=1}^{\infty} \frac{1}{n+10^{80}}$ (Hint: Rewrite using a change of index.)

23) $1+\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\cdots$

Answer

This is a convergent geometric series, since $r=\frac{1}{10}<1$

24) $1+\frac{e}{\pi}+\frac{e^2}{\pi^2}+\frac{e^3}{\pi^3}+\cdots$

25) $1+\frac{\pi}{e^2}+\frac{\pi^2}{e^4}+\frac{\pi^3}{e^6}+\frac{\pi^4}{e^8}+\cdots$

Answer

This is a convergent geometric series, since $r=\pi/e^2<1$

26) $1-\sqrt{\frac{\pi}{3}}+\sqrt{\frac{\pi^2}{9}}-\sqrt{\frac{\pi^3}{27}}+\cdots$

For each a_n in exercises 27 - 30, write its sum as a geometric series of the form $\sum_{n=1}^{\infty} ar^n$. State whether the series converges and if it does, find the exact value of its sum.

27) $a_1=-1$ and $\frac{a_n}{a_{n+1}}=-5$ for $n\geq 1$.

Answer

$\sum_{n=1}^{\infty} 5\cdot(-1/5)^n$, converges to $-5/6$

28) $a_1=2$ and $\frac{a_n}{a_{n+1}}=1/2$ for $n\geq 1$.

29) $a_1=10$ and $\frac{a_n}{a_{n+1}}=10$ for $n\geq 1$.

Answer

$\sum_{n=1}^{\infty} 100\cdot(1/10)^n$, converges to $\frac{100}{9}$

30) $a_1=\frac{1}{10}$ and $a_n/a_{n+1}=-10$ for $n\geq 1$.

In exercises 31 - 34, use the identity $\frac{1}{1-y}=\sum_{n=0}^{\infty} y^n$ (which is true for $|y|<1$) to express each function as a geometric series in the indicated term.

31) $\frac{x}{1+x}$ in x

Answer

$x\sum_{n=0}^{\infty}(-x)^n=\sum_{n=1}^{\infty}(-1)^{n-1}x^n$

32) $\frac{\sqrt{x}}{1-x^{3/2}}$ in $\sqrt{x}$

33) $\frac{1}{1+\sin^2 x}$ in $\sin x$

Answer

$$\sum_{n=0}^{\infty}(-1)^n \sin^{2n}(x)$$

34) $\sec^2 x$ in $\sin x$

In exercises 35 - 38, evaluate the telescoping series or state whether the series diverges.

35) $\sum_{n=1}^{\infty} 2^{1/n} - 2^{1/(n+1)}$

Answer

$S_k = 2 - 2^{1/(k+1)} \to 1$ as $k \to \infty$.

36) $\sum_{n=1}^{\infty} \frac{1}{n^{13}} - \frac{1}{(n+1)^{13}}$

37) $\sum_{n=1}^{\infty} (\sqrt{n} - \sqrt{n+1})$

Answer

$S_k = 1 - \sqrt{k+1}$ diverges

38) $\sum_{n=1}^{\infty} (\sin n - \sin(n+1))$

Express each series in exercises 39 - 42 as a telescoping sum and evaluate its nth partial sum.

39) $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

Answer

$\sum_{n=1}^{\infty} [\ln n - \ln(n+1)]$,
$S_k = -\ln(k+1)$

40) $\sum_{n=1}^{\infty} \frac{2n+1}{(n^2+n)^2}$ (Hint: Factor denominator and use partial fractions.)

41) $\sum_{n=2}^{\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{(\ln n)\ln(n+1)}$

Answer

$a_n = \frac{1}{\ln n} - \frac{1}{\ln(n+1)}$ and $S_k = \frac{1}{\ln(2)} - \frac{1}{\ln(k+1)} \to \frac{1}{\ln(2)}$

42) $\sum_{n=1}^{\infty} \frac{(n+2)}{n(n+1)2^{n+1}}$ (Hint: Look at $1/(n2^n)$.

A general telescoping series is one in which all but the first few terms cancel out after summing a given number of successive terms.

43) Let $a_n = f(n) - 2f(n+1) + f(n+2)$, in which $f(n) \to 0$ as $n \to \infty$. Find $\sum_{n=1}^{\infty} a_n$.

Answer

$\sum_{n=1}^{\infty} a_n = f(1) - f(2)$

44) $a_n = f(n) - f(n+1) - f(n+2) + f(n+3)$, in which $f(n) \to 0$ as $n \to \infty$. Find $\sum_{n=1}^{\infty} a_n$.

45) Suppose that $a_n = c_0 f(n) + c_1 f(n+1) + c_2 f(n+2) + c_3 f(n+3) + c_4 f(n+4)$, where $f(n) \to 0$ as $n \to \infty$. Find a condition on the coefficients $c_0, \ldots, c_4$ that make this a general telescoping series.

Answer

$c_0 + c_1 + c_2 + c_3 + c_4 = 0$

46) Evaluate $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$ (Hint: $\frac{1}{n(n+1)(n+2)} = \frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2(n+2)}$)

47) Evaluate $\sum_{n=2}^{\infty} \frac{2}{n^3 - n}$.

Answer

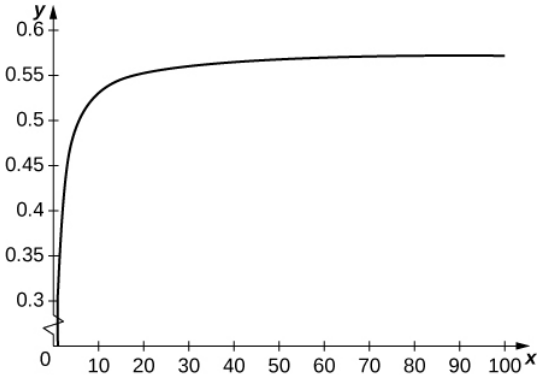
$\frac{2}{n^3-1} = \frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1}$,
$S_n = (1 - 1 + 1/3) + (1/2 - 2/3 + 1/4) + (1/3 - 2/4 + 1/5) + (1/4 - 2/5 + 1/6) + \cdots = 1/2$

48) Find a formula for $\sum_{n=1}^{\infty} \frac{1}{n(n+N)}$ where N is a positive integer.

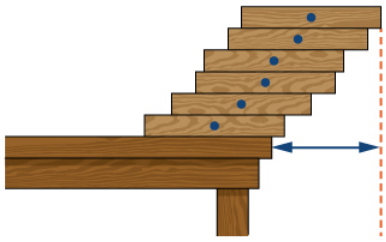
49) [T] Define a sequence $t_k = \sum_{n=1}^{k-1} (1/k) - \ln k$. Use the graph of $1/x$ to verify that t_k is increasing. Plot t_k for $k = 1 \ldots 100$ and state whether it appears that the sequence converges.

Answer

t_k converges to $0.57721 \ldots t_k$ is a sum of rectangles of height $1/k$ over the interval $[k, k+1]$ which lie above the graph of $1/x$.



50) [T] Suppose that N equal uniform rectangular blocks are stacked one on top of the other, allowing for some overhang. Archimedes' law of the lever implies that the stack of N blocks is stable as long as the center of mass of the top $(N-1)$ blocks lies at the edge of the bottom block. Let x denote the position of the edge of the bottom block, and think of its position as relative to the center of the next-to-bottom block. This implies that $(N-1)x = (\frac{1}{2} - x)$ or $x = 1/(2N)$. Use this expression to compute the maximum overhang (the position of the edge of the top block over the edge of the bottom block.) See the following figure.



Each of the following infinite series converges to the given multiple of π or $1/\pi$.

In each case, find the minimum value of N such that the Nth partial sum of the series accurately approximates the left-hand side to the given number of decimal places, and give the desired approximate value. Up to 15 decimals place, $\pi = 3.141592653589793\ldots$.

51) [T] $\pi = -3 + \sum_{n=1}^{\infty} \frac{n2^n n!^2}{(2n)!}$, error < 0.0001

Answer

$N = 22,$

$S_N = 6.1415$

52) [T] $\frac{\pi}{2} = \sum_{k=0}^{\infty} \frac{k!}{(2k+1)!!} = \sum_{k=0}^{\infty} \frac{2^k k!^2}{(2k+1)!}$, error $< 10^{-4}$

53) [T] $\frac{9801}{2\pi} = \frac{4}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103+26390k)}{(k!)^4 396^{4k}}$, error $< 10^{-12}$

Answer

$N = 3,$

$S_N = 1.559877597243667...$

54) [T] $\frac{1}{12\pi} = \sum_{k=0}^{\infty} \frac{(-1)^k (6k)!(13591409+545140134k)}{(3k)!(k!)^3 640320^{3k+3/2}}$, error $< 10^{-15}$

55) [T] A fair coin is one that has probability $1/2$ of coming up heads when flipped.

a. What is the probability that a fair coin will come up tails n times in a row?

b. Find the probability that a coin comes up heads for the first time after an even number of coin flips.

Answer

a. The probability of any given ordered sequence of outcomes for n coin flips is $1/2^n$.

b. The probability of coming up heads for the first time on the n th flip is the probability of the sequence $TT...TH$ which is $1/2^n$. The probability of coming up heads for the first time on an even flip is $\sum_{n=1}^{\infty} 1/2^{2n}$ or $1/3$.

56) [T] Find the probability that a fair coin is flipped a multiple of three times before coming up heads.

57) [T] Find the probability that a fair coin will come up heads for the second time after an even number of flips.

Answer

$5/9$

58) [T] Find a series that expresses the probability that a fair coin will come up heads for the second time on a multiple of three flips.

59) [T] The expected number of times that a fair coin will come up heads is defined as the sum over $n = 1, 2, ...$ of n times the probability that the coin will come up heads exactly n times in a row, or $\frac{n}{2^{n+1}}$. Compute the expected number of consecutive times that a fair coin will come up heads.

Answer

$E = \sum_{n=1}^{\infty} \frac{n}{2^{n+1}} = 1$, as can be shown using summation by parts

60) [T] A person deposits $\$10$ at the beginning of each quarter into a bank account that earns 4 annual interest compounded quarterly (four times a year).

a. Show that the interest accumulated after n quarters is $\$10(\frac{1.01^{n+1}-1}{0.01} - n)$.

b. Find the first eight terms of the sequence.

c. How much interest has accumulated after 2 years?

61) [T] Suppose that the amount of a drug in a patient's system diminishes by a multiplicative factor $r < 1$ each hour. Suppose that a new dose is administered every N hours. Find an expression that gives the amount $A(n)$ in the patient's system after n hours for

each n in terms of the dosage d and the ratio r. (Hint: Write $n = mN + k$, where $0 \le k < N$, and sum over values from the different doses administered.)

Answer

The part of the first dose after n hours is dr^n, the part of the second dose is dr^{n-N}, and, in general, the part remaining of the mth dose is dr^{n-mN}, so $A(n) = \sum_{l=0}^{m} dr^{n-lN} = \sum_{l=0}^{m} dr^{k+(m-l)N} = \sum_{q=0}^{m} dr^{k+qN} = dr^k \sum_{q=0}^{m} r^{Nq} = dr^k \frac{1-r^{(m+1)N}}{1-r^N}$, where $n = k + mN$.

62) [T] A certain drug is effective for an average patient only if there is at least 1 mg per kg in the patient's system, while it is safe only if there is at most 2 mg per kg in an average patient's system. Suppose that the amount in a patient's system diminishes by a multiplicative factor of 0.9 each hour after a dose is administered. Find the maximum interval N of hours between doses, and corresponding dose range d (in mg/kg) for this N that will enable use of the drug to be both safe and effective in the long term.

63) Suppose that $a_n \ge 0$ is a sequence of numbers. Explain why the sequence of partial sums of a_n is increasing.

Answer

$S_{N+1} = a_{N+1} + S_N \ge S_N$

64) [T] Suppose that a_n is a sequence of positive numbers and the sequence S_n of partial sums of a_n is bounded above. Explain why $\sum_{n=1}^{\infty} a_n$ converges. Does the conclusion remain true if we remove the hypothesis $a_n \ge 0$?

65) [T] Suppose that $a_1 = S_1 = 1$ and that, for given numbers $S > 1$ and $0 < k < 1$, one defines $a_{n+1} = k(S - S_n)$ and $S_{n+1} = a_{n+1} + S_n$. Does S_n converge? If so, to what? (Hint: First argue that $S_n < S$ for all n and S_n is increasing.)

Answer

Since $S > 1, a_2 > 0$, and since $k < 1, S_2 = 1 + a_2 < 1 + (S - 1) = S$. If $S_n > S$ for some n, then there is a smallest n. For this $n, S > S_{n-1}$, so $S_n = S_{n-1} + k(S - S_{n-1}) = kS + (1 - k)S_{n-1} < S$, a contradiction. Thus $S_n < S$ and $a_{n+1} > 0$ for all n, so S_n is increasing and bounded by S. Let $S_* = \lim S_n$. If $S_* < S$, then $\delta = k(S - S_*) > 0$, but we can find n such that $S_* - S_n < \delta/2$, which implies that $S_{n+1} = S_n + k(S - S_n) > S_* + \delta/2$, contradicting that Sn is increasing to S_*. Thus $S_n \to S$.

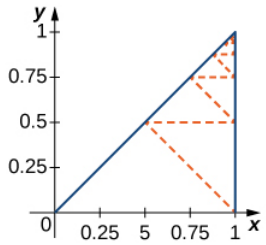
66) [T] A version of **von Bertalanffy growth** can be used to estimate the age of an individual in a homogeneous species from its length if the annual increase in year $n + 1$ satisfies $a_{n+1} = k(S - S_n)$, with S_n as the length at year n, S as a limiting length, and k as a relative growth constant. If $S_1 = 3, S = 9$, and $k = 1/2$, numerically estimate the smallest value of n such that $S_n \ge 8$. Note that $S_{n+1} = S_n + a_{n+1}$. Find the corresponding n when $k = 1/4$.

67) [T] Suppose that $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive terms. Explain why $\lim_{N\to\infty} \sum_{n=N+1}^{\infty} a_n = 0$.

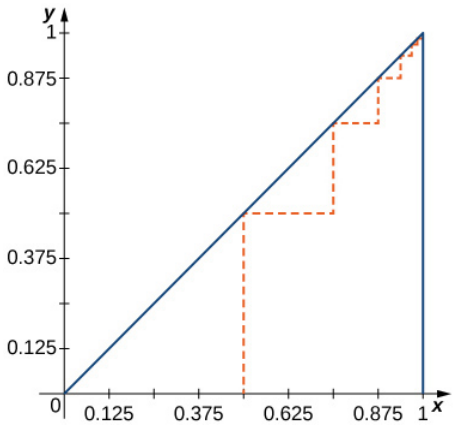
Answer

Let $S_k = \sum_{n=1}^{k} a_n$ and $S_k \to L$. Then S_k eventually becomes arbitrarily close to L, which means that $L - S_N = \sum_{n=N+1}^{\infty} a_n$ becomes arbitrarily small as $N \to \infty$.

68) [T] Find the length of the dashed zig-zag path in the following figure.



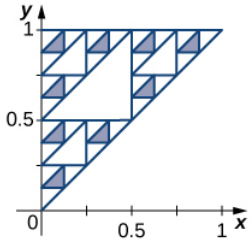
69) [T] Find the total length of the dashed path in the following figure.



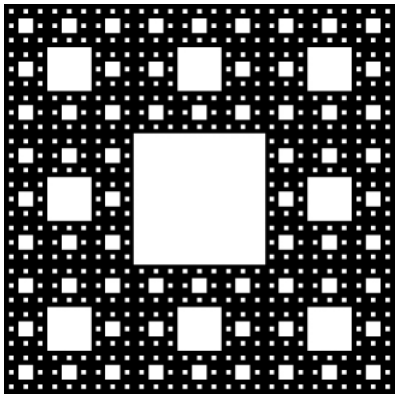
Answer

$$L = \left(1 + \frac{1}{2}\right) \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{3}{2}.$$

70) [T] The **Sierpinski triangle** is obtained from a triangle by deleting the middle fourth as indicated in the first step, by deleting the middle fourths of the remaining three congruent triangles in the second step, and in general deleting the middle fourths of the remaining triangles in each successive step. Assuming that the original triangle is shown in the figure, find the areas of the remaining parts of the original triangle after N steps and find the total length of all of the boundary triangles after N steps.



71) [T] The Sierpinski gasket is obtained by dividing the unit square into nine equal sub-squares, removing the middle square, then doing the same at each stage to the remaining sub-squares. The figure shows the remaining set after four iterations. Compute the total area removed after N stages, and compute the length the total perimeter of the remaining set after N stages.



Answer

At stage one a square of area $1/9$ is removed, at stage 2 one removes 8 squares of area $1/9^2$, at stage three one removes 8^2 squares of area $1/9^3$, and so on. The total removed area after N stages is $\sum_{n=0}^{N-1} \frac{8^N}{9^{N+1}} = \frac{1}{8} \cdot \frac{1-(8/9)^N}{1-8/9} \to 1$ as $N \to \infty$. The total perimeter is $4 + 4\sum_{n=0}^{\infty} \frac{8^N}{3^{N+1}} \to \infty$.

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9.3: The Divergence and Integral Tests

Learning Objectives

- Use the divergence test to determine whether a series converges or diverges.
- Use the integral test to determine the convergence of a series.
- Estimate the value of a series by finding bounds on its remainder term.

In the previous section, we determined the convergence or divergence of several series by explicitly calculating the limit of the sequence of partial sums S_k. In practice, explicitly calculating this limit can be difficult or impossible. Luckily, several tests exist that allow us to determine convergence or divergence for many types of series. In this section, we discuss two of these tests: the divergence test and the integral test. We will examine several other tests in the rest of this chapter and then summarize how and when to use them.

Divergence Test

For a series $\sum_{n=1}^{\infty} a_n$ to converge, the nth term a_n must satisfy $a_n \to 0$ as $n \to \infty$. Therefore, from the algebraic limit properties of sequences,

$$\begin{aligned} \lim_{k\to\infty} a_k &= \lim_{k\to\infty} (S_k - S_{k-1}) \\ &= \lim_{k\to\infty} S_k - \lim_{k\to\infty} S_{k-1} \\ &= S - S = 0. \end{aligned}$$

Therefore, if $\sum_{n=1}^{\infty} a_n$ converges, the nth term $a_n \to 0$ as $n \to \infty$. An important consequence of this fact is the following statement:

$$\text{If } a_n \nrightarrow 0 \text{ as } n \to \infty, \sum_{n=1}^{\infty} a_n \text{ diverges.}$$

This test is known as the divergence test because it provides a way of proving that a series diverges.

Definition: The Divergence Test

If $\lim_{n\to\infty} a_n = c \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

It is important to note that the inverse statement of this theorem is not true. That is, if $\lim_{n\to\infty} a_n = 0$, we cannot make any conclusion about the convergence of $\sum_{n=1}^{\infty} a_n$.

For example, $\lim_{n\to 0} \frac{1}{n} = 0$, but the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. In this section and the remaining sections of this chapter, we show many more examples of such series. Consequently, although we can use the divergence test to show that a series diverges, we cannot use it to prove that a series converges. Specifically, if $a_n \to 0$, the divergence test is inconclusive.

Example 9.3.1: Using the divergence test

For each of the following series, apply the divergence test. If the divergence test proves that the series diverges, state so. Otherwise, indicate that the divergence test is inconclusive.

a. $\sum_{n=1}^{\infty} \frac{n}{3n-1}$

b. $\sum_{n=1}^{\infty} \frac{1}{n^3}$

c. $\sum_{n=1}^{\infty} e^{1/n^2}$

Solution

a. Since $\lim_{n\to\infty} \dfrac{n}{3n-1} = \dfrac{1}{3} \neq 0$, by the divergence test, we can conclude that $\sum_{n=1}^{\infty} \dfrac{n}{3n-1}$ diverges.

b. Since $\lim_{n\to\infty} \dfrac{1}{n^3} = 0$, the divergence test is inconclusive.

c. Since $\lim_{n\to\infty} e^{1/n^2} = 1 \neq 0$, by the divergence test, the series $\sum_{n=1}^{\infty} e^{1/n^2}$ diverges.

? Exercise 9.3.1

What does the divergence test tell us about the series $\sum_{n=1}^{\infty} \cos(1/n^2)$?

Hint

Look at $\lim_{n\to\infty} \cos(1/n^2)$.

Answer

The series diverges.

Integral Test

In the previous section, we proved that the harmonic series diverges by looking at the sequence of partial sums S_k and showing that $S_{2^k} > 1 + k/2$ for all positive integers k. In this section we use a different technique to prove the divergence of the harmonic series. This technique is important because it is used to prove the divergence or convergence of many other series. This test, called the **integral test,** compares an infinite sum to an improper integral. It is important to note that this test can only be applied when we are considering a series whose terms are all positive.

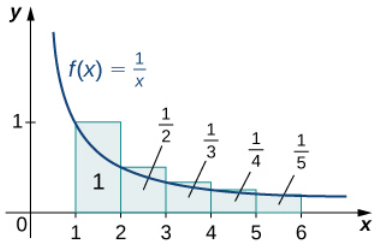

Figure 9.3.1: *The sum of the areas of the rectangles is greater than the area between the curve $f(x) = 1/x$ and the x-axis for $x \geq 1$. Since the area bounded by the curve is infinite (as calculated by an improper integral), the sum of the areas of the rectangles is also infinite.*

To illustrate how the integral test works, use the harmonic series as an example. In Figure 9.3.1, we depict the harmonic series by sketching a sequence of rectangles with areas $1, 1/2, 1/3, 1/4, \ldots$ along with the function $f(x) = 1/x$. From the graph, we see that

$$\sum_{n=1}^{k} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} > \int_1^{k+1} \frac{1}{x}\, dx.$$

Therefore, for each k, the kth partial sum S_k satisfies

$$\begin{aligned} S_k = \sum_{n=1}^{k} \frac{1}{n} > \int_1^{k+1} \frac{1}{x}\, dx &= \ln x\big|_1^{k+1} \\ &= \ln(k+1) - \ln(1) \\ &= \ln(k+1). \end{aligned}$$

Since $\lim_{k\to\infty} \ln(k+1) = \infty$, we see that the sequence of partial sums S_k is unbounded. Therefore, S_k diverges, and, consequently, the series $\sum_{n=1}^{\infty} \dfrac{1}{n}$ also diverges.

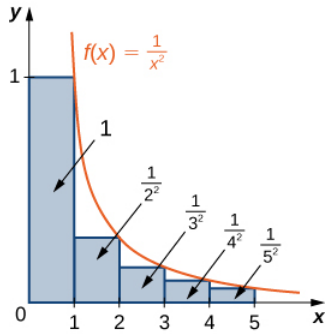

Figure 9.3.2: The sum of the areas of the rectangles is less than the sum of the area of the first rectangle and the area between the curve $f(x) = 1/x^2$ and the x-axis for $x \geq 1$. Since the area bounded by the curve is finite, the sum of the areas of the rectangles is also finite.

Now consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. We show how an integral can be used to prove that this series converges. In Figure 9.3.2, we sketch a sequence of rectangles with areas $1, 1/2^2, 1/3^2, \ldots$ along with the function $f(x) = \frac{1}{x^2}$. From the graph we see that

$$\sum_{n=1}^{k} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} < 1 + \int_1^k \frac{1}{x^2}\, dx.$$

Therefore, for each k, the kth partial sum S_k satisfies

$$\begin{aligned} S_k = \sum_{n=1}^{k} \frac{1}{n^2} < 1 + \int_1^k \frac{1}{x^2}\, dx &= 1 - \frac{1}{x}\Big|_1^k \\ &= 1 - \frac{1}{k} + 1 \\ &= 2 - \frac{1}{k} < 2. \end{aligned}$$

We conclude that the sequence of partial sums S_k is bounded. We also see that S_k is an increasing sequence:

$$S_k = S_{k-1} + \frac{1}{k^2}$$

for $k \geq 2$.

Since S_k is increasing and bounded, by the **Monotone Convergence Theorem**, it converges. Therefore, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

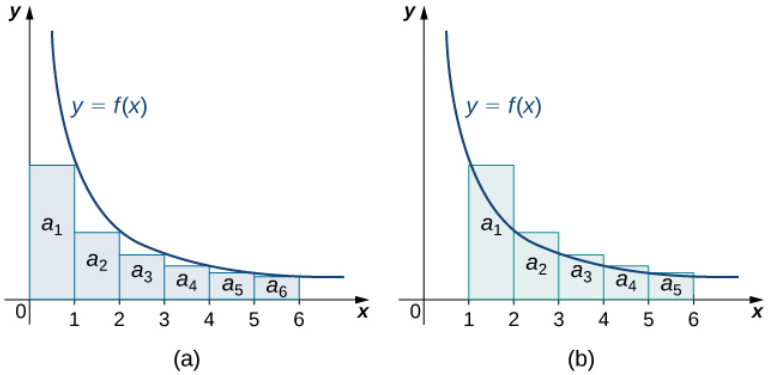

Figure 9.3.3: (a) If we can inscribe rectangles inside a region bounded by a curve $y = f(x)$ and the x-axis, and the area bounded by those curves for $x \geq 1$ is finite, then the sum of the areas of the rectangles is also finite. (b) If a set of rectangles circumscribes the region bounded by $y = f(x)$ and the x axis for $x \geq 1$ and the region has infinite area, then the sum of the areas of the rectangles is also infinite.

We can extend this idea to prove convergence or divergence for many different series. Suppose $\sum_{n=1}^{\infty} a_n$ is a series with positive terms a_n such that there exists a continuous, positive, decreasing function f where $f(n) = a_n$ for all positive integers. Then, as in Figure 9.3.3a, for any integer k, the kth partial sum S_k satisfies

$$S_k = a_1 + a_2 + a_3 + \cdots + a_k < a_1 + \int_1^k f(x)\,dx < 1 + \int_1^{\infty} f(x)\,dx.$$

Therefore, if $\int_1^{\infty} f(x)\,dx$ converges, then the sequence of partial sums S_k is bounded. Since S_k is an increasing sequence, if it is also a bounded sequence, then by the Monotone Convergence Theorem, it converges. We conclude that if $\int_1^{\infty} f(x)\,dx$ converges, then the series $\sum_{n=1}^{\infty} a_n$ also converges. On the other hand, from Figure 9.3.3b, for any integer k, the kth partial sum S_k satisfies

$$S_k = a_1 + a_2 + a_3 + \cdots + a_k > \int_1^{k+1} f(x)\,dx.$$

If

$$\lim_{k\to\infty} \int_1^{k+1} f(x)\,dx = \infty,$$

then S_k is an unbounded sequence and therefore diverges. As a result, the series $\sum_{n=1}^{\infty} a_n$ also diverges. Since f is a positive function, if $\int_1^{\infty} f(x)\,dx$ diverges, then

$$\lim_{k\to\infty} \int_1^{k+1} f(x)\,dx = \infty.$$

We conclude that if $\int_1^{\infty} f(x)\,dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Definition: The Integral Test

Suppose $\sum_{n=1}^{\infty} a_n$ is a series with positive terms a_n. Suppose there exists a function f and a positive integer N such that the following three conditions are satisfied:

i. f is continuous,
ii. f is decreasing, and
iii. $f(n) = a_n$ for all integers $n \geq N$.

Then

$$\sum_{n=1}^{\infty} a_n$$

and

$$\int_N^{\infty} f(x)\,dx$$

both converge or both diverge (Figure 9.3.3).

Although convergence of $\int_N^{\infty} f(x)\,dx$ implies convergence of the related series $\sum_{n=1}^{\infty} a_n$, it does not imply that the value of the integral and the series are the same. They may be different, and often are. For example,

$$\sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n = \frac{1}{e} + \left(\frac{1}{e}\right)^2 + \left(\frac{1}{e}\right)^3 + \cdots$$

is a geometric series with initial term $a = 1/e$ and ratio $r = 1/e$, which converges to

$$\frac{1/e}{1-(1/e)} = \frac{1/e}{(e-1)/e} = \frac{1}{e-1}.$$

However, the related integral $\int_1^\infty (1/e)^x\,dx$ satisfies

$$\int_1^\infty \left(\frac{1}{e}\right)^x dx = \int_1^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_1^b e^{-x}\,dx = \lim_{b\to\infty} -e^{-x}\Big|_1^b = \lim_{b\to\infty}[-e^{-b}+e^{-1}] = \frac{1}{e}.$$

Example 9.3.2: Using the Integral Test

For each of the following series, use the integral test to determine whether the series converges or diverges.

a. $\sum_{n=1}^\infty \frac{1}{n^3}$

b. $\sum_{n=1}^\infty \frac{1}{\sqrt{2n-1}}$

Solution

a. Compare

$$\sum_{n=1}^\infty \frac{1}{n^3} \text{ and } \int_1^\infty \frac{1}{x^3}\,dx.$$

We have

$$\int_1^\infty \frac{1}{x^3}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^3}\,dx = \lim_{b\to\infty}\left[-\frac{1}{2x^2}\Big|_1^b\right] = \lim_{b\to\infty}\left[-\frac{1}{2b^2}+\frac{1}{2}\right] = \frac{1}{2}.$$

Thus the integral $\int_1^\infty \frac{1}{x^3}\,dx$ converges, and therefore so does the series

$$\sum_{n=1}^\infty \frac{1}{n^3}.$$

b. Compare

$$\sum_{n=1}^\infty \frac{1}{\sqrt{2n-1}} \text{ and } \int_1^\infty \frac{1}{\sqrt{2x-1}}\,dx.$$

Since

$$\int_1^\infty \frac{1}{\sqrt{2x-1}}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{\sqrt{2x-1}}\,dx = \lim_{b\to\infty}\sqrt{2x-1}\Big|_1^b = \lim_{b\to\infty}\left[\sqrt{2b-1}-1\right] = \infty,$$

the integral $\int_1^\infty \frac{1}{\sqrt{2x-1}}\,dx$ diverges, and therefore

$$\sum_{n=1}^\infty \frac{1}{\sqrt{2n-1}}$$

diverges.

Exercise 9.3.2

Use the integral test to determine whether the series $\sum_{n=1}^\infty \frac{n}{3n^2+1}$ converges or diverges.

Hint

Compare to the integral $\int_1^\infty \frac{x}{3x^2+1}\,dx$.

Answer

The series diverges.

The p-Series

The harmonic series $\sum_{n=1}^{\infty} 1/n$ and the series $\sum_{n=1}^{\infty} 1/n^2$ are both examples of a type of series called a p-series.

Definition: p-series

For any real number p, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is called a *p-series*.

We know the p-series converges if $p = 2$ and diverges if $p = 1$. What about other values of p? In general, it is difficult, if not impossible, to compute the exact value of most p-series. However, we can use the tests presented thus far to prove whether a p-series converges or diverges.

If $p < 0$, then $1/n^p \to \infty$, and if $p = 0$, then $1/n^p \to 1$. Therefore, by the divergence test,

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

diverges if $p \leq 0$.

If $p > 0$, then $f(x) = 1/x^p$ is a positive, continuous, decreasing function. Therefore, for $p > 0$, we use the integral test, comparing

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

and

$$\int_1^{\infty} \frac{1}{x^p}\, dx.$$

We have already considered the case when $p = 1$. Here we consider the case when $p > 0, p \neq 1$. For this case,

$$\int_1^{\infty} \frac{1}{x^p}\, dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^p}\, dx = \lim_{b\to\infty} \frac{1}{1-p} x^{1-p} \,|_1^b = \lim_{b\to\infty} \frac{1}{1-p}[b^{1-p} - 1].$$

Because

$$b^{1-p} \to 0 \text{ if } p > 1 \text{ and } b^{1-p} \to \infty \text{ if } p < 1,$$

we conclude that

$$\int_1^{\infty} \frac{1}{x^p}\, dx = \begin{cases} \frac{1}{p-1}, \text{if } p > 1 \\ \infty, \text{if } p < 1. \end{cases}$$

Therefore, $\sum_{n=1}^{\infty} 1/n^p$ converges if $p > 1$ and diverges if $0 < p < 1$.

In summary,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1 \end{cases}$$

.

Example 9.3.3: Testing for Convergence of p-series

For each of the following series, determine whether it converges or diverges.

a. $\sum_{n=1}^{\infty} \frac{1}{n^4}$

b. $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$

Solution

a. This is a p-series with $p = 4 > 1$, so the series converges.

b. Since $p = 2/3 < 1$, the series diverges.

Exercise 9.3.3

Does the series $\sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$ converge or diverge?

Hint

$p = 5/4$

Answer

The series converges.

Estimating the Value of a Series

Suppose we know that a series $\sum_{n=1}^{\infty} a_n$ converges and we want to estimate the sum of that series. Certainly we can approximate that sum using any finite sum $\sum_{n=1}^{N} a_n$ where N is any positive integer. The question we address here is, for a convergent series $\sum_{n=1}^{\infty} a_n$, how good is the approximation $\sum_{n=1}^{N} a_n$?

More specifically, if we let

$$R_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n$$

be the remainder when the sum of an infinite series is approximated by the Nth partial sum, how large is R_N? For some types of series, we are able to use the ideas from the integral test to estimate R_N.

Note 9.3.1: Remainder Estimate from the Integral Test

Suppose $\sum_{n=1}^{\infty} a_n$ is a convergent series with positive terms. Suppose there exists a function f satisfying the following three conditions:

i. f is continuous,
ii. f is decreasing, and
iii. $f(n) = a_n$ for all integers $n \geq 1$.

Let S_N be the Nth partial sum of $\sum_{n=1}^{\infty} a_n$. For all positive integers N,

$$S_N + \int_{N+1}^{\infty} f(x)\,dx < \sum_{n=1}^{\infty} a_n < S_N + \int_{N}^{\infty} f(x)\,dx.$$

In other words, the remainder $R_N = \sum_{n=1}^{\infty} a_n - S_N = \sum_{n=N+1}^{\infty} a_n$ satisfies the following estimate:

$$\int_{N+1}^{\infty} f(x)\,dx < R_N < \int_{N}^{\infty} f(x)\,dx.$$

This is known as the **remainder estimate.**

We illustrate Note 9.3.1 in Figure 9.3.4. In particular, by representing the remainder $R_N = a_{N+1} + a_{N+2} + a_{N+3} + \cdots$ as the sum of areas of rectangles, we see that the area of those rectangles is bounded above by $\int_N^{\infty} f(x)\,dx$ and bounded below by $\int_{N+1}^{\infty} f(x)\,dx$. In other words,

$$R_N = a_{N+1} + a_{N+2} + a_{N+3} + \cdots > \int_{N+1}^{\infty} f(x)\,dx$$

and

$$R_N = a_{N+1} + a_{N+2} + a_{N+3} + \cdots < \int_{N}^{\infty} f(x)\,dx.$$

We conclude that

$$\int_{N+1}^{\infty} f(x)\,dx < R_N < \int_{N}^{\infty} f(x)\,dx.$$

Since

$$\sum_{n=1}^{\infty} a_n = S_N + R_N,$$

where S_N is the Nth partial sum, we conclude that

$$S_N + \int_{N+1}^{\infty} f(x)\,dx < \sum_{n=1}^{\infty} a_n < S_N + \int_{N}^{\infty} f(x)\,dx.$$

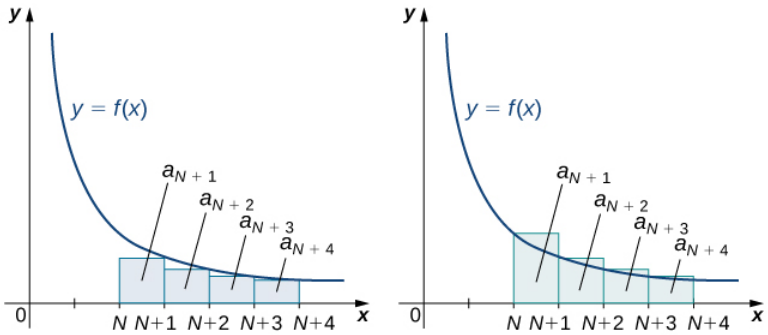

Figure 9.3.4: Given a continuous, positive, decreasing function f and a sequence of positive terms a_n such that $a_n = f(n)$ for all positive integers n, (a) the areas $a_{N+1} + a_{N+2} + a_{N+3} + \cdots < \int_N^{\infty} f(x)\,dx$, *or (b) the areas* $a_{N+1} + a_{N+2} + a_{N+3} + \cdots > \int_{N+1}^{\infty} f(x)\,dx$. *Therefore, the integral is either an overestimate or an underestimate of the error.*

Example 9.3.4: Estimating the Value of a Series

Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

a. Calculate $S_{10} = \sum_{n=1}^{10} \frac{1}{n^3}$ and estimate the error.

b. Determine the least value of N necessary such that S_N will estimate $\sum_{n=1}^{\infty} \frac{1}{n^3}$ to within 0.001.

Solution

a. Using a calculating utility, we have

$$S_{10} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots + \frac{1}{10^3} \approx 1.19753.$$

By the remainder estimate, we know

$$R_N < \int_N^\infty \frac{1}{x^3}\, dx.$$

We have

$$\int_{10}^\infty \frac{1}{x^3}\, dx = \lim_{b\to\infty} \int_{10}^b \frac{1}{x^3}\, dx = \lim_{b\to\infty} \left[-\frac{1}{2x^2}\right]_N^b = \lim_{b\to\infty} \left[-\frac{1}{2b^2} + \frac{1}{2N^2}\right] = \frac{1}{2N^2}.$$

Therefore, the error is $R_{10} < 1/2(10)^2 = 0.005$.

b. Find N such that $R_N < 0.001$. In part a. we showed that $R_N < 1/2N^2$. Therefore, the remainder $R_N < 0.001$ as long as $1/2N^2 < 0.001$. That is, we need $2N^2 > 1000$. Solving this inequality for N, we see that we need $N > 22.36$. To ensure that the remainder is within the desired amount, we need to round up to the nearest integer. Therefore, the minimum necessary value is $N = 23$.

? Exercise 9.3.4

For $\sum_{n=1}^\infty \frac{1}{n^4}$, calculate S_5 and estimate the error R_5.

Hint

Use the remainder estimate $R_N < \int_N^\infty \frac{1}{x^4}\, dx$.

Answer

$S_5 \approx 1.09035, R_5 < 0.00267$

Key Concepts

- If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^\infty a_n$ diverges.
- If $\lim_{n\to\infty} a_n = 0$, the series $\sum_{n=1}^\infty a_n$ may converge or diverge.
- If $\sum_{n=1}^\infty a_n$ is a series with positive terms a_n and f is a continuous, decreasing function such that $f(n) = a_n$ for all positive integers n, then

$$\sum_{n=1}^\infty a_n$$

and

$$\int_1^\infty f(x)\, dx$$

either both converge or both diverge. Furthermore, if $\sum_{n=1}^\infty a_n$ converges, then the Nth partial sum approximation S_N is accurate up to an error R_N where $\int_{N+1}^\infty f(x)\, dx < R_N < \int_N^\infty f(x)\, dx$.

- The p-series $\sum_{n=1}^\infty \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Key Equations

- **Divergence test**

If $a_n \nrightarrow 0$ as $n \to \infty$, $\sum_{n=1}^{\infty} a_n$ diverges.

- ***p*-series**

  $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $\begin{cases} \text{converges, if } p > 1 \\ \text{diverges, if } p \leq 1 \end{cases}$

- **Remainder estimate from the integral test**

  $\int_{N+1}^{\infty} f(x)\,dx < R_N < \int_{N}^{\infty} f(x)\,dx$

Glossary

divergence test

if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges

integral test

for a series $\sum_{n=1}^{\infty} a_n$ with positive terms a_n, if there exists a continuous, decreasing function f such that $f(n) = a_n$ for all positive integers n, then

$$\sum_{n=1}^{\infty} a_n$$

and

$$\int_{1}^{\infty} f(x)\,dx$$

either both converge or both diverge

***p*-series**

a series of the form $\sum_{n=1}^{\infty} 1/n^p$

remainder estimate

for a series $\sum_{n=}^{\infty} 1a_n$ with positive terms a_n and a continuous, decreasing function f such that $f(n) = a_n$ for all positive integers n, the remainder $R_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n$ satisfies the following estimate:

$$\int_{N+1}^{\infty} f(x)\,dx < R_N < \int_{N}^{\infty} f(x)\,dx$$

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9.3E: Exercises for Section 9.3

Divergence Test Problems

Consider the sequence for each series in exercises 1 - 14, if the divergence test applies, either state that $\lim_{n\to\infty} a_n$ does not exist or find $\lim_{n\to\infty} a_n$. If the divergence test does not apply, state why.

1) $\sum_{n=1}^{\infty} \frac{n}{n+2}$

2) $\sum_{n=1}^{\infty} \frac{n}{5n^2-3}$

Answer

$\lim_{n\to\infty} a_n = 0$. The Divergence Test does not apply.

3) $\sum_{n=1}^{\infty} \frac{n}{\sqrt{3n^2+2n+1}}$

4) $\sum_{n=1}^{\infty} \frac{(2n+1)(n-1)}{(n+1)^2}$

Answer

$\lim_{n\to\infty} a_n = 2$. So the series diverges by the nth-Term Test for Divergence.

5) $\sum_{n=1}^{\infty} \frac{(2n+1)^{2n}}{(3n^2+1)^n}$

6) $\sum_{n=1}^{\infty} \frac{2^n}{3^{n/2}}$

Answer

$\lim_{n\to\infty} a_n = \infty$ (does not exist). So the series diverges by the nth-Term Test for Divergence.

7) $\sum_{n=1}^{\infty} \frac{2^n+3^n}{10^{n/2}}$

8) $\sum_{n=1}^{\infty} e^{-2/n}$

Answer

$\lim_{n\to\infty} a_n = 1$. So the series diverges by the nth-Term Test for Divergence.

9) $\sum_{n=1}^{\infty} \cos n$

10) $\sum_{n=1}^{\infty} \tan n$

Answer

$\lim_{n\to\infty} a_n$ does not exist. So the series diverges by the nth-Term Test for Divergence.

11) $\sum_{n=1}^{\infty} \frac{1-\cos^2(1/n)}{\sin^2(2/n)}$

12) $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^{2n}$

Answer

$\lim_{n\to\infty} a_n = 1/e^2$. So the series diverges by the nth-Term Test for Divergence.

13) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

14) $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{\sqrt{n}}$

Answer

$\lim_{n\to\infty} a_n = 0$. The Divergence Test does not apply.

p-Series Problems & Integral Test Problems

In exercises 15 - 20, state whether the given p-series converges.

15) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

16) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

Answer

The series converges, since $p = 3/2 > 1$.

17) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2}}$

18) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^4}}$

Answer

The series converges, since $p = 4/3 > 1$.

19) $\sum_{n=1}^{\infty} \frac{n^e}{n^\pi}$

20) $\sum_{n=1}^{\infty} \frac{n^\pi}{n^{2e}}$

Answer

The series converges, since $p = 2e - \pi > 1$.

In exercises 21 - 27, use the integral test to determine whether the following sums converge.

21) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+5}}$

22) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n+5}}$

Answer

The series diverges by the Integral Test since $\int_1^{\infty} \frac{dx}{(x+5)^{1/3}}$ can be shown to diverge.

23) $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$

24) $\sum_{n=1}^{\infty} \frac{n}{1+n^2}$

Answer

The series diverges by the Integral Test since $\int_1^{\infty} \frac{x}{1+x^2}\,dx$ can be shown to diverge.

25) $\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}$

26) $\sum_{n=1}^{\infty} \frac{2n}{1+n^4}$

Answer

The series converges by the Integral Test since $\int_1^{\infty} \frac{2x}{1+x^4}\,dx$ converges.

27) $\sum_{n=2}^{\infty} \frac{1}{n \ln^2 n}$

Express the sums in exercises 28 - 31 as p-series and determine whether each converges.

28) $\sum_{n=1}^{\infty} 2^{-\ln n}$

$\left(\text{Hint: } 2^{-\ln n} = \frac{1}{n^{\ln 2}}.\right)$

Answer

$2^{-\ln n} = 1/n^{\ln 2}$. Since $p = \ln 2 < 1$, this series diverges by the p-series test.

29) $\sum_{n=1}^{\infty} 3^{-\ln n}$

$\left(\text{Hint: } 3^{-\ln n} = \frac{1}{n^{\ln 3}}.\right)$

30) $\sum_{n=1}^{n} 2^{-2\ln n}$

Answer

$2^{-2\ln n} = 1/n^{2\ln 2}$. Since $p = 2\ln 2 - 1 < 1$, this series diverges by the p-series test.

31) $\sum_{n=1}^{\infty} n3^{-2\ln n}$

In exercises 32 - 35, use the estimate $R_N \le \int_N^{\infty} f(t)\,dt$ to find a bound for the remainder $R_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n$ where $a_n = f(n)$.

32) $\sum_{n=1}^{1000} \frac{1}{n^2}$

Answer

$$R_{1000} \le \int_{1000}^{\infty} \frac{dt}{t^2} = \lim_{b\to\infty} -\frac{1}{t}\Big|_{1000}^{b} = \lim_{b\to\infty} \left(-\frac{1}{b} + \frac{1}{1000}\right) = 0.001$$

33) $\sum_{n=1}^{1000} \frac{1}{n^3}$

34) $\sum_{n=1}^{1000} \frac{1}{1+n^2}$

Answer

$$R_{1000} \le \int_{1000}^{\infty} \frac{dt}{1+t^2} = \lim_{b\to\infty} \left(\tan^{-1} b - \tan^{-1}(1000)\right) = \pi/2 - \tan^{-1}(1000) \approx 0.000999$$

35) $\sum_{n=1}^{100} \frac{n}{2^n}$

[T] In exercises 36 - 40, find the minimum value of N such that the remainder estimate $\int_{N+1}^{\infty} f(x)\,dx < R_N < \int_N^{\infty} f(x)\,dx$ guarantees that $\sum_{n=1}^{N} a_n$ estimates $\sum_{n=1}^{\infty} a_n$, accurate to within the given error.

36) $a_n = \dfrac{1}{n^2}$, error $< 10^{-4}$

Answer

$R_N < \displaystyle\int_N^\infty \frac{dx}{x^2} = 1/N, \text{ for } N > 10^4$

37) $a_n = \dfrac{1}{n^{1.1}}$, error $< 10^{-4}$

38) $a_n = \dfrac{1}{n^{1.01}}$, error $< 10^{-4}$

Answer

$R_N < \displaystyle\int_N^\infty \frac{dx}{x^{1.01}} = 100N^{-0.01}, \text{ for } N > 10^{600}$

39) $a_n = \dfrac{1}{n \ln^2 n}$, error $< 10^{-3}$

40) $a_n = \dfrac{1}{1+n^2}$, error $< 10^{-3}$

Answer

$R_N < \displaystyle\int_N^\infty \frac{dx}{1+x^2} = \pi/2 - \tan^{-1}(N), \text{ for } N > \tan(\pi/2 - 10^{-3}) \approx 1000$

In exercises 41 - 45, find a value of N such that R_N is smaller than the desired error. Compute the corresponding sum $\displaystyle\sum_{n=1}^{N} a_n$ and compare it to the given estimate of the infinite series.

41) $a_n = \dfrac{1}{n^{11}}$, error $< 10^{-4}, \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{11}} = 1.000494\ldots$

42) $a_n = \dfrac{1}{e^n}$, error $< 10^{-5}, \displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n} = \frac{1}{e-1} = 0.581976\ldots$

Answer

$R_N < \displaystyle\int_N^\infty \frac{dx}{e^x} = e^{-N}, \text{ for } N > 5\ln(10)$, okay if $N = 12; \displaystyle\sum_{n=1}^{12} e^{-n} = 0.581973\ldots$. Estimate agrees with $1/(e-1)$ to five decimal places.

43) $a_n = \dfrac{1}{e^{n^2}}$, error $< 10^{-5}, \displaystyle\sum_{n=1}^{\infty} \frac{1}{e^{n^2}} = 0.40488139857\ldots$

44) $a_n = \dfrac{1}{n^4}$, error $< 10^{-4}, \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} = 1.08232\ldots$

Answer

$R_N < \displaystyle\int_N^\infty \frac{dx}{x^4} = 4/N^3, \text{ for } N > (4.10^4)^{1/3}$, okay if $N = 35; \displaystyle\sum_{n=1}^{35} \frac{1}{n^4} = 1.08231\ldots$. Estimate agrees with the sum to four decimal places.

45) $a_n = \dfrac{1}{n^6}$, error $< 10^{-6}, \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945} = 1.01734306\ldots,$

46) Find the limit as $n \to \infty$ of $\dfrac{1}{n} + \dfrac{1}{n+1} + \cdots + \dfrac{1}{2n}$.
$\left(\text{Hint: Compare to } \displaystyle\int_n^{2n} \frac{1}{t}\, dt.\right)$

Answer

$\ln(2)$

47) Find the limit as $n \to \infty$ of $\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{3n}$

The next few exercises are intended to give a sense of applications in which partial sums of the harmonic series arise.

48) In certain applications of probability, such as the so-called Watterson estimator for predicting mutation rates in population genetics, it is important to have an accurate estimate of the number $H_k = (1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k})$. Recall that $T_k = H_k - \ln k$ is decreasing. Compute $T = \lim_{k \to \infty} T_k$ to four decimal places.

$\left(\text{Hint: } \frac{1}{k+1} < \int_k^{k+1} \frac{1}{x}\, dx.\right)$

Answer

$T = 0.5772...$

49) [T] Complete sampling with replacement, sometimes called the **coupon collector's problem**, is phrased as follows: Suppose you have N unique items in a bin. At each step, an item is chosen at random, identified, and put back in the bin. The problem asks what is the expected number of steps $E(N)$ that it takes to draw each unique item at least once. It turns out that $E(N) = N.H_N = N\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N}\right)$. Find $E(N)$ for $N = 10, 20,$ and 50.

50) [T] The simplest way to shuffle cards is to take the top card and insert it at a random place in the deck, called top random insertion, and then repeat. We will consider a deck to be randomly shuffled once enough top random insertions have been made that the card originally at the bottom has reached the top and then been randomly inserted. If the deck has n cards, then the probability that the insertion will be below the card initially at the bottom (call this card B) is $1/n$. Thus the expected number of top random insertions before B is no longer at the bottom is n. Once one card is below B, there are two places below B and the probability that a randomly inserted card will fall below B is $2/n$. The expected number of top random insertions before this happens is $n/2$. The two cards below B are now in random order. Continuing this way, find a formula for the expected number of top random insertions needed to consider the deck to be randomly shuffled.

Answer

The expected number of random insertions to get B to the top is $n + n/2 + n/3 + \cdots + n/(n-1)$. Then one more insertion puts B back in at random. Thus, the expected number of shuffles to randomize the deck is $n(1 + 1/2 + \cdots + 1/n)$.

51) Suppose a scooter can travel 100 km on a full tank of fuel. Assuming that fuel can be transferred from one scooter to another but can only be carried in the tank, present a procedure that will enable one of the scooters to travel $100H_N$ km, where $H_N = 1 + 1/2 + \cdots + 1/N$.

52) Show that for the remainder estimate to apply on $[N, \infty)$ it is sufficient that $f(x)$ be decreasing on $[N, \infty)$, but f need not be decreasing on $[1, \infty)$.

Answer

Set $b_n = a_{n+N}$ and $g(t) = f(t + N)$ such that f is decreasing on $[t, \infty)$.

53) [T] Use the remainder estimate and integration by parts to approximate $\sum_{n=1}^{\infty} \frac{n}{e^n}$ within an error smaller than 0.0001.

54) Does $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converge if p is large enough? If so, for which p?

Answer

The series converges for $p > 1$ by integral test using change of variable.

55) [T] Suppose a computer can sum one million terms per second of the divergent series $\sum_{n=1}^{N} \frac{1}{n}$. Use the integral test to approximate how many seconds it will take to add up enough terms for the partial sum to exceed 100.

56) [T] A fast computer can sum one million terms per second of the divergent series $\sum_{n=2}^{N} \frac{1}{n \ln n}$. Use the integral test to approximate how many seconds it will take to add up enough terms for the partial sum to exceed 100.

Answer

$N = e^{e^{100}} \approx e^{10^{43}}$ terms are needed.

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9.4: Comparison Tests

Learning Objectives

- Use the comparison test to test a series for convergence.
- Use the limit comparison test to determine convergence of a series.

We have seen that the integral test allows us to determine the convergence or divergence of a series by comparing it to a related improper integral. In this section, we show how to use comparison tests to determine the convergence or divergence of a series by comparing it to a series whose convergence or divergence is known. Typically these tests are used to determine convergence of series that are similar to geometric series or p-series.

Comparison Test

In the preceding two sections, we discussed two large classes of series: geometric series and p-series. We know exactly when these series converge and when they diverge. Here we show how to use the convergence or divergence of these series to prove convergence or divergence for other series, using a method called the **comparison test.**

For example, consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}.$$

This series looks similar to the convergent series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Since the terms in each of the series are positive, the sequence of partial sums for each series is monotone increasing. Furthermore, since

$$0 < \frac{1}{n^2+1} < \frac{1}{n^2}$$

for all positive integers n, the kth partial sum S_k of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ satisfies

$$S_k = \sum_{n=1}^{k} \frac{1}{n^2+1} < \sum_{n=1}^{k} \frac{1}{n^2} < \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

(See Figure $9.4.1a$ and Table $9.4.1$.) Since the series on the right converges, the sequence S_k is bounded above. We conclude that S_k is a monotone increasing sequence that is bounded above. Therefore, by the Monotone Convergence Theorem, S_k converges, and thus

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

converges.

Similarly, consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n-1/2}.$$

This series looks similar to the divergent series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

The sequence of partial sums for each series is monotone increasing and

$$\frac{1}{\quad} > \frac{1}{\quad} > 0$$

$$n-1/2 \quad n$$

for every positive integer n. Therefore, the kth partial sum S_k of

$$\sum_{n=1}^{\infty} \frac{1}{n-1/2}$$

satisfies

$$S_k = \sum_{n=1}^{k} \frac{1}{n-1/2} > \sum_{n=1}^{k} \frac{1}{n}.$$

(See Figure 9.4.1n and Table 9.4.1). Since the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to infinity, the sequence of partial sums $\sum_{n=1}^{k} \frac{1}{n}$ is unbounded. Consequently, S_k is an unbounded sequence, and therefore diverges. We conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n-1/2}$$

diverges.

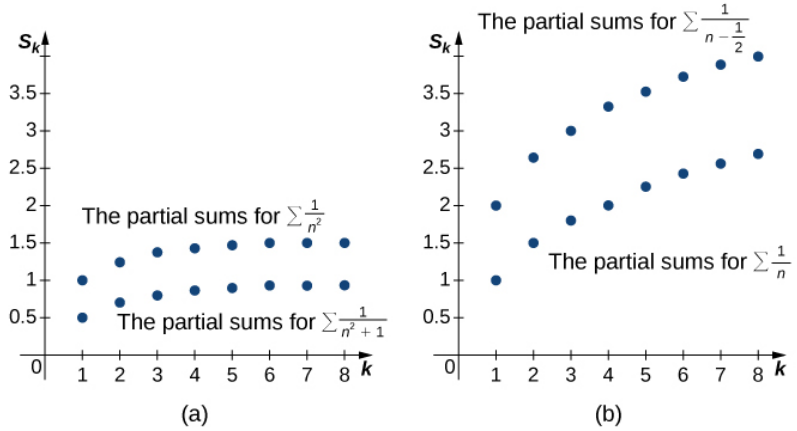

Figure 9.4.1: (a) Each of the partial sums for the given series is less than the corresponding partial sum for the converging $p-series$. (b) Each of the partial sums for the given series is greater than the corresponding partial sum for the diverging harmonic series.

Table 9.4.1: Comparing a series with a p-series ($p = 2$)

k	1	2	3	4	5	6	7	8
$\sum_{n=1}^{k} \frac{1}{n^2+1}$	0.5	0.7	0.8	0.8588	0.8973	0.9243	0.9443	0.9597
$\sum_{n=1}^{k} \frac{1}{n^2}$	1	1.25	1.3611	1.4236	1.4636	1.4914	1.5118	1.5274

Table 9.4.2: Comparing a series with the harmonic series

k	1	2	3	4	5	6	7	8
$\sum_{n=1}^{k} \frac{1}{n-1/2}$	2	2.6667	3.0667	3.3524	3.5746	3.7564	3.9103	4.0436
$\sum_{n=1}^{k} \frac{1}{n}$	1	1.5	1.8333	2.0933	2.2833	2.45	2.5929	2.7179

Comparison Test

i. Suppose there exists an integer N such that $0 \le a_n \le b_n$ for all $n \ge N$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

ii. Suppose there exists an integer N such that $a_n \ge b_n \ge 0$ for all $n \ge N$. If $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof

We prove part i. The proof of part ii. is the contrapositive of part i. Let S_k be the sequence of partial sums associated with $\sum_{n=1}^{\infty} a_n$, and let $L = \sum_{n=1}^{\infty} b_n$. Since the terms $a_n \ge 0$,

$$S_k = a_1 + a_2 + \cdots + a_k \le a_1 + a_2 + \cdots + a_k + a_{k+1} = S_{k+1}.$$

Therefore, the sequence of partial sums is increasing. Further, since $a_n \le b_n$ for all $n \ge N$, then

$$\sum_{n=N}^{k} a_n \le \sum_{n=N}^{k} b_n \le \sum_{n=1}^{\infty} b_n = L.$$

Therefore, for all $k \ge 1$,

$$S_k = (a_1 + a_2 + \cdots + a_{N-1}) + \sum_{n=N}^{k} a_n \le (a_1 + a_2 + \cdots + a_{N-1}) + L.$$

Since $a_1 + a_2 + \cdots + a_{N-1}$ is a finite number, we conclude that the sequence S_k is bounded above. Therefore, S_k is an increasing sequence that is bounded above. By the Monotone Convergence Theorem, we conclude that S_k converges, and therefore the series $\sum_{n=1}^{\infty} a_n$ converges.

□

To use the comparison test to determine the convergence or divergence of a series $\sum_{n=1}^{\infty} a_n$, it is necessary to find a suitable series with which to compare it. Since we know the convergence properties of geometric series and p-series, these series are often used. If there exists an integer N such that for all $n \ge N$, each term a_n is less than each corresponding term of a known convergent series, then $\sum_{n=1}^{\infty} a_n$ converges. Similarly, if there exists an integer N such that for all $n \ge N$, each term a_n is greater than each corresponding term of a known divergent series, then $\sum_{n=1}^{\infty} a_n$ diverges.

Example 9.4.1: Using the Comparison Test

For each of the following series, use the comparison test to determine whether the series converges or diverges.

a. $\sum_{n=1}^{\infty} \dfrac{1}{n^3 + 3n + 1}$

b. $\sum_{n=1}^{\infty} \dfrac{1}{2^n + 1}$

c. $\sum_{n=2}^{\infty} \dfrac{1}{\ln n}$

Solution

a. Compare to $\sum_{n=1}^{\infty} \dfrac{1}{n^3}$. Since $\sum_{n=1}^{\infty} \dfrac{1}{n^3}$ is a p-series with $p = 3$, it converges. Further,

$$\frac{1}{n^3 + 3n + 1} < \frac{1}{n^3}$$

for every positive integer n. Therefore, we can conclude that $\sum_{n=1}^{\infty} \dfrac{1}{n^3 + 3n + 1}$ converges.

b. Compare to $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$. Since $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$ is a geometric series with $r=\frac{1}{2}$ and $\left|\frac{1}{2}\right|<1$, it converges. Also,

$$\frac{1}{2^n+1}<\frac{1}{2^n}$$

for every positive integer n. Therefore, we see that $\sum_{n=1}^{\infty}\frac{1}{2^n+1}$ converges.

c. Compare to $\sum_{n=2}^{\infty}\frac{1}{n}$. Since

$$\frac{1}{\ln n}>\frac{1}{n}$$

for every integer $n\geq 2$ and $\sum_{n=2}^{\infty}\frac{1}{n}$ diverges, we have that $\sum_{n=2}^{\infty}\frac{1}{\ln n}$ diverges.

? Exercise 9.4.1

Use the comparison test to determine if the series $\sum_{n=1}^{\infty}\frac{n}{n^3+n+1}$ converges or diverges.

Hint

Find a value p such that $\frac{n}{n^3+n+1}\leq\frac{1}{n^p}$.

Answer

The series converges.

Limit Comparison Test

The comparison test works nicely if we can find a comparable series satisfying the hypothesis of the test. However, sometimes finding an appropriate series can be difficult. Consider the series

$$\sum_{n=2}^{\infty}\frac{1}{n^2-1}.$$

It is natural to compare this series with the convergent series

$$\sum_{n=2}^{\infty}\frac{1}{n^2}.$$

However, this series does not satisfy the hypothesis necessary to use the comparison test because

$$\frac{1}{n^2-1}>\frac{1}{n^2}$$

for all integers $n\geq 2$. Although we could look for a different series with which to compare $\sum_{n=2}^{\infty}\frac{1}{n^2-1}$, instead we show how we can use the **limit comparison test** to compare

$$\sum_{n=2}^{\infty}\frac{1}{n^2-1}$$

and

$$\sum_{n=2}^{\infty}\frac{1}{n^2}.$$

Let us examine the idea behind the limit comparison test. Consider two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$. with positive terms a_n and b_n and evaluate

$$\lim_{n\to\infty} \frac{a_n}{b_n}.$$

If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0,$$

then, for n sufficiently large, $a_n \approx Lb_n$. Therefore, either both series converge or both series diverge. For the series $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ and $\sum_{n=2}^{\infty} \frac{1}{n^2}$, we see that

$$\lim_{n\to\infty} \frac{1/(n^2-1)}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{n^2-1} = 1.$$

Since $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges, we conclude that

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}$$

converges.

The limit comparison test can be used in two other cases. Suppose

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

In this case, a_n/b_n is a bounded sequence. As a result, there exists a constant M such that $a_n \leq Mb_n$. Therefore, if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. On the other hand, suppose

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty.$$

In this case, a_n/b_n is an unbounded sequence. Therefore, for every constant M there exists an integer N such that $a_n \geq Mb_n$ for all $n \geq N$. Therefore, if $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges as well.

Limit Comparison Test

Let $a_n, b_n \geq 0$ for all $n \geq 1$.

i. If $\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.

ii. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

iii. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Note that if $\frac{a_n}{b_n} \to 0$ and $\sum_{n=1}^{\infty} b_n$ diverges, the limit comparison test gives no information. Similarly, if $\frac{a_n}{b_n} \to \infty$ and $\sum_{n=1}^{\infty} b_n$ converges, the test also provides no information. For example, consider the two series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$. These series are both p-series with $p = \frac{1}{2}$ and $p = 2$, respectively. Since $p = \frac{1}{2} < 1$, the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges. On the other hand, since $p = 2 > 1$, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$

converges. However, suppose we attempted to apply the limit comparison test, using the convergent p**−series** $\sum_{n=1}^{\infty} \frac{1}{n^3}$ as our comparison series. First, we see that

$$\frac{1/\sqrt{n}}{1/n^3} = \frac{n^3}{\sqrt{n}} = n^{5/2} \to \infty \text{ as } n \to \infty.$$

Similarly, we see that

$$\frac{1/n^2}{1/n^3} = n \to \infty \text{ as } n \to \infty.$$

Therefore, if $\frac{a_n}{b_n} \to \infty$ when $\sum_{n=1}^{\infty} b_n$ converges, we do not gain any information on the convergence or divergence of $\sum_{n=1}^{\infty} a_n$.

Example 9.4.2: Using the Limit Comparison Test

For each of the following series, use the limit comparison test to determine whether the series converges or diverges. If the test does not apply, say so.

a. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$
b. $\sum_{n=1}^{\infty} \frac{2^n+1}{3^n}$
c. $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$

Solution

a. Compare this series to $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. Calculate

$$\lim_{n\to\infty} \frac{1/(\sqrt{n}+1)}{1/\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+1} = \lim_{n\to\infty} \frac{1/\sqrt{n}}{1+1/\sqrt{n}} = 1.$$

By the limit comparison test, since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, then $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$ diverges.

b. Compare this series to $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$. We see that

$$\lim_{n\to\infty} \frac{(2^n+1)/3^n}{2^n/3^n} = \lim_{n\to\infty} \frac{2^n+1}{3^n} \cdot \frac{3^n}{2^n} = \lim_{n\to\infty} \frac{2^n+1}{2^n} = \lim_{n\to\infty} \left[1+\left(\tfrac{1}{2}\right)^n\right] = 1.$$

Therefore,

$$\lim_{n\to\infty} \frac{(2^n+1)/3^n}{2^n/3^n} = 1.$$

Since $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ converges, we conclude that $\sum_{n=1}^{\infty} \frac{2^n+1}{3^n}$ converges.

c. Since $\ln n < n$, compare with $\sum_{n=1}^{\infty} \frac{1}{n}$. We see that

$$\lim_{n\to\infty} \frac{\ln n/n^2}{1/n} = \lim_{n\to\infty} \frac{\ln n}{n^2} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{\ln n}{n}.$$

In order to evaluate $\lim_{n\to\infty} \ln n/n$, evaluate the limit as $x \to \infty$ of the real-valued function $\ln(x)/x$. These two limits are equal, and making this change allows us to use L'Hôpital's rule. We obtain

$$\lim_{x\to\infty} \frac{lnx}{x} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

Therefore, $\lim_{n\to\infty} \frac{\ln n}{n} = 0$, and, consequently,

$$\lim_{n\to\infty} \frac{(\ln n)/n^2}{1/n} = 0.$$

Since the limit is 0 but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the limit comparison test does not provide any information.

Compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$ instead. In this case,

$$\lim_{n\to\infty} \frac{(\ln n)/n^2}{1/n^2} = \lim_{n\to\infty} \frac{\ln n}{n^2} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \ln n = \infty.$$

Since the limit is ∞ but $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, the test still does not provide any information.

So now we try a series between the two we already tried. Choosing the series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, we see that

$$\lim_{n\to\infty} \frac{(\ln n)/n^2}{1/n^{3/2}} = \lim_{n\to\infty} \frac{\ln n}{n^2} \cdot \frac{n^{3/2}}{1} = \lim_{n\to\infty} \frac{\ln n}{\sqrt{n}}.$$

As above, in order to evaluate $\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}}$, evaluate the limit as $x \to \infty$ of the real-valued function $\frac{\ln n}{\sqrt{n}}$. Using L'Hôpital's rule,

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{2\sqrt{x}}{x} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0.$$

Since the limit is 0 and $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, we can conclude that $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges.

? Exercise 9.4.2

Use the limit comparison test to determine whether the series $\sum_{n=1}^{\infty} \frac{5^n}{3^n + 2}$ converges or diverges.

Hint

Compare with a geometric series.

Answer

The series diverges.

Key Concepts

- The comparison tests are used to determine convergence or divergence of series with positive terms.
- When using the comparison tests, a series $\sum_{n=1}^{\infty} a_n$ is often compared to a geometric or p-series.

Glossary

comparison test

If $0 \le a_n \le b_n$ for all $n \ge N$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges; if $a_n \ge b_n \ge 0$ for all $n \ge N$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

limit comparison test

Suppose $a_n, b_n \ge 0$ for all $n \ge 1$. If $\lim_{n\to\infty} a_n/b_n \to L \ne 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge; if $\lim_{n\to\infty} a_n/b_n \to 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. If $\lim_{n\to\infty} a_n/b_n \to \infty$, and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

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9.4E: Exercises for Section 9.4

Use the Comparison Test to determine whether each series in exercises 1 - 13 converges or diverges.

1) $\sum_{n=1}^{\infty} a_n$ where $a_n = \dfrac{2}{n(n+1)}$

2) $\sum_{n=1}^{\infty} a_n$ where $a_n = \dfrac{1}{n(n+1/2)}$

Answer

Converges by comparison with $\dfrac{1}{n^2}$.

3) $\sum_{n=1}^{\infty} \dfrac{1}{2(n+1)}$

4) $\sum_{n=1}^{\infty} \dfrac{1}{2n-1}$

Answer

Since $2n-1 < 2n$, we have $\dfrac{1}{2n} < \dfrac{1}{2n-1} \rightarrow \sum_{n=1}^{\infty} \dfrac{1}{2n} < \sum_{n=1}^{\infty} \dfrac{1}{2n-1}$.

But the series on the left is just half of the harmonic series, since $\sum_{n=1}^{\infty} \dfrac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \dfrac{1}{n}$. This clearly diverges, so the given series also diverges, by the Comparison Test.

5) $\sum_{n=2}^{\infty} \dfrac{1}{(n \ln n)^2}$

6) $\sum_{n=1}^{\infty} \dfrac{n!}{(n+2)!}$

Answer

$a_n = 1/(n+1)(n+2) < 1/n^2$. Converges by comparison with p-series, $p = 2 > 1$.

7) $\sum_{n=1}^{\infty} \dfrac{1}{n!}$

8) $\sum_{n=1}^{\infty} \dfrac{\sin(1/n)}{n}$

Answer

$\sin(1/n) \leq 1/n$, so converges by comparison with p-series, $p = 2 > 1$.

9) $\sum_{n=1}^{\infty} \dfrac{\sin^2 n}{n^2}$

10) $\sum_{n=1}^{\infty} \dfrac{\sin(1/n)}{\sqrt{n}}$

Answer

$\sin(1/n) \leq 1$, so converges by comparison with p-series, $p = 3/2 > 1$.

11) $\sum_{n=1}^{\infty} \dfrac{n^{1.2} - 1}{n^{2.3} + 1}$

12) $\sum_{n=1}^{\infty} \dfrac{\sqrt{n+1} - \sqrt{n}}{n}$

Answer

Since $\sqrt{n+1} - \sqrt{n} = 1/(\sqrt{n+1} + \sqrt{n}) \leq 2/\sqrt{n}$, series converges by comparison with p-series for $p = 1.5 > 1$.

13) $\sum_{n=1}^{\infty} \frac{\sqrt[4]{n}}{\sqrt[3]{n^4+n^2}}$

Use the Limit Comparison Test to determine whether each series in exercises 14 - 28 converges or diverges.

14) $\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^2$

Answer

Converges by limit comparison with p-series for $p > 1$.

15) $\sum_{n=1}^{\infty} \left(\frac{\ln n}{n^{0.6}}\right)^2$

16) $\sum_{n=1}^{\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{n}$

Answer

Converges by limit comparison with p-series, $p = 2 > 1$.

17) $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n^2}\right)$

18) $\sum_{n=1}^{\infty} \frac{1}{4^n - 3^n}$

Answer

Converges by limit comparison with 4^{-n}.

19) $\sum_{n=1}^{\infty} \frac{1}{n^2 - n\sin n}$

20) $\sum_{n=1}^{\infty} \frac{1}{e^{(1.1)n} - 3^n}$

Answer

Converges by limit comparison with $1/e^{1.1n}$.

21) $\sum_{n=1}^{\infty} \frac{1}{e^{(1.01)n} - 3^n}$

22) $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$

Answer

Diverges by limit comparison with harmonic series.

23) $\sum_{n=1}^{\infty} \frac{1}{2^{1+1/n}} n^{1+1/n}$

24) $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \sin\left(\frac{1}{n}\right)\right)$

Answer

Converges by limit comparison with p-series, $p = 3 > 1$.

25) $\sum_{n=1}^{\infty} \left(1 - \cos\left(\frac{1}{n}\right)\right)$

26) $\sum_{n=1}^{\infty} \frac{1}{n}\left(\tan^{-1} n - \frac{\pi}{2}\right)$

Answer

Converges by limit comparison with p-series, $p = 3 > 1$.

27) $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^{n.n}$ (Hint: $\left(1 - \frac{1}{n}\right)^{n} \to 1/e$.)

28) $\sum_{n=1}^{\infty} \left(1 - e^{-1/n}\right)$ (Hint:$1/e \approx (1 - 1/n)^n$, so $1 - e^{-1/n} \approx 1/n$.)

Answer

Diverges by limit comparison with $1/n$.

29) Does $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^p}$ converge if p is large enough? If so, for which p?

30) Does $\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^p$ converge if p is large enough? If so, for which p?

Answer

Converges for $p > 1$ by comparison with a p series for slightly smaller p.

31) For which p does the series $\sum_{n=1}^{\infty} 2^{pn}/3^n$ converge?

32) For which $p > 0$ does the series $\sum_{n=1}^{\infty} \frac{n^p}{2^n}$ converge?

Answer

Converges for all $p > 0$.

33) For which $r > 0$ does the series $\sum_{n=1}^{\infty} \frac{r^{n^2}}{2^n}$ converge?

34) For which $r > 0$ does the series $\sum_{n=1}^{\infty} \frac{2^n}{r^{n^2}}$ converge?

Answer

Converges for all $r > 1$. If $r > 1$ then $r^n > 4$, say, once $n > \ln(2)/\ln(r)$ and then the series converges by limit comparison with a geometric series with ratio $1/2$.

35) Find all values of p and q such that $\sum_{n=1}^{\infty} \frac{n^p}{(n!)^q}$ converges.

36) Does $\sum_{n=1}^{\infty} \frac{\sin^2(nr/2)}{n}$ converge or diverge? Explain.

Answer

The numerator is equal to 1 when n is odd and 0 when n is even, so the series can be rewritten $\sum_{n=1}^{\infty} \frac{1}{2n+1}$, which diverges by limit comparison with the harmonic series.

37) Explain why, for each n, at least one of $|\sin n|, |\sin(n+1)|, ..., |\sin(n+6)|$ is larger than $1/2$. Use this relation to test convergence of $\sum_{n=1}^{\infty} \frac{|\sin n|}{\sqrt{n}}$.

38) Suppose that $a_n \geq 0$ and $b_n \geq 0$ and that $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ converge. Prove that $\sum_{n=1}^{\infty} a_n b_n$ converges and $\sum_{n=1}^{\infty} a_n b_n \leq \frac{1}{2}\left(\sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2\right)$.

Answer

$(a-b)^2 = a^2 - 2ab + b^2$ or $a^2 + b^2 \geq 2ab$, so convergence follows from comparison of $2a_nb_n$ with $a_n^2 + b_n^2$. Since the partial sums on the left are bounded by those on the right, the inequality holds for the infinite series.

39) Does $\sum_{n=1}^{\infty} 2^{-\ln \ln n}$ converge? (Hint: Write $2^{\ln \ln n}$ as a power of $\ln n$.)

40) Does $\sum_{n=1}^{\infty} (\ln n)^{-\ln n}$ converge? (Hint: Use $t = e^{\ln(t)}$ to compare to a $p-$series.)

Answer

$(\ln n)^{-\ln n} = e^{-\ln(n)\ln\ln(n)}$. If n is sufficiently large, then $\ln \ln n > 2$, so $(\ln n)^{-\ln n} < 1/n^2$, and the series converges by comparison to a $p-$series.

41) Does $\sum_{n=2}^{\infty} (\ln n)^{-\ln \ln n}$ converge? (Hint: Compare a_n to $1/n$.)

42) Show that if $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^2$ converges. If $\sum_{n=1}^{\infty} a_n^2$ converges, does $\sum_{n=1}^{\infty} a_n$ necessarily converge?

Answer

$a_n \to 0$, so $a_n^2 \leq |a_n|$ for large n. Convergence follows from limit comparison. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, but $\sum_{n=1}^{\infty} \frac{1}{n}$ does not, so the fact that $\sum_{n=1}^{\infty} a_n^2$ converges does not imply that $\sum_{n=1}^{\infty} a_n$ converges.

43) Suppose that $a_n > 0$ for all n and that $\sum_{n=1}^{\infty} a_n$ converges. Suppose that b_n is an arbitrary sequence of zeros and ones. Does $\sum_{n=1}^{\infty} a_nb_n$ necessarily converge?

44) Suppose that $a_n > 0$ for all n and that $\sum_{n=1}^{\infty} a_n$ diverges. Suppose that b_n is an arbitrary sequence of zeros and ones with infinitely many terms equal to one. Does $\sum_{n=1}^{\infty} a_nb_n$ necessarily diverge?

Answer

No. $\sum_{n=1}^{\infty} 1/n$ diverges. Let $b_k = 0$ unless $k = n^2$ for some n. Then $\sum_k b_k/k = \sum 1/k^2$ converges.

45) Complete the details of the following argument: If $\sum_{n=1}^{\infty} \frac{1}{n}$ converges to a finite sum s, then $\frac{1}{2}s = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots$ and $s - \frac{1}{2}s = 1 + \frac{1}{3} + \frac{1}{5} + \cdots$. Why does this lead to a contradiction?

46) Show that if $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n^2$ converges, then $\sum_{n=1}^{\infty} \sin^2(a_n)$ converges.

Answer

$|\sin t| \leq |t|$, so the result follows from the comparison test.

47) Suppose that $a_n/b_n \to 0$ in the comparison test, where $a_n \geq 0$ and $b_n \geq 0$. Prove that if $\sum b_n$ converges, then $\sum a_n$ converges.

48) Let b_n be an infinite sequence of zeros and ones. What is the largest possible value of $x = \sum_{n=1}^{\infty} b_n/2^n$?

Answer

By the comparison test, $x = \sum_{n=1}^{\infty} b_n/2^n \leq \sum_{n=1}^{\infty} 1/2^n = 1$.

49) Let d_n be an infinite sequence of digits, meaning d_n takes values in $\{0, 1, \ldots, 9\}$. What is the largest possible value of $x = \sum_{n=1}^{\infty} d_n/10^n$ that converges?

50) Explain why, if $x > 1/2$, then x cannot be written $x = \sum_{n=2}^{\infty} \frac{b_n}{2^n} \quad (b_n = 0 \text{ or } 1,\ b_1 = 0)$.

Answer

If $b_1 = 0$, then, by comparison, $x \leq \sum_{n=2}^{\infty} 1/2^n = 1/2$.

51) [T] Evelyn has a perfect balancing scale, an unlimited number of 1-kg weights, and one each of $1/2$-kg, $1/4$-kg, $1/8$-kg, and so on weights. She wishes to weigh a meteorite of unspecified origin to arbitrary precision. Assuming the scale is big enough, can she do it? What does this have to do with infinite series?

52) [T] Robert wants to know his body mass to arbitrary precision. He has a big balancing scale that works perfectly, an unlimited collection of 1-kg weights, and nine each of 0.1-kg, 0.01-kg, 0.001-kg, and so on weights. Assuming the scale is big enough, can he do this? What does this have to do with infinite series?

Answer

Yes. Keep adding 1-kg weights until the balance tips to the side with the weights. If it balances perfectly, with Robert standing on the other side, stop. Otherwise, remove one of the 1-kg weights, and add 0.1-kg weights one at a time. If it balances after adding some of these, stop. Otherwise if it tips to the weights, remove the last 0.1-kg weight. Start adding 0.01-kg weights. If it balances, stop. If it tips to the side with the weights, remove the last 0.01-kg weight that was added. Continue in this way for the 0.001-kg weights, and so on. After a finite number of steps, one has a finite series of the form $A + \sum_{n=1}^{N} s_n/10^n$ where A is the number of full kg weights and d_n is the number of $1/10^n$-kg weights that were added. If at some state this series is Robert's exact weight, the process will stop. Otherwise it represents the Nth partial sum of an infinite series that gives Robert's exact weight, and the error of this sum is at most $1/10^N$.

53) The series $\sum_{n=1}^{\infty} \frac{1}{2n}$ is half the harmonic series and hence diverges. It is obtained from the harmonic series by deleting all terms in which n is odd. Let $m > 1$ be fixed. Show, more generally, that deleting all terms $1/n$ where $n = mk$ for some integer k also results in a divergent series.

54) In view of the previous exercise, it may be surprising that a subseries of the harmonic series in which about one in every five terms is deleted might converge. A depleted harmonic series is a series obtained from $\sum_{n=1}^{\infty} \frac{1}{n}$ by removing any term $1/n$ if a given digit, say 9, appears in the decimal expansion of n. Argue that this depleted harmonic series converges by answering the following questions.

a. How many whole numbers n have d digits?

b. How many d-digit whole numbers $h(d)$. do not contain 9 as one or more of their digits?

c. What is the smallest d-digit number $m(d)$?

d. Explain why the deleted harmonic series is bounded by $\sum_{d=1}^{\infty} \frac{h(d)}{m(d)}$.

e. Show that $\sum_{d=1}^{\infty} \frac{h(d)}{m(d)}$ converges.

Answer

a. $10^d - 10^{d-1} < 10^d$

b. $h(d) < 9^d$

c. $m(d) = 10^{d-1} + 1$

d. Group the terms in the deleted harmonic series together by number of digits. $h(d)$ bounds the number of terms, and each term is at most $\frac{1}{m(d)}$.

Then $\sum_{d=1}^{\infty} h(d)/m(d) \leq \sum_{d=1}^{\infty} 9^d/(10)^{d-1} \leq 90$. One can actually use comparison to estimate the value to smaller than 80. The actual value is smaller than 23.

55) Suppose that a sequence of numbers $a_n > 0$ has the property that $a_1 = 1$ and $a_{n+1} = \frac{1}{n+1} S_n$, where $S_n = a_1 + \cdots + a_n$. Can you determine whether $\sum_{n=1}^{\infty} a_n$ converges? (Hint: S_n is monotone.)

56) Suppose that a sequence of numbers $a_n > 0$ has the property that $a_1 = 1$ and $a_{n+1} = \frac{1}{(n+1)^2} S_n$, where $S_n = a_1 + \cdots + a_n$. Can you determine whether $\sum_{n=1}^{\infty} a_n$ converges? (Hint: $S_2 = a_2 + a_1 = a_2 + S_1 = a_2 + 1 = 1 + 1/4 = (1 + 1/4)S_1, S_3 = \frac{1}{3^2} S_2 + S_2 = (1 + 1/9)S_2 = (1 + 1/9)(1 + 1/4)S_1$, etc. Look at $\ln(S_n)$, and use $\ln(1 + t) \leq t, t > 0$.)

Answer

Continuing the hint gives $S_N = (1 + 1/N^2)(1 + 1/(N-1)^2 \ldots (1 + 1/4))$. Then $\ln(S_N) = \ln(1 + 1/N^2) + \ln(1 + 1/(N-1)^2) + \cdots + \ln(1 + 1/4)$. Since $\ln(1 + t)$ is bounded by a constant times t, when $0 < t < 1$ one has $\ln(S_N) \leq C \sum_{n=1}^{N} \frac{1}{n^2}$, which converges by comparison to the p-series for $p = 2$.

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9.5: Alternating Series

Learning Objectives

- Use the alternating series test to test an alternating series for convergence.
- Estimate the sum of an alternating series.
- Explain the meaning of absolute convergence and conditional convergence.

So far in this chapter, we have primarily discussed series with positive terms. In this section we introduce alternating series—those series whose terms alternate in sign. We will show in a later chapter that these series often arise when studying power series. After defining alternating series, we introduce the alternating series test to determine whether such a series converges.

The Alternating Series Test

A series whose terms alternate between positive and negative values is an **alternating series.** For example, the series

$$\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n = -\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-... \tag{9.5.1}$$

and

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+... \tag{9.5.2}$$

are both alternating series.

Definition: Alternating Series

Any series whose terms alternate between positive and negative values is called an *alternating series*. An alternating series can be written in the form

$$\sum_{n=1}^{\infty}(-1)^{n+1}b_n = b_1-b_2+b_3-b_4+... \tag{9.5.3}$$

or

$$\sum_{n-1}^{\infty}(-1)^{n}b_n = -b_1+b_2-b_3+b_4-... \tag{9.5.4}$$

Where $b_n \geq 0$ for all positive integers n.

Series (1), shown in Equation 9.5.1, is a *geometric series*. Since $|r| = |-1/2| < 1$, the series converges. Series (2), shown in Equation 9.5.2, is called the *alternating harmonic series*. We will show that whereas the harmonic series diverges, the alternating harmonic series converges. To prove this, we look at the sequence of partial sums $\{S_k\}$ (Figure 9.5.1).

Proof

Consider the odd terms S_{2k+1} for $k \geq 0$. Since $1/(2k+1) < 1/2k$,

$$S_{2k+1} = S_{2k-1}-\frac{1}{2k}+\frac{1}{2k+1} < S_{2k-1}.$$

Therefore, $\{S_{2k+1}\}$ is a decreasing sequence. Also,

$$S_{2k+1} = \left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+...+\left(\frac{1}{2k-1}-\frac{1}{2k}\right)+\frac{1}{2k+1} > 0.$$

Therefore, $\{S_{2k+1}\}$ is bounded below. Since $\{S_{2k+1}\}$ is a decreasing sequence that is bounded below, by the Monotone Convergence Theorem, $\{S_{2k+1}\}$ converges. Similarly, the even terms $\{S_{2k}\}$ form an increasing sequence that is bounded above

because

$$S_{2k} = S_{2k-2} + \frac{1}{2k-1} - \frac{1}{2k} > S_{2k-2}$$

and

$$S_{2k} = 1 + \left(-\frac{1}{2} + \frac{1}{3}\right) + \ldots + \left(-\frac{1}{2k-2} + \frac{1}{2k-1}\right) - \frac{1}{2k} < 1.$$

Therefore, by the Monotone Convergence Theorem, the sequence $\{S_{2k}\}$ also converges. Since

$$S_{2k+1} = S_{2k} + \frac{1}{2k+1},$$

we know that

$$\lim_{k\to\infty} S_{2k+1} = \lim_{k\to\infty} S_{2k} + \lim_{k\to\infty} \frac{1}{2k+1}.$$

Letting $S = \lim_{k\to\infty} S_{2k+1}$ and using the fact that $1/(2k+1) \to 0$, we conclude that $\lim_{k\to\infty} S_{2k} = S$. Since the odd terms and the even terms in the sequence of partial sums converge to the same limit S, it can be shown that the sequence of partial sums converges to S, and therefore the alternating harmonic series converges to S.

It can also be shown that $S = \ln 2$, and we can write

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + a \ldots = \ln(2).$$

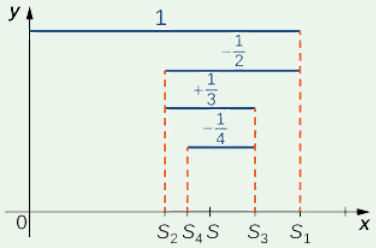

Figure 9.5.1: For the alternating harmonic series, the odd terms S_{2k+1} in the sequence of partial sums are decreasing and bounded below. The even terms S_{2k} are increasing and bounded above.

□

More generally, any alternating series of form (3) (Equation 9.5.3) or (4) (Equation 9.5.4) converges as long as $b_1 \geq b_2 \geq b_3 \geq \cdots$ and $b_n \to 0$ (Figure 9.5.2). The proof is similar to the proof for the alternating harmonic series.

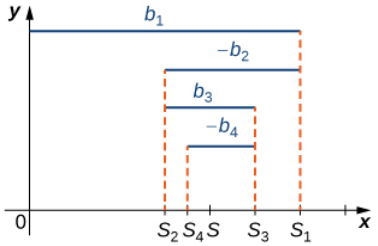

Figure 9.5.2: For an alternating series $b_1 - b_2 + b_3 - \cdots$ in which $b_1 > b_2 > b_3 > \cdots$, the odd terms S_{2k+1} in the sequence of partial sums are decreasing and bounded below. The even terms S_{2k} are increasing and bounded above.

Alternating Series Test

An alternating series of the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n$$

or

$$\sum_{n=1}^{\infty}(-1)^n b_n$$

converges if

i. $0 \le b_{n+1} \le b_n$ for all $n \ge 1$ and
ii. $\lim_{n\to\infty} b_n = 0$.

This is known as the *alternating series test*.

We remark that this theorem is true more generally as long as there exists some integer N such that $0 \le b_{n+1} \le b_n$ for all $n \ge N$.

Example 9.5.1: Convergence of Alternating Series

For each of the following alternating series, determine whether the series converges or diverges.

a. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$

b. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n+1}$

Solution

a. Since $\frac{1}{(n+1)^2} < \frac{1}{n^2}$ and $\frac{1}{n^2} \to 0$, the series converges.

b. Since $n/(n+1) \nrightarrow 0$ as $n \to \infty$, we cannot apply the alternating series test. Instead, we use the nth term test for divergence. Since $\lim_{n\to\infty}\frac{n}{n+1} = 1 \ne 0$, the series diverges.

Exercise 9.5.1

Determine whether the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{2^n}$ converges or diverges.

Hint

Is $\{\frac{n}{2^n}\}$ decreasing? What is $\lim_{n\to\infty}\frac{n}{2^n}$?

Answer

The series converges.

Remainder of an Alternating Series

It is difficult to explicitly calculate the sum of most alternating series, so typically the sum is approximated by using a partial sum. When doing so, we are interested in the amount of error in our approximation. Consider an alternating series

$$\sum_{n=1}^{\infty}(-1)^{n+1}b_n$$

satisfying the hypotheses of the alternating series test. Let S denote the sum of this series and S_k be the corresponding sequence of partial sums. From Figure 9.5.2, we see that for any integer $N \ge 1$, the remainder R_N satisfies

$$|R_N| = |S - S_N| \le |S_{N+1} - S_N| = b_{n+1}.$$

Remainders in Alternating Series

Consider an alternating series of the form

$$\sum_{n=1}^{\infty}(-1)^{n+1}b_n$$

or

$$\sum_{n=1}^{\infty}(-1)^{n}b_n$$

that satisfies the hypotheses of the alternating series test. Let S denote the sum of the series and S_N denote the Nth partial sum. For any integer $N \geq 1$, the remainder $R_N = S - S_N$ satisfies

$$|R_N| \leq b_{N+1}.$$

In other words, if the conditions of the alternating series test apply, then the error in approximating the infinite series by the Nth partial sum S_N is in magnitude at most the size of the next term b_{N+1}.

Example 9.5.2: Estimating the Remainder of an Alternating Series

Consider the alternating series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}.$$

Use the remainder estimate to determine a bound on the error R_{10} if we approximate the sum of the series by the partial sum S_{10}.

Solution

From the theorem stated above,

$$|R_{10}| \leq b_{11} = \frac{1}{11^2} \approx 0.008265.$$

Exercise 9.5.2

Find a bound for R_{20} when approximating $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ by S_{20}.

Hint

$|R_{20}| \leq b_{21}$

Answer

0.04762

Absolute and Conditional Convergence

Consider a series $\sum_{n=1}^{\infty}a_n$ and the related series $\sum_{n=1}^{\infty}|a_n|$. Here we discuss possibilities for the relationship between the convergence of these two series. For example, consider the alternating harmonic series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$. The series whose terms are the absolute value of these terms is the harmonic series, since $\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$. Since the alternating harmonic series converges, but the harmonic series diverges, we say the alternating harmonic series exhibits conditional convergence.

By comparison, consider the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$. The series whose terms are the absolute values of the terms of this series is the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$. Since both of these series converge, we say the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$ exhibits absolute convergence.

Definition: Absolute Convergence

A series $\sum_{n=1}^{\infty} a_n$ exhibits *absolute convergence* if $\sum_{n=1}^{\infty} |a_n|$ converges. A series $\sum_{n=1}^{\infty} a_n$ exhibits *conditional convergence* if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

As shown by the alternating harmonic series, a series $\sum_{n=1}^{\infty} a_n$ may converge, but $\sum_{n=1}^{\infty} |a_n|$ may diverge. In the following theorem, however, we show that if $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Absolute Convergence Implies Convergence

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Proof

Suppose that $\sum_{n=1}^{\infty} |a_n|$ converges. We show this by using the fact that $a_n = |a_n|$ or $a_n = -|a_n|$ and therefore $|a_n| + a_n = 2|a_n|$ or $|a_n| + a_n = 0$. Therefore, $0 \le |a_n| + a_n \le 2|a_n|$. Consequently, by the comparison test, since $2\sum_{n=1}^{\infty} |a_n|$ converges, the series

$$\sum_{n=1}^{\infty} (|a_n| + a_n)$$

converges. By using the algebraic properties for convergent series, we conclude that

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (|a_n| + a_n) - \sum_{n=1}^{\infty} |a_n|$$

converges.

□

Example 9.5.3: Absolute versus Conditional Convergence

For each of the following series, determine whether the series converges absolutely, converges conditionally, or diverges.

a. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n+1}$

b. $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$

Solution

a. We can see that

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{3n+1} \right| = \sum_{n=1}^{\infty} \frac{1}{3n+1}$$

diverges by using the limit comparison test with the harmonic series. In fact,

$$\lim_{n\to\infty} \frac{1/(3n+1)}{1/n} = \frac{1}{3}.$$

Therefore, the series does not converge absolutely. However, since

$$\frac{1}{3(n+1)+1} < \frac{1}{3n+1} \text{ and } \frac{1}{3n+1} \to 0,$$

the series converges. We can conclude that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n+1}$ converges conditionally.

b. Noting that $|\cos n| \leq 1$, to determine whether the series converges absolutely, compare

$$\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right|$$

with the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, by the comparison test, $\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right|$ converges, and therefore $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ converges absolutely.

? Exercise 9.5.3

Determine whether the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{2n^3+1}$ converges absolutely, converges conditionally, or diverges.

Hint

Check for absolute convergence first.

Answer

The series converges absolutely.

To see the difference between absolute and conditional convergence, look at what happens when we rearrange the terms of the alternating harmonic series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$. We show that we can rearrange the terms so that the new series diverges. Certainly if we rearrange the terms of a finite sum, the sum does not change. When we work with an infinite sum, however, interesting things can happen.

Begin by adding enough of the positive terms to produce a sum that is larger than some real number $M = 10$ For example, let $M = 10$, and find an integer k such that

$$1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2k-1} > 10$$

(We can do this because the series $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ diverges to infinity.) Then subtract $1/2$. Then add more positive terms until the sum reaches 100. That is, find another integer $j > k$ such that

$$(1 + \frac{1}{3} + \cdots + \frac{1}{2k-1} - \frac{1}{2} + \frac{1}{2k+1} + \ldots + \frac{1}{2j+1} > 100.$$

Then subtract $1/4$. Continuing in this way, we have found a way of rearranging the terms in the alternating harmonic series so that the sequence of partial sums for the rearranged series is unbounded and therefore diverges.

The terms in the alternating harmonic series can also be rearranged so that the new series converges to a different value. In Example, we show how to rearrange the terms to create a new series that converges to $3\ln(2)/2$. We point out that the alternating harmonic series can be rearranged to create a series that converges to any real number r; however, the proof of that fact is beyond the scope of this text.

In general, any series $\sum_{n=1}^{\infty} a_n$ that converges conditionally can be rearranged so that the new series diverges or converges to a different real number. A series that converges absolutely does not have this property. For any series $\sum_{n=1}^{\infty} a_n$ that converges absolutely, the value of $\sum_{n=1}^{\infty} a_n$ is the same for any rearrangement of the terms. This result is known as the **Riemann Rearrangement Theorem**, which is beyond the scope of this book.

Example 9.5.4: Rearranging Series

Use the fact that

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots=\ln 2$$

to rearrange the terms in the alternating harmonic series so the sum of the rearranged series is $3\ln(2)/2$.

Solution

Let

$$\sum_{n=1}^{\infty}a_n=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots.$$

Since $\sum_{n=1}^{\infty}a_n=\ln(2)$, by the algebraic properties of convergent series,

$$\sum_{n=1}^{\infty}\frac{1}{2}a_n=\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\cdots=\frac{1}{2}\sum_{n=1}^{\infty}a_n=\frac{\ln 2}{2}.$$

Now introduce the series $\sum_{n=1}^{\infty}b_n$ such that for all $n\geq 1, b_{2n-1}=0$ and $b_{2n}=a_n/2$. Then

$$\sum_{n=1}^{\infty}b_n=0+\frac{1}{2}+0-\frac{1}{4}+0+\frac{1}{6}+0-\frac{1}{8}+\cdots=\frac{\ln 2}{2}.$$

Then using the algebraic limit properties of convergent series, since $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ converge, the series $\sum_{n=1}^{\infty}(a_n+b_n)$ converges and

$$\sum_{n=1}^{\infty}(a_n+b_n)=\sum_{n=1}^{\infty}a_n+\sum_{n=1}^{\infty}b_n=\ln 2+\frac{\ln 2}{2}=\frac{3\ln 2}{2}.$$

Now adding the corresponding terms, a_n and b_n, we see that

$$\begin{aligned}\sum_{n=1}^{\infty}(a_n+b_n)&=(1+0)+\left(-\frac{1}{2}+\frac{1}{2}\right)+\left(\frac{1}{3}+0\right)+\left(-\frac{1}{4}-14\right)+\left(\frac{1}{5}+0\right)+\left(-\frac{1}{6}+\frac{1}{6}\right)\\&\quad+\left(\frac{1}{7}+0\right)+\left(\frac{1}{8}-\frac{1}{8}\right)+\cdots=1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots.\end{aligned}$$

We notice that the series on the right side of the equal sign is a rearrangement of the alternating harmonic series. Since $\sum_{n=1}^{\infty}(a_n+b_n)=3\ln(2)/2$, we conclude that

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots=\frac{3\ln(2)}{2}.$$

Therefore, we have found a rearrangement of the alternating harmonic series having the desired property.

Key Concepts

- For an alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$, if $b_{k+1}\leq b_k$ for all k and $b_k\to 0$ as $k\to\infty$, the alternating series converges.
- If $\sum_{n=1}^{\infty}|a_n|$ converges, then $\sum_{n=1}^{\infty}a_n$ converges.

Key Equations

- **Alternating series**

  $\sum_{n=1}^{\infty}(-1)^{n+1}b_n=b_1-b_2+b_3-b_4+\cdots$ or

$$\sum_{n=1}^{\infty}(-1)^n b_n = -b_1 + b_2 - b_3 + b_4 - \cdots$$

Glossary

absolute convergence

if the series $\sum_{n=1}^{\infty}|a_n|$ converges, the series $\sum_{n=1}^{\infty}a_n$ is said to converge absolutely

alternating series

a series of the form $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ or $\sum_{n=1}^{\infty}(-1)^n b_n$, where $b_n \geq 0$, is called an alternating series

alternating series test

for an alternating series of either form, if $b_{n+1} \leq b_n$ for all integers $n \geq 1$ and $b_n \to 0$, then an alternating series converges

conditional convergence

if the series $\sum_{n=1}^{\infty}a_n$ converges, but the series $\sum_{n=1}^{\infty}|a_n|$ diverges, the series $\sum_{n=1}^{\infty}a_n$ is said to converge conditionally

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9.5E: Exercises for Section 9.5

In exercises 1 - 30, state whether each of the following series converges absolutely, conditionally, or not at all.

1) $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{n}{n+3}$

2) $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{\sqrt{n}+1}{\sqrt{n}+3}$

Answer

This series diverges by the divergence test. Terms do not tend to zero.

3) $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{1}{\sqrt{n+3}}$

4) $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{\sqrt{n+3}}{n}$

Answer

Converges conditionally by alternating series test, since $\sqrt{n+3}/n$ is decreasing and its limit is 0. Does not converge absolutely by comparison with p-series, $p=1/2$.

5) $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{1}{n!}$

6) $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{3^n}{n!}$

Answer

Converges absolutely by limit comparison to $3^n/4^n$, for example.

7) $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\dfrac{n-1}{n}\right)^n$

8) $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\dfrac{n+1}{n}\right)^n$

Answer

Diverges by divergence test since $\lim_{n\to\infty}|a_n| = e$ and not 0.

9) $\sum_{n=1}^{\infty}(-1)^{n+1}\sin^2 n$

10) $\sum_{n=1}^{\infty}(-1)^{n+1}\cos^2 n$

Answer

Diverges by the divergence test, since its terms do not tend to zero. The limit of the sequence of its terms does not exist.

11) $\sum_{n=1}^{\infty}(-1)^{n+1}\sin^2(1/n)$

12) $\sum_{n=1}^{\infty}(-1)^{n+1}\cos^2(1/n)$

Answer

$\lim_{n\to\infty}\cos^2(1/n) = 1$. Diverges by divergence test.

13) $\sum_{n=1}^{\infty}(-1)^{n+1}\ln(1/n)$

14) $\sum_{n=1}^{\infty}(-1)^{n+1}\ln(1+\frac{1}{n})$

Answer

Converges by alternating series test.

15) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{1+n^4}$

16) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^e}{1+n^\pi}$

Answer

Converges conditionally by alternating series test. Does not converge absolutely by limit comparison with p-series, $p=\pi-e$

Solution:

17) $\sum_{n=1}^{\infty}(-1)^{n+1}2^{1/n}$

18) $\sum_{n=1}^{\infty}(-1)^{n+1}n^{1/n}$

Answer

Diverges; terms do not tend to zero.

19) $\sum_{n=1}^{\infty}(-1)^{n}(1-n^{1/n})$ (Hint: $n^{1/n}\approx 1+\ln(n)/n$ for large n.)

20) $\sum_{n=1}^{\infty}(-1)^{n+1}n\left(1-\cos\left(\frac{1}{n}\right)\right)$ (Hint: $\cos(1/n)\approx 1-1/n^2$ for large n.)

Answer

Converges by alternating series test. Does not converge absolutely by limit comparison with harmonic series.

21) $\sum_{n=1}^{\infty}(-1)^{n+1}(\sqrt{n+1}-\sqrt{n})$ (Hint: Rationalize the numerator.)

22) $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$ (Hint: Cross-multiply then rationalize numerator.)

Answer

Converges absolutely by limit comparison with p-series, $p=3/2$, after applying the hint.

23) $\sum_{n=1}^{\infty}(-1)^{n+1}(\ln(n+1)-\ln n)$

24) $\sum_{n=1}^{\infty}(-1)^{n+1}n(\tan^{-1}(n+1)-\tan^{-1}n)$ (Hint: Use Mean Value Theorem.)

Answer

Converges by alternating series test since $n(\tan^{-1}(n+1)-\tan^{-1}n)$ is decreasing to zero for large n.Does not converge absolutely by limit comparison with harmonic series after applying hint.

25) $\sum_{n=1}^{\infty}(-1)^{n+1}((n+1)^2-n^2)$

26) $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{n}-\frac{1}{n+1}\right)$

Answer

Converges absolutely, since $a_n = \frac{1}{n} - \frac{1}{n+1}$ are terms of a telescoping series.

27) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$

28) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^{1/n}}$

Answer

Terms do not tend to zero. Series diverges by divergence test.

29) $\sum_{n=1}^{\infty} \frac{1}{n}\sin(\frac{n\pi}{2})$

30) $\sum_{n=1}^{\infty} \sin(n\pi/2)\sin(1/n)$

Answer

Converges by alternating series test. Does not converge absolutely by limit comparison with harmonic series.

In exercises 31 - 36, use the estimate $|R_N| \le b_{N+1}$ to find a value of N that guarantees that the sum of the first N terms of the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ differs from the infinite sum by at most the given error. Calculate the partial sum S_N for this N.

31) [T] $b_n = 1/n$, error $< 10^{-5}$

32) [T] $b_n = 1/\ln(n), n \ge 2$, error $< 10^{-1}$

Answer

$\ln(N+1) > 10, N+1 > e^{10}, N \ge 22026; S_{22026} = 0.0257\ldots$

33) [T] $b_n = 1/\sqrt{n}$, error $< 10^{-3}$

34) [T] $b_n = 1/2^n$, error $< 10^{-6}$

Answer

$2^{N+1} > 10^6$ or $N + 1 > 6\ln(10)/\ln(2) = 19.93.$ or $N \ge 19; S_{19} = 0.333333969\ldots$

35) [T] $b_n = ln(1 + \frac{1}{n})$, error $< 10^{-3}$

36) [T] $b_n = 1/n^2$, error $< 10^{-6}$

Answer

$(N+1)^2 > 10^6$ or $N > 999; S_{1000} \approx 0.822466.$

For exercises 37 - 45, indicate whether each of the following statements is true or false. If the statement is false, provide an example in which it is false.

37) If $b_n \ge 0$ is decreasing and $\lim_{n\to\infty} b_n = 0$, then $\sum_{n=1}^{\infty}(b_{2n-1} - b_{2n})$ converges absolutely.

38) If $b_n \ge 0$ is decreasing, then $\sum_{n=1}^{\infty}(b_{2n-1} - b_{2n})$ converges absolutely.

Answer

True. b_n need not tend to zero since if $c_n = b_n - \lim b_n$, then $c_{2n-1} - c_{2n} = b_{2n-1} - b_{2n}$.

39) If $b_n \ge 0$ and $\lim_{n\to\infty} b_n = 0$ then $\sum_{n=1}^{\infty}(\frac{1}{2}(b_{3n-2} + b_{3n-1}) - b_{3n})$ converges.

40) If $b_n \geq 0$ is decreasing and $\sum_{n=1}^{\infty}(b_{3n-2}+b_{3n-1}-b_{3n})$ converges then $\sum_{n=1}^{\infty} b_{3n-2}$ converges.

Answer

True. $b_{3n-1}-b_{3n} \geq 0$, so convergence of $\sum b_{3n-2}$ follows from the comparison test.

41) If $b_n \geq 0$ is decreasing and $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ converges conditionally but not absolutely, then b_n does not tend to zero.

42) Let $a_n^+ = a_n$ if $a_n \geq 0$ and $a_n^- = -a_n$ if $a_n < 0$. (Also, $a_n^+ = 0$ if $a_n < 0$ and $a_n^- = 0$ if $a_n \geq 0$.) If $\sum_{n=1}^{\infty} a_n$ converges conditionally but not absolutely, then neither $\sum_{n=1}^{\infty} a_n^+$ nor $\sum_{n=1}^{\infty} a_n^-$ converge.

Answer

True. If one converges, then so must the other, implying absolute convergence.

43) Suppose that a_n is a sequence of positive real numbers and that $\sum_{n=1}^{\infty} a_n$ converges.

44) Suppose that b_n is an arbitrary sequence of ones and minus ones. Does $\sum_{n=1}^{\infty} a_n b_n$ necessarily converge?

45) Suppose that a_n is a sequence such that $\sum_{n=1}^{\infty} a_n b_n$ converges for every possible sequence b_n of zeros and ones. Does $\sum_{n=1}^{\infty} a_n$ converge absolutely?

Answer

Yes. Take $b_n = 1$ if $a_n \geq 0$ and $b_n = 0$ if $a_n < 0$. Then $\sum_{n=1}^{\infty} a_n b_n = \sum_{n:a_n \geq 0} a_n$ converges. Similarly, one can show $\sum_{n:a_n<0} a_n$ converges. Since both series converge, the series must converge absolutely.

In exercises 46 - 49, the series do not satisfy the hypotheses of the alternating series test as stated. In each case, state which hypothesis is not satisfied. State whether the series converges absolutely.

46) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin^2 n}{n}$

47) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos^2 n}{n}$

Answer

Not decreasing. Does not converge absolutely.

48) $1+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\cdots$

49) $1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}+\frac{1}{8}-\frac{1}{9}+\cdots$

Answer

Not alternating. Can be expressed as $\sum_{n=1}^{\infty}\left(\frac{1}{3n-2}+\frac{1}{3n-1}-\frac{1}{3n}\right)$, which diverges by comparison with $\sum_{n=1}^{\infty}\frac{1}{3n-2}$.

50) Show that the alternating series $1-\frac{1}{2}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}+\frac{1}{4}-\frac{1}{8}+\cdots$ does not converge. What hypothesis of the alternating series test is not met?

51) Suppose that $\sum a_n$ converges absolutely. Show that the series consisting of the positive terms a_n also converges.

Answer

Let $a_n^+ = a_n$ if $a_n \geq 0$ and $a_n^+ = 0$ if $a_n < 0$. Then $a_n^+ \leq |a_n|$ for all n so the sequence of partial sums of a_n^+ is increasing and bounded above by the sequence of partial sums of $|a_n|$, which converges; hence, $\sum_{n=1}^{\infty} a_n^+$ converges.

52) Show that the alternating series $\frac{2}{3} - \frac{3}{5} + \frac{4}{7} - \frac{5}{9} + \cdots$ does not converge. What hypothesis of the alternating series test is not met?

53) The formula $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$ will be derived in the next chapter. Use the remainder $|R_N| \leq b_{N+1}$ to find a bound for the error in estimating $\cos\theta$ by the fifth partial sum $1 - \theta^2/2! + \theta^4/4! - \theta^6/6! + \theta^8/8!$ for $\theta = 1, \theta = \pi/6$, and $\theta = \pi$.

Answer

For $N = 5$ one has $|\ R_N\ |\ b_6 = \theta^{10}/10!$. When $\theta = 1, R_5 \leq 1/10! \approx 2.75 \times 10^{-7}$. When $\theta = \pi/6$, $R_5 \leq (\pi/6)^{10}/10! \approx 4.26 \times 10^{-10}$. When $\theta = \pi, R_5 \leq \pi^{10}/10! = 0.0258$.

54) The formula $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$ will be derived in the next chapter. Use the remainder $|R_N| \leq b_{N+1}$ to find a bound for the error in estimating $\sin\theta$ by the fifth partial sum $\theta - \theta^3/3! + \theta^5/5! - \theta^7/7! + \theta^9/9!$ for $\theta = 1, \theta = \pi/6$, and $\theta = \pi$.

55) How many terms in $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$ are needed to approximate $\cos 1$ accurate to an error of at most 0.00001?

Answer

Let $b_n = 1/(2n-2)!$. Then $R_N \leq 1/(2N)! < 0.00001$ when $(2N)! > 10^5$ or $N = 5$ and $1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} = 0.540325\ldots$, whereas $\cos 1 = 0.5403023\ldots$

56) How many terms in $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$ are needed to approximate $\sin 1$ accurate to an error of at most 0.00001?

57) Sometimes the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges to a certain fraction of an absolutely convergent series $\sum_{n=1}^{\infty} b_n$ at a faster rate. Given that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, find $S = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$. Which of the series $6\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $S\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ gives a better estimation of π^2 using 1000 terms?

Answer

Let $T = \sum \frac{1}{n^2}$. Then $T - S = \frac{1}{2}T$, so $S = T/2$. $\sqrt{6 \times \sum_{n=1}^{1000} 1/n^2} = 3.140638\ldots$; $\sqrt{\frac{1}{2} \times \sum_{n=1}^{1000} (-1)^{n-1}/n^2} = 3.141591\ldots$; $\pi = 3.141592\ldots$. The alternating series is more accurate for 1000 terms.

The alternating series in exercises 58 & 59 converge to given multiples of π. Find the value of N predicted by the remainder estimate such that the Nth partial sum of the series accurately approximates the left-hand side to within the given error. Find the minimum N for which the error bound holds, and give the desired approximate value in each case. Up to 15 decimals places, $\pi = 3.141592653589793\ldots$.

58) [T] $\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$, error < 0.0001

59) [T] $\frac{\pi}{\sqrt{12}} = \sum_{k=0}^{\infty} \frac{(-3)^{-k}}{2k+1}$, error < 0.0001

Answer

$N = 6, S_N = 0.9068$

60) [T] The series $\sum_{n=0}^{\infty} \frac{\sin(x+\pi n)}{x+\pi n}$ plays an important role in signal processing. Show that $\sum_{n=0}^{\infty} \frac{\sin(x+\pi n)}{x+\pi n}$ converges whenever $0 < x < \pi$. (Hint: Use the formula for the sine of a sum of angles.)

61) [T] If $\sum_{n=1}^{N} (-1)^{n-1} \frac{1}{n} \to ln2$, what is $1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{8} - \frac{1}{10} - \frac{1}{12} + \cdots$?

Answer

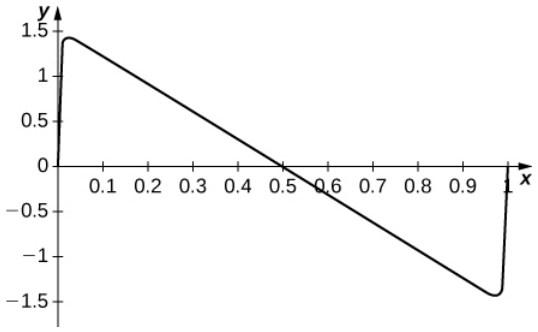
$\ln(2)$. The nth partial sum is the same as that for the alternating harmonic series.

62) [T] Plot the series $\sum_{n=1}^{100} \frac{\cos(2\pi nx)}{n}$ for $0 \le x < 1$. Explain why $\sum_{n=1}^{100} \frac{\cos(2\pi nx)}{n}$ diverges when $x = 0, 1$. How does the series behave for other x?

63) [T] Plot the series $\sum_{n=1}^{100} \frac{\sin(2\pi nx)}{n}$ for $0 \le x < 1$ and comment on its behavior

Answer

The series jumps rapidly near the endpoints. For x away from the endpoints, the graph looks like $\pi(1/2 - x)$.

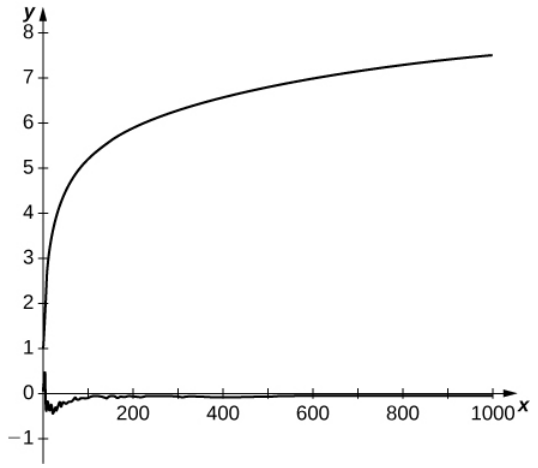


64) [T] Plot the series $\sum_{n=1}^{100} \frac{\cos(2\pi nx)}{n^2}$ for $0 \le x < 1$ and describe its graph.

65) [T] The alternating harmonic series converges because of cancelation among its terms. Its sum is known because the cancelation can be described explicitly. A random harmonic series is one of the form $\sum_{n=1}^{\infty} \frac{S_n}{n}$, where s_n is a randomly generated sequence of $\pm 1's$ in which the values ± 1 are equally likely to occur. Use a random number generator to produce 1000 random $\pm 1's$ and plot the partial sums $S_N = \sum_{n=1}^{N} \frac{s_n}{n}$ of your random harmonic sequence for $N = 1$ to 1000. Compare to a plot of the first 1000 partial sums of the harmonic series.

Answer

Here is a typical result. The top curve consists of partial sums of the harmonic series. The bottom curve plots partial sums of a random harmonic series.



66) [T] Estimates of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ can be accelerated by writing its partial sums as $\sum_{n=1}^{N} \frac{1}{n^2} = \sum_{n=1}^{N} \frac{1}{n(n+1)} + \sum_{n=1}^{N} \frac{1}{n^2(n+1)}$ and recalling that $\sum_{n=1}^{N} \frac{1}{n(n+1)} = 1 - \frac{1}{N+1}$ converges to one as $N \to \infty$. Compare the estimate of $\pi^2/6$ using the sums $\sum_{n=1}^{1000} \frac{1}{n^2}$ with the estimate using $1 + \sum_{n=1}^{1000} \frac{1}{n^2(n+1)}$.

67) [T] The **Euler transform** rewrites $S = \sum_{n=0}^{\infty} (-1)^n b_n$ as $S = \sum_{n=0}^{\infty} (-1)^n 2^{-n-1} \sum_{m=0}^{n} \binom{n}{m} b_{n-m}$. For the alternating harmonic series, it takes the form $\ln(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \sum_{n=1}^{\infty} \frac{1}{n2^n}$. Compute partial sums of $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ until they approximate $\ln(2)$ accurate to within 0.0001. How many terms are needed? Compare this answer to the number of terms of the alternating harmonic series are needed to estimate $\ln(2)$.

Answer

By the alternating series test, $|S_n - S| \leq b_{n+1}$, so one needs 10^4 terms of the alternating harmonic series to estimate $\ln(2)$ to within 0.0001. The first 10 partial sums of the series $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ are (up to four decimals)

$0.5000, 0.6250, 0.6667, 0.6823, 0.6885, 0.6911, 0.6923, 0.6928, 0.6930, 0.6931$ and the tenth partial sum is within 0.0001 of $\ln(2) = 0.6931 \ldots$.

68) [T] In the text it was stated that a conditionally convergent series can be rearranged to converge to any number. Here is a slightly simpler, but similar, fact. If $a_n \geq 0$ is such that $a_n \to 0$ as $n \to \infty$ but $\sum_{n=1}^{\infty} a_n$ diverges, then, given any number A there is a sequence s_n of $\pm 1's$ such that $\sum_{n=1}^{\infty} a_n s_n \to A$. Show this for $A > 0$ as follows.

a. Recursively define s_n by $s_n = 1$ if $S_{n-1} = \sum_{k=1}^{n-1} a_k s_k < A$ and $s_n = -1$ otherwise.

b. Explain why eventually $S_n \geq A$, and for any m larger than this n, $A - a_m \leq S_m \leq A + a_m$.

c. Explain why this implies that $S_n \to A$ as $n \to \infty$.

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9.6: Ratio and Root Tests

Learning Objectives

- Use the ratio test to determine absolute convergence of a series.
- Use the root test to determine absolute convergence of a series.
- Describe a strategy for testing the convergence of a given series.

In this section, we prove the last two series convergence tests: the ratio test and the root test. These tests are particularly nice because they do not require us to find a comparable series. The ratio test will be especially useful in the discussion of power series in the next chapter. Throughout this chapter, we have seen that no single convergence test works for all series. Therefore, at the end of this section we discuss a strategy for choosing which convergence test to use for a given series.

Ratio Test

Consider a series $\sum_{n=1}^{\infty} a_n$. From our earlier discussion and examples, we know that $\lim_{n\to\infty} a_n = 0$ is not a sufficient condition for the series to converge. Not only do we need $a_n \to 0$, but we need $a_n \to 0$ quickly enough. For example, consider the series $\sum_{n=1}^{\infty} \frac{1}{n}$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. We know that $\frac{1}{n} \to 0$ and $\frac{1}{n^2} \to 0$. However, only the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges because the terms in the sequence $\left\{\frac{1}{n}\right\}$ do not approach zero fast enough as $n \to \infty$. Here we introduce the **ratio test,** which provides a way of measuring how fast the terms of a series approach zero.

Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with nonzero terms. Let

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

i. If $0 \le \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

ii. If $\rho > 1$ or $\rho = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

iii. If $\rho = 1$, the test does not provide any information.

Proof

Let $\sum_{n=1}^{\infty} a_n$ be a series with nonzero terms.

We begin with the proof of part i. In this case, $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$. Since $0 \le \rho < 1$, there exists R such that $0 \le \rho < R < 1$. Let $\varepsilon = R - \rho > 0$. By the definition of limit of a sequence, there exists some integer N such that

$$\left|\left|\frac{a_{n+1}}{a_n}\right| - \rho\right| < \varepsilon, \text{ for all } n \ge N.$$

Therefore,

$$\left|\frac{a_{n+1}}{a_n}\right| < \rho + \varepsilon = R, \text{ for all } n \ge N$$

and, thus,

$$|a_{N+1}| < R|a_N|$$

$$|a_{N+2}| < R|a_{N+1}| < R^2|a_N|$$

$$|a_{N+3}| < R|a_{N+2}| < R^2|a_{N+1}| < R^3|a_N|$$

$$|a_{N+4}| < R|a_{N+3}| < R^2|a_{N+2}| < R^3|a_{N+1}| < R^4|a_N|$$

$$\vdots$$

Since $R < 1$, the geometric series

$$R|a_N| + R^2|a_N| + R^3|a_N| + \cdots$$

converges. Given the inequalities above, we can apply the comparison test and conclude that the series

$$|a_{N+1}| + |a_{N+2}| + |a_{N+3}| + |a_{N+4}| + \cdots$$

converges. Therefore, since

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{N} |a_n| + \sum_{n=N+1}^{\infty} |a_n|$$

where $\sum_{n=1}^{N} |a_n|$ is a finite sum and $\sum_{n=N+1}^{\infty} |a_n|$ converges, we conclude that $\sum_{n=1}^{\infty} |a_n|$ converges.

For part ii.

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1.$$

Since $\rho > 1$, there exists R such that $\rho > R > 1$. Let $\varepsilon = \rho - R > 0$. By the definition of the limit of a sequence, there exists an integer N such that

$$\left|\left|\frac{a_{n+1}}{a_n}\right| - \rho\right| < \varepsilon, \text{ for all } n \geq N.$$

Therefore,

$$R = \rho - \varepsilon < \left|\frac{a_{n+1}}{a_n}\right|, \text{ for all } n \geq N,$$

and, thus,

$$|a_{N+1}| > R|a_N|$$

$$|a_{N+2}| > R|a_{N+1}| > R^2|a_N|$$

$$|a_{N+3}| > R|a_{N+2}| > R^2|a_{N+1}| > R^3|a_N|$$

$$|a_{N+4}| > R|a_{N+3}| > R^2|a_{N+2}| > R^3|a_{N+1}| > R^4|a_N|.$$

Since $R > 1$, the geometric series

$$R|a_N| + R^2|a_N| + R^3|a_N| + \cdots$$

diverges. Applying the comparison test, we conclude that the series

$$|a_{N+1}| + |a_{N+2}| + |a_{N+3}| + \cdots$$

diverges, and therefore the series $\sum_{n=1}^{\infty} |a_n|$ diverges.

For part iii. we show that the test does not provide any information if $\rho = 1$ by considering the $p - series$ $\sum_{n=1}^{\infty} \frac{1}{n^p}$. For any real number p,

$$\rho = \lim_{n\to\infty} \frac{1/(n+1)^p}{1/n^p} = \lim_{n\to\infty} \frac{n^p}{(n+1)^p} = 1.$$

However, we know that if $p \leq 1$, the **p−series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges, whereas $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$.

□

The ratio test is particularly useful for series whose terms contain factorials or exponential, where the ratio of terms simplifies the expression. The ratio test is convenient because it does not require us to find a comparative series. The drawback is that the test sometimes does not provide any information regarding convergence.

Example 9.6.1: Using the Ratio Test

For each of the following series, use the ratio test to determine whether the series converges or diverges.

a. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

b. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

c. $\sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2}{(2n)!}$

Solution

a. From the ratio test, we can see that

$$\rho = \lim_{n\to\infty} \frac{2^{n+1}/(n+1)!}{2^n/n!} = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}.$$

Since $(n+1)! = (n+1)\cdot n!$,

$$\rho = \lim_{n\to\infty} \frac{2}{n+1} = 0.$$

Since $\rho < 1$, the series converges.

b. We can see that

$$\rho = \lim_{n\to\infty} \frac{(n+1)^{n+1}/(n+1)!}{n^n/n!} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e.$$

Since $\rho > 1$, the series diverges.

c. Since

$$\left|\frac{(-1)^{n+1}((n+1)!)^2/(2(n+1))!}{(-1)^n (n!)^2/(2n)!}\right| = \frac{(n+1)!(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{n!n!} = \frac{(n+1)(n+1)}{(2n+2)(2n+1)}$$

we see that

$$\rho = \lim_{n\to\infty} \frac{(n+1)(n+1)}{(2n+2)(2n+1)} = \frac{1}{4}.$$

Since $\rho < 1$, the series converges.

Exercise 9.6.1

Use the ratio test to determine whether the series $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ converges or diverges.

Hint

Evaluate $\lim_{n\to\infty} \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}$.

Answer

The series converges.

Root Test

The approach of the **root test** is similar to that of the ratio test. Consider a series $\sum_{n=1}^{\infty} a_n$ such that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho$ for some real number ρ. Then for N sufficiently large, $|a_N| \approx \rho^N$. Therefore, we can approximate $\sum_{n=N}^{\infty} |a_n|$ by writing

$$|a_N| + |a_{N+1}| + |a_{N+2}| + \cdots \approx \rho^N + \rho^{N+1} + \rho^{N+2} + \cdots.$$

The expression on the right-hand side is a geometric series. As in the ratio test, the series $\sum_{n=1}^{\infty} a_n$ converges absolutely if $0 \le \rho < 1$ and the series diverges if $\rho \ge 1$. If $\rho = 1$, the test does not provide any information. For example, for any p-series, $\sum_{n=1}^{\infty} \frac{1}{n^p}$, we see that

$$\rho = \lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{n^p}\right|} = \lim_{n\to\infty} \frac{1}{n^{p/n}}$$

.

To evaluate this limit, we use the natural logarithm function. Doing so, we see that

$$\ln \rho = \ln\left(\lim_{n\to\infty} \frac{1}{n^{p/n}}\right) = \lim_{n\to\infty} \ln\left(\frac{1}{n}\right)^{p/n} = \lim_{n\to\infty} \frac{p}{n} \cdot \ln\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{p\ln(1/n)}{n}.$$

Using L'Hôpital's rule, it follows that $\ln \rho = 0$, and therefore $\rho = 1$ for all p. However, we know that the p-series only converges if $p > 1$ and diverges if $p < 1$.

Root Test

Consider the series $\sum_{n=1}^{\infty} a_n$. Let

$$\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$$

.

i. If $0 \le \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

ii. If $\rho > 1$ or $\rho = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

iii. If $\rho = 1$, the test does not provide any information.

The root test is useful for series whose terms involve exponentials. In particular, for a series whose terms a_n satisfy $|a_n| = (b_n)^n$, then $\sqrt[n]{|a_n|} = b_n$ and we need only evaluate $\lim_{n\to\infty} b_n$.

Example 9.6.2: Using the Root Test

For each of the following series, use the root test to determine whether the series converges or diverges.

a. $\sum_{n=1}^{\infty} \frac{(n^2+3n)^n}{(4n^2+5)^n}$

b. $\sum_{n=1}^{\infty} \frac{n^n}{(\ln(n))^n}$

Solution

a. To apply the root test, we compute

$$\rho = \lim_{n\to\infty} \sqrt[n]{\frac{(n^2+3n)^n}{(4n^2+5)^n}} = \lim_{n\to\infty} \frac{n^2+3n}{4n^2+5} = \frac{1}{4}.$$

Since $\rho < 1$, the series converges absolutely.

b. We have

$$\rho = \lim_{n\to\infty} \sqrt[n]{\frac{n^n}{(\ln(n))^n}} = \lim_{n\to\infty} \frac{n}{\ln n} = \infty \quad \text{by L'Hôpital's rule.}$$

Since $\rho = \infty$, the series diverges.

? Exercise 9.6.2

Use the root test to determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n^n}$ converges or diverges.

Hint

Evaluate $\lim_{n\to\infty} \sqrt[n]{\frac{1}{n^n}}$.

Answer

The series converges.

Choosing a Convergence Test

At this point, we have a long list of convergence tests. However, not all tests can be used for all series. When given a series, we must determine which test is the best to use. Here is a strategy for finding the best test to apply.

Problem-Solving Strategy: Choosing a Convergence Test for a Series

Consider a series $\sum_{n=1}^{\infty} a_n$. In the steps below, we outline a strategy for determining whether the series converges.

1. Is $\sum_{n=1}^{\infty} a_n$ a familiar series? For example, is it the harmonic series (which diverges) or the alternating harmonic series (which converges)? Is it a **p−series** or geometric series? If so, check the power p or the ratio r to determine if the series converges.
2. Is it an alternating series? Are we interested in absolute convergence or just convergence? If we are just interested in whether the series converges, apply the alternating series test. If we are interested in absolute convergence, proceed to step 3, considering the series of absolute values $\sum_{n=1}^{\infty} |a_n|$.
3. Is the series similar to a **p−series** or geometric series? If so, try the comparison test or limit comparison test.
4. Do the terms in the series contain a factorial or power? If the terms are powers such that $a_n = (b_n)^n$, try the root test first. Otherwise, try the ratio test first.
5. Use the divergence test. If this test does not provide any information, try the integral test.

✓ Example 9.6.3: Using Convergence Tests

For each of the following series, determine which convergence test is the best to use and explain why. Then determine if the series converges or diverges. If the series is an alternating series, determine whether it converges absolutely, converges conditionally, or diverges.

a. $\sum_{n=1}^{\infty} \frac{n^2+2n}{n^3+3n^2+1}$

b. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(3n+1)}{n!}$

c. $\sum_{n=1}^{\infty} \frac{e^n}{n^3}$

d. $\sum_{n=1}^{\infty} \frac{3^n}{(n+1)^n}$

Solution

a. Step 1. The series is not a **p–series** or geometric series.

Step 2. The series is not alternating.

Step 3. For large values of n, we approximate the series by the expression

$$\frac{n^2+2n}{n^3+3n^2+1} \approx \frac{n^2}{n^3} = \frac{1}{n}.$$

Therefore, it seems reasonable to apply the comparison test or limit comparison test using the series $\sum_{n=1}^{\infty} 1/n$. Using the limit comparison test, we see that

$$\lim_{n\to\infty} \frac{(n^2+2n)/(n^3+3n^2+1)}{1/n} = \lim_{n\to\infty} \frac{n^3+2n^2}{n^3+3n^2+1} = 1.$$

Since the series $\sum_{n=1}^{\infty} 1/n$

diverges, this series diverges as well.

b. Step 1.The series is not a familiar series.

Step 2. The series is alternating. Since we are interested in absolute convergence, consider the series

$$\sum_{n=1}^{\infty} \frac{3n}{(n+1)!}.$$

Step 3. The series is not similar to a p-series or geometric series.

Step 4. Since each term contains a factorial, apply the ratio test. We see that

$$\lim_{n\to\infty} \frac{(3(n+1))/(n+1)!}{(3n+1)/n!} = \lim_{n\to\infty} \frac{3n+3}{(n+1)!} \cdot \frac{n!}{3n+1} = \lim_{n\to\infty} \frac{3n+3}{(n+1)(3n+1)} = 0.$$

Therefore, this series converges, and we conclude that the original series converges absolutely, and thus converges.

c. Step 1. The series is not a familiar series.

Step 2. It is not an alternating series.

Step 3. There is no obvious series with which to compare this series.

Step 4. There is no factorial. There is a power, but it is not an ideal situation for the root test.

Step 5. To apply the divergence test, we calculate that

$$\lim_{n\to\infty} \frac{e^n}{n^3} = \infty.$$

Therefore, by the divergence test, the series diverges.

d. Step 1. This series is not a familiar series.

Step 2. It is not an alternating series.

Step 3. There is no obvious series with which to compare this series.

Step 4. Since each term is a power of n,we can apply the root test. Since

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{3}{n+1}\right)^n} = \lim_{n\to\infty} \frac{3}{n+1} = 0,$$

by the root test, we conclude that the series converges.

? Exercise 9.6.3

For the series $\sum_{n=1}^{\infty} \frac{2^n}{3^n+n}$, determine which convergence test is the best to use and explain why.

Hint

The series is similar to the geometric series $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$.

Answer

The comparison test because $\frac{2^n}{3^n+n} < \frac{2^n}{3^n}$ for all positive integers n. The limit comparison test could also be used.

In Table, we summarize the convergence tests and when each can be applied. Note that while the comparison test, limit comparison test, and integral test require the series $\sum_{n=1}^{\infty} a_n$ to have nonnegative terms, if $\sum_{n=1}^{\infty} a_n$ has negative terms, these tests can be applied to $\sum_{n=1}^{\infty} |a_n|$ to test for absolute convergence.

Summary of Convergence Tests

Series or Test	Conclusions	Comments
Divergence Test For any series $\sum_{n=1}^{\infty} a_n$, evaluate $\lim_{n\to\infty} a_n$.	If $\lim_{n\to\infty} a_n = 0$, the test is inconclusive. If $\lim_{n\to\infty} a_n \neq 0$, the series diverges.	This test cannot prove convergence of a series.
Geometric Series $\sum_{n=1}^{\infty} ar^{n-1}$	If $\lvert r\rvert < 1$, the series converges to $a/(1-r)$. If $\lvert r\rvert \geq 1$, the series diverges.	Any geometric series can be reindexed to be written in the form $a + ar + ar^2 + \cdots$, where a is the initial term and r is the ratio.
p-Series $\sum_{n=1}^{\infty} \frac{1}{n^p}$	If $p > 1$, the series converges. If $p \leq 1$, the series diverges.	For $p = 1$, we have the harmonic series $\sum_{n=1}^{\infty} 1/n$.
Comparison Test For $\sum_{n=1}^{\infty} a_n$ with nonnegative terms, compare with a known series $\sum_{n=1}^{\infty} b_n$.	If $a_n \leq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. If $a_n \geq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.	Typically used for a series similar to a geometric or p-series. It can sometimes be difficult to find an appropriate series.
Limit Comparison Test For $\sum_{n=1}^{\infty} a_n$ with positive terms, compare with a series $\sum_{n=1}^{\infty} b_n$ by evaluating $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.	If L is a real number and $L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge. If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. If $L = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.	Typically used for a series similar to a geometric or p-series. Often easier to apply than the comparison test.
Integral Test If there exists a positive, continuous, decreasing function f such that $a_n = f(n)$ for all $n \geq N$, evaluate $\int_N^{\infty} f(x)dx$.	$\int_N^{\infty} f(x)dx$ and $\sum_{n=1}^{\infty} a_n$ both converge or both diverge.	Limited to those series for which the corresponding function can be easily integrated.
Alternating Series $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ or $\sum_{n=1}^{\infty} (-1)^n b_n$	If $b_{n+1} \leq b_n$ for all $n \geq 1$ and $b_n \to 0$, then the series converges.	Only applies to alternating series.
Ratio Test	If $0 \leq \rho < 1$, the series converges absolutely.	Often used for series involving factorials or exponentials.

Series or Test	Conclusions	Comments
For any series $\sum_{n=1}^{\infty} a_n$ with nonzero terms, let $\rho = \lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert$	If $\rho > 1$ or $\rho = \infty$, the series diverges.	
	If $\rho = 1$, the test is inconclusive.	
Root Test For any series $\sum_{n=1}^{\infty} a_n$, let $\rho = \lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert}$.	If $0 \le \rho < 1$, the series converges absolutely.	Often used for series where $\lvert a_n \rvert = (b_n)^n$.
	If $\rho > 1$ or $\rho = \infty$, the series diverges.	
	If $\rho = 1$, the test is inconclusive.	

Series Converging to π and $1/\pi$

Dozens of series exist that converge to π or an algebraic expression containing π. Here we look at several examples and compare their rates of convergence. By rate of convergence, we mean the number of terms necessary for a partial sum to be within a certain amount of the actual value. The series representations of π in the first two examples can be explained using Maclaurin series, which are discussed in the next chapter. The third example relies on material beyond the scope of this text.

1. The series

$$\pi = 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \cdots$$

was discovered by Gregory and Leibniz in the late 1600s. This result follows from the Maclaurin series for $f(x) = \tan^{-1} x$. We will discuss this series in the next chapter.

a. Prove that this series converges.

b. Evaluate the partial sums S_n for $n = 10, 20, 50, 100$.

c. Use the remainder estimate for alternating series to get a bound on the error R_n.

d. What is the smallest value of N that guarantees $|R_N| < 0.01$? Evaluate S_N.

2. The series

$$\pi = 6\sum_{n=0}^{\infty} \frac{(2n)!}{2^{4n+1}(n!)^2(2n+1)}$$
$$= 6\left(\frac{1}{2} + \frac{1}{2\cdot 3}\left(\frac{1}{2}\right)^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5}\cdot\left(\frac{1}{2}\right)^5 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}\left(\frac{1}{2}\right)^7 + \cdots\right)$$

has been attributed to Newton in the late 1600s. The proof of this result uses the Maclaurin series for $f(x) = \sin^{-1} x$.

a. Prove that the series converges.

b. Evaluate the partial sums S_n for $n = 5, 10, 20$.

c. Compare S_n to π for $n = 5, 10, 20$ and discuss the number of correct decimal places.

3. The series

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801}\sum_{n=0}^{\infty} \frac{(4n)!(1103 + 26390n)}{(n!)^4 396^{4n}}$$

was discovered by **Ramanujan** in the early 1900s. William Gosper, Jr., used this series to calculate π to an accuracy of more than 17 million digits in the mid-1980s. At the time, that was a world record. Since that time, this series and others by Ramanujan have led mathematicians to find many other series representations for π and $1/\pi$.

a. Prove that this series converges.

b. Evaluate the first term in this series. Compare this number with the value of π from a calculating utility. To how many decimal places do these two numbers agree? What if we add the first two terms in the series?

c. Investigate the life of Srinivasa Ramanujan (1887–1920) and write a brief summary. Ramanujan is one of the most fascinating stories in the history of mathematics. He was basically self-taught, with no formal training in mathematics, yet he contributed in highly original ways to many advanced areas of mathematics.

Key Concepts

- For the ratio test, we consider

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

If $\rho < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges absolutely. If $\rho > 1$, the series diverges. If $\rho = 1$, the test does not provide any information. This test is useful for series whose terms involve factorials.

- For the root test, we consider

$$\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

If $\rho < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges absolutely. If $\rho > 1$, the series diverges. If $\rho = 1$, the test does not provide any information. The root test is useful for series whose terms involve powers.

- For a series that is similar to a geometric series or **p−series,** consider one of the comparison tests.

Glossary

ratio test

for a series $\sum_{n=1}^{\infty} a_n$ with nonzero terms, let $\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$; if $0 \le \rho < 1$, the series converges absolutely; if $\rho > 1$, the series diverges; if $\rho = 1$, the test is inconclusive

root test

for a series $\sum_{n=1}^{\infty} a_n$, let $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$; if $0 \le \rho < 1$, the series converges absolutely; if $\rho > 1$, the series diverges; if $\rho = 1$, the test is inconclusive

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9.6E: Exercises for Section 9.6

In exercises 1 - 11, use the ratio test to determine whether each series $\sum_{n=1}^{\infty} a_n$ converges or diverges. State if the ratio test is inconclusive.

1) $\sum_{n=1}^{\infty} \frac{1}{n!}$

Answer

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$. Converges by the Ratio Test.

2) $\sum_{n=1}^{\infty} \frac{10^n}{n!}$

3) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

Answer

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{2}\left(\frac{n+1}{n}\right)^2 = \frac{1}{2} < 1$. Converges by the Ratio Test.

4) $\sum_{n=1}^{\infty} \frac{n^{10}}{2^n}$

5) $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$

Answer

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{27} < 1$. Converges by the Ratio Test.

6) $\sum_{n=1}^{\infty} \frac{2^{3n}(n!)^3}{(3n)!}$

7) $\sum_{n=1}^{\infty} \frac{(2n)!}{n^{2n}}$

Answer

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{4}{e^2} < 1$. Converges by the Ratio Test.

8) $\sum_{n=1}^{\infty} \frac{(2n)!}{(2n)^n}$

9) $\sum_{n=1}^{\infty} \frac{n!}{(n/e)^n}$

Answer

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. Ratio test is inconclusive.

10) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n/e)^{2n}}$

11) $\sum_{n=1}^{\infty} \frac{(2^n n!)^2}{(2n)^{2n}}$

Answer

$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{1}{e^2} < 1$. Converges by the Ratio Test.

In exercises 12 - 21, use the root test to determine whether $\sum_{n=1}^{\infty} a_n$ converges, where a_n is as follows.

12) $a_k = \left(\frac{k-1}{2k+3}\right)^k$

13) $a_k = \left(\frac{2k^2-1}{k^2+3}\right)^k$

Answer

$\lim_{k\to\infty} (a_k)^{1/k} = 2 > 1$. Diverges by the Root Test.

14) $a_n = \frac{(\ln n)^{2n}}{n^n}$

15) $a_n = n/2^n$

Answer

$\lim_{n\to\infty} (a_n)^{1/n} = 1/2 < 1$. Converges by the Root Test.

16) $a_n = n/e^n$

17) $a_k = \frac{k^e}{e^k}$

Answer

$\lim_{k\to\infty} (a_k)^{1/k} = 1/e < 1$. Converges by the Root Test.

18) $a_k = \frac{\pi^k}{k^\pi}$

19) $a_n = \left(\frac{1}{e} + \frac{1}{n}\right)^n$

Answer

$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \frac{1}{e} + \frac{1}{n} = \frac{1}{e} < 1$. Converges by the Root Test.

20) $a_k = \frac{1}{(1+\ln k)^k}$

21) $a_n = \frac{(\ln(1+\ln n))^n}{(\ln n)^n}$

Answer

$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \frac{(\ln(1+\ln n))}{(\ln n)} = 0$ by L'Hôpital's rule. Converges by the Root Test.

In exercises 22 - 28, use either the ratio test or the root test as appropriate to determine whether the series $\sum_{k=1}^{\infty} a_k$ with given terms a_k converges, or state if the test is inconclusive.

22) $a_k = \frac{k!}{1\cdot 3\cdot 5\cdots(2k-1)}$

23) $a_k = \frac{2\cdot 4\cdot 6\cdots 2k}{(2k)!}$

Answer

$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{1}{2k+1} = 0$. Converges by the Ratio Test.

24) $a_k = \frac{1\cdot 4\cdot 7\cdots(3k-2)}{3^k k!}$

25) $a_n = \left(1 - \frac{1}{n}\right)^{n^2}$

Answer

$\lim_{n\to\infty}(a_n)^{1/n} = 1/e$. Converges by the Root Test.

26) $a_k = \left(\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{2k}\right)^k$ $\left(\text{Hint: Compare } a_k^{1/k} \text{ to } \int_k^{2k} \frac{dt}{t}.\right)$

27) $a_k = \left(\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{3k}\right)^k$

Answer

$\lim_{k\to\infty} a_k^{1/k} = \ln(3) > 1$. Diverges by the Root Test.

28) $a_n = \left(n^{1/n} - 1\right)^n$

In exercises 29 - 30, use the ratio test to determine whether $\sum_{n=1}^{\infty} a_n$ converges, or state if the ratio test is inconclusive.

29) $\sum_{n=1}^{\infty} \frac{3^{n^2}}{2^{n^3}}$

Answer

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{3^{2n+1}}{2^{3n^2+3n+1}} = 0$. Converges by the Ratio Test.

30) $\sum_{n=1}^{\infty} \frac{2^{n^2}}{n^n n!}$

In exercise 31, use the root and limit comparison tests to determine whether $\sum_{n=1}^{\infty} a_n$ converges.

31) $\sum_{n=1}^{\infty} \frac{1}{x_n^n}$ where $x_{n+1} = \frac{1}{2}x_n + \frac{1}{x_n}, x_1 = 1$ (Hint: Find limit of x_n.)

Answer

Converges by the Root Test and Limit Comparison Test since $\lim_{n\to\infty} x_n = \sqrt{2}$.

In exercises 32 - 43, use an appropriate test to determine whether the series converges.

32) $\sum_{n=1}^{\infty} \frac{n+1}{n^3+n^2+n+1}$

33) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{n^3+3n^2+3n+1}$

Answer

Converges absolutely by limit comparison with p**−series,** $p = 2$.

34) $\sum_{n=1}^{\infty} \frac{(n+1)^2}{n^3+(1.1)^n}$

35) $\sum_{n=1}^{\infty} \frac{(n-1)^n}{(n+1)^n}$

Answer

$\lim_{n\to\infty} a_n = 1/e^2 \neq 0$. Series diverges by the Divergence Test.

36) $a_n = \left(1 + \frac{1}{n^2}\right)^n$ $\left(\text{Hint: } \left(1 + \frac{1}{n^2}\right)^{n^2} \approx e.\right)$

37) $a_k = 1/2^{\sin^2 k}$

Answer

Terms do not tend to zero: $a_k \geq 1/2$, since $\sin^2 x \leq 1$.

38) $a_k = 2^{-\sin(1/k)}$

39) $a_n = 1/\binom{n+2}{n}$ where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Answer

$a_n = \dfrac{2}{(n+1)(n+2)}$, which converges by comparison with p**−series** for $p = 2$.

40) $a_k = 1/\binom{2k}{k}$

41) $a_k = 2^k/\binom{3k}{k}$

Answer

$a_k = \dfrac{2^k 1 \cdot 2 \cdots k}{(2k+1)(2k+2)\cdots 3k} \leq (2/3)^k$ converges by comparison with geometric series.

42) $a_k = \left(\dfrac{k}{k+\ln k}\right)^k$ $\left(\text{Hint: } a_k = \left(1 + \dfrac{\ln k}{k}\right)^{-(k/\ln k)\ln k} \approx e^{-\ln k}.\right)$

43) $a_k = \left(\dfrac{k}{k+\ln k}\right)^{2k}$ $\left(\text{Hint: } a_k = \left(1 + \dfrac{\ln k}{k}\right)^{-(k/\ln k)\ln k^2}.\right)$

Answer

$a_k \approx e^{-\ln k^2} = 1/k^2$. Series converges by limit comparison with p**−series**, $p = 2$.

The series in exercises 44 - 47 converge by the ratio test. Use summation by parts, $\sum_{k=1}^{n} a_k(b_{k+1} - b_k) = [a_{n+1}b_{n+1} - a_1 b_1] - \sum_{k=1}^{n} b_{k+1}(a_{k+1} - a_k)$**, to find the sum of the given series.**

44) $\sum_{k=1}^{\infty} \dfrac{k}{2^k}$ (Hint: Take $a_k = k$ and $b_k = 2^{1-k}$.)

45) $\sum_{k=1}^{\infty} \dfrac{k}{c^k}$, where $c > 1$ (Hint: Take $a_k = k$ and $b_k = c^{1-k}/(c-1)$.)

Answer

If $b_k = c^{1-k}/(c-1)$ and $a_k = k$, then $b_{k+1} - b_k = -c^{-k}$ and $\sum_{n=1}^{\infty} \dfrac{k}{c^k} = a_1 b_1 + \dfrac{1}{c-1}\sum_{k=1}^{\infty} c^{-k} = \dfrac{c}{(c-1)^2}$.

46) $\sum_{n=1}^{\infty} \dfrac{n^2}{2^n}$

47) $\sum_{n=1}^{\infty} \dfrac{(n+1)^2}{2^n}$

Answer

$6 + 4 + 1 = 11$

The kth term of each of the following series has a factor x^k. Find the range of x for which the ratio test implies that the series converges.

48) $\sum_{k=1}^{\infty} \dfrac{x^k}{k^2}$

49) $\sum_{k=1}^{\infty} \dfrac{x^{2k}}{k^2}$

Answer

$|x| \leq 1$

50) $\sum_{k=1}^{\infty} \frac{x^{2k}}{3^k}$

51) $\sum_{k=1}^{\infty} \frac{x^k}{k!}$

Answer

$|x| < \infty$

52) Does there exist a number p such that $\sum_{n=1}^{\infty} \frac{2^n}{n^p}$ converges?

53) Let $0 < r < 1$. For which real numbers p does $\sum_{n=1}^{\infty} n^p r^n$ converge?

Answer

All real numbers p by the Ratio Test.

54) Suppose that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = p$. For which values of p must $\sum_{n=1}^{\infty} 2^n a_n$ converge?

55) Suppose that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = p$. For which values of $r > 0$ is $\sum_{n=1}^{\infty} r^n a_n$ guaranteed to converge?

Answer

$r < 1/p$

56) Suppose that $\left|\frac{a_{n+1}}{a_n}\right| \leq (n+1)^p$ for all $n = 1, 2, \ldots$ where p is a fixed real number. For which values of p is $\sum_{n=1}^{\infty} n! a_n$ guaranteed to converge?

57) For which values of $r > 0$, if any, does $\sum_{n=1}^{\infty} r^{\sqrt{n}}$ converge? $\left(\text{Hint: } sum_{n=1}^{\infty} a_n = \sum_{k=1}^{\infty} \sum_{n=k^2}^{(k+1)^2-1} a_n.\right)$

Answer

$0 < r < 1$. Note that the ratio and root tests are inconclusive. Using the hint, there are $2k$ terms $r^{\sqrt{n}}$ for $k^2 \leq n < (k+1)^2$, and for $r < 1$ each term is at least r^k. Thus, $\sum_{n=1}^{\infty} r^{\sqrt{n}} = \sum_{k=1}^{\infty} \sum_{n=k^2}^{(k+1)^2-1} r^{\sqrt{n}} \geq \sum_{k=1}^{\infty} 2kr^k$, which converges by the ratio test for $r < 1$. For $r \geq 1$ the series diverges by the divergence test.

58) Suppose that $\left|\frac{a_{n+2}}{a_n}\right| \leq r < 1$ for all n. Can you conclude that $\sum_{n=1}^{\infty} a_n$ converges?

59) Let $a_n = 2^{-[n/2]}$ where $[x]$ is the greatest integer less than or equal to x. Determine whether $\sum_{n=1}^{\infty} a_n$ converges and justify your answer.

Answer

One has $a_1 = 1, a_2 = a_3 = 1/2, \ldots a_{2n} = a_{2n+1} = 1/2^n$. The ratio test does not apply because $a_{n+1}/a_n = 1$ if n is even. However, $a_{n+2}/a_n = 1/2$, so the series converges according to the previous exercise. Of course, the series is just a duplicated geometric series.

The following *advanced exercises* use a generalized ratio test to determine convergence of some series that arise in particular applications when tests in this chapter, including the ratio and root test, are not powerful enough to determine their convergence. The test states that if $\lim_{n\to\infty} \frac{a_{2n}}{a_n} < 1/2$, then $\sum a_n$ converges, while if $\lim_{n\to\infty} \frac{a_{2n+1}}{a_n} > 1/2$, then $\sum a_n$ diverges.

60) Let $a_n = \frac{1}{4}\frac{3}{6}\frac{5}{8}\cdots\frac{2n-1}{2n+2} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n(n+1)!}$. Explain why the ratio test cannot determine convergence of $\sum_{n=1}^{\infty} a_n$. Use the fact that $1-1/(4k)$ is increasing k to estimate $\lim_{n\to\infty}\frac{a_{2n}}{a_n}$.

61) Let $a_n = \frac{1}{1+x}\frac{2}{2+x}\cdots\frac{n}{n+x}\frac{1}{n} = \frac{(n-1)!}{(1+x)(2+x)\cdots(n+x).}$ Show that $a_{2n}/a_n \le e^{-x/2}/2$. For which $x>0$ does the generalized ratio test imply convergence of $\sum_{n=1}^{\infty} a_n$? (Hint: Write $2a_{2n}/a_n$ as a product of n factors each smaller than $1/(1+x/(2n)).$)

Answer

$a_{2n}/a_n = \frac{1}{2}\cdot\frac{n+1}{n+1+x}\frac{n+2}{n+2+x}\cdots\frac{2n}{2n+x}$. The inverse of the kth factor is $(n+k+x)/(n+k) > 1+x/(2n)$ so the product is less than $(1+x/(2n))^{-n} \approx e^{-x/2}$. Thus for $x>0, \frac{a_{2n}}{a_n} \le \frac{1}{2}e^{-x/2}$. The series converges for $x>0$.

62) Let $a_n = \frac{n^{\ln n}}{(\ln n)^n}$. Show that $\frac{a_{2n}}{a_n} \to 0$ as $n\to\infty$.

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9.R: Chapter 9 Review Exercises

***True or False?* Justify your answer with a proof or a counterexample.**

1) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

Answer

false

2) If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

3) If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Answer

true

4) If $\sum_{n=1}^{\infty} 2^n a_n$ converges, then $\sum_{n=1}^{\infty} (-2)^n a_n$ converges.

Is the sequence bounded, monotone, and convergent or divergent? If it is convergent, find the limit.

5) $a_n = \dfrac{3+n^2}{1-n}$

Answer

unbounded, not monotone, divergent

6) $a_n = \ln\left(\frac{1}{n}\right)$

7) $a_n = \dfrac{\ln(n+1)}{\sqrt{n+1}}$

Answer

bounded, monotone, convergent, 0

8) $a_n = \dfrac{2^{n+1}}{5^n}$

9) $a_n = \dfrac{\ln(\cos n)}{n}$

Answer

unbounded, not monotone, divergent

Is the series convergent or divergent?

10) $\sum_{n=1}^{\infty} \dfrac{1}{n^2+5n+4}$

11) $\sum_{n=1}^{\infty} \ln\left(\dfrac{n+1}{n}\right)$

Answer

diverges

12) $\sum_{n=1}^{\infty} \dfrac{2^n}{n^4}$

13) $\sum_{n=1}^{\infty} \dfrac{e^n}{n!}$

Answer

converges

14) $\sum_{n=1}^{\infty} n^{-(n+1/n)}$

Is the series convergent or divergent? If convergent, is it absolutely convergent?

15) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

Answer

converges, but not absolutely

16) $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{3^n}$

17) $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^n}$

Answer

converges absolutely

18) $\sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2}\right)$

19) $\sum_{n=1}^{\infty} \cos(\pi n) e^{-n}$

Answer

converges absolutely

Evaluate.

20) $\sum_{n=1}^{\infty} \frac{2^{n+4}}{7^n}$

21) $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$

Answer

$\frac{1}{2}$

22) A legend from India tells that a mathematician invented chess for a king. The king enjoyed the game so much he allowed the mathematician to demand any payment. The mathematician asked for one grain of rice for the first square on the chessboard, two grains of rice for the second square on the chessboard, and so on. Find an exact expression for the total payment (in grains of rice) requested by the mathematician. Assuming there are $30,000$ grains of rice in 1 pound, and 2000 pounds in 1 ton, how many tons of rice did the mathematician attempt to receive?

The following problems consider a simple population model of the housefly, which can be exhibited by the recursive formula $x_{n+1} = bx_n$, where x_n is the population of houseflies at generation n, and b is the average number of offspring per housefly who survive to the next generation. Assume a starting population x_0.

23) Find $\lim_{n\to\infty} x_n$ if $b > 1,\ b < 1$, and $b = 1$.

Answer

$\infty,\ 0,\ x_0$

24) Find an expression for $S_n = \sum_{i=0}^{n} x_i$ in terms of b and x_0. What does it physically represent?

25) If $b = \frac{3}{4}$ and $x_0 = 100$, find S_{10} and $\lim_{n\to\infty} S_n$

Answer

$S_{10} \approx 383, \quad \lim_{n\to\infty} S_n = 400$

26) For what values of b will the series converge and diverge? What does the series converge to?

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CHAPTER OVERVIEW

10: Power Series

A power series (in one variable) is an infinite series. Any polynomial can be easily expressed as a power series around any center c, although most of the coefficients will be zero since a power series has infinitely many terms by definition. One can view power series as being like "polynomials of infinite degree," although power series are not polynomials. The content in this Textmap's chapter is complemented by Guichard's Calculus Textmap.

10.0: Prelude to Power Series

10.1: Power Series and Functions

10.1E: Exercises for Section 10.1

10.2: Properties of Power Series

10.2E: Exercises for Section 10.2

10.3: Taylor and Maclaurin Series

10.3E: Exercises for Section 10.3

10.4: Working with Taylor Series

10.4E: Exercises for Section 10.4

10.R: Chapter 10 Review Exercises

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10.0: Prelude to Power Series

When winning a lottery, sometimes an individual has an option of receiving winnings in one lump-sum payment or receiving smaller payments over fixed time intervals. For example, you might have the option of receiving 20 million dollars today or receiving 1.5 million dollars each year for the next 20 years. Which is the better deal? Certainly 1.5 million dollars over 20 years is equivalent to 30 million dollars. However, receiving the 20 million dollars today would allow you to invest the money.

Figure 10.0.1: If you win a lottery, do you get more money by taking a lump-sum payment or by accepting fixed payments over time? (credit: modification of work by Robert Huffstutter, Flickr)

Alternatively, what if you were guaranteed to receive 1 million dollars every year indefinitely (extending to your heirs) or receive 20 million dollars today. Which would be the better deal? To answer these questions, you need to know how to use infinite series to calculate the value of periodic payments over time in terms of today's dollars.

An infinite series of the form

$$\sum_{n=0}^{\infty} c_n x^n$$

is known as a **power series**. Since the terms contain the variable x, power series can be used to define functions. They can be used to represent given functions, but they are also important because they allow us to write functions that cannot be expressed any other way than as "infinite polynomials." In addition, power series can be easily differentiated and integrated, thus being useful in solving differential equations and integrating complicated functions. An infinite series can also be truncated, resulting in a finite polynomial that we can use to approximate functional values. Power series have applications in a variety of fields, including physics, chemistry, biology, and economics. As we will see in this chapter, representing functions using power series allows us to solve mathematical problems that cannot be solved with other techniques.

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10.1: Power Series and Functions

Learning Objectives

- Identify a power series and provide examples of them.
- Determine the radius of convergence and interval of convergence of a power series.
- Use a power series to represent a function.

A power series is a type of series with terms involving a variable. More specifically, if the variable is x, then all the terms of the series involve powers of x. As a result, a power series can be thought of as an infinite polynomial. Power series are used to represent common functions and also to define new functions. In this section we define power series and show how to determine when a power series converges and when it diverges. We also show how to represent certain functions using power series.

Form of a Power Series

A series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots,$$

where x is a variable and the coefficients c_n are constants, is known as a **power series.** The series

$$1 + x + x^2 + \ldots = \sum_{n=0}^{\infty} x^n$$

is an example of a power series. Since this series is a geometric series with ratio $r = |x|$, we know that it converges if $|x| < 1$ and diverges if $|x| \geq 1$.

Definition 10.1.1: Power Series

A series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots$$

is a power series centered at $x = 0$. A series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \ldots$$

is a *power series* centered at $x = a$.

To make this definition precise, we stipulate that $x^0 = 1$ and $(x-a)^0 = 1$ even when $x = 0$ and $x = a$, respectively.

The series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

and

$$\sum_{n=0}^{\infty} n! x^n = 1 + x + 2! x^2 + 3! x^3 + \ldots$$

are both power series centered at $x = 0$. The series

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)3^n} = 1 + \frac{x-2}{2 \cdot 3} + \frac{(x-2)^2}{3 \cdot 3^2} + \frac{(x-2)^3}{4 \cdot 3^3} + \ldots$$

is a power series centered at $x = 2$.

Convergence of a Power Series

Since the terms in a power series involve a variable x, the series may converge for certain values of x and diverge for other values of x. For a power series centered at $x = a$, the value of the series at $x = a$ is given by c_0. Therefore, a power series always converges at its center. Some power series converge only at that value of x. Most power series, however, converge for more than one value of x. In that case, the power series either converges for all real numbers x or converges for all x in a finite interval. For example, the geometric series $\sum_{n=0}^{\infty} x^n$ converges for all x in the interval $(-1, 1)$, but diverges for all x outside that interval. We now summarize these three possibilities for a general power series.

Note 10.1.1: Convergence of a Power Series

Consider the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$. The series satisfies exactly one of the following properties:

i. The series converges at $x = a$ and diverges for all $x \neq a$.
ii. The series converges for all real numbers x.
iii. There exists a real number $R > 0$ such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$. At the values x where $|x - a| = R$, the series may converge or diverge.

Proof

Suppose that the power series is centered at $a = 0$. (For a series centered at a value of a other than zero, the result follows by letting $y = x - a$ and considering the series

$$\sum_{n=1}^{\infty} c_n y^n.$$

We must first prove the following fact:

If there exists a real number $d \neq 0$ such that $\sum_{n=0}^{\infty} c_n d^n$ converges, then the series $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely for all x such that $|x| < |d|$.

Since $\sum_{n=0}^{\infty} c_n d^n$ converges, the nth term $c_n d^n \to 0$ as $n \to \infty$. Therefore, there exists an integer N such that $|c_n d^n| \leq 1$ for all $n \geq N$. Writing

$$|c_n x^n| = |c_n d^n| \left|\frac{x}{d}\right|^n,$$

we conclude that, for all n≥N,

$$|c_n x^n| \leq \left|\frac{x}{d}\right|^n.$$

The series

$$\sum_{n=N}^{\infty} \left|\frac{x}{d}\right|^n$$

is a geometric series that converges if $|\frac{x}{d}| < 1$. Therefore, by the comparison test, we conclude that $\sum_{n=N}^{\infty} c_n x^n$ also converges for $|x| < |d|$. Since we can add a finite number of terms to a convergent series, we conclude that $\sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < |d|$.

With this result, we can now prove the theorem. Consider the series

$$\sum_{n=0}^{\infty} a_n x^n$$

and let S be the set of real numbers for which the series converges. Suppose that the set $S = 0$. Then the series falls under case i.

Suppose that the set S is the set of all real numbers. Then the series falls under case ii. Suppose that $S \neq 0$ and S is not the set of real numbers. Then there exists a real number $x* \neq 0$ such that the series does not converge. Thus, the series cannot converge for any x such that $|x| > |x*|$. Therefore, the set S must be a bounded set, which means that it must have a smallest upper bound. (This fact follows from the **Least Upper Bound Property** for the real numbers, which is beyond the scope of this text and is covered in real analysis courses.) Call that smallest upper bound R. Since $S \neq 0$, the number $R > 0$. Therefore, the series converges for all x such that $|x| < R$, and the series falls into case iii.

□

If a series $\sum_{n=0}^{\infty} c_n(x-a)^n$ falls into case iii. of Note 10.1.1, then the series converges for all x such that $|x-a| < R$ for some $R > 0$, and diverges for all x such that $|x-a| > R$. The series may converge or diverge at the values x where $|x-a| = R$. The set of values x for which the series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges is known as the interval of convergence. Since the series diverges for all values x where $|x-a| > R$, the length of the interval is $2R$, and therefore, the radius of the interval is R. The value R is called the radius of convergence. For example, since the series $\sum_{n=0}^{\infty} x^n$ converges for all values x in the interval $(-1, 1)$ and diverges for all values x such that $|x| \geq 1$, the interval of convergence of this series is $(-1, 1)$. Since the length of the interval is 2, the radius of convergence is 1.

Definition: Radius of Convergence

Consider the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$. The set of real numbers x where the series converges is the interval of convergence. If there exists a real number $R > 0$ such that the series converges for $|x-a| < R$ and diverges for $|x-a| > R$, then R is the radius of convergence. If the series converges only at $x = a$, we say the radius of convergence is $R = 0$. If the series converges for all real numbers x, we say the radius of convergence is $R = \infty$ (Figure 10.1.1).

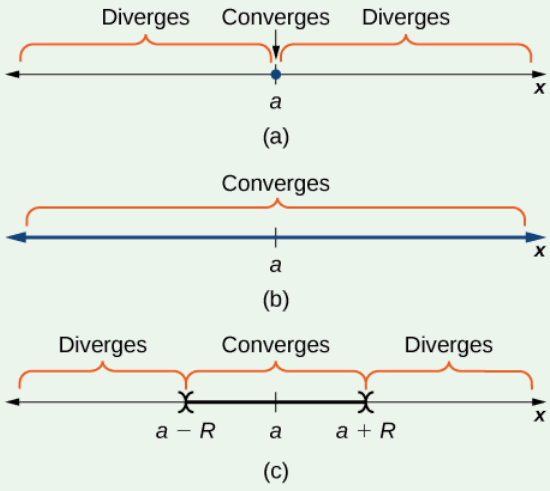


Figure 10.1.1: For a series $\sum_{n=0}^{\infty} c_n(x-a)^n$ graph (a) shows a radius of convergence at $R = 0$, graph (b) shows a radius of convergence at $R = \infty$, and graph (c) shows a radius of convergence at R. For graph (c) we note that the series may or may not converge at the endpoints $x = a + R$ and $x = a - R$.

To determine the interval of convergence for a power series, we typically apply the ratio test. In Example 10.1.1, we show the three different possibilities illustrated in Figure 10.1.1.

Example 10.1.1: Finding the Interval and Radius of Convergence

For each of the following series, find the interval and radius of convergence.

a. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

b. $\sum_{n=0}^{\infty} n!x^n$

c. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)3^n}$

Solution

a. To check for convergence, apply the ratio test. We have

$$\begin{aligned}\rho &= \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| \\ &= \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| \\ &= \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)\cdot n!} \cdot \frac{n!}{x^n} \right| \\ &= \lim_{n\to\infty} \left| \frac{x}{n+1} \right| \\ &= |x| \lim_{n\to\infty} \frac{1}{n+1} \\ &= 0 < 1\end{aligned}$$

for all values of x. Therefore, the series converges for all real numbers x. The interval of convergence is $(-\infty, \infty)$ and the radius of convergence is $R = \infty$.

b. Apply the ratio test. For $x \neq 0$, we see that

$$\begin{aligned}\rho &= \lim_{n\to\infty} \left| \frac{(n+1)!x^{n+1}}{n!x^n} \right| \\ &= \lim_{n\to\infty} |(n+1)x| \\ &= |x| \lim_{n\to\infty} (n+1) \\ &= \infty.\end{aligned}$$

Therefore, the series diverges for all $x \neq 0$. Since the series is centered at $x = 0$, it must converge there, so the series converges only for $x \neq 0$. The interval of convergence is the single value $x = 0$ and the radius of convergence is $R = 0$.

c. In order to apply the ratio test, consider

$$\begin{aligned}\rho &= \lim_{n\to\infty} \left| \frac{\frac{(x-2)^{n+1}}{(n+2)3^{n+1}}}{\frac{(x-2)^n}{(n+1)3^n}} \right| \\ &= \lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(n+2)3^{n+1}} \cdot \frac{(n+1)3^n}{(x-2)^n} \right| \\ &= \lim_{n\to\infty} \left| \frac{(x-2)(n+1)}{3(n+2)} \right| \\ &= \frac{|x-2|}{3}.\end{aligned}$$

The ratio $\rho < 1$ if $|x-2| < 3$. Since $|x-2| < 3$ implies that $-3 < x-2 < 3$, the series converges absolutely if $-1 < x < 5$. The ratio $\rho > 1$ if $|x-2| > 3$. Therefore, the series diverges if $x < -1$ or $x > 5$. The ratio test is inconclusive if $\rho = 1$. The ratio $\rho = 1$

if and only if $x = -1$ or $x = 5$. We need to test these values of x separately. For $x = -1$, the series is given by

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots .$$

Since this is the alternating harmonic series, it converges. Thus, the series converges at $x = -1$. For $x = 5$, the series is given by

$$\sum_{n=0}^{\infty} \frac{1}{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots .$$

This is the harmonic series, which is divergent. Therefore, the power series diverges at $x = 5$. We conclude that the interval of convergence is $[-1, 5)$ and the radius of convergence is $R = 3$.

? Exercise 10.1.1

Find the interval and radius of convergence for the series

$$\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}.$$

Hint

Apply the ratio test to check for absolute convergence.

Answer

The interval of convergence is $[-1, 1)$. The radius of convergence is $R = 1$.

Representing Functions as Power Series

Being able to represent a function by an "infinite polynomial" is a powerful tool. Polynomial functions are the easiest functions to analyze, since they only involve the basic arithmetic operations of addition, subtraction, multiplication, and division. If we can represent a complicated function by an infinite polynomial, we can use the polynomial representation to differentiate or integrate it. In addition, we can use a truncated version of the polynomial expression to approximate values of the function. So, the question is, when can we represent a function by a power series?

Consider again the geometric series

$$1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n.$$

Recall that the geometric series

$$a + ar + ar^2 + ar^3 + \ldots$$

converges if and only if $|r| < 1$. In that case, it converges to $\frac{a}{1-r}$. Therefore, if $|x| < 1$, the series in Example 10.1.1 converges to $\frac{1}{1-x}$ and we write

$$1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x} \text{ for } |x| < 1.$$

As a result, we are able to represent the function $f(x) = \frac{1}{1-x}$ by the power series

$$1 + x + x^2 + x^3 + \ldots \text{ when } |x| < 1.$$

We now show graphically how this series provides a representation for the function $f(x) = \frac{1}{1-x}$ by comparing the graph of f with the graphs of several of the partial sums of this infinite series.

Example 10.1.2: Graphing a Function and Partial Sums of its Power Series

Sketch a graph of $f(x) = \dfrac{1}{1-x}$ and the graphs of the corresponding partial sums $S_N(x) = \sum_{n=0}^{N} x^n$ for $N = 2, 4, 6$ on the interval $(-1, 1)$. Comment on the approximation S_N as N increases.

Solution

From the graph in Figure 10.1.2, you see that as N increases, S_N becomes a better approximation for $f(x) = \dfrac{1}{1-x}$ for x in the interval $(-1, 1)$.

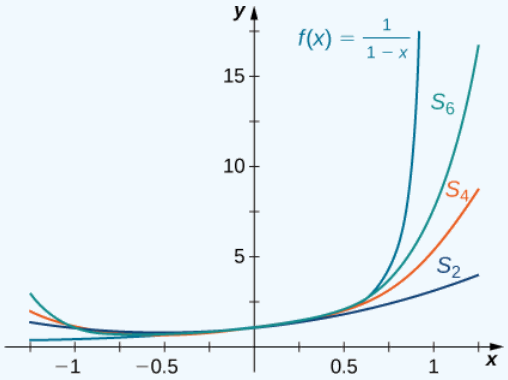


Figure 10.1.2: The graph shows a function and three approximations of it by partial sums of a power series.

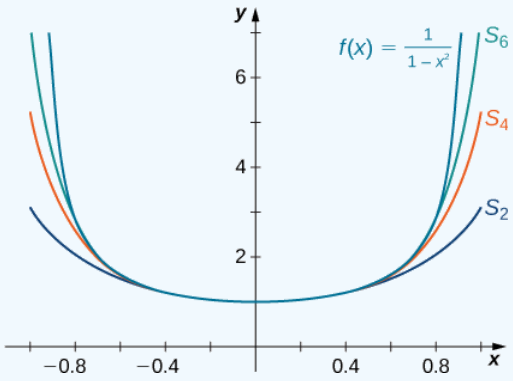
Exercise 10.1.2

Sketch a graph of $f(x) = \dfrac{1}{1-x^2}$ and the corresponding partial sums $S_N(x) = \sum_{n=0}^{N} x^{2n}$ for $N = 2, 4, 6$ on the interval $(-1, 1)$.

Hint

$$S_N(x) = 1 + x^2 + \ldots + x^{2N} = \frac{1 - x^{2(N+1)}}{1 - x^2}$$

Answer



Next we consider functions involving an expression similar to the sum of a geometric series and show how to represent these functions using power series.

Example 10.1.3: Representing a Function with a Power Series

Use a power series to represent each of the following functions f. Find the interval of convergence.

a. $f(x) = \dfrac{1}{1+x^3}$

b. $f(x) = \dfrac{x^2}{4 - x^2}$

Solution

a. You should recognize this function f as the sum of a geometric series, because

$$\frac{1}{1+x^3} = \frac{1}{1-(-x^3)}.$$

Using the fact that, for $|r| < 1$, $\dfrac{a}{1-r}$ is the sum of the geometric series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \ldots,$$

we see that, for $|-x^3| < 1$,

$$\begin{aligned}\frac{1}{1+x^3} &= \frac{1}{1-(-x^3)} \\ &= \sum_{n=0}^{\infty} (-x^3)^n \\ &= 1 - x^3 + x^6 - x^9 + \ldots.\end{aligned}$$

Since this series converges if and only if $|-x^3| < 1$, the interval of convergence is $(-1, 1)$, and we have

$$\frac{1}{1+x^3} = 1 - x^3 + x^6 - x^9 + \ldots \text{ for } |x| < 1.$$

b. This function is not in the exact form of a sum of a geometric series. However, with a little algebraic manipulation, we can relate f to a geometric series. By factoring 4 out of the two terms in the denominator, we obtain

$$\begin{aligned}\frac{x^2}{4-x^2} &= \frac{x^2}{4\left(\dfrac{1-x^2}{4}\right)} \\ &= \frac{x^2}{4\left(1-\left(\dfrac{x}{2}\right)^2\right)}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\frac{x^2}{4-x^2} &= \frac{x^2}{4\left(1-\left(\dfrac{x}{2}\right)^2\right)} \\ &= \frac{\dfrac{x^2}{4}}{1-\left(\dfrac{x}{2}\right)^2} \\ &= \sum_{n=0}^{\infty} \frac{x^2}{4}\left(\frac{x}{2}\right)^{2n}.\end{aligned}$$

The series converges as long as $\left|\left(\dfrac{x}{2}\right)^2\right| < 1$ (note that when $\left|\left(\dfrac{x}{2}\right)^2\right| = 1$ the series does not converge). Solving this inequality, we conclude that the interval of convergence is $(-2, 2)$ and

$$\begin{aligned}\frac{x^2}{4-x^2} &= \sum_{n=0}^{\infty} \frac{x^{2n+2}}{4^{n+1}} \\ &= \frac{x^2}{4} + \frac{x^4}{4^2} + \frac{x^6}{4^3} + \ldots\end{aligned}$$

for $|x| < 2$.

? Exercise 10.1.3

Represent the function $f(x) = \dfrac{x^3}{2-x}$ using a power series and find the interval of convergence.

Hint

Rewrite f in the form $f(x) = \dfrac{g(x)}{1-h(x)}$ for some functions g and h.

Answer

$\displaystyle\sum_{n=0}^{\infty} \frac{x^{n+3}}{2^{n+1}}$ with interval of convergence $(-2, 2)$

In the remaining sections of this chapter, we will show ways of deriving power series representations for many other functions, and how we can make use of these representations to evaluate, differentiate, and integrate various functions.

Key Concepts

- For a power series centered at $x = a$, one of the following three properties hold:
  - i. The power series converges only at $x = a$. In this case, we say that the radius of convergence is $R = 0$.
  - ii. The power series converges for all real numbers x. In this case, we say that the radius of convergence is $R = \infty$.
  - iii. There is a real number R such that the series converges for $|x - a| < R$ and diverges for $|x - a| > R$. In this case, the radius of convergence is R.
- If a power series converges on a finite interval, the series may or may not converge at the endpoints.
- The ratio test may often be used to determine the radius of convergence.
- The geometric series $\displaystyle\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$ allows us to represent certain functions using geometric series.

Key Equations

- **Power series centered at** $x = 0$

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots n$$

- **Power series centered at** $x = a$

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \ldots$$

Glossary

interval of convergence

the set of real numbers x for which a power series converges

power series

a series of the form $\displaystyle\sum_{n=0}^{\infty} c_n x^n$ is a power series centered at $x = 0$; a series of the form $\displaystyle\sum_{n=0}^{\infty} c_n (x-a)^n$ is a power series centered at $x = a$

radius of convergence

if there exists a real number $R > 0$ such that a power series centered at $x = a$ converges for $|x - a| < R$ and diverges for $|x - a| > R$, then R is the radius of convergence; if the power series only converges at $x = a$, the radius of convergence is $R = 0$; if the power series converges for all real numbers x, the radius of convergence is $R = \infty$

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10.1E: Exercises for Section 10.1

In exercises 1 - 4, state whether each statement is true, or give an example to show that it is false.

1) If $\sum_{n=1}^{\infty} a_n x^n$ converges, then $a_n x^n \to 0$ as $n \to \infty$.

Answer

True. If a series converges then its terms tend to zero.

2) $\sum_{n=1}^{\infty} a_n x^n$ converges at $x = 0$ for any real numbers a_n.

3) Given any sequence a_n, there is always some $R > 0$, possibly very small, such that $\sum_{n=1}^{\infty} a_n x^n$ converges on $(-R, R)$.

Answer

False. It would imply that $a_n x^n \to 0$ for $|x| < R$. If $a_n = n^n$, then $a_n x^n = (nx)^n$ does not tend to zero for any $x \neq 0$.

4) If $\sum_{n=1}^{\infty} a_n x^n$ has radius of convergence $R > 0$ and if $|b_n| \leq |a_n|$ for all n, then the radius of convergence of $\sum_{n=1}^{\infty} b_n x^n$ is greater than or equal to R.

5) Suppose that $\sum_{n=0}^{\infty} a_n (x-3)^n$ converges at $x = 6$. At which of the following points must the series also converge? Use the fact that if $\sum a_n (x-c)^n$ converges at x, then it converges at any point closer to c than x.

a. $x = 1$

b. $x = 2$

c. $x = 3$

d. $x = 0$

e. $x = 5.99$

f. $x = 0.000001$

Answer

It must converge on $(0, 6]$ and hence at: a. $x = 1$; b. $x = 2$; c. $x = 3$; d. $x = 0$; e. $x = 5.99$; and f. $x = 0.000001$.

6) Suppose that $\sum_{n=0}^{\infty} a_n (x+1)^n$ converges at $x = -2$. At which of the following points must the series also converge? Use the fact that if $\sum a_n (x-c)^n$ converges at x, then it converges at any point closer to c than x.

a. $x = 2$

b. $x = -1$

c. $x = -3$

d. $x = 0$

e. $x = 0.99$

f. $x = 0.000001$

In the following exercises, suppose that $\left|\frac{a_{n+1}}{a_n}\right| \to 1$ as $n \to \infty$. Find the radius of convergence for each series.

7) $\sum_{n=0}^{\infty} a_n 2^n x^n$

Answer

$$\left|\frac{a_{n+1}2^{n+1}x^{n+1}}{a_n 2^n x^n}\right| = 2|x|\left|\frac{a_{n+1}}{a_n}\right| \to 2|x| \text{ so } R = \tfrac{1}{2}$$

8) $\sum_{n=0}^{\infty} \frac{a_n x^n}{2^n}$

9) $\sum_{n=0}^{\infty} \frac{a_n \pi^n x^n}{e^n}$

Answer

$$\left|\frac{a_{n+1}(\frac{\pi}{e})^{n+1}x^{n+1}}{a_n(\frac{\pi}{e})^n x^n}\right| = \frac{\pi|x|}{e}\left|\frac{a_{n+1}}{a_n}\right| \to \frac{\pi|x|}{e} \text{ so } R = \tfrac{e}{\pi}$$

10) $\sum_{n=0}^{\infty} \frac{a_n(-1)^n x^n}{10^n}$

11) $\sum_{n=0}^{\infty} a_n(-1)^n x^{2n}$

Answer

$$\left|\frac{a_{n+1}(-1)^{n+1}x^{2n+2}}{a_n(-1)^n x^{2n}}\right| = |x^2|\left|\frac{a_{n+1}}{a_n}\right| \to |x^2| \text{ so } R = 1$$

12) $\sum_{n=0}^{\infty} a_n(-4)^n x^{2n}$

In exercises 13 - 22, find the radius of convergence R and interval of convergence for $\sum a_n x^n$ with the given coefficients a_n.

13) $\sum_{n=1}^{\infty} \frac{(2x)^n}{n}$

Answer

$a_n = \frac{2^n}{n}$ so $\frac{a_{n+1}x}{a_n} \to 2x$. so $R = \frac{1}{2}$. When $x = \frac{1}{2}$ the series is harmonic and diverges. When $x = -\frac{1}{2}$ the series is alternating harmonic and converges. The interval of convergence is $I = \left[-\frac{1}{2}, \frac{1}{2}\right)$.

14) $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{\sqrt{n}}$

15) $\sum_{n=1}^{\infty} \frac{nx^n}{2^n}$

Answer

$a_n = \frac{n}{2^n}$ so $\frac{a_{n+1}x}{a_n} \to \frac{x}{2}$ so $R = 2$. When $x = \pm 2$ the series diverges by the divergence test. The interval of convergence is $I = (-2, 2)$.

16) $\sum_{n=1}^{\infty} \frac{nx^n}{e^n}$

17) $\sum_{n=1}^{\infty} \frac{n^2 x^n}{2^n}$

Answer

$a_n = \frac{n^2}{2^n}$ so $R = 2$. When $x = \pm 2$ the series diverges by the divergence test. The interval of convergence is $I = (-2, 2)$.

18) $\sum_{k=1}^{\infty} \frac{k^e x^k}{e^k}$

19) $\sum_{k=1}^{\infty} \frac{\pi^k x^k}{k^\pi}$

Answer

$a_k = \frac{\pi^k}{k^\pi}$ so $R = \frac{1}{\pi}$. When $x = \pm\frac{1}{\pi}$ the series is an absolutely convergent p-series. The interval of convergence is $I = \left[-\frac{1}{\pi}, \frac{1}{\pi}\right]$.

20) $\sum_{n=1}^{\infty} \frac{x^n}{n!}$

21) $\sum_{n=1}^{\infty} \frac{10^n x^n}{n!}$

Answer

$a_n = \frac{10^n}{n!}, \frac{a_{n+1}x}{a_n} = \frac{10x}{n+1} \to 0 < 1$ so the series converges for all x by the ratio test and $I = (-\infty, \infty)$.

22) $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{\ln(2n)}$

In exercises 23 - 28, find the radius of convergence of each series.

23) $\sum_{k=1}^{\infty} \frac{(k!)^2 x^k}{(2k)!}$

Answer

$a_k = \frac{(k!)^2}{(2k)!}$ so $\frac{a_{k+1}}{a_k} = \frac{(k+1)^2}{(2k+2)(2k+1)} \to \frac{1}{4}$ so $R = 4$

24) $\sum_{n=1}^{\infty} \frac{(2n)! x^n}{n^{2n}}$

25) $\sum_{k=1}^{\infty} \frac{k!}{1 \cdot 3 \cdot 5 \cdots (2k-1)} x^k$

Answer

$a_k = \frac{k!}{1 \cdot 3 \cdot 5 \cdots (2k-1)}$ so $\frac{a_{k+1}}{a_k} = \frac{k+1}{2k+1} \to \frac{1}{2}$ so $R = 2$

26) $\sum_{k=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdots 2k}{(2k)!} x^k$

27) $\sum_{n=1}^{\infty} \frac{x^n}{\binom{2n}{n}}$ where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Answer

$a_n = \frac{1}{\binom{2n}{n}}$ so $\frac{a_{n+1}}{a_n} = \frac{\left((n+1)!\right)^2}{(2n+2)!} \frac{2n!}{(n!)^2} = \frac{(n+1)^2}{(2n+2)(2n+1)} \to \frac{1}{4}$ so $R = 4$

28) $\sum_{n=1}^{\infty} \sin^2 n x^n$

In exercises 29 - 32, use the ratio test to determine the radius of convergence of each series.

29) $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!} x^n$

Answer

$\frac{a_{n+1}}{a_n} = \frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)} \to \frac{1}{27}$ so $R = 27$

30) $\sum_{n=1}^{\infty} \frac{2^{3n}(n!)^3}{(3n)!} x^n$

31) $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$

Answer

$a_n = \dfrac{n!}{n^n}$ so $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)!}{n!}\dfrac{n^n}{(n+1)^{n+1}} = (\dfrac{n}{n+1})^n \to \dfrac{1}{e}$ so $R = e$

32) $\sum_{n=1}^{\infty} \dfrac{(2n)!}{n^{2n}} x^n$

In the following exercises, given that $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ with convergence in $(-1,1)$, find the power series for each function with the given center a, and identify its interval of convergence.

33) $f(x) = \dfrac{1}{x}; a = 1$ (Hint: $\dfrac{1}{x} = \dfrac{1}{1-(1-x)}$)

Answer

$f(x) = \sum_{n=0}^{\infty} (1-x)^n$ on $I = (0,2)$

34) $f(x) = \dfrac{1}{1-x^2}; a = 0$

35) $f(x) = \dfrac{x}{1-x^2}; a = 0$

Answer

$\sum_{n=0}^{\infty} x^{2n+1}$ on $I = (-1,1)$

36) $f(x) = \dfrac{1}{1+x^2}; a = 0$

37) $f(x) = \dfrac{x^2}{1+x^2}; a = 0$

Answer

$\sum_{n=0}^{\infty} (-1)^n x^{2n+2}$ on $I = (-1,1)$

38) $f(x) = \dfrac{1}{2-x}; a = 1$

39) $f(x) = \dfrac{1}{1-2x}; a = 0.$

Answer

$\sum_{n=0}^{\infty} 2^n x^n$ on $(-\frac{1}{2}, \frac{1}{2})$

40) $f(x) = \dfrac{1}{1-4x^2}; a = 0$

41) $f(x) = \dfrac{x^2}{1-4x^2}; a = 0$

Answer

$\sum_{n=0}^{\infty} 4^n x^{2n+2}$ on $(-\frac{1}{2}, \frac{1}{2})$

42) $f(x) = \dfrac{x^2}{5-4x+x^2}; a = 2$

Use the result of exercise 43 to find the radius of convergence of the given series in the subsequent exercises (44 - 47).

43) Explain why, if $|a_n|^{1/n} \to r > 0$, then $|a_n x^n|^{1/n} \to |x| r < 1$ whenever $|x| < \frac{1}{r}$ and, therefore, the radius of convergence of $\sum_{n=1}^{\infty} a_n x^n$ is $R = \frac{1}{r}$.

Answer

$|a_n x^n|^{1/n} = |a_n|^{1/n}|x| \to |x|r$ as $n \to \infty$ and $|x|r < 1$ when $|x| < \frac{1}{r}$. Therefore, $\sum_{n=1}^{\infty} a_n x^n$ converges when $|x| < \frac{1}{r}$ by the nth root test.

44) $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$

45) $\sum_{k=1}^{\infty} \left(\frac{k-1}{2k+3}\right)^k x^k$

Answer

$a_k = \left(\frac{k-1}{2k+3}\right)^k$ so $(a_k)^{1/k} \to \frac{1}{2} < 1$ so $R = 2$

46) $\sum_{k=1}^{\infty} (\frac{2k^2-1}{k^2+3})^k x^k$

47) $\sum_{n=1}^{\infty} a_n = (n^{1/n} - 1)^n x^n$

Answer

$a_n = (n^{1/n} - 1)^n$ so $(a_n)^{1/n} \to 0$ so $R = \infty$

48) Suppose that $p(x) = \sum_{n=0}^{\infty} a_n x^n$ such that $a_n = 0$ if n is even. Explain why $p(x) = p(-x)$.

49) Suppose that $p(x) = \sum_{n=0}^{\infty} a_n x^n$ such that $a_n = 0$ if n is odd. Explain why $p(x) = -p(-x)$.

Answer

We can rewrite $p(x) = \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}$ and $p(x) = p(-x)$ since $x^{2n+1} = -(-x)^{2n+1}$.

50) Suppose that $p(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on $(-1, 1]$. Find the interval of convergence of $p(Ax)$.

51) Suppose that $p(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on $(-1, 1]$. Find the interval of convergence of $p(2x - 1)$.

Answer

If $x \in [0, 1]$, then $y = 2x - 1 \in [-1, 1]$ so $p(2x - 1) = p(y) = \sum_{n=0}^{\infty} a_n y^n$ converges.

In the following exercises, suppose that $p(x) = \sum_{n=0}^{\infty} a_n x^n$ satisfies $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ where $a_n \geq 0$ for each n. State whether each series converges on the full interval $(-1, 1)$, or if there is not enough information to draw a conclusion. Use the comparison test when appropriate.

52) $\sum_{n=0}^{\infty} a_n x^{2n}$

53) $\sum_{n=0}^{\infty} a_{2n} x^{2n}$

Answer

Converges on $(-1, 1)$ by the ratio test

54) $\sum_{n=0}^{\infty} a_{2n} x^n$ (Hint:$x = \pm\sqrt{x^2}$)

55) $\sum_{n=0}^{\infty} a_{n^2} x^{n^2}$ (Hint: Let $b_k = a_k$ if $k = n^2$ for some n, otherwise $b_k = 0$.)

Answer

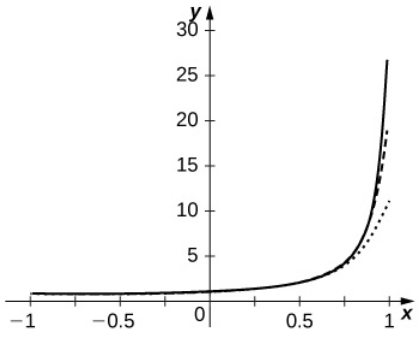
Consider the series $\sum b_k x^k$ where $b_k = a_k$ if $k = n^2$ and $b_k = 0$ otherwise. Then $b_k \le a_k$ and so the series converges on $(-1, 1)$ by the comparison test.

56) Suppose that $p(x)$ is a polynomial of degree N. Find the radius and interval of convergence of $\sum_{n=1}^{\infty} p(n) x^n$.

57) [T] Plot the graphs of $\dfrac{1}{1-x}$ and of the partial sums $S_N = \sum_{n=0}^{N} x^n$ for $n = 10, 20, 30$ on the interval $[-0.99, 0.99]$. Comment on the approximation of $\dfrac{1}{1-x}$ by S_N near $x = -1$ and near $x = 1$ as N increases.

Answer

The approximation is more accurate near $x = -1$. The partial sums follow $\dfrac{1}{1-x}$ more closely as N increases but are never accurate near $x = 1$ since the series diverges there.

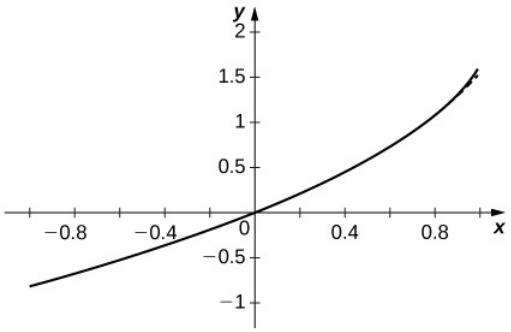


58) [T] Plot the graphs of $-\ln(1-x)$ and of the partial sums $S_N = \sum_{n=1}^{N} \dfrac{x^n}{n}$ for $n = 10, 50, 100$ on the interval $[-0.99, 0.99]$. Comment on the behavior of the sums near $x = -1$ and near $x = 1$ as N increases.

59) [T] Plot the graphs of the partial sums $S_n = \sum_{n=1}^{N} \dfrac{x^n}{n^2}$ for $n = 10, 50, 100$ on the interval $[-0.99, 0.99]$. Comment on the behavior of the sums near $x = -1$ and near $x = 1$ as N increases.

Answer

The approximation appears to stabilize quickly near both $x = \pm 1$.

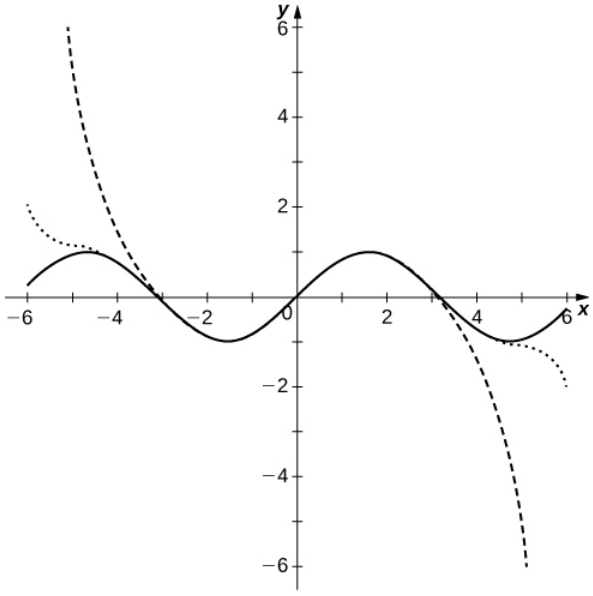


60) [T] Plot the graphs of the partial sums $S_N = \sum_{n=1}^{N} (\sin n) x^n$ for $n = 10, 50, 100$ on the interval $[-0.99, 0.99]$. Comment on the behavior of the sums near $x = -1$ and near $x = 1$ as N increases.

61) [T] Plot the graphs of the partial sums $S_N = \sum_{n=0}^{N} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for $n = 3, 5, 10$ on the interval $[-2\pi, 2\pi]$. Comment on how these plots approximate $\sin x$ as N increases.

Answer

The polynomial curves have roots close to those of $\sin x$ up to their degree and then the polynomials diverge from $\sin x$.



62) [T] Plot the graphs of the partial sums $S_N = \sum_{n=0}^{N} (-1)^n \frac{x^{2n}}{(2n)!}$ for $n = 3, 5, 10$ on the interval $[-2\pi, 2\pi]$. Comment on how these plots approximate $\cos x$ as N increases.

Contributors

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10.2: Properties of Power Series

Learning Objectives

- Combine power series by addition or subtraction.
- Create a new power series by multiplication by a power of the variable or a constant, or by substitution.
- Multiply two power series together.
- Differentiate and integrate power series term-by-term.

In the preceding section on power series and functions we showed how to represent certain functions using power series. In this section we discuss how power series can be combined, differentiated, or integrated to create new power series. This capability is particularly useful for a couple of reasons. First, it allows us to find power series representations for certain elementary functions, by writing those functions in terms of functions with known power series. For example, given the power series representation for $f(x) = \dfrac{1}{1-x}$, we can find a power series representation for $f'(x) = \dfrac{1}{(1-x)^2}$. Second, being able to create power series allows us to define new functions that cannot be written in terms of elementary functions. This capability is particularly useful for solving differential equations for which there is no solution in terms of elementary functions.

Combining Power Series

If we have two power series with the same interval of convergence, we can add or subtract the two series to create a new power series, also with the same interval of convergence. Similarly, we can multiply a power series by a power of x or evaluate a power series at x^m for a positive integer m to create a new power series. Being able to do this allows us to find power series representations for certain functions by using power series representations of other functions. For example, since we know the power series representation for $f(x) = \frac{1}{1-x}$, we can find power series representations for related functions, such as

$$y = \frac{3x}{1-x^2}$$

and

$$y = \frac{1}{(x-1)(x-3)}.$$

In Theorem 10.2.1, we state results regarding addition or subtraction of power series, composition of a power series, and multiplication of a power series by a power of the variable. For simplicity, we state the theorem for power series centered at $x = 0$. Similar results hold for power series centered at $x = a$.

Theorem 10.2.1: Combining Power Series

Suppose that the two power series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge to the functions f and g, respectively, on a common interval I.

i. The power series $\sum_{n=0}^{\infty}(c_n x^n \pm d_n x^n)$ converges to $f \pm g$ on I.

ii. For any integer $m \geq 0$ and any real number b, the power series $\sum_{n=0}^{\infty} bx^m c_n x^n$ converges to $bx^m f(x)$ on I.

iii. For any integer $m \geq 0$ and any real number b, the series $\sum_{n=0}^{\infty} c_n (bx^m)^n$ converges to $f(bx^m)$ for all x such that bx^m is in I.

Proof

We prove $i.$ in the case of the series $\sum_{n=0}^{\infty}(c_n x^n + d_n x^n)$. Suppose that $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge to the functions f and g, respectively, on the interval I. Let x be a point in I and let $S_N(x)$ and $T_N(x)$ denote the Nth partial sums of the series $\sum_{n=0}^{\infty} c_n x^n$

and $\sum_{n=0}^{\infty} d_n x^n$, respectively. Then the sequence $S_N(x)$ converges to $f(x)$ and the sequence $T_N(x)$ converges to $g(x)$. Furthermore, the Nth partial sum of $\sum_{n=0}^{\infty}(c_n x^n + d_n x^n)$ is

$$\begin{aligned}\sum_{n=0}^{N}(c_n x^n + d_n x^n) &= \sum_{n=0}^{N} c_n x^n + \sum_{n=0}^{N} d_n x^n \\ &= S_N(x) + T_N(x).\end{aligned}$$

Because

$$\begin{aligned}\lim_{N\to\infty}(S_N(x) + T_N(x)) &= \lim_{N\to\infty} S_N(x) + \lim_{N\to\infty} T_N(x) \\ &= f(x) + g(x),\end{aligned}$$

we conclude that the series $\sum_{n=0}^{\infty}(c_n x^n + d_n x^n)$ converges to $f(x) + g(x)$.

□

We examine products of power series in a later theorem. First, we show several applications of Theorem 10.2.1 and how to find the interval of convergence of a power series given the interval of convergence of a related power series.

Example 10.2.1: Combining Power Series

Suppose that $\sum_{n=0}^{\infty} a_n x^n$ is a power series whose interval of convergence is $(-1, 1)$, and suppose that $\sum_{n=0}^{\infty} b_n x^n$ is a power series whose interval of convergence is $(-2, 2)$.

a. Find the interval of convergence of the series $\sum_{n=0}^{\infty}(a_n x^n + b_n x^n)$.

b. Find the interval of convergence of the series $\sum_{n=0}^{\infty} a_n 3^n x^n$.

Solution

a. Since the interval $(-1, 1)$ is a common interval of convergence of the series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$, the interval of convergence of the series $\sum_{n=0}^{\infty}(a_n x^n + b_n x^n)$ is $(-1, 1)$.

b. Since $\sum_{n=0}^{\infty} a_n x^n$ is a power series centered at zero with radius of convergence 1, it converges for all x in the interval $(-1, 1)$. By Theorem 10.2.1, the series

$$\sum_{n=0}^{\infty} a_n 3^n x^n = \sum_{n=0}^{\infty} a_n (3x)^n$$

converges if $3x$ is in the interval $(-1, 1)$. Therefore, the series converges for all x in the interval $\left(-\frac{1}{3}, \frac{1}{3}\right)$.

Exercise 10.2.1

Suppose that $\sum_{n=0}^{\infty} a_n x^n$ has an interval of convergence of $(-1, 1)$. Find the interval of convergence of $\sum_{n=0}^{\infty} a_n (\frac{x}{2})^n$.

Hint

Find the values of x such that $\frac{x}{2}$ is in the interval $(-1, 1)$.

Answer

Interval of convergence is $(-2, 2)$.

In the next example, we show how to use Theorem 10.2.1 and the power series for a function f to construct power series for functions related to f. Specifically, we consider functions related to the function $f(x) = \dfrac{1}{1-x}$ and we use the fact that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$$

for $|x| < 1$.

Example 10.2.2: Constructing Power Series from Known Power Series

Use the power series representation for $f(x) = \dfrac{1}{1-x}$ combined with Theorem 10.2.1 to construct a power series for each of the following functions. Find the interval of convergence of the power series.

a. $f(x) = \dfrac{3x}{1+x^2}$

b. $f(x) = \dfrac{1}{(x-1)(x-3)}$

Solution a

First write $f(x)$ as

$$f(x) = 3x\left(\frac{1}{1-(-x^2)}\right).$$

Using the power series representation for $f(x) = \dfrac{1}{1-x}$ and parts ii. and iii. of Theorem 10.2.1, we find that a power series representation for f is given by

$$\sum_{n=0}^{\infty} 3x(-x^2)^n = \sum_{n=0}^{\infty} 3(-1)^n x^{2n+1}.$$

Since the interval of convergence of the series for $\dfrac{1}{1-x}$ is $(-1,1)$, the interval of convergence for this new series is the set of real numbers x such that $|\,x^2\,| < 1$. Therefore, the interval of convergence is $(-1,1)$.

Solution b

To find the power series representation, use partial fractions to write $f(x) = \dfrac{1}{(x-1)(x-3)}$ as the sum of two fractions. We have

$$\frac{1}{(x-1)(x-3)} = \frac{-1/2}{x-1} + \frac{1/2}{x-3} = \frac{1/2}{1-x} - \frac{1/2}{3-x} = \frac{1/2}{1-x} - \frac{1/6}{1-\dfrac{x}{3}}.$$

First, using part ii. of Theorem 10.2.1, we obtain

$$\frac{1/2}{1-x} = \sum_{n=0}^{\infty} \frac{1}{2}x^n \quad \text{for } |x| < 1.$$

Then, using parts ii. and iii. of Theorem 10.2.1, we have

$$\frac{1/6}{1-x/3} = \sum_{n=0}^{\infty} \frac{1}{6}\left(\frac{x}{3}\right)^n \quad \text{for } |x| < 3.$$

Since we are combining these two power series, the interval of convergence of the difference must be the smaller of these two intervals. Using this fact and part i. of Theorem 10.2.1, we have

$$\frac{1}{(x-1)(x-3)} = \sum_{n=0}^{\infty} \left(\frac{1}{2} - \frac{1}{6\cdot 3^n}\right) x^n$$

where the interval of convergence is $(-1,1)$.

? Exercise Runaway argument for \eatSpaces?

Use the series for $f(x) = \dfrac{1}{1-x}$ on $|x| < 1$ to construct a series for $\dfrac{1}{(1-x)(x-2)}$. Determine the interval of convergence.

Hint

Use partial fractions to rewrite $\dfrac{1}{(1-x)(x-2)}$ as the difference of two fractions.

Answer

$\displaystyle\sum_{n=0}^{\infty}\left(-1+\frac{1}{2^{n+1}}\right)x^n$. The interval of convergence is $(-1,1)$.

In Example 10.2.2, we showed how to find power series for certain functions. In Example 10.2.3 we show how to do the opposite: given a power series, determine which function it represents.

✓ Example 10.2.3: Finding the Function Represented by a Given Power Series

Consider the power series $\displaystyle\sum_{n=0}^{\infty}2^n x^n$. Find the function f represented by this series. Determine the interval of convergence of the series.

Solution

Writing the given series as

$$\sum_{n=0}^{\infty}2^n x^n = \sum_{n=0}^{\infty}(2x)^n,$$

we can recognize this series as the power series for

$$f(x) = \frac{1}{1-2x}.$$

Since this is a geometric series, the series converges if and only if $|2x| < 1$. Therefore, the interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

? Exercise 10.2.3

Find the function represented by the power series $\displaystyle\sum_{n=0}^{\infty}\frac{1}{3^n}x^n$.

Determine its interval of convergence.

Hint

Write $\dfrac{1}{3^n}x^n = \left(\dfrac{x}{3}\right)^n$.

Answer

$f(x) = \dfrac{3}{3-x}$. The interval of convergence is $(-3,3)$.

Recall the questions posed in the chapter opener about which is the better way of receiving payouts from lottery winnings. We now revisit those questions and show how to use series to compare values of payments over time with a lump sum payment today. We will compute how much future payments are worth in terms of today's dollars, assuming we have the ability to invest winnings and earn interest. The value of future payments in terms of today's dollars is known as the *present value* of those payments.

Example 10.2.4: Present Value of Future Winnings

Suppose you win the lottery and are given the following three options:

- Receive 20 million dollars today;
- Receive 1.5 million dollars per year over the next 20 years; or
- Receive 1 million dollars per year indefinitely (being passed on to your heirs).

Which is the best deal, assuming that the annual interest rate is 5%? We answer this by working through the following sequence of questions.

a. How much is the 1.5 million dollars received annually over the course of 20 years worth in terms of today's dollars, assuming an annual interest rate of 5%?
b. Use the answer to part a. to find a general formula for the **present value** of payments of C dollars received each year over the next n years, assuming an average annual interest rate r.
c. Find a formula for the present value if annual payments of C dollars continue indefinitely, assuming an average annual interest rate r.
d. Use the answer to part c. to determine the present value of 1 million dollars paid annually indefinitely.
e. Use your answers to parts a. and d. to determine which of the three options is best.

Figure 10.2.1: (credit: modification of work by Robert Huffstutter, Flickr)

Solution a

Consider the payment of 1.5 million dollars made at the end of the first year. If you were able to receive that payment today instead of one year from now, you could invest that money and earn 5% interest. Therefore, the present value of that money P_1 satisfies $P_1(1+0.05)=1.5$ **million dollars.** We conclude that

$$P_1 = \frac{1.5}{1.05} = 1.429 \text{ million dollars.}$$

Similarly, consider the payment of 1.5 million dollars made at the end of the second year. If you were able to receive that payment today, you could invest that money for two years, earning 5% interest, compounded annually. Therefore, the present value of that money P_2 satisfies $P_2(1+0.05)^2=1.5$ **million dollars**. We conclude that

$$P_2 = 1.5(1.05)^2 = 1.361 \text{ million dollars.}$$

The value of the future payments today is the sum of the present values $P_1, P_2, \ldots, P_{20}$ of each of those annual payments. The present value P_k satisfies

$$P_k = \frac{1.5}{(1.05)^k}.$$

Therefore,

$$P = \frac{1.5}{1.05} + \frac{1.5}{(1.05)^2} + \ldots + \frac{1.5}{(1.05)^{20}} = 18.693 \text{ million dollars.}$$

Solution b

Using the result from part a. we see that the present value P of C dollars paid annually over the course of n years, assuming an annual interest rate r, is given by

$$P = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \ldots + \frac{C}{(1+r)^n} \text{ dollars.}$$

Solution c

Using the result from part b. we see that the present value of an annuity that continues indefinitely is given by the infinite series

$$P = \sum_{n=0}^{\infty} \frac{C}{(1+r)^{n+1}}.$$

We can view the present value as a power series in r, which converges as long as $\left|\frac{1}{1+r}\right| < 1$. Since $r > 0$, this series converges. Rewriting the series as

$$P = \frac{C}{(1+r)} \sum_{n=0}^{\infty} \left(\frac{1}{1+r}\right)^n,$$

we recognize this series as the power series for

$$f(r) = \frac{1}{1-\left(\frac{1}{1+r}\right)} = \frac{1}{\left(\frac{r}{1+r}\right)} = \frac{1+r}{r}.$$

We conclude that the present value of this annuity is

$$P = \frac{C}{1+r} \cdot \frac{1+r}{r} = \frac{C}{r}.$$

Solution d

From the result to part c. we conclude that the present value P of $C = 1$ **million dollars** paid out every year indefinitely, assuming an annual interest rate $r = 0.05$, is given by

$$P = \frac{1}{0.05} = 20 \textbf{ million dollars.}$$

Solution e

From part a. we see that receiving \$1.5 million dollars over the course of 20 years is worth \$18.693 million dollars in today's dollars. From part d. we see that receiving \$1 million dollars per year indefinitely is worth \$20 million dollars in today's dollars. Therefore, either receiving a lump-sum payment of \$20 million dollars today or receiving \$1 million dollars indefinitely have the same present value.

Multiplication of Power Series

We can also create new power series by multiplying power series. Being able to multiply two power series provides another way of finding power series representations for functions. The way we multiply them is similar to how we multiply polynomials. For example, suppose we want to multiply

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots$$

and

$$\sum_{n=0}^{\infty} d_n x^n = d_0 + d_1 x + d_2 x^2 + \ldots.$$

It appears that the product should satisfy

$$\begin{aligned}\left(\sum_{n=0}^{\infty} c_n x^n\right)\left(\sum_{n=-0}^{\infty} d_n x^n\right) &= (c_0 + c_1 x + c_2 x^2 + \ldots) \cdot (d_0 + d_1 x + d_2 x^2 + \ldots) \\ &= c_0 d_0 + (c_1 d_0 + c_0 d_1) x + (c_2 d_0 + c_1 d_1 + c_0 d_2) x^2 + \ldots.\end{aligned}$$

In Theorem 10.2.2, we state the main result regarding multiplying power series, showing that if $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge on a common interval I, then we can multiply the series in this way, and the resulting series also converges on the interval I.

Theorem 10.2.2: Multiplying Power Series

Suppose that the power series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge to f and g, respectively, on a common interval I. Let

$$e_n = c_0 d_n + c_1 d_{n-1} + c_2 d_{n-2} + ... + c_{n-1} d_1 + c_n d_0 = \sum_{k=0}^{n} c_k d_{n-k}.$$

Then

$$\left(\sum_{n=0}^{\infty} c_n x^n\right)\left(\sum_{n=0}^{\infty} d_n x^n\right) = \sum_{n=0}^{\infty} e_n x^n$$

and

$$\sum_{n=0}^{\infty} e_n x^n \text{ converges to } f(x) \cdot g(x) \text{ on } I.$$

The series $\sum_{n=0}^{\infty} e_n x^n$ is known as the *Cauchy product* of the series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$.

We omit the proof of this theorem, as it is beyond the level of this text and is typically covered in a more advanced course. We now provide an example of this theorem by finding the power series representation for

$$f(x) = \frac{1}{(1-x)(1-x^2)}$$

using the power series representations for

$$y = \frac{1}{1-x} \text{ and } y = \frac{1}{1-x^2}$$

.

Example 10.2.5: Multiplying Power Series

Multiply the power series representation

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + ...$$

for $|x| < 1$ with the power series representation

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} \left(x^2\right)^n = 1 + x^2 + x^4 + x^6 + ...$$

for $|x| < 1$ to construct a power series for $f(x) = \frac{1}{(1-x)(1-x^2)}$ on the interval $(-1, 1)$.

Solution

We need to multiply

$$(1 + x + x^2 + x^3 + ...)(1 + x^2 + x^4 + x^6 + ...).$$

Writing out the first several terms, we see that the product is given by

$$\begin{aligned} &= (1 + x^2 + x^4 + x^6 + ...) + (x + x^3 + x^5 + x^7 + ...) \\ &\quad + (x^2 + x^4 + x^6 + x^8 + ...) + (x^3 + x^5 + x^7 + x^9 + ...) \\ &= 1 + x + (1+1)x^2 + (1+1)x^3 + (1+1+1)x^4 + (1+1+1)x^5 + ... \\ &= 1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + ... \end{aligned}$$

Since the series for $y = \dfrac{1}{1-x}$ and $y = \dfrac{1}{1-x^2}$ both converge on the interval $(-1, 1)$, the series for the product also converges on the interval $(-1, 1)$.

? Exercise 10.2.4

Multiply the series $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$ by itself to construct a series for $\dfrac{1}{(1-x)(1-x)}$.

Hint

Multiply the first few terms of $(1 + x + x^2 + x^3 + ...)(1 + x + x^2 + x^3 + ...)$

Answer

$1 + 2x + 3x^2 + 4x^3 + ...$

Differentiating and Integrating Power Series

Consider a power series

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + ...$$

that converges on some interval I, and let f be the function defined by this series. Here we address two questions about f.

- Is f differentiable, and if so, how do we determine the derivative f'?
- How do we evaluate the indefinite integral $\displaystyle\int f(x)\,dx$?

We know that, for a polynomial with a finite number of terms, we can evaluate the derivative by differentiating each term separately. Similarly, we can evaluate the indefinite integral by integrating each term separately. Here we show that we can do the same thing for convergent power series. That is, if

$$f(x) = c_n x^n = c_0 + c_1 x + c_2 x^2 + ...$$

converges on some interval I, then

$$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + ...$$

and

$$\int f(x)\,dx = C + c_0 x + c_1 \frac{x^2}{2} + c_2 \frac{x^3}{3} + ...$$

Evaluating the derivative and indefinite integral in this way is called **term-by-term differentiation of a power series** and **term-by-term integration of a power series**, respectively. The ability to differentiate and integrate power series term-by-term also allows us to use known power series representations to find power series representations for other functions. For example, given the power series for $f(x) = \dfrac{1}{1-x}$, we can differentiate term-by-term to find the power series for $f'(x) = \dfrac{1}{(1-x)^2}$. Similarly, using the power series for $g(x) = \dfrac{1}{1+x}$, we can integrate term-by-term to find the power series for $G(x) = \ln(1 + x)$, an antiderivative of g. We show how to do this in Example 10.2.6 and Example 10.2.7. First, we state Theorem 10.2.3, which provides the main result regarding differentiation and integration of power series.

Theorem 10.2.3: Term-by-Term Differentiation and Integration for Power Series

Suppose that the power series $\displaystyle\sum_{n=0}^{\infty} c_n (x-a)^n$ converges on the interval $(a - R, a + R)$ for some $R > 0$. Let f be the function defined by the series

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + c_3 (x-a)^3 + ...$$

for $|x-a|<R$. Then f is differentiable on the interval $(a-R, a+R)$ and we can find f' by differentiating the series term-by-term:

$$f'(x)=\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}=c_1+2c_2(x-a)+3c_3(x-a)^2+...$$

for $|x-a|<R$. Also, to find $\int f(x)\,dx$, we can integrate the series term-by-term. The resulting series converges on $(a-R, a+R)$, and we have

$$\int f(x)\,dx=C+\sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}=C+c_0(x-a)+c_1\frac{(x-a)^2}{2}+c_2\frac{(x-a)^3}{3}+...$$

for $|x-a|<R$.

The proof of this result is beyond the scope of the text and is omitted. Note that although Theorem 10.2.3 guarantees the same radius of convergence when a power series is differentiated or integrated term-by-term, it says nothing about what happens at the endpoints. It is possible that the differentiated and integrated power series have different behavior at the endpoints than does the original series. We see this behavior in the next examples.

Example 10.2.6: Differentiating Power Series

a. Use the power series representation

$$f(x)=\frac{1}{1-x}=\sum_{n=0}^{\infty} x^n=1+x+x^2+x^3+...$$

for $|x|<1$ to find a power series representation for

$$g(x)=\frac{1}{(1-x)^2}$$

on the interval $(-1, 1)$. Determine whether the resulting series converges at the endpoints.

b. Use the result of part a. to evaluate the sum of the series $\sum_{n=0}^{\infty}\frac{n+1}{4^n}$.

Solution a

Since $g(x)=\frac{1}{(1-x)^2}$ is the derivative of $f(x)=\frac{1}{1-x}$, we can find a power series representation for g by differentiating the power series for f term-by-term. The result is

$$\begin{aligned} g(x)=\frac{1}{(1-x)^2}&=\frac{d}{dx}\left(\frac{1}{1-x}\right)\\ &=\sum_{n=0}^{\infty}\frac{d}{dx}(x^n)=\frac{d}{dx}(1+x+x^2+x^3+...)=0+1+2x+3x^2+4x^3+...\\ &=\sum_{n=0}^{\infty}(n+1)x^n \end{aligned}$$

for $|x|<1$.

Theorem 10.2.1 does not guarantee anything about the behavior of this series at the endpoints. Testing the endpoints by using the divergence test, we find that the series diverges at both endpoints $x=\pm 1$. Note that this is the same result found in Example 10.2.5.

Solution b

From part a. we know that

$$\sum_{n=0}^{\infty}(n+1)x^n=\frac{1}{(1-x)^2}.$$

Therefore,

$$\sum_{n=0}^{\infty}\frac{n+1}{4^n}=\sum_{n=0}^{\infty}(n+1)\left(\frac{1}{4}\right)^n$$
$$=\frac{1}{\left(1-\frac{1}{4}\right)^2}$$
$$=\frac{1}{\left(\frac{3}{4}\right)^2}$$
$$=\frac{16}{9}$$

Exercise 10.2.5

Differentiate the series $\frac{1}{(1-x)^2}=\sum_{n=0}^{\infty}(n+1)x^n$ term-by-term to find a power series representation for $\frac{2}{(1-x)^3}$ on the interval $(-1,1)$.

Hint

Write out the first several terms and apply the power rule.

Answer

$$\sum_{n=0}^{\infty}(n+2)(n+1)x^n$$

Example 10.2.7: Integrating Power Series

For each of the following functions f, find a power series representation for f by integrating the power series for f' and find its interval of convergence.

a. $f(x)=\ln(1+x)$
b. $f(x)=\tan^{-1}x$

Solution a

For $f(x)=\ln(1+x)$,the derivative is $f'(x)=\frac{1}{1+x}$. We know that

$$\frac{1}{1+x}=\frac{1}{1-(-x)}=\sum_{n=0}^{\infty}(-x)^n=1-x+x^2-x^3+\ldots$$

for $|x|<1$. To find a power series for $f(x)=\ln(1+x)$, we integrate the series term-by-term.

$$\int f'(x)\,dx=\int(1-x+x^2-x^3+\ldots)\,dx=C+x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$$

Since $f(x)=\ln(1+x)$ is an antiderivative of $\frac{1}{1+x}$, it remains to solve for the constant C. Since $\ln(1+0)=0$, we have $C=0$. Therefore, a power series representation for $f(x)=\ln(1+x)$ is

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}\text{ for }|x|<1.$$

Theorem 10.2.3 does not guarantee anything about the behavior of this power series at the endpoints. However, checking the endpoints, we find that at $x=1$ the series is the alternating harmonic series, which converges. Also, at $x=-1$, the series is the harmonic series, which diverges. It is important to note that, even though this series converges at $x=1$, Theorem 10.2.3 does not guarantee that the series actually converges to $\ln(2)$. In fact, the series does converge to $\ln(2)$, but showing this fact requires more advanced techniques. (Abel's theorem, covered in more advanced texts, deals with this more technical point.) The interval of convergence is $(-1,1]$.

Solution b

The derivative of $f(x) = \tan^{-1} x$ is $f'(x) = \dfrac{1}{1+x^2}$. We know that

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = 1 - x^2 + x^4 - x^6 + ...$$

for $|x| < 1$. To find a power series for $f(x) = \tan^{-1} x$, we integrate this series term-by-term.

$$\int f'(x)\,dx = \int (1 - x^2 + x^4 - x^6 + ...)\,dx = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$$

Since $\tan^{-1}(0) = 0$, we have $C = 0$. Therefore, a power series representation for $f(x) = \tan^{-1} x$ is

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

for $|x| < 1$. Again, Theorem 10.2.3 does not guarantee anything about the convergence of this series at the endpoints. However, checking the endpoints and using the alternating series test, we find that the series converges at $x = 1$ and $x = -1$. As discussed in part a., using Abel's theorem, it can be shown that the series actually converges to $\tan^{-1}(1)$ and $\tan^{-1}(-1)$ at $x = 1$ and $x = -1$, respectively. Thus, the interval of convergence is $[-1, 1]$.

? Exercise 10.2.6

Integrate the power series $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{x^n}{n}$ term-by-term to evaluate $\int \ln(1+x)\,dx$.

Hint

Use the fact that $\dfrac{x^{n+1}}{(n+1)n}$ is an antiderivative of $\dfrac{x^n}{n}$.

Answer

$$\sum_{n=2}^{\infty} \frac{(-1)^n x^n}{n(n-1)}$$

Up to this point, we have shown several techniques for finding power series representations for functions. However, how do we know that these power series are unique? That is, given a function f and a power series for f at a, is it possible that there is a different power series for f at a that we could have found if we had used a different technique? The answer to this question is no. This fact should not seem surprising if we think of power series as polynomials with an infinite number of terms. Intuitively, if

$$c_0 + c_1 x + c_2 x^2 + ... = d_0 + d_1 x + d_2 x^2 + ...$$

for all values x in some open interval I about zero, then the coefficients c_n should equal d_n for $n \geq 0$. We now state this result formally.

Theorem 10.2.4: Uniqueness of Power Series

Let $\sum_{n=0}^{\infty} c_n (x-a)^n$ and $\sum_{n=0}^{\infty} d_n (x-a)^n$ be two convergent power series such that

$$\sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=0}^{\infty} d_n (x-a)^n$$

for all x in an open interval containing a. Then $c_n = d_n$ for all $n \geq 0$.

Proof

Let

$$\begin{aligned} f(x) &= c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots \\ &= d_0 + d_1(x-a) + d_2(x-a)^2 + d_3(x-a)^3 + \ldots . \end{aligned}$$

Then $f(a) = c_0 = d_0$. By Theorem 10.2.3, we can differentiate both series term-by-term. Therefore,

$$\begin{aligned} f'(x) &= c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \ldots \\ &= d_1 + 2d_2(x-a) + 3d_3(x-a)^2 + \ldots , \end{aligned}$$

and thus, $f'(a) = c_1 = d_1$. Similarly,

$$\begin{aligned} f''(x) &= 2c_2 + 3 \cdot 2c_3(x-a) + \ldots \\ &= 2d_2 + 3 \cdot 2d_3(x-a) + \ldots \end{aligned}$$

implies that $f''(a) = 2c_2 = 2d_2$, and therefore, $c_2 = d_2$. More generally, for any integer $n \geq 0, f^{(n)}(a) = n!c_n = n!d_n$, and consequently, $c_n = d_n$ for all $n \geq 0$.

□

In this section we have shown how to find power series representations for certain functions using various algebraic operations, differentiation, or integration. At this point, however, we are still limited as to the functions for which we can find power series representations. Next, we show how to find power series representations for many more functions by introducing Taylor series.

Key Concepts

- Given two power series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ that converge to functions f and g on a common interval I, the sum and difference of the two series converge to $f \pm g$, respectively, on I. In addition, for any real number b and integer $m \geq 0$, the series $\sum_{n=0}^{\infty} bx^m c_n x^n$ converges to $bx^m f(x)$ and the series $\sum_{n=0}^{\infty} c_n (bx^m)^n$ converges to $f(bx^m)$ whenever bx^m is in the interval I.
- Given two power series that converge on an interval $(-R, R)$, the Cauchy product of the two power series converges on the interval $(-R, R)$.
- Given a power series that converges to a function f on an interval $(-R, R)$, the series can be differentiated term-by-term and the resulting series converges to f' on $(-R, R)$. The series can also be integrated term-by-term and the resulting series converges to $\int f(x)\, dx$ on $(-R, R)$.

Glossary

term-by-term differentiation of a power series

a technique for evaluating the derivative of a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ by evaluating the derivative of each term separately to create the new power series $\sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$

term-by-term integration of a power series

a technique for integrating a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ by integrating each term separately to create the new power series $C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$

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10.2E: Exercises for Section 10.2

1) If $f(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ and $g(x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!}$, find the power series of $\frac{1}{2}\big(f(x)+g(x)\big)$ and of $\frac{1}{2}\big(f(x)-g(x)\big)$.

Answer

$$\frac{1}{2}\big(f(x)+g(x)\big)=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!} \text{ and } \frac{1}{2}\big(f(x)-g(x)\big)=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}.$$

2) If $C(x)=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$ and $S(x)=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$, find the power series of $C(x)+S(x)$ and of $C(x)-S(x)$.

In exercises 3 - 6, use partial fractions to find the power series of each function.

3) $\frac{4}{(x-3)(x+1)}$

Answer

$$\frac{4}{(x-3)(x+1)}=\frac{1}{x-3}-\frac{1}{x+1}=-\frac{1}{3(1-\frac{x}{3})}-\frac{1}{1-(-x)}=-\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n-\sum_{n=0}^{\infty}(-1)^nx^n=\sum_{n=0}^{\infty}\left((-1)^{n+1}-\frac{1}{3^{n+1}}\right)x^n$$

4) $\frac{3}{(x+2)(x-1)}$

5) $\frac{5}{(x^2+4)(x^2-1)}$

Answer

$$\frac{5}{(x^2+4)(x^2-1)}=\frac{1}{x^2-1}-\frac{1}{4}\frac{1}{1+(\frac{x}{2})^2}=-\sum_{n=0}^{\infty}x^{2n}-\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{2}\right)^n=\sum_{n=0}^{\infty}\left((-1)+(-1)^{n+1}\frac{1}{2^{n+2}}\right)x^{2n}$$

6) $\frac{30}{(x^2+1)(x^2-9)}$

In exercises 7 - 10, express each series as a rational function.

7) $\sum_{n=1}^{\infty}\frac{1}{x^n}$

Answer

$$\frac{1}{x}\sum_{n=0}^{\infty}\frac{1}{x^n}=\frac{1}{x}\cdot\frac{1}{1-\frac{1}{x}}=\frac{1}{x-1}$$

8) $\sum_{n=1}^{\infty}\frac{1}{x^{2n}}$

9) $\sum_{n=1}^{\infty}\frac{1}{(x-3)^{2n-1}}$

Answer

$$\frac{1}{x-3}\cdot\frac{1}{1-\frac{1}{(x-3)^2}}=\frac{x-3}{(x-3)^2-1}$$

10) $\sum_{n=1}^{\infty}\left(\frac{1}{(x-3)^{2n-1}}-\frac{1}{(x-2)^{2n-1}}\right)$

Exercises 11 - 16 explore applications of *annuities*.

11) Calculate the present values P of an annuity in which \$10,000 is to be paid out annually for a period of 20 years, assuming interest rates of $r=0.03$, $r=0.05$, and $r=0.07$.

Answer

$P = P_1 + \cdots + P_{20}$ where $P_k = 10,000\dfrac{1}{(1+r)^k}$. Then $P = 10,000\displaystyle\sum_{k=1}^{20}\frac{1}{(1+r)^k} = 10,000\frac{1-(1+r)^{-20}}{r}$. When $r = 0.03,\ P \approx 10,000 \times 14.8775 = 148,775$. When $r = 0.05,\ P \approx 10,000 \times 12.4622 = 124,622$. When $r = 0.07,\ P \approx 105,940$.

12) Calculate the present values P of annuities in which \$9,000 is to be paid out annually perpetually, assuming interest rates of $r = 0.03$, $r = 0.05$ and $r = 0.07$.

13) Calculate the annual payouts C to be given for 20 years on annuities having present value \$100,000 assuming respective interest rates of $r = 0.03$, $r = 0.05$, and $r = 0.07$.

Answer

In general, $P = \dfrac{C(1-(1+r)^{-N})}{r}$ for N years of payouts, or $C = \dfrac{Pr}{1-(1+r)^{-N}}$. For $N = 20$ and $P = 100,000$, one has $C = 6721.57$ when $r = 0.03$; $C = 8024.26$ when $r = 0.05$; and $C \approx 9439.29$ when $r = 0.07$.

14) Calculate the annual payouts C to be given perpetually on annuities having present value \$100,000 assuming respective interest rates of $r = 0.03$, $r = 0.05$, and $r = 0.07$.

15) Suppose that an annuity has a present value $P = 1$ **million dollars.** What interest rate r would allow for perpetual annual payouts of \$50,000?

Answer

In general, $P = \dfrac{C}{r}$. Thus, $r = \dfrac{C}{P} = 5 \times \frac{10^4}{10^6} = 0.05$.

16) Suppose that an annuity has a present value $P = 10$ **million dollars.** What interest rate r would allow for perpetual annual payouts of \$100,000?

In exercises 17 - 20, express the sum of each power series in terms of geometric series, and then express the sum as a rational function.

17) $x + x^2 - x^3 + x^4 + x^5 - x^6 + \cdots$ (Hint: Group powers x^{3k}, x^{3k-1}, and x^{3k-2}.)

Answer

$$(x + x^2 - x^3)(1 + x^3 + x^6 + \cdots) = \frac{x + x^2 - x^3}{1 - x^3}$$

18) $x + x^2 - x^3 - x^4 + x^5 + x^6 - x^7 - x^8 + \cdots$ (Hint: Group powers x^{4k}, x^{4k-1}, etc.)

19) $x - x^2 - x^3 + x^4 - x^5 - x^6 + x^7 - \cdots$ (Hint: Group powers x^{3k}, x^{3k-1}, and x^{3k-2}.)

Answer

$$(x - x^2 - x^3)(1 + x^3 + x^6 + \cdots) = \frac{x - x^2 - x^3}{1 - x^3}$$

20) $\dfrac{x}{2} + \dfrac{x^2}{4} - \dfrac{x^3}{8} + \dfrac{x^4}{16} + \dfrac{x^5}{32} - \dfrac{x^6}{64} + \cdots$ (Hint: Group powers $\left(\dfrac{x}{2}\right)^{3k}$, $\left(\dfrac{x}{2}\right)^{3k-1}$, and $\left(\dfrac{x}{2}\right)^{3k-2}$.)

In exercises 21 - 24, find the power series of $f(x)g(x)$ given f and g as defined.

21) $f(x) = 2\displaystyle\sum_{n=0}^{\infty} x^n, g(x) = \sum_{n=0}^{\infty} nx^n$

Answer

$a_n = 2$, $b_n = n$ so $c_n = \displaystyle\sum_{k=0}^{n} b_k a_{n-k} = 2\sum_{k=0}^{n} k = (n)(n+1)$ and $f(x)g(x) = \displaystyle\sum_{n=1}^{\infty} n(n+1)x^n$

22) $f(x) = \displaystyle\sum_{n=1}^{\infty} x^n,\ g(x) = \sum_{n=1}^{\infty} \frac{1}{n}x^n$. Express the coefficients of $f(x)g(x)$ in terms of $H_n = \displaystyle\sum_{k=1}^{n}\frac{1}{k}$.

23) $f(x) = g(x) = \displaystyle\sum_{n=1}^{\infty}\left(\frac{x}{2}\right)^n$

Answer

$a_n = b_n = 2^{-n}$ so $c_n = \sum_{k=1}^{n} b_k a_{n-k} = 2^{-n} \sum_{k=1}^{n} 1 = \frac{n}{2^n}$ and $f(x)g(x) = \sum_{n=1}^{\infty} n \left(\frac{x}{2}\right)^n$

24) $f(x) = g(x) = \sum_{n=1}^{\infty} nx^n$

In exercises 25 - 26, differentiate the given series expansion of f term-by-term to obtain the corresponding series expansion for the derivative of f.

25) $f(x) = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$

Answer

The derivative of f is $-\frac{1}{(1+x)^2} = -\sum_{n=0}^{\infty} (-1)^n (n+1) x^n$.

26) $f(x) = \frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n}$

In exercises 27 - 28, integrate the given series expansion of f term-by-term from zero to x to obtain the corresponding series expansion for the indefinite integral of f.

27) $f(x) = \frac{2x}{(1+x^2)^2} = \sum_{n=1}^{\infty} (-1)^n (2n) x^{2n-1}$

Answer

The indefinite integral of f is $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$.

28) $f(x) = \frac{2x}{1+x^2} = 2\sum_{n=0}^{\infty} (-1)^n x^{2n+1}$

In exercises 29 - 32, evaluate each infinite series by identifying it as the value of a derivative or integral of geometric series.

29) Evaluate $\sum_{n=1}^{\infty} \frac{n}{2^n}$ as $f'(\frac{1}{2})$ where $f(x) = \sum_{n=0}^{\infty} x^n$.

Answer

$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$; $f'(\frac{1}{2}) = \sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = \frac{d}{dx}(1-x)^{-1}\Big|_{x=1/2} = \frac{1}{(1-x)^2}\Big|_{x=1/2} = 4$ so $\sum_{n=1}^{\infty} \frac{n}{2^n} = 2$.

30) Evaluate $\sum_{n=1}^{\infty} \frac{n}{3^n}$ as $f'(\frac{1}{3})$ where $f(x) = \sum_{n=0}^{\infty} x^n$.

31) Evaluate $\sum_{n=2}^{\infty} \frac{n(n-1)}{2^n}$ as $f''(\frac{1}{2})$ where $f(x) = \sum_{n=0}^{\infty} x^n$.

Answer

$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$; $f''\left(\frac{1}{2}\right) = \sum_{n=2}^{\infty} \frac{n(n-1)}{2^{n-2}} = \frac{d^2}{dx^2}(1-x)^{-1}\Big|_{x=1/2} = \frac{2}{(1-x)^3}\Big|_{x=1/2} = 16$ so $\sum_{n=2}^{\infty} n\frac{(n-1)}{2^n} = 4$.

32) Evaluate $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ as $\int_0^1 f(t)\,dt$ where $f(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2}$.

In exercises 33 - 39, given that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, use term-by-term differentiation or integration to find power series for each function centered at the given point.

33) $f(x) = \ln x$ centered at $x = 1$ (Hint: $x = 1 - (1-x)$)

Answer

$$\int \sum (1-x)^n\,dx = \int \sum (-1)^n (x-1)^n\,dx = \sum \frac{(-1)^n (x-1)^{n+1}}{n+1}$$

34) $\ln(1-x)$ at $x=0$

35) $\ln(1-x^2)$ at $x=0$

Answer

$$-\int_{t=0}^{x^2} \frac{1}{1-t}dt = -\sum_{n=0}^{\infty} \int_0^{x^2} t^n dx - \sum_{n=0}^{\infty} \frac{x^{2(n+1)}}{n+1} = -\sum_{n=1}^{\infty} \frac{x^{2n}}{n}$$

36) $f(x) = \dfrac{2x}{(1-x^2)^2}$ at $x=0$

37) $f(x) = \tan^{-1}(x^2)$ at $x=0$

Answer

$$\int_0^{x^2} \frac{dt}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n \int_0^{x^2} t^{2n} dt = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{2n+1} \Big|_{t=0}^{x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}$$

38) $f(x) = \ln(1+x^2)$ at $x=0$

39) $f(x) = \displaystyle\int_0^x \ln t\,dt$ where $\ln(x) = \displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-1)^n}{n}$

Answer

Term-by-term integration gives $\displaystyle\int_0^x \ln t\,dt = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-1)^{n+1}}{n(n+1)} = \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{1}{n} - \frac{1}{n+1}\right)(x-1)^{n+1} = (x-1)\ln x + \displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{(x-1)^n}{n} = x\ln x - x.$

40) [T] Evaluate the power series expansion $\ln(1+x) = \displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ at $x=1$ to show that $\ln(2)$ is the sum of the alternating harmonic series. Use the alternating series test to determine how many terms of the sum are needed to estimate $\ln(2)$ accurate to within 0.001, and find such an approximation.

41) [T] Subtract the infinite series of $\ln(1-x)$ from $\ln(1+x)$ to get a power series for $\ln\left(\dfrac{1+x}{1-x}\right)$. Evaluate at $x=\frac{1}{3}$. What is the smallest N such that the Nth partial sum of this series approximates $\ln(2)$ with an error less than 0.0001?

Answer

We have $\ln(1-x) = -\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$ so $\ln(1+x) = \displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$. Thus, $\ln\left(\dfrac{1+x}{1-x}\right) = \displaystyle\sum_{n=1}^{\infty} \left(1+(-1)^{n-1}\right)\frac{x^n}{n} = 2\sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1}$.

When $x=\frac{1}{3}$ we obtain $\ln(2) = 2\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^{2n-1}(2n-1)}$. We have $2\displaystyle\sum_{n=1}^{3} \frac{1}{3^{2n-1}(2n-1)} = 0.69300\ldots$, while $2\displaystyle\sum_{n=1}^{4} \frac{1}{3^{2n-1}(2n-1)} = 0.69313\ldots$ and $\ln(2) = 0.69314\ldots$; therefore, $N=4$.

In exercises 42 - 45, using a substitution if indicated, express each series in terms of elementary functions and find the radius of convergence of the sum.

42) $\displaystyle\sum_{k=0}^{\infty} (x^k - x^{2k+1})$

43) $\displaystyle\sum_{k=1}^{\infty} \frac{x^{3k}}{6k}$

Answer

$\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k} = -\ln(1-x)$ so $\displaystyle\sum_{k=1}^{\infty} \frac{x^{3k}}{6k} = -\frac{1}{6}\ln(1-x^3)$. The radius of convergence is equal to 1 by the ratio test.

44) $\sum_{k=1}^{\infty}(1+x^2)^{-k}$ using $y=\dfrac{1}{1+x^2}$

45) $\sum_{k=1}^{\infty}2^{-kx}$ using $y=2^{-x}$

Answer

If $y=2^{-x}$, then $\sum_{k=1}^{\infty}y^k=\dfrac{y}{1-y}=\dfrac{2^{-x}}{1-2^{-x}}=\dfrac{1}{2^x-1}$. If $a_k=2^{-kx}$, then $\dfrac{a_{k+1}}{a_k}=2^{-x}<1$ when $x>0$. So the series converges for all $x>0$.

46) Show that, up to powers x^3 and y^3, $E(x)=\sum_{n=0}^{\infty}\dfrac{x^n}{n!}$ satisfies $E(x+y)=E(x)E(y)$.

47) Differentiate the series $E(x)=\sum_{n=0}^{\infty}\dfrac{x^n}{n!}$ term-by-term to show that $E(x)$ is equal to its derivative.

Answer

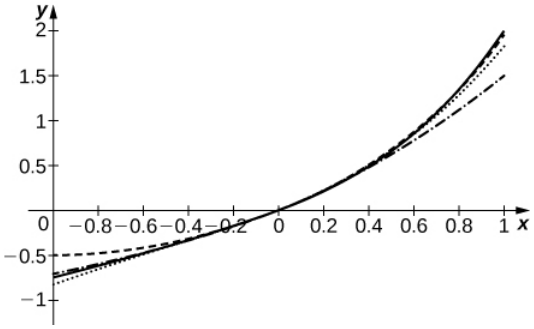
Answers will vary.

48) Show that if $f(x)=\sum_{n=0}^{\infty}a_nx^n$ is a sum of even powers, that is, $a_n=0$ if n is odd, then $F=\int_0^x f(t)\,dt$ is a sum of odd powers, while if I is a sum of odd powers, then F is a sum of even powers.

49) [T] Suppose that the coefficients an of the series $\sum_{n=0}^{\infty}a_nx^n$ are defined by the recurrence relation $a_n=\dfrac{a_{n-1}}{n}+\dfrac{a_{n-2}}{n(n-1)}$. For $a_0=0$ and $a_1=1$, compute and plot the sums $S_N=\sum_{n=0}^{N}a_nx^n$ for $N=2,3,4,5$ on $[-1,1]$.

Answer

The solid curve is S_5. The dashed curve is S_2, dotted is S_3, and dash-dotted is S_4



50) [T] Suppose that the coefficients an of the series $\sum_{n=0}^{\infty}a_nx^n$ are defined by the recurrence relation $a_n=\dfrac{a_{n-1}}{\sqrt{n}}-\dfrac{a_{n-2}}{\sqrt{n(n-1)}}$. For $a_0=1$ and $a_1=0$, compute and plot the sums $S_N=\sum_{n=0}^{N}a_nx^n$ for $N=2,3,4,5$ on $[-1,1]$.

51) [T] Given the power series expansion $\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{x^n}{n}$, determine how many terms N of the sum evaluated at $x=-1/2$ are needed to approximate $\ln(2)$ accurate to within $1/1000$. Evaluate the corresponding partial sum $\sum_{n=1}^{N}(-1)^{n-1}\dfrac{x^n}{n}$.

Answer

When $x=-\dfrac{1}{2}$, $-\ln(2)=\ln\left(\dfrac{1}{2}\right)=-\sum_{n=1}^{\infty}\dfrac{1}{n2^n}$. Since $\sum_{n=11}^{\infty}\dfrac{1}{n2^n}<\sum_{n=11}^{\infty}\dfrac{1}{2^n}=\dfrac{1}{2^{10}}$, one has $\sum_{n=1}^{10}\dfrac{1}{n2^n}=0.69306\ldots$ whereas $\ln(2)=0.69314\ldots$; therefore, $N=10$.

52) [T] Given the power series expansion $\tan^{-1}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$, use the alternating series test to determine how many terms N of the sum evaluated at $x = 1$ are needed to approximate $\tan^{-1}(1) = \frac{\pi}{4}$ accurate to within $1/1000$. Evaluate the corresponding partial sum $\sum_{k=0}^{N} (-1)^k \frac{x^{2k+1}}{2k+1}$.

53) [T] Recall that $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$. Assuming an exact value of $\frac{1}{\sqrt{3}}$), estimate $\frac{\pi}{6}$ by evaluating partial sums $S_N\left(\frac{1}{\sqrt{3}}\right)$ of the power series expansion $\tan^{-1}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$ at $x = \frac{1}{\sqrt{3}}$. What is the smallest number N such that $6S_N\left(\frac{1}{\sqrt{3}}\right)$ approximates π accurately to within 0.001? How many terms are needed for accuracy to within 0.00001?

Answer

$6S_N\left(\frac{1}{\sqrt{3}}\right) = 2\sqrt{3}\sum_{n=0}^{N} (-1)^n \frac{1}{3^n(2n+1).}$ One has $\pi - 6S_4\left(\frac{1}{\sqrt{3}}\right) = 0.00101\ldots$ and $\pi - 6S_5\left(\frac{1}{\sqrt{3}}\right) = 0.00028\ldots$ so $N = 5$ is the smallest partial sum with accuracy to within 0.001. Also, $\pi - 6S_7\left(\frac{1}{\sqrt{3}}\right) = 0.00002\ldots$ while $\pi - 6S_8\left(\frac{1}{\sqrt{3}}\right) = -0.000007\ldots$ so $N = 8$ is the smallest N to give accuracy to within 0.00001.

Contributors

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10.3: Taylor and Maclaurin Series

Learning Objectives

- Describe the procedure for finding a Taylor polynomial of a given order for a function.
- Explain the meaning and significance of Taylor's theorem with remainder.
- Estimate the remainder for a Taylor series approximation of a given function.

In the previous two sections we discussed how to find power series representations for certain types of functions—specifically, functions related to geometric series. Here we discuss power series representations for other types of functions. In particular, we address the following questions: Which functions can be represented by power series and how do we find such representations? If we can find a power series representation for a particular function f and the series converges on some interval, how do we prove that the series actually converges to f?

Overview of Taylor/Maclaurin Series

Consider a function f that has a power series representation at $x = a$. Then the series has the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots . \tag{10.3.1}$$

What should the coefficients be? For now, we ignore issues of convergence, but instead focus on what the series should be, if one exists. We return to discuss convergence later in this section. If the series Equation 10.3.1 is a representation for f at $x = a$, we certainly want the series to equal $f(a)$ at $x = a$. Evaluating the series at $x = a$, we see that

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(a-a) + c_2(a-a)^2 + \cdots = c_0. \tag{10.3.2}$$

Thus, the series equals $f(a)$ if the coefficient $c_0 = f(a)$. In addition, we would like the first derivative of the power series to equal $f'(a)$ at $x = a$. Differentiating Equation 10.3.2 term-by-term, we see that

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots . \tag{10.3.3}$$

Therefore, at $x = a$, the derivative is

$$f'(a) = c_1 + 2c_2(a-a) + 3c_3(a-a)^2 + \cdots = c_1. \tag{10.3.4}$$

Therefore, the derivative of the series equals $f'(a)$ if the coefficient $c_1 = f'(a)$. Continuing in this way, we look for coefficients c_n such that all the derivatives of the power series Equation 10.3.4 will agree with all the corresponding derivatives of f at $x = a$. The second and third derivatives of Equation 10.3.3 are given by

$$\frac{d^2}{dx^2}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = 2c_2 + 3\cdot 2c_3(x-a) + 4\cdot 3c_4(x-a)^2 + \dots \tag{10.3.5}$$

and

$$\frac{d^3}{dx^3}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4(x-a) + 5\cdot 4\cdot 3c_5(x-a)^2 + \cdots . \tag{10.3.6}$$

Therefore, at $x = a$, the second and third derivatives

$$f''(a) = 2c_2 + 3\cdot 2c_3(a-a) + 4\cdot 3c_4(a-a)^2 + \cdots = 2c_2 \tag{10.3.7}$$

and

$$f'''(a) \ = 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4(a-a) + 5\cdot 4\cdot 3c_5(a-a)^2 + \cdots = 3\cdot 2c_3 \tag{10.3.8}$$

equal $f''(a)$ and $f'''(a)$, respectively, if $c_2 = \dfrac{f''(a)}{2}$ and $c_3 = \dfrac{f'''(a)}{3 \cdot 2}$. More generally, we see that if f has a power series representation at $x = a$, then the coefficients should be given by $c_n = \dfrac{f^{(n)}(a)}{n!}$. That is, the series should be

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

This power series for f is known as the Taylor series for f at a. If $a = 0$, then this series is known as the Maclaurin series for f.

Definition 10.3.1: Maclaurin and Taylor series

If f has derivatives of all orders at $x = a$, then the *Taylor series* for the function f at a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

The Taylor series for f at 0 is known as the *Maclaurin series* for f.

Later in this section, we will show examples of finding Taylor series and discuss conditions under which the Taylor series for a function will converge to that function. Here, we state an important result. Recall that power series representations are unique. Therefore, if a function f has a power series at a, then it must be the Taylor series for f at a.

Theorem 10.3.1: Uniqueness of Taylor Series

If a function f has a power series at a that converges to f on some open interval containing a, then that power series is the Taylor series for f at a.

The proof follows directly from that discussed previously.

To determine if a Taylor series converges, we need to look at its sequence of partial sums. These partial sums are finite polynomials, known as **Taylor polynomials.**

Taylor Polynomials

The nth partial sum of the Taylor series for a function f at a is known as the nth-degree Taylor polynomial. For example, the 0th, 1st, 2nd, and 3rd partial sums of the Taylor series are given by

$$\begin{aligned}
p_0(x) &= f(a) \\
p_1(x) &= f(a) + f'(a)(x-a) \\
p_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\
p_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3
\end{aligned}$$

respectively. These partial sums are known as the 0th, 1st, 2nd, and 3rd degree Taylor polynomials of f at a, respectively. If $a = 0$, then these polynomials are known as *Maclaurin polynomials* for f. We now provide a formal definition of Taylor and Maclaurin polynomials for a function f.

Definition 10.3.2: Maclaurin polynomial

If f has n derivatives at $x = a$, then the nth-degree Taylor polynomial of f at a is

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The nth-degree Taylor polynomial for f at 0 is known as the nth-degree Maclaurin polynomial for f.

We now show how to use this definition to find several Taylor polynomials for $f(x) = \ln x$ at $x = 1$.

Example 10.3.1: Finding Taylor Polynomials

Find the Taylor polynomials p_0, p_1, p_2 and p_3 for $f(x) = \ln x$ at $x = 1$. Use a graphing utility to compare the graph of f with the graphs of p_0, p_1, p_2 and p_3.

Solution

To find these Taylor polynomials, we need to evaluate f and its first three derivatives at $x = 1$.

$$\begin{aligned} f(x) &= \ln x & f(1) &= 0 \\ f'(x) &= \frac{1}{x} & f'(1) &= 1 \\ f''(x) &= -\frac{1}{x^2} & f''(1) &= -1 \\ f'''(x) &= \frac{2}{x^3} & f'''(1) &= 2 \end{aligned}$$

Therefore,

$$\begin{aligned} p_0(x) &= f(1) = 0, \\ p_1(x) &= f(1) + f'(1)(x-1) = x - 1, \\ p_2(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 = (x-1) - \frac{1}{2}(x-1)^2 \\ p_3(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 \\ &= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 \end{aligned}$$

The graphs of $y = f(x)$ and the first three Taylor polynomials are shown in Figure 10.3.1.

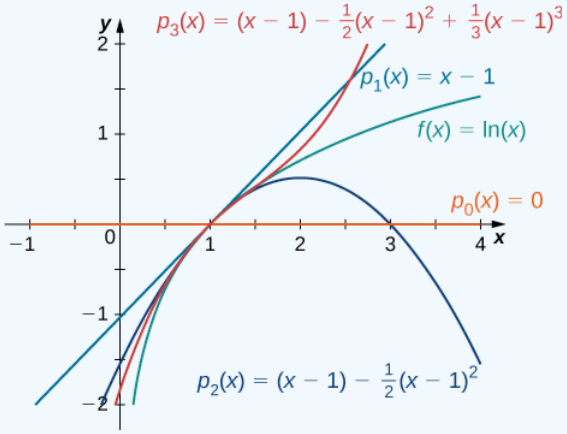

Figure 10.3.1: The function $y = \ln x$ and the Taylor polynomials p_0, p_1, p_2 and p_3 at $x = 1$ are plotted on this graph.

Exercise 10.3.1

Find the Taylor polynomials p_0, p_1, p_2 and p_3 for $f(x) = \frac{1}{x^2}$ at $x = 1$.

Hint

Find the first three derivatives of f and evaluate them at $x = 1$.

Answer

$$\begin{aligned} p_0(x) &= 1 \\ p_1(x) &= 1 - 2(x-1) \\ p_2(x) &= 1 - 2(x-1) + 3(x-1)^2 \\ p_3(x) &= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 \end{aligned}$$

We now show how to find Maclaurin polynomials for $e^x, \sin x,$ and $\cos x$. As stated above, Maclaurin polynomials are Taylor polynomials centered at zero.

Example 10.3.2: Finding Maclaurin Polynomials

For each of the following functions, find formulas for the Maclaurin polynomials p_0, p_1, p_2 and p_3. Find a formula for the nth-degree Maclaurin polynomial and write it using sigma notation. Use a graphing utility to compare the graphs of p_0, p_1, p_2 and p_3 with f.

a. $f(x) = e^x$
b. $f(x) = \sin x$
c. $f(x) = \cos x$

Solution

Since $f(x) = e^x$,we know that $f(x) = f'(x) = f''(x) = \cdots = f^{(n)}(x) = e^x$ for all positive integers n. Therefore,

$$f(0) = f'(0) = f''(0) = \cdots = f^{(n)}(0) = 1$$

for all positive integers n. Therefore, we have

$$p_0(x) = f(0) = 1,$$

$$p_1(x) = f(0) + f'(0)x = 1 + x,$$

$$p_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 + x + \frac{1}{2}x^2,$$

$$p_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3,$$

$$\begin{aligned} p_n(x) &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} \\ &= \sum_{k=0}^{n} \frac{x^k}{k!} \end{aligned}.$$

The function and the first three Maclaurin polynomials are shown in Figure 10.3.2.

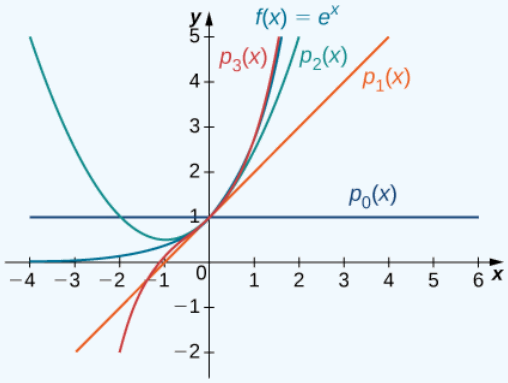

Figure 10.3.2: *The graph shows the function $y = e^x$ and the Maclaurin polynomials p_0, p_1, p_2 and p_3.*

b. For $f(x) = \sin x$, the values of the function and its first four derivatives at $x = 0$ are given as follows:

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0. \end{aligned}$$

Since the fourth derivative is $\sin x$, the pattern repeats. That is, $f^{(2m)}(0) = 0$ and $f^{(2m+1)}(0) = (-1)^m$ for $m \geq 0$. Thus, we have

$$\begin{aligned}
p_0(x) &= 0,\\
p_1(x) &= 0 + x = x,\\
p_2(x) &= 0 + x + 0 = x,\\
p_3(x) &= 0 + x + 0 - \frac{1}{3!}x^3 = x - \frac{x^3}{3!},\\
p_4(x) &= 0 + x + 0 - \frac{1}{3!}x^3 + 0 = x - \frac{x^3}{3!},\\
p_5(x) &= 0 + x + 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 = x - \frac{x^3}{3!} + \frac{x^5}{5!},
\end{aligned}$$

and for $m \geq 0$,

$$\begin{aligned}
p_{2m+1}(x) = p_{2m+2}(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^m \frac{x^{2m+1}}{(2m+1)!}\\
&= \sum_{k=0}^{m} (-1)^k \frac{x^{2k+1}}{(2k+1)!}.
\end{aligned}$$

Graphs of the function and its Maclaurin polynomials are shown in Figure 10.3.3.

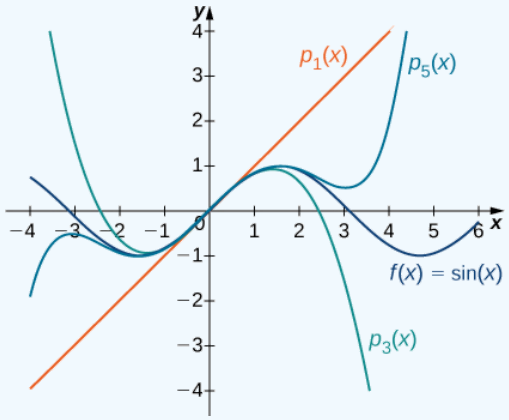

Figure 10.3.3: *The graph shows the function $y = \sin x$ and the Maclaurin polynomials p_1, p_3 and p_5.*

c. For $f(x) = \cos x$, the values of the function and its first four derivatives at $x = 0$ are given as follows:

$$\begin{aligned}
f(x) &= \cos x & f(0) &= 1\\
f'(x) &= -\sin x & f'(0) &= 0\\
f''(x) &= -\cos x & f''(0) &= -1\\
f'''(x) &= \sin x & f'''(0) &= 0\\
f^{(4)}(x) &= \cos x & f^{(4)}(0) &= 1.
\end{aligned}$$

Since the fourth derivative is $\sin x$, the pattern repeats. In other words, $f^{(2m)}(0) = (-1)^m$ and $f^{(2m+1)} = 0$ for $m \geq 0$. Therefore,

$$\begin{aligned}
p_0(x) &= 1,\\
p_1(x) &= 1 + 0 = 1,\\
p_2(x) &= 1 + 0 - \frac{1}{2!}x^2 = 1 - \frac{x^2}{2!},\\
p_3(x) &= 1 + 0 - \frac{1}{2!}x^2 + 0 = 1 - \frac{x^2}{2!},\\
p_4(x) &= 1 + 0 - \frac{1}{2!}x^2 + 0 + \frac{1}{4!}x^4 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!},\\
p_5(x) &= 1 + 0 - \frac{1}{2!}x^2 + 0 + \frac{1}{4!}x^4 + 0 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!},
\end{aligned}$$

and for $n \geq 0$,

$$\begin{aligned} p_{2m}(x) &= p_{2m+1}(x) \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^m \frac{x^{2m}}{(2m)!} \\ &= \sum_{k=0}^{m} (-1)^k \frac{x^{2k}}{(2k)!}. \end{aligned}$$

Graphs of the function and the Maclaurin polynomials appear in Figure 10.3.4.

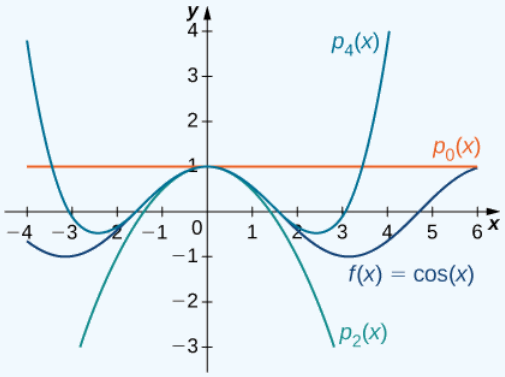

Figure 10.3.4: *The function $y = \cos x$ and the Maclaurin polynomials p_0, p_2 and p_4 are plotted on this graph.*

? Exercise 10.3.2

Find formulas for the Maclaurin polynomials p_0, p_1, p_2 and p_3 for $f(x) = \dfrac{1}{1+x}$.

Find a formula for the nth-degree Maclaurin polynomial. Write your answer using sigma notation.

Hint

Evaluate the first four derivatives of f and look for a pattern.

Answer

$$p_0(x) = 1,\ p_1(x) = 1 - x,\ p_2(x) = 1 - x + x^2,\ p_3(x) = 1 - x + x^2 - x^3,\ p_n(x) = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n = \sum_{k=0}^{n} (-1)^k x^k$$

Taylor's Theorem with Remainder

Recall that the nth-degree Taylor polynomial for a function f at a is the nth partial sum of the Taylor series for f at a. Therefore, to determine if the Taylor series converges, we need to determine whether the sequence of Taylor polynomials p_n converges. However, not only do we want to know if the sequence of Taylor polynomials converges, we want to know if it converges to f. To answer this question, we define the remainder $R_n(x)$ as

$$R_n(x) = f(x) - p_n(x).$$

For the sequence of Taylor polynomials to converge to f, we need the remainder R_n to converge to zero. To determine if R_n converges to zero, we introduce **Taylor's theorem with remainder**. Not only is this theorem useful in proving that a Taylor series converges to its related function, but it will also allow us to quantify how well the nth-degree Taylor polynomial approximates the function.

Here we look for a bound on $|R_n|$. Consider the simplest case: $n = 0$. Let p_0 be the 0th Taylor polynomial at a for a function f. The remainder R_0 satisfies

$$R_0(x) = f(x) - p_0(x) = f(x) - f(a).$$

If f is differentiable on an interval I containing a and x, then by the Mean Value Theorem there exists a real number c between a and x such that $f(x) - f(a) = f'(c)(x - a)$. Therefore,

$$R_0(x) = f'(c)(x - a).$$

Using the Mean Value Theorem in a similar argument, we can show that if f is n times differentiable on an interval I containing a and x, then the nth remainder R_n satisfies

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some real number c between a and x. It is important to note that the value c in the numerator above is not the center a, but rather an unknown value c between a and x. This formula allows us to get a bound on the remainder R_n. If we happen to know that $| f^{(n+1)}(x) |$ is bounded by some real number M on this interval I, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

for all x in the interval I.

We now state Taylor's theorem, which provides the formal relationship between a function f and its nth-degree Taylor polynomial $p_n(x)$. This theorem allows us to bound the error when using a Taylor polynomial to approximate a function value, and will be important in proving that a Taylor series for f converges to f.

Theorem 10.3.2: Taylor's Theorem with Remainder

Let f be a function that can be differentiated $n+1$ times on an interval I containing the real number a. Let p_n be the nth-degree Taylor polynomial of f at a and let

$$R_n(x) = f(x) - p_n(x)$$

be the nth remainder. Then for each x in the interval I, there exists a real number c between a and x such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

.

If there exists a real number M such that $| f^{(n+1)}(x) | \leq M$ for all $x \in I$, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

for all x in I.

Proof

Fix a point $x \in I$ and introduce the function g such that

$$g(t) = f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2!}(x-t)^2 - \cdots - \frac{f^{(n)}(t)}{n!}(x-t)^n - R_n(x)\frac{(x-t)^{n+1}}{(x-a)^{n+1}}.$$

We claim that g satisfies the criteria of Rolle's theorem. Since g is a polynomial function (in t), it is a differentiable function. Also, g is zero at $t=a$ and $t=x$ because

$$\begin{aligned} g(a) &= f(x) - f(a) - f'(a)(x-a) - \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n - R_n(x) \\ &= f(x) - p_n(x) - R_n(x) \\ &= 0, \\ g(x) &= f(x) - f(x) - 0 - \cdots - 0 \\ &= 0. \end{aligned}$$

Therefore, g satisfies Rolle's theorem, and consequently, there exists c between a and x such that $g'(c) = 0$. We now calculate g'. Using the product rule, we note that

$$\frac{d}{dt}\left[\frac{f^{(n)}(t)}{n!}(x-t)^n\right] = -\frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} + \frac{f^{(n+1)}(t)}{n!}(x-t)^n.$$

Consequently,

$$g'(t) = -f'(t) + [f'(t) - f''(t)(x-t)] + \left[f''(t)(x-t) - \frac{f'''(t)}{2!}(x-t)^2\right] + \cdots$$
$$+ \left[\frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n+1)}(t)}{n!}(x-t)^n\right] + (n+1)R_n(x)\frac{(x-t)^n}{(x-a)^{n+1}}$$

.

Notice that there is a telescoping effect. Therefore,

$$g'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n + (n+1)R_n(x)\frac{(x-t)^n}{(x-a)^{n+1}}$$

.

By Rolle's theorem, we conclude that there exists a number c between a and x such that $g'(c) = 0$. Since

$$g'(c) = -\frac{f^{(n+1)}(c)}{n!}(x-c)^n + (n+1)R_n(x)\frac{(x-c)^n}{(x-a)^{n+1}}$$

we conclude that

$$-\frac{f^{(n+1)}(c)}{n!}(x-c)^n + (n+1)R_n(x)\frac{(x-c)^n}{(x-a)^{n+1}} = 0.$$

Adding the first term on the left-hand side to both sides of the equation and dividing both sides of the equation by $n+1$, we conclude that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

as desired. From this fact, it follows that if there exists M such that $|\, f^{(n+1)}(x)| \leq M$ for all x in I, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

.

□

Not only does Taylor's theorem allow us to prove that a Taylor series converges to a function, but it also allows us to estimate the accuracy of Taylor polynomials in approximating function values. We begin by looking at linear and quadratic approximations of $f(x) = \sqrt[3]{x}$ at $x = 8$ and determine how accurate these approximations are at estimating $\sqrt[3]{11}$.

Example 10.3.3: Using Linear and Quadratic Approximations to Estimate Function Values

Consider the function $f(x) = \sqrt[3]{x}$.

a. Find the first and second Taylor polynomials for f at $x = 8$. Use a graphing utility to compare these polynomials with f near $x = 8$.
b. Use these two polynomials to estimate $\sqrt[3]{11}$.
c. Use Taylor's theorem to bound the error.

Solution:

a. For $f(x) = \sqrt[3]{x}$, the values of the function and its first two derivatives at $x = 8$ are as follows:

$$f(x) = \sqrt[3]{x}, \qquad f(8) = 2$$
$$f'(x) = \frac{1}{3x^{2/3}}, \qquad f'(8) = \frac{1}{12}$$
$$f''(x) = \frac{-2}{9x^{5/3}}, \qquad f''(8) = -\frac{1}{144}.$$

Thus, the first and second Taylor polynomials at $x = 8$ are given by

$$\begin{aligned} p_1(x) &= f(8) + f'(8)(x-8) \\ &= 2 + \frac{1}{12}(x-8) \\ p_2(x) &= f(8) + f'(8)(x-8) + \frac{f''(8)}{2!}(x-8)^2 \\ &= 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2. \end{aligned}$$

The function and the Taylor polynomials are shown in Figure 10.3.5.

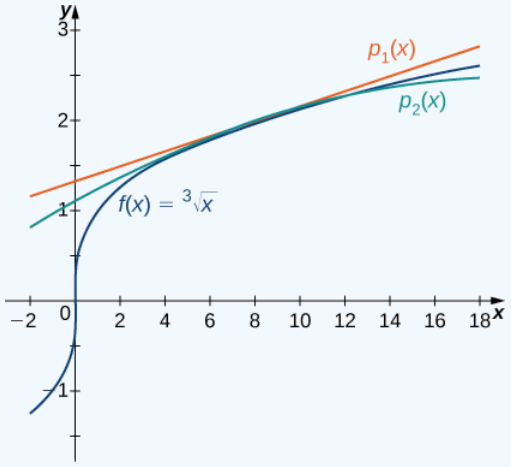

Figure 10.3.5: *The graphs of $f(x) = \sqrt[3]{x}$ and the linear and quadratic approximations $p_1(x)$ and $p_2(x)$*

b. Using the first Taylor polynomial at $x = 8$, we can estimate

$$\sqrt[3]{11} \approx p_1(11) = 2 + \frac{1}{12}(11-8) = 2.25.$$

Using the second Taylor polynomial at $x = 8$, we obtain

$$\sqrt[3]{11} \approx p_2(11) = 2 + \frac{1}{12}(11-8) - \frac{1}{288}(11-8)^2 = 2.21875.$$

c. By Theorem 10.3.2, there exists a c in the interval $(8, 11)$ such that the remainder when approximating $\sqrt[3]{11}$ by the first Taylor polynomial satisfies

$$R_1(11) = \frac{f''(c)}{2!}(11-8)^2.$$

We do not know the exact value of c, so we find an upper bound on $R_1(11)$ by determining the maximum value of f'' on the interval $(8, 11)$. Since $f''(x) = -\frac{2}{9x^{5/3}}$, the largest value for $|f''(x)|$ on that interval occurs at $x = 8$. Using the fact that $f''(8) = -\frac{1}{144}$, we obtain

$$|R_1(11)| \leq \frac{1}{144 \cdot 2!}(11-8)^2 = 0.03125.$$

Similarly, to estimate $R_2(11)$, we use the fact that

$$R_2(11) = \frac{f'''(c)}{3!}(11-8)^3.$$

Since $f'''(x) = \frac{10}{27x^{8/3}}$, the maximum value of f''' on the interval $(8, 11)$ is $f'''(8) \approx 0.0014468$. Therefore, we have

$$|R_2(11)| \leq \frac{0.0011468}{3!}(11-8)^3 \approx 0.0065104.$$

Exercise 10.3.3:

Find the first and second Taylor polynomials for $f(x) = \sqrt{x}$ at $x = 4$. Use these polynomials to estimate $\sqrt{6}$. Use Taylor's theorem to bound the error.

Hint

Evaluate $f(4)$, $f'(4)$, and $f''(4)$.

Answer

$p_1(x) = 2 + \frac{1}{4}(x-4); p_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2; p_1(6) = 2.5; p_2(6) = 2.4375;$

$|R_1(6)| \leq 0.0625; |R_2(6)| \leq 0.015625$

Example 10.3.4: Approximating $\sin x$ Using Maclaurin Polynomials

From Example 10.3.2*b*, the Maclaurin polynomials for $\sin x$ are given by

$$p_{2m+1}(x) = p_{2m+2}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^m \frac{x^{2m+1}}{(2m+1)!}$$

for $m = 0, 1, 2, \ldots$.

a. Use the fifth Maclaurin polynomial for $\sin x$ to approximate $\sin\left(\frac{\pi}{18}\right)$ and bound the error.

b. For what values of x does the fifth Maclaurin polynomial approximate $\sin x$ to within 0.0001?

Solution

a.

The fifth Maclaurin polynomial is

$$p_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

.

Using this polynomial, we can estimate as follows:

$$\sin\left(\frac{\pi}{18}\right) \approx p_5\left(\frac{\pi}{18}\right) = \frac{\pi}{18} - \frac{1}{3!}\left(\frac{\pi}{18}\right)^3 + \frac{1}{5!}\left(\frac{\pi}{18}\right)^5 \approx 0.173648.$$

To estimate the error, use the fact that the sixth Maclaurin polynomial is $p_6(x) = p_5(x)$ and calculate a bound on $R_6(\frac{\pi}{18})$. By Theorem 10.3.2, the remainder is

$$R_6\left(\frac{\pi}{18}\right) = \frac{f^{(7)}(c)}{7!}\left(\frac{\pi}{18}\right)^7$$

for some c between 0 and $\frac{\pi}{18}$. Using the fact that $|\, f^{(7)}(x)\, | \leq 1$ for all x, we find that the magnitude of the error is at most

$$\frac{1}{7!} \cdot \left(\frac{\pi}{18}\right)^7 \leq 9.8 \times 10^{-10}.$$

b.

We need to find the values of x such that

$$\frac{1}{7!}|x|^7 \leq 0.0001.$$

Solving this inequality for x, we have that the fifth Maclaurin polynomial gives an estimate to within 0.0001 as long as $|x| < 0.907$.

Exercise 10.3.4

Use the fourth Maclaurin polynomial for $\cos x$ to approximate $\cos\left(\frac{\pi}{12}\right)$.

Hint

The fourth Maclaurin polynomial is $p_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$.

Answer

0.96593

Now that we are able to bound the remainder $R_n(x)$, we can use this bound to prove that a Taylor series for f at a converges to f.

Representing Functions with Taylor and Maclaurin Series

We now discuss issues of convergence for Taylor series. We begin by showing how to find a Taylor series for a function, and how to find its interval of convergence.

Example 10.3.5: Finding a Taylor Series

Find the Taylor series for $f(x) = \frac{1}{x}$ at $x = 1$. Determine the interval of convergence.

Solution

For $f(x) = \frac{1}{x}$, the values of the function and its first four derivatives at $x = 1$ are

$$f(x) = \frac{1}{x} \qquad f(1) = 1$$

$$f'(x) = -\frac{1}{x^2} \qquad f'(1) = -1$$

$$f''(x) = \frac{2}{x^3} \qquad f''(1) = 2!$$

$$f'''(x) = -\frac{3 \cdot 2}{x^4} \qquad f'''(1) = -3!$$

$$f^{(4)}(x) = \frac{4 \cdot 3 \cdot 2}{x^5} \qquad f^{(4)}(1) = 4!.$$

That is, we have $f^{(n)}(1) = (-1)^n n!$ for all $n \geq 0$. Therefore, the Taylor series for f at $x = 1$ is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \sum_{n=0}^{\infty} (-1)^n (x-1)^n.$$

To find the interval of convergence, we use the ratio test. We find that

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|\,(-1)^{n+1}(x-1)n^{+1}\,|}{|(-1)^n(x-1)^n|} = |x-1|.$$

Thus, the series converges if $|x-1| < 1$. That is, the series converges for $0 < x < 2$. Next, we need to check the endpoints. At $x = 2$, we see that

$$\sum_{n=0}^{\infty} (-1)^n (2-1)^n = \sum_{n=0}^{\infty} (-1)^n$$

diverges by the divergence test. Similarly, at $x = 0$,

$$\sum_{n=0}^{\infty} (-1)^n (0-1)^n = \sum_{n=0}^{\infty} (-1)^{2n} = \sum_{n=0}^{\infty} 1$$

diverges. Therefore, the interval of convergence is $(0, 2)$.

Exercise 10.3.5

Find the Taylor series for $f(x) = \dfrac{1}{2x}$ at $x = 2$ and determine its interval of convergence.

Hint

$f^{(n)}(2) = \dfrac{(-1)^n n!}{2^{n+2}}$

Answer

$\dfrac{1}{4}\displaystyle\sum_{n=0}^{\infty}\left(\dfrac{2-x}{2}\right)^n$. The interval of convergence is $(0, 4)$.

We know that the Taylor series found in this example converges on the interval $(0, 2)$, but how do we know it actually converges to f? We consider this question in more generality in a moment, but for this example, we can answer this question by writing

$$f(x) = \frac{1}{x} = \frac{1}{1-(1-x)}.$$

That is, f can be represented by the geometric series $\displaystyle\sum_{n=0}^{\infty}(1-x)^n$. Since this is a geometric series, it converges to $\dfrac{1}{x}$ as long as $|1 - x| < 1$. Therefore, the Taylor series found in Example 10.3.5 does converge to $f(x) = \dfrac{1}{x}$ on $(0, 2)$.

We now consider the more general question: if a Taylor series for a function f converges on some interval, how can we determine if it actually converges to f? To answer this question, recall that a series converges to a particular value if and only if its sequence of partial sums converges to that value. Given a Taylor series for f at a, the nth partial sum is given by the nth-degree Taylor polynomial p_n. Therefore, to determine if the Taylor series converges to f, we need to determine whether

$$\lim_{n\to\infty} p_n(x) = f(x).$$

Since the remainder $R_n(x) = f(x) - p_n(x)$, the Taylor series converges to f if and only if

$$\lim_{n\to\infty} R_n(x) = 0.$$

We now state this theorem formally.

Theorem 10.3.3: Convergence of Taylor Series

Suppose that f has derivatives of all orders on an interval I containing a. Then the Taylor series

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

converges to $f(x)$ for all x in I if and only if

$$\lim_{n\to\infty} R_n(x) = 0$$

for all x in I.

With this theorem, we can prove that a Taylor series for f at a converges to f if we can prove that the remainder $R_n(x) \to 0$. To prove that $R_n(x) \to 0$, we typically use the bound

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$$

from Taylor's theorem with remainder.

In the next example, we find the Maclaurin series for e^x and $\sin x$ and show that these series converge to the corresponding functions for all real numbers by proving that the remainders $R_n(x) \to 0$ for all real numbers x.

Example 10.3.6: Finding Maclaurin Series

For each of the following functions, find the Maclaurin series and its interval of convergence. Use Theorem 10.3.2 to prove that the Maclaurin series for f converges to f on that interval.

a. e^x
b. $\sin x$

Solution

a. Using the nth-degree Maclaurin polynomial for e^x found in Example 10.3.2a, we find that the Maclaurin series for e^x is given by

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

To determine the interval of convergence, we use the ratio test. Since

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x|^{n+1}}{(n+1)!} \cdot \frac{n!}{|x|^n} = \frac{|x|}{n+1},$$

we have

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0$$

for all x. Therefore, the series converges absolutely for all x, and thus, the interval of convergence is $(-\infty, \infty)$. To show that the series converges to e^x for all x, we use the fact that $f^{(n)}(x) = e^x$ for all $n \geq 0$ and e^x is an increasing function on $(-\infty, \infty)$. Therefore, for any real number b, the maximum value of e^x for all $|x| \leq b$ is e^b. Thus,

$$|R_n(x)| \leq \frac{e^b}{(n+1)!}|x|^{n+1}.$$

Since we just showed that

$$\sum_{n=0}^{\infty} \frac{|x|^n}{n!}$$

converges for all x, by the divergence test, we know that

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

for any real number x. By combining this fact with the squeeze theorem, the result is $\lim_{n\to\infty} R_n(x) = 0$.

b. Using the nth-degree Maclaurin polynomial for $\sin x$ found in Example 10.3.2b, we find that the Maclaurin series for $\sin x$ is given by

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

In order to apply the ratio test, consider

$$\begin{aligned} \frac{|a_{n+1}|}{|a_n|} &= \frac{|x|^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{|x|^{2n+1}} \\ &= \frac{|x|^2}{(2n+3)(2n+2)} \end{aligned}.$$

Since

$$\lim_{n\to\infty} \frac{|x|^2}{(2n+3)(2n+2)} = 0$$

for all x, we obtain the interval of convergence as $(-\infty, \infty)$. To show that the Maclaurin series converges to $\sin x$, look at $R_n(x)$. For each x there exists a real number c between 0 and x such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}.$$

Since $|f^{(n+1)}(c)| \leq 1$ for all integers n and all real numbers c, we have

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$$

for all real numbers x. Using the same idea as in part a., the result is $\lim_{n\to\infty} R_n(x) = 0$ for all x, and therefore, the Maclaurin series for $\sin x$ converges to $\sin x$ for all real x.

Exercise 10.3.6

Find the Maclaurin series for $f(x) = \cos x$. Use the ratio test to show that the interval of convergence is $(-\infty, \infty)$. Show that the Maclaurin series converges to $\cos x$ for all real numbers x.

Hint

Use the Maclaurin polynomials for $\cos x$.

Answer

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

By the ratio test, the interval of convergence is $(-\infty, \infty)$. Since $|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$, the series converges to $\cos x$ for all real x.

Project: Proving that e is Irrational

In this project, we use the Maclaurin polynomials for e^x to prove that e is irrational. The proof relies on supposing that e is rational and arriving at a contradiction. Therefore, in the following steps, we suppose $e = r/s$ for some integers r and s where $s \neq 0$.

1. Write the Maclaurin polynomials $p_0(x), p_1(x), p_2(x), p_3(x), p_4(x)$ for e^x. Evaluate $p_0(1), p_1(1), p_2(1), p_3(1), p_4(1)$ to estimate e.
2. Let $R_n(x)$ denote the remainder when using $p_n(x)$ to estimate e^x. Therefore, $R_n(x) = e^x - p_n(x)$, and $R_n(1) = e - p_n(1)$. Assuming that $e = \frac{r}{s}$ for integers r and s, evaluate $R_0(1), R_1(1), R_2(1), R_3(1), R_4(1)$.
3. Using the results from part 2, show that for each remainder $R_0(1), R_1(1), R_2(1), R_3(1), R_4(1)$, we can find an integer k such that $kR_n(1)$ is an integer for $n = 0, 1, 2, 3, 4$.
4. Write down the formula for the nth-degree Maclaurin polynomial $p_n(x)$ for e^x and the corresponding remainder $R_n(x)$. Show that $sn!R_n(1)$ is an integer.
5. Use Taylor's theorem to write down an explicit formula for $R_n(1)$. Conclude that $R_n(1) \neq 0$, and therefore, $sn!R_n(1) \neq 0$.
6. Use Taylor's theorem to find an estimate on $R_n(1)$. Use this estimate combined with the result from part 5 to show that $|sn!R_n(1)| < \frac{se}{n+1}$. Conclude that if n is large enough, then $|sn!R_n(1)| < 1$. Therefore, $sn!R_n(1)$ is an integer with magnitude less than 1. Thus, $sn!R_n(1) = 0$. But from part 5, we know that $sn!R_n(1) \neq 0$. We have arrived at a contradiction, and consequently, the original supposition that e is rational must be false.

Key Concepts

- Taylor polynomials are used to approximate functions near a value $x = a$. Maclaurin polynomials are Taylor polynomials at $x = 0$.
- The nth-degree Taylor polynomials for a function f are the partial sums of the Taylor series for f.
- If a function f has a power series representation at $x = a$, then it is given by its Taylor series at $x = a$.
- A Taylor series for f converges to f if and only if $\lim_{n\to\infty} R_n(x) = 0$ where $R_n(x) = f(x) - p_n(x)$.
- The Taylor series for $e^x, \sin x$, and $\cos x$ converge to the respective functions for all real x.

Key Equations

- **Taylor series for the function f at the point $x = a$**

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

Glossary

Maclaurin polynomial

a Taylor polynomial centered at 0; the nth-degree Taylor polynomial for f at 0 is the nth-degree Maclaurin polynomial for f

Maclaurin series

a Taylor series for a function f at $x = 0$ is known as a Maclaurin series for f

Taylor polynomials

the nth-degree Taylor polynomial for f at $x = a$ is $p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$

Taylor series

a power series at a that converges to a function f on some open interval containing a.

Taylor's theorem with remainder

for a function f and the nth-degree Taylor polynomial for f at $x = a$, the remainder $R_n(x) = f(x) - p_n(x)$ satisfies $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

for some$c$ between x and a; if there exists an interval I containing a and a real number M such that $|\, f^{(n+1)}(x)\,| \leq M$ for all x in I, then $|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$

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10.3E: Exercises for Section 10.3

Taylor Polynomials

In exercises 1 - 8, find the Taylor polynomials of degree two approximating the given function centered at the given point.

1) $f(x) = 1 + x + x^2$ at $a = 1$

2) $f(x) = 1 + x + x^2$ at $a = -1$

Answer

$f(-1) = 1;\ f'(-1) = -1;\ f''(-1) = 2; \quad p_2(x) = 1 - (x+1) + (x+1)^2$

3) $f(x) = \cos(2x)$ at $a = \pi$

4) $f(x) = \sin(2x)$ at $a = \frac{\pi}{2}$

Answer

$f'(x) = 2\cos(2x);\ f''(x) = -4\sin(2x); \quad p_2(x) = -2(x - \frac{\pi}{2})$

5) $f(x) = \sqrt{x}$ at $a = 4$

6) $f(x) = \ln x$ at $a = 1$

Answer

$f'(x) = \dfrac{1}{x};\ f''(x) = -\dfrac{1}{x^2}; \quad p_2(x) = 0 + (x-1) - \frac{1}{2}(x-1)^2$

7) $f(x) = \dfrac{1}{x}$ at $a = 1$

8) $f(x) = e^x$ at $a = 1$

Answer

$p_2(x) = e + e(x-1) + \dfrac{e}{2}(x-1)^2$

Taylor Remainder Theorem

In exercises 9 - 14, verify that the given choice of n in the remainder estimate $|R_n| \le \dfrac{M}{(n+1)!}(x-a)^{n+1}$, where M is the maximum value of $| f^{(n+1)}(z) |$ on the interval between a and the indicated point, yields $|R_n| \le \frac{1}{1000}$. Find the value of the Taylor polynomial p_n of f at the indicated point.

9) [T] $\sqrt{10};\ a = 9,\ n = 3$

10) [T] $(28)^{1/3};\ a = 27,\ n = 1$

Answer

$\dfrac{d^2}{dx^2}x^{1/3} = -\dfrac{2}{9x^{5/3}} \ge -0.00092\ldots$ when $x \ge 28$ so the remainder estimate applies to the linear approximation $x^{1/3} \approx p_1(27) = 3 + \dfrac{x-27}{27}$, which gives $(28)^{1/3} \approx 3 + \frac{1}{27} = 3.0\overline{37}$, while $(28)^{1/3} \approx 3.03658$.

11) [T] $\sin(6);\ a = 2\pi,\ n = 5$

12) [T] $e^2;\ a = 0,\ n = 9$

Answer

Using the estimate $\dfrac{2^{10}}{10!} < 0.000283$ we can use the Taylor expansion of order 9 to estimate e^x at $x = 2$. as $e^2 \approx p_9(2) = 1 + 2 + \frac{2^2}{2} + \frac{2^3}{6} + \cdots + \frac{2^9}{9!} = 7.3887\ldots$ whereas $e^2 \approx 7.3891$.

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13) [T] $\cos(\frac{\pi}{5});\ a=0,\ n=4$

14) [T] $\ln(2);\ a=1,\ n=1000$

Answer

Since $\frac{d^n}{dx^n}(\ln x)=(-1)^{n-1}\frac{(n-1)!}{x^n}$, $R_{1000}\approx\frac{1}{1001}$. One has $p_{1000}(1)=\sum_{n=1}^{1000}\frac{(-1)^{n-1}}{n}\approx 0.6936$ whereas $\ln(2)\approx 0.6931\cdots$.

Approximating Definite Integrals Using Taylor Series

15) Integrate the approximation $\sin t\approx t-\frac{t^3}{6}+\frac{t^5}{120}-\frac{t^7}{5040}$ evaluated at πt to approximate $\int_0^1\frac{\sin\pi t}{\pi t}\,dt$.

16) Integrate the approximation $e^x\approx 1+x+\frac{x^2}{2}+\cdots+\frac{x^6}{720}$ evaluated at $-x^2$ to approximate $\int_0^1 e^{-x^2}\,dx$.

Answer

$\int_0^1\left(1-x^2+\frac{x^4}{2}-\frac{x^6}{6}+\frac{x^8}{24}-\frac{x^{10}}{120}+\frac{x^{12}}{720}\right)dx=1-\frac{1^3}{3}+\frac{1^5}{10}-\frac{1^7}{42}+\frac{1^9}{9\cdot 24}-\frac{1^{11}}{120\cdot 11}+\frac{1^{13}}{720\cdot 13}\approx 0.74683$ whereas $\int_0^1 e^{-x^2}dx\approx 0.74682$.

More Taylor Remainder Theorem Problems

In exercises 17 - 20, find the smallest value of n such that the remainder estimate $|R_n|\le\frac{M}{(n+1)!}(x-a)^{n+1}$, where M is the maximum value of $|\,f^{(n+1)}(z)\,|$ on the interval between a and the indicated point, yields $|R_n|\le\frac{1}{1000}$ on the indicated interval.

17) $f(x)=\sin x$ on $[-\pi,\pi],\ a=0$

18) $f(x)=\cos x$ on $[-\frac{\pi}{2},\frac{\pi}{2}],\ a=0$

Answer

Since $f^{(n+1)}(z)$ is $\sin z$ or $\cos z$, we have $M=1$. Since $|x-0|\le\frac{\pi}{2}$, we seek the smallest n such that $\frac{\pi^{n+1}}{2^{n+1}(n+1)!}\le 0.001$. The smallest such value is $n=7$. The remainder estimate is $R_7\le 0.00092$.

19) $f(x)=e^{-2x}$ on $[-1,1], a=0$

20) $f(x)=e^{-x}$ on $[-3,3], a=0$

Answer

Since $f^{(n+1)}(z)=\pm e^{-z}$ one has $M=e^3$. Since $|x-0|\le 3$, one seeks the smallest n such that $\frac{3^{n+1}e^3}{(n+1)!}\le 0.001$. The smallest such value is $n=14$. The remainder estimate is $R_{14}\le 0.000220$.

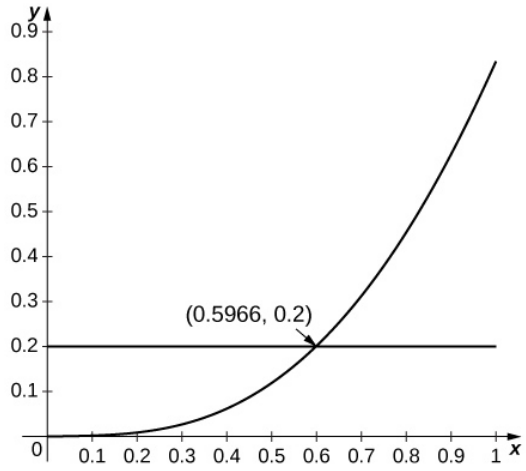
In exercises 21 - 24, the maximum of the right-hand side of the remainder estimate $|R_1|\le\frac{max|f''(z)|}{2}R^2$ on $[a-R,a+R]$ occurs at a or $a\pm R$. Estimate the maximum value of R such that $\frac{max|f''(z)|}{2}R^2\le 0.1$ on $[a-R,a+R]$ by plotting this maximum as a function of R.

21) [T] e^x approximated by $1+x,\ a=0$

22) [T] $\sin x$ approximated by $x,\ a=0$

Answer

Since $\sin x$ is increasing for small x and since $\frac{d^2}{dx^2}(\sin x)=-\sin x$, the estimate applies whenever $R^2\sin(R)\le 0.2$, which applies up to $R=0.596$.

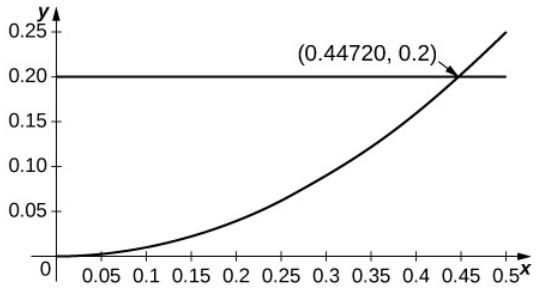


23) [T] $\ln x$ approximated by $x-1,\ a=1$

24) [T] $\cos x$ approximated by $1,\ a=0$

Answer

Since the second derivative of $\cos x$ is $-\cos x$ and since $\cos x$ is decreasing away from $x=0$, the estimate applies when $R^2 \cos R \leq 0.2$ or $R \leq 0.447$.



Taylor Series

In exercises 25 - 35, find the Taylor series of the given function centered at the indicated point.

25) $f(x)=x^4$ at $a=-1$

26) $f(x)=1+x+x^2+x^3$ at $a=-1$

Answer

$(x+1)^3-2(x+1)^2+2(x+1)$

27) $f(x)=\sin x$ at $a=\pi$

28) $f(x)=\cos x$ at $a=2\pi$

Answer

Values of derivatives are the same as for $x=0$ so $\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{(x-2\pi)^{2n}}{(2n)!}$

29) $f(x)=\sin x$ at $x=\frac{\pi}{2}$

30) $f(x)=\cos x$ at $x=\frac{\pi}{2}$

Answer

$\cos(\frac{\pi}{2}) = 0,\ -\sin(\frac{\pi}{2}) = -1$ so $\cos x = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-\frac{\pi}{2})^{2n+1}}{(2n+1)!}$, which is also $-\cos(x-\frac{\pi}{2})$.

31) $f(x) = e^x$ at $a = -1$

32) $f(x) = e^x$ at $a = 1$

Answer

The derivatives are $f^{(n)}(1) = e$, so $e^x = e\sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$.

33) $f(x) = \frac{1}{(x-1)^2}$ at $a = 0$ (Hint: Differentiate the Taylor Series for $\frac{1}{1-x}$.)

34) $f(x) = \frac{1}{(x-1)^3}$ at $a = 0$

Answer

$$\frac{1}{(x-1)^3} = -\frac{1}{2}\frac{d^2}{dx^2}\left(\frac{1}{1-x}\right) = -\sum_{n=0}^{\infty}\left(\frac{(n+2)(n+1)x^n}{2}\right)$$

35) $F(x) = \int_0^x \cos(\sqrt{t})\,dt;$ where $f(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{(2n)!}$ at a=0 (Note: f is the Taylor series of $\cos(\sqrt{t})$.)

In exercises 36 - 44, compute the Taylor series of each function around $x = 1$.

36) $f(x) = 2 - x$

Answer

$2 - x = 1 - (x-1)$

37) $f(x) = x^3$

38) $f(x) = (x-2)^2$

Answer

$((x-1)-1)^2 = (x-1)^2 - 2(x-1) + 1$

39) $f(x) = \ln x$

40) $f(x) = \frac{1}{x}$

Answer

$$\frac{1}{1-(1-x)} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

41) $f(x) = \frac{1}{2x - x^2}$

42) $f(x) = \frac{x}{4x - 2x^2 - 1}$

Answer

$$x\sum_{n=0}^{\infty} 2^n (1-x)^{2n} = \sum_{n=0}^{\infty} 2^n (x-1)^{2n+1} + \sum_{n=0}^{\infty} 2^n (x-1)^{2n}$$

43) $f(x) = e^{-x}$

44) $f(x) = e^{2x}$

Answer

$$e^{2x} = e^{2(x-1)+2} = e^2 \sum_{n=0}^{\infty} \frac{2^n (x-1)^n}{n!}$$

[T] In exercises 45 - 48, identify the value of x such that the given series $\sum_{n=0}^{\infty} a_n$ is the value of the Maclaurin series of $f(x)$ at x. Approximate the value of $f(x)$ using $S_{10} = \sum_{n=0}^{10} a_n$.

45) $\sum_{n=0}^{\infty} \frac{1}{n!}$

46) $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

Answer

$x = e^2; \quad S_{10} = \frac{34,913}{4725} \approx 7.3889947$

47) $\sum_{n=0}^{\infty} \frac{(-1)^n (2\pi)^{2n}}{(2n)!}$

48) $\sum_{n=0}^{\infty} \frac{(-1)^n (2\pi)^{2n+1}}{(2n+1)!}$

Answer

$\sin(2\pi) = 0; \quad S_{10} = 8.27 \times 10^{-5}$

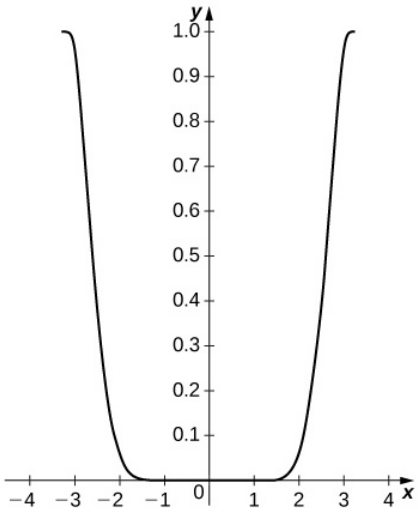
In exercises 49 - 52 use the functions $S_5(x) = x - \frac{x^3}{6} + \frac{x^5}{120}$ and $C_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$ on $[-\pi, \pi]$.

49) [T] Plot $\sin^2 x - (S_5(x))^2$ on $[-\pi, \pi]$. Compare the maximum difference with the square of the Taylor remainder estimate for $\sin x$.

50) [T] Plot $\cos^2 x - (C_4(x))^2$ on $[-\pi, \pi]$. Compare the maximum difference with the square of the Taylor remainder estimate for $\cos x$.

Answer

The difference is small on the interior of the interval but approaches 1 near the endpoints. The remainder estimate is $|R_4| = \frac{\pi^5}{120} \approx 2.552$.

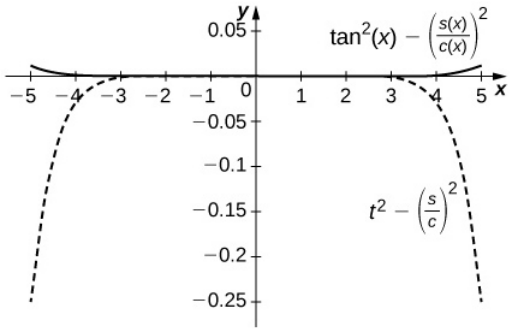


51) [T] Plot $|2S_5(x)C_4(x) - \sin(2x)|$ on $[-\pi, \pi]$.

52) [T] Compare $\frac{S_5(x)}{C_4(x)}$ on $[-1,1]$ to $\tan x$. Compare this with the Taylor remainder estimate for the approximation of $\tan x$ by $x + \frac{x^3}{3} + \frac{2x^5}{15}$.

Answer

The difference is on the order of 10^{-4} on $[-1,1]$ while the Taylor approximation error is around 0.1 near ± 1. The top curve is a plot of $\tan^2 x - \left(\frac{S_5(x)}{C_4(x)}\right)^2$ and the lower dashed plot shows $t^2 - \left(\frac{S_5}{C_4}\right)^2$.



53) [T] Plot $e^x - e_4(x)$ where $e_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$ on $[0,2]$. Compare the maximum error with the Taylor remainder estimate.

54) (Taylor approximations and root finding.) Recall that Newton's method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ approximates solutions of $f(x) = 0$ near the input x_0.

a. If f and g are inverse functions, explain why a solution of $g(x) = a$ is the value $f(a)$ of f.

b. Let $p_N(x)$ be the Nth degree Maclaurin polynomial of e^x. Use Newton's method to approximate solutions of $p_N(x) - 2 = 0$ for $N = 4, 5, 6$.

c. Explain why the approximate roots of $p_N(x) - 2 = 0$ are approximate values of $\ln(2)$.

Answer

a. Answers will vary.

b. The following are the x_n values after 10 iterations of Newton's method to approximation a root of $p_N(x) - 2 = 0$: for $N = 4, x = 0.6939...$; for $N = 5, x = 0.6932...$; for $N = 6, x = 0.69315...$; . (Note: $\ln(2) = 0.69314...$)

c. Answers will vary.

Evaluating Limits using Taylor Series

In exercises 55 - 58, use the fact that if $q(x) = \sum_{n=1}^{\infty} a_n(x-c)^n$ converges in an interval containing c, then $\lim_{x\to c} q(x) = a_0$ to evaluate each limit using Taylor series.

55) $\lim_{x\to 0} \frac{\cos x - 1}{x^2}$

56) $\lim_{x\to 0} \frac{\ln(1-x^2)}{x^2}$

Answer

$\frac{\ln(1-x^2)}{x^2} \to -1$

57) $\lim_{x\to 0} \frac{e^{x^2} - x^2 - 1}{x^4}$

58) $\lim_{x\to 0^+} \frac{\cos(\sqrt{x}) - 1}{2x}$

Answer

$$\frac{\cos(\sqrt{x})-1}{2x} \approx \frac{(1-\frac{x}{2}+\frac{x^2}{4!}-\cdots)-1}{2x} \to -\frac{1}{4}$$

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10.4: Working with Taylor Series

Learning Objectives

- Write the terms of the binomial series.
- Recognize the Taylor series expansions of common functions.
- Recognize and apply techniques to find the Taylor series for a function.
- Use Taylor series to solve differential equations.
- Use Taylor series to evaluate non-elementary integrals.

In the preceding section, we defined Taylor series and showed how to find the Taylor series for several common functions by explicitly calculating the coefficients of the Taylor polynomials. In this section we show how to use those Taylor series to derive Taylor series for other functions. We then present two common applications of power series. First, we show how power series can be used to solve differential equations. Second, we show how power series can be used to evaluate integrals when the antiderivative of the integrand cannot be expressed in terms of elementary functions. In one example, we consider $\int e^{-x^2}dx$, an integral that arises frequently in probability theory.

The Binomial Series

Our first goal in this section is to determine the Maclaurin series for the function $f(x) = (1+x)^r$ for all real numbers r. The Maclaurin series for this function is known as the **binomial series.** We begin by considering the simplest case: r is a nonnegative integer. We recall that, for $r = 0,\ 1,\ 2,\ 3,\ 4,\ f(x) = (1+x)^r$ can be written as

$$\begin{aligned} f(x) &= (1+x)^0 = 1, \\ f(x) &= (1+x)^1 = 1+x, \\ f(x) &= (1+x)^2 = 1+2x+x^2, \\ f(x) &= (1+x)^3 = 1+3x+3x^2+x^3 \\ f(x) &= (1+x)^4 = 1+4x+6x^2+4x^3+x^4. \end{aligned}$$

The expressions on the right-hand side are known as binomial expansions and the coefficients are known as binomial coefficients. More generally, for any nonnegative integer r, the binomial coefficient of x^n in the binomial expansion of $(1+x)^r$ is given by

$$\binom{r}{n} = \frac{r!}{n!(r-n)!} \tag{10.4.1}$$

and

$$\begin{aligned} f(x) &= (1+x)^r \\ &= \binom{r}{0} + \binom{r}{1}x + \binom{r}{2}x^2 + \binom{r}{3}x^3 + \cdots + \binom{r}{r-1}x^{r-1} + \binom{r}{r}x^r \\ &= \sum_{n=0}^{r}\binom{r}{n}x^n. \end{aligned} \tag{10.4.2}$$

For example, using this formula for $r = 5$, we see that

$$\begin{aligned} f(x) &= (1+x)^5 \\ &= \binom{5}{0}1 + \binom{5}{1}x + \binom{5}{2}x^2 + \binom{5}{3}x^3 + \binom{5}{4}x^4 + \binom{5}{5}x^5 \\ &= \frac{5!}{0!5!}1 + \frac{5!}{1!4!}x + \frac{5!}{2!3!}x^2 + \frac{5!}{3!2!}x^3 + \frac{5!}{4!1!}x^4 + \frac{5!}{5!0!}x^5 \\ &= 1+5x+10x^2+10x^3+5x^4+x^5. \end{aligned}$$

We now consider the case when the exponent r is any real number, not necessarily a nonnegative integer. If r is not a nonnegative integer, then $f(x) = (1+x)^r$ cannot be written as a finite polynomial. However, we can find a power series for f. Specifically, we

look for the Maclaurin series for f. To do this, we find the derivatives of f and evaluate them at $x = 0$.

$$\begin{aligned} f(x) &= (1+x)^r & f(0) &= 1 \\ f'(x) &= r(1+x)^{r-1} & f'(0) &= r \\ f''(x) &= r(r-1)(1+x)^{r-2} & f''(0) &= r(r-1) \\ f'''(x) &= r(r-1)(r-2)(1+x)^{r-3} & f'''(0) &= r(r-1)(r-2) \\ f^{(n)}(x) &= r(r-1)(r-2)\cdots(r-n+1)(1+x)^{r-n} & f^{(n)}(0) &= r(r-1)(r-2)\cdots(r-n+1) \end{aligned}$$

We conclude that the coefficients in the binomial series are given by

$$\frac{f^{(n)}(0)}{n!} = \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}. \tag{10.4.3}$$

We note that if r is a nonnegative integer, then the $(r+1)^{\text{st}}$ derivative $f^{(r+1)}$ is the zero function, and the series terminates. In addition, if r is a nonnegative integer, then Equation 10.4.3 for the coefficients agrees with Equation 10.4.1 for the coefficients, and the formula for the binomial series agrees with Equation 10.4.2 for the finite binomial expansion. More generally, to denote the binomial coefficients for any real number r, we define

$$\binom{r}{n} = \frac{(r-1)(r-2)\cdots(r-n+1)}{n!}.$$

With this notation, we can write the binomial series for $(1+x)^r$ as

$$\sum_{n=0}^{\infty} \binom{r}{n} x^n = 1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots + \frac{r(r-1)\cdots(r-n+1)}{n!}x^n + \cdots. \tag{10.4.4}$$

We now need to determine the interval of convergence for the binomial series Equation 10.4.4. We apply the ratio test. Consequently, we consider

$$\begin{aligned} \frac{|a_{n+1}|}{|a_n|} &= \frac{|r(r-1)(r-2)\cdots(r-n)||x||^{n+1}}{(n+1)!} \cdot \frac{n!}{|r(r-1)(r-2)\cdots(r-n+1)||x|^n} \\ &= \frac{|r-n||x|}{|n+1|} \end{aligned}$$

.

Since

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = |x| < 1$$

if and only if $|x| < 1$, we conclude that the interval of convergence for the binomial series is $(-1, 1)$. The behavior at the endpoints depends on r. It can be shown that for $r \geq 0$ the series converges at both endpoints; for $-1 < r < 0$, the series converges at $x = 1$ and diverges at $x = -1$; and for $r < -1$, the series diverges at both endpoints. The binomial series does converge to $(1+x)^r$ in $(-1, 1)$ for all real numbers r, but proving this fact by showing that the remainder $R_n(x) \to 0$ is difficult.

Definition: Binomial Series

For any real number r, the Maclaurin series for $f(x) = (1+x)^r$ is the binomial series. It converges to f for $|x| < 1$, and we write

$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n = 1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots + r\frac{(r-1)\cdots(r-n+1)}{n!}x^n + \cdots$$

for $|x| < 1$.

We can use this definition to find the binomial series for $f(x) = \sqrt{1+x}$ and use the series to approximate $\sqrt{1.5}$.

Example 10.4.1: Finding Binomial Series

a. Find the binomial series for $f(x) = \sqrt{1+x}$.

b. Use the third-order Maclaurin polynomial $p_3(x)$ to estimate $\sqrt{1.5}$. Use Taylor's theorem to bound the error. Use a graphing utility to compare the graphs of f and p_3.

Solution

a. Here $r = \frac{1}{2}$. Using the definition for the binomial series, we obtain

$$\begin{aligned}\sqrt{1+x} &= 1 + \frac{1}{2}x + \frac{(1/2)(-1/2)}{2!}x^2 + \frac{(1/2)(-1/2)(-3/2)}{3!}x^3 + \cdots \\ &= 1 + \frac{1}{2}x - \frac{1}{2!}\frac{1}{2^2}x^2 + \frac{1}{3!}\frac{1\cdot 3}{2^3}x^3 - \cdots + \frac{(-1)^{n+1}}{n!}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n}x^n + \cdots \\ &= 1 + \frac{1}{2}x + \sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n!}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n}x^n.\end{aligned}$$

b. From the result in part a. the third-order Maclaurin polynomial is

$$p_3(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3.$$

Therefore,

$$\sqrt{1.5} = \sqrt{1+0.5} \approx 1 + \frac{1}{2}(0.5) - \frac{1}{8}(0.5)^2 + \frac{1}{16}(0.5)^3 \approx 1.2266.$$

From Taylor's theorem, the error satisfies

$$R_3(0.5) = \frac{f^{(4)}(c)}{4!}(0.5)^4$$

for some c between 0 and 0.5. Since $f^{(4)}(x) = -\dfrac{15}{2^4(1+x)^{7/2}}$, and the maximum value of $|\, f^{(4)}(x)\,|$ on the interval $(0, 0.5)$ occurs at $x = 0$, we have

$$|R_3(0.5)| \le \frac{15}{4!2^4}(0.5)^4 \approx 0.00244.$$

The function and the Maclaurin polynomial p_3 are graphed in Figure 10.4.1.

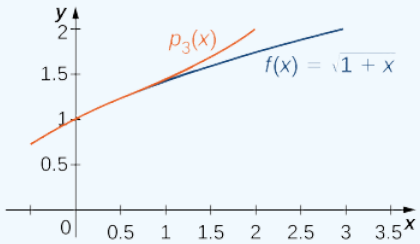

Figure 10.4.1: The third-order Maclaurin polynomial $p_3(x)$ provides a good approximation for $f(x) = \sqrt{1+x}$ for x near zero.

Exercise 10.4.1

Find the binomial series for $f(x) = \dfrac{1}{(1+x)^2}$.

Hint

Use the definition of binomial series for $r = -2$.

Answer

$$\sum_{n=0}^{\infty}(-1)^n(n+1)x^n$$

Common Functions Expressed as Taylor Series

At this point, we have derived Maclaurin series for exponential, trigonometric, and logarithmic functions, as well as functions of the form $f(x) = (1+x)^r$. In Table 10.4.1, we summarize the results of these series. We remark that the convergence of the Maclaurin series for $f(x) = \ln(1+x)$ at the endpoint $x = 1$ and the Maclaurin series for $f(x) = \tan^{-1} x$ at the endpoints $x = 1$ and $x = -1$ relies on a more advanced theorem than we present here. (Refer to Abel's theorem for a discussion of this more technical point.)

Table 10.4.1: Maclaurin Series for Common Functions

Function	Maclaurin Series	Interval of Convergence
$f(x) = \dfrac{1}{1-x}$	$\displaystyle\sum_{n=0}^{\infty} x^n$	$-1 < x < 1$
$f(x) = e^x$	$\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$-\infty < x < \infty$
$f(x) = \sin x$	$\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$-\infty < x < \infty$
$f(x) = \cos x$	$\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$-\infty < x < \infty$
$f(x) = \ln(1+x)$	$\displaystyle\sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^n}{n}$	$-1 < x \leq 1$
$f(x) = \tan^{-1} x$	$\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$-1 \leq x \leq 1$
$f(x) = (1+x)^r$	$\displaystyle\sum_{n=0}^{\infty} \binom{r}{n} x^n$	$-1 < x < 1$ Convergence at the endpoints depends on the value of r

Earlier in the chapter, we showed how you could combine power series to create new power series. Here we use these properties, combined with the Maclaurin series in Table 10.4.1, to create Maclaurin series for other functions.

Example 10.4.2: Deriving Maclaurin Series from Known Series

Find the Maclaurin series of each of the following functions by using one of the series listed in Table 10.4.1.

a. $f(x) = \cos\sqrt{x}$
b. $f(x) = \sinh x$

Solution

a. Using the Maclaurin series for $\cos x$ we find that the Maclaurin series for $\cos\sqrt{x}$ is given by

$$\sum_{n=0}^{\infty} \frac{(-1)^n (\sqrt{x})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!} = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \frac{x^4}{8!} - \cdots.$$

This series converges to $\cos\sqrt{x}$ for all x in the domain of $\cos\sqrt{x}$; that is, for all $x \geq 0$.

b. To find the Maclaurin series for $\sinh x$, we use the fact that

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

Using the Maclaurin series for e^x, we see that the nth term in the Maclaurin series for $\sinh x$ is given by

$$\frac{x^n}{n!} - \frac{(-x)^n}{n!}.$$

For n even, this term is zero. For n odd, this term is $\dfrac{2x^n}{n!}$. Therefore, the Maclaurin series for $\sinh x$ has only odd-order terms and is given by

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots.$$

Exercise 10.4.2

Find the Maclaurin series for $\sin(x^2)$.

Hint

Use the Maclaurin series for $\sin x$.

Answer

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

We also showed previously in this chapter how power series can be differentiated term by term to create a new power series. In Example 10.4.3, we differentiate the binomial series for $\sqrt{1+x}$ term by term to find the binomial series for $\frac{1}{\sqrt{1+x}}$. Note that we could construct the binomial series for $\frac{1}{\sqrt{1+x}}$ directly from the definition, but differentiating the binomial series for $\sqrt{1+x}$ is an easier calculation.

Example 10.4.3: Differentiating a Series to Find a New Series

Use the binomial series for $\sqrt{1+x}$ to find the binomial series for $\frac{1}{\sqrt{1+x}}$.

Solution

The two functions are related by

$$\frac{d}{dx}\sqrt{1+x} = \frac{1}{2\sqrt{1+x}},$$

so the binomial series for $\frac{1}{\sqrt{1+x}}$ is given by

$$\frac{1}{\sqrt{1+x}} = 2\frac{d}{dx}\sqrt{1+x} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} x^n.$$

Exercise 10.4.3

Find the binomial series for $f(x) = \frac{1}{(1+x)^{3/2}}$

Hint

Differentiate the series for $\frac{1}{\sqrt{1+x}}$

Answer

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{2^n} x^n$$

In this example, we differentiated a known Taylor series to construct a Taylor series for another function. The ability to differentiate power series term by term makes them a powerful tool for solving differential equations. We now show how this is accomplished.

Solving Differential Equations with Power Series

Consider the differential equation

$$y'(x) = y.$$

Recall that this is a first-order separable equation and its solution is $y = Ce^x$. This equation is easily solved using techniques discussed earlier in the text. For most differential equations, however, we do not yet have analytical tools to solve them. Power series are an extremely useful tool for solving many types of differential equations. In this technique, we look for a solution of the form $y = \sum_{n=0}^{\infty} c_n x^n$ and determine what the coefficients would need to be. In the next example, we consider an initial-value problem involving $y' = y$ to illustrate the technique.

Example 10.4.4: Power Series Solution of a Differential Equation

Use power series to solve the initial-value problem $y' = y, \quad y(0) = 3$.

Solution

Suppose that there exists a power series solution

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots.$$

Differentiating this series term by term, we obtain

$$y' = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots.$$

If y satisfies the differential equation, then

$$c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots = c_1 + 2c_2 x + 3c_3 x^2 + 4c_3 x^3 + \cdots.$$

Using the uniqueness of power series representations, we know that these series can only be equal if their coefficients are equal. Therefore,

$$c_0 = c_1,$$
$$c_1 = 2c_2,$$
$$c_2 = 3c_3,$$
$$c_3 = 4c_4,$$
$$\vdots$$

Using the initial condition $y(0) = 3$ combined with the power series representation

$$y(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots,$$

we find that $c_0 = 3$. We are now ready to solve for the rest of the coefficients. Using the fact that $c_0 = 3$, we have

$$c_1 = c_0 = 3 = \frac{3}{1!},$$
$$c_2 = \frac{c_1}{2} = \frac{3}{2} = \frac{3}{2!},$$
$$c_3 = \frac{c_2}{3} = \frac{3}{3 \cdot 2} = \frac{3}{3!},$$
$$c_4 = \frac{c_3}{4} = \frac{3}{4 \cdot 3 \cdot 2} = \frac{3}{4!}.$$

Therefore,

$$y = 3\left[1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 \frac{1}{4!}x^4 + \cdots\right] = 3\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

You might recognize

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

as the Taylor series for e^x. Therefore, the solution is $y = 3e^x$.

Exercise 10.4.4

Use power series to solve $y' = 2y, \quad y(0) = 5.$

Hint

The equations for the first several coefficients c_n will satisfy $c_0 = 2c_1,\ c_1 = 2 \cdot 2c_2,\ c_2 = 2 \cdot 3c_3,\ \ldots$. In general, for all $n \geq 0$, $c_n = 2(n+1)C_{n+1}$.

Answer

$y = 5e^{2x}$

We now consider an example involving a differential equation that we cannot solve using previously discussed methods. This differential equation

$$y'' - xy = 0$$

is known as **Airy's equation**. It has many applications in mathematical physics, such as modeling the diffraction of light. Here we show how to solve it using power series.

Example 10.4.5: Power Series Solution of Airy's Equation

Use power series to solve $y'' - xy = 0$ with the initial conditions $y(0) = a$ and $y'(0) = b$.

Solution

We look for a solution of the form

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$$

Differentiating this function term by term, we obtain

$$y' = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots,$$
$$y'' = 2 \cdot 1c_2 + 3 \cdot 2c_3 x + 4 \cdot 3c_4 x^2 + \cdots.$$

If y satisfies the equation $y'' = xy$, then

$$2 \cdot 1c_2 + 3 \cdot 2c_3 x + 4 \cdot 3c_4 x^2 + \cdots = x(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots).$$

Using the Uniqueness of Power Series Theorem (from an earlier Section) on the uniqueness of power series representations, we know that coefficients of the same degree must be equal. Therefore,

$$2 \cdot 1c_2 = 0,$$
$$3 \cdot 2c_3 = c_0,$$
$$4 \cdot 3c_4 = c_1,$$
$$5 \cdot 4c_5 = c_2,$$
$$\vdots$$

More generally, for $n \geq 3$, we have $n \cdot (n-1)c_n = c_{n-3}$. In fact, all coefficients can be written in terms of c_0 and c_1. To see this, first note that $c_2 = 0$. Then

$$c_3 = \frac{c_0}{3 \cdot 2},$$
$$c_4 = \frac{c_1}{4 \cdot 3}.$$

For $c_5,\ c_6,\ c_7$, we see that

$$c_5 = \frac{c_2}{5 \cdot 4} = 0,$$
$$c_6 = \frac{c_3}{6 \cdot 5} = \frac{c_0}{6 \cdot 5 \cdot 3 \cdot 2},$$
$$c_7 = \frac{c_4}{7 \cdot 6} = \frac{c_1}{7 \cdot 6 \cdot 4 \cdot 3}.$$

Therefore, the series solution of the differential equation is given by

$$y = c_0 + c_1 x + 0 \cdot x^2 + \frac{c_0}{3 \cdot 2} x^3 + \frac{c_1}{4 \cdot 3} x^4 + 0 \cdot x^5 + \frac{c_0}{6 \cdot 5 \cdot 3 \cdot 2} x^6 + \frac{c_1}{7 \cdot 6 \cdot 4 \cdot 3} x^7 + \cdots.$$

The initial condition $y(0) = a$ implies $c_0 = a$. Differentiating this series term by term and using the fact that $y'(0) = b$, we conclude that $c_1 = b$.

Therefore, the solution of this initial-value problem is

$$y = a\left(1 + \frac{x^3}{3 \cdot 2} + \frac{x^6}{6 \cdot 5 \cdot 3 \cdot 2} + \cdots\right) + b\left(x + \frac{x^4}{4 \cdot 3} + \frac{x^7}{7 \cdot 6 \cdot 4 \cdot 3} + \cdots\right).$$

? Exercise 10.4.5

Use power series to solve $y'' + x^2 y = 0$ with the initial condition $y(0) = a$ and $y'(0) = b$.

Hint

The coefficients satisfy $c_0 = a,\ c_1 = b,\ c_2 = 0,\ c_3 = 0$, and for $n \geq 4,\ n(n-1)c_n = -c_{n-4}$.

Answer

$$y = a\left(1 - \frac{x^4}{3 \cdot 4} + \frac{x^8}{3 \cdot 4 \cdot 7 \cdot 8} - \cdots\right) + b\left(x - \frac{x^5}{4 \cdot 5} + \frac{x^9}{4 \cdot 5 \cdot 8 \cdot 9} - \cdots\right)$$

Evaluating Non-elementary Integrals

Solving differential equations is one common application of power series. We now turn to a second application. We show how power series can be used to evaluate integrals involving functions whose antiderivatives cannot be expressed using elementary functions.

One integral that arises often in applications in probability theory is $\int e^{-x^2}\,dx$. Unfortunately, the antiderivative of the integrand e^{-x^2} is not an elementary function. By elementary function, we mean a function that can be written using a finite number of algebraic combinations or compositions of exponential, logarithmic, trigonometric, or power functions. We remark that the term "elementary function" is not synonymous with noncomplicated function. For example, the function $f(x) = \sqrt{x^2 - 3x} + e^{x^3} - \sin(5x + 4)$ is an elementary function, although not a particularly simple-looking function. Any integral of the form $\int f(x)\,dx$ where the antiderivative of f cannot be written as an elementary function is considered a **non-elementary integral.**

Non-elementary integrals cannot be evaluated using the basic integration techniques discussed earlier. One way to evaluate such integrals is by expressing the integrand as a power series and integrating term by term. We demonstrate this technique by considering $\int e^{-x^2}\,dx$.

✓ Example 10.4.6: Using Taylor Series to Evaluate a Definite Integral

a. Express $\int e^{-x^2} dx$ as an infinite series.

b. Evaluate $\int_0^1 e^{-x^2} dx$ to within an error of 0.01.

Solution

a. The Maclaurin series for e^{-x^2} is given by

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}$$
$$= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}.$$

Therefore,

$$\int e^{-x^2}\,dx = \int \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots\right) dx$$
$$= C + x - \frac{x^3}{3} + \frac{x^5}{5.2!} - \frac{x^7}{7.3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)n!} + \cdots.$$

b. Using the result from part a. we have

$$\int_0^1 e^{-x^2}\,dx = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \cdots.$$

The sum of the first four terms is approximately 0.74. By the alternating series test, this estimate is accurate to within an error of less than $\frac{1}{216} \approx 0.0046296 < 0.01$.

Exercise 10.4.6

Express $\int \cos\sqrt{x}\,dx$ as an infinite series. Evaluate $\int_0^1 \cos\sqrt{x}\,dx$ to within an error of 0.01.

Hint

Use the series found in Example 10.4.2.

Answer

$C + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n(2n-2)!}$ The definite integral is approximately 0.764 to within an error of 0.01.

As mentioned above, the integral $\int e^{-x^2}\,dx$ arises often in probability theory. Specifically, it is used when studying data sets that are normally distributed, meaning the data values lie under a bell-shaped curve. For example, if a set of data values is normally distributed with mean μ and standard deviation σ, then the probability that a randomly chosen value lies between $x = a$ and $x = b$ is given by

$$\frac{1}{\sigma\sqrt{2\pi}} \int_a^b e^{-(x-\mu)^2/(2\sigma^2)}\,dx. \tag{10.4.5}$$

(See Figure 10.4.2.)

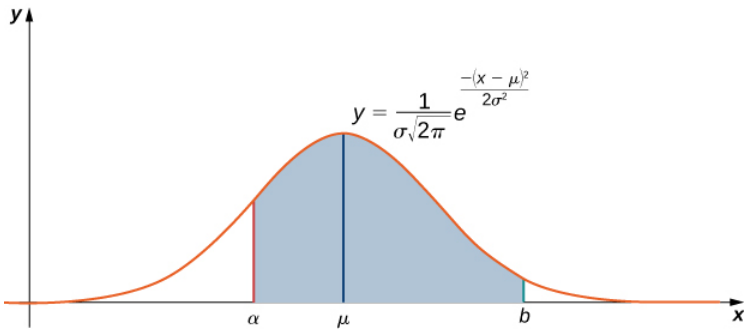

Figure 10.4.2: If data values are normally distributed with mean μ and standard deviation σ, the probability that a randomly selected data value is between a and b is the area under the curve $y = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$ between $x = a$ and $x = b$.

To simplify this integral, we typically let $z = \frac{x-\mu}{\sigma}$. This quantity z is known as the z score of a data value. With this simplification, integral Equation 10.4.5 becomes

$$\frac{1}{\sqrt{2\pi}} \int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} e^{-z^2/2}\, dz.$$

In Example 10.4.7, we show how we can use this integral in calculating probabilities.

Example 10.4.7: Using Maclaurin Series to Approximate a Probability

Suppose a set of standardized test scores are normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 50$. Use Equation 10.4.5 and the first six terms in the Maclaurin series for $e^{-x^2/2}$ to approximate the probability that a randomly selected test score is between $x = 100$ and $x = 200$. Use the alternating series test to determine how accurate your approximation is.

Solution

Since $\mu = 100, \sigma = 50$, and we are trying to determine the area under the curve from $a = 100$ to $b = 200$, integral Equation 10.4.5 becomes

$$\frac{1}{\sqrt{2\pi}} \int_0^2 e^{-z^2/2}\, dz. \tag{10.4.6}$$

The Maclaurin series for $e^{-x^2/2}$ is given by

$$\begin{aligned} e^{-x^2/2} &= \sum_{n=0}^{\infty} \frac{\left(-\frac{x^2}{2}\right)^n}{n!} \\ &= 1 - \frac{x^2}{2^1 \cdot 1!} + \frac{x^4}{2^2 \cdot 2!} - \frac{x^6}{2^3 \cdot 3!} + \cdots + (-1)^n \frac{x^{2n}}{2^n \cdot n}! + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^n \cdot n!}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int e^{-z^2/2}\, dz &= \frac{1}{\sqrt{2\pi}} \\ &\quad \int \left(1 - \frac{z^2}{2^1 \cdot 1!} + \frac{z^4}{2^2 \cdot 2!} - \frac{z^6}{2^3 \cdot 3!} + \cdots + (-1)^n \frac{z^{2n}}{2^n \cdot n!} + \cdots\right) dz \\ &= \frac{1}{\sqrt{2\pi}} \left(C + z - \frac{z^3}{3 \cdot 2^1 \cdot 1!} + \frac{z^5}{5 \cdot 2^2 \cdot 2!} - \frac{z^7}{7 \cdot 2^3 \cdot 3!} + \cdots \right. \\ &\quad \left. + (-1)^n \frac{z^{2n+1}}{(2n+1) 2^n \cdot n!} + \cdots\right) \end{aligned}$$

We now use this result to evaluate the definite integral from Equation 10.4.6:

$$\frac{1}{\sqrt{2\pi}}\int_0^2 e^{-z^2/2}\,dz = \frac{1}{\sqrt{2\pi}}\left(2-\frac{8}{6}+\frac{32}{40}-\frac{128}{336}+\frac{512}{3456}-\frac{2^{11}}{11\cdot 2^5\cdot 5!}+\cdots\right)$$

Using the first five terms, we estimate that the probability is approximately **0.4729.** By the alternating series test, we see that this estimate is accurate to within

$$\frac{1}{\sqrt{2\pi}}\frac{2^{13}}{13\cdot 2^6\cdot 6!}\approx 0.00546.$$

Analysis

If you are familiar with probability theory, you may know that the probability that a data value is within two standard deviations of the mean is approximately 95%. Here we calculated the probability that a data value is between the mean and two standard deviations above the mean, so the estimate should be around 47.5%. The estimate, combined with the bound on the accuracy, falls within this range.

Alternative method. If term-by-term integration is done instead, the result of evaluating the definite integral is

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}\int_0^2 z^{2n}\,dz &= \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)2^n n!}\left[2^{2n+1}-0^{2n+1}\right] = \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n 2^{n+1}}{n!(2n+1)}\\ &= \frac{1}{\sqrt{2\pi}}\left(\frac{2}{1\cdot 0!}-\frac{4}{3\cdot 1!}+\frac{8}{5\cdot 2!}-\frac{16}{7\cdot 3!}+\frac{32}{9\cdot 4!}-\frac{64}{11\cdot 5!}+\cdots\right)\\ &= \frac{1}{\sqrt{2\pi}}\left(2-\frac{4}{3}+\frac{4}{5}-\frac{8}{21}+\frac{4}{27}-\frac{8}{165}+\cdots\right)\end{aligned}$$

Exercise 10.4.7

Given a set of normally distributed standardized test scores with mean $\mu = 100$ and standard deviation $\sigma = 50$, use the first five terms of the Maclaurin series for $e^{-x^2/2}$ to estimate the probability that a randomly selected test score is between 100 and 150. Use the alternating series test to determine the accuracy of this estimate.

Hint

Evaluate $\int_0^1 e^{-z^2/2}\,dz$ using the first five terms of the Maclaurin series for $e^{-z^2/2}$.

Answer

The estimate is approximately 0.3414. This estimate is accurate to within 0.0000094.

Another application in which a non-elementary integral arises involves the period of a pendulum. The integral is

$$\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

.

An integral of this form is known as an **elliptic integral** of the first kind. Elliptic integrals originally arose when trying to calculate the arc length of an ellipse. We now show how to use power series to approximate this integral.

Example 10.4.8: Period of a Pendulum

The period of a pendulum is the time it takes for a pendulum to make one complete back-and-forth swing. For a pendulum with length L that makes a maximum angle θ_{max} with the vertical, its period T is given by

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

where g is the acceleration due to gravity and $k=\sin\left(\frac{\theta_{max}}{2}\right)$ (see Figure 10.4.3). (We note that this formula for the period arises from a non-linearized model of a pendulum. In some cases, for simplification, a linearized model is used and $\sin\theta$ is

approximated by θ.)

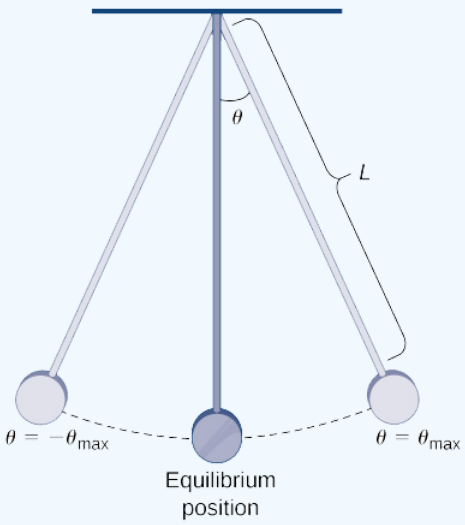

Figure 10.4.3: This pendulum has length L and makes a maximum angle θ_{max} with the vertical.

Use the binomial series

$$\frac{1}{\sqrt{1+x}} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} x^n$$

to estimate the period of this pendulum. Specifically, approximate the period of the pendulum if

a. you use only the first term in the binomial series, and
b. you use the first two terms in the binomial series.

Solution

We use the binomial series, replacing x with $-k^2 \sin^2 \theta$. Then we can write the period as

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \left(1 + \frac{1}{2}k^2 \sin^2 \theta + \frac{1 \cdot 3}{2!2^2} k^4 \sin^4 \theta + \cdots\right) d\theta.$$

a. Using just the first term in the integrand, the first-order estimate is

$$T \approx 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} d\theta = 2\pi\sqrt{\frac{L}{g}}.$$

If θ_{max} is small, then $k = \sin\left(\frac{\theta_{max}}{2}\right)$ is small. We claim that when k is small, this is a good estimate. To justify this claim, consider

$$\int_0^{\pi/2} \left(1 + \frac{1}{2}k^2 \sin^2 \theta + \frac{1 \cdot 3}{2!2^2} k^4 \sin^4 \theta + \cdots\right) d\theta.$$

Since $|\sin x| \leq 1$, this integral is bounded by

$$\int_0^{\pi/2} \left(\frac{1}{2}k^2 + \frac{1.3}{2!2^2} k^4 + \cdots\right) d\theta \; < \; \frac{\pi}{2}\left(\frac{1}{2}k^2 + \frac{1 \cdot 3}{2!2^2} k^4 + \cdots\right).$$

Furthermore, it can be shown that each coefficient on the right-hand side is less than 1 and, therefore, that this expression is bounded by

$$\frac{\pi k^2}{2}(1 + k^2 + k^4 + \cdots) = \frac{\pi k^2}{2} \cdot \frac{1}{1-k^2},$$

which is small for k small.

b. For larger values of θ_{max}, we can approximate T by using more terms in the integrand. By using the first two terms in the integral, we arrive at the estimate

$$T \approx 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left(1+\frac{1}{2}k^2\sin^2\theta\right)\,d\theta = 2\pi\sqrt{\frac{L}{g}}\left(1+\frac{k^2}{4}\right).$$

The applications of Taylor series in this section are intended to highlight their importance. In general, Taylor series are useful because they allow us to represent known functions using polynomials, thus providing us a tool for approximating function values and estimating complicated integrals. In addition, they allow us to define new functions as power series, thus providing us with a powerful tool for solving differential equations.

Key Concepts

- The binomial series is the Maclaurin series for $f(x) = (1+x)^r$. It converges for $|x| < 1$.
- Taylor series for functions can often be derived by algebraic operations with a known Taylor series or by differentiating or integrating a known Taylor series.
- Power series can be used to solve differential equations.
- Taylor series can be used to help approximate integrals that cannot be evaluated by other means.

Glossary

binomial series

the Maclaurin series for $f(x) = (1+x)^r$; it is given by $(1+x)^r = \sum_{n=0}^{\infty}\binom{r}{n}x^n = 1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots + \frac{r(r-1)\cdots(r-n+1)}{n!}x^n + \cdots$ for $|x| < 1$

non-elementary integral

an integral for which the antiderivative of the integrand cannot be expressed as an elementary function

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10.4E: Exercises for Section 10.4

In exercises 1 - 4, use appropriate substitutions to write down the Maclaurin series for the given binomial.

1) $(1-x)^{1/3}$

2) $(1+x^2)^{-1/3}$

Answer

$$(1+x^2)^{-1/3} = \sum_{n=0}^{\infty} \left(n^{-\frac{1}{3}}\right) x^{2n}$$

3) $(1-x)^{1.01}$

4) $(1-2x)^{2/3}$

Answer

$$(1-2x)^{2/3} = \sum_{n=0}^{\infty} (-1)^n 2^n \left(n^{\frac{2}{3}}\right) x^n$$

In exercises 5 - 12, use the substitution $(b+x)^r = (b+a)^r \left(1+\dfrac{x-a}{b+a}\right)^r$ in the binomial expansion to find the Taylor series of each function with the given center.

5) $\sqrt{x+2}$ at $a=0$

6) $\sqrt{x^2+2}$ at $a=0$

Answer

$$\sqrt{2+x^2} = \sum_{n=0}^{\infty} 2^{(1/2)-n} \left(n^{\frac{1}{2}}\right) x^{2n}; (|x^2|<2)$$

7) $\sqrt{x+2}$ at $a=1$

8) $\sqrt{2x-x^2}$ at $a=1$ (Hint: $2x-x^2 = 1-(x-1)^2$)

Answer

$$\sqrt{2x-x^2} = \sqrt{1-(x-1)^2} \text{ so } \sqrt{2x-x^2} = \sum_{n=0}^{\infty} (-1)^n \left(n^{\frac{1}{2}}\right) (x-1)^{2n}$$

9) $(x-8)^{1/3}$ at $a=9$

10) $\sqrt{x}$ at $a=4$

Answer

$$\sqrt{x} = 2\sqrt{1+\tfrac{x-4}{4}} \text{ so } \sqrt{x} = \sum_{n=0}^{\infty} 2^{1-2n} \left(n^{\frac{1}{2}}\right) (x-4)^n$$

11) $x^{1/3}$ at $a=27$

12) $\sqrt{x}$ at $x=9$

Answer

$$\sqrt{x} = \sum_{n=0}^{\infty} 3^{1-3n} \left(n^{\frac{1}{2}}\right) (x-9)^n$$

In exercises 13 - 14, use the binomial theorem to estimate each number, computing enough terms to obtain an estimate accurate to an error of at most 1/1000.

13) [T] $(15)^{1/4}$ using $(16-x)^{1/4}$

14) [T] $(1001)^{1/3}$ using $(1000+x)^{1/3}$

Answer

$10(1+\frac{x}{1000})^{1/3}=\sum_{n=0}^{\infty}10^{1-3n}(\frac{1}{n}^{\frac{1}{3}})x^n$. Using, for example, a fourth-degree estimate at $x=1$ gives $(1001)^{1/3}$ $\approx 10\left(1+\left(1^{\frac{1}{3}}\right)10^{-3}+\left(2^{\frac{1}{3}}\right)10^{-6}+\left(3^{\frac{1}{3}}\right)10^{-9}+\left(3^{\frac{1}{3}}\right)10^{-12}\right)=10\left(1+\frac{1}{3.10^3}-\frac{1}{9.10^6}+\frac{5}{81.10^9}-\frac{10}{243.10^{12}}\right)=10.00333222...$ whereas $(1001)^{1/3}=10.00332222839093....$ Two terms would suffice for three-digit accuracy.

In exercises 15 - 18, use the binomial approximation $\sqrt{1-x}\approx 1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}-\frac{5x^4}{128}-\frac{7x^5}{256}$ for $|x|<1$ to approximate each number. Compare this value to the value given by a scientific calculator.

15) [T] $\frac{1}{\sqrt{2}}$ using $x=\frac{1}{2}$ in $(1-x)^{1/2}$

16) [T] $\sqrt{5}=5\times\frac{1}{\sqrt{5}}$ using $x=\frac{4}{5}$ in $(1-x)^{1/2}$

Answer

The approximation is 2.3152; the CAS value is $2.23\ldots$.

17) [T] $\sqrt{3}=\frac{3}{\sqrt{3}}$ using $x=\frac{2}{3}$ in $(1-x)^{1/2}$

18) [T] $\sqrt{6}$ using $x=\frac{5}{6}$ in $(1-x)^{1/2}$

Answer

The approximation is $2.583\ldots$; the CAS value is $2.449\ldots$.

19) Integrate the binomial approximation of $\sqrt{1-x}$ to find an approximation of $\int_0^x\sqrt{1-t}\,dt$.

20) [T] Recall that the graph of $\sqrt{1-x^2}$ is an upper semicircle of radius 1. Integrate the binomial approximation of $\sqrt{1-x^2}$ up to order 8 from $x=-1$ to $x=1$ to estimate $\frac{\pi}{2}$.

Answer

$\sqrt{1-x^2}=1-\frac{x^2}{2}-\frac{x^4}{8}-\frac{x^6}{16}-\frac{5x^8}{128}+\cdots$. Thus $\int_{-1}^{1}\sqrt{1-x^2}\,dx=\left[x-\frac{x^3}{6}-\frac{x^5}{40}-\frac{x^7}{7\cdot 16}-\frac{5x^9}{9\cdot 128}+\cdots\right]\Big|_{-1}^{1}\approx 2-\frac{1}{3}-\frac{1}{20}-\frac{1}{56}-\frac{10}{9\cdot 128}+error=1.590...$ whereas $\frac{\pi}{2}=1.570...$

In exercises 21 - 24, use the expansion $(1+x)^{1/3}=1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3-\frac{10}{243}x^4+\cdots$ to write the first five terms (not necessarily a quartic polynomial) of each expression.

21) $(1+4x)^{1/3};\ a=0$

22) $(1+4x)^{4/3};\ a=0$

Answer

$(1+x)^{4/3}=(1+x)(1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3-\frac{10}{243}x^4+\cdots)=1+\frac{4x}{3}+\frac{2x^2}{9}-\frac{4x^3}{81}+\frac{5x^4}{243}+\cdots$

23) $(3+2x)^{1/3};\ a=-1$

24) $(x^2+6x+10)^{1/3};\ a=-3$

Answer

$(1+(x+3)^2)^{1/3}=1+\frac{1}{3}(x+3)^2-\frac{1}{9}(x+3)^4+\frac{5}{81}(x+3)^6-\frac{10}{243}(x+3)^8+\cdots$

25) Use $(1+x)^{1/3}=1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3-\frac{10}{243}x^4+\cdots$ with $x=1$ to approximate $2^{1/3}$.

26) Use the approximation $(1-x)^{2/3}=1-\frac{2x}{3}-\frac{x^2}{9}-\frac{4x^3}{81}-\frac{7x^4}{243}-\frac{14x^5}{729}+\cdots$ for $|x|<1$ to approximate $2^{1/3}=2.2^{-2/3}$.

Answer

Twice the approximation is $1.260\ldots$ whereas $2^{1/3}=1.2599....$

27) Find the 25th derivative of $f(x) = (1 + x^2)^{13}$ at $x = 0$.

28) Find the 99th derivative of $f(x) = (1 + x^4)^{25}$.

Answer

$f^{(99)}(0) = 0$

In exercises 29 - 36, find the Maclaurin series of each function.

29) $f(x) = xe^{2x}$

30) $f(x) = 2^x$

Answer

$$\sum_{n=0}^{\infty} \frac{(\ln(2)x)^n}{n!}$$

31) $f(x) = \dfrac{\sin x}{x}$

32) $f(x) = \dfrac{\sin(\sqrt{x})}{\sqrt{x}}, (x > 0),$

Answer

$$\text{For } x > 0,\ \sin(\sqrt{x}) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{(2n+1)/2}}{\sqrt{x}(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n+1)!}.$$

33) $f(x) = \sin(x^2)$

34) $f(x) = e^{x^3}$

Answer

$$e^{x^3} = \sum_{n=0}^{\infty} \frac{x^{3n}}{n!}$$

35) $f(x) = \cos^2 x$ using the identity $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$

36) $f(x) = \sin^2 x$ using the identity $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$

Answer

$$\sin^2 x = -\sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k-1} x^{2k}}{(2k)!}$$

In exercises 37 - 44, find the Maclaurin series of $F(x) = \displaystyle\int_0^x f(t)\,dt$ by integrating the Maclaurin series of f term by term. If f is not strictly defined at zero, you may substitute the value of the Maclaurin series at zero.

37) $F(x) = \displaystyle\int_0^x e^{-t^2}\,dt;\ f(t) = e^{-t^2} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{n!}$

38) $F(x) = \tan^{-1} x;\ f(t) = \dfrac{1}{1+t^2} = \displaystyle\sum_{n=0}^{\infty} (-1)^n t^{2n}$

Answer

$$\tan^{-1} x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$$

39) $F(x) = \tanh^{-1} x;\ f(t) = \dfrac{1}{1-t^2} = \displaystyle\sum_{n=0}^{\infty} t^{2n}$

40) $F(x) = \sin^{-1} x;\ f(t) = \dfrac{1}{\sqrt{1-t^2}} = \displaystyle\sum_{k=0}^{\infty} \left(k^{\frac{1}{2}}\right) \frac{t^{2k}}{k!}$

Answer

$$\sin^{-1} x = \sum_{n=0}^{\infty} \left(n^{\frac{1}{2}}\right) \frac{x^{2n+1}}{(2n+1)n!}$$

41) $F(x) = \displaystyle\int_0^x \frac{\sin t}{t}\, dt;\ f(t) = \frac{\sin t}{t} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n+1)!}$

42) $F(x) = \displaystyle\int_0^x \cos\left(\sqrt{t}\right)\, dt;\ f(t) = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n)!}$

Answer

$$F(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{(n+1)(2n)!}$$

43) $F(x) = \displaystyle\int_0^x \frac{1-\cos t}{t^2}\, dt;\ f(t) = \frac{1-\cos t}{t^2} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n+2)!}$

44) $F(x) = \displaystyle\int_0^x \frac{\ln(1+t)}{t}\, dt;\ f(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n+1}$

Answer

$$F(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n^2}$$

In exercises 45 - 52, compute at least the first three nonzero terms (not necessarily a quadratic polynomial) of the Maclaurin series of f.

45) $f(x) = \sin\left(x + \frac{\pi}{4}\right) = \sin x \cos\left(\frac{\pi}{4}\right) + \cos x \sin\left(\frac{\pi}{4}\right)$

46) $f(x) = \tan x$

Answer

$$x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

47) $f(x) = \ln(\cos x)$

48) $f(x) = e^x \cos x$

Answer

$$1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots$$

49) $f(x) = e^{\sin x}$

50) $f(x) = \sec^2 x$

Answer

$$1 + x^2 + \frac{2x^4}{3} + \frac{17x^6}{45} + \cdots$$

51) $f(x) = \tanh x$

52) $f(x) = \dfrac{\tan\sqrt{x}}{\sqrt{x}}$ (see expansion for $\tan x$)

Answer

Using the expansion for $\tan x$ gives $1 + \dfrac{x}{3} + \dfrac{2x^2}{15}$.

In exercises 53 - 56, find the radius of convergence of the Maclaurin series of each function.

53) $\ln(1+x)$

54) $\dfrac{1}{1+x^2}$

Answer

$\dfrac{1}{1+x^2} = \displaystyle\sum_{n=0}^{\infty} (-1)^n x^{2n}$ so $R = 1$ by the ratio test.

55) $\tan^{-1} x$

56) $\ln(1+x^2)$

Answer

$\ln(1+x^2) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^{2n}$ so $R = 1$ by the ratio test.

57) Find the Maclaurin series of $\sinh x = \dfrac{e^x - e^{-x}}{2}$.

58) Find the Maclaurin series of $\cosh x = \dfrac{e^x + e^{-x}}{2}$.

Answer

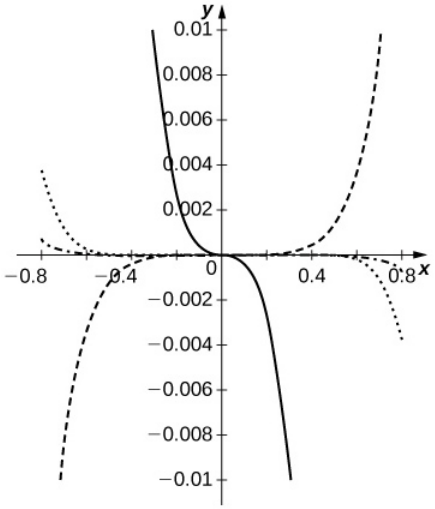
Add series of e^x and e^{-x} term by term. Odd terms cancel and $\cosh x = \displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$.

59) Differentiate term by term the Maclaurin series of $\sinh x$ and compare the result with the Maclaurin series of $\cosh x$.

60) [T] Let $S_n(x) = \displaystyle\sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$ and $C_n(x) = \displaystyle\sum_{n=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!}$ denote the respective Maclaurin polynomials of degree $2n+1$ of $\sin x$ and degree $2n$ of $\cos x$. Plot the errors $\dfrac{S_n(x)}{C_n(x)} - \tan x$ for $n = 1, .., 5$ and compare them to $x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \dfrac{17x^7}{315} - \tan x$ on $(-\frac{\pi}{4}, \frac{\pi}{4})$.

Answer

The ratio $\dfrac{S_n(x)}{C_n(x)}$ approximates $\tan x$ better than does $p_7(x) = x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \dfrac{17x^7}{315}$ for $N \geq 3$. The dashed curves are $\dfrac{S_n}{C_n} - \tan x$ for $n = 1,\ 2$. The dotted curve corresponds to $n = 3$, and the dash-dotted curve corresponds to $n = 4$. The solid curve is $p_7 - \tan x$.



61) Use the identity $2 \sin x \cos x = \sin(2x)$ to find the power series expansion of $\sin^2 x$ at $x = 0$. (Hint: Integrate the Maclaurin series of $\sin(2x)$ term by term.)

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62) If $y=\sum_{n=0}^{\infty} a_n x^n$, find the power series expansions of xy' and x^2y''.

Answer

By the term-by-term differentiation theorem, $y'=\sum_{n=1}^{\infty} na_n x^{n-1}$ so $y'=\sum_{n=1}^{\infty} na_n x^{n-1} xy'=\sum_{n=1}^{\infty} na_n x^n$, whereas $y'=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$ so $xy''=\sum_{n=2}^{\infty} n(n-1)a_n x^n$.

63) [T] Suppose that $y=\sum_{k=0}^{\infty} a^k x^k$ satisfies $y'=-2xy$ and $y(0)=0$. Show that $a_{2k+1}=0$ for all k and that $a_{2k+2}=\frac{-a_{2k}}{k+1}$. Plot the partial sum S_{20} of y on the interval $[-4,4]$.

64) [T] Suppose that a set of standardized test scores is normally distributed with mean $\mu=100$ and standard deviation $\sigma=10$. Set up an integral that represents the probability that a test score will be between 90 and 110 and use the integral of the degree 10 Maclaurin polynomial of $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ to estimate this probability.

Answer

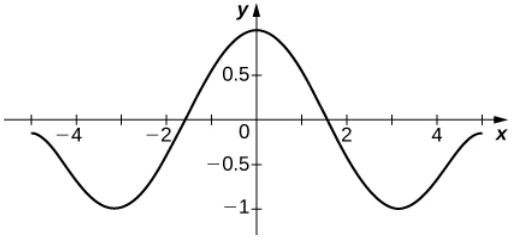
The probability is $p=\frac{1}{\sqrt{2\pi}}\int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} e^{-x^2/2}\,dx$ where $a=90$ and $b=100$, that is, $p=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-x^2/2}\,dx$ $=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}\sum_{n=0}^{5}(-1)^n\frac{x^{2n}}{2^n n!}\,dx=\frac{2}{\sqrt{2\pi}}\sum_{n=0}^{5}(-1)^n\frac{1}{(2n+1)2^n n!}\approx 0.6827.$

65) [T] Suppose that a set of standardized test scores is normally distributed with mean $\mu=100$ and standard deviation $\sigma=10$. Set up an integral that represents the probability that a test score will be between 70 and 130 and use the integral of the degree 50 Maclaurin polynomial of $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ to estimate this probability.

66) [T] Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges to a function $f(x)$ such that $f(0)=1,\ f'(0)=0$, and $f''(x)=-f(x)$. Find a formula for a_n and plot the partial sum S_N for $N=20$ on $[-5,5]$.

Answer

As in the previous problem one obtains $a_n=0$ if n is odd and $a_n=-(n+2)(n+1)a_{n+2}$ if n is even, so $a_0=1$ leads to $a_{2n}=\frac{(-1)^n}{(2n)!}$.



67) [T] Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges to a function $f(x)$ such that $f(0)=0,\ f'(0)=1$, and $f''(x)=-f(x)$. Find a formula for an and plot the partial sum S_N for $N=10$ on $[-5,5]$.

68) Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges to a function y such that $y''-y'+y=0$ where $y(0)=1$ and $y'(0)=0$. Find a formula that relates $a_{n+2},\ a_{n+1}$, and an and compute $a_0,\ldots,a_5$.

Answer

$y''=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$ and $y'=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$ so $y''-y'+y=0$ implies that $(n+2)(n+1)a_{n+2}-(n+1)a_{n+1}+a_n$ $=0$ or $a_n=\frac{a_{n-1}}{n}-\frac{a_{n-2}}{n(n-1)}$ for all $n\cdot y(0)=a_0=1$ and $y'(0)=a_1=0$, so $a_2=\frac{1}{2},\ a_3=\frac{1}{6}$, $a_4=0$, and $a_5=-\frac{1}{120}$.

69) Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges to a function y such that $y'' - y' + y = 0$ where $y(0) = 0$ and $y'(0) = 1$. Find a formula that relates a_{n+2}, a_{n+1}, and an and compute $a_1, ..., a_5$.

The error in approximating the integral $\int_a^b f(t)\,dt$ by that of a Taylor approximation $\int_a^b Pn(t)\,dt$ is at most $\int_a^b R_n(t)\,dt$. In exercises 70 - 71, the Taylor remainder estimate $R_n \le \frac{M}{(n+1)!}|x-a|^{n+1}$ guarantees that the integral of the Taylor polynomial of the given order approximates the integral of f with an error less than $\frac{1}{10}$.

a. Evaluate the integral of the appropriate Taylor polynomial and verify that it approximates the CAS value with an error less than $\frac{1}{100}$.

b. Compare the accuracy of the polynomial integral estimate with the remainder estimate.

70) [T] $\int_0^{\pi} \frac{\sin t}{t}\,dt; \quad P_s = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!}$ (You may assume that the absolute value of the ninth derivative of $\frac{\sin t}{t}$ is bounded by 0.1.)

Answer

a. (Proof)

b. We have $R_s \le \frac{0.1}{(9)!}\pi^9 \approx 0.0082 < 0.01$. We have $\int_0^{\pi}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!}\right)dx = \pi - \frac{\pi^3}{3\cdot 3!} + \frac{\pi^5}{5\cdot 5!} - \frac{\pi^7}{7\cdot 7!} + \frac{\pi^9}{9\cdot 9!}$ $= 1.852...$, whereas $\int_0^{\pi} \frac{\sin t}{t}\,dt = 1.85194...$, so the actual error is approximately 0.00006.

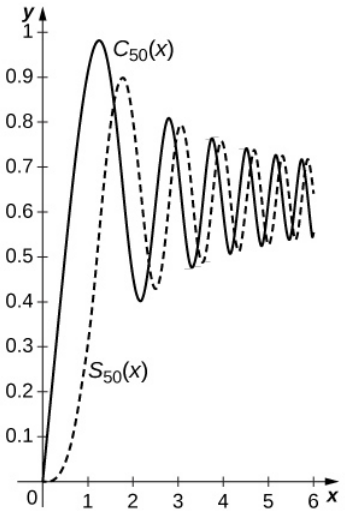
71) [T] $\int_0^2 e^{-x^2}\,dx;\ p_{11} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \cdots - \frac{x^{22}}{11!}$ (You may assume that the absolute value of the 23rd derivative of e^{-x^2} is less than 2×10^{14}.)

The following exercises (72-73) deal with *Fresnel integrals*.

72) The Fresnel integrals are defined by $C(x) = \int_0^x \cos(t^2)\,dt$ and $S(x) = \int_0^x \sin(t^2)\,dt$. Compute the power series of $C(x)$ and $S(x)$ and plot the sums $C_N(x)$ and $S_N(x)$ of the first $N = 50$ nonzero terms on $[0, 2\pi]$.

Answer

Since $\cos(t^2) = \sum_{n=0}^{\infty}(-1)^n \frac{t^{4n}}{(2n)!}$ and $\sin(t^2) = \sum_{n=0}^{\infty}(-1)^n \frac{t^{4n+2}}{(2n+1)!}$, one has $S(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+3}}{(4n+3)(2n+1)!}$ and $C(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+1}}{(4n+1)(2n)!}$. The sums of the first 50 nonzero terms are plotted below with $C_{50}(x)$ the solid curve and $S_{50}(x)$ the dashed curve.



73) [T] The Fresnel integrals are used in design applications for roadways and railways and other applications because of the curvature properties of the curve with coordinates $(C(t), S(t))$. Plot the curve (C_{50}, S_{50}) for $0 \le t \le 2\pi$, the coordinates of which were computed in the previous exercise.

74) Estimate $\int_0^{1/4} \sqrt{x - x^2}\, dx$ by approximating $\sqrt{1-x}$ using the binomial approximation $1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{2128} - \frac{7x^5}{256}$.

Answer

$\int_0^{1/4} \sqrt{x}\left(1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \frac{7x^5}{256}\right) dx = \frac{2}{3}2^{-3} - \frac{1}{2}\frac{2}{5}2^{-5} - \frac{1}{8}\frac{2}{7}2^{-7} - \frac{1}{16}\frac{2}{9}2^{-9} - \frac{5}{128}\frac{2}{11}2^{-11} - \frac{7}{256}\frac{2}{13}2^{-13}$
$= 0.0767732...$ whereas $\int_0^{1/4} \sqrt{x - x^2}\, dx = 0.076773.$

75) [T] Use Newton's approximation of the binomial $\sqrt{1-x^2}$ to approximate π as follows. The circle centered at $(\frac{1}{2}, 0)$ with radius $\frac{1}{2}$ has upper semicircle $y = \sqrt{x}\sqrt{1-x}$. The sector of this circle bounded by the x-axis between $x = 0$ and $x = \frac{1}{2}$ and by the line joining $(\frac{1}{4}, \frac{\sqrt{3}}{4})$ corresponds to $\frac{1}{6}$ of the circle and has area $\frac{\pi}{24}$. This sector is the union of a right triangle with height $\frac{\sqrt{3}}{4}$ and base $\frac{1}{4}$ and the region below the graph between $x = 0$ and $x = \frac{1}{4}$. To find the area of this region you can write $y = \sqrt{x}\sqrt{1-x} = \sqrt{x}$ $\times$ (binomial expansion of$\sqrt{1-x}$) and integrate term by term. Use this approach with the binomial approximation from the previous exercise to estimate π.

76) Use the approximation $T \approx 2\pi\sqrt{\frac{L}{g}}(1 + \frac{k^2}{4})$ to approximate the period of a pendulum having length 10 meters and maximum angle $\theta_{max} = \frac{\pi}{6}$ where $k = \sin\left(\frac{\theta_{max}}{2}\right)$. Compare this with the small angle estimate $T \approx 2\pi\sqrt{\frac{L}{g}}$.

Answer

$T \approx 2\pi\sqrt{\frac{10}{9.8}}\left(1 + \frac{\sin^2(\theta/12)}{4}\right) \approx 6.453$ seconds. The small angle estimate is $T \approx 2\pi\sqrt{\frac{10}{9.8}} \approx 6.347$. The relative error is around 2 percent.

77) Suppose that a pendulum is to have a period of 2 seconds and a maximum angle of $\theta_{max} = \frac{\pi}{6}$. Use $T \approx 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{k^2}{4}\right)$ to approximate the desired length of the pendulum. What length is predicted by the small angle estimate $T \approx 2\pi\sqrt{\frac{L}{g}}$?

78) Evaluate $\int_0^{\pi/2} \sin^4\theta\, d\theta$ in the approximation $T = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left(1 + \frac{1}{2}k^2\sin^2\theta + \frac{3}{8}k^4\sin^4\theta + \cdots\right) d\theta$ to obtain an improved estimate for T.

Answer

$\int_0^{\pi/2} \sin^4\theta\, d\theta = \frac{3\pi}{16}$. Hence $T \approx 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{k^2}{4} + \frac{9}{256}k^4\right)$.

79) [T] An equivalent formula for the period of a pendulum with amplitude θ_{max} is $T(\theta_{max}) = 2\sqrt{2}\sqrt{\frac{L}{g}}\int_0^{\theta_{max}} \frac{d\theta}{\sqrt{\cos\theta - \cos(\theta_{max})}}$ where L is the pendulum length and g is the gravitational acceleration constant. When $\theta_{max} = \frac{\pi}{3}$ we get $\frac{1}{\sqrt{\cos t - 1/2}}$ $\approx \sqrt{2}\left(1 + \frac{t^2}{2} + \frac{t^4}{3} + \frac{181t^6}{720}\right)$. Integrate this approximation to estimate $T(\frac{\pi}{3})$ in terms of L and g. Assuming $g = 9.806$ meters per second squared, find an approximate length L such that $T(\frac{\pi}{3}) = 2$ seconds.

Contributors

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10.R: Chapter 10 Review Exercises

True or False? In exercises 1 - 4, justify your answer with a proof or a counterexample.

1) If the radius of convergence for a power series $\sum_{n=0}^{\infty} a_n x^n$ is 5, then the radius of convergence for the series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ is also 5.

Answer

True

2) Power series can be used to show that the derivative of e^x is e^x. (Hint: Recall that $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$.)

3) For small values of x, $\sin x \approx x$.

Answer

True

4) The radius of convergence for the Maclaurin series of $f(x) = 3^x$ is 3.

In exercises 5 - 8, find the radius of convergence and the interval of convergence for the given series.

5) $\sum_{n=0}^{\infty} n^2 (x-1)^n$

Answer

ROC: 1; IOC: $(0, 2)$

6) $\sum_{n=0}^{\infty} \frac{x^n}{n^n}$

7) $\sum_{n=0}^{\infty} \frac{3n x^n}{12^n}$

Answer

ROC: 12; IOC: $(-16, 8)$

8) $\sum_{n=0}^{\infty} \frac{2^n}{e^n} (x-e)^n$

In exercises 9 - 10, find the power series representation for the given function. Determine the radius of convergence and the interval of convergence for that series.

9) $f(x) = \frac{x^2}{x+3}$

Answer

$\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} x^n$; ROC: 3; IOC: $(-3, 3)$

10) $f(x) = \frac{8x+2}{2x^2 - 3x + 1}$

In exercises 11 - 12, find the power series for the given function using term-by-term differentiation or integration.

11) $f(x) = \tan^{-1}(2x)$

Answer

integration: $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (2x)^{2n+1}$

12) $f(x) = \frac{x}{(2+x^2)^2}$

In exercises 13 - 14, evaluate the Taylor series expansion of degree four for the given function at the specified point. What is the error in the approximation?

13) $f(x) = x^3 - 2x^2 + 4, \quad a = -3$

Answer

$p_4(x) = (x+3)^3 - 11(x+3)^2 + 39(x+3) - 41$; exact

14) $f(x) = e^{1/(4x)}, \quad a = 4$

In exercises 15 - 16, find the Maclaurin series for the given function.

15) $f(x) = \cos(3x)$

Answer

$$\sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n}}{2n!}$$

16) $f(x) = \ln(x+1)$

In exercises 17 - 18, find the Taylor series at the given value.

17) $f(x) = \sin x, \quad a = \frac{\pi}{2}$

Answer

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(x - \frac{\pi}{2}\right)^{2n}$$

18) $f(x) = \dfrac{3}{x}, \quad a = 1$

In exercises 19 - 20, find the Maclaurin series for the given function.

19) $f(x) = e^{-x^2} - 1$

Answer

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} x^{2n}$$

20) $f(x) = \cos x - x \sin x$

In exercises 21 - 23, find the Maclaurin series for $F(x) = \int_0^x f(t)\, dt$ by integrating the Maclaurin series of $f(x)$ term by term.

21) $f(x) = \dfrac{\sin x}{x}$

Answer

$$F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!} x^{2n+1}$$

22) $f(x) = 1 - e^x$

23) Use power series to prove **Euler's formula:** $e^{ix} = cosx + isinx$

Answer

Answers may vary.

Exercises 24 - 26 consider problems of *annuity payments*.

24) For annuities with a present value of $\$1$ million, calculate the annual payouts given over 25 years assuming interest rates of $1\%, 5\%$, and 10%.

25) A lottery winner has an annuity that has a present value of $\$10$ million. What interest rate would they need to live on perpetual annual payments of $\$250,000$?

Answer

2.5%

26) Calculate the necessary present value of an annuity in order to support annual payouts of $\$15,000$ given over 25 years assuming interest rates of $1\%, 5\%$,and 10%.

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CHAPTER OVERVIEW

11: Parametric Equations and Polar Coordinates

Parametric equations define a group of quantities as functions of one or more independent variables called parameters. Parametric equations are commonly used to express the coordinates of the points that make up a geometric object such as a curve or surface, in which case the equations are collectively called a parametric representation or parameterization. The polar coordinate system is a two-dimensional coordinate system in which each point on a plane is determined by a distance from a reference point and an angle from a reference direction. The reference point (analogous to the origin of a Cartesian system) is called the pole, and the ray from the pole in the reference direction is the polar axis. The distance from the pole is called the radial coordinate or radius, and the angle is called the angular coordinate, polar angle, or azimuth

11.0: Prelude to Parametric Equations and Polar Coordinates

11.1: Parametric Equations

11.1E: Exercises for Section 11.1

11.2: Calculus of Parametric Curves

11.2E: Exercises for Section 11.2

11.3: Polar Coordinates

11.3E: Exercises for Section 11.3

11.4: Area and Arc Length in Polar Coordinates

11.4E: Exercises for Section 11.4

11.5: Conic Sections

11.5E: Exercises for Section 11.5

11.R: Chapter 11 Review Exercises

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11.0: Prelude to Parametric Equations and Polar Coordinates

The **chambered nautilus** is a fascinating creature. This animal feeds on hermit crabs, fish, and other crustaceans. It has a hard outer shell with many chambers connected in a spiral fashion, and it can retract into its shell to avoid predators. When part of the shell is cut away, a perfect spiral is revealed, with chambers inside that are somewhat similar to growth rings in a tree.

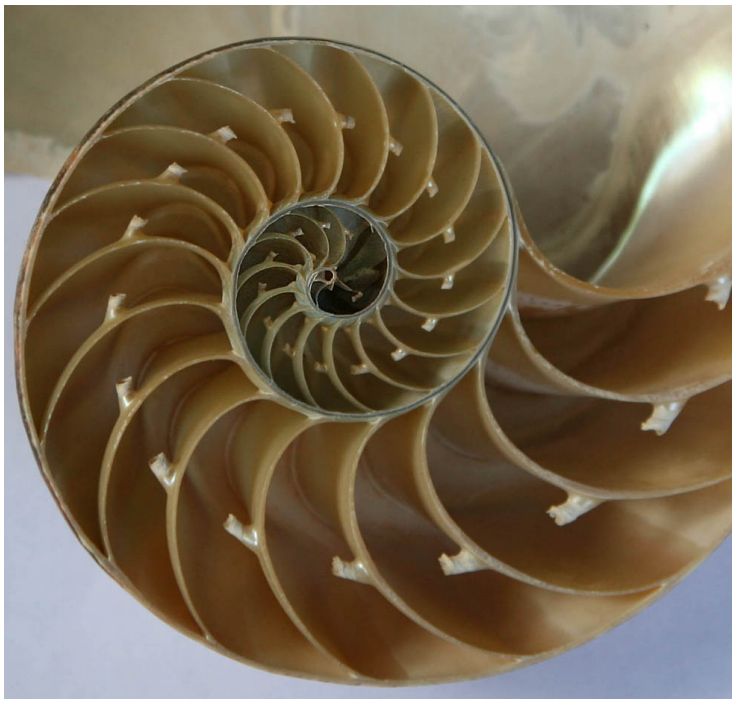

Figure 11.0.1: The chambered nautilus is a marine animal that lives in the tropical Pacific Ocean. Scientists think they have existed mostly unchanged for about 500 million years.(credit: modification of work by Jitze Couperus, Flickr)

The mathematical function that describes a spiral can be expressed using rectangular (or Cartesian) coordinates. However, if we change our coordinate system to something that works a bit better with circular patterns, the function becomes much simpler to describe. The polar coordinate system is well suited for describing curves of this type. How can we use this coordinate system to describe spirals and other radial figures?

In this chapter we also study parametric equations, which give us a convenient way to describe curves, or to study the position of a particle or object in two dimensions as a function of time. We will use parametric equations and polar coordinates for describing many topics later in this text.

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11.1: Parametric Equations

Learning Objectives

- Plot a curve described by parametric equations.
- Convert the parametric equations of a curve into the form $y = f(x)$.
- Recognize the parametric equations of basic curves, such as a line and a circle.
- Recognize the parametric equations of a cycloid.

In this section we examine parametric equations and their graphs. In the two-dimensional coordinate system, parametric equations are useful for describing curves that are not necessarily functions. The parameter is an independent variable that both x and y depend on, and as the parameter increases, the values of x and y trace out a path along a plane curve. For example, if the parameter is t (a common choice), then t might represent time. Then x and y are defined as functions of time, and $(x(t), y(t))$ can describe the position in the plane of a given object as it moves along a curved path.

Parametric Equations and Their Graphs

Consider the orbit of Earth around the Sun. Our year lasts approximately 365.25 days, but for this discussion we will use 365 days. On January 1 of each year, the physical location of Earth with respect to the Sun is nearly the same, except for leap years, when the lag introduced by the extra $\frac{1}{4}$ day of orbiting time is built into the calendar. We call January 1 "day 1" of the year. Then, for example, day 31 is January 31, day 59 is February 28, and so on.

The number of the day in a year can be considered a variable that determines Earth's position in its orbit. As Earth revolves around the Sun, its physical location changes relative to the Sun. After one full year, we are back where we started, and a new year begins. According to Kepler's laws of planetary motion, the shape of the orbit is elliptical, with the Sun at one focus of the ellipse. We study this idea in more detail in Conic Sections.

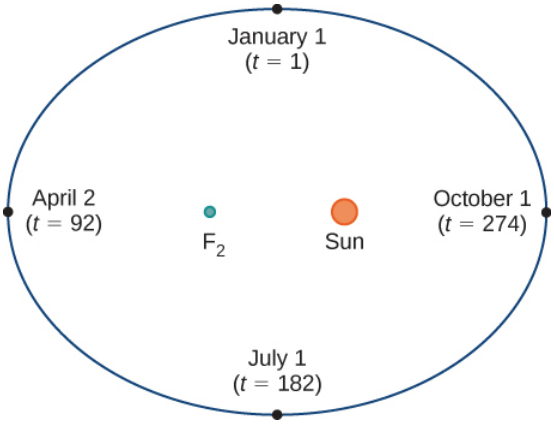


Figure 11.1.1: Earth's orbit around the Sun in one year.

Figure 11.1.1 depicts **Earth's orbit** around the Sun during one year. The point labeled F_2 is one of the foci of the ellipse; the other focus is occupied by the Sun. If we superimpose coordinate axes over this graph, then we can assign ordered pairs to each point on the ellipse (Figure 11.1.2). Then each x value on the graph is a value of position as a function of time, and each y value is also a value of position as a function of time. Therefore, each point on the graph corresponds to a value of Earth's position as a function of time.

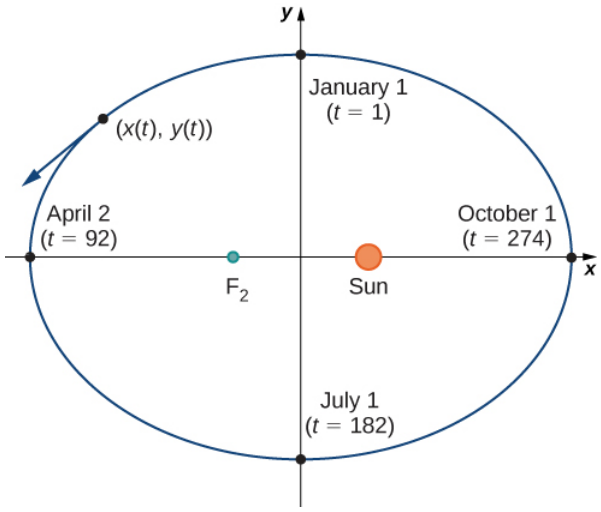


Figure 11.1.2: Coordinate axes superimposed on the orbit of Earth.

We can determine the functions for $x(t)$ and $y(t)$, thereby parameterizing the orbit of Earth around the Sun. The variable t is called an independent parameter and, in this context, represents time relative to the beginning of each year.

A curve in the (x, y) plane can be represented parametrically. The equations that are used to define the curve are called *parametric equations.*

Definition: Parametric Equations

If x and y are continuous functions of t on an interval I, then the equations

$$x = x(t)$$

and

$$y = y(t)$$

are called parametric equations and t is called the parameter. The set of points (x, y) obtained as t varies over the interval I is called the graph of the parametric equations. The graph of parametric equations is called a *parametric curve* or plane curve, and is denoted by C.

Notice in this definition that x and y are used in two ways. The first is as functions of the independent variable t. As t varies over the interval I, the functions $x(t)$ and $y(t)$ generate a set of ordered pairs (x, y). This set of ordered pairs generates the graph of the parametric equations. In this second usage, to designate the ordered pairs, x and y are variables. It is important to distinguish the variables x and y from the functions $x(t)$ and $y(t)$.

Example 11.1.1: Graphing a Parametrically Defined Curve

Sketch the curves described by the following parametric equations:

a. $x(t) = t - 1, \quad y(t) = 2t + 4, \quad \text{for } -3 \le t \le 2$
b. $x(t) = t^2 - 3, \quad y(t) = 2t + 1, \quad \text{for } -2 \le t \le 3$
c. $x(t) = 4\cos t, \quad y(t) = 4\sin t, \quad \text{for } 0 \le t \le 2\pi$

Solution a

To create a graph of this curve, first set up a table of values. Since the independent variable in both $x(t)$ and $y(t)$ is t, let t appear in the first column. Then $x(t)$ and $y(t)$ will appear in the second and third columns of the table.

t	$x(t)$	$y(t)$
−3	−4	−2
−2	−3	0

t	$x(t)$	$y(t)$
−1	−2	2
0	−1	4
1	0	6
2	1	8

The second and third columns in this table provide a set of points to be plotted. The graph of these points appears in Figure 11.1.3. The arrows on the graph indicate the **orientation** of the graph, that is, the direction that a point moves on the graph as t varies from −3 to 2.

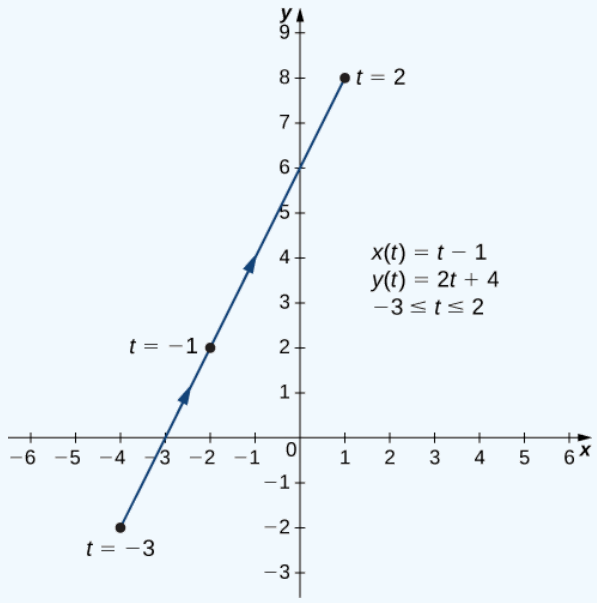


Figure 11.1.3: Graph of the plane curve described by the parametric equations in part a.

Solution b

To create a graph of this curve, again set up a table of values.

t	$x(t)$	$y(t)$
−2	1	−3
−1	−2	−1
0	−3	1
1	−2	3
2	1	5
3	6	7

The second and third columns in this table give a set of points to be plotted (Figure 11.1.4). The first point on the graph (corresponding to $t = -2$) has coordinates $(1, -3)$, and the last point (corresponding to $t = 3$) has coordinates $(6, 7)$. As t progresses from -2 to 3, the point on the curve travels along a parabola. The direction the point moves is again called the orientation and is indicated on the graph.

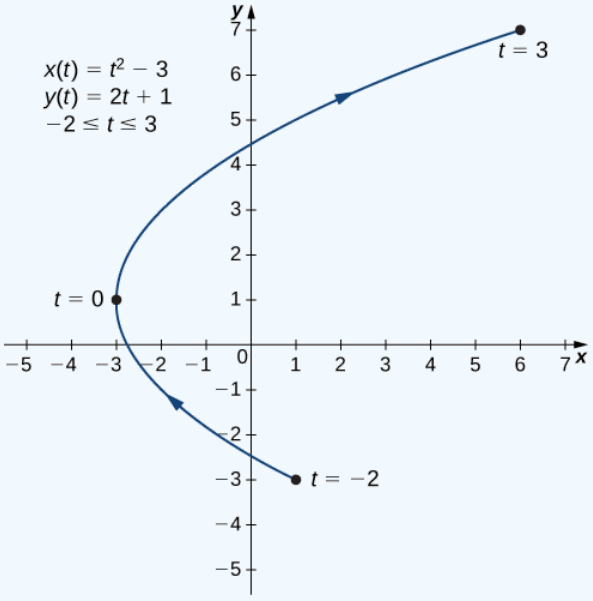


Figure 11.1.4: Graph of the plane curve described by the parametric equations in part b.

Solution c

In this case, use multiples of $\pi/6$ for t and create another table of values:

t	$x(t)$	$y(t)$	t	$x(t)$	$y(t)$
0	4	0	$\frac{7\pi}{6}$	$-2\sqrt{3} \approx -3.5$	-2
$\frac{\pi}{6}$	$2\sqrt{3} \approx 3.5$	2	$\frac{4\pi}{3}$	-2	$-2\sqrt{3} \approx -3.5$
$\frac{\pi}{3}$	2	$2\sqrt{3} \approx 3.5$	$\frac{3\pi}{2}$	0	-4
$\frac{\pi}{2}$	0	4	$\frac{5\pi}{3}$	2	$-2\sqrt{3} \approx -3.5$
$\frac{2\pi}{3}$	-2	$2\sqrt{3} \approx 3.5$	$\frac{11\pi}{6}$	$2\sqrt{3} \approx 3.5$	-2
$\frac{5\pi}{6}$	$-2\sqrt{3} \approx -3.5$	2	2π	4	0
π	-4	0			

The graph of this plane curve appears in the following graph.

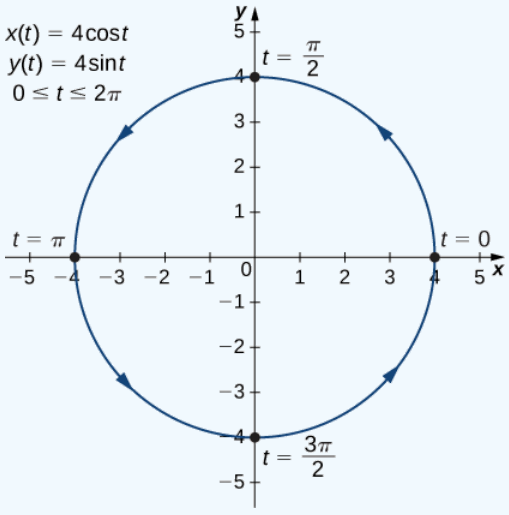


Figure 11.1.5: Graph of the plane curve described by the parametric equations in part c.

This is the graph of a circle with radius 4 centered at the origin, with a counterclockwise orientation. The starting point and ending points of the curve both have coordinates $(4, 0)$.

? Exercise 11.1.1

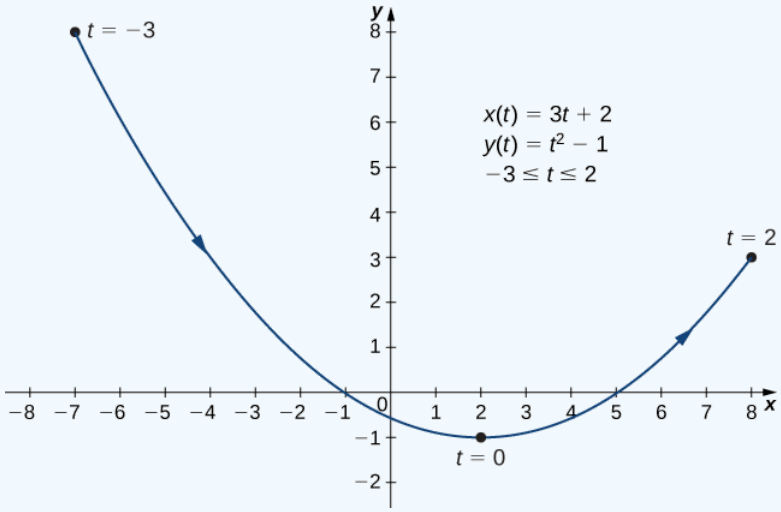
Sketch the curve described by the parametric equations

$$x(t) = 3t + 2, \quad y(t) = t^2 - 1, \quad \text{for } -3 \le t \le 2.$$

Hint

Make a table of values for $x(t)$ and $y(t)$ using t values from -3 to 2.

Answer



Eliminating the Parameter

To better understand the graph of a curve represented parametrically, it is useful to rewrite the two equations as a single equation relating the variables x and y. Then we can apply any previous knowledge of equations of curves in the plane to identify the curve. For example, the equations describing the plane curve in Example $11.1.1b$ are

$$x(t) = t^2 - 3 \tag{11.1.1}$$
$$y(t) = 2t + 1 \tag{11.1.2}$$

over the region $-2 \le t \le 3$.

Solving Equation 11.1.2 for t gives

$$t = \frac{y-1}{2}.$$

This can be substituted into Equation 11.1.1:

$$x = \left(\frac{y-1}{2}\right)^2 - 3 \tag{11.1.3}$$
$$= \frac{y^2 - 2y + 1}{4} - 3 \tag{11.1.4}$$
$$= \frac{y^2 - 2y - 11}{4}. \tag{11.1.5}$$

Equation 11.1.5 describes x as a function of y. These steps give an example of eliminating the parameter. The graph of this function is a parabola opening to the right (Figure $11.1.4$). Recall that the plane curve started at $(1, -3)$ and ended at $(6, 7)$. These terminations were due to the restriction on the parameter t.

Example 11.1.2: Eliminating the Parameter

Eliminate the parameter for each of the plane curves described by the following parametric equations and describe the resulting graph.

a. $x(t) = \sqrt{2t+4}, \quad y(t) = 2t+1, \quad \text{for } -2 \le t \le 6$
b. $x(t) = 4\cos t, \quad y(t) = 3\sin t, \quad \text{for } 0 \le t \le 2\pi$

Solution a

To eliminate the parameter, we can solve either of the equations for t. For example, solving the first equation for t gives

$$\begin{aligned} x &= \sqrt{2t+4} \\ x^2 &= 2t+4 \\ x^2 - 4 &= 2t \\ t &= \frac{x^2-4}{2}. \end{aligned}$$

Note that when we square both sides it is important to observe that $x \ge 0$. Substituting $t = \dfrac{x^2-4}{2}$ into $y(t)$ yields

$$\begin{aligned} y(t) &= 2t+1 \\ y &= 2\left(\frac{x^2-4}{2}\right)+1 \\ y &= x^2-4+1 \\ y &= x^2-3. \end{aligned}$$

This is the equation of a parabola opening upward. There is, however, a domain restriction because of the limits on the parameter t. When $t=-2$, $x = \sqrt{2(-2)+4} = 0$, and when $t=6$, $x = \sqrt{2(6)+4} = 4$. The graph of this plane curve follows.

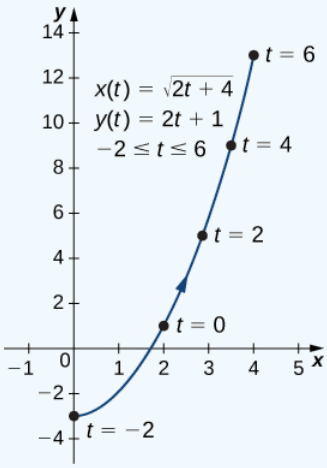


Figure 11.1.6: Graph of the plane curve described by the parametric equations in part a.

Solution b

Sometimes it is necessary to be a bit creative in eliminating the parameter. The parametric equations for this example are

$$x(t) = 4\cos t$$

and

$$y(t) = 3\sin t$$

Solving either equation for t directly is not advisable because sine and cosine are not one-to-one functions. However, dividing the first equation by 4 and the second equation by 3 (and suppressing the t) gives us

$$\cos t = \frac{x}{4}$$

and

$$\sin t = \frac{y}{3}.$$

Now use the Pythagorean identity $\cos^2 t + \sin^2 t = 1$ and replace the expressions for $\sin t$ and $\cos t$ with the equivalent expressions in terms of x and y. This gives

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

$$\frac{x^2}{16} + \frac{y^2}{9} = 1.$$

This is the equation of a horizontal ellipse centered at the origin, with semi-major axis 4 and semi-minor axis 3 as shown in the following graph.

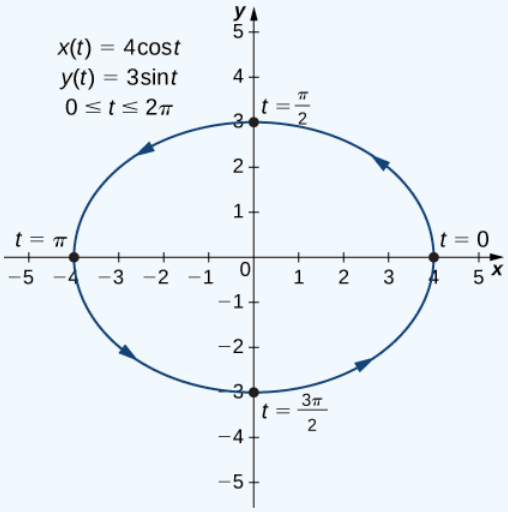


Figure 11.1.7: Graph of the plane curve described by the parametric equations in part b.

As t progresses from 0 to 2π, a point on the curve traverses the ellipse once, in a counterclockwise direction. Recall from the section opener that the orbit of Earth around the Sun is also elliptical. This is a perfect example of using parameterized curves to model a real-world phenomenon.

? Exercise 11.1.2

Eliminate the parameter for the plane curve defined by the following parametric equations and describe the resulting graph.

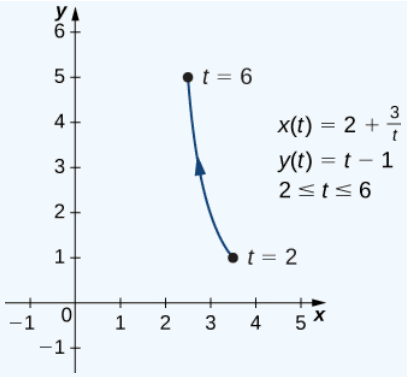
$$x(t) = 2 + \frac{3}{t}, \quad y(t) = t - 1, \quad \text{for } 2 \le t \le 6$$

Hint

Solve one of the equations for t and substitute into the other equation.

Answer

$x = 2 + \frac{3}{y+1}$, or $y = -1 + \frac{3}{x-2}$. This equation describes a portion of a rectangular hyperbola centered at $(2, -1)$.



So far we have seen the method of eliminating the parameter, assuming we know a set of parametric equations that describe a plane curve. What if we would like to start with the equation of a curve and determine a pair of parametric equations for that curve? This is certainly possible, and in fact it is possible to do so in many different ways for a given curve. The process is known as **parameterization of a curve.**

Example 11.1.3: Parameterizing a Curve

Find two different pairs of parametric equations to represent the graph of $y = 2x^2 - 3$.

Solution

First, it is always possible to parameterize a curve by defining $x(t) = t$, then replacing x with t in the equation for $y(t)$. This gives the parameterization

$$x(t) = t, \quad y(t) = 2t^2 - 3.$$

Since there is no restriction on the domain in the original graph, there is no restriction on the values of t.

We have complete freedom in the choice for the second parameterization. For example, we can choose $x(t) = 3t - 2$. The only thing we need to check is that there are no restrictions imposed on x; that is, the range of $x(t)$ is all real numbers. This is the case for $x(t) = 3t - 2$. Now since $y = 2x^2 - 3$, we can substitute $x(t) = 3t - 2$ for x. This gives

$$y(t) = 2(3t - 2)^2 - 2 = 2(9t^2 - 12t + 4) - 2 = 18t^2 - 24t + 8 - 2 = 18t^2 - 24t + 6.$$

Therefore, a second parameterization of the curve can be written as

$$x(t) = 3t - 2 \text{ and } y(t) = 18t^2 - 24t + 6.$$

Exercise 11.1.3

Find two different sets of parametric equations to represent the graph of $y = x^2 + 2x$.

Hint

Follow the steps in Example 11.1.3. Remember we have freedom in choosing the parameterization for $x(t)$.

Answer

One possibility is $x(t) = t, \quad y(t) = t^2 + 2t$. Another possibility is $x(t) = 2t - 3, \quad y(t) = (2t - 3)^2 + 2(2t - 3) = 4t^2 - 8t + 3$. There are, in fact, an infinite number of possibilities.

Cycloids and Other Parametric Curves

Imagine going on a bicycle ride through the country. The tires stay in contact with the road and rotate in a predictable pattern. Now suppose a very determined ant is tired after a long day and wants to get home. So he hangs onto the side of the tire and gets a free ride. The path that this ant travels down a straight road is called a *cycloid* (Figure 11.1.8). A cycloid generated by a circle (or bicycle wheel) of radius a is given by the parametric equations

$$x(t) = a(t - \sin t), \quad y(t) = a(1 - \cos t).$$

To see why this is true, consider the path that the center of the wheel takes. The center moves along the x-axis at a constant height equal to the radius of the wheel. If the radius is a, then the coordinates of the center can be given by the equations

$$x(t) = at, \quad y(t) = a$$

for any value of t. Next, consider the ant, which rotates around the center along a circular path. If the bicycle is moving from left to right then the wheels are rotating in a clockwise direction. A possible parameterization of the circular motion of the ant (relative to the center of the wheel) is given by

$$x(t) = -a \sin t$$
$$y(t) = -a \cos t.$$

(The negative sign is needed to reverse the orientation of the curve. If the negative sign were not there, we would have to imagine the wheel rotating counterclockwise.) Adding these equations together gives the equations for the cycloid.

$$x(t) = a(t - \sin t)$$
$$y(t) = a(1 - \cos t)$$

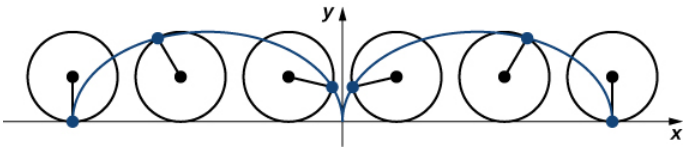


Figure 11.1.8: A wheel traveling along a road without slipping; the point on the edge of the wheel traces out a cycloid.

Now suppose that the bicycle wheel doesn't travel along a straight road but instead moves along the inside of a larger wheel, as in Figure 11.1.9. In this graph, the green circle is traveling around the blue circle in a counterclockwise direction. A point on the edge of the green circle traces out the red graph, which is called a *hypocycloid*.

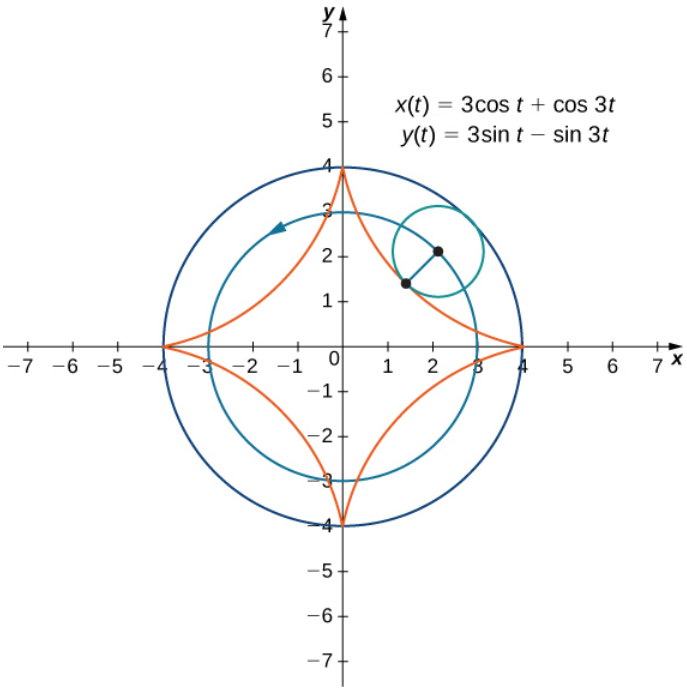


Figure 11.1.9: Graph of the hypocycloid described by the parametric equations shown.

The general parametric equations for a hypocycloid are

$$x(t) = (a - b)\cos t + b\cos(\frac{a - b}{b})t$$

$$y(t) = (a - b)\sin t - b\sin(\frac{a - b}{b})t.$$

These equations are a bit more complicated, but the derivation is somewhat similar to the equations for the cycloid. In this case we assume the radius of the larger circle is a and the radius of the smaller circle is b. Then the center of the wheel travels along a circle of radius $a-b$. This fact explains the first term in each equation above. The period of the second trigonometric function in both $x(t)$ and $y(t)$ is equal to $\dfrac{2\pi b}{a-b}$.

The ratio $\dfrac{a}{b}$ is related to the number of **cusps** on the graph (cusps are the corners or pointed ends of the graph), as illustrated in Figure 11.1.10. This ratio can lead to some very interesting graphs, depending on whether or not the ratio is rational. Figure 11.1.9 corresponds to $a=4$ and $b=1$. The result is a hypocycloid with four cusps. Figure 11.1.10 shows some other possibilities. The last two hypocycloids have irrational values for $\dfrac{a}{b}$. In these cases the hypocycloids have an infinite number of cusps, so they never return to their starting point. These are examples of what are known as **space-filling curves.**

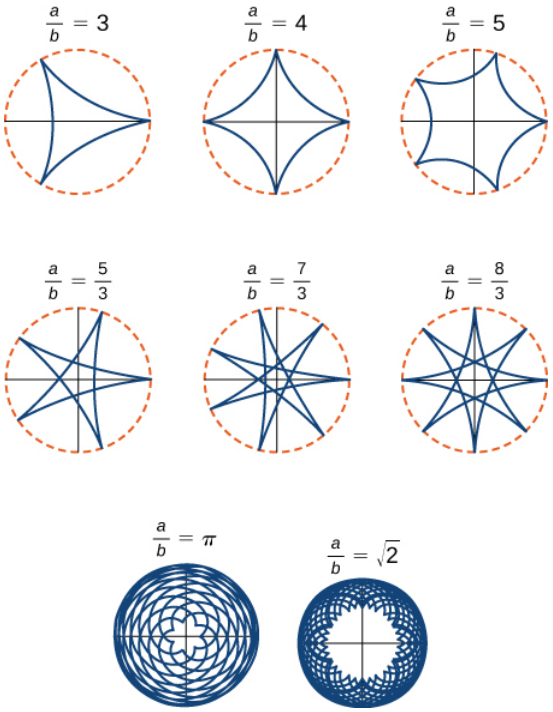


Figure 11.1.10: Graph of various hypocycloids corresponding to different values of a/b.

The Witch of Agnesi

Many plane curves in mathematics are named after the people who first investigated them, like the folium of Descartes or the spiral of Archimedes. However, perhaps the strangest name for a curve is the **witch of Agnesi.** Why a witch?

Maria Gaetana Agnesi (1718–1799) was one of the few recognized women mathematicians of eighteenth-century Italy. She wrote a popular book on analytic geometry, published in 1748, which included an interesting curve that had been studied by Fermat in 1630. The mathematician Guido Grandi showed in 1703 how to construct this curve, which he later called the "versoria," a Latin term for a rope used in sailing. Agnesi used the Italian term for this rope, "versiera," but in Latin, this same word means a "female goblin." When Agnesi's book was translated into English in 1801, the translator used the term "witch" for the curve, instead of rope. The name "witch of Agnesi" has stuck ever since.

The witch of Agnesi is a curve defined as follows: Start with a circle of radius a so that the points $(0,0)$ and $(0,2a)$ are points on the circle (Figure 11.1.11). Let O denote the origin. Choose any other point A on the circle, and draw the secant line OA. Let B denote the point at which the line OA intersects the horizontal line through $(0,2a)$. The vertical line through B intersects the horizontal line through A at the point P. As the point A varies, the path that the point P travels is the witch of Agnesi curve for the given circle.

Witch of Agnesi curves have applications in physics, including modeling water waves and distributions of spectral lines. In probability theory, the curve describes the probability density function of the *Cauchy distribution*. In this project you will parameterize these curves.

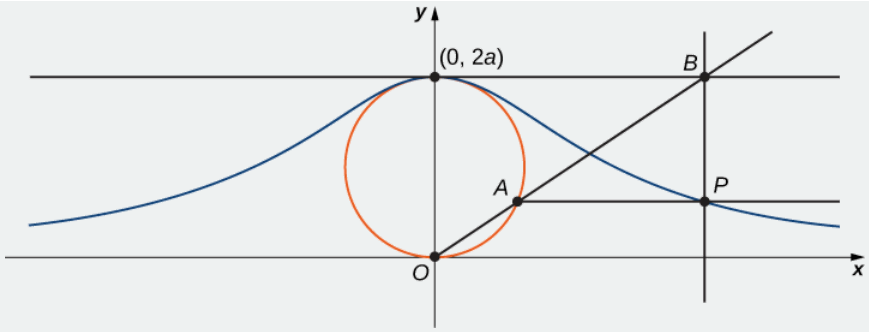


Figure 11.1.11: As the point A moves around the circle, the point P traces out the witch of Agnesi curve for the given circle.

1. On the figure, label the following points, lengths, and angle:
   a. C is the point on the x-axis with the same x-coordinate as A.
   b. x is the x-coordinate of P, and y is the y-coordinate of P.
   c. E is the point $(0, a)$.
   d. F is the point on the line segment OA such that the line segment EF is perpendicular to the line segment OA.
   e. b is the distance from O to F.
   f. c is the distance from F to A.
   g. d is the distance from O to B.
   h. θ is the measure of angle $\angle COA$.

The goal of this project is to parameterize the witch using θ as a parameter. To do this, write equations for x and y in terms of only θ.

2. Show that $d = \dfrac{2a}{\sin\theta}$.
3. Note that $x = d\cos\theta$. Show that $x = 2a\cot\theta$. When you do this, you will have parameterized the x-coordinate of the curve with respect to θ. If you can get a similar equation for y, you will have parameterized the curve.
4. In terms of θ, what is the angle $\angle EOA$?
5. Show that $b + c = 2a\cos\left(\frac{\pi}{2} - \theta\right)$.
6. Show that $y = 2a\cos\left(\frac{\pi}{2} - \theta\right)\sin\theta$.
7. Show that $y = 2a\sin^2\theta$. You have now parameterized the y-coordinate of the curve with respect to θ.
8. Conclude that a parameterization of the given witch curve is

$$x = 2a\cot\theta, \quad y = 2a\sin^2\theta, \quad \text{for } -\infty < \theta < \infty.$$

9. Use your parameterization to show that the given witch curve is the graph of the function $f(x) = \dfrac{8a^3}{x^2 + 4a^2}$.

Travels with My Ant: The Curtate and Prolate Cycloids

Earlier in this section, we looked at the parametric equations for a cycloid, which is the path a point on the edge of a wheel traces as the wheel rolls along a straight path. In this project we look at two different variations of the cycloid, called the curtate and prolate cycloids.

First, let's revisit the derivation of the parametric equations for a cycloid. Recall that we considered a tenacious ant trying to get home by hanging onto the edge of a bicycle tire. We have assumed the ant climbed onto the tire at the very edge, where the tire touches the ground. As the wheel rolls, the ant moves with the edge of the tire (Figure 11.1.12).

As we have discussed, we have a lot of flexibility when parameterizing a curve. In this case we let our parameter t represent the angle the tire has rotated through. Looking at Figure 11.1.12, we see that after the tire has rotated through an angle of t, the position of the center of the wheel, $C = (x_C, y_C)$, is given by

$$x_C = at \text{ and } y_C = a.$$

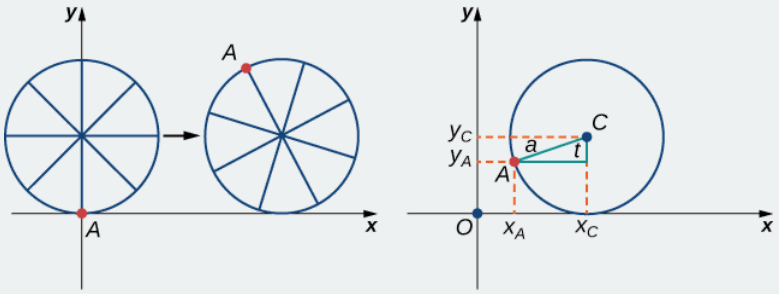
Furthermore, letting $A = (x_A, y_A)$ denote the position of the ant, we note that

$$x_C - x_A = a\sin t \text{ and } y_C - y_A = a\cos t$$

Then

$$x_A = x_C - a\sin t = at - a\sin t = a(t - \sin t)$$

$$y_A = y_C - a\cos t = a - a\cos t = a(1 - \cos t).$$

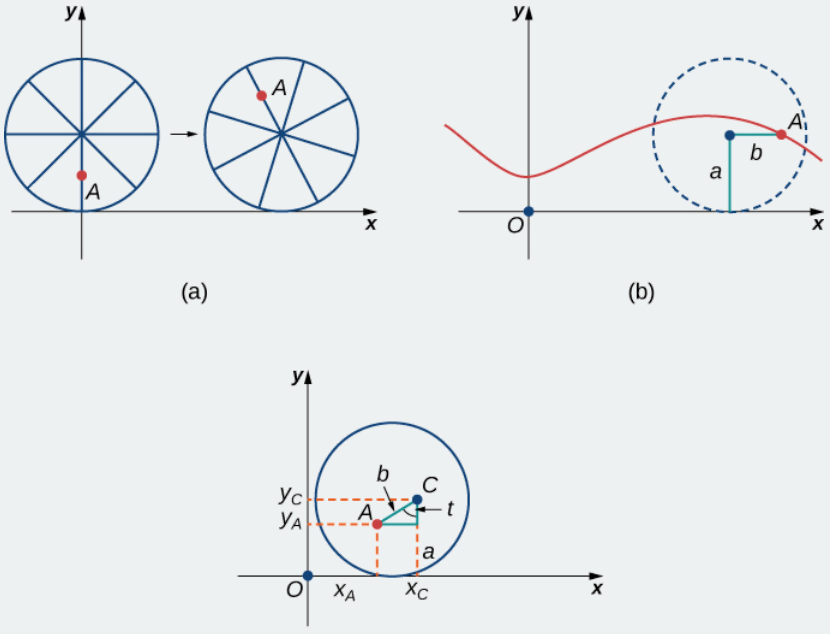


(a) (b)

Figure 11.1.12: (a) The ant clings to the edge of the bicycle tire as the tire rolls along the ground. (b) Using geometry to determine the position of the ant after the tire has rotated through an angle of t.

Note that these are the same parametric representations we had before, but we have now assigned a physical meaning to the parametric variable t.

After a while the ant is getting dizzy from going round and round on the edge of the tire. So he climbs up one of the spokes toward the center of the wheel. By climbing toward the center of the wheel, the ant has changed his path of motion. The new path has less up-and-down motion and is called a **curtate cycloid** (Figure 11.1.13). As shown in the figure, we let b denote the distance along the spoke from the center of the wheel to the ant. As before, we let t represent the angle the tire has rotated through. Additionally, we let $C = (x_C, y_C)$ represent the position of the center of the wheel and $A = (x_A, y_A)$ represent the position of the ant.



(a) (b)

(c)

Figure 11.1.13: (a) The ant climbs up one of the spokes toward the center of the wheel. (b) The ant's path of motion after he climbs closer to the center of the wheel. This is called a curtate cycloid. (c) The new setup, now that the ant has moved closer to the center of the wheel.

1. What is the position of the center of the wheel after the tire has rotated through an angle of t?
2. Use geometry to find expressions for $x_C - x_A$ and for $y_C - y_A$.
3. On the basis of your answers to parts 1 and 2, what are the parametric equations representing the curtate cycloid?

Once the ant's head clears, he realizes that the bicyclist has made a turn, and is now traveling away from his home. So he drops off the bicycle tire and looks around. Fortunately, there is a set of train tracks nearby, headed back in the right direction. So the ant heads over to the train tracks to wait. After a while, a train goes by, heading in the right direction, and he manages to jump up and just catch the edge of the train wheel (without getting squished!).

The ant is still worried about getting dizzy, but the train wheel is slippery and has no spokes to climb, so he decides to just hang on to the edge of the wheel and hope for the best. Now, train wheels have a flange to keep the wheel running on the tracks. So, in this case, since the ant is hanging on to the very edge of the flange, the distance from the center of the wheel to the ant is actually greater than the radius of the wheel (Figure 11.1.14).

The setup here is essentially the same as when the ant climbed up the spoke on the bicycle wheel. We let b denote the distance from the center of the wheel to the ant, and we let t represent the angle the tire has rotated through. Additionally, we let $C = (x_C, y_C)$ represent the position of the center of the wheel and $A = (x_A, y_A)$ represent the position of the ant (Figure 11.1.14).

When the distance from the center of the wheel to the ant is greater than the radius of the wheel, his path of motion is called a prolate cycloid. A graph of a **prolate cycloid** is shown in the figure.

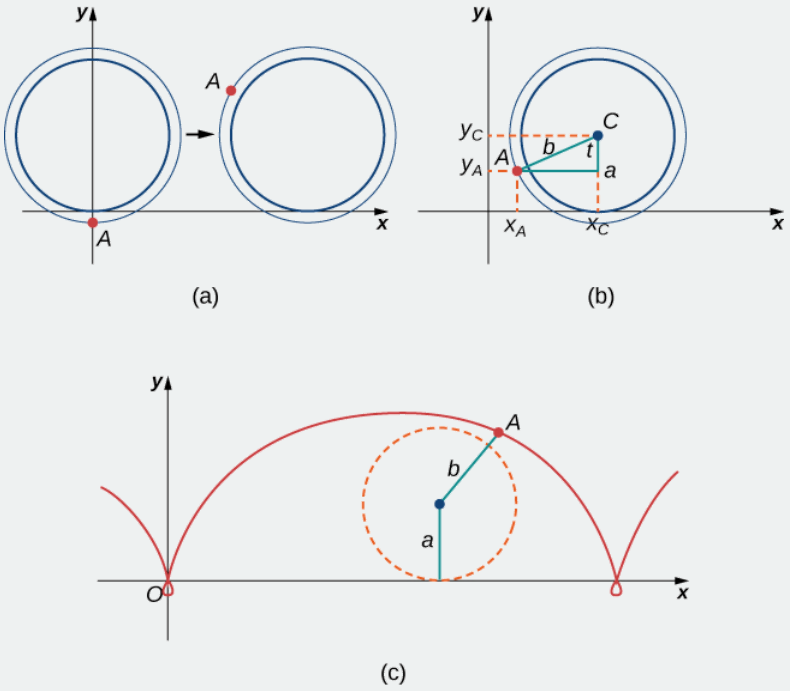


Figure 11.1.14: (a) The ant is hanging onto the flange of the train wheel. (b) The new setup, now that the ant has jumped onto the train wheel. (c) The ant travels along a prolate cycloid.

4. Using the same approach you used in parts 1– 3, find the parametric equations for the path of motion of the ant.
5. What do you notice about your answer to part 3 and your answer to part 4?

Notice that the ant is actually traveling backward at times (the "loops" in the graph), even though the train continues to move forward. He is probably going to be really dizzy by the time he gets home!

Key Concepts

- Parametric equations provide a convenient way to describe a curve. A parameter can represent time or some other meaningful quantity.
- It is often possible to eliminate the parameter in a parameterized curve to obtain a function or relation describing that curve.
- There is always more than one way to parameterize a curve.
- Parametric equations can describe complicated curves that are difficult or perhaps impossible to describe using rectangular coordinates.

Glossary

cycloid
the curve traced by a point on the rim of a circular wheel as the wheel rolls along a straight line without slippage

cusp
a pointed end or part where two curves meet

orientation
the direction that a point moves on a graph as the parameter increases

parameter
an independent variable that both x and y depend on in a parametric curve; usually represented by the variable t

parametric curve
the graph of the parametric equations $x(t)$ and $y(t)$ over an interval $a \leq t \leq b$ combined with the equations

parametric equations
the equations $x = x(t)$ and $y = y(t)$ that define a parametric curve

parameterization of a curve
rewriting the equation of a curve defined by a function $y = f(x)$ as parametric equations

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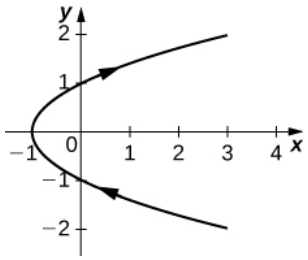
11.1E: EXERCISES FOR SECTION 11.1

In exercises 1 - 4, sketch the curves below by eliminating the parameter t. Give the orientation of the curve.

1) $x = t^2 + 2t, \quad y = t + 1$

Answer

Orientation: bottom to top

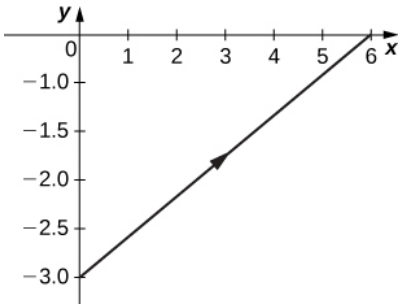


2) $x = \cos(t), \quad y = \sin(t), \quad \text{for } (0, 2\pi]$

3) $x = 2t + 4, \quad y = t - 1$

Answer

Orientation: left to right



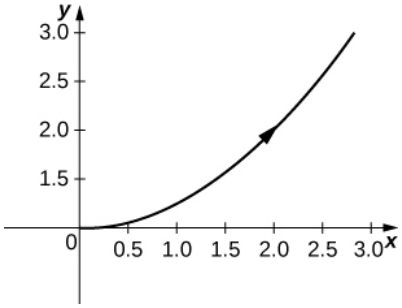
4) $x = 3 - t, \quad y = 2t - 3, \quad \text{for } 1.5 \le t \le 3$

In exercise 5, eliminate the parameter and sketch the graph.

5) $x = 2t^2, \quad y = t^4 + 1$

Answer

$$y = \frac{x^2}{4} + 1$$

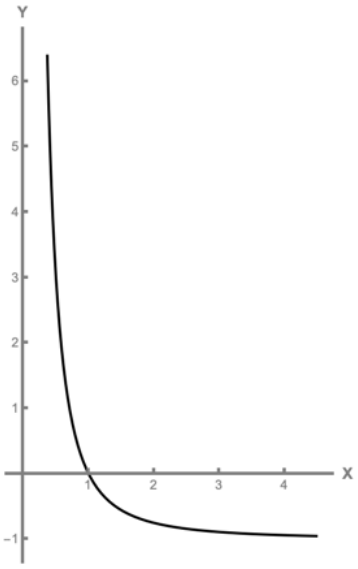


In exercises 6 - 9, use technology (CAS or calculator) to sketch the parametric equations.

6) [T] $x = t^2 + t, \quad y = t^2 - 1$

7) [T] $x = e^{-t}, \quad y = e^{2t} - 1$

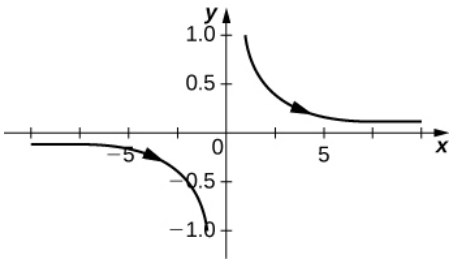
Answer



8) [T] $x = 3\cos t, \quad y = 4\sin t$

9) [T] $x = \sec t, \quad y = \cos t$

Answer

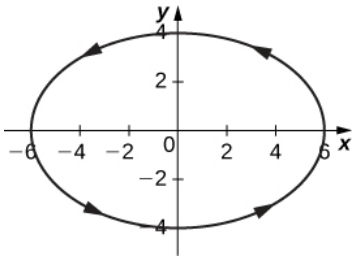


In exercises 10 - 20, sketch the parametric equations by eliminating the parameter. Indicate any asymptotes of the graph.

10) $x = e^t, \quad y = e^{2t} + 1$

11) $x = 6\sin(2\theta), \quad y = -4\cos(2\theta)$

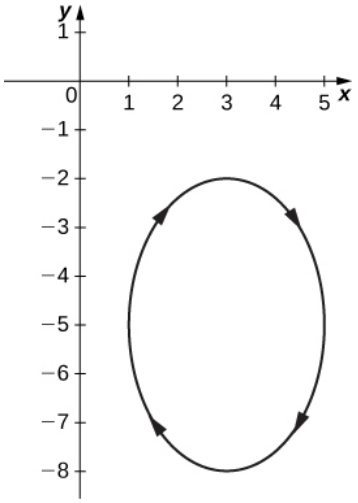
Answer



12) $x = \cos\theta, \quad y = 2\sin(2\theta)$

13) $x = 3 - 2\cos\theta, \quad y = -5 + 3\sin\theta$

Answer

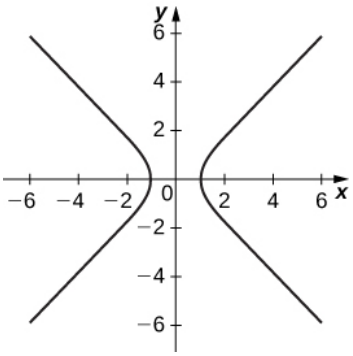


14) $x = 4 + 2\cos\theta, \quad y = -1 + \sin\theta$

15) $x = \sec t, \quad y = \tan t$

Answer

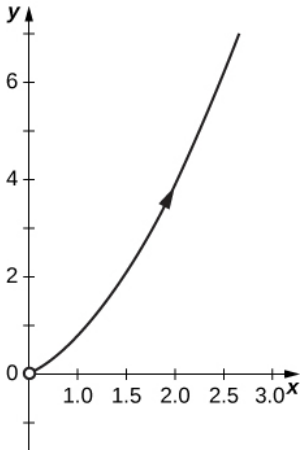
Asymptotes are $y = x$ and $y = -x$



16) $x = \ln(2t), \quad y = t^2$

17) $x = e^t, \quad y = e^{2t}$

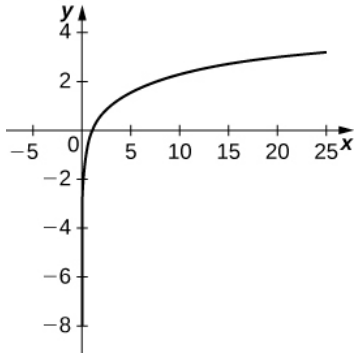
Answer



18) $x = e^{-2t}, \quad y = e^{3t}$

19) $x = t^3, \quad y = 3\ln t$

Answer



20) $x = 4\sec\theta, \quad y = 3\tan\theta$

In exercises 21 - 38, convert the parametric equations of a curve into rectangular form. No sketch is necessary. State the domain of the rectangular form.

21) $x = t^2 - 1, \quad y = \dfrac{t}{2}$

Answer

$x = 4y^2 - 1$; domain: $x \in [1, \infty)$.

22) $x = \dfrac{1}{\sqrt{t+1}}, \quad y = \dfrac{t}{1+t}, \quad \text{for } t > -1$

23) $x = 4\cos\theta, \quad y = 3\sin\theta, \quad \text{for } t \in (0, 2\pi]$

Answer

$\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$; domain $x \in [-4, 4]$.

24) $x = \cosh t, \quad y = \sinh t$

25) $x = 2t - 3, \quad y = 6t - 7$

Answer

$y = 3x + 2$; domain: all real numbers.

26) $x = t^2, \quad y = t^3$

27) $x = 1 + \cos t, \quad y = 3 - \sin t$

Answer

$(x-1)^2 + (y-3)^2 = 1$; domain: $x \in [0, 2]$.

28) $x = \sqrt{t}, \quad y = 2t + 4$

29) $x = \sec t, \quad y = \tan t, \quad \text{for } \pi \le t < \frac{3\pi}{2}$

Answer

$y = \sqrt{x^2 - 1}$; domain: $x \in (-\infty, -1]$.

30) $x = 2\cosh t, \quad y = 4\sinh t$

31) $x = \cos(2t), \quad y = \sin t$

Answer

$y^2 = \dfrac{1-x}{2}$; domain: $x \in [-1, 1]$.

32) $x = 4t + 3, \quad y = 16t^2 - 9$

33) $x = t^2, \quad y = 2\ln t, \quad \text{for } t \ge 1$

Answer

$y = \ln x$; domain: $x \in [1, \infty)$.

34) $x = t^3, \quad y = 3 \ln t, \quad \text{for } t \geq 1$

35) $x = t^n, \quad y = n \ln t, \quad \text{for } t \geq 1$, where n is a natural number

Answer

$y = \ln x$; domain: $x \in (0, \infty)$.

36) $x = \ln(5t), \quad y = \ln(t^2)$ where $1 \leq t \leq e$

37) $x = 2 \sin(8t), \quad y = 2 \cos(8t)$

Answer

$x^2 + y^2 = 4$; domain: $x \in [-2, 2]$.

38) $x = \tan t, \quad y = \sec^2 t - 1$

In exercises 39 - 48, the pairs of parametric equations represent lines, parabolas, circles, ellipses, or hyperbolas. Name the type of basic curve that each pair of equations represents.

39) $x = 3t + 4, \quad y = 5t - 2$

Answer

line

40) $x - 4 = 5t, \quad y + 2 = t$

41) $x = 2t + 1, \quad y = t^2 - 3$

Answer

parabola

42) $x = 3 \cos t, \quad y = 3 \sin t$

43) $x = 2 \cos(3t), \quad y = 2 \sin(3t)$

Answer

circle

44) $x = \cosh t, \quad y = \sinh t$

45) $x = 3 \cos t, \quad y = 4 \sin t$

Answer

ellipse

46) $x = 2 \cos(3t), \quad y = 5 \sin(3t)$

47) $x = 3 \cosh(4t) \quad y = 4 \sinh(4t)$

Answer

the right branch of a horizontally opening hyperbola

48) $x = 2 \cosh t, \quad y = 2 \sinh t$

49) Show that $x = h + r \cos \theta, \quad y = k + r \sin \theta$ represents the equation of a circle.

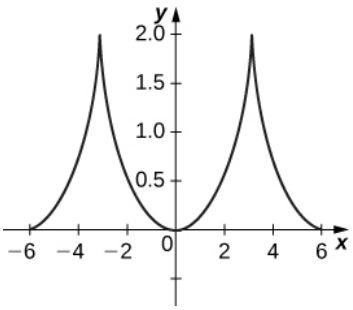
50) Use the equations in the preceding problem to find a set of parametric equations for a circle whose radius is 5 and whose center is $(-2, 3)$.

In exercises 51 - 53, use a graphing utility to graph the curve represented by the parametric equations and identify the curve from its equation.

51) [T] $x = \theta + \sin \theta, \quad y = 1 - \cos \theta$

Answer

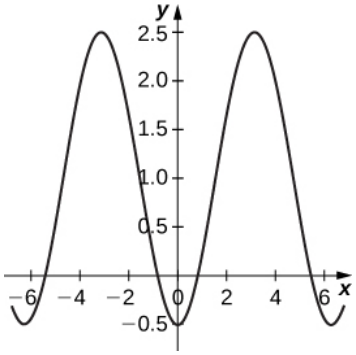
The equations represent a cycloid.



52) [T] $x = 2t - 2\sin t, \quad y = 2 - 2\cos t$

53) [T] $x = t - 0.5\sin t, \quad y = 1 - 1.5\cos t$

Answer



54) An airplane traveling horizontally at 100 m/s over flat ground at an elevation of 4000 meters must drop an emergency package on a target on the ground. The trajectory of the package is given by $x = 100t, \quad y = -4.9t^2 + 4000, \quad \text{where } t \geq 0$ where the origin is the point on the ground directly beneath the plane at the moment of release. How many horizontal meters before the target should the package be released in order to hit the target?

55) The trajectory of a bullet is given by $x = v_0(\cos\alpha)t, \quad y = v_0(\sin\alpha)t - \frac{1}{2}gt^2$ where $v_0 = 500$ m/s, $g = 9.8 = 9.8\ \text{m/s}^2$, and $\alpha = 30$ degrees. When will the bullet hit the ground? How far from the gun will the bullet hit the ground?

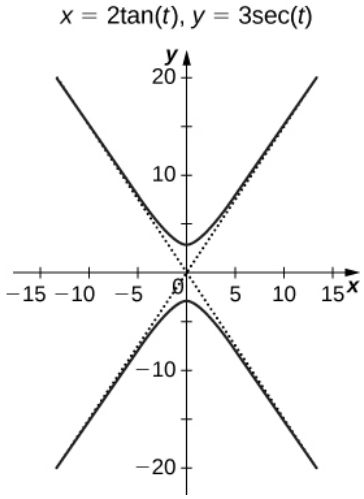
Answer

22,092 meters at approximately 51 seconds.

56) [T] Use technology to sketch the curve represented by $x = \sin(4t), \quad y = \sin(3t), \quad \text{for } 0 \leq t \leq 2\pi$.

57) [T] Use technology to sketch $x = 2\tan(t), \quad y = 3\sec(t), \quad \text{for } -\pi < t < \pi$.

Answer



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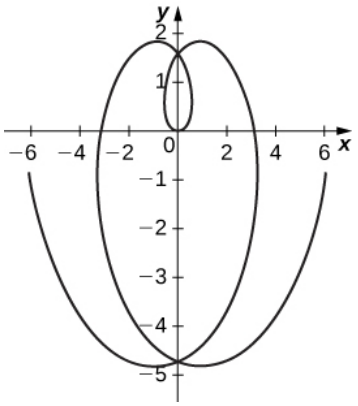
58) Sketch the curve known as an **epitrochoid**, which gives the path of a point on a circle of radius b as it rolls on the outside of a circle of radius a. The equations are

$x = (a+b)\cos t - c \cdot \cos\left[\frac{(a+b)t}{b}\right], \quad y = (a+b)\sin t - c \cdot \sin\left[\frac{(a+b)t}{b}\right].$

Let $a = 1,\ b = 2,\ c = 1$.

59) [T] Use technology to sketch the spiral curve given by $x = t\cos(t), \quad y = t\sin(t)$ for $-2\pi \le t \le 2\pi$.

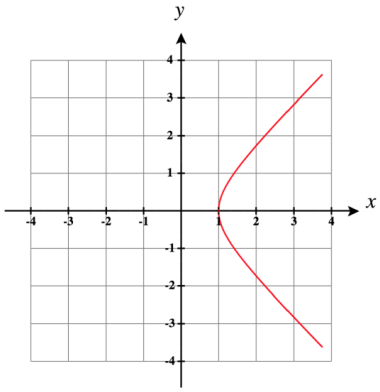
Answer



60) [T] Use technology to graph the curve given by the parametric equations $x = 2\cot(t), \quad y = 1 - \cos(2t), \quad \text{for } -\pi/2 \le t \le \pi/2$. This curve is known as the witch of Agnesi.

61) [T] Sketch the curve given by parametric equations $x = \cosh(t), \quad y = \sinh(t)$, for $-2 \le t \le 2$.

Answer



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11.2: Calculus of Parametric Curves

Learning Objectives

- Determine derivatives and equations of tangents for parametric curves.
- Find the area under a parametric curve.
- Use the equation for arc length of a parametric curve.
- Apply the formula for surface area to a volume generated by a parametric curve.

Now that we have introduced the concept of a parameterized curve, our next step is to learn how to work with this concept in the context of calculus. For example, if we know a parameterization of a given curve, is it possible to calculate the slope of a tangent line to the curve? How about the arc length of the curve? Or the area under the curve?

Another scenario: Suppose we would like to represent the location of a baseball after the ball leaves a pitcher's hand. If the position of the baseball is represented by the plane curve $(x(t), y(t))$ then we should be able to use calculus to find the speed of the ball at any given time. Furthermore, we should be able to calculate just how far that ball has traveled as a function of time.

Derivatives of Parametric Equations

We start by asking how to calculate the slope of a line tangent to a parametric curve at a point. Consider the plane curve defined by the parametric equations

$$x(t) = 2t + 3 \tag{11.2.1}$$
$$y(t) = 3t - 4 \tag{11.2.2}$$

within $-2 \leq t \leq 3$.

The graph of this curve appears in Figure 11.2.1. It is a line segment starting at $(-1, -10)$ and ending at $(9, 5)$.

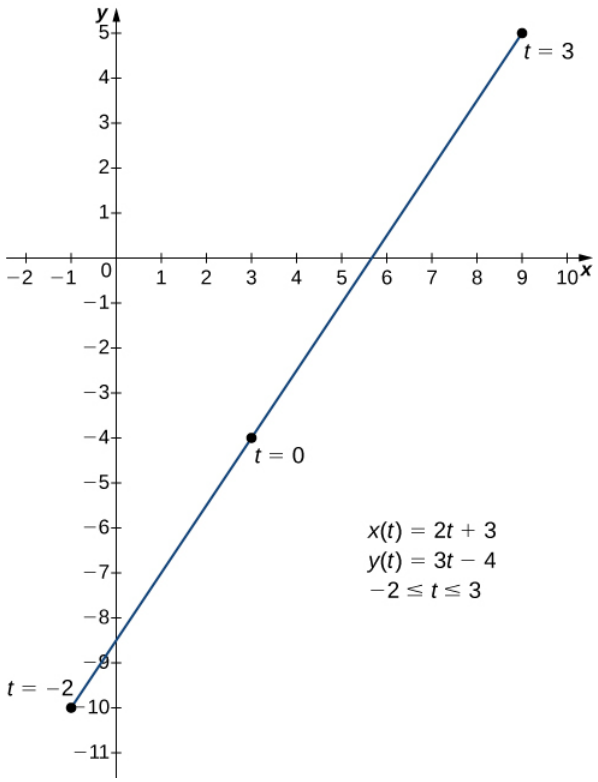


Figure 11.2.1: Graph of the line segment described by the given parametric equations.

We can eliminate the parameter by first solving Equation 11.2.1 for t:

$$x(t) = 2t + 3$$

$$x - 3 = 2t$$
$$t = \frac{x-3}{2}.$$

Substituting this into $y(t)$ (Equation 11.2.2), we obtain

$$y(t) = 3t - 4$$
$$y = 3\left(\frac{x-3}{2}\right) - 4$$
$$y = \frac{3x}{2} - \frac{9}{2} - 4$$
$$y = \frac{3x}{2} - \frac{17}{2}.$$

The slope of this line is given by $\frac{dy}{dx} = \frac{3}{2}$. Next we calculate $x'(t)$ and $y'(t)$. This gives $x'(t) = 2$ and $y'(t) = 3$. Notice that

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3}{2}.$$

This is no coincidence, as outlined in the following theorem.

Derivative of Parametric Equations

Consider the plane curve defined by the parametric equations $x = x(t)$ and $y = y(t)$. Suppose that $x'(t)$ and $y'(t)$ exist, and assume that $x'(t) \neq 0$. Then the derivative $\frac{dy}{dx}$ is given by

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}. \tag{11.2.3}$$

Proof

This theorem can be proven using the Chain Rule. In particular, assume that the parameter t can be eliminated, yielding a differentiable function $y = F(x)$. Then $y(t) = F(x(t))$. Differentiating both sides of this equation using the Chain Rule yields

$$y'(t) = F'\big(x(t)\big)x'(t),$$

so

$$F'\big(x(t)\big) = \frac{y'(t)}{x'(t)}.$$

But $F'\big(x(t)\big) = \frac{dy}{dx}$, which proves the theorem.

□

Equation 11.2.3 can be used to calculate derivatives of plane curves, as well as critical points. Recall that a critical point of a differentiable function $y = f(x)$ is any point $x = x_0$ such that either $f'(x_0) = 0$ or $f'(x_0)$ does not exist. Equation 11.2.3 gives a formula for the slope of a tangent line to a curve defined parametrically regardless of whether the curve can be described by a function $y = f(x)$ or not.

Example 11.2.1: Finding the Derivative of a Parametric Curve

Calculate the derivative $\frac{dy}{dx}$ for each of the following parametrically defined plane curves, and locate any critical points on their respective graphs.

a. $x(t) = t^2 - 3, \quad y(t) = 2t - 1, \quad \text{for } -3 \leq t \leq 4$
b. $x(t) = 2t + 1, \quad y(t) = t^3 - 3t + 4, \quad \text{for } -2 \leq t \leq 2$
c. $x(t) = 5\cos t, \quad y(t) = 5\sin t, \quad \text{for } 0 \leq t \leq 2\pi$

Solution

a. To apply Equation 11.2.3, first calculate $x'(t)$ and $y'(t)$:

$$x'(t) = 2t$$

$$y'(t) = 2.$$

Next substitute these into the equation:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{dy}{dx} = \frac{2}{2t}$$

$$\frac{dy}{dx} = \frac{1}{t}.$$

This derivative is undefined when $t = 0$. Calculating $x(0)$ and $y(0)$ gives $x(0) = (0)^2 - 3 = -3$ and $y(0) = 2(0) - 1 = -1$, which corresponds to the point $(-3, -1)$ on the graph. The graph of this curve is a parabola opening to the right, and the point $(-3, -1)$ is its vertex as shown.

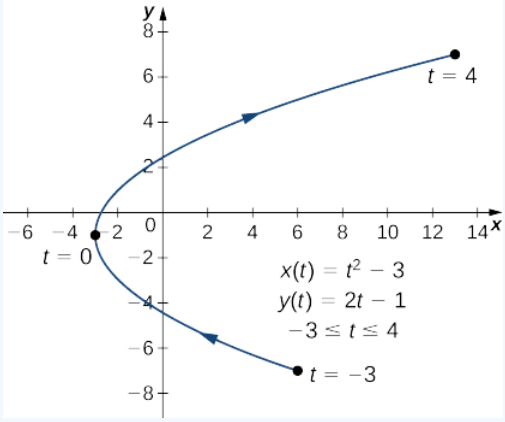


Figure 11.2.2: Graph of the parabola described by parametric equations in part a.

b. To apply Equation 11.2.3, first calculate $x'(t)$ and $y'(t)$:

$$x'(t) = 2$$

$$y'(t) = 3t^2 - 3.$$

Next substitute these into the equation:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2}.$$

This derivative is zero when $t = \pm 1$. When $t = -1$ we have

$$x(-1) = 2(-1) + 1 = -1 \text{ and } y(-1) = (-1)^3 - 3(-1) + 4 = -1 + 3 + 4 = 6,$$

which corresponds to the point $(-1, 6)$ on the graph. When $t = 1$ we have

$$x(1) = 2(1) + 1 = 3 \text{ and } y(1) = (1)^3 - 3(1) + 4 = 1 - 3 + 4 = 2,$$

which corresponds to the point $(3, 2)$ on the graph. The point $(3, 2)$ is a relative minimum and the point $(-1, 6)$ is a relative maximum, as seen in the following graph.

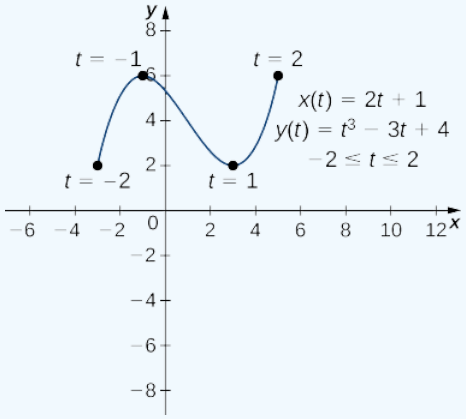


Figure 11.2.3: Graph of the curve described by parametric equations in part b.

c. To apply Equation 11.2.3, first calculate $x'(t)$ and $y'(t)$:

$$x'(t) = -5\sin t$$

$$y'(t) = 5\cos t.$$

Next substitute these into the equation:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{dy}{dx} = \frac{5\cos t}{-5\sin t}$$

$$\frac{dy}{dx} = -\cot t.$$

This derivative is zero when $\cos t = 0$ and is undefined when $\sin t = 0$. This gives $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π as critical points for t. Substituting each of these into $x(t)$ and $y(t)$, we obtain

t	$x(t)$	$y(t)$
0	5	0
$\frac{\pi}{2}$	0	5
π	−5	0
$\frac{3\pi}{2}$	0	−5
2π	5	0

These points correspond to the sides, top, and bottom of the circle that is represented by the parametric equations (Figure 11.2.4). On the left and right edges of the circle, the derivative is undefined, and on the top and bottom, the derivative equals zero.

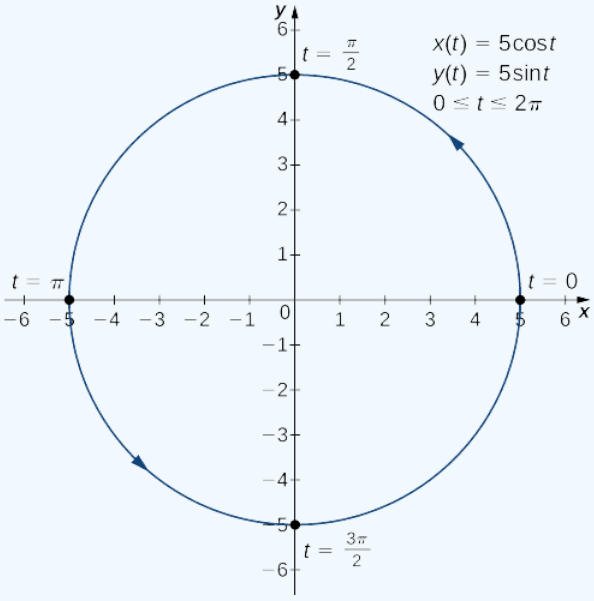


Figure 11.2.4: Graph of the curve described by parametric equations in part c.

Exercise 11.2.1

Calculate the derivative dy/dx for the plane curve defined by the equations

$$x(t) = t^2 - 4t, \quad y(t) = 2t^3 - 6t, \quad \text{for } -2 \le t \le 3$$

and locate any critical points on its graph.

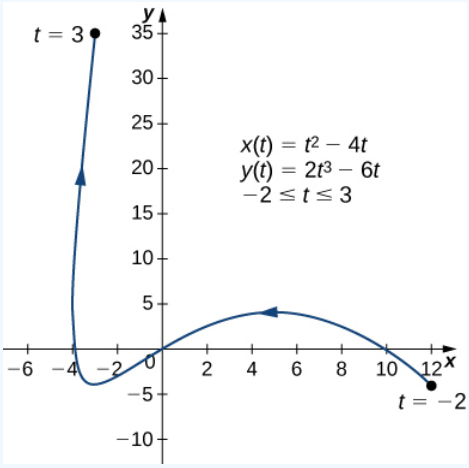
Hint

Calculate $x'(t)$ and $y'(t)$ and use Equation 11.2.3.

Answer

$x'(t) = 2t - 4$ and $y'(t) = 6t^2 - 6$, so $\dfrac{dy}{dx} = \dfrac{6t^2 - 6}{2t - 4} = \dfrac{3t^2 - 3}{t - 2}$.

This expression is undefined when $t = 2$ and equal to zero when $t = \pm 1$.



Example 11.2.2: Finding a Tangent Line

Find the equation of the tangent line to the curve defined by the equations

$$x(t) = t^2 - 3, \quad y(t) = 2t - 1, \quad \text{for } -3 \leq t \leq 4$$

when $t = 2$.

Solution

First find the slope of the tangent line using Equation 11.2.3, which means calculating $x'(t)$ and $y'(t)$:

$$x'(t) = 2t$$

$$y'(t) = 2.$$

Next substitute these into the equation:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{dy}{dx} = \frac{2}{2t}$$

$$\frac{dy}{dx} = \frac{1}{t}.$$

When $t = 2$, $\frac{dy}{dx} = \frac{1}{2}$, so this is the slope of the tangent line. Calculating $x(2)$ and $y(2)$ gives

$$x(2) = (2)^2 - 3 = 1 \text{ and } y(2) = 2(2) - 1 = 3,$$

which corresponds to the point $(1, 3)$ on the graph (Figure 11.2.5). Now use the point-slope form of the equation of a line to find the equation of the tangent line:

$$y - y_0 = m(x - x_0)$$

$$y - 3 = \frac{1}{2}(x - 1)$$

$$y - 3 = \frac{1}{2}x - \frac{1}{2}$$

$$y = \frac{1}{2}x + \frac{5}{2}.$$

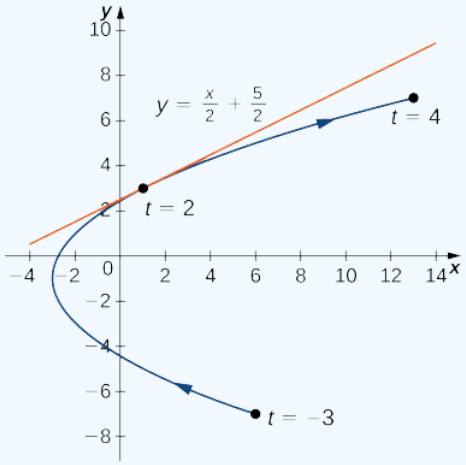


Figure 11.2.5: Tangent line to the parabola described by the given parametric equations when $t = 2$.

Exercise 11.2.2

Find the equation of the tangent line to the curve defined by the equations

$$x(t) = t^2 - 4t, \quad y(t) = 2t^3 - 6t, \quad \text{for } -2 \leq t \leq 6 \text{ when } t = 5.$$

Hint

Calculate $x'(t)$ and $y'(t)$ and use Equation 11.2.3.

Answer

The equation of the tangent line is $y = 24x + 100$.

Second-Order Derivatives

Our next goal is to see how to take the second derivative of a function defined parametrically. The second derivative of a function $y = f(x)$ is defined to be the derivative of the first derivative; that is,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right]. \tag{11.2.4}$$

Since

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt},$$

we can replace the y on both sides of Equation 11.2.4 with $\frac{dy}{dx}$. This gives us

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{(d/dt)(dy/dx)}{dx/dt}. \tag{11.2.5}$$

If we know dy/dx as a function of t, then this formula is straightforward to apply

Example 11.2.3: Finding a Second Derivative

Calculate the second derivative d^2y/dx^2 for the plane curve defined by the parametric equations $x(t) = t^2 - 3, \quad y(t) = 2t - 1,$ for $-3 \le t \le 4$.

Solution

From Example 11.2.1 we know that $\frac{dy}{dx} = \frac{2}{2t} = \frac{1}{t}$. Using Equation 11.2.5, we obtain

$$\frac{d^2y}{dx^2} = \frac{(d/dt)(dy/dx)}{dx/dt} = \frac{(d/dt)(1/t)}{2t} = \frac{-t^{-2}}{2t} = -\frac{1}{2t^3}.$$

Exercise 11.2.3

Calculate the second derivative d^2y/dx^2 for the plane curve defined by the equations

$$x(t) = t^2 - 4t, \quad y(t) = 2t^3 - 6t, \quad \text{for } -2 \le t \le 3$$

and locate any critical points on its graph.

Hint

Start with the solution from the previous exercise, and use Equation 11.2.5.

Answer

$\frac{d^2y}{dx^2} = \frac{3t^2 - 12t + 3}{2(t-2)^3}$. Critical points $(5, 4)$, $(-3, -4)$,and $(-4, 6)$.

Integrals Involving Parametric Equations

Now that we have seen how to calculate the derivative of a plane curve, the next question is this: How do we find the area under a curve defined parametrically? Recall the cycloid defined by these parametric equations

$$x(t) = t - \sin t$$
$$y(t) = 1 - \cos t.$$

Suppose we want to find the area of the shaded region in the following graph.

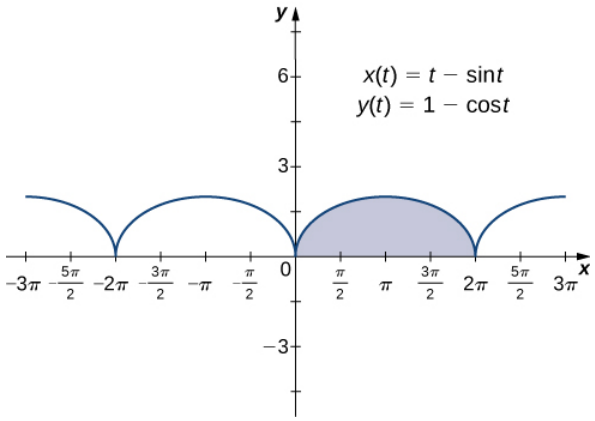


Figure 11.2.6: Graph of a cycloid with the arch over $[0, 2\pi]$ highlighted.

To derive a formula for the area under the curve defined by the functions

$$x = x(t)$$
$$y = y(t)$$

where $a \leq t \leq b$.

We assume that $x(t)$ is differentiable and start with an equal partition of the interval $a \leq t \leq b$. Suppose $t_0 = a < t_1 < t_2 < \cdots < t_n = b$ and consider the following graph.

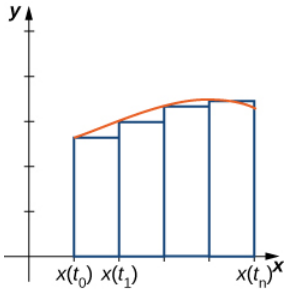


Figure 11.2.7: Approximating the area under a parametrically defined curve.

We use rectangles to approximate the area under the curve. The height of a typical rectangle in this parametrization is $y(x(\bar{t}_i))$ for some value $\bar{t}_i$ in the ith subinterval, and the width can be calculated as $x(t_i) - x(t_{i-1})$. Thus the area of the ith rectangle is given by

$$A_i = y(x(\bar{t}_i))(x(t_i) - x(t_{i-1})).$$

Then a Riemann sum for the area is

$$A_n = \sum_{i=1}^{n} y(x(\bar{t}_i))(x(t_i) - x(t_{i-1})).$$

Multiplying and dividing each area by $t_i - t_{i-1}$ gives

$$\begin{aligned} A_n &= \sum_{i=1}^{n} y(x(\bar{t}_i)) \left(\frac{x(t_i) - x(t_{i-1})}{t_i - t_{i-1}} \right) (t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} y(x(\bar{t}_i)) \left(\frac{x(t_i) - x(t_{i-1})}{\Delta t} \right) \Delta t. \end{aligned}$$

Taking the limit as n approaches infinity gives

$$A = \lim_{n \to \infty} A_n = \int_a^b y(t)x'(t)\, dt.$$

This leads to the following theorem.

Area under a Parametric Curve

Consider the non-self-intersecting plane curve defined by the parametric equations

$$x = x(t), \quad y = y(t), \quad \text{for } a \le t \le b$$

and assume that $x(t)$ is differentiable. The area under this curve is given by

$$A = \int_a^b y(t)x'(t)\,dt. \tag{11.2.6}$$

Example 11.2.4: Finding the Area under a Parametric Curve

Find the area under the curve of the cycloid defined by the equations

$$x(t) = t - \sin t, \quad y(t) = 1 - \cos t, \quad \text{for } 0 \le t \le 2\pi.$$

Solution

Using Equation 11.2.6, we have

$$\begin{aligned} A &= \int_a^b y(t)x'(t)\,dt \\ &= \int_0^{2\pi} (1-\cos t)(1-\cos t)\,dt \\ &= \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)\,dt \\ &= \int_0^{2\pi} \left(1 - 2\cos t + \frac{1+\cos(2t)}{2}\right) dt \\ &= \int_0^{2\pi} \left(\frac{3}{2} - 2\cos t + \frac{\cos(2t)}{2}\right) dt \\ &= \frac{3t}{2} - 2\sin t + \frac{\sin(2t)}{4} \Big|_0^{2\pi} \\ &= 3\pi \end{aligned}$$

Exercise 11.2.4

Find the area under the curve of the hypocycloid defined by the equations

$$x(t) = 3\cos t + \cos(3t), \quad y(t) = 3\sin t - \sin(3t), \quad \text{for } 0 \le t \le \pi.$$

Hint

Use Equation 11.2.6, along with the identities $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$ and $\sin^2 t = \frac{1-\cos(2t)}{2}$.

Answer

$A = 3\pi$ (Note that the integral formula actually yields a negative answer. This is due to the fact that $x(t)$ is a decreasing function over the interval $[0, \pi]$; that is, the curve is traced from right to left.)

Arc Length of a Parametric Curve

In addition to finding the area under a parametric curve, we sometimes need to find the arc length of a parametric curve. In the case of a line segment, arc length is the same as the distance between the endpoints. If a particle travels from point A to point B along a curve, then the distance that particle travels is the arc length. To develop a formula for arc length, we start with an approximation by line segments as shown in the following graph.

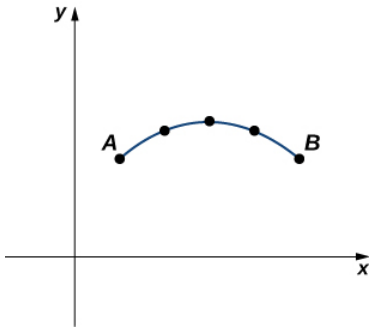


Figure 11.2.7: Approximation of a curve by line segments.

Given a plane curve defined by the functions $x = x(t), \quad y = y(t), \quad$ for $a \le t \le b$, we start by partitioning the interval $[a, b]$ into n equal subintervals: $t_0 = a < t_1 < t_2 < \cdots < t_n = b$. The width of each subinterval is given by $\Delta t = (b - a)/n$. We can calculate the length of each line segment:

$$d_1 = \sqrt{(x(t_1) - x(t_0))^2 + (y(t_1) - y(t_0))^2}$$

$$d_2 = \sqrt{(x(t_2) - x(t_1))^2 + (y(t_2) - y(t_1))^2}$$

etc.

Then add these up. We let s denote the exact arc length and s_n denote the approximation by n line segments:

$$s \approx \sum_{k=1}^{n} s_k = \sum_{k=1}^{n} \sqrt{(x(t_k) - x(t_{k-1}))^2 + (y(t_k) - y(t_{k-1}))^2}. \tag{11.2.7}$$

If we assume that $x(t)$ and $y(t)$ are differentiable functions of t, then the Mean Value Theorem applies, so in each subinterval $[t_{k-1}, t_k]$ there exist $\hat{t_k}$ and $\tilde{t_k}$ such that

$$x(t_k) - x(t_{k-1}) = x'(\hat{t_k})(t_k - t_{k-1}) = x'(\hat{t_k})\,\Delta t$$

$$y(t_k) - y(t_{k-1}) = y'(\tilde{t_k})(t_k - t_{k-1}) = y'(\tilde{t_k})\,\Delta t.$$

Therefore Equation 11.2.7 becomes

$$\begin{aligned} s \approx \sum_{k=1}^{n} s_k &= \sum_{k=1}^{n} \sqrt{(x'(\hat{t_k})\Delta t)^2 + (y'(\tilde{t_k})\Delta t)^2} \\ &= \sum_{k=1}^{n} \sqrt{(x'(\hat{t_k}))^2(\Delta t)^2 + (y'(\tilde{t_k}))^2(\Delta t)^2} \\ &= \sum_{k=1}^{n} \sqrt{(x'(\hat{t_k}))^2 + (y'(\tilde{t_k}))^2}\Delta t. \end{aligned}$$

This is a Riemann sum that approximates the arc length over a partition of the interval $[a, b]$. If we further assume that the derivatives are continuous and let the number of points in the partition increase without bound, the approximation approaches the exact arc length. This gives

$$\begin{aligned} s &= \lim_{n\to\infty} \sum_{k=1}^{n} s_k \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{(x'(\hat{t_k}))^2 + (y'(\tilde{t_k}))^2}\Delta t \\ &= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt. \end{aligned}$$

When taking the limit, the values of $\hat{t_k}$ and $\tilde{t_k}$ are both contained within the same ever-shrinking interval of width Δt, so they must converge to the same value.

We can summarize this method in the following theorem.

Arc Length of a Parametric Curve

Consider the plane curve defined by the parametric equations

$$x = x(t), \quad y = y(t), \quad \text{for } t_1 \le t \le t_2$$

and assume that $x(t)$ and $y(t)$ are differentiable functions of t. Then the arc length of this curve is given by

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \tag{11.2.8}$$

At this point a side derivation leads to a previous formula for arc length. In particular, suppose the parameter can be eliminated, leading to a function $y = F(x)$. Then $y(t) = F(x(t))$ and the Chain Rule gives

$$y'(t) = F'\big(x(t)\big)x'(t).$$

Substituting this into Equation 11.2.8 gives

$$\begin{aligned} s &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(F'(x)\frac{dx}{dt}\right)^2}\, dt \\ &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 \left(1 + (F'(x))^2\right)}\, dt \\ &= \int_{t_1}^{t_2} x'(t)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dt. \end{aligned}$$

Here we have assumed that $x'(t) > 0$, which is a reasonable assumption. The Chain Rule gives $dx = x'(t)\, dt$, and letting $a = x(t_1)$ and $b = x(t_2)$ we obtain the formula

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$

which is the formula for arc length obtained in the Introduction to the Applications of Integration.

Example 11.2.5: Finding the Arc Length of a Parametric Curve

Find the arc length of the semicircle defined by the equations

$$x(t) = 3\cos t, \quad y(t) = 3\sin t, \quad \text{for } 0 \le t \le \pi.$$

Solution

The values $t = 0$ to $t = \pi$ trace out the blue curve in Figure 11.2.8. To determine its length, use Equation 11.2.8:

$$\begin{aligned} s &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^{\pi} \sqrt{(-3\sin t)^2 + (3\cos t)^2}\, dt \\ &= \int_0^{\pi} \sqrt{9\sin^2 t + 9\cos^2 t}\, dt \\ &= \int_0^{\pi} \sqrt{9(\sin^2 t + \cos^2 t)}\, dt \\ &= \int_0^{\pi} 3\, dt = 3t\Big|_0^{\pi} \\ &= 3\pi \text{ units}. \end{aligned}$$

Note that the formula for the arc length of a semicircle is πr and the radius of this circle is 3. This is a great example of using calculus to derive a known formula of a geometric quantity.

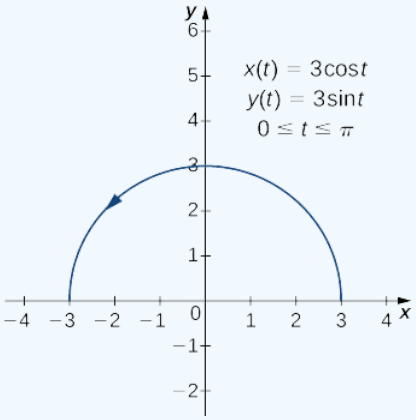


Figure 11.2.8: The arc length of the semicircle is equal to its radius times π.

Exercise 11.2.5

Find the arc length of the curve defined by the equations

$$x(t) = 3t^2, \quad y(t) = 2t^3, \quad \text{for } 1 \le t \le 3.$$

Hint

Use Equation 11.2.8.

Answer

$s = 2(10^{3/2} - 2^{3/2}) \approx 57.589$ units

We now return to the problem posed at the beginning of the section about a baseball leaving a pitcher's hand. Ignoring the effect of air resistance (unless it is a curve ball!), the ball travels a parabolic path. Assuming the pitcher's hand is at the origin and the ball travels left to right in the direction of the positive x-axis, the parametric equations for this curve can be written as

$$x(t) = 140t, \quad y(t) = -16t^2 + 2t$$

where t represents time. We first calculate the distance the ball travels as a function of time. This distance is represented by the arc length. We can modify the arc length formula slightly. First rewrite the functions $x(t)$ and $y(t)$ using v as an independent variable, so as to eliminate any confusion with the parameter t:

$$x(v) = 140v, \quad y(v) = -16v^2 + 2v.$$

Then we write the arc length formula as follows:

$$\begin{aligned} s(t) &= \int_0^t \sqrt{(\frac{dx}{dv})^2 + (\frac{dy}{dv})^2}\, dv \\ &= \int_0^t \sqrt{140^2 + (-32v + 2)^2}\, dv \end{aligned}$$

The variable v acts as a dummy variable that disappears after integration, leaving the arc length as a function of time t. To integrate this expression we can use a formula from Appendix A,

$$\int \sqrt{a^2 + u^2}\, du = \frac{u}{2}\sqrt{a^2 + u^2} + \frac{a^2}{2} \ln | u + \sqrt{a^2 + u^2} | + C.$$

We set $a = 140$ and $u = -32v + 2$. This gives $du = -32\, dv$, so $dv = -\frac{1}{32}\, du$. Therefore

$$\int \sqrt{140^2+(-32v+2)^2}\,dv = -\frac{1}{32}\int \sqrt{a^2+u^2}\,du$$
$$= -\frac{1}{32}\left[\frac{(-32v+2)}{2}\sqrt{140^2+(-32v+2)^2}+\frac{140^2}{2}\ln$$
$$|\,(-32v+2)+\sqrt{140^2+(-32v+2)^2}|+C\right]$$

and

$$s(t) = -\frac{1}{32}\left[\frac{(-32t+2)}{2}\sqrt{140^2+(-32t+2)^2}+\frac{140^2}{2}\ln\left|(-32t+2)+\sqrt{140^2+(-32t+2)^2}\right|\right]$$
$$+\frac{1}{32}\left[\sqrt{140^2+2^2}+\frac{140^2}{2}\ln\left|2+\sqrt{140^2+2^2}\right|\right]$$
$$= \left(\frac{t}{2}-\frac{1}{32}\right)\sqrt{1024t^2-128t+19604}-\frac{1225}{4}\ln\left|(-32t+2)+\sqrt{1024t^2-128t+19604}\right|$$
$$+\frac{\sqrt{19604}}{32}+\frac{1225}{4}\ln(2+\sqrt{19604})$$

This function represents the distance traveled by the ball as a function of time. To calculate the speed, take the derivative of this function with respect to t. While this may seem like a daunting task, it is possible to obtain the answer directly from the Fundamental Theorem of Calculus:

$$\frac{d}{dx}\int_a^x f(u)\,du = f(x).$$

Therefore

$$s'(t) = \frac{d}{dt}\Big[s(t)\Big]$$
$$= \frac{d}{dt}\left[\int_0^t \sqrt{140^2+(-32v+2)^2}\,dv\right]$$
$$= \sqrt{140^2+(-32t+2)^2}$$
$$= \sqrt{1024t^2-128t+19604}$$
$$= 2\sqrt{256t^2-32t+4901}.$$

One third of a second after the ball leaves the pitcher's hand, the distance it travels is equal to

$$s\left(\frac{1}{3}\right) = \left(\frac{1/3}{2}-\frac{1}{32}\right)\sqrt{1024\left(\frac{1}{3}\right)^2-128\left(\frac{1}{3}\right)+19604}$$
$$-\frac{1225}{4}\ln\left|\left(-32\left(\frac{1}{3}\right)+2\right)+\sqrt{1024\left(\frac{1}{3}\right)^2-128\left(\frac{1}{3}\right)+19604}\right|$$
$$+\frac{\sqrt{19604}}{32}+\frac{1225}{4}\ln(2+\sqrt{19604})$$
$$\approx 46.69 \text{ feet}.$$

This value is just over three quarters of the way to home plate. The speed of the ball is

$$s'\left(\tfrac{1}{3}\right) = 2\sqrt{256\left(\tfrac{1}{3}\right)^2-32\left(\tfrac{1}{3}\right)+4901} \approx 140.27 \text{ ft/s}.$$

This speed translates to approximately 95 mph—a major-league fastball.

Surface Area Generated by a Parametric Curve

Recall the problem of finding the surface area of a volume of revolution. In Curve Length and Surface Area, we derived a formula for finding the surface area of a volume generated by a function $y=f(x)$ from $x=a$ to $x=b$, revolved around the x-axis:

$$S = 2\pi\int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx.$$

We now consider a volume of revolution generated by revolving a parametrically defined curve $x = x(t), \quad y = y(t), \quad \text{for } a \le t \le b$ around the x-axis as shown in Figure 11.2.9.

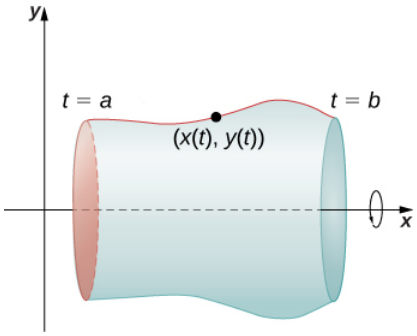


Figure 11.2.9: A surface of revolution generated by a parametrically defined curve.

The analogous formula for a parametrically defined curve is

$$S = 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt \tag{11.2.9}$$

provided that $y(t)$ is not negative on $[a, b]$.

Example 11.2.6: Finding Surface Area

Find the surface area of a sphere of radius r centered at the origin.

Solution

We start with the curve defined by the equations

$$x(t) = r\cos t, \quad y(t) = r\sin t, \quad \text{for } 0 \le t \le \pi.$$

This generates an upper semicircle of radius r centered at the origin as shown in the following graph.

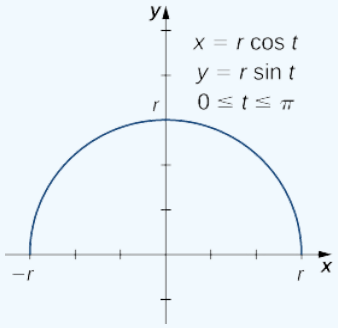


Figure 11.2.10: A semicircle generated by parametric equations.

When this curve is revolved around the x-axis, it generates a sphere of radius r. To calculate the surface area of the sphere, we use Equation 11.2.9:

$$
\begin{aligned}
S &= 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt \\
&= 2\pi \int_0^{\pi} r\sin t\sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt \\
&= 2\pi \int_0^{\pi} r\sin t\sqrt{r^2\sin^2 t + r^2\cos^2 t}\, dt \\
&= 2\pi \int_0^{\pi} r\sin t\sqrt{r^2(\sin^2 t + \cos^2 t)}\, dt \\
&= 2\pi \int_0^{\pi} r^2 \sin t\, dt \\
&= 2\pi r^2 \left(-\cos t\Big|_0^{\pi}\right) \\
&= 2\pi r^2(-\cos\pi + \cos 0) \\
&= 4\pi r^2 \text{ units}^2.
\end{aligned}
$$

This is, in fact, the formula for the surface area of a sphere.

? Exercise 11.2.6

Find the surface area generated when the plane curve defined by the equations

$$x(t) = t^3, \quad y(t) = t^2, \quad \text{for } 0 \le t \le 1$$

is revolved around the x-axis.

Hint

Use Equation 11.2.9. When evaluating the integral, use a u-substitution.

Answer

$$A = \frac{\pi(494\sqrt{13} + 128)}{1215} \text{ units}^2$$

Key Concepts

- The derivative of the parametrically defined curve $x = x(t)$ and $y = y(t)$ can be calculated using the formula $\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$. Using the derivative, we can find the equation of a tangent line to a parametric curve.
- The area between a parametric curve and the x-axis can be determined by using the formula $A = \int_{t_1}^{t_2} y(t)x'(t)\, dt$.
- The arc length of a parametric curve can be calculated by using the formula

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

- The surface area of a volume of revolution revolved around the x-axis is given by

$$S = 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

- If the curve is revolved around the y-axis, then the formula is

$$S = 2\pi \int_a^b x(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

Key Equations

- **Derivative of parametric equations**

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}$$

- **Second-order derivative of parametric equations**

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{(d/dt)(dy/dx)}{dx/dt}$$

- **Area under a parametric curve**

$$A = \int_a^b y(t)x'(t)\,dt$$

- **Arc length of a parametric curve**

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

- **Surface area generated by a parametric curve**

$$S = 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

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11.2E: EXERCISES FOR SECTION 11.2

In exercises 1 - 4, each set of parametric equations represents a line. Without eliminating the parameter, find the slope of each line.

1) $x = 3 + t, \quad y = 1 - t$

2) $x = 8 + 2t, \quad y = 1$

Answer

$m = 0$

3) $x = 4 - 3t, \quad y = -2 + 6t$

4) $x = -5t + 7, \quad y = 3t - 1$

Answer

$m = -\frac{3}{5}$

In exercises 5 - 9, determine the slope of the tangent line, then find the equation of the tangent line at the given value of the parameter.

5) $x = 3\sin t, \quad y = 3\cos t, \quad \text{for } t = \frac{\pi}{4}$

6) $x = \cos t, \quad y = 8\sin t, \quad \text{for } t = \frac{\pi}{2}$

Answer

Slope $= 0; y = 8.$

7) $x = 2t, \quad y = t^3, \quad \text{for } t = -1$

8) $x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}, \quad \text{for } t = 1$

Answer

Slope is undefined; $x = 2$.

9) $x = \sqrt{t}, \quad y = 2t, \quad \text{for } t = 4$

In exercises 10 - 13, find all points on the curve that have the given slope.

10) $x = 4\cos t, \quad y = 4\sin t$, slope = 0.5

Solution

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4\cos t}{-4\sin t} = -\cot t.$

Setting this derivative equal to 0.5, we obtain the equation, $\tan t = -2$.

$\tan t = -2 \implies \frac{y}{x} = -2 \implies y = -2x.$

Note also that this pair of parametric equations represents the circle $x^2 + y^2 = 16$.

By substitution, we find that this curve has a slope of 0.5 at the points:

$\left(\frac{4\sqrt{5}}{5}, \frac{-8\sqrt{5}}{5}\right)$ and $\left(\frac{-4\sqrt{5}}{5}, \frac{8\sqrt{5}}{5}\right)$.

11) $x = 2\cos t, \quad y = 8\sin t$, slope= -1

12) $x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}$, slope= 1

Answer

No points possible; undefined expression.

13) $x = 2 + \sqrt{t}, \quad y = 2 - 4t$, slope= 0

In exercises 14 - 16, write the equation of the tangent line in Cartesian coordinates for the given parameter t.

14) $x = e^{\sqrt{t}}, \quad y = 1 - \ln t^2, \quad \text{for } t = 1$

Answer

$$y = -\left(\frac{4}{e}\right)x + 5$$

15) $x = t\ln t, \quad y = \sin^2 t, \quad \text{for } t = \frac{\pi}{4}$

16) $x = e^t, \quad y = (t-1)^2$, at $(1,1)$

Answer

$$y = -2x + 3$$

17) For $x = \sin(2t), \quad y = 2\sin t$ where $0 \le t < 2\pi$. Find all values of t at which a horizontal tangent line exists.

18) For $x = \sin(2t), \quad y = 2\sin t$ where $0 \le t < 2\pi$. Find all values of t at which a vertical tangent line exists.

Answer

A vertical tangent line exists at $t = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$

19) Find all points on the curve $x = 4\cos(t), \quad y = 4\sin(t)$ that have the slope of $\frac{1}{2}$.

20) Find $\frac{dy}{dx}$ for $x = \sin(t), \quad y = \cos(t)$.

Answer

$$\frac{dy}{dx} = -\tan(t)$$

21) Find the equation of the tangent line to $x = \sin(t), \quad y = \cos(t)$ at $t = \frac{\pi}{4}$.

22) For the curve $x = 4t, \quad y = 3t - 2$, find the slope and concavity of the curve at $t = 3$.

Answer

$\frac{dy}{dx} = \frac{3}{4}$ and $\frac{d^2y}{dx^2} = 0$, so the curve is neither concave up nor concave down at $t = 3$. Therefore the graph is linear and has a constant slope but no concavity.

23) For the parametric curve whose equation is $x = 4\cos\theta, \quad y = 4\sin\theta$, find the slope and concavity of the curve at $\theta = \frac{\pi}{4}$.

24) Find the slope and concavity for the curve whose equation is $x = 2 + \sec\theta, \quad y = 1 + 2\tan\theta$ at $\theta = \frac{\pi}{6}$.

Answer

$\frac{dy}{dx} = 4, \quad \frac{d^2y}{dx^2} = -4\sqrt{3}$; the curve is concave down at $\theta = \frac{\pi}{6}$.

25) Find all points on the curve $x = t + 4, \quad y = t^3 - 3t$ at which there are vertical and horizontal tangents.

26) Find all points on the curve $x = \sec\theta, \quad y = \tan\theta$ at which horizontal and vertical tangents exist.

Answer

No horizontal tangents. Vertical tangents at $(1,0)$ and $(-1,0)$.

In exercises 27 - 29, find d^2y/dx^2.

27) $x = t^4 - 1, \quad y = t - t^2$

28) $x = \sin(\pi t), \quad y = \cos(\pi t)$

Answer

$$d^2y/dx^2 = -\sec^3(\pi t)$$

29) $x = e^{-t}, \quad y = te^{2t}$

In exercises 30 - 31, find points on the curve at which tangent line is horizontal or vertical.

30) $x = t(t^2 - 3), \quad y = 3(t^2 - 3)$

Answer

Horizontal $(0, -9)$;
Vertical $(\pm 2, -6)$.

31) $x = \dfrac{3t}{1+t^3}, \quad y = \dfrac{3t^2}{1+t^3}$

In exercises 32 - 34, find dy/dx at the value of the parameter.

32) $x = \cos t, y = \sin t, \quad \text{for } t = \frac{3\pi}{4}$

Answer

$dy/dx = 1$

33) $x = \sqrt{t}, \quad y = 2t + 4, t = 9$

34) $x = 4\cos(2\pi s), \quad y = 3\sin(2\pi s), \quad \text{for } s = -\frac{1}{4}$

Answer

$dy/dx = 0$

In exercises 35 - 36, find d^2y/dx^2 at the given point without eliminating the parameter.

35) $x = \frac{1}{2}t^2, \quad y = \frac{1}{3}t^3, \quad \text{for } t = 2$

36) $x = \sqrt{t}, \quad y = 2t + 4, \quad \text{for } t = 1$

Answer

$d^2y/dx^2 = 4$

37) Find intervals for t on which the curve $x = 3t^2, \quad y = t^3 - t$ is concave up as well as concave down.

38) Determine the concavity of the curve $x = 2t + \ln t, \quad y = 2t - \ln t$.

Answer

Concave up on $t > 0$.

39) Sketch and find the area under one arch of the cycloid $x = r(\theta - \sin\theta), \quad y = r(1 - \cos\theta)$.

40) Find the area bounded by the curve $x = \cos t, \quad y = e^t, \quad \text{for } 0 \le t \le \frac{\pi}{2}$ and the lines $y = 1$ and $x = 0$.

Answer

1 unit^2

41) Find the area enclosed by the ellipse $x = a\cos\theta, \quad y = b\sin\theta, \quad \text{for } 0 \le \theta < 2\pi$.

42) Find the area of the region bounded by $x = 2\sin^2\theta, \quad y = 2\sin^2\theta\tan\theta$, for $0 \le \theta \le \frac{\pi}{2}$.

Answer

$\frac{3\pi}{2} \text{ units}^2$

In exercises 43 - 46, find the area of the regions bounded by the parametric curves and the indicated values of the parameter.

43) $x = 2\cot\theta, \quad y = 2\sin^2\theta, \quad \text{for } 0 \le \theta \le \pi$

44) [T] $x = 2a\cos t - a\cos(2t), \quad y = 2a\sin t - a\sin(2t), \quad \text{for } 0 \le t < 2\pi$

Answer

$6\pi a^2 \text{ units}^2$

45) [T] $x = a\sin(2t), \quad y = b\sin(t), \quad \text{for } 0 \le t < 2\pi$ (the “hourglass”)

46) [T] $x = 2a\cos t - a\sin(2t), \quad y = b\sin t, \quad \text{for } 0 \le t < 2\pi$ (the “teardrop”)

Answer

$2\pi ab \text{ units}^2$

In exercises 47 - 52, find the arc length of the curve on the indicated interval of the parameter.

47) $x = 4t + 3, \quad y = 3t - 2, \quad \text{for } 0 \le t \le 2$

48) $x = \frac{1}{3}t^3, \quad y = \frac{1}{2}t^2, \quad \text{for } 0 \le t \le 1$

Answer

$s = \frac{1}{3}(2\sqrt{2} - 1)$ units

49) $x = \cos(2t), \quad y = \sin(2t), \quad \text{for } 0 \le t \le \frac{\pi}{2}$

50) $x = 1 + t^2, \quad y = (1 + t)^3, \quad \text{for } 0 \le t \le 1$

Answer

$s = 7.075$ units

51) $x = e^t \cos t, \quad y = e^t \sin t, \quad \text{for } 0 \le t \le \frac{\pi}{2}$ (express answer as a decimal rounded to three places)

52) $x = a\cos^3\theta, \quad y = a\sin^3\theta$ on the interval $[0, 2\pi)$ (the hypocycloid)

Answer

$s = 6a$ units

53) Find the length of one arch of the cycloid $x = 4(t - \sin t), \quad y = 4(1 - \cos t)$.

54) Find the distance traveled by a particle with position (x, y) as t varies in the given time interval: $x = \sin^2 t, \quad y = \cos^2 t, \quad \text{for } 0 \le t \le 3\pi$.

Answer

$6\sqrt{2}$ units

55) Find the length of one arch of the cycloid $x = \theta - \sin\theta, \quad y = 1 - \cos\theta$.

56) Show that the total length of the ellipse $x = 4\sin\theta, \quad y = 3\cos\theta$ is $L = 16\int_0^{\pi/2} \sqrt{1 - e^2\sin^2\theta}\, d\theta$, where $e = \frac{c}{a}$ and $c = \sqrt{a^2 - b^2}$.

57) Find the length of the curve $x = e^t - t, \quad y = 4e^{t/2}, \quad \text{for } -8 \le t \le 3$.

In exercises 58 - 59, find the area of the surface obtained by rotating the given curve about the x-axis.

58) $x = t^3, \quad y = t^2, \quad \text{for } 0 \le t \le 1$

Answer

$\frac{2\pi(247\sqrt{13} + 64)}{1215}$ units^2

59) $x = a\cos^3\theta, \quad y = a\sin^3\theta, \quad \text{for } 0 \le \theta \le \frac{\pi}{2}$

60) [T] Use a CAS to find the area of the surface generated by rotating $x = t + t^3, \quad y = t - \frac{1}{t^2}, \quad \text{for } 1 \le t \le 2$ about the x-axis. (Answer to three decimal places.)

Answer

59.101 units^2

61) Find the surface area obtained by rotating $x = 3t^2, \quad y = 2t^3, \quad \text{for } 0 \le t \le 5$ about the y-axis.

62) Find the area of the surface generated by revolving $x = t^2, \quad y = 2t, \quad \text{for } 0 \le t \le 4$ about the x-axis.

Answer

$\frac{8\pi}{3}(17\sqrt{17} - 1)$ units^2

63) Find the surface area generated by revolving $x = t^2, \quad y = 2t^2, \quad \text{for } 0 \le t \le 1$ about the y-axis.

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11.3: Polar Coordinates

Learning Objectives

- Locate points in a plane by using polar coordinates.
- Convert points between rectangular and polar coordinates.
- Sketch polar curves from given equations.
- Convert equations between rectangular and polar coordinates.
- Identify symmetry in polar curves and equations.

The rectangular coordinate system (or Cartesian plane) provides a means of mapping points to ordered pairs and ordered pairs to points. This is called a *one-to-one mapping* from points in the plane to ordered pairs. The polar coordinate system provides an alternative method of mapping points to ordered pairs. In this section we see that in some circumstances, polar coordinates can be more useful than rectangular coordinates.

Defining Polar Coordinates

To find the coordinates of a point in the polar coordinate system, consider Figure 11.3.1. The point P has Cartesian coordinates (x, y). The line segment connecting the origin to the point P measures the distance from the origin to P and has length r. The angle between the positive x-axis and the line segment has measure θ. This observation suggests a natural correspondence between the coordinate pair (x, y) and the values r and θ. This correspondence is the basis of the **polar coordinate system**. Note that every point in the Cartesian plane has two values (hence the term ordered pair) associated with it. In the polar coordinate system, each point also has two values associated with it: r and θ.

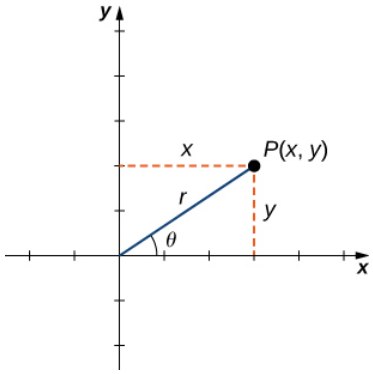


Figure 11.3.1: An arbitrary point in the Cartesian plane.

Using right-triangle trigonometry, the following equations are true for the point P:

$$\cos\theta = \frac{x}{r} \text{ so } x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \text{ so } y = r\sin\theta.$$

Furthermore,

$$r^2 = x^2 + y^2$$

and

$$\tan\theta = \frac{y}{x}.$$

Each point (x, y) in the Cartesian coordinate system can therefore be represented as an ordered pair (r, θ) in the polar coordinate system. The first coordinate is called the **radial coordinate** and the second coordinate is called the **angular coordinate**. Every point in the plane can be represented in this form.

Note that the equation $\tan\theta = y/x$ has an infinite number of solutions for any ordered pair (x, y). However, if we restrict the solutions to values between 0 and 2π then we can assign a unique solution to the quadrant in which the original point (x, y) is

located. Then the corresponding value of r is positive, so $r^2 = x^2 + y^2$.

Converting Points between Coordinate Systems

Given a point P in the plane with Cartesian coordinates (x, y) and polar coordinates (r, θ), the following conversion formulas hold true:

$$x = r\cos\theta \tag{11.3.1}$$
$$y = r\sin\theta \tag{11.3.2}$$

and

$$r^2 = x^2 + y^2 \tag{11.3.3}$$
$$\tan\theta = \frac{y}{x} \tag{11.3.4}$$

These formulas can be used to convert from rectangular to polar or from polar to rectangular coordinates. Notice that Equation 11.3.3 is the *Pythagorean theorem*. (Figure 11.3.1).

Example 11.3.1: Converting between Rectangular and Polar Coordinates

Convert each of the following points into polar coordinates.

a. $(1, 1)$
b. $(-3, 4)$
c. $(0, 3)$
d. $(5\sqrt{3}, -5)$

Convert each of the following points into rectangular coordinates.

e. $(3, \pi/3)$
f. $(2, 3\pi/2)$
g. $(6, -5\pi/6)$

Solution

a. Use $x = 1$ and $y = 1$ in Equation 11.3.3:

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= 1^2 + 1^2 \\ r &= \sqrt{2} \end{aligned}$$

and via Equation 11.3.4

$$\begin{aligned} \tan\theta = \frac{y}{x} = \frac{1}{1} = 1 \\ \theta = \frac{\pi}{4}. \end{aligned}$$

Therefore this point can be represented as $(\sqrt{2}, \frac{\pi}{4})$ in polar coordinates.

b. Use $x = -3$ and $y = 4$ in Equation 11.3.3:

$$\begin{aligned} r^2 = x^2 + y^2 = (-3)^2 + (4)^2 \\ r = 5 \end{aligned}$$

and via Equation 11.3.4

$$\begin{aligned} \tan\theta = \frac{y}{x} = -\frac{4}{3} \\ \theta = \arctan(-\frac{4}{3}) + \pi \approx 2.21. \end{aligned}$$

Therefore this point can be represented as $(5, 2.21)$ in polar coordinates.

c. Use $x = 0$ and $y = 3$ in Equation 11.3.3:

$$r^2 = x^2 + y^2 = (3)^2 + (0)^2 = 9 + 0 \; r = 3$$

and via Equation 11.3.4

$$\tan\theta = \frac{y}{x} = \frac{3}{0}.$$

Direct application of the second equation leads to division by zero. Graphing the point $(0, 3)$ on the rectangular coordinate system reveals that the point is located on the positive y-axis. The angle between the positive x-axis and the positive y-axis is $\frac{\pi}{2}$. Therefore this point can be represented as $(3, \frac{\pi}{2})$ in polar coordinates.

d. Use $x = 5\sqrt{3}$ and $y = -5$ in Equation 11.3.3:

$$r^2 = x^2 + y^2 = (5\sqrt{3})^2 + (-5)^2 = 75 + 25$$

$$r = 10$$

and via Equation 11.3.4

$$\tan\theta = \frac{y}{x} = \frac{-5}{5\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

$$\theta = -\frac{\pi}{6}.$$

Therefore this point can be represented as $(10, -\frac{\pi}{6})$ in polar coordinates.

e. Use $r = 3$ and $\theta = \frac{\pi}{3}$ in Equation 11.3.1:

$$x = r\cos\theta = 3\cos(\frac{\pi}{3}) = 3(\frac{1}{2}) = \frac{3}{2}$$

and

$$y = r\sin\theta = 3\sin(\frac{\pi}{3}) = 3(\frac{\sqrt{3}}{2}) = \frac{3\sqrt{3}}{2}.$$

Therefore this point can be represented as $(\frac{3}{2}, \frac{3\sqrt{3}}{2})$ in rectangular coordinates.

f. Use $r = 2$ and $\theta = \frac{3\pi}{2}$ in Equation 11.3.1:

$$x = r\cos\theta = 2\cos(\frac{3\pi}{2}) = 2(0) = 0$$

and

$$y = r\sin\theta = 2\sin(\frac{3\pi}{2}) = 2(-1) = -2.$$

Therefore this point can be represented as $(0, -2)$ in rectangular coordinates.

g. Use $r = 6$ and $\theta = -\frac{5\pi}{6}$ in Equation 11.3.1:

$$x = r\cos\theta = 6\cos(-\frac{5\pi}{6}) = 6(-\frac{\sqrt{3}}{2}) = -3\sqrt{3}$$

and

$$y = r\sin\theta = 6\sin(-\frac{5\pi}{6}) = 6(-\frac{1}{2}) = -3.$$

Therefore this point can be represented as $(-3\sqrt{3}, -3)$ in rectangular coordinates.

? Exercise 11.3.1

Convert $(-8, -8)$ into polar coordinates and $(4, \frac{2\pi}{3})$ into rectangular coordinates.

Hint

Use Equation 11.3.3 and Equation 11.3.1. Make sure to check the quadrant when calculating θ.

Answer

$(8\sqrt{2}, \frac{5\pi}{4})$ and $(-2, 2\sqrt{3})$

The polar representation of a point is not unique. For example, the polar coordinates $(2, \frac{\pi}{3})$ and $(2, \frac{7\pi}{3})$ both represent the point $(1, \sqrt{3})$ in the rectangular system. Also, the value of r can be negative. Therefore, the point with polar coordinates $(-2, \frac{4\pi}{3})$ also represents the point $(1, \sqrt{3})$ in the rectangular system, as we can see by using Equation 11.3.1:

$$x = r\cos\theta = -2\cos(\frac{4\pi}{3}) = -2(-\frac{1}{2}) = 1$$

and

$$y = r\sin\theta = -2\sin(\frac{4\pi}{3}) = -2(-\frac{\sqrt{3}}{2}) = \sqrt{3}.$$

Every point in the plane has an infinite number of representations in polar coordinates. However, each point in the plane has only one representation in the rectangular coordinate system.

Note that the polar representation of a point in the plane also has a visual interpretation. In particular, r is the directed distance that the point lies from the origin, and θ measures the angle that the line segment from the origin to the point makes with the positive x-axis. Positive angles are measured in a counterclockwise direction and negative angles are measured in a clockwise direction. The polar coordinate system appears in Figure 11.3.2.

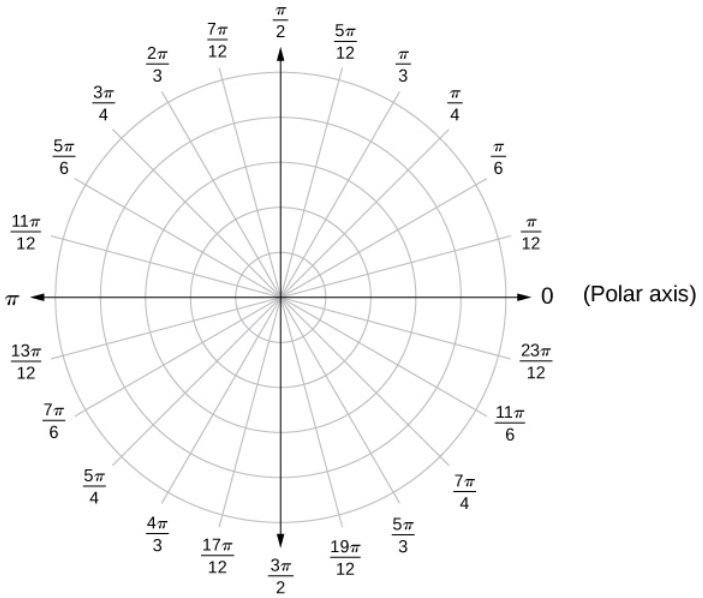


Figure 11.3.2: The polar coordinate system.

The line segment starting from the center of the graph going to the right (called the positive x-axis in the Cartesian system) is the **polar axis.** The center point is the **pole**, or origin, of the coordinate system, and corresponds to $r = 0$. The innermost circle shown in Figure 11.3.2 contains all points a distance of 1 unit from the pole, and is represented by the equation $r = 1$. Then $r = 2$ is the set of points 2 units from the pole, and so on. The line segments emanating from the pole correspond to fixed angles. To plot a point in the polar coordinate system, start with the angle. If the angle is positive, then measure the angle from the polar axis in a counterclockwise direction. If it is negative, then measure it clockwise. If the value of r is positive, move that distance along the terminal ray of the angle. If it is negative, move along the ray that is opposite the terminal ray of the given angle.

Example 11.3.2: Plotting Points in the Polar Plane

Plot each of the following points on the polar plane.

a. $(2, \frac{\pi}{4})$

b. $(-3, \frac{2\pi}{3})$
c. $(4, \frac{5\pi}{4})$

Solution

The three points are plotted in Figure 11.3.3.

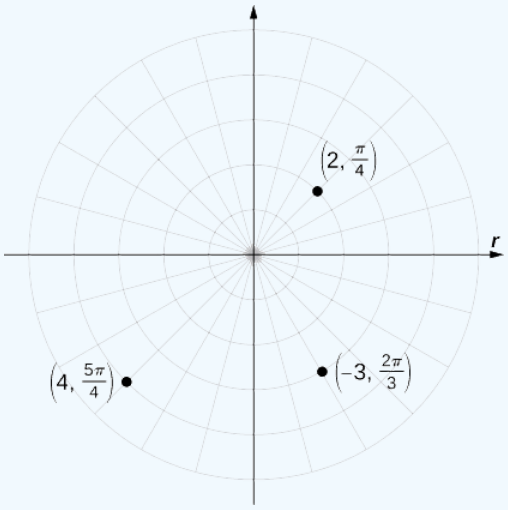


Figure 11.3.3: Three points plotted in the polar coordinate system.

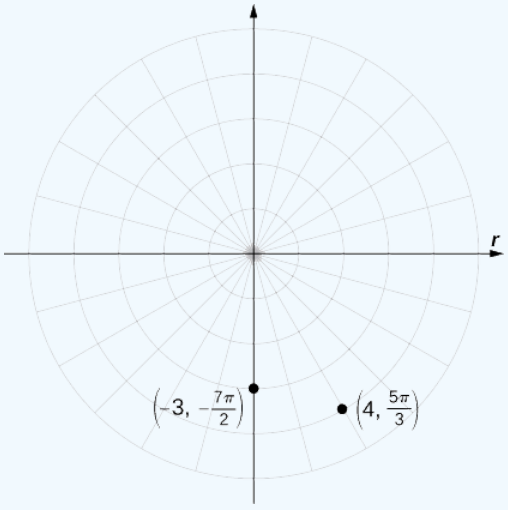
? Exercise 11.3.2

Plot $(4, \frac{5\pi}{3})$ and $(-3, -\frac{7\pi}{2})$ on the polar plane.

Hint

Start with θ, then use r.

Answer



Polar Curves

Now that we know how to plot points in the polar coordinate system, we can discuss how to plot curves. In the rectangular coordinate system, we can graph a function $y = f(x)$ and create a curve in the Cartesian plane. In a similar fashion, we can graph a curve that is generated by a function $r = f(\theta)$.

The general idea behind graphing a function in polar coordinates is the same as graphing a function in rectangular coordinates. Start with a list of values for the independent variable (θ in this case) and calculate the corresponding values of the dependent variable r. This process generates a list of ordered pairs, which can be plotted in the polar coordinate system. Finally, connect the points, and take advantage of any patterns that may appear. The function may be periodic, for example, which indicates that only a limited number of values for the independent variable are needed.

Problem-Solving Strategy: Plotting a Curve in Polar Coordinates

1. Create a table with two columns. The first column is for θ, and the second column is for r.
2. Create a list of values for θ.
3. Calculate the corresponding r values for each θ.
4. Plot each ordered pair (r, θ) on the coordinate axes.
5. Connect the points and look for a pattern.

Example 11.3.3: Graphing a Function in Polar Coordinates

Graph the curve defined by the function $r = 4 \sin \theta$. Identify the curve and rewrite the equation in rectangular coordinates.

Solution

Because the function is a multiple of a sine function, it is periodic with period 2π, so use values for θ between 0 and 2π. The result of steps 1–3 appear in the following table. Figure 11.3.4 shows the graph based on this table.

Table 11.3.1: Table of values for this polar function.

θ	$r = 4 \sin \theta$	θ	$r = 4 \sin \theta$
0	0	π	0
$\frac{\pi}{6}$	2	$\frac{7\pi}{6}$	-2
$\frac{\pi}{4}$	$2\sqrt{2} \approx 2.8$	$\frac{5\pi}{4}$	$-2\sqrt{2} \approx -2.8$
$\frac{\pi}{3}$	$2\sqrt{3} \approx 3.4$	$\frac{4\pi}{3}$	$-2\sqrt{3} \approx -3.4$
$\frac{\pi}{2}$	4	$\frac{3\pi}{2}$	-4
$\frac{2\pi}{3}$	$2\sqrt{3} \approx 3.4$	$\frac{5\pi}{3}$	$-2\sqrt{3} \approx -3.4$
$\frac{3\pi}{4}$	$2\sqrt{2} \approx 2.8$	$\frac{7\pi}{4}$	$-2\sqrt{2} \approx -2.8$
$\frac{5\pi}{6}$	2	$\frac{11\pi}{6}$	-2
		2π	0

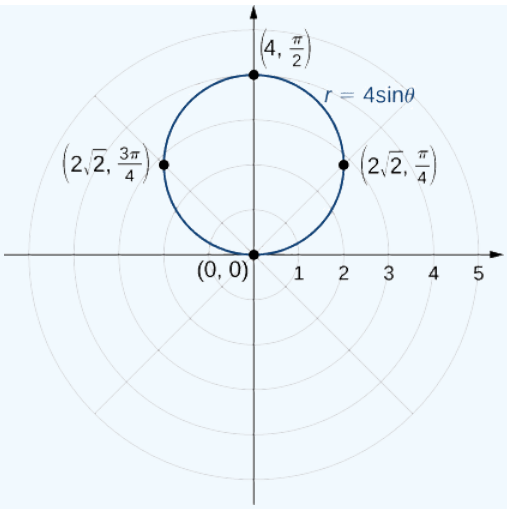


Figure 11.3.4: The graph of the function $r = 4\sin\theta$ is a circle.

This is the graph of a circle. The equation $r = 4\sin\theta$ can be converted into rectangular coordinates by first multiplying both sides by r. This gives the equation $r^2 = 4r\sin\theta$. Next use the facts that $r^2 = x^2 + y^2$ and $y = r\sin\theta$. This gives $x^2 + y^2 = 4y$. To put this equation into standard form, subtract $4y$ from both sides of the equation and complete the square:

$$x^2 + y^2 - 4y = 0$$
$$x^2 + (y^2 - 4y) = 0$$
$$x^2 + (y^2 - 4y + 4) = 0 + 4$$
$$x^2 + (y - 2)^2 = 4$$

This is the equation of a circle with radius 2 and center $(0, 2)$ in the rectangular coordinate system.

? Exercise 11.3.3

Create a graph of the curve defined by the function $r = 4 + 4\cos\theta$.

Hint

Follow the problem-solving strategy for creating a graph in polar coordinates.

Answer

The name of this shape is a cardioid, which we will study further later in this section.

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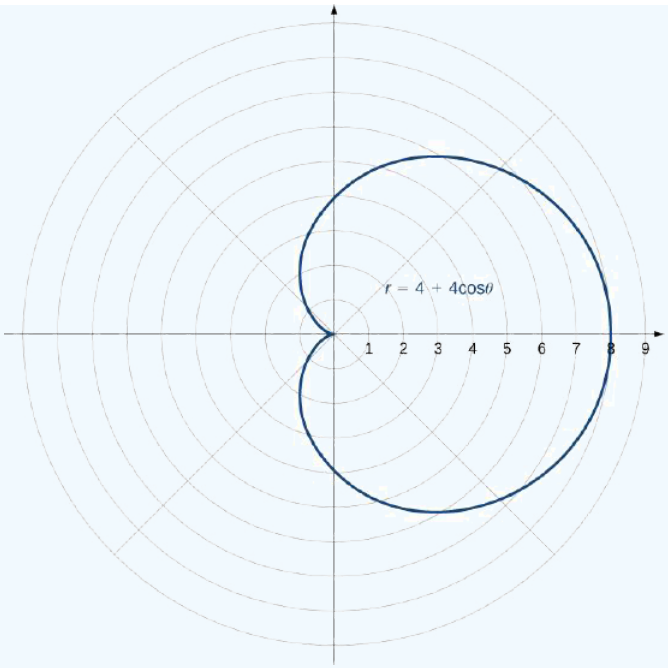


The graph in Example 11.3.3 was that of a circle. The equation of the circle can be transformed into rectangular coordinates using the coordinate transformation formulas in Equation 11.3.1. Example 11.3.4 gives some more examples of functions for transforming from polar to rectangular coordinates.

Example 11.3.4: Transforming Polar Equations to Rectangular Coordinates

Rewrite each of the following equations in rectangular coordinates and identify the graph.

a. $\theta = \dfrac{\pi}{3}$
b. $r = 3$
c. $r = 6\cos\theta - 8\sin\theta$

Solution:

a. Take the tangent of both sides. This gives $\tan\theta = \tan(\pi/3) = \sqrt{3}$. Since $\tan\theta = y/x$ we can replace the left-hand side of this equation by y/x. This gives $y/x = \sqrt{3}$, which can be rewritten as $y = x\sqrt{3}$. This is the equation of a straight line passing through the origin with slope $\sqrt{3}$. In general, any polar equation of the form $\theta = K$ represents a straight line through the pole with slope equal to $\tan K$.

b. First, square both sides of the equation. This gives $r^2 = 9$. Next replace r^2 with $x^2 + y^2$. This gives the equation $x^2 + y^2 = 9$, which is the equation of a circle centered at the origin with radius 3. In general, any polar equation of the form $r = k$ where *k* is a positive constant represents a circle of radius *k* centered at the origin. (Note: when squaring both sides of an equation it is possible to introduce new points unintentionally. This should always be taken into consideration. However, in this case we do not introduce new points. For example, $(-3, \dfrac{\pi}{3})$ is the same point as $(3, \dfrac{4\pi}{3})$.)

c. Multiply both sides of the equation by r. This leads to $r^2 = 6r\cos\theta - 8r\sin\theta$. Next use the formulas

$$r^2 = x^2 + y^2,\ x = r\cos\theta,\ y = r\sin\theta.$$

This gives

$$r^2 = 6(r\cos\theta) - 8(r\sin\theta)$$
$$x^2 + y^2 = 6x - 8y.$$

To put this equation into standard form, first move the variables from the right-hand side of the equation to the left-hand side, then complete the square.

$$x^2 + y^2 = 6x - 8y$$

$$x^2 - 6x + y^2 + 8y = 0$$

$$(x^2 - 6x) + (y^2 + 8y) = 0$$

$$(x^2 - 6x + 9) + (y^2 + 8y + 16) = 9 + 16$$

$$(x - 3)^2 + (y + 4)^2 = 25.$$

This is the equation of a circle with center at $(3, -4)$ and radius 5. Notice that the circle passes through the origin since the center is 5 units away.

? Exercise 11.3.4

Rewrite the equation $r = \sec\theta \tan\theta$ in rectangular coordinates and identify its graph.

Hint

Convert to sine and cosine, then multiply both sides by cosine.

Answer

$y = x^2$, which is the equation of a parabola opening upward.

We have now seen several examples of drawing graphs of curves defined by **polar equations.** A summary of some common curves is given in the tables below. In each equation, *a* and *b* are arbitrary constants.

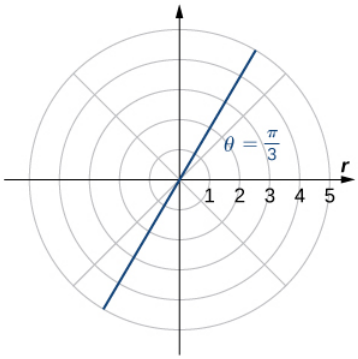
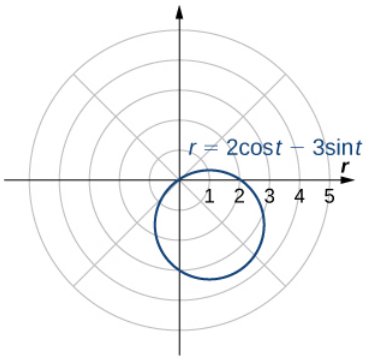
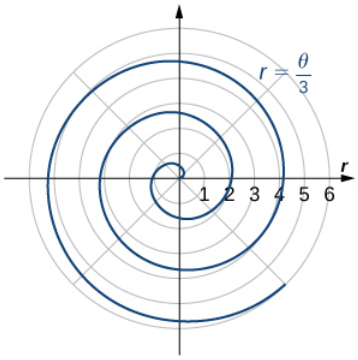
Name	Equation	Example
Line passing through the pole with slope tan K	$\theta = K$	
Circle	$r = a\cos\theta + b\sin\theta$	
Spiral	$r = a + b\theta$	

Figure 11.3.5

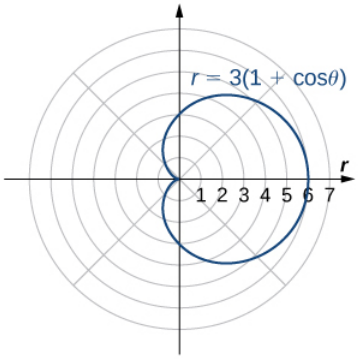
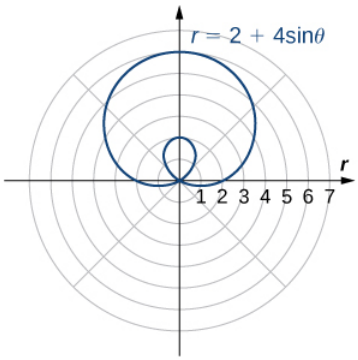
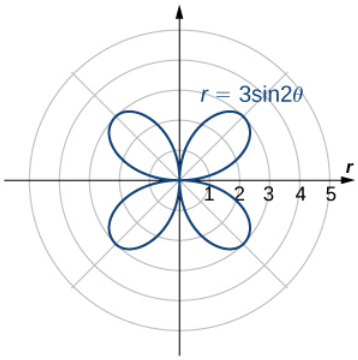
Name	Equation	Example
Cardioid	$r = a(1 + \cos\theta)$ $r = a(1 - \cos\theta)$ $r = a(1 + \sin\theta)$ $r = a(1 - \sin\theta)$	
Limaçon	$r = a\cos\theta + b$ $r = a\sin\theta + b$	
Rose	$r = a\cos(b\theta)$ $r = a\sin(b\theta)$	

Figure 11.3.6

A **cardioid** is a special case of a **limaçon** (pronounced “lee-mah-son”), in which $a = b$ or $a = -b$. The **rose i**s a very interesting curve. Notice that the graph of $r = 3\sin 2\theta$ has four petals. However, the graph of $r = 3\sin 3\theta$ has three petals as shown.

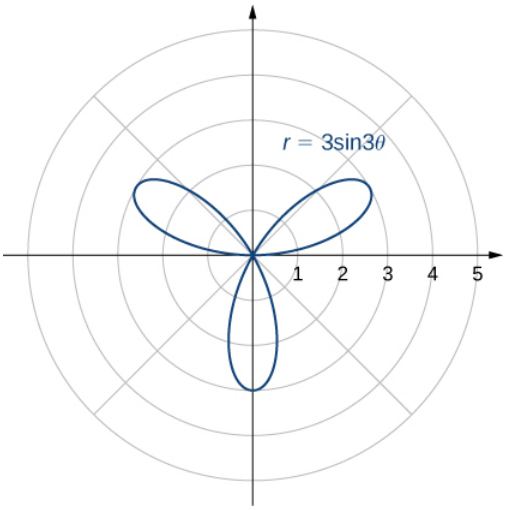


Figure 11.3.7: Graph of $r = 3\sin 3\theta$.

If the coefficient of θ is even, the graph has twice as many petals as the coefficient. If the coefficient of θ is odd, then the number of petals equals the coefficient. You are encouraged to explore why this happens. Even more interesting graphs emerge when the coefficient of θ is not an integer. For example, if it is rational, then the curve is closed; that is, it eventually ends where it started (Figure 11.3.8a). However, if the coefficient is irrational, then the curve never closes (Figure 11.3.8b). Although it may appear that the curve is closed, a closer examination reveals that the petals just above the positive x axis are slightly thicker. This is because the petal does not quite match up with the starting point.

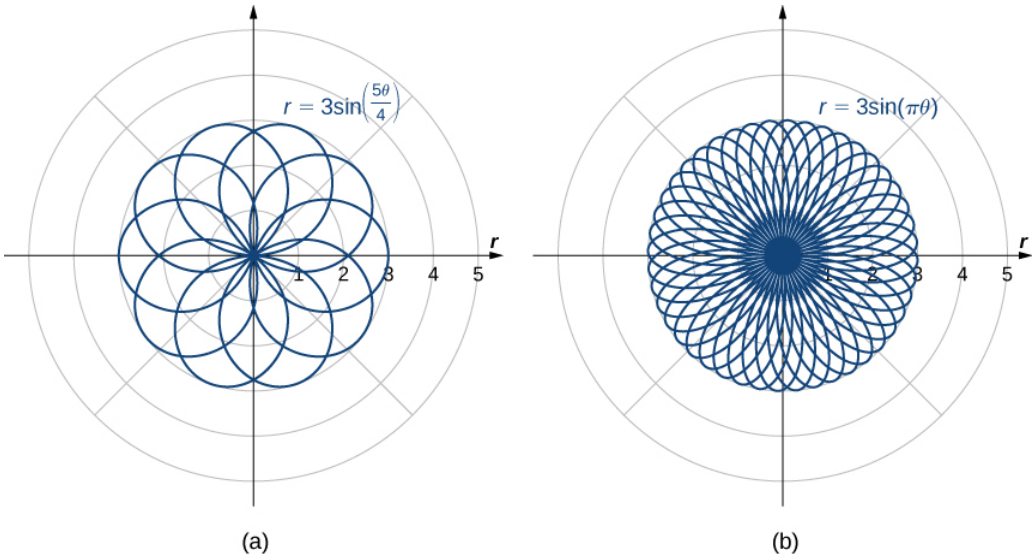


Figure 11.3.8: Polar rose graphs of functions with (a) rational coefficient and (b) irrational coefficient. Note that the rose in part (b) would actually fill the entire circle if plotted in full.

Since the curve defined by the graph of $r = 3\sin(\pi\theta)$ never closes, the curve depicted in Figure 11.3.8b is only a partial depiction. In fact, this is an example of a **space-filling curve.** A space-filling curve is one that in fact occupies a two-dimensional subset of the real plane. In this case the curve occupies the circle of radius 3 centered at the origin.

Example 11.3.5: Describing a Spiral

Recall the **chambered nautilus** introduced in the chapter prelude. This creature displays a spiral when half the outer shell is cut away. It is possible to describe a spiral using rectangular coordinates. Figure 11.3.9 shows a spiral in rectangular coordinates. How can we describe this curve mathematically?

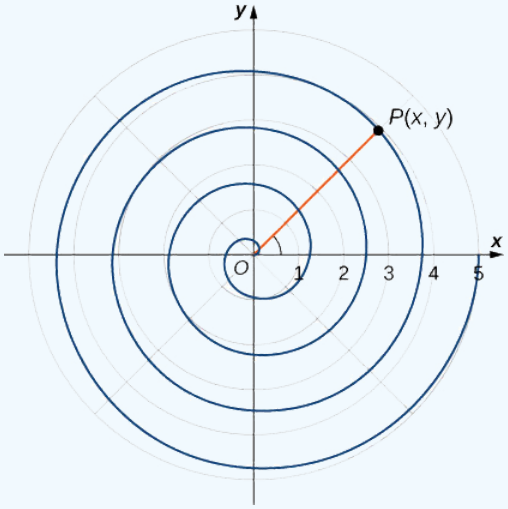


Figure 11.3.9: How can we describe a spiral graph mathematically?

Solution

As the point *P* travels around the spiral in a counterclockwise direction, its distance d from the origin increases. Assume that the distance d is a constant multiple k of the angle θ that the line segment OP makes with the positive x-axis. Therefore $d(P,O) = k\theta$, where *O* is the origin. Now use the distance formula and some trigonometry:

$$d(P,O) = k\theta$$

$$\sqrt{(x-0)^2+(y-0)^2} = k\arctan(\frac{y}{x})$$

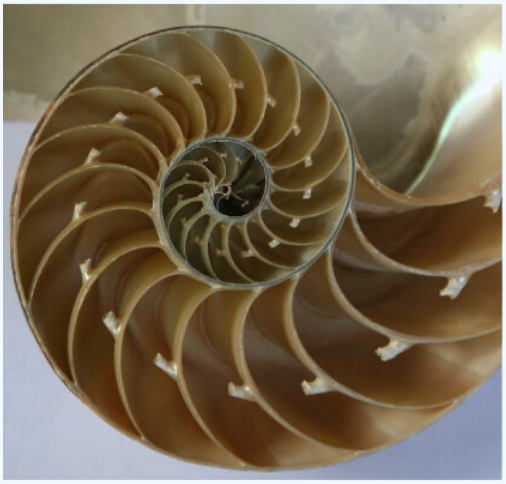
$$\sqrt{x^2+y^2} = k\arctan(\frac{y}{x})$$

$$\arctan(\frac{y}{x}) = \frac{\sqrt{x^2+y^2}}{k}$$

$$y = x\tan(\frac{\sqrt{x^2+y^2}}{k}).$$

Although this equation describes the spiral, it is not possible to solve it directly for either *x* or *y*. However, if we use polar coordinates, the equation becomes much simpler. In particular, $d(P,O) = r$, and θ is the second coordinate. Therefore the equation for the spiral becomes $r = k\theta$. Note that when $\theta = 0$ we also have $r = 0$, so the spiral emanates from the origin. We can remove this restriction by adding a constant to the equation. Then the equation for the spiral becomes $r = a + k\theta$ for arbitrary constants a and k. This is referred to as an **Archimedean spiral**, after the Greek mathematician Archimedes.

Another type of spiral is the logarithmic spiral, described by the function $r = a \cdot b^{\theta}$. A graph of the function $r = 1.2(1.25^{\theta})$ is given in Figure 11.3.10. This spiral describes the shell shape of the chambered nautilus.



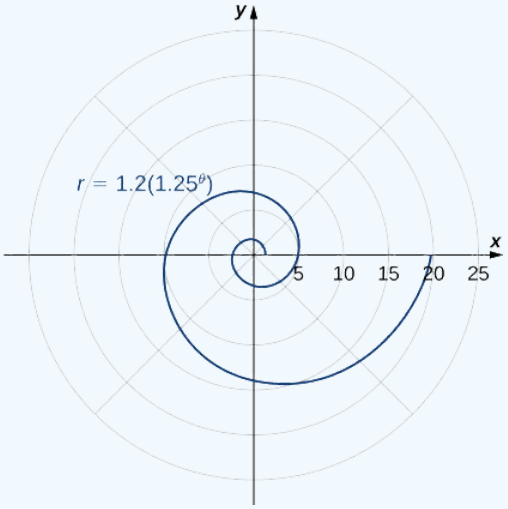


Figure 11.3.10: A logarithmic spiral is similar to the shape of the chambered nautilus shell. (credit: modification of work by Jitze Couperus, Flickr)

Suppose a curve is described in the polar coordinate system via the function $r = f(\theta)$. Since we have conversion formulas from polar to rectangular coordinates given by

$$x = r\cos\theta$$

$$y = r\sin\theta$$

,

it is possible to rewrite these formulas using the function

$$x = f(\theta)\cos\theta$$

$$y = f(\theta)\sin\theta.$$

This step gives a parameterization of the curve in rectangular coordinates using θ as the parameter. For example, the spiral formula $r = a + b\theta$ from Figure becomes

$$x = (a + b\theta)\cos\theta$$

$$y = (a + b\theta)\sin\theta.$$

Letting θ range from $-\infty$ to ∞ generates the entire spiral.

Symmetry in Polar Coordinates

When studying **symmetry** of functions in rectangular coordinates (i.e., in the form $y = f(x)$), we talk about symmetry with respect to the *y*-axis and symmetry with respect to the origin. In particular, if $f(-x) = f(x)$ for all x in the domain of f, then f is an even function and its graph is symmetric with respect to the *y*-axis. If $f(-x) = -f(x)$ for all x in the domain of f, then f is an odd function and its graph is symmetric with respect to the origin. By determining which types of symmetry a graph exhibits, we can learn more about the shape and appearance of the graph. Symmetry can also reveal other properties of the function that generates the graph. Symmetry in polar curves works in a similar fashion.

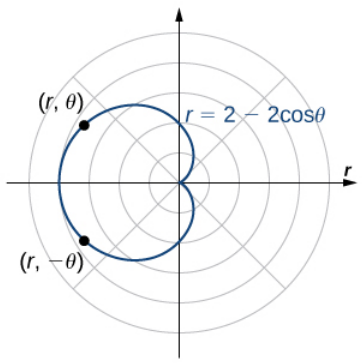
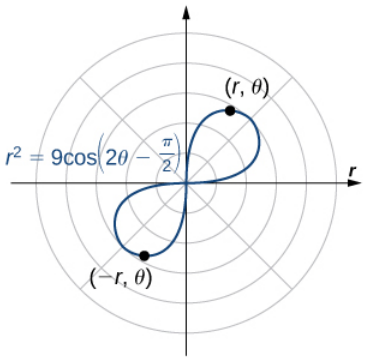
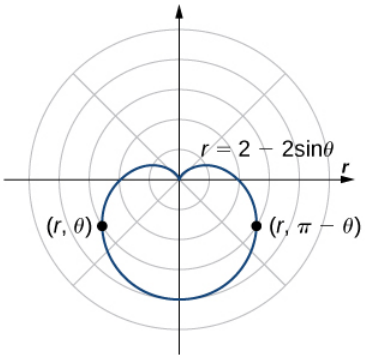
Symmetry in Polar Curves and Equations

Consider a curve generated by the function $r = f(\theta)$ in polar coordinates.

i. The curve is symmetric about the polar axis if for every point (r, θ) on the graph, the point $(r, -\theta)$ is also on the graph. Similarly, the equation $r = f(\theta)$ is unchanged by replacing θ with $-\theta$.
ii. The curve is symmetric about the pole if for every point (r, θ) on the graph, the point $(r, \pi + \theta)$ is also on the graph. Similarly, the equation $r = f(\theta)$ is unchanged when replacing r with $-r$, or θ with $\pi + \theta$.

iii. The curve is symmetric about the vertical line $\theta = \frac{\pi}{2}$ if for every point (r, θ) on the graph, the point $(r, \pi - \theta)$ is also on the graph. Similarly, the equation $r = f(\theta)$ is unchanged when θ is replaced by $\pi - \theta$.

The following table shows examples of each type of symmetry.

Symmetry with respect to the polar axis: For every point (r, θ) on the graph, there is also a point reflected directly across the horizontal (polar) axis.	
Symmetry with respect to the pole: For every point (r, θ) on the graph, there is also a point on the graph that is reflected through the pole as well.	
Symmetry with respect to the vertical line $\theta = \frac{\pi}{2}$: For every point (r, θ) on the graph, there is also a point reflected directly across the vertical axis.	

Example 11.3.6: Using Symmetry to Graph a Polar Equation

Find the symmetry of the rose defined by the equation $r = 3\sin(2\theta)$ and create a graph.

Solution

Suppose the point (r, θ) is on the graph of $r = 3\sin(2\theta)$.

i. To test for symmetry about the polar axis, first try replacing θ with $-\theta$. This gives $r = 3\sin(2(-\theta)) = -3\sin(2\theta)$. Since this changes the original equation, this test is not satisfied. However, returning to the original equation and replacing r with $-r$ and θ with $\pi - \theta$ yields

$$-r = 3\sin(2(\pi - \theta))$$
$$-r = 3\sin(2\pi - 2\theta)$$
$$-r = 3\sin(-2\theta)$$
$$-r = -3\sin 2\theta.$$

Multiplying both sides of this equation by -1 gives $r = 3\sin 2\theta$, which is the original equation. This demonstrates that the graph is symmetric with respect to the polar axis.

ii. To test for symmetry with respect to the pole, first replace r with $-r$, which yields $-r = 3\sin(2\theta)$. Multiplying both sides by -1 gives $r = -3\sin(2\theta)$, which does not agree with the original equation. Therefore the equation does not pass the test for this symmetry. However, returning to the original equation and replacing θ with $\theta + \pi$ gives

$$\begin{aligned} r &= 3\sin(2(\theta + \pi)) \\ &= 3\sin(2\theta + 2\pi) \\ &= 3(\sin 2\theta \cos 2\pi + \cos 2\theta \sin 2\pi) \\ &= 3\sin 2\theta. \end{aligned}$$

Since this agrees with the original equation, the graph is symmetric about the pole.

iii. To test for symmetry with respect to the vertical line $\theta = \dfrac{\pi}{2}$, first replace both r with $-r$ and θ with $-\theta$.

$$-r = 3\sin(2(-\theta))$$
$$-r = 3\sin(-2\theta)$$
$$-r = -3\sin 2\theta.$$

Multiplying both sides of this equation by -1 gives $r = 3\sin 2\theta$, which is the original equation. Therefore the graph is symmetric about the vertical line $\theta = \dfrac{\pi}{2}$.

This graph has symmetry with respect to the polar axis, the origin, and the vertical line going through the pole. To graph the function, tabulate values of θ between 0 and $\pi/2$ and then reflect the resulting graph.

Table 11.3.2: Table of values for this polar function.

θ	$r = 3\sin(2\theta)$
0	0
$\dfrac{\pi}{6}$	$\dfrac{3\sqrt{3}}{2} \approx 2.6$
$\dfrac{\pi}{4}$	3
$\dfrac{\pi}{3}$	$\dfrac{3\sqrt{3}}{2} \approx 2.6$
$\dfrac{\pi}{2}$	0

This gives one petal of the rose, as shown in the following graph.

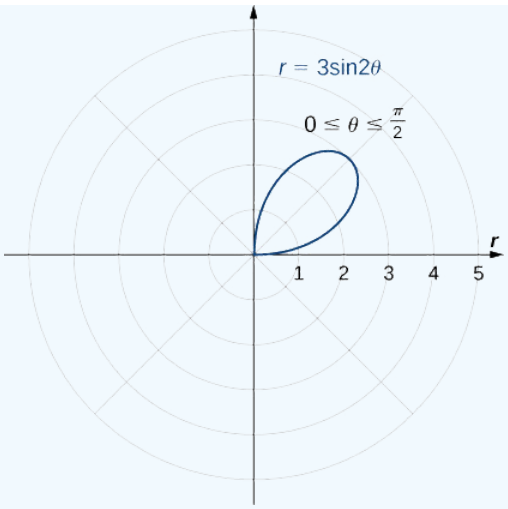


Figure 11.3.11: The graph of the equation between $\theta = 0$ and $\theta = \pi/2$.

Reflecting this image into the other three quadrants gives the entire graph as shown.

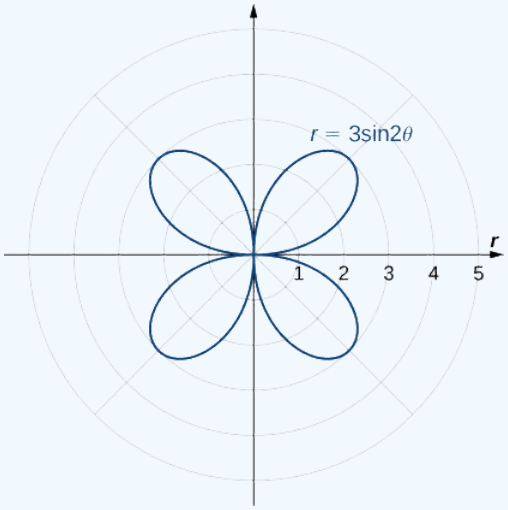


Figure 11.3.12: The entire graph of the equation is called a four-petaled rose.

? Exercise 11.3.5Symmetry

Determine the symmetry of the graph determined by the equation $r = 2\cos(3\theta)$ and create a graph.

Hint

Use Note.

Answer

Symmetric with respect to the polar axis.

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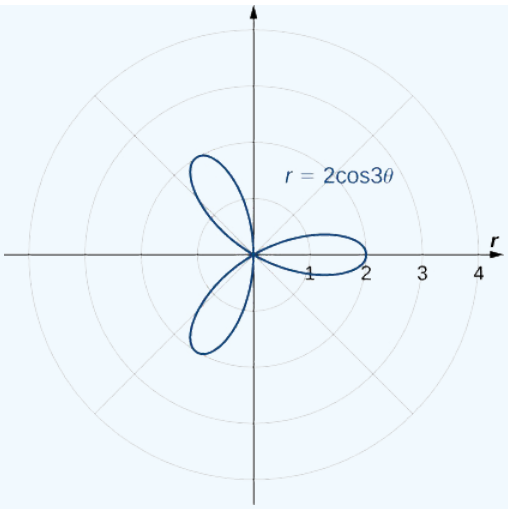

Key Concepts

- The polar coordinate system provides an alternative way to locate points in the plane.
- Convert points between rectangular and polar coordinates using the formulas

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

and

$$r = \sqrt{x^2 + y^2} \text{ and } \tan\theta = \frac{y}{x}.$$

- To sketch a polar curve from a given polar function, make a table of values and take advantage of periodic properties.
- Use the conversion formulas to convert equations between rectangular and polar coordinates.
- Identify symmetry in polar curves, which can occur through the pole, the horizontal axis, or the vertical axis.

Glossary

angular coordinate

θ the angle formed by a line segment connecting the origin to a point in the polar coordinate system with the positive radial x-axis, measured counterclockwise

cardioid

a plane curve traced by a point on the perimeter of a circle that is rolling around a fixed circle of the same radius; the equation of a cardioid is $r = a(1 + \sin\theta)$ or $r = a(1 + \cos\theta)$

limaçon

the graph of the equation $r = a + b\sin\theta$ or $r = a + b\cos\theta$. If $a = b$ then the graph is a cardioid

polar axis

the horizontal axis in the polar coordinate system corresponding to $r \geq 0$

polar coordinate system

a system for locating points in the plane. The coordinates are r, the radial coordinate, and θ, the angular coordinate

polar equation

an equation or function relating the radial coordinate to the angular coordinate in the polar coordinate system

pole

the central point of the polar coordinate system, equivalent to the origin of a Cartesian system

radial coordinate

r the coordinate in the polar coordinate system that measures the distance from a point in the plane to the pole

rose

graph of the polar equation $r = a\cos 2\theta$ or $r = a\sin 2\theta$for a positive constant a

space-filling curve

a curve that completely occupies a two-dimensional subset of the real plane

---

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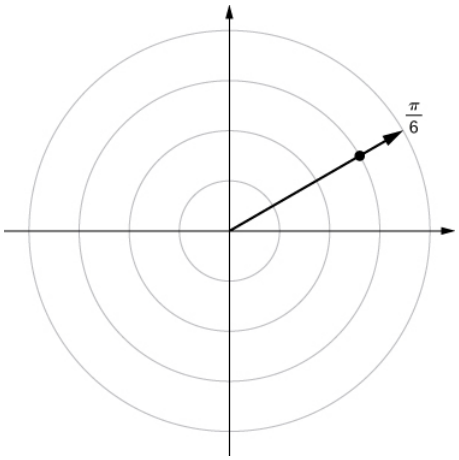
11.3E: Exercises for Section 11.3

In exercises 1 - 7, plot the point whose polar coordinates are given by first constructing the angle θ and then marking off the distance r along the ray.

1) $\left(3, \frac{\pi}{6}\right)$

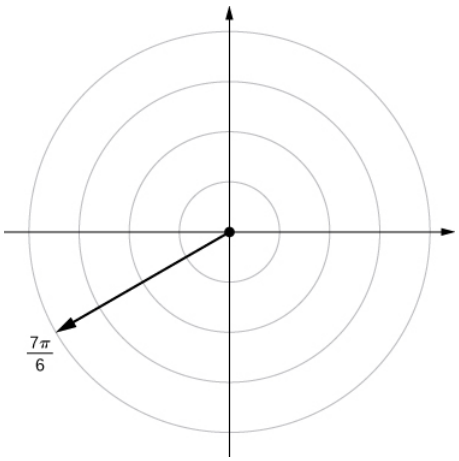
Answer



2) $\left(-2, \frac{5\pi}{3}\right)$

3) $\left(0, \frac{7\pi}{6}\right)$

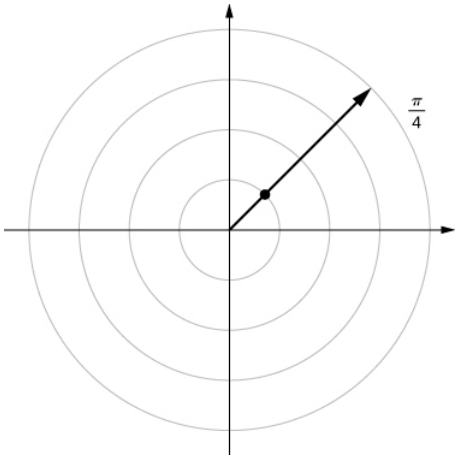
Answer



4) $\left(-4, \frac{3\pi}{4}\right)$

5) $\left(1, \frac{\pi}{4}\right)$

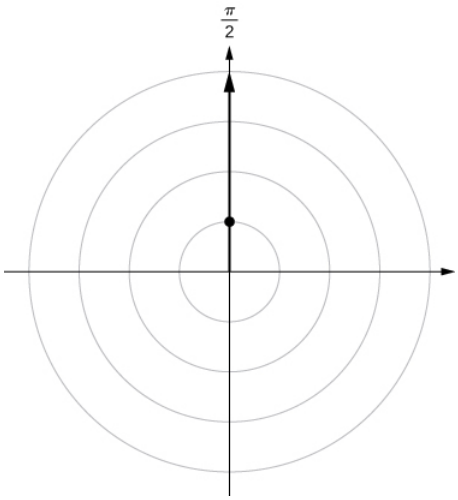
Answer



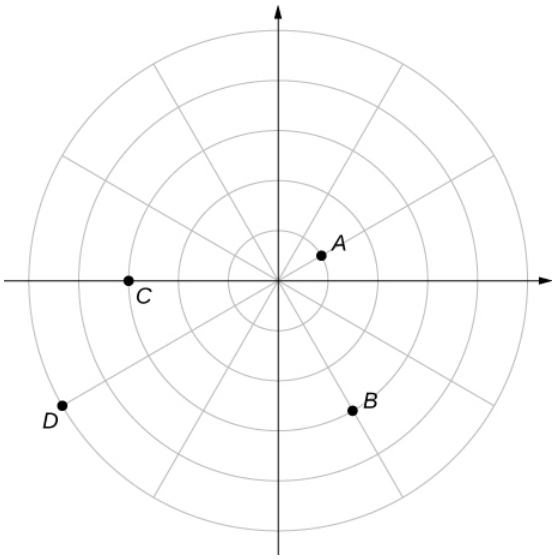
6) $(2, \frac{5\pi}{6})$

7) $(1, \frac{\pi}{2})$

Answer



In exercises 8 - 11, consider the polar graph below. Give two sets of polar coordinates for each point.



8) Coordinates of point A.

9) Coordinates of point B.

Answer

$B\left(3, \frac{-\pi}{3}\right)\ B\left(-3, \frac{2\pi}{3}\right)$

10) Coordinates of point C.

11) Coordinates of point D.

Answer

$D\left(5, \frac{7\pi}{6}\right)\ D\left(-5, \frac{\pi}{6}\right)$

In exercises 12 - 17, the rectangular coordinates of a point are given. Find two sets of polar coordinates for the point in $(0, 2\pi]$. Round to three decimal places.

12) $(2, 2)$

13) $(3, -4)$

Answer

$(5, -0.927),\ (-5, -0.927 + \pi)$

14) $(8, 15)$

15) $(-6, 8)$

Answer

$(10, -0.927),\ (-10, -0.927 + \pi)$

16) $(4, 3)$

17) $(3, -\sqrt{3})$

Answer

$(2\sqrt{3}, -0.524),\ (-2\sqrt{3}, -0.524 + \pi)$

In exercises 18 - 24, find rectangular coordinates for the given point in polar coordinates.

18) $\left(2, \frac{5\pi}{4}\right)$

19) $\left(-2, \frac{\pi}{6}\right)$

Answer

$(-\sqrt{3}, -1)$

20) $\left(5, \frac{\pi}{3}\right)$

21) $\left(1, \frac{7\pi}{6}\right)$

Answer

$\left(-\frac{\sqrt{3}}{2}, \frac{-1}{2}\right)$

22) $\left(-3, \frac{3\pi}{4}\right)$

23) $\left(0, \frac{\pi}{2}\right)$

Answer

$(0, 0)$

24) $(-4.5, 6.5)$

In exercises 25 - 29, determine whether the graphs of the polar equation are symmetric with respect to the x-axis, the y-axis, or the origin.

25) $r = 3\sin(2\theta)$

Answer

Symmetry with respect to the *x*-axis, *y*-axis, and origin.

26) $r^2 = 9\cos\theta$

27) $r = \cos\left(\frac{\theta}{5}\right)$

Answer

Symmetric with respect to *x*-axis only.

28) $r = 2\sec\theta$

29) $r = 1 + \cos\theta$

Answer

Symmetry with respect to *x*-axis only.

In exercises 30 - 33, describe the graph of each polar equation. Confirm each description by converting into a rectangular equation.

30) $r = 3$

31) $\theta = \frac{\pi}{4}$

Answer

Line $y = x$

32) $r = \sec\theta$

33) $r = \csc\theta$

Answer

$y = 1$

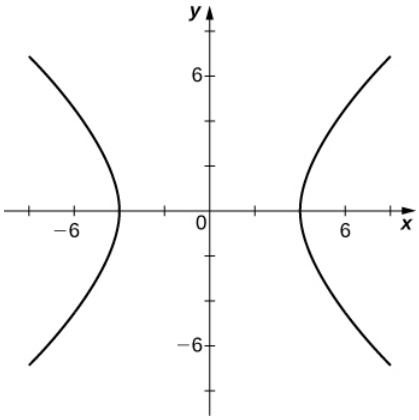
In exercises 34 - 36, convert the rectangular equation to polar form and sketch its graph.

34) $x^2 + y^2 = 16$

35) $x^2 - y^2 = 16$

Answer

Hyperbola; polar form $r^2\cos(2\theta) = 16$ or $r^2 = 16\sec\theta$.



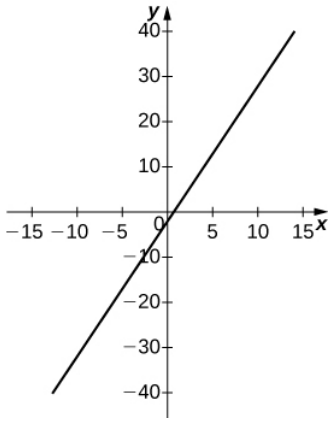
36) $x = 8$

In exercises 37 - 38, convert the rectangular equation to polar form and sketch its graph.

37) $3x - y = 2$

Answer

$r = \frac{2}{3\cos\theta - \sin\theta}$



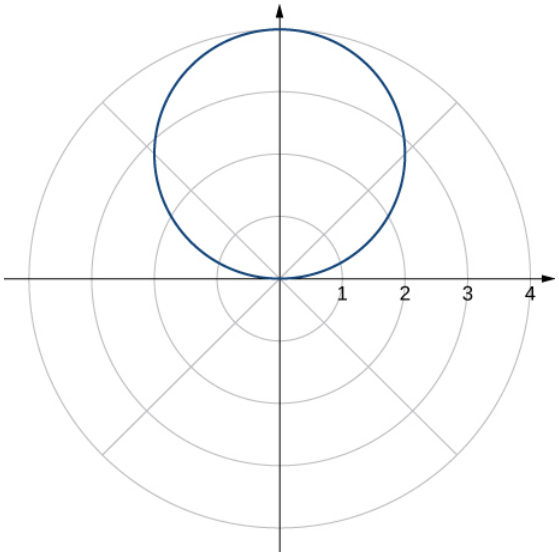
38) $y^2 = 4x$

In exercises 39 - 43, convert the polar equation to rectangular form and sketch its graph.

39) $r = 4\sin\theta$

40) $x^2 + y^2 = 4y$

Answer



41) $r = 6\cos\theta$

42) $r = \theta$

Answer

$x \tan\sqrt{x^2 + y^2} = y$

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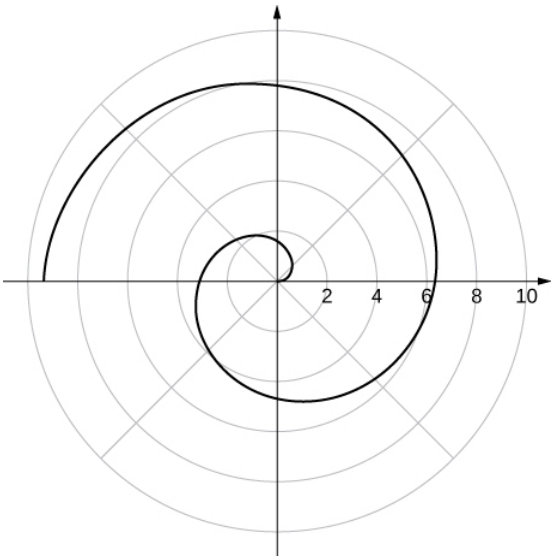


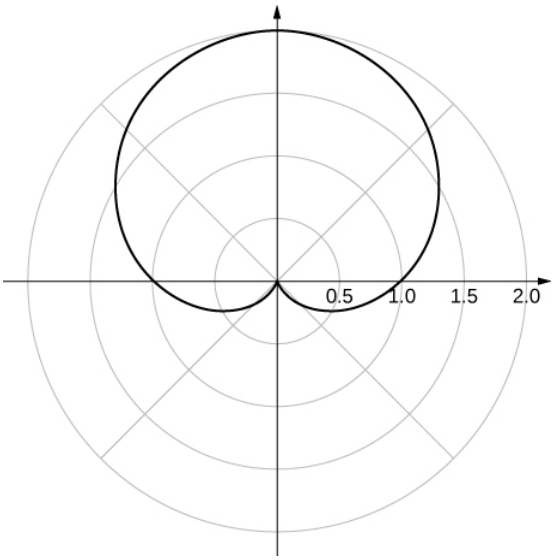
43) $r = \cot\theta \csc\theta$

In exercises 44 - 54, sketch a graph of the polar equation and identify any symmetry.

44) $r = 1 + \sin\theta$

Answer

y-axis symmetry

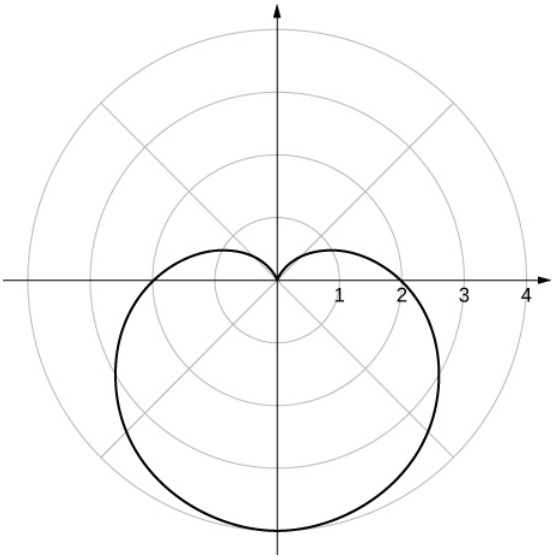


45) $r = 3 - 2\cos\theta$

46) $r = 2 - 2\sin\theta$

Answer

y-axis symmetry

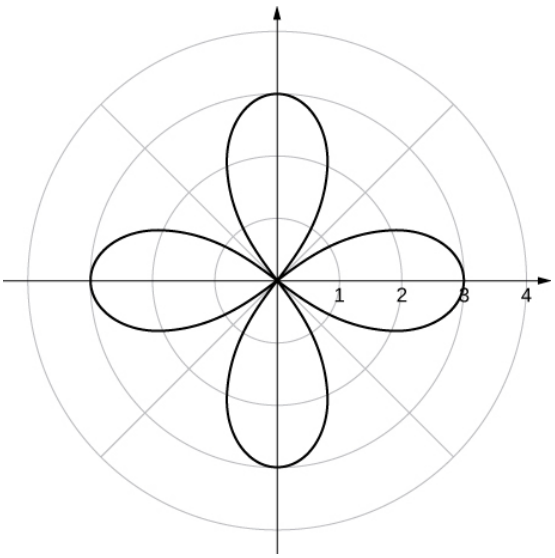


47) $r = 5 - 4\sin\theta$

48) $r = 3\cos(2\theta)$

Answer

x-and y-axis symmetry and symmetry about the pole

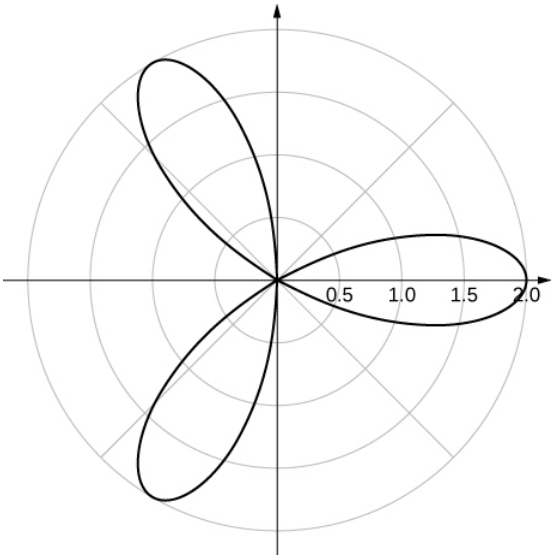


49) $r = 3\sin(2\theta)$

50) $r = 2\cos(3\theta)$

Answer

x-axis symmetry

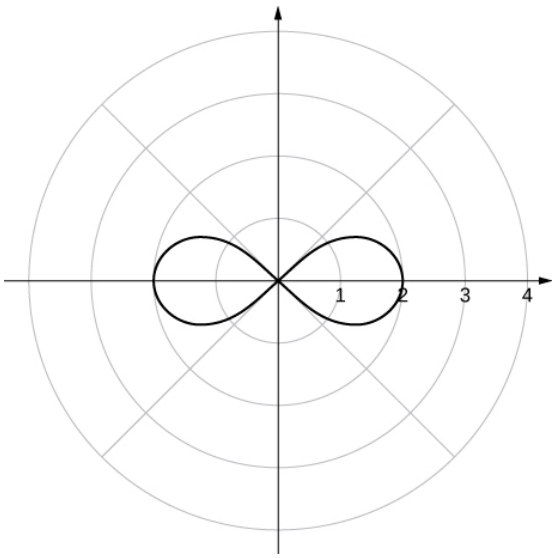


51) $r = 3\cos\left(\frac{\theta}{2}\right)$

52) $r^2 = 4\cos\left(\frac{2}{\theta}\right)$

Answer

x-and y-axis symmetry and symmetry about the pole

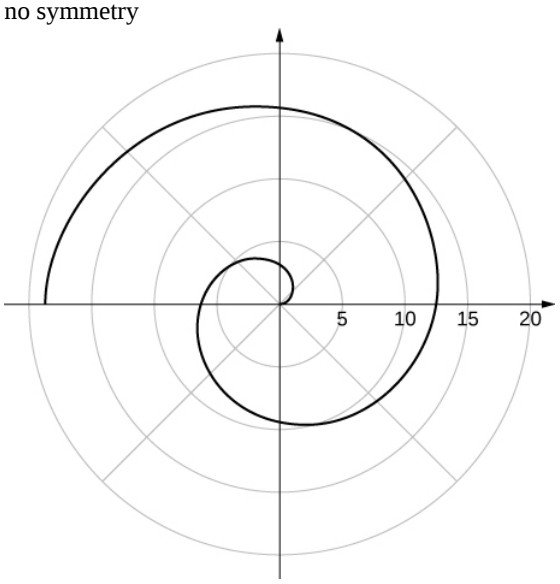


53) $r^2 = 4\sin\theta$

54) $r = 2\theta$

Answer

no symmetry

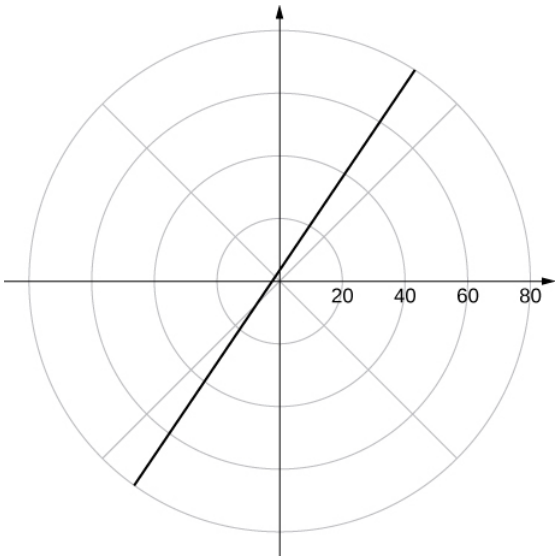


55) [T] The graph of $r = 2\cos(2\theta)\sec(\theta).$ is called a strophoid. Use a graphing utility to sketch the graph, and, from the graph, determine the asymptote.

56) [T] Use a graphing utility and sketch the graph of $r = \dfrac{6}{2\sin\theta - 3\cos\theta}$.

Answer

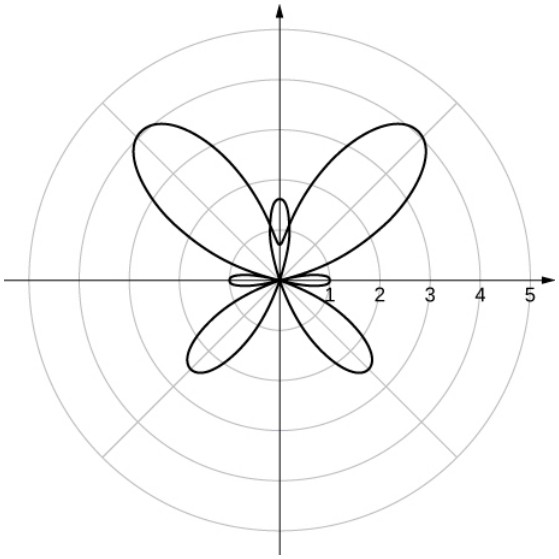
a line



57) [T] Use a graphing utility to graph $r = \frac{1}{1-\cos\theta}$.

58) [T] Use technology to graph $r = e^{\sin(\theta)} - 2\cos(4\theta)$.

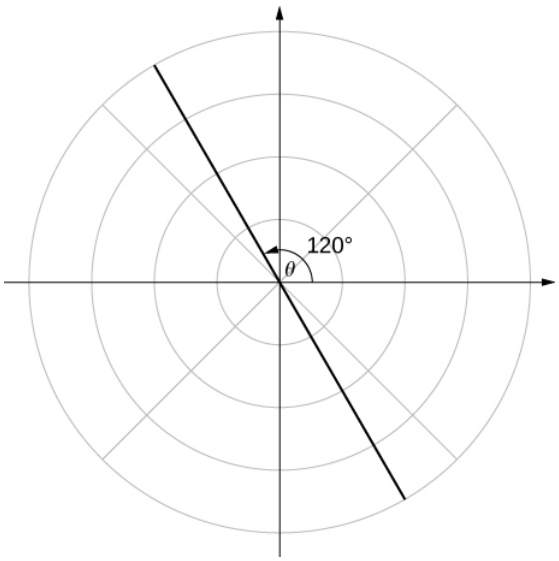
Answer



59) [T] Use technology to plot $r = \sin(\frac{3\theta}{7})$ (use the interval $0 \leq \theta \leq 14\pi$).

60) Without using technology, sketch the polar curve $\theta = \frac{2\pi}{3}$.

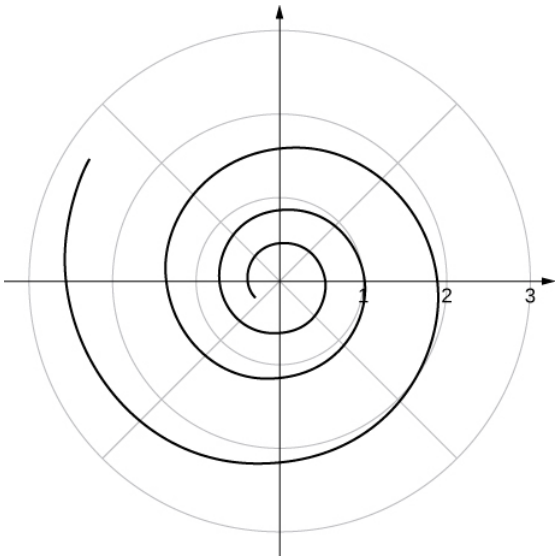
Answer



61) [T] Use a graphing utility to plot $r = \theta \sin \theta$ for $-\pi \leq \theta \leq \pi$.

62) [T] Use technology to plot $r = e^{-0.1\theta}$ for $-10 \leq \theta \leq 10$.

Answer



63) [T] There is a curve known as the “Black Hole.” Use technology to plot $r = e^{-0.01\theta}$ for $-100 \leq \theta \leq 100$.

64) [T] Use the results of the preceding two problems to explore the graphs of $r = e^{-0.001\theta}$ and $r = e^{-0.0001\theta}$ for $|\theta| > 100$.

Answer

Answers vary. One possibility is the spiral lines become closer together and the total number of spirals increases.

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11.4: Area and Arc Length in Polar Coordinates

Learning Objectives

- Apply the formula for area of a region in polar coordinates.
- Determine the arc length of a polar curve.

In the rectangular coordinate system, the definite integral provides a way to calculate the area under a curve. In particular, if we have a function $y = f(x)$ defined from $x = a$ to $x = b$ where $f(x) > 0$ on this interval, the area between the curve and the x-axis is given by

$$A = \int_a^b f(x)dx.$$

This fact, along with the formula for evaluating this integral, is summarized in the Fundamental Theorem of Calculus. Similarly, the arc length of this curve is given by

$$L = \int_a^b \sqrt{1 + (f'(x))^2}dx.$$

In this section, we study analogous formulas for area and arc length in the polar coordinate system.

Areas of Regions Bounded by Polar Curves

We have studied the formulas for area under a curve defined in rectangular coordinates and parametrically defined curves. Now we turn our attention to deriving a formula for the area of a region bounded by a polar curve. Recall that the proof of the Fundamental Theorem of Calculus used the concept of a Riemann sum to approximate the area under a curve by using rectangles. For polar curves we use the Riemann sum again, but the rectangles are replaced by sectors of a circle.

Consider a curve defined by the function $r = f(\theta)$, where $\alpha \leq \theta \leq \beta$. Our first step is to partition the interval $[\alpha, \beta]$ into n equal-width subintervals. The width of each subinterval is given by the formula $\Delta\theta = (\beta - \alpha)/n$, and the ith partition point θ_i is given by the formula $\theta_i = \alpha + i\Delta\theta$. Each partition point $\theta = \theta_i$ defines a line with slope $\tan\theta_i$ passing through the pole as shown in the following graph.

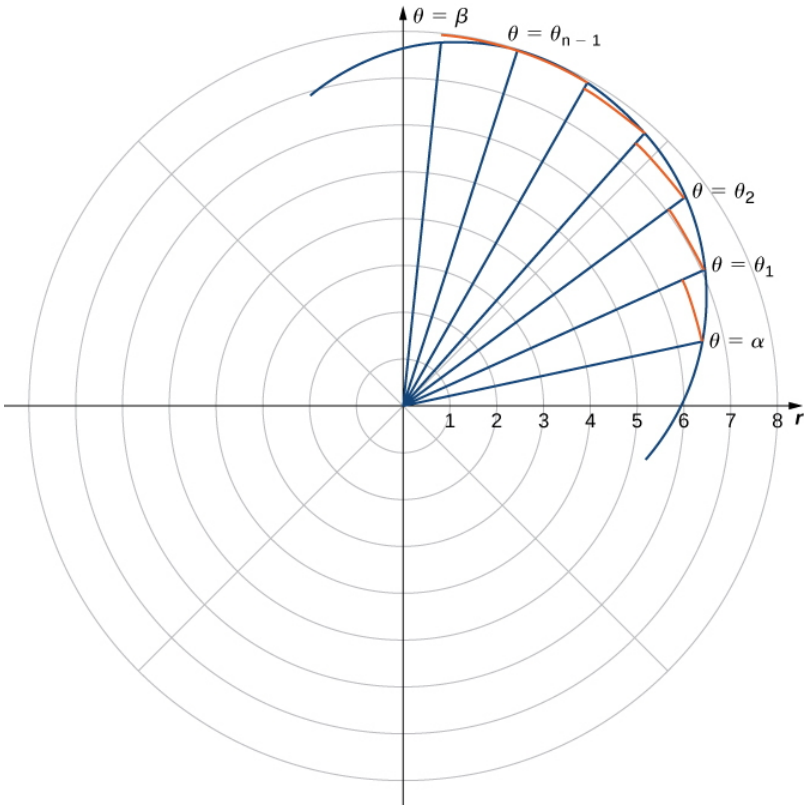

Figure 11.4.1: A partition of a typical curve in polar coordinates.

The line segments are connected by arcs of constant radius. This defines sectors whose areas can be calculated by using a geometric formula. The area of each sector is then used to approximate the area between successive line segments. We then sum the areas of the sectors to approximate the total area. This approach gives a Riemann sum approximation for the total area. The formula for the area of a sector of a circle is illustrated in the following figure.

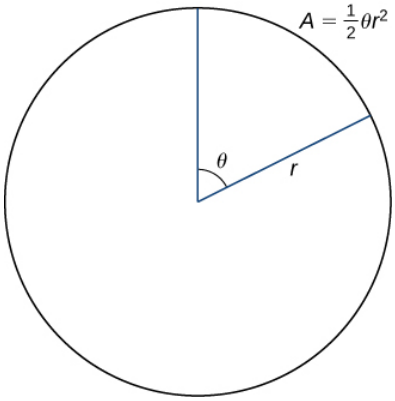

Figure 11.4.2: *The area of a sector of a circle is given by* $A = \frac{1}{2}\theta r^2$.

Recall that the area of a circle is $A = \pi r^2$. When measuring angles in radians, 360 degrees is equal to 2π radians. Therefore a fraction of a circle can be measured by the central angle θ. The fraction of the circle is given by $\frac{\theta}{2\pi}$, so the area of the sector is this fraction multiplied by the total area:

$$A = (\frac{\theta}{2\pi})\pi r^2 = \frac{1}{2}\theta r^2.$$

Since the radius of a typical sector in Figure 11.4.1 is given by $r_i = f(\theta_i)$, the area of the ith sector is given by

$$A_i = \frac{1}{2}(\Delta\theta)(f(\theta_i))^2.$$

Therefore a Riemann sum that approximates the area is given by

$$A_n = \sum_{i=1}^{n} A_i \approx \sum_{i=1}^{n} \frac{1}{2}(\Delta\theta)(f(\theta_i))^2.$$

We take the limit as $n \to \infty$ to get the exact area:

$$A = \lim_{n\to\infty} A_n = \frac{1}{2}\int_{\alpha}^{\beta} (f(\theta))^2 d\theta.$$

This gives the following theorem.

Area of a Region Bounded by a Polar Curve

Suppose f is continuous and nonnegative on the interval $\alpha \le \theta \le \beta$ with $0 < \beta - \alpha \le 2\pi$. The area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 d\theta \tag{11.4.1}$$

$$= \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta. \tag{11.4.2}$$

Example 11.4.1: Finding an Area of a Polar Region

Find the area of one petal of the rose defined by the equation $r = 3\sin(2\theta)$.

Solution

The graph of $r = 3\sin(2\theta)$ follows.

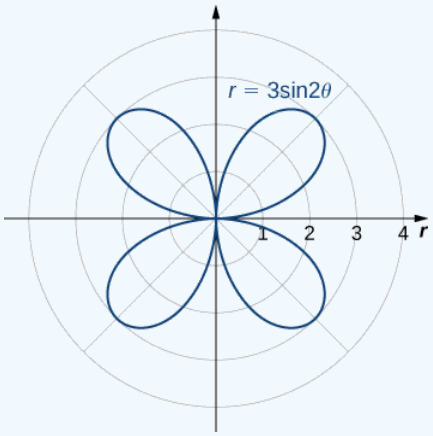

Figure 11.4.3: The graph of $r = 3\sin(2\theta)$.

When $\theta = 0$ we have $r = 3\sin(2(0)) = 0$. The next value for which $r = 0$ is $\theta = \pi/2$. This can be seen by solving the equation $3\sin(2\theta) = 0$ for θ. Therefore the values $\theta = 0$ to $\theta = \pi/2$ trace out the first petal of the rose. To find the area inside this petal, use Equation 11.4.2 with $f(\theta) = 3\sin(2\theta), \alpha = 0$, and $\beta = \pi/2$:

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 d\theta$$

$$= \frac{1}{2}\int_{0}^{\pi/2} [3\sin(2\theta)]^2 d\theta$$

$$= \frac{1}{2}\int_{0}^{\pi/2} 9\sin^2(2\theta) d\theta.$$

To evaluate this integral, use the formula $\sin^2\alpha = (1 - \cos(2\alpha))/2$ with $\alpha = 2\theta$:

$$
\begin{aligned}
A &= \frac{1}{2}\int_0^{\pi/2} 9\sin^2(2\theta)d\theta \\
&= \frac{9}{2}\int_0^{\pi/2} \frac{(1-\cos(4\theta))}{2}d\theta \\
&= \frac{9}{4}(\int_0^{\pi/2} 1-\cos(4\theta)d\theta) \\
&= \frac{9}{4}(\theta - \frac{\sin(4\theta)}{4}\,|_0^{\pi/2} \\
&= \frac{9}{4}(\frac{\pi}{2} - \frac{\sin 2\pi}{4}) - \frac{9}{4}(0 - \frac{\sin 4(0)}{4}) \\
&= \frac{9\pi}{8}
\end{aligned}
$$

? Exercise 11.4.1

Find the area inside the cardioid defined by the equation $r = 1 - \cos\theta$.

Hint

Use Equation 11.4.2. Be sure to determine the correct limits of integration before evaluating.

Answer

$A = 3\pi/2$

Example 11.4.1 involved finding the area inside one curve. We can also use Equation 11.4.2 to find the area between two polar curves. However, we often need to find the points of intersection of the curves and determine which function defines the outer curve or the inner curve between these two points.

✓ Example 11.4.2: Finding the Area between Two Polar Curves

Find the area outside the cardioid $r = 2 + 2\sin\theta$ and inside the circle $r = 6\sin\theta$.

Solution

First draw a graph containing both curves as shown.

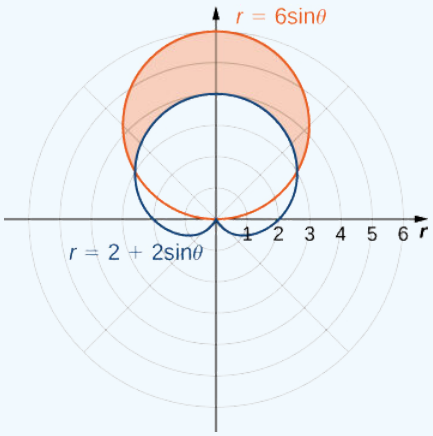

Figure 11.4.4: *The region between the curves* $r = 2 + 2\sin\theta$ *and* $r = 6\sin\theta$.

To determine the limits of integration, first find the points of intersection by setting the two functions equal to each other and solving for θ:

$$6\sin\theta = 2 + 2\sin\theta$$
$$4\sin\theta = 2$$
$$\sin\theta = \frac{1}{2}.$$

This gives the solutions $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$, which are the limits of integration. The circle $r = 3\sin\theta$ is the red graph, which is the outer function, and the cardioid $r = 2 + 2\sin\theta$ is the blue graph, which is the inner function. To calculate the area between the curves, start with the area inside the circle between $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$, then subtract the area inside the cardioid between $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$:

$$\begin{aligned} A &= \text{circle} - \text{cardioid} \\ &= \frac{1}{2}\int_{\pi/6}^{5\pi/6}[6\sin\theta]^2 d\theta - \frac{1}{2}\int_{\pi/6}^{5\pi/6}[2+2\sin\theta]^2 d\theta \\ &= \frac{1}{2}\int_{\pi/6}^{5\pi/6} 36\sin^2\theta\, d\theta - \frac{1}{2}\int_{\pi/6}^{5\pi/6} 4 + 8\sin\theta + 4\sin^2\theta\, d\theta \\ &= 18\int_{\pi/6}^{5\pi/6}\frac{1-\cos(2\theta)}{2}d\theta - 2\int_{\pi/6}^{5\pi/6} 1 + 2\sin\theta + \frac{1-\cos(2\theta)}{2}d\theta \\ &= 9[\theta - \frac{\sin(2\theta)}{2}]_{\pi/6}^{5\pi/6} - 2[\frac{3\theta}{2} - 2\cos\theta - \frac{\sin(2\theta)}{4}]_{\pi/6}^{5\pi/6} \\ &= 9(\frac{5\pi}{6} - \frac{\sin(10\pi/6)}{2}) - 9(\frac{\pi}{6} - \frac{\sin(2\pi/6)}{2}) - (3(\frac{5\pi}{6}) - 4\cos\frac{5\pi}{6} - \frac{\sin(10\pi/6)}{2}) + (3(\frac{\pi}{6}) - 4\cos\frac{\pi}{6} - \frac{\sin(2\pi/6)}{2}) \\ &= 4\pi. \end{aligned}$$

Exercise 11.4.2

Find the area inside the circle $r = 4\cos\theta$ and outside the circle $r = 2$.

Hint

Use Equation 11.4.2 and take advantage of symmetry.

Answer

$$A = \frac{4\pi}{3} + 2\sqrt{3}$$

In Example 11.4.2 we found the area inside the circle and outside the cardioid by first finding their intersection points. Notice that solving the equation directly for θ yielded two solutions: $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$. However, in the graph there are three intersection points. The third intersection point is the origin. The reason why this point did not show up as a solution is because the origin is on both graphs but for different values of θ. For example, for the cardioid we get

$$2 + 2\sin\theta = 0$$
$$\sin\theta = -1,$$

so the values for θ that solve this equation are $\theta = \frac{3\pi}{2} + 2n\pi$, where n is any integer. For the circle we get

$$6\sin\theta = 0.$$

The solutions to this equation are of the form $\theta = n\pi$ for any integer value of n. These two solution sets have no points in common. Regardless of this fact, the curves intersect at the origin. This case must always be taken into consideration.

Arc Length in Polar Curves

Here we derive a formula for the arc length of a curve defined in polar coordinates. In rectangular coordinates, the arc length of a parameterized curve $(x(t), y(t))$ for $a \le t \le b$ is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

In polar coordinates we define the curve by the equation $r = f(\theta)$, where $\alpha \leq \theta \leq \beta$. In order to adapt the arc length formula for a polar curve, we use the equations

$$x = r\cos\theta = f(\theta)\cos\theta$$

and

$$y = r\sin\theta = f(\theta)\sin\theta,$$

and we replace the parameter t by θ. Then

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta$$

$$\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta.$$

We replace dt by $d\theta$, and the lower and upper limits of integration are α and β, respectively. Then the arc length formula becomes

$$\begin{aligned} L &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \\ &= \int_\alpha^\beta \sqrt{(f'(\theta)\cos\theta - f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2}\, d\theta \\ &= \int_\alpha^\beta \sqrt{(f'(\theta))^2(\cos^2\theta + \sin^2\theta) + (f(\theta))^2(\cos^2\theta + \sin^2\theta)}\, d\theta \\ &= \int_\alpha^\beta \sqrt{(f'(\theta))^2 + (f(\theta))^2}\, d\theta \\ &= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \end{aligned}$$

This gives us the following theorem.

Arc Length of a Curve Defined by a Polar Function

Let f be a function whose derivative is continuous on an interval $\alpha \leq \theta \leq \beta$. The arc length of the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$L = \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta \tag{11.4.3}$$

$$= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta. \tag{11.4.4}$$

Example 11.4.3: Finding the Arc Length of a cardioid

Find the arc length of the cardioid $r = 2 + 2\cos\theta$.

Solution

When $\theta = 0, r = 2 + 2\cos 0 = 4$. Furthermore, as θ goes from 0 to 2π, the cardioid is traced out exactly once. Therefore these are the limits of integration. Using $f(\theta) = 2 + 2\cos\theta, \alpha = 0$, and $\beta = 2\pi$, Equation 11.4.3 becomes

$$
\begin{aligned}
L &= \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta \\
&= \int_{0}^{2\pi} \sqrt{[2 + 2\cos\theta]^2 + [-2\sin\theta]^2}\, d\theta \\
&= \int_{0}^{2\pi} \sqrt{4 + 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}\, d\theta \\
&= \int_{0}^{2\pi} \sqrt{4 + 8\cos\theta + 4(\cos^2\theta + \sin^2\theta)}\, d\theta \\
&= \int_{0}^{2\pi} \sqrt{8 + 8\cos\theta}\, d\theta \\
&= 2\int_{0}^{2\pi} \sqrt{2 + 2\cos\theta}\, d\theta.
\end{aligned}
$$

Next, using the identity $\cos(2\alpha) = 2\cos^2\alpha - 1$, add 1 to both sides and multiply by 2. This gives $2 + 2\cos(2\alpha) = 4\cos^2\alpha$. Substituting $\alpha = \theta/2$ gives $2 + 2\cos\theta = 4\cos^2(\theta/2)$, so the integral becomes

$$
\begin{aligned}
L &= 2\int_{0}^{2\pi} \sqrt{2 + 2\cos\theta}\, d\theta \\
&= 2\int_{0}^{2\pi} \sqrt{4\cos^2(\frac{\theta}{2})}\, d\theta \\
&= 4\int_{0}^{2\pi} |\cos(\frac{\theta}{2})|\, d\theta.
\end{aligned}
$$

The absolute value is necessary because the cosine is negative for some values in its domain. To resolve this issue, change the limits from 0 to π and double the answer. This strategy works because cosine is positive between 0 and $\frac{\pi}{2}$. Thus,

$$
\begin{aligned}
L &= 4\int_{0}^{2\pi} |\cos(\frac{\theta}{2})|\, d\theta \\
&= 8\int_{0}^{\pi} \cos(\frac{\theta}{2})\, d\theta \\
&= 8(2\sin(\frac{\theta}{2}))\,|_{0}^{\pi} \\
&= 16
\end{aligned}
$$

? Exercise 11.4.3

Find the total arc length of the graph of $r = 3\sin\theta$.

Hint

Use Equation 11.4.3. To determine the correct limits, make a table of values.

Answer

$s = 3\pi$

Key Concepts

- The area of a region in polar coordinates defined by the equation $r = f(\theta)$ with $\alpha \le \theta \le \beta$ is given by the integral $A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 d\theta$.
- To find the area between two curves in the polar coordinate system, first find the points of intersection, then subtract the corresponding areas.
- The arc length of a polar curve defined by the equation $r = f(\theta)$ with $\alpha \le \theta \le \beta$ is given by the integral $L = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + (\frac{dr}{d\theta})^2} d\theta$.

Key Equations

- **Area of a region bounded by a polar curve**

$$A = \frac{1}{2}\int_{\alpha}^{\beta}[f(\theta)]^2 d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$$

- **Arc length of a polar curve**

$$L = \int_{\alpha}^{\beta}\sqrt{[f(\theta)]^2 + [f'(\theta)]^2}d\theta = \int_{\alpha}^{\beta}\sqrt{r^2 + (\frac{dr}{d\theta})^2}d\theta$$

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11.4E: Exercises for Section 11.4

In exercises 1 -13, determine a definite integral that represents the area.

1) Region enclosed by $r = 4$

2) Region enclosed by $r = 3\sin\theta$

Answer

$$\frac{9}{2}\int_0^{\pi} \sin^2\theta\, d\theta$$

3) Region in the first quadrant within the cardioid $r = 1 + \sin\theta$

4) Region enclosed by one petal of $r = 8\sin(2\theta)$

Answer

$$32\int_0^{\pi/2} \sin^2(2\theta)\, d\theta$$

5) Region enclosed by one petal of $r = cos(3\theta)$

6) Region below the polar axis and enclosed by $r = 1 - \sin\theta$

Answer

$$\frac{1}{2}\int_{\pi}^{2\pi} (1 - \sin\theta)^2\, d\theta$$

7) Region in the first quadrant enclosed by $r = 2 - \cos\theta$

8) Region enclosed by the inner loop of $r = 2 - 3\sin\theta$

Answer

$$\int_{\sin^{-1}(2/3)}^{\pi/2} (2 - 3\sin\theta)^2\, d\theta$$

9) Region enclosed by the inner loop of $r = 3 - 4\cos\theta$

10) Region enclosed by $r = 1 - 2\cos\theta$ and outside the inner loop

Answer

$$\int_0^{\pi} (1 - 2\cos\theta)^2\, d\theta - \int_0^{\pi/3} (1 - 2\cos\theta)^2\, d\theta$$

11) Region common to $r = 3\sin\theta$ and $r = 2 - \sin\theta$

12) Region common to $r = 2$ and $r = 4\cos\theta$

Answer

$$4\int_0^{\pi/3} d\theta + 16\int_{\pi/3}^{\pi/2} (\cos^2\theta)\, d\theta$$

13) Region common to $r = 3\cos\theta$ and $r = 3\sin\theta$

In exercises 14 -26, find the area of the described region.

14) Enclosed by $r = 6\sin\theta$

Answer

9π units^2

15) Above the polar axis enclosed by $r = 2 + \sin\theta$

16) Below the polar axis and enclosed by $r = 2 - \cos\theta$

Answer

$\frac{9\pi}{4}$ units^2

17) Enclosed by one petal of $r = 4\cos(3\theta)$

18) Enclosed by one petal of $r = 3\cos(2\theta)$

Answer

$\frac{9\pi}{8}$ units^2

19) Enclosed by $r = 1 + \sin\theta$

20) Enclosed by the inner loop of $r = 3 + 6\cos\theta$

Answer

$\frac{18\pi - 27\sqrt{3}}{2}$ units^2

21) Enclosed by $r = 2 + 4\cos\theta$ and outside the inner loop

22) Common interior of $r = 4\sin(2\theta)$ and $r = 2$

Answer

$\frac{4}{3}(4\pi - 3\sqrt{3})$ units^2

23) Common interior of $r = 3 - 2\sin\theta$ and $r = -3 + 2\sin\theta$

24) Common interior of $r = 6\sin\theta$ and $r = 3$

Answer

$\frac{3}{2}(4\pi - 3\sqrt{3})$ units^2

25) Inside $r = 1 + \cos\theta$ and outside $r = \cos\theta$

26) Common interior of $r = 2 + 2\cos\theta$ and $r = 2\sin\theta$

Answer

$(2\pi - 4)$ units^2

In exercises 27 - 30, find a definite integral that represents the arc length.

27) $r = 4\cos\theta$ on the interval $0 \le \theta \le \frac{\pi}{2}$

28) $r = 1 + \sin\theta$ on the interval $0 \le \theta \le 2\pi$

Answer

$$\int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\, d\theta$$

29) $r = 2\sec\theta$ on the interval $0 \le \theta \le \frac{\pi}{3}$

30) $r = e^{\theta}$ on the interval $0 \le \theta \le 1$

Answer

$$\sqrt{2}\int_0^1 e^{\theta}\, d\theta$$

In exercises 31 - 35, find the length of the curve over the given interval.

31) $r = 6$ on the interval $0 \le \theta \le \frac{\pi}{2}$

32) $r = e^{3\theta}$ on the interval $0 \le \theta \le 2$

Answer

$\frac{\sqrt{10}}{3}(e^6 - 1)$ units

33) $r = 6\cos\theta$ on the interval $0 \le \theta \le \frac{\pi}{2}$

34) $r = 8 + 8\cos\theta$ on the interval $0 \le \theta \le \pi$

Answer

32 units

35) $r = 1 - \sin\theta$ on the interval $0 \le \theta \le 2\pi$

In exercises 36 - 40, use the integration capabilities of a calculator to approximate the length of the curve.

36) [T] $r = 3\theta$ on the interval $0 \le \theta \le \frac{\pi}{2}$

Answer

6.238 units

37) [T] $r = \dfrac{2}{\theta}$ on the interval $\pi \le \theta \le 2\pi$

38) [T] $r = \sin^2\left(\frac{\theta}{2}\right)$ on the interval $0 \le \theta \le \pi$

Answer

2 units

39) [T] $r = 2\theta^2$ on the interval $0 \le \theta \le \pi$

40) [T] $r = \sin(3\cos\theta)$ on the interval $0 \le \theta \le \pi$

Answer

4.39 units

In exercises 41 - 43, use the familiar formula from geometry to find the area of the region described and then confirm by using the definite integral.

41) $r = 3\sin\theta$ on the interval $0 \le \theta \le \pi$

42) $r = \sin\theta + \cos\theta$ on the interval $0 \le \theta \le \pi$

Answer

$A = \pi\left(\frac{\sqrt{2}}{2}\right)^2 = \dfrac{\pi}{2}$ units^2 and $\dfrac{1}{2}\displaystyle\int_0^{\pi}(1 + 2\sin\theta\cos\theta)\,d\theta = \dfrac{\pi}{2}$ units^2

43) $r = 6\sin\theta + 8\cos\theta$ on the interval $0 \le \theta \le \pi$

In exercises 44 - 46, use the familiar formula from geometry to find the length of the curve and then confirm using the definite integral.

44) $r = 3\sin\theta$ on the interval $0 \le \theta \le \pi$

Answer

$C = 2\pi\left(\frac{3}{2}\right) = 3\pi$ units and $\displaystyle\int_0^{\pi} 3\,d\theta = 3\pi$ units

45) $r = \sin\theta + \cos\theta$ on the interval $0 \le \theta \le \pi$

46) $r = 6\sin\theta + 8\cos\theta$ on the interval $0 \le \theta \le \pi$

Answer

$C = 2\pi(5) = 10\pi$ units and $\int_0^{\pi} 10\,d\theta = 10\pi$ units

47) Verify that if $y = r\sin\theta = f(\theta)\sin\theta$ then $\dfrac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$.

In exercises 48 - 56, find the slope of a tangent line to a polar curve $r = f(\theta)$. Let $x = r\cos\theta = f(\theta)\cos\theta$ and $y = r\sin\theta = f(\theta)\sin\theta$, so the polar equation $r = f(\theta)$ is now written in parametric form.

48) Use the definition of the derivative $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta}$ and the product rule to derive the derivative of a polar equation.

Answer

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

49) $r = 1 - \sin\theta;\ \left(\frac{1}{2}, \frac{\pi}{6}\right)$

50) $r = 4\cos\theta;\ \left(2, \frac{\pi}{3}\right)$

Answer

The slope is $\frac{1}{\sqrt{3}}$.

51) $r = 8\sin\theta;\ \left(4, \frac{5\pi}{6}\right)$

52) $r = 4 + \sin\theta;\ \left(3, \frac{3\pi}{2}\right)$

Answer

The slope is 0.

53) $r = 6 + 3\cos\theta;\ (3, \pi)$

54) $r = 4\cos(2\theta)$; tips of the leaves

Answer

At $(4, 0)$, the slope is undefined. At $(-4, \frac{\pi}{2})$, the slope is 0.

55) $r = 2\sin(3\theta)$; tips of the leaves

56) $r = 2\theta;\ \left(\frac{\pi}{2}, \frac{\pi}{4}\right)$

Answer

The slope is undefined at $\theta = \frac{\pi}{4}$.

57) Find the points on the interval $-\pi \le \theta \le \pi$ at which the cardioid $r = 1 - \cos\theta$ has a vertical or horizontal tangent line.

58) For the cardioid $r = 1 + \sin\theta$, find the slope of the tangent line when $\theta = \frac{\pi}{3}$.

Answer

Slope = -1.

In exercises 59 - 62, find the slope of the tangent line to the given polar curve at the point given by the value of θ.

59) $r = 3\cos\theta,\ \theta = \frac{\pi}{3}$

60) $r = \theta,\ \theta = \frac{\pi}{2}$

Answer

Slope is $\frac{-2}{\pi}$.

61) $r = \ln\theta,\ \theta = e$

62) [T] Use technology: $r = 2 + 4\cos\theta$ at $\theta = \frac{\pi}{6}$

Answer

Calculator answer: -0.836.

In exercises 63 - 66, find the points at which the following polar curves have a horizontal or vertical tangent line.

63) $r = 4\cos\theta$

64) $r^2 = 4\cos(2\theta)$

Answer

Horizontal tangent at $\left(\pm\sqrt{2}, \frac{\pi}{6}\right),\ \left(\pm\sqrt{2}, -\frac{\pi}{6}\right)$.

65) $r = 2\sin(2\theta)$

66) The cardioid $r = 1 + \sin\theta$

Answer

Horizontal tangents at $\frac{\pi}{2},\ \frac{7\pi}{6},\ \frac{11\pi}{6}$.
Vertical tangents at $\frac{\pi}{6},\ \frac{5\pi}{6}$ and also at the pole $(0, 0)$.

67) Show that the curve $r = \sin\theta\tan\theta$ (called a **cissoid of Diocles**) has the line $x = 1$ as a vertical asymptote.

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11.5: Conic Sections

Learning Objectives

- Identify the equation of a parabola in standard form with given focus and directrix.
- Identify the equation of an ellipse in standard form with given foci.
- Identify the equation of a hyperbola in standard form with given foci.
- Recognize a parabola, ellipse, or hyperbola from its eccentricity value.
- Write the polar equation of a conic section with eccentricity e.
- Identify when a general equation of degree two is a parabola, ellipse, or hyperbola.

Conic sections have been studied since the time of the ancient Greeks, and were considered to be an important mathematical concept. As early as 320 BCE, such Greek mathematicians as Menaechmus, Appollonius, and Archimedes were fascinated by these curves. Appollonius wrote an entire eight-volume treatise on conic sections in which he was, for example, able to derive a specific method for identifying a conic section through the use of geometry. Since then, important applications of conic sections have arisen (for example, in astronomy), and the properties of conic sections are used in radio telescopes, satellite dish receivers, and even architecture. In this section we discuss the three basic conic sections, some of their properties, and their equations.

Conic sections get their name because they can be generated by intersecting a plane with a cone. A cone has two identically shaped parts called **nappes.** One nappe is what most people mean by "cone," having the shape of a party hat. A right circular cone can be generated by revolving a line passing through the origin around the y-axis as shown in Figure 11.5.1.

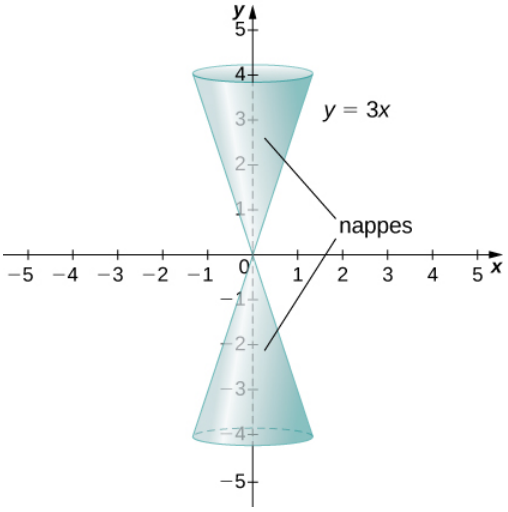


Figure 11.5.1: A cone generated by revolving the line $y = 3x$ around the y-axis.

Conic sections are generated by the intersection of a plane with a cone (Figure 11.5.2). If the plane is parallel to the axis of revolution (the y-axis), then the **conic section** is a hyperbola. If the plane is parallel to the generating line, the conic section is a parabola. If the plane is perpendicular to the axis of revolution, the conic section is a circle. If the plane intersects one nappe at an angle to the axis (other than **90°**), then the conic section is an ellipse.

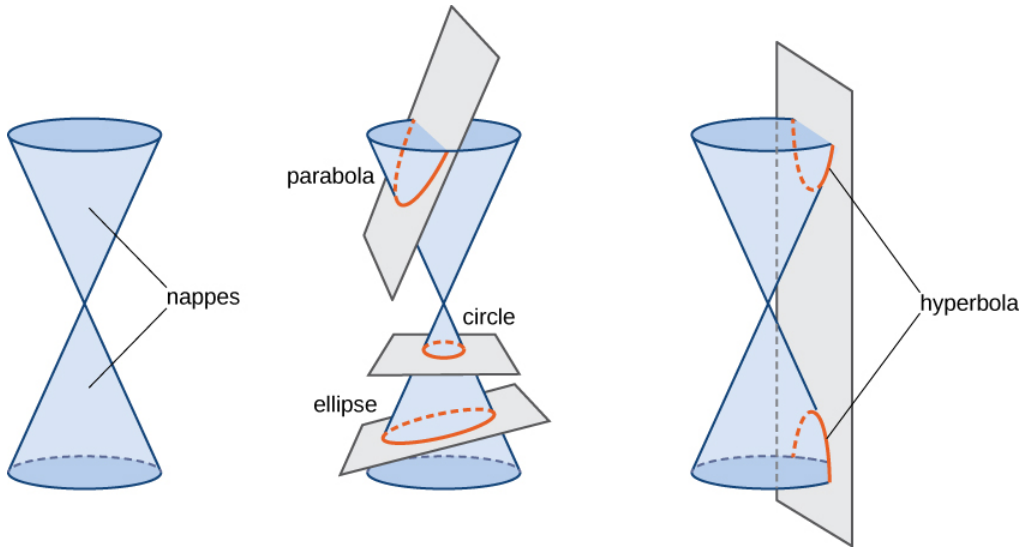


Figure 11.5.2: The four conic sections. Each conic is determined by the angle the plane makes with the axis of the cone.

Parabolas

A parabola is generated when a plane intersects a cone parallel to the generating line. In this case, the plane intersects only one of the nappes. A parabola can also be defined in terms of distances.

Definitions: The Focus, Directrix and Vertex

A parabola is the set of all points whose distance from a fixed point, called the *focus*, is equal to the distance from a fixed line, called the *directrix*. The point halfway between the focus and the directrix is called the *vertex* of the parabola.

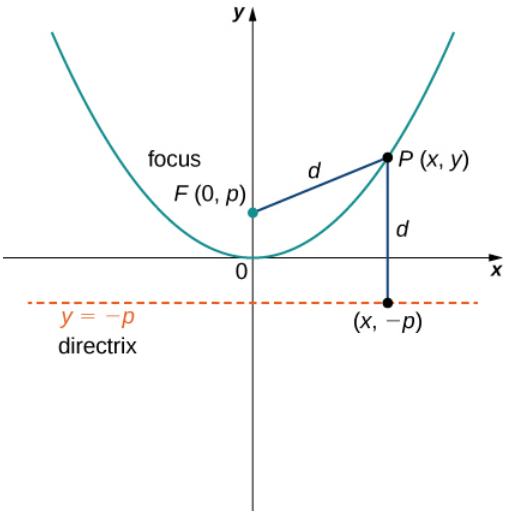


Figure 11.5.3: A typical parabola in which the distance from the focus to the vertex is represented by the variable p.

A graph of a typical parabola appears in Figure 11.5.3. Using this diagram in conjunction with the distance formula, we can derive an equation for a parabola. Recall the distance formula: Given point P with coordinates (x_1, y_1) and point Q with coordinates (x_2, y_2), the distance between them is given by the formula

$$d(P,Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Then from the definition of a parabola and Figure 11.5.3, we get

$$d(F,P) = d(P,Q)$$

$$\sqrt{(0-x)^2 + (p-y)^2} = \sqrt{(x-x)^2 + (-p-y)^2}.$$

Squaring both sides and simplifying yields

$$x^2 + (p-y)^2 = 0^2 + (-p-y)^2 \tag{11.5.1}$$
$$x^2 + p^2 - 2py + y^2 = p^2 + 2py + y^2 \tag{11.5.2}$$
$$x^2 - 2py = 2py \tag{11.5.3}$$
$$x^2 = 4py. \tag{11.5.4}$$

Now suppose we want to relocate the vertex. We use the variables (h, k) to denote the coordinates of the vertex. Then if the focus is directly above the vertex, it has coordinates $(h,\ k+p)$ and the directrix has the equation $y = k - p$. Going through the same derivation yields the formula $(x-h)^2 = 4p(y-k)$. Solving this equation for y leads to the following theorem.

Equations for Parabolas: standard form

Given a parabola opening upward with vertex located at (h, k) and focus located at $(h, k+p)$, where p is a constant, the equation for the parabola is given by

$$y = \frac{1}{4p}(x-h)^2 + k. \tag{11.5.5}$$

This is the *standard form* of a parabola.

We can also study the cases when the parabola opens down or to the left or the right. The equation for each of these cases can also be written in standard form as shown in the following graphs.

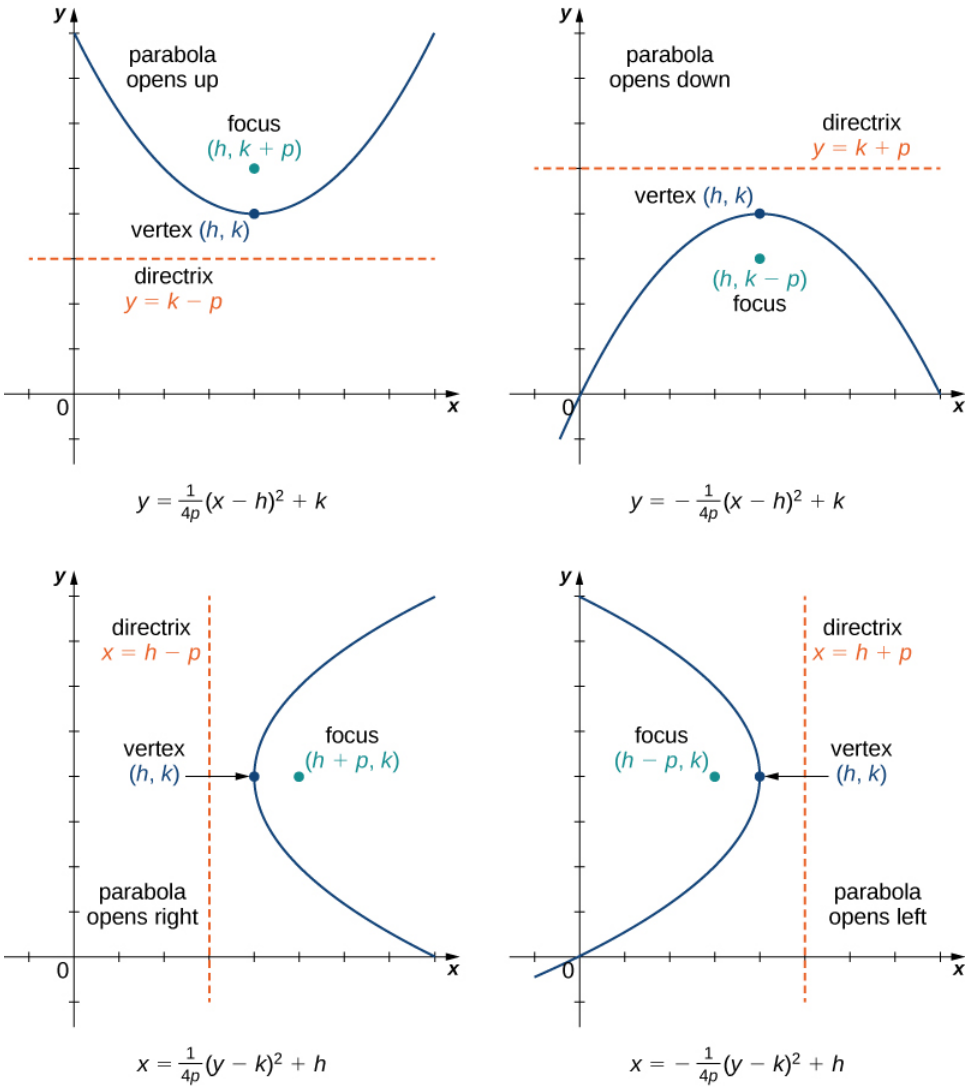


Figure 11.5.4: Four parabolas, opening in various directions, along with their equations in standard form.

In addition, the equation of a parabola can be written in the **general form**, though in this form the values of h, k, and p are not immediately recognizable. The general form of a parabola is written as

$$ax^2 + bx + cy + d = 0 \tag{11.5.6}$$

or

$$ay^2 + bx + cy + d = 0. \tag{11.5.7}$$

Equation 11.5.6 represents a parabola that opens either up or down. Equation 11.5.7 represents a parabola that opens either to the left or to the right. To put the equation into standard form, use the method of completing the square.

Example 11.5.1: Converting the Equation of a Parabola from General into Standard Form

Put the equation

$$x^2 - 4x - 8y + 12 = 0$$

into standard form and graph the resulting parabola.

Solution

Since y is not squared in this equation, we know that the parabola opens either upward or downward. Therefore we need to solve this equation for y, which will put the equation into standard form. To do that, first add $8y$ to both sides of the equation:

$$8y = x^2 - 4x + 12.$$

The next step is to complete the square on the right-hand side. Start by grouping the first two terms on the right-hand side using parentheses:

$$8y = (x^2 - 4x) + 12.$$

Next determine the constant that, when added inside the parentheses, makes the quantity inside the parentheses a perfect square trinomial. To do this, take half the coefficient of x and square it. This gives $\left(\frac{-4}{2}\right)^2 = 4$. Add 4 inside the parentheses and subtract 4 outside the parentheses, so the value of the equation is not changed:

$$8y = (x^2 - 4x + 4) + 12 - 4.$$

Now combine like terms and factor the quantity inside the parentheses:

$$8y = (x - 2)^2 + 8.$$

Finally, divide by 8:

$$y = \frac{1}{8}(x - 2)^2 + 1.$$

This equation is now in standard form. Comparing this to Equation 11.5.5 gives $h = 2$, $k = 1$, and $p = 2$. The parabola opens up, with vertex at $(2, 1)$, focus at $(2, 3)$, and directrix $y = -1$. The graph of this parabola appears as follows.

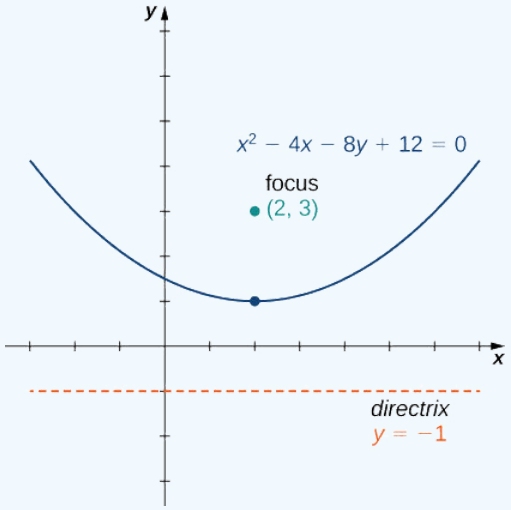


Figure 11.5.5: The parabola in Example 11.5.1.

Exercise 11.5.1

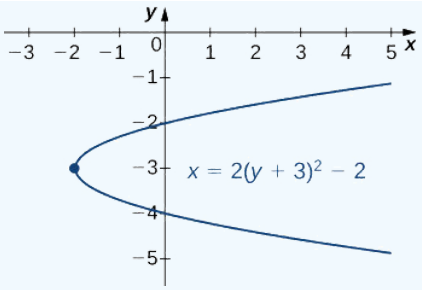
Put the equation $2y^2 - x + 12y + 16 = 0$ into standard form and graph the resulting parabola.

Hint

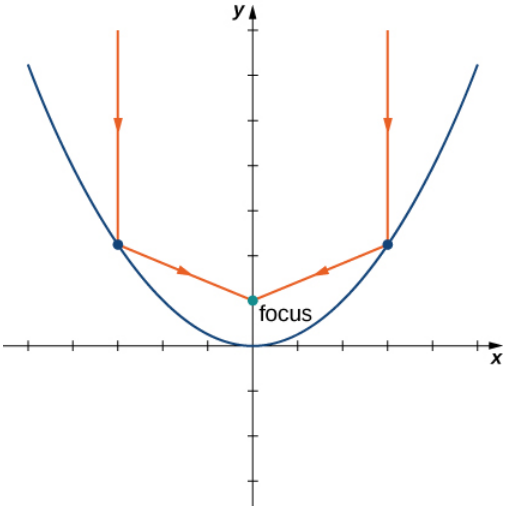
Solve for x. Check which direction the parabola opens.

Answer

$$x = 2(y + 3)^2 - 2$$



The axis of symmetry of a vertical (opening up or down) parabola is a vertical line passing through the vertex. The parabola has an interesting reflective property. Suppose we have a satellite dish with a parabolic cross section. If a beam of electromagnetic waves, such as light or radio waves, comes into the dish in a straight line from a satellite (parallel to the axis of symmetry), then the waves reflect off the dish and collect at the focus of the parabola as shown.



Consider a parabolic dish designed to collect signals from a satellite in space. The dish is aimed directly at the satellite, and a receiver is located at the focus of the parabola. Radio waves coming in from the satellite are reflected off the surface of the parabola to the receiver, which collects and decodes the digital signals. This allows a small receiver to gather signals from a wide angle of sky. Flashlights and headlights in a car work on the same principle, but in reverse: the source of the light (that is, the light bulb) is located at the focus and the reflecting surface on the parabolic mirror focuses the beam straight ahead. This allows a small light bulb to illuminate a wide angle of space in front of the flashlight or car.

Ellipses

An ellipse can also be defined in terms of distances. In the case of an ellipse, there are two foci (plural of focus), and two directrices (plural of directrix). We look at the directrices in more detail later in this section.

Definition: Ellipse

An ellipse is the set of all points for which the sum of their distances from two fixed points (the foci) is constant.

A graph of a typical ellipse is shown in Figure 11.5.6. In this figure the foci are labeled as F and F'. Both are the same fixed distance from the origin, and this distance is represented by the variable c. Therefore the coordinates of F are $(c, 0)$ and the coordinates of F' are $(-c, 0)$. The points P and P' are located at the ends of the **major axis** of the ellipse, and have coordinates $(a, 0)$ and $(-a, 0)$, respectively. The major axis is always the longest distance across the ellipse, and can be horizontal or vertical. Thus, the length of the major axis in this ellipse is $2a$. Furthermore, P and P' are called the vertices of the ellipse. The points Q and Q' are located at the ends of the **minor axis** of the ellipse, and have coordinates $(0, b)$ and $(0, -b)$, respectively. The minor axis is the shortest distance across the ellipse. The minor axis is perpendicular to the major axis.

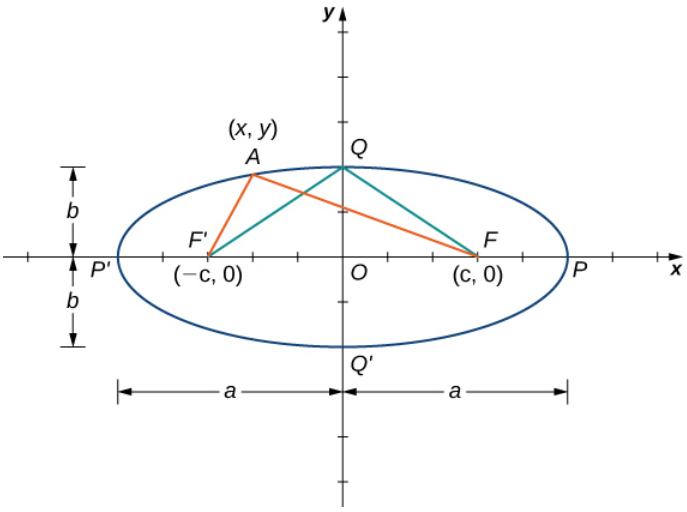


Figure 11.5.6: A typical ellipse in which the sum of the distances from any point on the ellipse to the foci is constant.

According to the definition of the ellipse, we can choose any point on the ellipse and the sum of the distances from this point to the two foci is constant. Suppose we choose the point P. Since the coordinates of point P are $(a, 0)$, the sum of the distances is

$$d(P, F) + d(P, F') = (a - c) + (a + c) = 2a.$$

Therefore the sum of the distances from an arbitrary point A with coordinates (x, y) is also equal to $2a$. Using the distance formula, we get

$$d(A, F) + d(A, F') = 2a.$$

$$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a$$

Subtract the second radical from both sides and square both sides:

$$\sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2}$$

$$(x-c)^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2$$

$$x^2 - 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2cx + c^2 + y^2$$

$$-2cx = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + 2cx.$$

Now isolate the radical on the right-hand side and square again:

$$-2cx = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + 2cx$$

$$4a\sqrt{(x+c)^2 + y^2} = 4a^2 + 4cx$$

$$\sqrt{(x+c)^2 + y^2} = a + \frac{cx}{a}$$

$$(x+c)^2 + y^2 = a^2 + 2cx + \frac{c^2x^2}{a^2}$$

$$x^2 + 2cx + c^2 + y^2 = a^2 + 2cx + \frac{c^2x^2}{a^2}$$

$$x^2 + c^2 + y^2 = a^2 + \frac{c^2x^2}{a^2}.$$

Isolate the variables on the left-hand side of the equation and the constants on the right-hand side:

$$x^2 - \frac{c^2x^2}{a^2} + y^2 = a^2 - c^2$$

$$x^2 - \frac{}{a^2} + y^2 = a^2 - c^2$$

$$\frac{(a^2 - c^2)x^2}{a^2} + y^2 = a^2 - c^2.$$

Divide both sides by $a^2 - c^2$. This gives the equation

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

If we refer back to Figure 11.5.6, then the length of each of the two green line segments is equal to a. This is true because the sum of the distances from the point Q to the foci F and F' is equal to $2a$, and the lengths of these two line segments are equal. This line segment forms a right triangle with hypotenuse length a and leg lengths b and c. From the Pythagorean theorem, $b^2 + c^2 = a^2$ and $b^2 = a^2 - c^2$. Therefore the equation of the ellipse becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Finally, if the center of the ellipse is moved from the origin to a point (h, k), we have the following standard form of an ellipse.

Equation of an Ellipse in Standard Form

Consider the ellipse with center (h, k), a horizontal major axis with length $2a$, and a vertical minor axis with length $2b$. Then the equation of this ellipse in standard form is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \tag{11.5.8}$$

and the foci are located at $(h \pm c, k)$, where $c^2 = a^2 - b^2$. The equations of the directrices are $x = h \pm \frac{a^2}{c}$.

If the major axis is vertical, then the equation of the ellipse becomes

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1 \tag{11.5.9}$$

and the foci are located at $(h, k \pm c)$, where $c^2 = a^2 - b^2$. The equations of the directrices in this case are $y = k \pm \frac{a^2}{c}$.

If the major axis is horizontal, then the ellipse is called horizontal, and if the major axis is vertical, then the ellipse is called vertical. The equation of an ellipse is in general form if it is in the form

$$Ax^2 + By^2 + Cx + Dy + E = 0,$$

where A and B are either both positive or both negative. To convert the equation from general to standard form, use the method of **completing the square**.

Example 11.5.2: Finding the Standard Form of an Ellipse

Put the equation

$$9x^2 + 4y^2 - 36x + 24y + 36 = 0$$

into standard form and graph the resulting ellipse.

Solution

First subtract 36 from both sides of the equation:

$$9x^2 + 4y^2 - 36x + 24y = -36.$$

Next group the x terms together and the y terms together, and factor out the common factor:

$$(9x^2 - 36x) + (4y^2 + 24y) = -36$$

$$9(x^2 - 4x) + 4(y^2 + 6y) = -36.$$

We need to determine the constant that, when added inside each set of parentheses, results in a perfect square. In the first set of parentheses, take half the coefficient of x and square it. This gives $\left(\frac{-4}{2}\right)^2 = 4$. In the second set of parentheses, take half the coefficient of y and square it. This gives $\left(\frac{6}{2}\right)^2 = 9$. Add these inside each pair of parentheses. Since the first set of parentheses has a 9 in front, we are actually adding 36 to the left-hand side. Similarly, we are adding 36 to the second set as well. Therefore the equation becomes

$$9(x^2 - 4x + 4) + 4(y^2 + 6y + 9) = -36 + 36 + 36$$

$$9(x^2 - 4x + 4) + 4(y^2 + 6y + 9) = 36.$$

Now factor both sets of parentheses and divide by 36:

$$9(x-2)^2 + 4(y+3)^2 = 36$$

$$\frac{9(x-2)^2}{36} + \frac{4(y+3)^2}{36} = 1$$

$$\frac{(x-2)^2}{4} + \frac{(y+3)^2}{9} = 1.$$

The equation is now in standard form. Comparing this to Equation 11.5.9 gives $h = 2, k = -3, a = 3,$ and $b = 2$. This is a vertical ellipse with center at $(2, -3)$, major axis 6, and minor axis 4. The graph of this ellipse appears as follows.

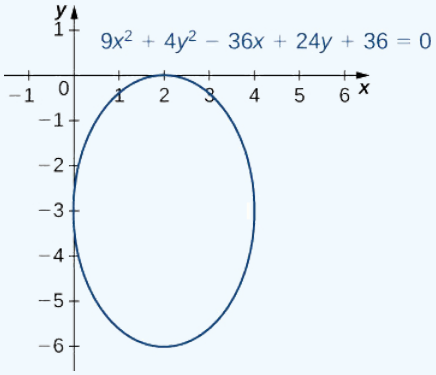


Figure 11.5.7: The ellipse in Example 11.5.2.

Exercise 11.5.2

Put the equation

$$9x^2 + 16y^2 + 18x - 64y - 71 = 0$$

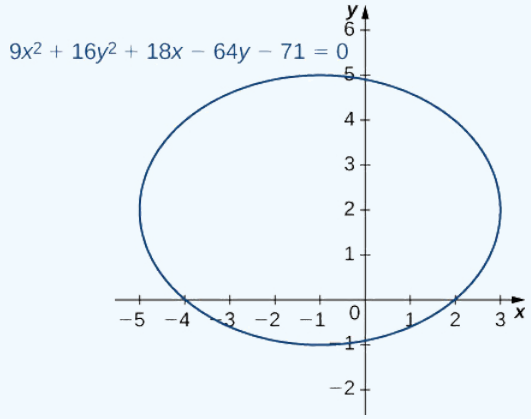
into standard form and graph the resulting ellipse.

Hint

Move the constant over and complete the square.

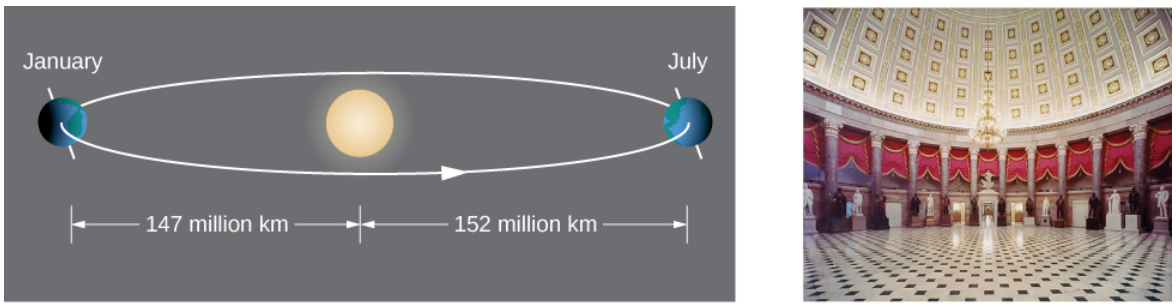
Answer

$$\frac{(x+1)^2}{16} + \frac{(y-2)^2}{9} = 1$$



According to Kepler's first law of planetary motion, the orbit of a planet around the Sun is an ellipse with the Sun at one of the foci as shown in Figure 11.5.8*A*. Because Earth's orbit is an ellipse, the distance from the Sun varies throughout the year. A commonly held misconception is that Earth is closer to the Sun in the summer. In fact, in summer for the northern hemisphere, Earth is farther from the Sun than during winter. The difference in season is caused by the tilt of Earth's axis in the orbital plane. Comets that orbit the Sun, such as Halley's Comet, also have elliptical orbits, as do moons orbiting the planets and satellites orbiting Earth.

Ellipses also have interesting reflective properties: A light ray emanating from one focus passes through the other focus after mirror reflection in the ellipse. The same thing occurs with a sound wave as well. The National Statuary Hall in the U.S. Capitol in Washington, DC, is a famous room in an elliptical shape as shown in Figure 11.5.8*B*. This hall served as the meeting place for the U.S. House of Representatives for almost fifty years. The location of the two foci of this semi-elliptical room are clearly identified by marks on the floor, and even if the room is full of visitors, when two people stand on these spots and speak to each other, they can hear each other much more clearly than they can hear someone standing close by. Legend has it that John Quincy Adams had his desk located on one of the foci and was able to eavesdrop on everyone else in the House without ever needing to stand. Although this makes a good story, it is unlikely to be true, because the original ceiling produced so many echoes that the entire room had to be hung with carpets to dampen the noise. The ceiling was rebuilt in 1902 and only then did the now-famous whispering effect emerge. Another famous whispering gallery—the site of many marriage proposals—is in Grand Central Station in New York City.



(a) (b)

Figure 11.5.8: (a) Earth's orbit around the Sun is an ellipse with the Sun at one focus. (b) Statuary Hall in the U.S. Capitol is a whispering gallery with an elliptical cross section.

Hyperbolas

A hyperbola can also be defined in terms of distances. In the case of a hyperbola, there are two foci and two directrices. Hyperbolas also have two asymptotes.

Definition: hyperbola

A hyperbola is the set of all points where the difference between their distances from two fixed points (the foci) is constant.

A graph of a typical hyperbola appears as follows.

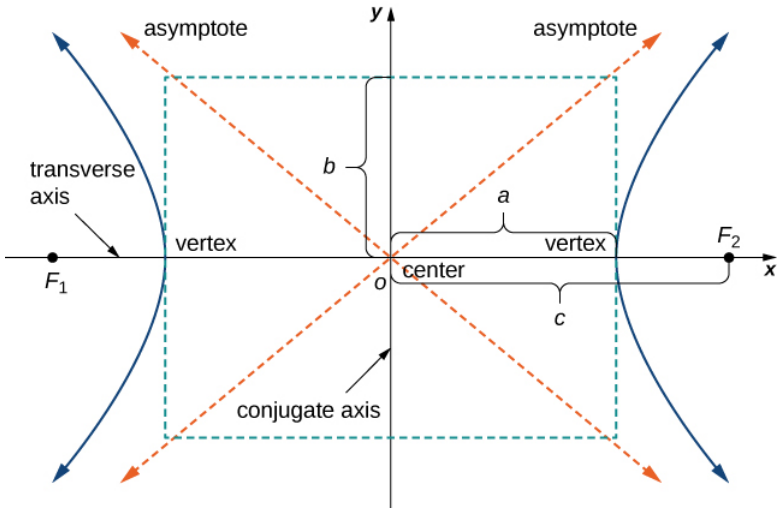


Figure 11.5.9: A typical hyperbola in which the difference of the distances from any point on the hyperbola to the foci is constant. The transverse axis is also called the major axis, and the conjugate axis is also called the minor axis.

The derivation of the equation of a hyperbola in standard form is virtually identical to that of an ellipse. One slight hitch lies in the definition: The difference between two numbers is always positive. Let P be a point on the hyperbola with coordinates (x, y). Then the definition of the hyperbola gives $|d(P, F_1) - d(P, F_2)| = constant$. To simplify the derivation, assume that P is on the right branch of the hyperbola, so the absolute value bars drop. If it is on the left branch, then the subtraction is reversed. The vertex of the right branch has coordinates $(a, 0)$, so

$$d(P, F_1) - d(P, F_2) = (c + a) - (c - a) = 2a.$$

This equation is therefore true for any point on the hyperbola. Returning to the coordinates (x, y) for P:

$$d(P, F_1) - d(P, F_2) = 2a$$

$$\sqrt{(x + c)^2 + y^2} - \sqrt{(x - c)^2 + y^2} = 2a.$$

Isolate the second radical and square both sides:

$$\sqrt{(x - c)^2 + y^2} = -2a + \sqrt{(x + c)^2 + y^2}$$

$$(x - c)^2 + y^2 = 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + (x + c)^2 + y^2$$

$$x^2 - 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + x^2 + 2cx + c^2 + y^2$$

$$-2cx = 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + 2cx.$$

Now isolate the radical on the right-hand side and square again:

$$-2cx = 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + 2cx$$

$$-4a\sqrt{(x + c)^2 + y^2} = -4a^2 - 4cx$$

$$-\sqrt{(x + c)^2 + y^2} = -a - \frac{cx}{a}$$

$$(x + c)^2 + y^2 = a^2 + 2cx + \frac{c^2x^2}{a^2}$$

$$x^2 + 2cx + c^2 + y^2 = a^2 + 2cx + \frac{c^2x^2}{a^2}$$

$$x^2 + c^2 + y^2 = a^2 + \frac{c^2x^2}{a^2}.$$

Isolate the variables on the left-hand side of the equation and the constants on the right-hand side:

$$x^2 - \frac{c^2x^2}{a^2} + y^2 = a^2 - c^2$$

$$a^2$$

$$\frac{(a^2-c^2)x^2}{a^2}+y^2=a^2-c^2.$$

Finally, divide both sides by a^2-c^2. This gives the equation

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1.$$

We now define *b* so that $b^2=c^2-a^2$. This is possible because $c>a$. Therefore the equation of the hyperbola becomes

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

Finally, if the center of the hyperbola is moved from the origin to the point (h,k), we have the following standard form of a hyperbola.

Equation of a Hyperbola in Standard Form

Consider the hyperbola with center (h,k), a horizontal major axis, and a vertical minor axis. Then the equation of this hyperbola is

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1 \tag{11.5.10}$$

and the foci are located at $(h\pm c,k)$, where $c^2=a^2+b^2$. The equations of the asymptotes are given by $y=k\pm\frac{b}{a}(x-h)$. The equations of the directrices are

$$x=h\pm\frac{a^2}{\sqrt{a^2+b^2}}=h\pm\frac{a^2}{c}$$

If the major axis is vertical, then the equation of the hyperbola becomes

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

and the foci are located at $(h,k\pm c)$, where $c^2=a^2+b^2$. The equations of the asymptotes are given by $y=k\pm\frac{a}{b}(x-h)$. The equations of the directrices are

$$y=k\pm\frac{a^2}{\sqrt{a^2+b^2}}=k\pm\frac{a^2}{c}.$$

If the major axis (transverse axis) is horizontal, then the hyperbola is called horizontal, and if the major axis is vertical then the hyperbola is called vertical. The equation of a hyperbola is in general form if it is in the form

$$Ax^2+By^2+Cx+Dy+E=0,$$

where A and B have opposite signs. In order to convert the equation from general to standard form, use the method of completing the square.

Example 11.5.3: Finding the Standard Form of a Hyperbola

Put the equation $9x^2-16y^2+36x+32y-124=0$ into standard form and graph the resulting hyperbola. What are the equations of the asymptotes?

Solution

First add 124 to both sides of the equation:

$$9x^2-16y^2+36x+32y=124.$$

Next group the x terms together and the y terms together, then factor out the common factors:

$$(9x^2+36x)-(16y^2-32y)=124$$

$$9(x^2+4x)-16(y^2-2y)=124.$$

We need to determine the constant that, when added inside each set of parentheses, results in a perfect square. In the first set of parentheses, take half the coefficient of x and square it. This gives $\left(\frac{4}{2}\right)^2=4$. In the second set of parentheses, take half the coefficient of y and square it. This gives $\left(\frac{-2}{2}\right)^2=1$. Add these inside each pair of parentheses. Since the first set of parentheses has a 9 in front, we are actually adding 36 to the left-hand side. Similarly, we are subtracting 16 from the second set of parentheses. Therefore the equation becomes

$$9(x^2+4x+4)-16(y^2-2y+1)=124+36-16$$

$$9(x^2+4x+4)-16(y^2-2y+1)=144.$$

Next factor both sets of parentheses and divide by 144:

$$9(x+2)^2-16(y-1)^2=144$$

$$\frac{9(x+2)^2}{144}-\frac{16(y-1)^2}{144}=1$$

$$\frac{(x+2)^2}{16}-\frac{(y-1)^2}{9}=1.$$

The equation is now in standard form. Comparing this to Equation 11.5.10 gives $h=-2$, $k=1$, $a=4$, and $b=3$. This is a horizontal hyperbola with center at $(-2,1)$ and asymptotes given by the equations $y=1\pm\frac{3}{4}(x+2)$. The graph of this hyperbola appears in Figure 11.5.10.

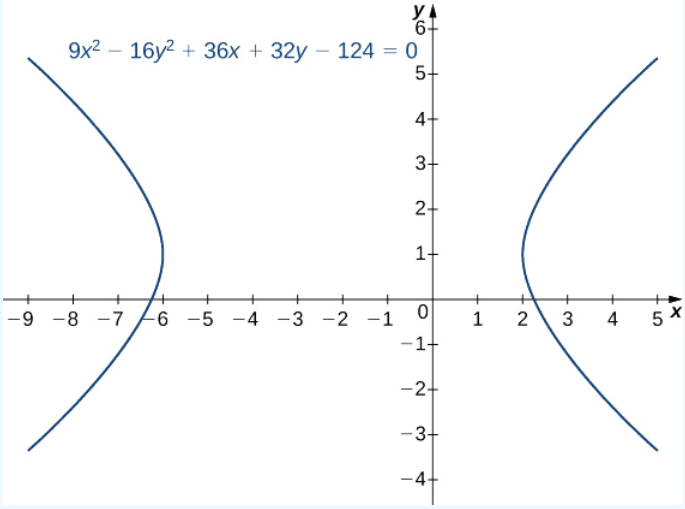


Figure 11.5.10: Graph of the hyperbola in Example 11.5.3.

? Exercise 11.5.3

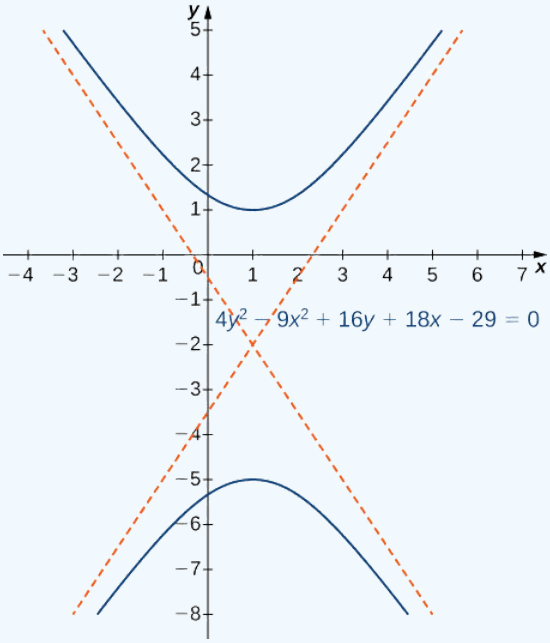
Put the equation $4y^2-9x^2+16y+18x-29=0$ into standard form and graph the resulting hyperbola. What are the equations of the asymptotes?

Hint

Move the constant over and complete the square. Check which direction the hyperbola opens

Answer

$\frac{(y+2)^2}{9}-\frac{(x-1)^2}{4}=1$. This is a vertical hyperbola. Asymptotes $y=-2\pm\frac{3}{2}(x-1)$.



Hyperbolas also have interesting reflective properties. A ray directed toward one focus of a hyperbola is reflected by a hyperbolic mirror toward the other focus. This concept is illustrated in Figure 11.5.11.

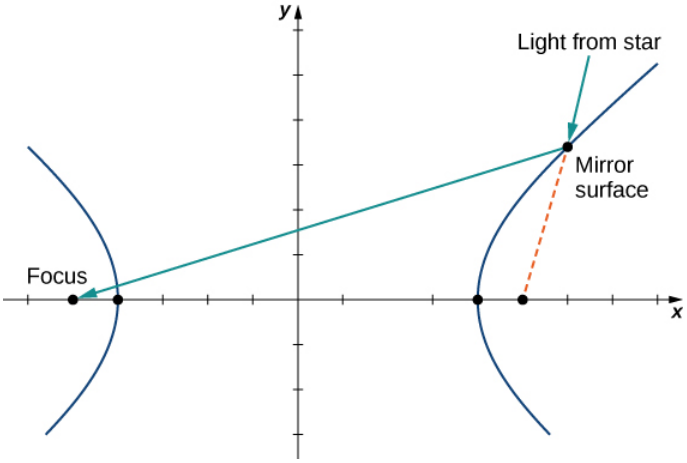


Figure 11.5.11: A hyperbolic mirror used to collect light from distant stars.

This property of the hyperbola has important applications. It is used in radio direction finding (since the difference in signals from two towers is constant along hyperbolas), and in the construction of mirrors inside telescopes (to reflect light coming from the parabolic mirror to the eyepiece). Another interesting fact about hyperbolas is that for a comet entering the solar system, if the speed is great enough to escape the Sun's gravitational pull, then the path that the comet takes as it passes through the solar system is hyperbolic.

Eccentricity and Directrix

An alternative way to describe a conic section involves the directrices, the foci, and a new property called eccentricity. We will see that the value of the eccentricity of a conic section can uniquely define that conic.

Definition: Eccentricity and Directrices

The *eccentricity* e of a conic section is defined to be the distance from any point on the conic section to its focus, divided by the perpendicular distance from that point to the nearest directrix. This value is constant for any conic section, and can define the conic section as well:

1. If $e = 1$, the conic is a parabola.
2. If $e < 1$, it is an ellipse.
3. If $e > 1$, it is a hyperbola.

The eccentricity of a circle is zero. The *directrix* of a conic section is the line that, together with the point known as the focus, serves to define a conic section. Hyperbolas and noncircular ellipses have two foci and two associated directrices. Parabolas have one focus and one directrix.

The three conic sections with their directrices appear in Figure 11.5.12.

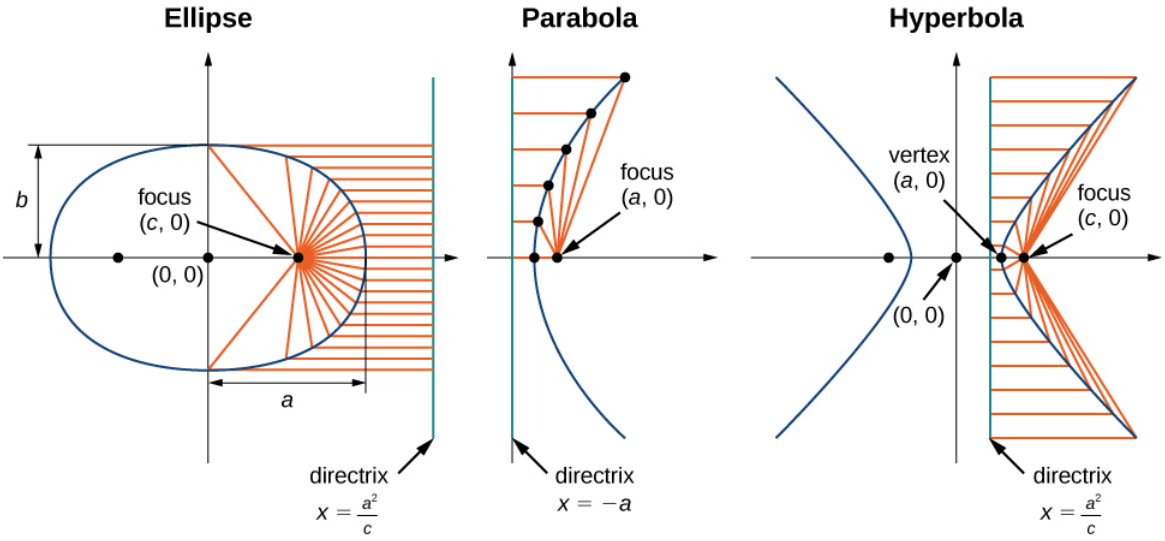


Figure 11.5.12: The three conic sections with their foci and directrices.

Recall from the definition of a parabola that the distance from any point on the parabola to the focus is equal to the distance from that same point to the directrix. Therefore, by definition, the eccentricity of a parabola must be 1. The equations of the directrices of a horizontal ellipse are $x = \pm\frac{a^2}{c}$. The right vertex of the ellipse is located at $(a, 0)$ and the right focus is $(c, 0)$. Therefore the distance from the vertex to the focus is $a - c$ and the distance from the vertex to the right directrix is $\frac{a^2}{c} - c$. This gives the eccentricity as

$$e = \frac{a-c}{\frac{a^2}{c} - a} = \frac{c(a-c)}{a^2 - ac} = \frac{c(a-c)}{a(a-c)} = \frac{c}{a}.$$

Since $c < a$, this step proves that the eccentricity of an ellipse is less than 1. The directrices of a horizontal hyperbola are also located at $x = \pm\frac{a^2}{c}$, and a similar calculation shows that the eccentricity of a hyperbola is also $e = \frac{c}{a}$. However in this case we have $c > a$, so the eccentricity of a hyperbola is greater than 1.

Example 11.5.4: Determining Eccentricity of a Conic Section

Determine the eccentricity of the ellipse described by the equation

$$\frac{(x-3)^2}{16} + \frac{(y+2)^2}{25} = 1.$$

Solution

From the equation we see that $a = 5$ and $b = 4$. The value of c can be calculated using the equation $a^2 = b^2 + c^2$ for an ellipse. Substituting the values of a and b and solving for c gives $c = 3$. Therefore the eccentricity of the ellipse is $e = \frac{c}{a} = \frac{3}{5} = 0.6$.

Exercise 11.5.4

Determine the eccentricity of the hyperbola described by the equation

$$\frac{(y-3)^2}{49} - \frac{(x+2)^2}{25} = 1.$$

Hint

First find the values of a and b, then determine c using the equation $c^2 = a^2 + b^2$.

Answer

$$e = \frac{c}{a} = \frac{\sqrt{74}}{7} \approx 1.229$$

Polar Equations of Conic Sections

Sometimes it is useful to write or identify the equation of a conic section in polar form. To do this, we need the concept of the focal parameter. The **focal parameter** of a conic section p is defined as the distance from a focus to the nearest directrix. The following table gives the focal parameters for the different types of conics, where a is the length of the semi-major axis (i.e., half the length of the major axis), c is the distance from the origin to the focus, and e is the eccentricity. In the case of a parabola, a represents the distance from the vertex to the focus.

Table 11.5.1: Eccentricities and Focal Parameters of the Conic Sections

Conic	e	p
Ellipse	$0 < e < 1$	$\frac{a^2 - c^2}{c} = \frac{a(1-e^2)}{c}$
Parabola	$e = 1$	$2a$
Hyperbola	$e > 1$	$\frac{c^2 - a^2}{c} = \frac{a(e^2-1)}{c}$

Using the definitions of the focal parameter and eccentricity of the conic section, we can derive an equation for any conic section in polar coordinates. In particular, we assume that one of the foci of a given conic section lies at the pole. Then using the definition of the various conic sections in terms of distances, it is possible to prove the following theorem.

Polar Equation of Conic Sections

The polar equation of a conic section with focal parameter p is given by

$$r = \frac{ep}{1 \pm e\cos\theta} \text{ or } r = \frac{ep}{1 \pm e\sin\theta}.$$

In the equation on the left, the major axis of the conic section is horizontal, and in the equation on the right, the major axis is vertical. To work with a conic section written in polar form, first make the constant term in the denominator equal to 1. This can be done by dividing both the numerator and the denominator of the fraction by the constant that appears in front of the plus or minus in the denominator. Then the coefficient of the sine or cosine in the denominator is the eccentricity. This value identifies the conic. If cosine appears in the denominator, then the conic is horizontal. If sine appears, then the conic is vertical. If both appear then the axes are rotated. The center of the conic is not necessarily at the origin. The center is at the origin only if the conic is a circle (i.e., $e = 0$).

Example 11.5.5: Graphing a Conic Section in Polar Coordinates

Identify and create a graph of the conic section described by the equation

$$r = \frac{3}{1 + 2\cos\theta}.$$

Solution

The constant term in the denominator is 1, so the eccentricity of the conic is 2. This is a hyperbola. The focal parameter p can be calculated by using the equation $ep = 3$. Since $e = 2$, this gives $p = \frac{3}{2}$. The cosine function appears in the denominator, so the hyperbola is horizontal. Pick a few values for θ and create a table of values. Then we can graph the hyperbola (Figure 11.5.13).

θ	r	θ	r
0	1	π	-3
$\frac{\pi}{4}$	$\frac{3}{1+\sqrt{2}} \approx 1.2426$	$\frac{5\pi}{4}$	$\frac{3}{1-\sqrt{2}} \approx -7.2426$
$\frac{\pi}{2}$	3	$\frac{3\pi}{2}$	3
$\frac{3\pi}{4}$	$\frac{3}{1-\sqrt{2}} \approx -7.2426$	$\frac{7\pi}{4}$	$\frac{3}{1+\sqrt{2}} \approx 1.2426$

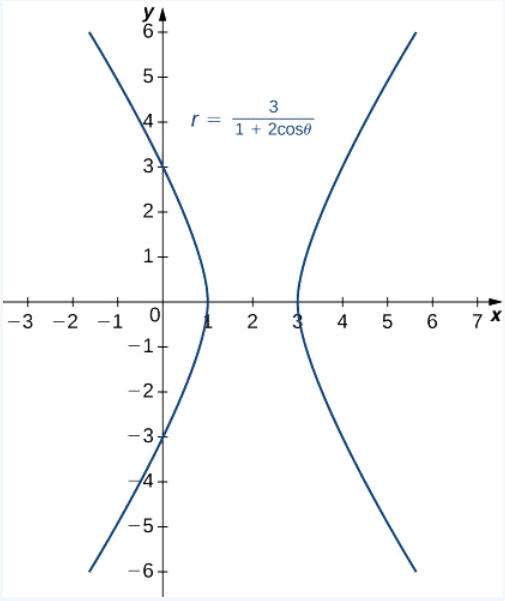


Figure 11.5.13: Graph of the hyperbola described in Example 11.5.5.

Exercise 11.5.5

Identify and create a graph of the conic section described by the equation

$$r = \frac{4}{1 - 0.8\sin\theta}.$$

Hint

First find the values of e and p, and then create a table of values.

Answer

Here $e = 0.8$ and $p = 5$. This conic section is an ellipse.

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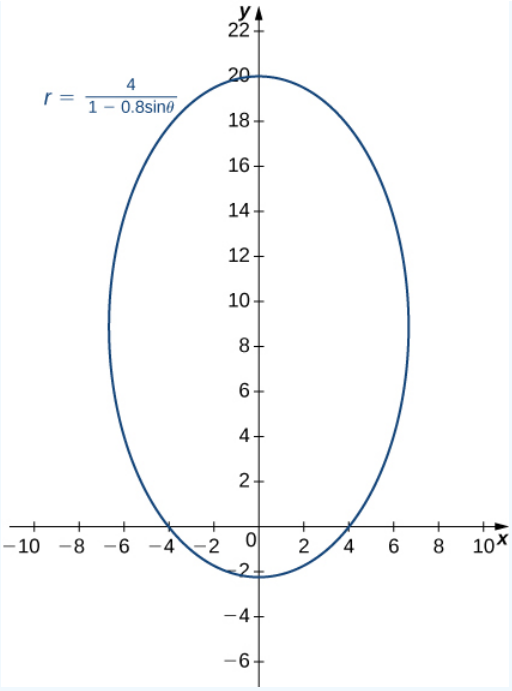

General Equations of Degree Two

A general equation of degree two can be written in the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

The graph of an equation of this form is a conic section. If $B \neq 0$ then the coordinate axes are rotated. To identify the conic section, we use the discriminant of the conic section $4AC - B^2$.

Identifying the Conic Section

One of the following cases must be true:

1. $4AC - B^2 > 0$. If so, the graph is an ellipse.
2. $4AC - B^2 = 0$. If so, the graph is a parabola.
3. $4AC - B^2 < 0$. If so, the graph is a hyperbola.

The simplest example of a second-degree equation involving a cross term is $xy = 1$. This equation can be solved for y to obtain $y = \frac{1}{x}$. The graph of this function is called a rectangular hyperbola as shown.

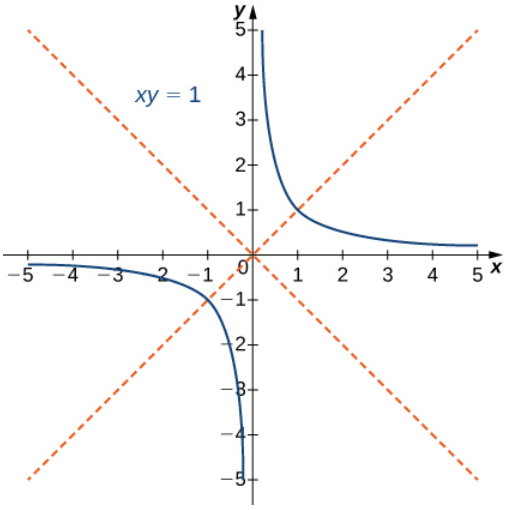

Figure 11.5.14: Graph of the equation $xy = 1$; The red lines indicate the rotated axes.

The asymptotes of this hyperbola are the x and y coordinate axes. To determine the angle θ of rotation of the conic section, we use the formula $\cot 2\theta = \frac{A-C}{B}$. In this case $A = C = 0$ and $B = 1$, so $\cot 2\theta = (0-0)/1 = 0$ and $\theta = 45°$. The method for graphing a conic section with rotated axes involves determining the coefficients of the conic in the rotated coordinate system. The new coefficients are labeled A', B', C', D', E', and F', and are given by the formulas

$$\begin{aligned} A' &= A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta \\ B' &= 0 \\ C' &= A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta \\ D' &= D\cos\theta + E\sin\theta \\ E' &= -D\sin\theta + E\cos\theta \\ F' &= F. \end{aligned}$$

Procedure: graphing a rotated conic

The procedure for graphing a rotated conic is the following:

1. Identify the conic section using the discriminant $4AC - B^2$.
2. Determine θ using the formula

$$\cot 2\theta = \frac{A-C}{B}. \tag{11.5.11}$$

3. Calculate A', B', C', D', E', and F'.
4. Rewrite the original equation using A', B', C', D', E', and F'.
5. Draw a graph using the rotated equation.

Example 11.5.6: Identifying a Rotated Conic

Identify the conic and calculate the angle of rotation of axes for the curve described by the equation

$$13x^2 - 6\sqrt{3}xy + 7y^2 - 256 = 0.$$

Solution

In this equation, $A = 13,\ B = -6\sqrt{3},\ C = 7,\ D = 0,\ E = 0$, and $F = -256$. The discriminant of this equation is

$$4AC - B^2 = 4(13)(7) - (-6\sqrt{3})^2 = 364 - 108 = 256.$$

Therefore this conic is an ellipse.

To calculate the angle of rotation of the axes, use Equation 11.5.11

$$\cot 2\theta = \frac{A-C}{B}.$$

This gives

$$\cot 2\theta = \frac{A-C}{B} = \frac{13-7}{-6\sqrt{3}} = -\frac{\sqrt{3}}{3}.$$

Therefore $2\theta = 120^o$ and $\theta = 60^o$, which is the angle of the rotation of the axes.

To determine the rotated coefficients, use the formulas given above:

$$\begin{aligned}
A' &= A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta \\
&= 13\cos^2 60 + (-6\sqrt{3})\cos 60\sin 60 + 7\sin^2 60 \\
&= 13\left(\frac{1}{2}\right)^2 - 6\sqrt{3}\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + 7\left(\frac{\sqrt{3}}{2}\right)^2 \\
&= 4, \\
B' &= 0 \\
C' &= A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta \\
&= 13\sin^2 60 + (6\sqrt{3})\sin 60\cos 60 + 7\cos^2 60 \\
&= 13\left(\frac{\sqrt{3}}{2}\right)^2 + 6\sqrt{3}\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) + 7\left(\frac{1}{2}\right)^2 \\
&= 16, \\
D' &= D\cos\theta + E\sin\theta \\
&= (0)\cos 60 + (0)\sin 60 \\
&= 0, \\
E' &= -D\sin\theta + E\cos\theta \\
&= -(0)\sin 60 + (0)\cos 60 \\
&= 0 \\
F' &= F \\
&= -256.
\end{aligned}$$

The equation of the conic in the rotated coordinate system becomes

$$\begin{aligned}
4(x')^2 + 16(y')^2 &= 256 \\
\frac{(x')^2}{64} + \frac{(y')^2}{16} &= 1.
\end{aligned}$$

A graph of this conic section appears as follows.

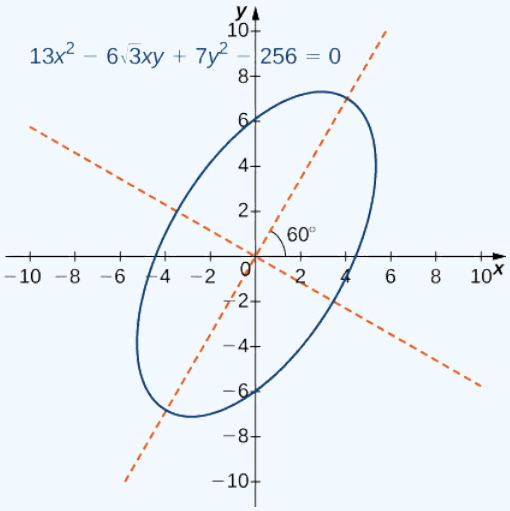

Figure 11.5.15: Graph of the ellipse described by the equation $13x^2 - 6\sqrt{3}xy + 7y^2 - 256 = 0$. The axes are rotated $60°$. The red dashed lines indicate the rotated axes.

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Exercise 11.5.6

Identify the conic and calculate the angle of rotation of axes for the curve described by the equation

$$3x^2 + 5xy - 2y^2 - 125 = 0.$$

Hint

Follow steps 1 and 2 of the five-step method outlined above

Answer

The conic is a hyperbola and the angle of rotation of the axes is $\theta = 22.5°$.

Key Concepts

- The equation of a vertical parabola in standard form with given focus and directrix is $y = \frac{1}{4p}(x-h)^2 + k$ where p is the distance from the vertex to the focus and (h, k) are the coordinates of the vertex.
- The equation of a horizontal ellipse in standard form is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ where the center has coordinates (h, k), the major axis has length $2a$, the minor axis has length $2b$, and the coordinates of the foci are $(h \pm c, k)$, where $c^2 = a^2 - b^2$.
- The equation of a horizontal hyperbola in standard form is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ where the center has coordinates (h, k), the vertices are located at $(h \pm a, k)$, and the coordinates of the foci are $(h \pm c, k)$, where $c^2 = a^2 + b^2$.
- The eccentricity of an ellipse is less than 1, the eccentricity of a parabola is equal to 1, and the eccentricity of a hyperbola is greater than 1. The eccentricity of a circle is 0.
- The polar equation of a conic section with eccentricity e is $r = \frac{ep}{1 \pm e\cos\theta}$ or $r = \frac{ep}{1 \pm e\sin\theta}$, where p represents the focal parameter.
- To identify a conic generated by the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$,first calculate the discriminant $D = 4AC - B^2$. If $D > 0$ then the conic is an ellipse, if $D = 0$ then the conic is a parabola, and if $D < 0$ then the conic is a hyperbola.

Glossary

conic section

a conic section is any curve formed by the intersection of a plane with a cone of two nappes

directrix

a directrix (plural: directrices) is a line used to construct and define a conic section; a parabola has one directrix; ellipses and hyperbolas have two

discriminant

the value $4AC - B^2$, which is used to identify a conic when the equation contains a term involving xy, is called a discriminant

focus

a focus (plural: foci) is a point used to construct and define a conic section; a parabola has one focus; an ellipse and a hyperbola have two

eccentricity

the eccentricity is defined as the distance from any point on the conic section to its focus divided by the perpendicular distance from that point to the nearest directrix

focal parameter

the focal parameter is the distance from a focus of a conic section to the nearest directrix

general form

an equation of a conic section written as a general second-degree equation

major axis

the major axis of a conic section passes through the vertex in the case of a parabola or through the two vertices in the case of an ellipse or hyperbola; it is also an axis of symmetry of the conic; also called the transverse axis

minor axis

the minor axis is perpendicular to the major axis and intersects the major axis at the center of the conic, or at the vertex in the case of the parabola; also called the conjugate axis

nappe

a nappe is one half of a double cone

standard form

an equation of a conic section showing its properties, such as location of the vertex or lengths of major and minor axes

vertex

a vertex is an extreme point on a conic section; a parabola has one vertex at its turning point. An ellipse has two vertices, one at each end of the major axis; a hyperbola has two vertices, one at the turning point of each branch

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11.5E: Exercises for Section 11.5

In exercises 1 - 8, determine the equation of the parabola using the information given.

1) Focus $(4,0)$ and directrix $x=-4$

Answer

$y^2=16x$

2) Focus $(0,-3)$ and directrix $y=3$

3) Focus $(0,0.5)$ and directrix $y=-0.5$

Answer

$x^2=2y$

4) Focus $(2,3)$ and directrix $x=-2$

5) Focus $(0,2)$ and directrix $y=4$

Answer

$x^2=-4(y-3)$

6) Focus $(-1,4)$ and directrix $x=5$

7) Focus $(-3,5)$ and directrix $y=1$

Answer

$(x+3)^2=8(y-3)$

8) Focus $\left(\frac{5}{2},-4\right)$ and directrix $x=\frac{7}{2}$

In exercises 9 - 16, determine the equation of the ellipse using the information given.

9) Endpoints of major axis at $(4,0),\ (-4,0)$ and foci located at $(2,0),\ (-2,0)$

Answer

$\frac{x^2}{16}+\frac{y^2}{12}=1$

10) Endpoints of major axis at $(0,5),\ (0,-5)$ and foci located at $(0,3),\ (0,-3)$

11) Endpoints of major axis at $(0,2),\ (0,-2)$ and foci located at $(3,0),\ (-3,0)$

Answer

$\frac{x^2}{13}+\frac{y^2}{4}=1$

12) Endpoints of major axis at $(-3,3),\ (7,3)$ and foci located at $(-2,3),\ (6,3)$

13) Endpoints of major axis at $(-3,5),\ (-3,-3)$ and foci located at $(-3,3),\ (-3,-1)$

Answer

$\frac{(y-1)^2}{16}+\frac{(x+3)^2}{12}=1$

14) Endpoints of major axis at $(0,0),\ (0,4)$ and foci located at $(5,2),\ (-5,2)$

15) Foci located at $(2,0),\ (-2,0)$ and eccentricity of $\frac{1}{2}$

Answer

$$\frac{x^2}{16}+\frac{y^2}{12}=1$$

16) Foci located at $(0,-3),\ (0,3)$ and eccentricity of $\frac{3}{4}$

In exercises 17 - 24, determine the equation of the hyperbola using the information given.

17) Vertices located at $(5,0),\ (-5,0)$ and foci located at $(6,0),\ (-6,0)$

Answer

$\frac{x^2}{25}-\frac{y^2}{11}=1$

18) Vertices located at $(0,2),\ (0,-2)$ and foci located at $(0,3),\ (0,-3)$

19) Endpoints of the conjugate axis located at $(0,3),\ (0,-3)$ and foci located $(4,0),\ (-4,0)$

Answer

$$\frac{x^2}{7}-\frac{y^2}{9}=1$$

20) Vertices located at $(0,1),\ (6,1)$ and focus located at $(8,1)$

21) Vertices located at $(-2,0),\ (-2,-4)$ and focus located at $(-2,-8)$

Answer

$$\frac{(y+2)^2}{4}-\frac{(x+2)^2}{32}=1$$

22) Endpoints of the conjugate axis located at $(3,2),\ (3,4)$ and focus located at $(3,7)$

23) Foci located at $(6,-0),\ (6,0)$ and eccentricity of 3

Answer

$$\frac{x^2}{4}-\frac{y^2}{32}=1$$

24) $(0,10),\ (0,-10)$ and eccentricity of 2.5

In exercises 25 - 30, consider the following polar equations of conics. Determine the eccentricity and identify the conic.

25) $r=\dfrac{-1}{1+\cos\theta}$

Answer

$e=1$, parabola

26) $r=\dfrac{8}{2-\sin\theta}$

27) $r=\dfrac{5}{2+\sin\theta}$

Answer

$e=\frac{1}{2}$, ellipse

28) $r=\dfrac{5}{-1+2\sin\theta}$

29) $r=\dfrac{3}{2-6\sin\theta}$

Answer

$e=3$, hyperbola

30) $r=\dfrac{3}{-4+3\sin\theta}$

In exercises 31 - 34, find a polar equation of the conic with focus at the origin and eccentricity and directrix as given.

31) Directrix: $x = 4;\ e = \frac{1}{5}$

Answer

$$r = \frac{4}{5 + \cos\theta}$$

32) Directrix: $x = -4;\ e = 5$

33) Directrix: $y = 2;\ e = 2$

Answer

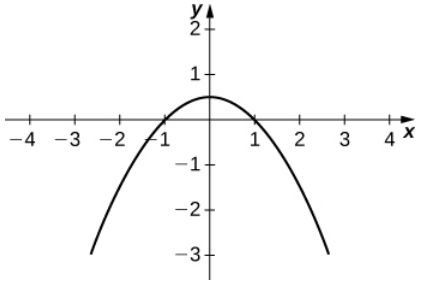
$$r = \frac{4}{1 + 2\sin\theta}$$

34) Directrix: $y = -2;\ e = \frac{1}{2}$

In exercises 35 - 51, sketch the graph of each conic.

35) $r = \dfrac{1}{1 + \sin\theta}$

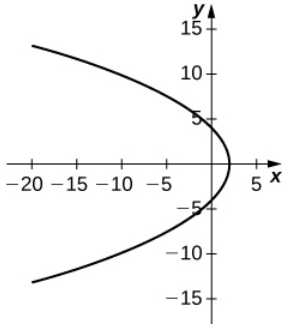
Answer



36) $r = \dfrac{1}{1 - \cos\theta}$

37) $r = \dfrac{4}{1 + \cos\theta}$

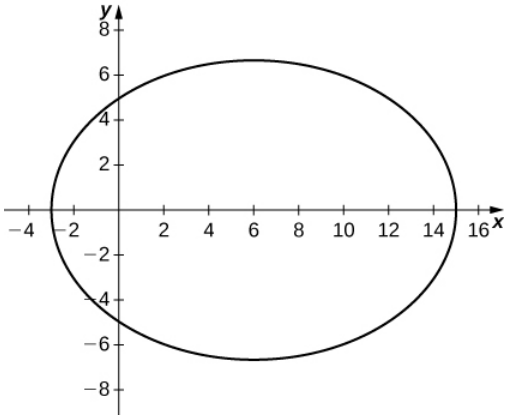
Answer



38) $r = \dfrac{10}{5 + 4\sin\theta}$

39) $r = \dfrac{15}{3 - 2\cos\theta}$

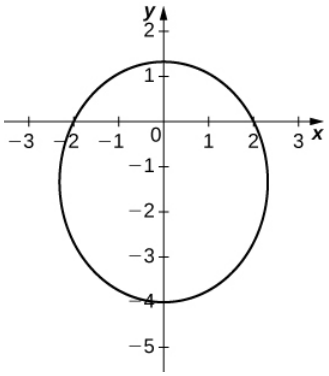
Answer



40) $r = \dfrac{32}{3 + 5\sin\theta}$

41) $r(2 + \sin\theta) = 4$

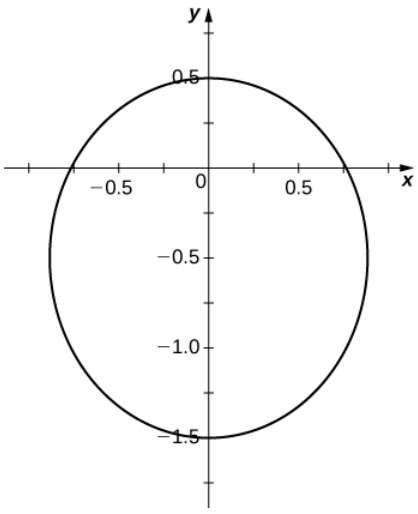
Answer



42) $r = \dfrac{3}{2 + 6\sin\theta}$

43) $r = \dfrac{3}{-4 + 2\sin\theta}$

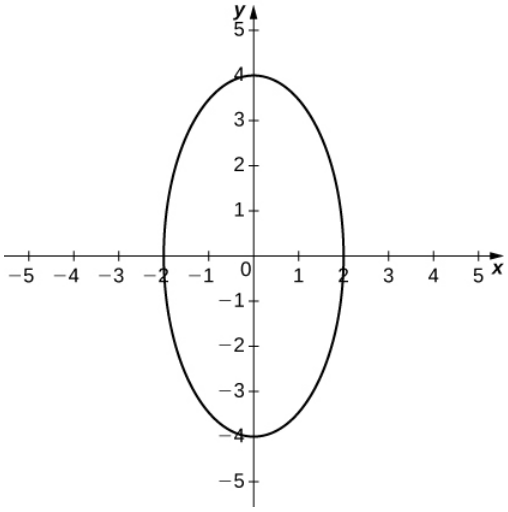
Answer



44) $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

45) $\frac{x^2}{4} + \frac{y^2}{16} = 1$

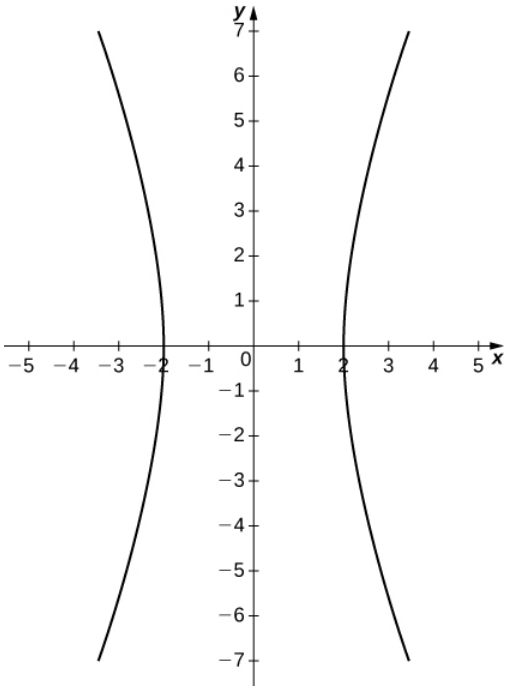
Answer



46) $4x^2 + 9y^2 = 36$

47) $25x^2 - 4y^2 = 100$

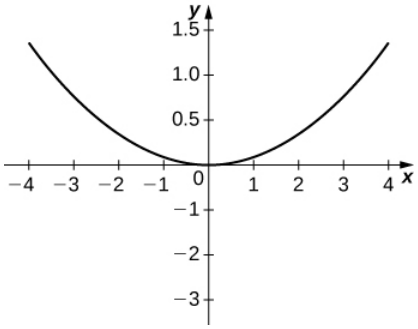
Answer



48) $\frac{x^2}{16} - \frac{y^2}{9} = 1$

49) $x^2 = 12y$

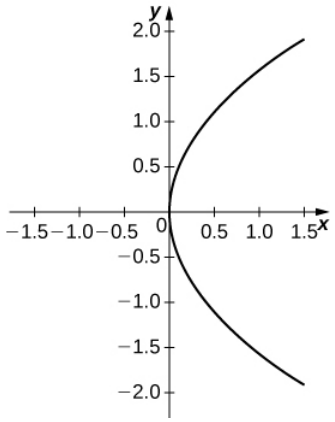
Answer



50) $y^2 = 20x$

51) $12x = 5y^2$

Answer



For the equations in exercises 52 - 57, determine which of the conic sections is described.

52) $xy = 4$

53) $x^2 + 4xy - 2y^2 - 6 = 0$

Answer

Hyperbola

54) $x^2 + 2\sqrt{3}xy + 3y^2 - 6 = 0$

55) $x^2 - xy + y^2 - 2 = 0$

Answer

Ellipse

56) $34x^2 - 24xy + 41y^2 - 25 = 0$

57) $52x^2 - 72xy + 73y^2 + 40x + 30y - 75 = 0$

Answer

Ellipse

58) The mirror in an automobile headlight has a parabolic cross section, with the lightbulb at the focus. On a schematic, the equation of the parabola is given as $x^2 = 4y$. At what coordinates should you place the lightbulb?

59) A satellite dish is shaped like a paraboloid of revolution. The receiver is to be located at the focus. If the dish is 12 feet across at its opening and 4 feet deep at its center, where should the receiver be placed?

Answer

At the point 2.25 feet above the vertex.

60) Consider the satellite dish of the preceding problem. If the dish is 8 feet across at the opening and 2 feet deep, where should we place the receiver?

61) A searchlight is shaped like a paraboloid of revolution. A light source is located 1 foot from the base along the axis of symmetry. If the opening of the searchlight is 3 feet across, find the depth.

Answer

0.5625 feet

62) Whispering galleries are rooms designed with elliptical ceilings. A person standing at one focus can whisper and be heard by a person standing at the other focus because all the sound waves that reach the ceiling are reflected to the other person. If a whispering gallery has a length of 120 feet and the foci are located 30 feet from the center, find the height of the ceiling at the center.

63) A person is standing 8 feet from the nearest wall in a whispering gallery. If that person is at one focus and the other focus is 80 feet away, what is the length and the height at the center of the gallery?

Answer

Length is 96 feet and height is approximately 26.53 feet.

In exercises 64 - 67, determine the polar equation form of the orbit given the length of the major axis and eccentricity for the orbits of the comets or planets. Distance is given in astronomical units (AU).

64) Halley's Comet: length of major axis = 35.88, eccentricity = 0.967

65) Hale-Bopp Comet: length of major axis = 525.91, eccentricity = 0.995

Answer

$$r = \frac{2.616}{1 + 0.995 \cos \theta}$$

66) Mars: length of major axis = 3.049, eccentricity = 0.0934

67) Jupiter: length of major axis = 10.408, eccentricity = 0.0484

Answer

$$r = \frac{5.192}{1 + 0.0484 \cos \theta}$$

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11.R: Chapter 11 Review Exercises

***True or False?* Justify your answer with a proof or a counterexample.**

1) The rectangular coordinates of the point $(4, \frac{5\pi}{6})$ are $\left(2\sqrt{3}, -2\right)$.

2) The equations $x = \cosh(3t),\ y = 2\sinh(3t)$ represent a hyperbola.

Answer

True

3) The arc length of the spiral given by $r = \dfrac{\theta}{2}$ for $0 \le \theta \le 3\pi$ is $\frac{9}{4}\pi^3$ units.

4) Given $x = f(t)$ and $y = g(t)$, if $\dfrac{dx}{dy} = \dfrac{dy}{dx}$, then $f(t) = g(t) + C$, where C is a constant.

Answer

False. Imagine $y = t + 1,\ x = -t + 1$.

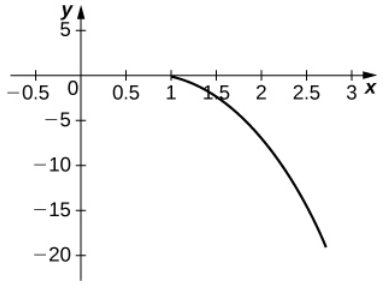
In exercises 5 -8, sketch the parametric curve and eliminate the parameter to find the Cartesian equation of the curve.

5) $x = 1 + t,\ y = t^2 - 1, \quad -1 \le t \le 1$

6) $x = e^t,\ y = 1 - e^{3t}, \quad 0 \le t \le 1$

Answer

$y = 1 - x^3$

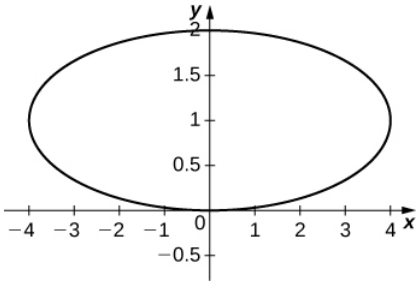


7) $x = \sin\theta,\ y = 1 - \csc\theta, \quad 0 \le \theta \le 2\pi$

8) $x = 4\cos\phi,\ y = 1 - \sin\phi, \quad 0 \le \phi \le 2\pi$

Answer

$\dfrac{x^2}{16} + (y-1)^2 = 1$



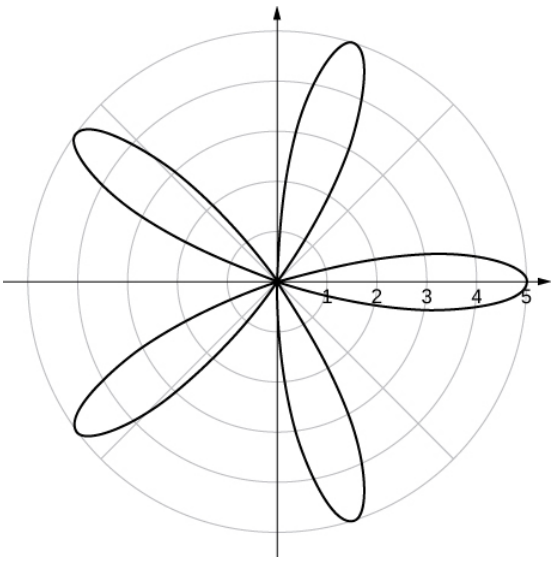
In exercises 9 - 10, sketch the polar curve and determine what type of symmetry exists, if any.

9) $r = 4\sin\left(\frac{\theta}{3}\right)$

10) $r = 5\cos(5\theta)$

Answer

Symmetric about polar axis



In exercises 11 - 12, find the polar equation for the curve given as a Cartesian equation.

11) $x + y = 5$

12) $y^2 = 4 + x^2$

Answer

$$r^2 = \frac{4}{\sin^2\theta - \cos^2\theta}$$

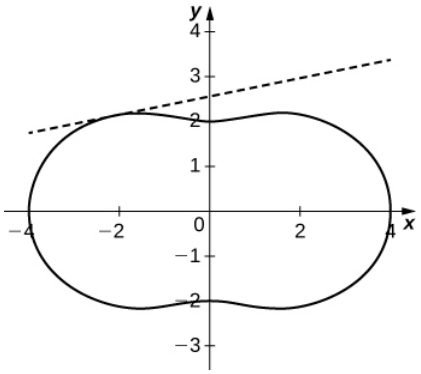
In exercises 13 - 14, find the equation of the tangent line to the given curve. Graph both the function and its tangent line.

13) $x = \ln(t),\ y = t^2 - 1,\ t = 1$

14) $r = 3 + \cos(2\theta),\ \theta = \frac{3\pi}{4}$

Answer

$y = \frac{3\sqrt{2}}{2} + \frac{1}{5}\left(x + \frac{3\sqrt{2}}{2}\right)$



15) Find $\frac{dy}{dx}$, $\frac{dx}{dy}$, and $\frac{d^2x}{dy^2}$ of $y = (2 + e^{-t}),\ x = 1 - \sin t$

In exercises 16 -17, find the area of the region.

16) $x = t^2,\ y = \ln(t), \quad 0 \le t \le e$

Answer

$\frac{e^2}{2}$ units^2

17) $r = 1 - \sin\theta$ in the first quadrant

In exercises 18 - 19, find the arc length of the curve over the given interval.

18) $x = 3t + 4,\ y = 9t - 2, \quad 0 \le t \le 3$

Answer

$9\sqrt{10}$ units

19) $r = 6\cos\theta, \quad 0 \le \theta \le 2\pi$. Check your answer by geometry.

In exercises 20 - 22, find the Cartesian equation describing the given shapes.

20) A parabola with focus $(2, -5)$ and directrix $x = 6$

Answer

$(y+5)^2 = -8x + 32$

21) An ellipse with a major axis length of 10 and foci at $(-7, 2)$ and $(1, 2)$

22) A hyperbola with vertices at $(3, -2)$ and $(-5, -2)$ and foci at $(-2, -6)$ and $(-2, 4)$

Answer

$\frac{(y+1)^2}{16} - \frac{(x+2)^2}{9} = 1$

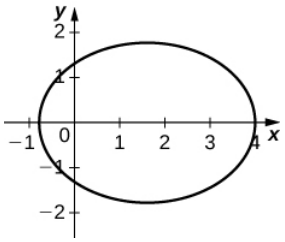
In exercises 23 - 25, determine the eccentricity and identify the conic. Sketch the conic.

23) $r = \frac{6}{1 + 3\cos\theta}$

24) $r = \frac{4}{3 - 2\cos\theta}$

Answer

$e = \frac{2}{3}$, ellipse



25) $r = \frac{7}{5 - 5\cos\theta}$

26) Determine the Cartesian equation describing the orbit of Pluto, the most eccentric orbit around the Sun. The length of the major axis is 39.26 AU and minor axis is 38.07 AU. What is the eccentricity?

Answer

$\frac{y^2}{19.03^2} + \frac{x^2}{19.63^2} = 1, \quad e = 0.2447$

27) The C/1980 E1 comet was observed in 1980. Given an eccentricity of 1.057 and a perihelion (point of closest approach to the Sun) of 3.364 AU, find the Cartesian equations describing the comet's trajectory. Are we guaranteed to see this comet again? (Hint: Consider the Sun at point $(0, 0)$.)

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CHAPTER OVERVIEW

12: Vectors in Space

A quantity that has magnitude and direction is called a **vector**. Vectors have many real-life applications, including situations involving force or velocity. For example, consider the forces acting on a boat crossing a river. The boat's motor generates a force in one direction, and the current of the river generates a force in another direction. Both forces are vectors. We must take both the magnitude and direction of each force into account if we want to know where the boat will go.

12.0: Prelude to Vectors in Space
12.1: Vectors in the Plane
12.1E: Exercises for Section 12.1
12.2: Vectors in Three Dimensions
12.2E: Exercises for Section 12.2
12.3: The Dot Product
12.3E: Exercises for Section 12.3
12.4: The Cross Product
12.4E: Exercises for Section 12.4
12.5: Equations of Lines and Planes in Space
12.5E: Exercises for Section 12.5
12.6: Quadric Surfaces
12.6E: Exercises for Section 12.6
12.7: Cylindrical and Spherical Coordinates
12.7E: Exercises for Section 12.7
12.R: Chapter 12 Review Exercises

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12.0: Prelude to Vectors in Space

Modern astronomical observatories often consist of a large number of parabolic reflectors, connected by computers, used to analyze radio waves. Each dish focuses the incoming parallel beams of radio waves to a precise focal point, where they can be synchronized by computer. If the surface of one of the parabolic reflectors is described by the equation $\frac{x^2}{100} + \frac{y^2}{100} = \frac{z}{4}$, where is the focal point of the reflector?

Figure 12.0.1: The Karl G. Jansky Very Large Array, located in Socorro, New Mexico, consists of a large number of radio telescopes that can collect radio waves and collate them as if they were gathering waves over a huge area with no gaps in coverage. (credit: modification of work by CGP Grey, Wikimedia Commons)

We are now about to begin a new part of the calculus course, when we study functions of two or three independent variables in multidimensional space. Many of the computations are similar to those in the study of single-variable functions, but there are also a lot of differences. In this first chapter, we examine coordinate systems for working in three-dimensional space, along with vectors, which are a key mathematical tool for dealing with quantities in more than one dimension. Let's start here with the basic ideas and work our way up to the more general and powerful tools of mathematics in later chapters.

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12.1: Vectors in the Plane

Learning Objectives

- Describe a plane vector, using correct notation.
- Perform basic vector operations (scalar multiplication, addition, subtraction).
- Express a vector in component form.
- Explain the formula for the magnitude of a vector.
- Express a vector in terms of unit vectors.
- Give two examples of vector quantities.

When describing the movement of an airplane in flight, it is important to communicate two pieces of information: the direction in which the plane is traveling and the plane's speed. Together these form the plane's velocity. When measuring a force, such as the thrust of the plane's engines, it is important to describe not only the strength of that force, but also the direction in which it is applied. Some quantities, such as velocity and force, are defined in terms of both size (also called magnitude) and direction. A quantity that has magnitude and direction is called a **vector**. In textbooks, vectors are often denoted by boldface letters, such as $\mathbf{v}$. Since it is difficult to write in clear boldface when we write vectors by hand, we will also include an arrow or harpoon above the letter representing the vector. To make the vectors clearer in this textbook (and to reinforce the way you should write them by hand), we will generally use arrows or harpoons above boldface (or italic) letters to represent vectors, giving us $\vec{v}$ or $\vec{\mathbf{v}}$. Note that some figures will still only use boldface letters to denote vectors.

Definition: Vector

A vector is a quantity that has both magnitude and direction.

Vector Representation

A vector in a plane is represented by a directed line segment (an arrow). The endpoints of the segment are called the **initial point** and the **terminal point** of the vector. An arrow from the initial point to the terminal point indicates the direction of the vector. The length of the line segment represents its **magnitude**. We use the notation $\|\vec{\mathbf{v}}\|$ to denote the magnitude of the vector $\vec{\mathbf{v}}$. A vector with an initial point and terminal point that are the same is called the **zero vector**, denoted $\vec{0}$. The zero vector is the only vector without a direction, and by convention can be considered to have any direction convenient to the problem at hand.

Vectors with the same magnitude and direction are called equivalent vectors. We treat equivalent vectors as equal, even if they have different initial points. Thus, if $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ are equivalent, we write

$$\vec{\mathbf{v}} = \vec{\mathbf{w}}.$$

Definition: Equivalent Vectors

Vectors are said to be *equivalent vectors* if they have the same magnitude and direction.

The arrows in Figure 12.1.1 are equivalent. Each arrow has the same length and direction. A closely related concept is the idea of parallel vectors. Two vectors are said to be parallel if they have the same or opposite directions. We explore this idea in more detail later in the chapter. A vector is defined by its magnitude and direction, regardless of where its initial point is located.

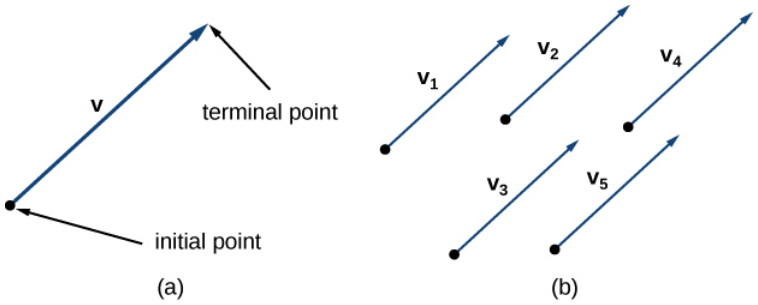

Figure 12.1.1: (a) A vector is represented by a directed line segment from its initial point to its terminal point. (b) Vectors $\vec{v}_1$ through $\vec{v}_5$ are equivalent.

The use of boldface, lowercase letters to name vectors is a common representation in print, but there are alternative notations. When writing the name of a vector by hand, for example, it is easier to sketch an arrow over the variable than to show it is a vector: $\vec{v}$. When a vector has initial point P and terminal point Q, the notation $\overrightarrow{PQ}$ is useful because it indicates the direction and location of the vector.

Example 12.1.1: Sketching Vectors

Sketch a vector in the plane from initial point $P(1,1)$ to terminal point $Q(8,5)$.

Solution

See Figure 12.1.2. Because the vector goes from point P to point Q, we name it $\overrightarrow{PQ}$.

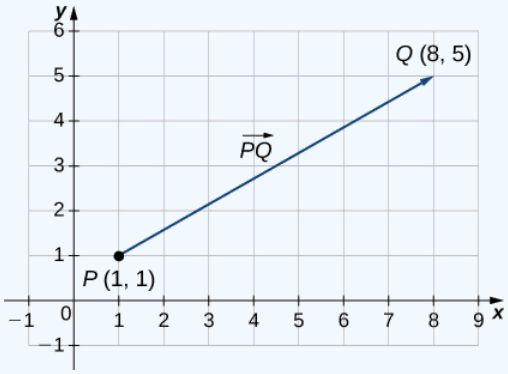

Figure 12.1.2: The vector with initial point $(1,1)$ and terminal point $(8,5)$ is named $\overrightarrow{PQ}$.

Exercise 12.1.1

Sketch the vector $\overrightarrow{ST}$ where S is point $(3,-1)$ and T is point $(-2,3)$.

Hint

The first point listed in the name of the vector is the initial point of the vector.

Answer

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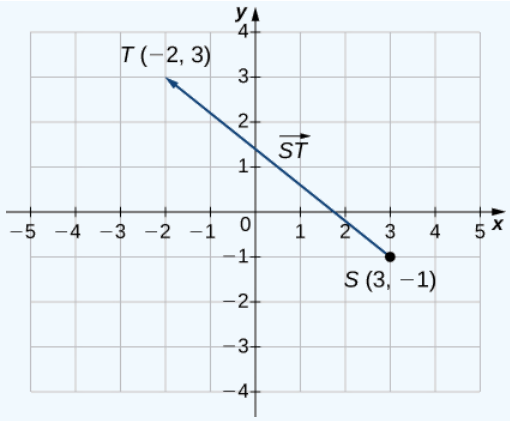


Combining Vectors

Vectors have many real-life applications, including situations involving force or velocity. For example, consider the forces acting on a boat crossing a river. The boat's motor generates a force in one direction, and the current of the river generates a force in another direction. Both forces are vectors. We must take both the magnitude and direction of each force into account if we want to know where the boat will go.

A second example that involves vectors is a quarterback throwing a football. The quarterback does not throw the ball parallel to the ground; instead, he aims up into the air. The velocity of his throw can be represented by a vector. If we know how hard he throws the ball (magnitude—in this case, speed), and the angle (direction), we can tell how far the ball will travel down the field.

A real number is often called a **scalar** in mathematics and physics. Unlike vectors, scalars are generally considered to have a magnitude only, but no direction. Multiplying a vector by a scalar changes the vector's magnitude. This is called scalar multiplication. Note that changing the magnitude of a vector does not indicate a change in its direction. For example, wind blowing from north to south might increase or decrease in speed while maintaining its direction from north to south.

Definition: Scalar Multiplication

The product $k\vec{\mathrm{v}}$ of a vector $\vec{\mathrm{v}}$ and a scalar k is a vector with a magnitude that is $|k|$ times the magnitude of $\vec{\mathrm{v}}$, and with a direction that is the same as the direction of $\vec{\mathrm{v}}$ if $k > 0$, and opposite the direction of $\vec{\mathrm{v}}$ if $k < 0$. This is called *scalar multiplication*. If $k = 0$ or $\vec{\mathrm{v}} = \vec{0}$, then $k\vec{\mathrm{v}} = \vec{0}$.

As you might expect, if $k = -1$, we denote the product $k\vec{\mathrm{v}}$ as

$$k\vec{\mathrm{v}} = (-1)\vec{\mathrm{v}} = -\vec{\mathrm{v}}.$$

Note that $-\vec{\mathrm{v}}$ has the same magnitude as $\vec{\mathrm{v}}$, but has the opposite direction (Figure 12.1.3).

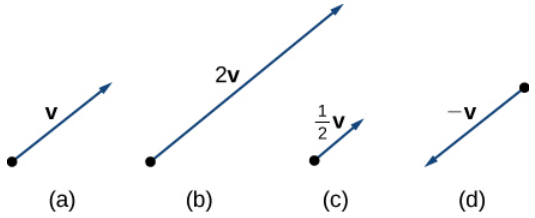


Figure 12.1.3: (a) The original vector $\vec{\mathrm{v}}$ has length n units. (b) The length of $2\vec{\mathrm{v}}$ equals $2n$ units. (c) The length of $\vec{\mathrm{v}}/2$ is $n/2$ units. (d) The vectors $\vec{\mathrm{v}}$ and $-\vec{\mathrm{v}}$ have the same length but opposite directions.

Another operation we can perform on vectors is to add them together in vector addition, but because each vector may have its own direction, the process is different from adding two numbers. The most common graphical method for adding two vectors is to place the initial point of the second vector at the terminal point of the first, as in Figure 12.1.4. To see why this makes sense, suppose, for example, that both vectors represent displacement. If an object moves first from the initial point to the terminal point of vector $\vec{\mathrm{v}}$, then from the initial point to the terminal point of vector $\vec{\mathrm{w}}$, the overall displacement is the same as if the object had made just one movement from the initial point to the terminal point of the vector $\vec{\mathrm{v}} + \vec{\mathrm{w}}$. For obvious reasons, this approach is called the **triangle**

method. Notice that if we had switched the order, so that $\vec{w}$ was our first vector and $\vec{v}$ was our second vector, we would have ended up in the same place. (Again, see Figure 12.1.4.) Thus,

$$\vec{v} + \vec{w} = \vec{w} + \vec{v}.$$

A second method for adding vectors is called the **parallelogram method**. With this method, we place the two vectors so they have the same initial point, and then we draw a parallelogram with the vectors as two adjacent sides, as in Figure 12.1.4. The length of the diagonal of the parallelogram is the sum. Comparing Figure 12.1.4 and Figure 12.1.4, we can see that we get the same answer using either method. The vector $\vec{v} + \vec{w}$ is called the **vector sum.**

Definition: Vector Addition

The sum of two vectors $\vec{v}$ and $\vec{w}$ can be constructed graphically by placing the initial point of $\vec{w}$ at the terminal point of $\vec{v}$. Then, the vector sum, $\vec{v} + \vec{w}$, is the vector with an initial point that coincides with the initial point of $\vec{v}$ and has a terminal point that coincides with the terminal point of $\vec{w}$. This operation is known as *vector addition.*

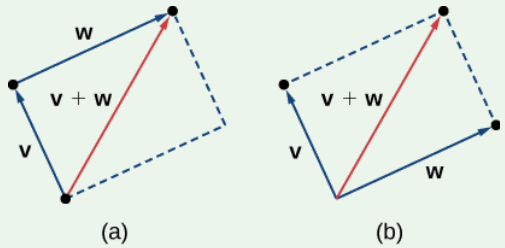


Figure 12.1.4: (a) When adding vectors by the triangle method, the initial point of $\vec{w}$ is the terminal point of $\vec{v}$. (b) When adding vectors by the parallelogram method, the vectors $\vec{v}$ and $\vec{w}$ have the same initial point.

It is also appropriate here to discuss vector subtraction. We define $\vec{v} - \vec{w}$ as $\vec{v} + (-\vec{w}) = \vec{v} + (-1)\vec{w}$. The vector $\vec{v} - \vec{w}$ is called the **vector difference**. Graphically, the vector $\vec{v} - \vec{w}$ is depicted by drawing a vector from the terminal point of $\vec{w}$ to the terminal point of $\vec{v}$ (Figure 12.1.5).

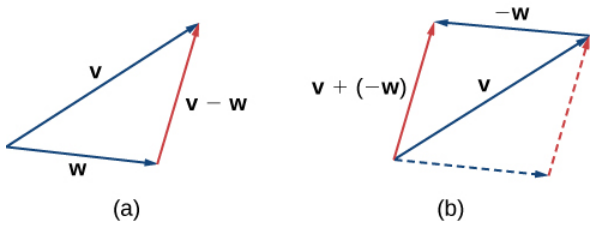


Figure 12.1.5: (a) The vector difference $\vec{v} - \vec{w}$ is depicted by drawing a vector from the terminal point of $\vec{w}$ to the terminal point of $\vec{v}$. (b) The vector$\vec{v} - \vec{w}$ is equivalent to the vector $\vec{v} + (-\vec{w})$.

In Figure 12.1.4, the initial point of $\vec{v} + \vec{w}$ is the initial point of $\vec{v}$. The terminal point of $\vec{v} + \vec{w}$ is the terminal point of $\vec{w}$. These three vectors form the sides of a triangle. It follows that the length of any one side is less than the sum of the lengths of the remaining sides. So we have

$$\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|.$$

This is known more generally as the **triangle inequality**. There is one case, however, when the resultant vector $\vec{u} + \vec{v}$ has the same magnitude as the sum of the magnitudes of $\vec{u}$ and $\vec{v}$. This happens only when $\vec{u}$ and $\vec{v}$ have the same direction.

Example 12.1.2: Combining Vectors

Given the vectors $\vec{v}$ and $\vec{w}$ shown in Figure 12.1.6, sketch the vectors

a. $3\vec{w}$
b. $\vec{v} + \vec{w}$
c. $2\vec{v} - \vec{w}$

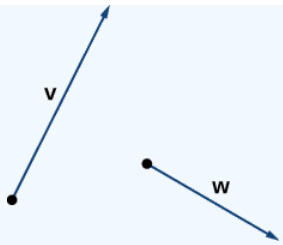
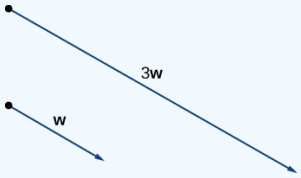


Figure 12.1.6: Vectors $\vec{v}$ and $\vec{w}$ lie in the same plane.

Solution

a. The vector $3\vec{w}$ has the same direction as $\vec{w}$; it is three times as long as $\vec{w}$.



Vector $3\vec{w}$ has the same direction as $\vec{w}$ and is three times as long.

b. Use either addition method to find $\vec{v} + \vec{w}$.

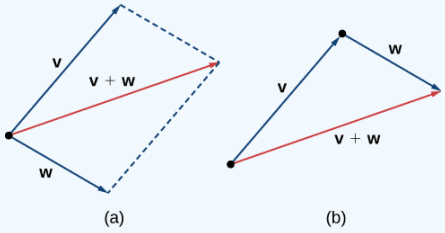


Figure 12.1.7: To find $\vec{v} + \vec{w}$, align the vectors at their initial points or place the initial point of one vector at the terminal point of the other. (a) The vector$\vec{v} + \vec{w}$ is the diagonal of the parallelogram with sides $\vec{v}$ and $\vec{w}$. (b) The vector $\vec{v} + \vec{w}$ is the third side of a triangle formed with $\vec{w}$ placed at the terminal point of $\vec{v}$.

c. To find $2\vec{v} - \vec{w}$, we can first rewrite the expression as $2\vec{v} + (-\vec{w})$. Then we can draw the vector $-\vec{w}$, then add it to the vector $2\vec{v}$.

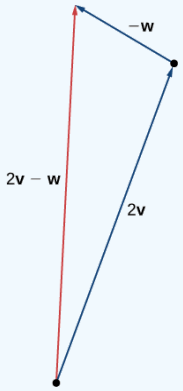


Figure 12.1.8: To find $2\vec{v} - \vec{w}$, simply add $2\vec{v} + (-\vec{w})$.

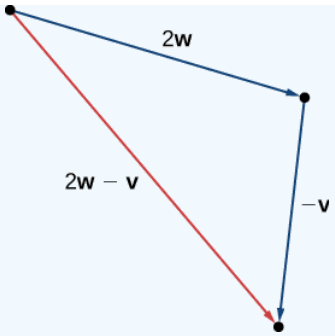
? Exercise 12.1.2

Using vectors $\vec{v}$ and $\vec{w}$ from Example 12.1.2, sketch the vector $2\vec{w} - \vec{v}$.

Hint

First sketch vectors $2\vec{w}$ and $-\vec{v}$.

Answer



Vector Components

Working with vectors in a plane is easier when we are working in a coordinate system. When the initial points and terminal points of vectors are given in Cartesian coordinates, computations become straightforward.

Example 12.1.3: Comparing Vectors

Are $\vec{v}$ and $\vec{w}$ equivalent vectors?

a.

- $\vec{v}$ has initial point $(3, 2)$ and terminal point $(7, 2)$
- $\vec{w}$ has initial point $(1, -4)$ and terminal point $(1, 0)$

b.

- $\vec{v}$ has initial point $(0, 0)$ and terminal point $(1, 1)$
- $\vec{w}$ has initial point $(-2, 2)$ and terminal point $(-1, 3)$

Solution

a. The vectors are each 4 units long, but they are oriented in different directions. So $\vec{v}$ and $\vec{w}$ are not equivalent (Figure 12.1.9).

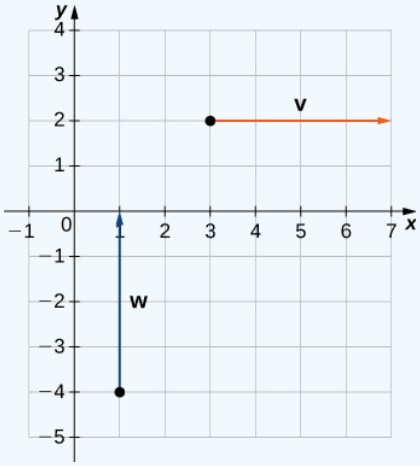


Figure 12.1.9: These vectors are not equivalent.

b. Based on Figure 12.1.10, and using a bit of geometry, it is clear these vectors have the same length and the same direction, so $\vec{v}$ and $\vec{w}$ are equivalent.

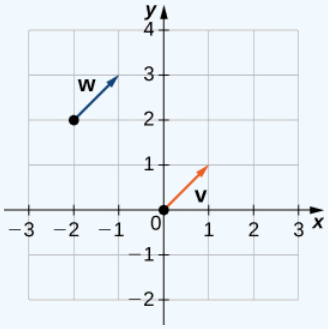
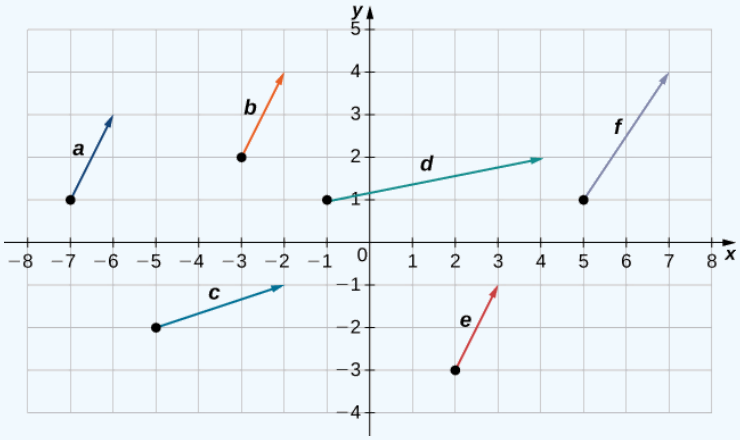


Figure 12.1.10: These vectors are equivalent.

Exercise 12.1.3

Which of the following vectors are equivalent?



Hint

Equivalent vectors have both the same magnitude and the same direction.

Answer

Vectors $\vec{a}$, $\vec{b}$, and $\vec{e}$ are equivalent.

We have seen how to plot a vector when we are given an initial point and a terminal point. However, because a vector can be placed anywhere in a plane, it may be easier to perform calculations with a vector when its initial point coincides with the origin. We call a vector with its initial point at the origin a **standard-position vector.** Because the initial point of any vector in standard position is known to be $(0, 0)$, we can describe the vector by looking at the coordinates of its terminal point. Thus, if vector $\vec{v}$ has its initial point at the origin and its terminal point at (x, y), we write the vector in component form as

$$\vec{v} = \langle x, y \rangle.$$

When a vector is written in component form like this, the scalars x and y are called the **components** of $\vec{v}$.

Definition: Vector components

The vector with initial point $(0, 0)$ and terminal point (x, y) can be written in component form as

$$\vec{v} = \langle x, y \rangle.$$

The scalars x and y are called the components of $\vec{v}$.

Recall that vectors are named with lowercase letters in bold type or by drawing an arrow over their name. We have also learned that we can name a vector by its component form, with the coordinates of its terminal point in angle brackets. However, when writing

the component form of a vector, it is important to distinguish between $\langle x, y\rangle$ and (x, y). The first ordered pair uses angle brackets to describe a vector, whereas the second uses parentheses to describe a point in a plane. The initial point of $\langle x, y\rangle$ is $(0, 0)$; the terminal point of $\langle x, y\rangle$ is (x, y).

When we have a vector not already in standard position, we can determine its component form in one of two ways. We can use a geometric approach, in which we sketch the vector in the coordinate plane, and then sketch an equivalent standard-position vector. Alternatively, we can find it algebraically, using the coordinates of the initial point and the terminal point. To find it algebraically, we subtract the x-coordinate of the initial point from the x-coordinate of the terminal point to get the x-component, and we subtract the y-coordinate of the initial point from the y-coordinate of the terminal point to get the y-component.

Rule: Component Form of a Vector

Let $\vec{v}$ be a vector with initial point (x_i, y_i) and terminal point (x_t, y_t). Then we can express $\vec{v}$ in component form as $\vec{v} = \langle x_t - x_i, y_t - y_i\rangle$.

Example 12.1.4: Expressing Vectors in Component Form

Express vector $\vec{v}$ with initial point $(-3, 4)$ and terminal point $(1, 2)$ in component form.

Solution:

a. Geometric

1. Sketch the vector in the coordinate plane (Figure 12.1.11).
2. The terminal point is 4 units to the right and 2 units down from the initial point.
3. Find the point that is 4 units to the right and 2 units down from the origin.
4. In standard position, this vector has initial point $(0, 0)$ and terminal point $(4, -2)$:

$$\vec{v} = \langle 4, -2\rangle.$$

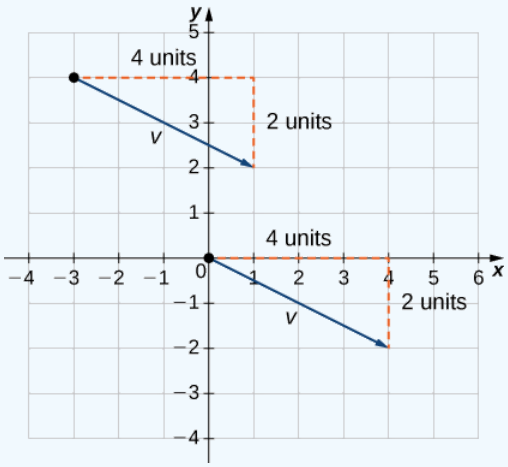


Figure 12.1.11: These vectors are equivalent.

b. Algebraic

In the first solution, we used a sketch of the vector to see that the terminal point lies 4 units to the right. We can accomplish this algebraically by finding the difference of the x-coordinates:

$$x_t - x_i = 1 - (-3) = 4.$$

Similarly, the difference of the y-coordinates shows the vertical length of the vector.

$$y_t - y_i = 2 - 4 = -2.$$

So, in component form,

$$\vec{v} = \langle x_t - x_i, y_t - y_i\rangle = \langle 1 - (-3), 2 - 4\rangle = \langle 4, -2\rangle.$$

Exercise 12.1.4

Vector $\vec{\mathbf{w}}$ has initial point $(-4,-5)$ and terminal point $(-1,2)$. Express $\vec{\mathbf{w}}$ in component form.

Hint

You may use either geometric or algebraic method.

Answer

$\langle 3,7\rangle$

To find the magnitude of a vector, we calculate the distance between its initial point and its terminal point. The **magnitude** of vector $\vec{\mathbf{v}}=\langle x,y\rangle$ is denoted $\|\vec{\mathbf{v}}\|$, or $|\vec{\mathbf{v}}|$, and can be computed using the formula

$$\|\vec{\mathbf{v}}\|=\sqrt{x^2+y^2}.$$

Note that because this vector is written in component form, it is equivalent to a vector in standard position, with its initial point at the origin and terminal point (x,y). Thus, it suffices to calculate the magnitude of the vector in standard position. Using the distance formula to calculate the distance between initial point $(0,0)$ and terminal point (x,y), we have

$$\|\vec{\mathbf{v}}\|=\sqrt{(x-0)^2+(y-0)^2}=\sqrt{x^2+y^2}.$$

Based on this formula, it is clear that for any vector $\vec{\mathbf{v}}$, $\|\vec{\mathbf{v}}\|\geq 0$, and $\|\vec{\mathbf{v}}\|=0$ if and only if $\vec{\mathbf{v}}=\vec{\mathbf{0}}$.

The magnitude of a vector can also be derived using the Pythagorean theorem, as in the following figure.

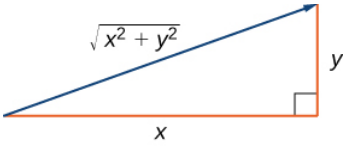


Figure 12.1.12: If you use the components of a vector to define a right triangle, the magnitude of the vector is the length of the triangle's hypotenuse.

We have defined scalar multiplication and vector addition geometrically. Expressing vectors in component form allows us to perform these same operations algebraically.

Definition: Scalar multiplication and Vector addition

Let $\vec{\mathbf{v}}=\langle x_1,y_1\rangle$ and $\vec{\mathbf{w}}=\langle x_2,y_2\rangle$ be vectors, and let k be a scalar.

- *Scalar multiplication:*

$$k\vec{\mathbf{v}}=\langle kx_1,ky_1\rangle$$

- *Vector addition:*

$$\vec{\mathbf{v}}+\vec{\mathbf{w}}=\langle x_1,y_1\rangle+\langle x_2,y_2\rangle=\langle x_1+x_2,y_1+y_2\rangle$$

Example 12.1.5: Performing Operations in Component Form

Let $\vec{\mathbf{v}}$ be the vector with initial point $(2,5)$ and terminal point $(8,13)$, and let $\vec{\mathbf{w}}=\langle -2,4\rangle$.

a. Express $\vec{\mathbf{v}}$ in component form and find $\|\vec{\mathbf{v}}\|$. Then, using algebra, find
b. $\vec{\mathbf{v}}+\vec{\mathbf{w}}$,
c. $3\vec{\mathbf{v}}$, and
d. $\vec{\mathbf{v}}-2\vec{\mathbf{w}}$.

Solution

a. To place the initial point of $\vec{\mathbf{v}}$ at the origin, we must translate the vector **2** units to the left and **5** units down (Figure 12.1.13). Using the algebraic method, we can express $\vec{\mathbf{v}}$ as $\vec{\mathbf{v}}=\langle 8-2,13-5\rangle=\langle 6,8\rangle$:

$$\|\vec{v}\| = \sqrt{6^2 + 8^2} = \sqrt{36 + 64} = \sqrt{100} = 10.$$

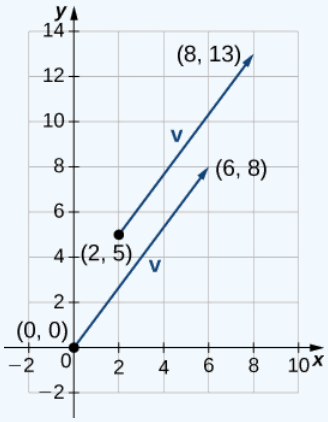


Figure 12.1.13: In component form, $\vec{v} = \langle 6, 8 \rangle$.

b. To find $\vec{v} + \vec{w}$, add the x-components and the y-components separately:

$$\vec{v} + \vec{w} = \langle 6, 8 \rangle + \langle -2, 4 \rangle = \langle 4, 12 \rangle.$$

c. To find $3\vec{v}$, multiply $\vec{v}$ by the scalar $k = 3$:

$$3\vec{v} = 3 \cdot \langle 6, 8 \rangle = \langle 3 \cdot 6, 3 \cdot 8 \rangle = \langle 18, 24 \rangle.$$

d. To find $\vec{v} - 2\vec{w}$, find $-2\vec{w}$ and add it to $\vec{v}$:

$$\vec{v} - 2\vec{w} = \langle 6, 8 \rangle - 2 \cdot \langle -2, 4 \rangle = \langle 6, 8 \rangle + \langle 4, -8 \rangle = \langle 10, 0 \rangle.$$

Exercise 12.1.5A

Let $\vec{a} = \langle 7, 1 \rangle$ and let $\vec{b}$ be the vector with initial point $(3, 2)$ and terminal point $(-1, -1)$.

a. Find $\|\vec{a}\|$.
b. Express $\vec{b}$ in component form.
c. Find $3\vec{a} - 4\vec{b}$.

Hint

Use the Pythagorean Theorem to find $\|\vec{a}\|$. To find $3\vec{a} - 4\vec{b}$, start by finding the scalar multiples $3\vec{a}$ and $-4\vec{b}$.

Answer a

$\|\vec{a}\| = 5\sqrt{2}$,

Answer b

$\vec{b} = \langle -4, -3 \rangle$,

Answer c

$3\vec{a} - 4\vec{b} = \langle 37, 15 \rangle$

Now that we have established the basic rules of vector arithmetic, we can state the properties of vector operations. We will prove two of these properties. The others can be proved in a similar manner.

Properties of Vector Operations

Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors in a plane. Let r and s be scalars.

i. *Commutative property*

$$\vec{u} + \vec{v} = \vec{v} + \vec{u} \tag{12.1.1}$$

ii. *Associative property*

$$(\vec{\mathbf{u}} + \vec{\mathbf{v}}) + \vec{\mathbf{w}} = \vec{\mathbf{u}} + (\vec{\mathbf{v}} + \vec{\mathbf{w}})$$

iii. *Additive identity property*

$$\vec{\mathbf{u}} + \vec{\mathbf{0}} = \vec{\mathbf{u}}$$

iv. *Additive inverse property*

$$\vec{\mathbf{u}} + (-\vec{\mathbf{u}}) = \vec{\mathbf{0}}$$

v. *Associativity of scalar multiplication*

$$r(s\vec{\mathbf{u}}) = (rs)\vec{\mathbf{u}}$$

vi. *Distributive property*

$$(r + s)\vec{\mathbf{u}} = r\vec{\mathbf{u}} + s\vec{\mathbf{u}} \tag{12.1.2}$$

vii. *Distributive property*

$$r(\vec{\mathbf{u}} + \vec{\mathbf{v}}) = r\vec{\mathbf{u}} + r\vec{\mathbf{v}}$$

viii. *Identity and zero properties*

$$1\vec{\mathbf{u}} = \vec{\mathbf{u}},\ 0\vec{\mathbf{u}} = \vec{\mathbf{0}}$$

Proof of Commutative Property

Let $\vec{\mathbf{u}} = \langle x_1, y_1 \rangle$ and $\vec{\mathbf{v}} = \langle x_2, y_2 \rangle$. Apply the commutative property for real numbers:

$$\begin{aligned} \vec{\mathbf{u}} + \vec{\mathbf{v}} &= \langle x_1 + x_2, y_1 + y_2 \rangle \\ &= \langle x_2 + x_1, y_2 + y_1 \rangle \\ &= \vec{\mathbf{v}} + \vec{\mathbf{u}}. \end{aligned}$$

□

Proof of Distributive Property

Apply the distributive property for real numbers:

$$\begin{aligned} r(\vec{\mathbf{u}} + \vec{\mathbf{v}}) &= r \cdot \langle x_1 + x_2, y_1 + y_2 \rangle \\ &= \langle r(x_1 + x_2), r(y_1 + y_2) \rangle \\ &= \langle rx_1 + rx_2, ry_1 + ry_2 \rangle \\ &= \langle rx_1, ry_1 \rangle + \langle rx_2, ry_2 \rangle \\ &= r\vec{\mathbf{u}} + r\vec{\mathbf{v}}. \end{aligned}$$

□

Exercise 12.1.5B

Prove the additive inverse property.

Hint

Use the component form of the vectors.

We have found the components of a vector given its initial and terminal points. In some cases, we may only have the magnitude and direction of a vector, not the points. For these vectors, we can identify the horizontal and vertical components using trigonometry (Figure 12.1.14).

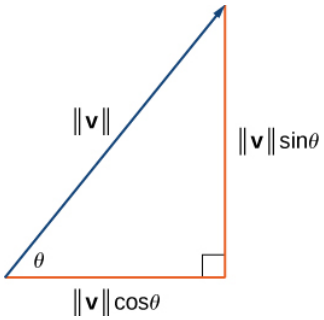


Figure 12.1.14: The components of a vector form the legs of a right triangle, with the vector as the hypotenuse.

Consider the angle θ formed by the vector $\vec{v}$ and the positive x-axis. We can see from the triangle that the components of vector $\vec{v}$ are $\langle \|\vec{v}\| \cos\theta, \|\vec{v}\| \sin\theta \rangle$. Therefore, given an angle and the magnitude of a vector, we can use the cosine and sine of the angle to find the components of the vector.

Example 12.1.6: Finding the Component Form of a Vector Using Trigonometry

Find the component form of a vector with magnitude 4 that forms an angle of $-45°$ with the x-axis.

Solution

Let x and y represent the components of the vector (Figure 12.1.15). Then $x = 4\cos(-45°) = 2\sqrt{2}$ and $y = 4\sin(-45°) = -2\sqrt{2}$. The component form of the vector is $\langle 2\sqrt{2}, -2\sqrt{2} \rangle$.

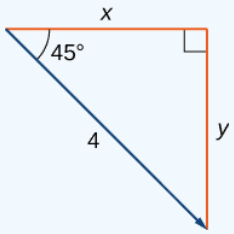


Figure 12.1.15: Use trigonometric ratios, $x = \|\vec{v}\| \cos\theta$ and $y = \|\vec{v}\| \sin\theta$, to identify the components of the vector.

Exercise 12.1.6

Find the component form of vector $\vec{v}$ with magnitude 10 that forms an angle of $120°$ with the positive x-axis.

Hint

$x = \|\vec{v}\| \cos\theta$ and $y = \|\vec{v}\| \sin\theta$

Answer

$\vec{v} = \langle -5, 5\sqrt{3} \rangle$

Unit Vectors

A **unit vector** is a vector with magnitude 1. For any nonzero vector $\vec{v}$, we can use scalar multiplication to find a unit vector $\vec{u}$ that has the same direction as $\vec{v}$. To do this, we multiply the vector by the reciprocal of its magnitude:

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}.$$

Recall that when we defined scalar multiplication, we noted that $\|k\vec{v}\| = |k| \cdot \|\vec{v}\|$. For $\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$, it follows that $\|\vec{u}\| = \frac{1}{\|\vec{v}\|}(\|\vec{v}\|) = 1$. We say that $\vec{u}$ is the *unit vector in the direction of* $\vec{v}$ (Figure 12.1.16). The process of using scalar multiplication to find a unit vector with a given direction is called **normalization.**

Figure 12.1.16: The vector $\vec{v}$ and associated unit vector $\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$. In this case, $\|\vec{v}\| > 1$.

Example 12.1.7: Finding a Unit Vector

Let $\vec{v} = \langle 1, 2\rangle$.

a. Find a unit vector with the same direction as $\vec{v}$.
b. Find a vector $\vec{w}$ with the same direction as $\vec{v}$ such that $\|\vec{w}\| = 7$.

Solution:

a. First, find the magnitude of $\vec{v}$, then divide the components of $\vec{v}$ by the magnitude:

$$\|\vec{v}\| = \sqrt{1^2 + 2^2} = \sqrt{1+4} = \sqrt{5}$$

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{\sqrt{5}}\langle 1, 2\rangle = \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\rangle.$$

b. The vector $\vec{u}$ is in the same direction as $\vec{v}$ and $\|\vec{u}\| = 1$. Use scalar multiplication to increase the length of $\vec{u}$ without changing direction:

$$\vec{w} = 7\vec{u} = 7\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\rangle = \langle \frac{7}{\sqrt{5}}, \frac{14}{\sqrt{5}}\rangle.$$

Exercise 12.1.7

Let $\vec{v} = \langle 9, 2\rangle$. Find a vector with magnitude 5 in the opposite direction as $\vec{v}$.

Hint

First, find a unit vector in the same direction as $\vec{v}$.

Answer

$$\langle -\frac{45}{\sqrt{85}}, -\frac{10}{\sqrt{85}}\rangle$$

We have seen how convenient it can be to write a vector in component form. Sometimes, though, it is more convenient to write a vector as a sum of a horizontal vector and a vertical vector. To make this easier, let's look at standard unit vectors. The **standard unit vectors** are the vectors $\hat{\mathbf{i}} = \langle 1, 0\rangle$ and $\hat{\mathbf{j}} = \langle 0, 1\rangle$ (Figure 12.1.17).

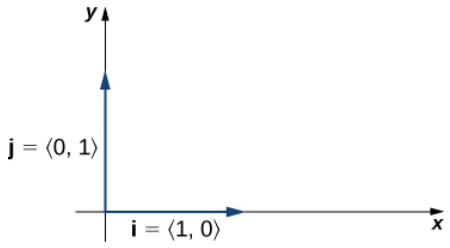


Figure 12.1.17: The standard unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$.

By applying the properties of vectors, it is possible to express any vector in terms of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ in what we call a *linear combination:*

$$\vec{v} = \langle x, y\rangle = \langle x, 0\rangle + \langle 0, y\rangle = x\langle 1, 0\rangle + y\langle 0, 1\rangle = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}.$$

Thus, $\vec{v}$ is the sum of a horizontal vector with magnitude x, and a vertical vector with magnitude y, as in Figure 12.1.18.

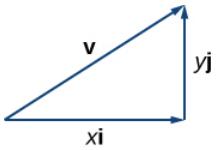


Figure 12.1.18: The vector $\vec{\mathbf{v}}$ is the sum of $x\hat{\mathbf{i}}$ and $y\hat{\mathbf{j}}$.

Example 12.1.8: Using Standard Unit Vectors

a. Express the vector $\vec{\mathbf{w}} = \langle 3, -4 \rangle$ in terms of standard unit vectors.

b. Vector $\vec{\mathbf{u}}$ is a unit vector that forms an angle of $60°$ with the positive x-axis. Use standard unit vectors to describe $\vec{\mathbf{u}}$.

Solution:

a. Resolve vector $\vec{\mathbf{w}}$ into a vector with a zero y-component and a vector with a zero x-component:

$$\vec{\mathbf{w}} = \langle 3, -4 \rangle = 3\hat{\mathbf{i}} - 4\hat{\mathbf{j}}.$$

b. Because $\vec{\mathbf{u}}$ is a unit vector, the terminal point lies on the unit circle when the vector is placed in standard position (Figure 12.1.19).

$$\begin{aligned} \vec{\mathbf{u}} &= \langle \cos 60°, \sin 60° \rangle \\ &= \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle \\ &= \frac{1}{2}\hat{\mathbf{i}} + \frac{\sqrt{3}}{2}\hat{\mathbf{j}}. \end{aligned}$$

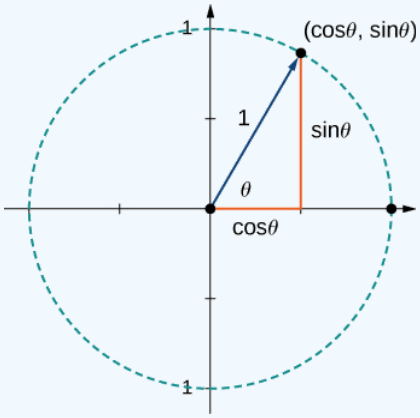


Figure 12.1.19: The terminal point of $\vec{\mathbf{u}}$ lies on the unit circle $(\cos\theta, \sin\theta)$.

Exercise 12.1.8

Let $\vec{\mathbf{a}} = \langle 16, -11 \rangle$ and let $\vec{\mathbf{b}}$ be a unit vector that forms an angle of $225°$ with the positive x-axis. Express $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ in terms of the standard unit vectors.

Hint

Use sine and cosine to find the components of $\vec{\mathbf{b}}$.

Answer

$$\vec{\mathbf{a}} = 16\hat{\mathbf{i}} - 11\hat{\mathbf{j}}, \quad \vec{\mathbf{b}} = -\frac{\sqrt{2}}{2}\hat{\mathbf{i}} - \frac{\sqrt{2}}{2}\hat{\mathbf{j}}$$

Applications of Vectors

Because vectors have both direction and magnitude, they are valuable tools for solving problems involving such applications as motion and **force.** Recall the boat example and the quarterback example we described earlier. Here we look at two other examples in detail.

Example 12.1.9A: Finding Resultant Force

Jane's car is stuck in the mud. Lisa and Jed come along in a truck to help pull her out. They attach one end of a tow strap to the front of the car and the other end to the truck's trailer hitch, and the truck starts to pull. Meanwhile, Jane and Jed get behind the car and push. The truck generates a horizontal force of **300** lb on the car. Jane and Jed are pushing at a slight upward angle and generate a force of **150** lb on the car. These forces can be represented by vectors, as shown in Figure 12.1.20. The angle between these vectors is **15°**. Find the resultant force (the vector sum) and give its magnitude to the nearest tenth of a pound and its direction angle from the positive x-axis.

Figure 12.1.20: Two forces acting on a car in different directions.

Solution

To find the effect of combining the two forces, add their representative vectors. First, express each vector in component form or in terms of the standard unit vectors. For this purpose, it is easiest if we align one of the vectors with the positive x-axis. The horizontal vector, then, has initial point $(0, 0)$ and terminal point $(300, 0)$. It can be expressed as $\langle 300, 0\rangle$ or $300\hat{\mathbf{i}}$.

The second vector has magnitude 150 and makes an angle of $15°$ with the first, so we can express it as $\langle 150\cos(15°), 150\sin(15°)\rangle$, or $150\cos(15°)\hat{\mathbf{i}} + 150\sin(15°)\hat{\mathbf{j}}$. Then, the sum of the vectors, or resultant vector, is $\vec{\mathbf{r}} = \langle 300, 0\rangle + \langle 150\cos(15°), 150\sin(15°)\rangle$, and we have

$$\|\vec{\mathbf{r}}\| = \sqrt{(300 + 150\cos(15°))^2 + (150\sin(15°))^2} \approx 446.6.$$

The angle θ made by $\vec{\mathbf{r}}$ and the positive x-axis has $\tan\theta = \dfrac{150\sin 15°}{(300 + 150\cos 15°)} \approx 0.09$, so $\theta \approx \tan^{-1}(0.09) \approx 5°$, which means the resultant force $\vec{\mathbf{r}}$ has an angle of $5°$ above the horizontal axis.

Example 12.1.9B: Finding Resultant Velocity

An airplane flies due west at an airspeed of 425 mph. The wind is blowing from the northeast at 40 mph. What is the ground speed of the airplane? What is the bearing of the airplane?

Solution

Let's start by sketching the situation described (Figure 12.1.21).

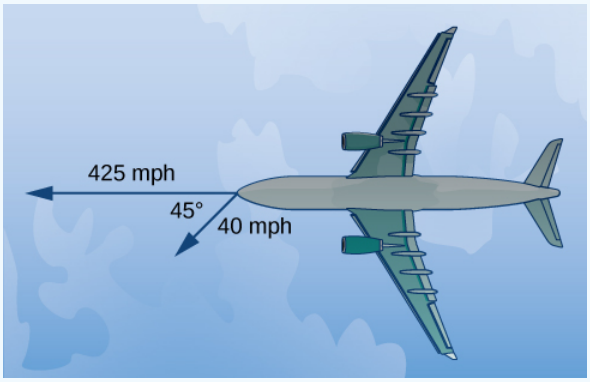


Figure 12.1.21: Initially, the plane travels due west. The wind is from the northeast, so it is blowing to the southwest. The angle between the plane's course and the wind is $45°$. (Figure not drawn to scale.)

Set up a sketch so that the initial points of the vectors lie at the origin. Then, the plane's velocity vector is $\vec{\mathbf{p}} = -425\hat{\mathbf{i}}$. The vector describing the wind makes an angle of $225°$ with the positive x-axis:

$$\vec{\mathbf{w}} = \langle 40\cos(225°), 40\sin(225°)\rangle = \langle -\frac{40}{\quad}, -\frac{40}{\quad}\rangle = -\frac{40}{\quad}\hat{\mathbf{i}} - \frac{40}{\quad}\hat{\mathbf{i}}.$$

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When the airspeed and the wind act together on the plane, we can add their vectors to find the resultant force:

$$\vec{\mathbf{p}} + \vec{\mathbf{w}} = -425\hat{\mathbf{i}} + (-\frac{40}{\sqrt{2}}\hat{\mathbf{i}} - \frac{40}{\sqrt{2}}\hat{\mathbf{j}}) = (-425 - \frac{40}{\sqrt{2}})\hat{\mathbf{i}} - \frac{40}{\sqrt{2}}\hat{\mathbf{j}}.$$

The magnitude of the resultant vector shows the effect of the wind on the ground speed of the airplane:

$$\|\vec{\mathbf{p}} + \vec{\mathbf{w}}\| = \sqrt{(-425 - \frac{40}{\sqrt{2}})^2 + (-\frac{40}{\sqrt{2}})^2} \approx 454.17 \text{ mph}$$

As a result of the wind, the plane is traveling at approximately 454 mph relative to the ground.

To determine the bearing of the airplane, we want to find the direction of the vector $\vec{\mathbf{p}} + \vec{\mathbf{w}}$:

$$\tan\theta = \frac{-\frac{40}{\sqrt{2}}}{(-425 - \frac{40}{\sqrt{2}})} \approx 0.06$$

$$\theta \approx 3.57°.$$

The overall direction of the plane is $3.57°$ south of west.

? Exercise 12.1.9

An airplane flies due north at an airspeed of 550 mph. The wind is blowing from the northwest at 50 mph. What is the ground speed of the airplane?

Hint

Sketch the vectors with the same initial point and find their sum.

Answer

Approximately 516 mph

Key Concepts

- Vectors are used to represent quantities that have both magnitude and direction.
- We can add vectors by using the parallelogram method or the triangle method to find the sum. We can multiply a vector by a scalar to change its length or give it the opposite direction.
- Subtraction of vectors is defined in terms of adding the negative of the vector.
- A vector is written in component form as $\vec{\mathbf{v}} = \langle x, y\rangle$.
- The magnitude of a vector is a scalar: $\|\vec{\mathbf{v}}\| = \sqrt{x^2 + y^2}$.
- A unit vector $\vec{\mathbf{u}}$ has magnitude 1 and can be found by dividing a vector by its magnitude: $\vec{\mathbf{u}} = \frac{1}{\|\vec{\mathbf{v}}\|}\vec{\mathbf{v}}$. The standard unit vectors are $\hat{\mathbf{i}} = \langle 1, 0\rangle$ and $\hat{\mathbf{j}} = \langle 0, 1\rangle$. A vector $\vec{\mathbf{v}} = \langle x, y\rangle$ can be expressed in terms of the standard unit vectors as $\vec{\mathbf{v}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$.
- Vectors are often used in physics and engineering to represent forces and velocities, among other quantities.

Glossary

component

a scalar that describes either the vertical or horizontal direction of a vector

equivalent vectors

vectors that have the same magnitude and the same direction

initial point

the starting point of a vector

magnitude

the length of a vector

normalization
using scalar multiplication to find a unit vector with a given direction

parallelogram method
a method for finding the sum of two vectors; position the vectors so they share the same initial point; the vectors then form two adjacent sides of a parallelogram; the sum of the vectors is the diagonal of that parallelogram

scalar
a real number

scalar multiplication
a vector operation that defines the product of a scalar and a vector

standard-position vector
a vector with initial point $(0, 0)$

standard unit vectors
unit vectors along the coordinate axes: $\hat{\mathrm{i}} = \langle 1, 0 \rangle, \hat{\mathrm{j}} = \langle 0, 1 \rangle$

terminal point
the endpoint of a vector

triangle inequality
the length of any side of a triangle is less than the sum of the lengths of the other two sides

triangle method
a method for finding the sum of two vectors; position the vectors so the terminal point of one vector is the initial point of the other; these vectors then form two sides of a triangle; the sum of the vectors is the vector that forms the third side; the initial point of the sum is the initial point of the first vector; the terminal point of the sum is the terminal point of the second vector

unit vector
a vector with magnitude 1

vector
a mathematical object that has both magnitude and direction

vector addition
a vector operation that defines the sum of two vectors

vector difference
the vector difference $\vec{\mathrm{v}} - \vec{\mathrm{w}}$ is defined as $\vec{\mathrm{v}} + (-\vec{\mathrm{w}}) = \vec{\mathrm{v}} + (-1)\vec{\mathrm{w}}$

vector sum
the sum of two vectors, $\vec{\mathrm{v}}$ and $\vec{\mathrm{w}}$, can be constructed graphically by placing the initial point of $\vec{\mathrm{w}}$ at the terminal point of $\vec{\mathrm{v}}$; then the vector sum $\vec{\mathrm{v}} + \vec{\mathrm{w}}$ is the vector with an initial point that coincides with the initial point of $\vec{\mathrm{v}}$, and with a terminal point that coincides with the terminal point of $\vec{\mathrm{w}}$

zero vector
the vector with both initial point and terminal point $(0, 0)$

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12.1E: Exercises for Section 12.1

For exercises 1 - 10, consider points $P(-1,3)$, $Q(1,5)$, and $R(-3,7)$. Determine the requested vectors and express each of them

a. in component form and

b. by using standard unit vectors.

1) $\overrightarrow{PQ}$

Answer

a. $\overrightarrow{PQ} = \langle 2,2 \rangle$

b. $\overrightarrow{PQ} = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$

2) $\overrightarrow{PR}$

3) $\overrightarrow{QP}$

Answer

a. $\overrightarrow{QP} = \langle -2,-2 \rangle$

b. $\overrightarrow{QP} = -2\hat{\mathbf{i}} - 2\hat{\mathbf{j}}$

4) $\overrightarrow{RP}$

5) $\overrightarrow{PQ} + \overrightarrow{PR}$

Answer

a. $\overrightarrow{PQ} + \overrightarrow{PR} = \langle 0,6 \rangle$

b. $\overrightarrow{PQ} + \overrightarrow{PR} = 6\hat{\mathbf{j}}$

6) $\overrightarrow{PQ} - \overrightarrow{PR}$

7) $2\overrightarrow{PQ} - 2\overrightarrow{PR}$

Answer

a. $2\overrightarrow{PQ} \rightarrow -2\overrightarrow{PR} = \langle 8,-4 \rangle$

b. $2\overrightarrow{PQ} - 2\overrightarrow{PR} = 8\hat{\mathbf{i}} - 4\hat{\mathbf{j}}$

8) $2\overrightarrow{PQ} + \frac{1}{2}\overrightarrow{PR}$

9) The unit vector in the direction of $\overrightarrow{PQ}$

Answer

a. $\left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$

b. $\frac{\sqrt{2}}{2}\hat{\mathbf{i}} + \frac{\sqrt{2}}{2}\hat{\mathbf{j}}$

10) The unit vector in the direction of $\overrightarrow{PR}$

11) A vector $\vec{\mathbf{v}}$ has initial point $(-1,-3)$ and terminal point $(2,1)$. Find the unit vector in the direction of $\vec{\mathbf{v}}$. Express the answer in component form.

Answer

$\langle \frac{3}{5}, \frac{4}{5} \rangle$

12) A vector $\vec{\mathbf{v}}$ has initial point $(-2,5)$ and terminal point $(3,-1)$. Find the unit vector in the direction of $\vec{\mathbf{v}}$. Express the answer in component form.

13) The vector $\vec{v}$ has initial point $P(1,0)$ and terminal point Q that is on the y-axis and above the initial point. Find the coordinates of terminal point Q such that the magnitude of the vector $\vec{v}$ is $\sqrt{5}$.

Answer

$Q(0,2)$

14) The vector $\vec{v}$ has initial point $P(1,1)$ and terminal point Q that is on the x-axis and left of the initial point. Find the coordinates of terminal point Q such that the magnitude of the vector $\vec{v}$ is $\sqrt{10}$.

For exercises 15 and 16, use the given vectors $\vec{a}$ and $\vec{b}$.

a. Determine the vector sum $\vec{a}+\vec{b}$ and express it in both the component form and by using the standard unit vectors.

b. Find the vector difference $\vec{a}-\vec{b}$ and express it in both the component form and by using the standard unit vectors.

c. Verify that the vectors $\vec{a}$, $\vec{b}$, and $\vec{a}+\vec{b}$, and, respectively, $\vec{a}$, $\vec{b}$, and $\vec{a}-\vec{b}$ satisfy the triangle inequality.

d. Determine the vectors $2\vec{a}$, $-\vec{b}$, and $2\vec{a}-\vec{b}$. Express the vectors in both the component form and by using standard unit vectors.

15) $\vec{a}=2\hat{i}+\hat{j}, \quad \vec{b}=\hat{i}+3\hat{j}$

Answer

a. $\vec{a}+\vec{b}=\langle 3,4\rangle, \quad \vec{a}+\vec{b}=3\hat{i}+4\hat{j}$
b. $\vec{a}-\vec{b}=\langle 1,-2\rangle, \quad \vec{a}-\vec{b}=\hat{i}-2\hat{j}$
c. Answers will vary
d. $2\vec{a}=\langle 4,2\rangle, \quad 2\vec{a}=4\hat{i}+2\hat{j}, \quad -\vec{b}=\langle -1,-3\rangle, \quad -\vec{b}=-\hat{i}-3\hat{j}, \quad 2\vec{a}-\vec{b}=\langle 3,-1\rangle, \quad 2\vec{a}-\vec{b}=3\hat{i}-\hat{j}$

16) $\vec{a}=2\hat{i}, \quad \vec{b}=-2\hat{i}+2\hat{j}$

17) Let $\vec{a}$ be a standard-position vector with terminal point $(-2,-4)$. Let $\vec{b}$ be a vector with initial point $(1,2)$ and terminal point $(-1,4)$. Find the magnitude of vector $-3\vec{a}+\vec{b}-4\hat{i}+\hat{j}$.

Answer

15

18) Let $\vec{a}$ be a standard-position vector with terminal point at $(2,5)$. Let $\vec{b}$ be a vector with initial point $(-1,3)$ and terminal point $(1,0)$. Find the magnitude of vector $\vec{a}-3\vec{b}+14\hat{i}-14\hat{j}$.

19) Let $\vec{u}$ and $\vec{v}$ be two nonzero vectors that are nonequivalent. Consider the vectors $\vec{a}=4\vec{u}+5\vec{v}$ and $\vec{b}=\vec{u}+2\vec{v}$ defined in terms of $\vec{u}$ and $\vec{v}$. Find the scalar λ such that vectors $\vec{a}+\lambda\vec{b}$ and $\vec{u}-\vec{v}$ are equivalent.

Answer

$\lambda=-3$

20) Let $\vec{u}$ and $\vec{v}$ be two nonzero vectors that are nonequivalent. Consider the vectors $\vec{a}=2\vec{u}-4\vec{v}$ and $\vec{b}=3\vec{u}-7\vec{v}$ defined in terms of $\vec{u}$ and $\vec{v}$. Find the scalars α and β such that vectors $\alpha\vec{a}+\beta\vec{b}$ and $\vec{u}-\vec{v}$ are equivalent.

21) Consider the vector $\vec{a}(t)=\langle \cos t, \sin t\rangle$ with components that depend on a real number t. As the number t varies, the components of $\vec{a}(t)$ change as well, depending on the functions that define them.

a. Write the vectors $\vec{a}(0)$ and $\vec{a}(\pi)$ in component form.

b. Show that the magnitude $\| \vec{a}(t) \|$ of vector $\vec{a}(t)$ remains constant for any real number t.

c. As t varies, show that the terminal point of vector $\vec{a}(t)$ describes a circle centered at the origin of radius 1.

Answer

a. $\vec{\mathbf{a}}(0) = \langle 1, 0\rangle, \quad \vec{\mathbf{a}}(\pi) = \langle -1, 0\rangle$

b. Answers may vary

c. Answers may vary

22) Consider vector $\vec{\mathbf{a}}(x) = \langle x, \sqrt{1-x^2}\rangle$ with components that depend on a real number $x \in [-1, 1]$. As the number x varies, the components of $\vec{\mathbf{a}}(x)$ change as well, depending on the functions that define them.

a. Write the vectors $\vec{\mathbf{a}}(0)$ and $\vec{\mathbf{a}}(1)$ in component form.

b. Show that the magnitude $\| \vec{\mathbf{a}}(x) \|$ of vector $\vec{\mathbf{a}}(x)$ remains constant for any real number x.

c. As x varies, show that the terminal point of vector $\vec{\mathbf{a}}(x)$ describes a circle centered at the origin of radius 1.

23) Show that vectors $\vec{\mathbf{a}}(t) = \langle \cos t, \sin t\rangle$ and $\vec{\mathbf{a}}(x) = \langle x, \sqrt{1-x^2}\rangle$ are equivalent for $x = 1$ and $t = 2k\pi$, where k is an integer.

Answer Answers may vary

24) Show that vectors $\vec{\mathbf{a}}(t) = \langle \cos t, \sin t\rangle$ and $\vec{\mathbf{a}}(x) = \langle x, \sqrt{1-x^2}\rangle$ are opposite for $x = 1$ and $t = \pi + 2k\pi$, where k is an integer.

For exercises 25-28, find a vector $\vec{\mathbf{v}}$ with the given magnitude and in the same direction as the vector $\vec{\mathbf{u}}$.

25) $\|\vec{\mathbf{v}}\| = 7, \quad \vec{\mathbf{u}} = \langle 3, 4\rangle$

Answer

$\vec{\mathbf{v}} = \langle \frac{21}{5}, \frac{28}{5}\rangle$

26) $\|\vec{\mathbf{v}}\| = 3, \quad \vec{\mathbf{u}} = \langle -2, 5\rangle$

27) $\|\vec{\mathbf{v}}\| = 7, \quad \vec{\mathbf{u}} = \langle 3, -5\rangle$

Answer

$\vec{\mathbf{v}} = \langle \frac{21\sqrt{34}}{34}, -\frac{35\sqrt{34}}{34}\rangle$

28) $\|\vec{\mathbf{v}}\| = 10, \quad \vec{\mathbf{u}} = \langle 2, -1\rangle$

For exercises 29-34, find the component form of vector $\vec{\mathbf{u}}$, given its magnitude and the angle the vector makes with the positive x-axis. Give exact answers when possible.

29) $\|\vec{\mathbf{u}}\| = 2, \theta = 30°$

Answer

$\vec{\mathbf{u}} = \langle \sqrt{3}, 1\rangle$

30) $\|\vec{\mathbf{u}}\| = 6, \theta = 60°$

31) $\|\vec{\mathbf{u}}\| = 5, \theta = \frac{\pi}{2}$

Answer

$\vec{\mathbf{u}} = \langle 0, 5\rangle$

32) $\|\vec{\mathbf{u}}\| = 8, \theta = \pi$

33) $\|\vec{\mathbf{u}}\| = 10, \theta = \frac{5\pi}{6}$

Answer

$\vec{\mathbf{u}} = \langle -5\sqrt{3}, 5\rangle$

34) $\|\vec{\mathbf{u}}\| = 50, \theta = \frac{3\pi}{4}$

For exercises 35 and 36, vector $\vec{\mathbf{u}}$ is given. Find the angle $\theta \in [0, 2\pi)$ that vector $\vec{\mathbf{u}}$ makes with the positive direction of the x-axis, in a counter-clockwise direction.

35) $\vec{\mathbf{u}} = 5\sqrt{2}\hat{\mathbf{i}} - 5\sqrt{2}\hat{\mathbf{j}}$

Answer

$\theta = \frac{7\pi}{4}$

36) $\vec{\mathbf{u}} = -\sqrt{3}\hat{\mathbf{i}} - \hat{\mathbf{j}}$

37) Let $\vec{\mathbf{a}} = \langle a_1, a_2 \rangle$, $\vec{\mathbf{b}} = \langle b_1, b_2 \rangle$, and $\vec{\mathbf{c}} = \langle c_1, c_2 \rangle$ be three nonzero vectors. If $a_1 b_2 - a_2 b_1 \neq 0$, then show there are two scalars, α and β, such that $\vec{\mathbf{c}} = \alpha\vec{\mathbf{a}} + \beta\vec{\mathbf{b}}$.

Answer Answers may vary

38) Consider vectors $\vec{\mathbf{a}} = \langle 2, -4 \rangle$, $\vec{\mathbf{b}} = \langle -1, 2 \rangle$, and $\vec{\mathbf{c}} = \vec{\mathbf{0}}$ Determine the scalars α and β such that $\vec{\mathbf{c}} = \alpha\vec{\mathbf{a}} + \beta\vec{\mathbf{b}}$.

39) Let $P(x_0, f(x_0))$ be a fixed point on the graph of the differentiable function f with a domain that is the set of real numbers.

a. Determine the real number z_0 such that point $Q(x_0 + 1, z_0)$ is situated on the line tangent to the graph of f at point P.

b. Determine the unit vector $\vec{\mathbf{u}}$ with initial point P and terminal point Q.

Answer

$a.\quad z_0 = f(x_0) + f'(x_0); \quad b.\quad \vec{\mathbf{u}} = \frac{1}{\sqrt{1+[f'(x_0)]^2}}\langle 1, f'(x_0) \rangle$

40) Consider the function $f(x) = x^4$, where $x \in R$.

a. Determine the real number z_0 such that point $Q(2, z_0)$ s situated on the line tangent to the graph of f at point $P(1, 1)$.

b. Determine the unit vector $\vec{\mathbf{u}}$ with initial point P and terminal point Q.

41) Consider f and g two functions defined on the same set of real numbers D. Let $\vec{\mathbf{a}} = \langle x, f(x) \rangle$ and $\vec{\mathbf{b}} = \langle x, g(x) \rangle$ be two vectors that describe the graphs of the functions, where $x \in D$. Show that if the graphs of the functions f and g do not intersect, then the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are not equivalent.

42) Find $x \in R$ such that vectors $\vec{\mathbf{a}} = \langle x, \sin x \rangle$ and $\vec{\mathbf{b}} = \langle x, \cos x \rangle$ are equivalent.

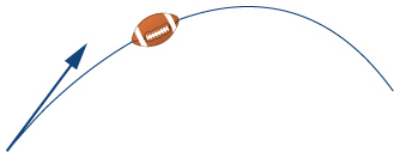
43) Calculate the coordinates of point D such that $ABCD$ is a parallelogram, with $A(1, 1)$, $B(2, 4)$, and $C(7, 4)$.

Answer

$D(6, 1)$

44) Consider the points $A(2, 1), B(10, 6), C(13, 4)$, and $D(16, -2)$. Determine the component form of vector $\overrightarrow{AD}$.

45) The **speed** of an object is the magnitude of its related velocity vector. A football thrown by a quarterback has an initial speed of 70 mph and an angle of elevation of $30°$. Determine the velocity vector in mph and express it in component form. (Round to two decimal places.)

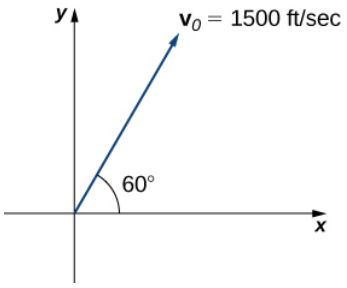


Answer

$\langle 60.62, 35 \rangle$

46) A baseball player throws a baseball at an angle of $30°$ with the horizontal. If the initial speed of the ball is 100 mph, find the horizontal and vertical components of the initial velocity vector of the baseball. (Round to two decimal places.)

47) A bullet is fired with an initial velocity of 1500 ft/sec at an angle of $60°$ with the horizontal. Find the horizontal and vertical components of the velocity vector of the bullet. (Round to two decimal places.)

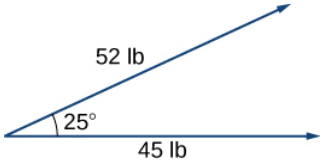


Answer

The horizontal and vertical components are 750 ft/sec and 1299.04 ft/sec, respectively.

48) [T] A 65-kg sprinter exerts a force of 798 N at a $19°$ angle with respect to the ground on the starting block at the instant a race begins. Find the horizontal component of the force. (Round to two decimal places.)

49) [T] Two forces, a horizontal force of 45 lb and another of 52 lb, act on the same object. The angle between these forces is $25°$. Find the magnitude and direction angle from the positive x-axis of the resultant force that acts on the object. (Round to two decimal places.)



Answer

The magnitude of resultant force is 94.71 lb; the direction angle is $13.42°$.

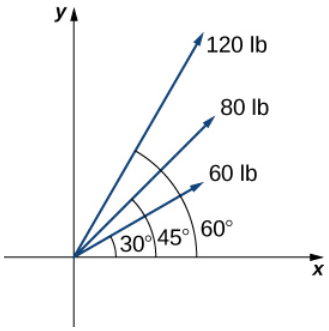
50) [T] Two forces, a vertical force of 26 lb and another of 45 lb, act on the same object. The angle between these forces is $55°$. Find the magnitude and direction angle from the positive x-axis of the resultant force that acts on the object. (Round to two decimal places.)

51) [T] Three forces act on object. Two of the forces have the magnitudes 58 N and 27 N, and make angles $53°$ and $152°$, respectively, with the positive x-axis. Find the magnitude and the direction angle from the positive x-axis of the third force such that the resultant force acting on the object is zero. (Round to two decimal places.)

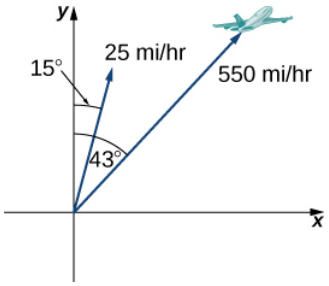
Answer

The magnitude of the third vector is 60.03 N; the direction angle is $259.38°$.

52) Three forces with magnitudes **80** lb, **120** lb, and **60** lb act on an object at angles of $45°, 60°$ and $30°$, respectively, with the positive x-axis. Find the magnitude and direction angle from the positive x-axis of the resultant force. (Round to two decimal places.)



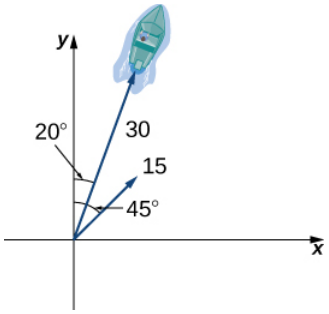
53) [T] An airplane is flying in the direction of $43°$ east of north (also abbreviated as $N43E$ at a speed of 550 mph. A wind with speed 25 mph comes from the southwest at a bearing of $N15E$. What are the ground speed and new direction of the airplane?



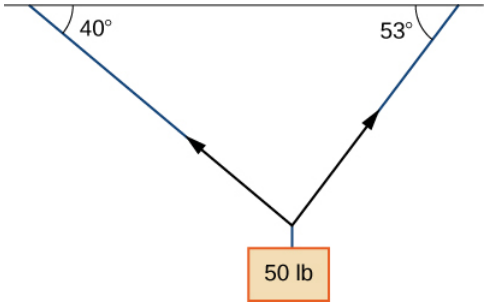
Answer

The new ground speed of the airplane is 572.19 mph; the new direction is $N41.82E$.

54) [T] A boat is traveling in the water at 30 mph in a direction of $N20E$ (that is, $20°$ east of north). A strong current is moving at 15 mph in a direction of $N45E$. What are the new speed and direction of the boat?



55) [T] A 50-lb weight is hung by a cable so that the two portions of the cable make angles of $40°$ and $53°$, respectively, with the horizontal. Find the magnitudes of the forces of tension $\vec{\mathbf{T}}_1$ and $\vec{\mathbf{T}}_2$ in the cables if the resultant force acting on the object is zero. (Round to two decimal places.)

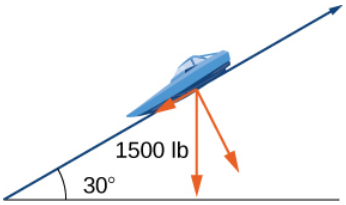


Answer

$\|\vec{\mathbf{T}}_1\| = 30.13\,lb, \quad \|\vec{\mathbf{T}}_2\| = 38.35\,lb$

56) [T] A 62-lb weight hangs from a rope that makes the angles of $29°$ and $61°$, respectively, with the horizontal. Find the magnitudes of the forces of tension $\vec{\mathbf{T}}_1$ and $\vec{\mathbf{T}}_2$ in the cables if the resultant force acting on the object is zero. (Round to two decimal places.)

57) [T] A 1500-lb boat is parked on a ramp that makes an angle of $30°$ with the horizontal. The boat's weight vector points downward and is a sum of two vectors: a horizontal vector $\vec{\mathbf{v}}_1$ that is parallel to the ramp and a vertical vector $\vec{\mathbf{v}}_2$ that is perpendicular to the inclined surface. The magnitudes of vectors $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ are the horizontal and vertical component, respectively, of the boat's weight vector. Find the magnitudes of $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$. (Round to the nearest integer.)

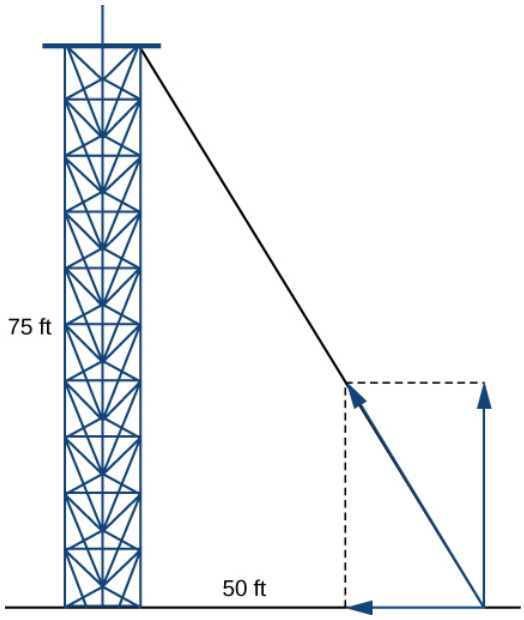


Answer

$\|\vec{v}_1\| = 750\,lb, \quad \|\vec{v}_2\| = 1299\,lb$

58) [T] An 85-lb box is at rest on a $26°$ incline. Determine the magnitude of the force parallel to the incline necessary to keep the box from sliding. (Round to the nearest integer.)

59) A guy-wire supports a pole that is 75 ft high. One end of the wire is attached to the top of the pole and the other end is anchored to the ground 50 ft from the base of the pole. Determine the horizontal and vertical components of the force of tension in the wire if its magnitude is 50 lb. (Round to the nearest integer.)



Answer

The two horizontal and vertical components of the force of tension are 28 lb and 42 lb, respectively.

60) A telephone pole guy-wire has an angle of elevation of $35°$ with respect to the ground. The force of tension in the guy-wire is 120 lb. Find the horizontal and vertical components of the force of tension. (Round to the nearest integer.)

Contributors

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12.2: Vectors in Three Dimensions

Learning Objectives

- Describe three-dimensional space mathematically.
- Locate points in space using coordinates.
- Write the distance formula in three dimensions.
- Write the equations for simple planes and spheres.
- Perform vector operations in $\mathbb{R}^3$.

Vectors are useful tools for solving two-dimensional problems. Life, however, happens in three dimensions. To expand the use of vectors to more realistic applications, it is necessary to create a framework for describing three-dimensional space. For example, although a two-dimensional map is a useful tool for navigating from one place to another, in some cases the topography of the land is important. Does your planned route go through the mountains? Do you have to cross a river? To appreciate fully the impact of these geographic features, you must use three dimensions. This section presents a natural extension of the two-dimensional Cartesian coordinate plane into three dimensions.

Three-Dimensional Coordinate Systems

As we have learned, the two-dimensional rectangular coordinate system contains two perpendicular axes: the horizontal x-axis and the vertical y-axis. We can add a third dimension, the z-axis, which is perpendicular to both the x-axis and the y-axis. We call this system the three-dimensional rectangular coordinate system. It represents the three dimensions we encounter in real life.

Definition: Three-dimensional Rectangular Coordinate System

The three-dimensional rectangular coordinate system consists of three perpendicular axes: the x-axis, the y-axis, and the z-axis. Because each axis is a number line representing all real numbers in $\mathbb{R}$, the three-dimensional system is often denoted by $\mathbb{R}^3$.

In Figure 12.2.1a, the positive z-axis is shown above the plane containing the x- and y-axes. The positive x-axis appears to the left and the positive y-axis is to the right. A natural question to ask is: How was this arrangement determined? The system displayed follows the **right-hand rule**. If we take our right hand and align the fingers with the positive x-axis, then curl the fingers so they point in the direction of the positive y-axis, our thumb points in the direction of the positive z-axis (Figure 12.2.1b). In this text, we always work with coordinate systems set up in accordance with the right-hand rule. Some systems do follow a left-hand rule, but the right-hand rule is considered the standard representation.

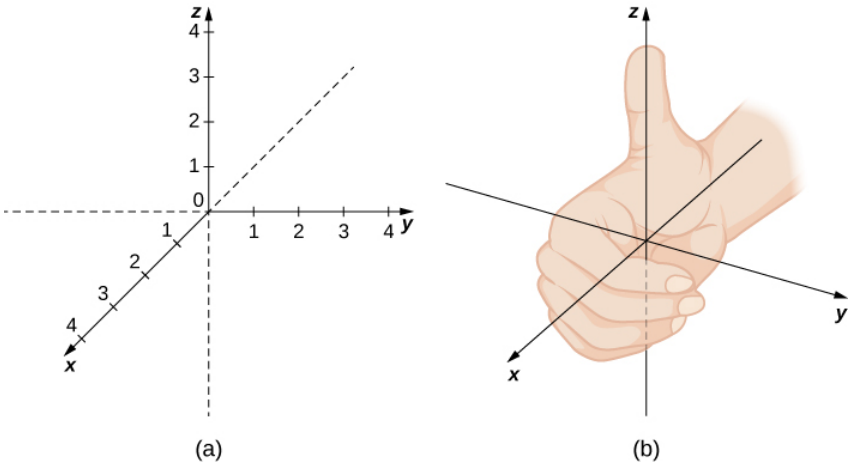


Figure 12.2.1: (a) We can extend the two-dimensional rectangular coordinate system by adding a third axis, the z-axis, that is perpendicular to both the x-axis and the y-axis. (b) The right-hand rule is used to determine the placement of the coordinate axes in the standard Cartesian plane.

In two dimensions, we describe a point in the plane with the coordinates (x, y). Each coordinate describes how the point aligns with the corresponding axis. In three dimensions, a new coordinate, z, is appended to indicate alignment with the z-axis: (x, y, z). A point in space is identified by all three coordinates (Figure 12.2.2). To plot the point (x, y, z), go x units along the x-axis, then y units in the direction of the y-axis, then z units in the direction of the z-axis.

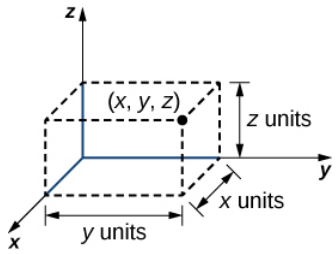


Figure 12.2.2: To plot the point (x, y, z) go x units along the x-axis, then y units in the direction of the y-axis, then z units in the direction of the z-axis.

Example 12.2.1: Locating Points in Space

Sketch the point $(1, -2, 3)$ in three-dimensional space.

Solution

To sketch a point, start by sketching three sides of a rectangular prism along the coordinate axes: one unit in the positive x direction, 2 units in the negative y direction, and 3 units in the positive z direction. Complete the prism to plot the point (Figure 12.2.3).

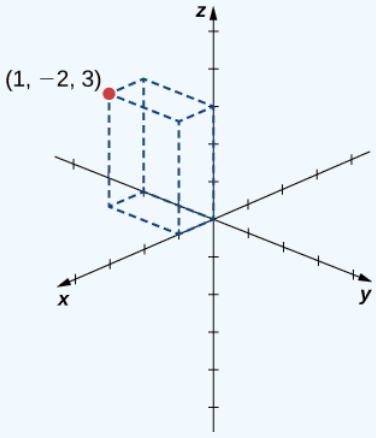


Figure 12.2.3: Sketching the point $(1, -2, 3)$.

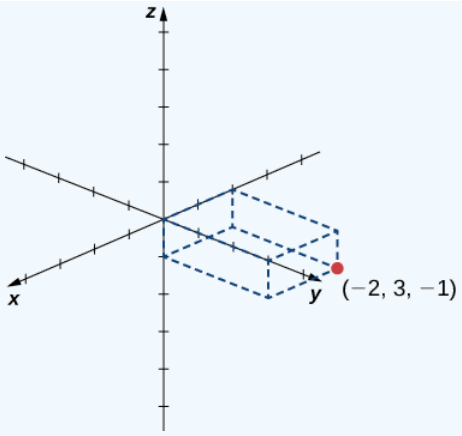
Exercise 12.2.1

Sketch the point $(-2, 3, -1)$ in three-dimensional space.

Hint

Start by sketching the coordinate axes. e.g., Figure 12.2.3. Then sketch a rectangular prism to help find the point in space.

Answer



In two-dimensional space, the coordinate plane is defined by a pair of perpendicular axes. These axes allow us to name any location within the plane. In three dimensions, we define **coordinate planes** by the coordinate axes, just as in two dimensions. There are three axes now, so there are three intersecting pairs of axes. Each pair of axes forms a coordinate plane: the xy-plane, the xz-plane, and the yz-plane (Figure 12.2.4). We define the xy-plane formally as the following set: $\{(x, y, 0) : x, y \in \mathbb{R}\}$. Similarly, the xz-plane and the yz-plane are defined as $\{(x, 0, z) : x, z \in \mathbb{R}\}$ and $\{(0, y, z) : y, z \in \mathbb{R}\}$, respectively.

To visualize this, imagine you're building a house and are standing in a room with only two of the four walls finished. (Assume the two finished walls are adjacent to each other.) If you stand with your back to the corner where the two finished walls meet, facing out into the room, the floor is the xy-plane, the wall to your right is the xz-plane, and the wall to your left is the yz-plane.

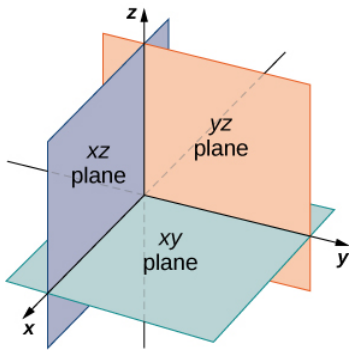


Figure 12.2.4: The plane containing the x- *and* y-axes is called the xy-plane. The plane containing the x- and z-axes is called the xz-plane, and the y- and z-axes define the yz-plane.

In two dimensions, the coordinate axes partition the plane into four quadrants. Similarly, the coordinate planes divide space between them into eight regions about the origin, called **octants.** The octants fill $\mathbb{R}^3$ in the same way that quadrants fill $\mathbb{R}^2$, as shown in Figure 12.2.5.

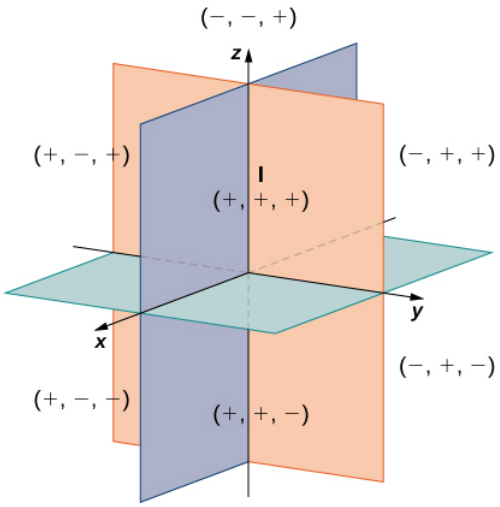


Figure 12.2.5: Points that lie in octants have three nonzero coordinates.

Most work in three-dimensional space is a comfortable extension of the corresponding concepts in two dimensions. In this section, we use our knowledge of circles to describe spheres, then we expand our understanding of vectors to three dimensions. To accomplish these goals, we begin by adapting the distance formula to three-dimensional space.

If two points lie in the same coordinate plane, then it is straightforward to calculate the distance between them. We know that the distance d between two points (x_1, y_1) and (x_2, y_2) in the xy-coordinate plane is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

The formula for the distance between two points in space is a natural extension of this formula.

The Distance between Two Points in Space

The distance d between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \tag{12.2.1}$$

The proof of this theorem is left as an exercise. (Hint: First find the distance d_1 between the points (x_1, y_1, z_1) and (x_2, y_2, z_1) as shown in Figure 12.2.6.)

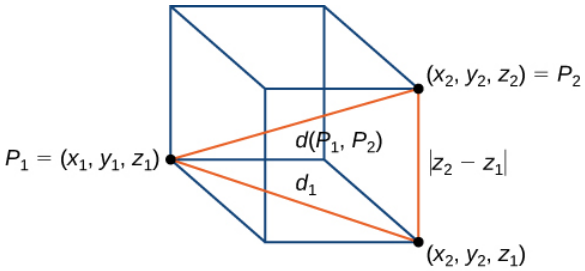


Figure 12.2.6: The distance between P_1 and P_2 is the length of the diagonal of the rectangular prism having P_1 and P_2 as opposite corners.

Example 12.2.2: Distance in Space

Find the distance between points $P_1 = (3, -1, 5)$ and $P_2 = (2, 1, -1)$.

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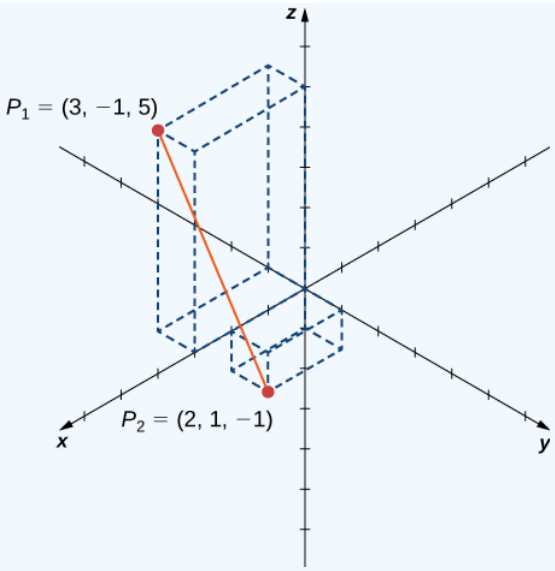


Figure 12.2.7: Find the distance between the two points.

Solution

Substitute values directly into the distance formula (Equation 12.2.1):

$$\begin{aligned} d(P_1, P_2) &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ &= \sqrt{(2-3)^2 + (1-(-1))^2 + (-1-5)^2} \\ &= \sqrt{(-1)^2 + 2^2 + (-6)^2} \\ &= \sqrt{41}. \end{aligned}$$

? Exercise 12.2.2

Find the distance between points $P_1 = (1, -5, 4)$ and $P_2 = (4, -1, -1)$.

Hint

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Answer

$$5\sqrt{2}$$

Before moving on to the next section, let's get a feel for how $\mathbb{R}^3$ differs from $\mathbb{R}^2$. For example, in $\mathbb{R}^2$, lines that are not parallel must always intersect. This is not the case in $\mathbb{R}^3$. For example, consider the lines shown in Figure 12.2.8. These two lines are not parallel, nor do they intersect.

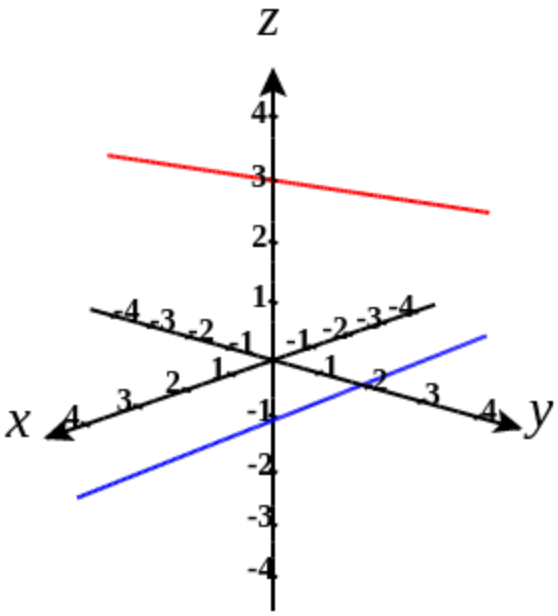


Figure 12.2.8: These two lines are not parallel, but still do not intersect.

You can also have circles that are interconnected but have no points in common, as in Figure 12.2.9.

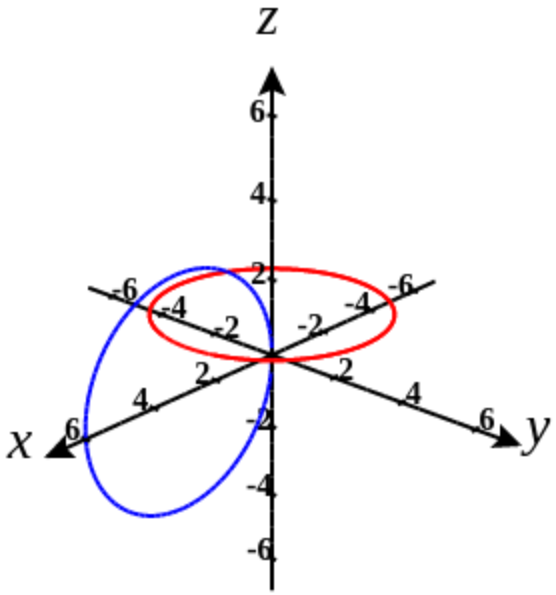


Figure 12.2.9: These circles are interconnected, but have no points in common.

We have a lot more flexibility working in three dimensions than we do if we stuck with only two dimensions.

Writing Equations in $\mathbb{R}^3$

Now that we can represent points in space and find the distance between them, we can learn how to write equations of geometric objects such as lines, planes, and curved surfaces in $\mathbb{R}^3$. First, we start with a simple equation. Compare the graphs of the equation $x = 0$ in $\mathbb{R}$, $\mathbb{R}^2$,and $\mathbb{R}^3$ (Figure 12.2.10). From these graphs, we can see the same equation can describe a point, a line, or a plane.

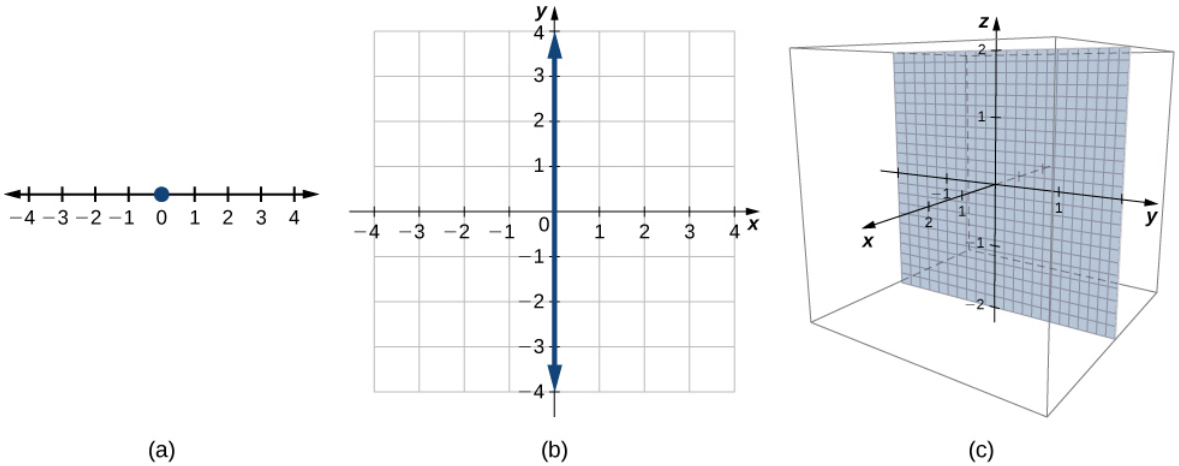


Figure 12.2.10: (a) In $\mathbb{R}$, the equation $x = 0$ describes a single point. (b) In $\mathbb{R}^2$, the equation $x = 0$ describes a line, the y-axis. (c) In $\mathbb{R}^3$, the equation $x = 0$ describes a plane, the yz-plane.

In space, the equation $x = 0$ describes all points $(0, y, z)$. This equation defines the yz-plane. Similarly, the xy-plane contains all points of the form $(x, y, 0)$. The equation $z = 0$ defines the xy-plane and the equation $y = 0$ describes the xz-plane (Figure 12.2.11).

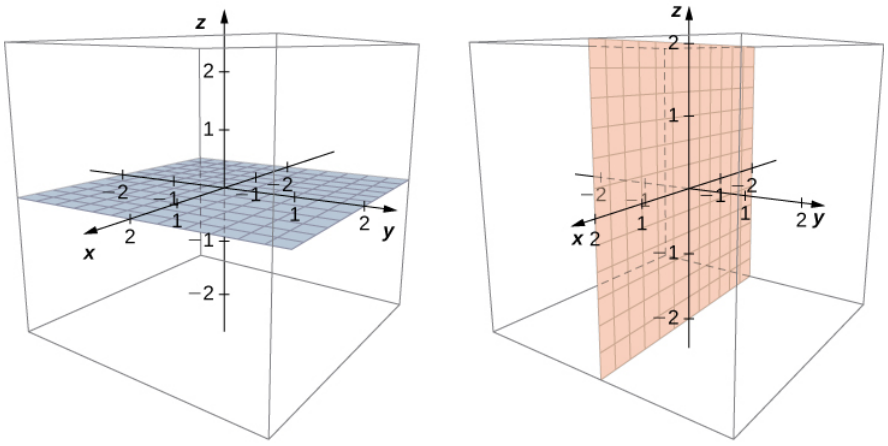


Figure 12.2.11: (a) In space, the equation $z = 0$ describes the xy-plane. (b) All points in the xz-plane satisfy the equation $y = 0$.

Understanding the equations of the coordinate planes allows us to write an equation for any plane that is parallel to one of the coordinate planes. When a plane is parallel to the xy-plane, for example, the z-coordinate of each point in the plane has the same constant value. Only the x- and y-coordinates of points in that plane vary from point to point.

Equations of Planes Parallel to Coordinate Planes

1. The plane in space that is parallel to the xy-plane and contains point (a, b, c) can be represented by the equation $z = c$.
2. The plane in space that is parallel to the xz-plane and contains point (a, b, c) can be represented by the equation $y = b$.
3. The plane in space that is parallel to the yz-plane and contains point (a, b, c) can be represented by the equation $x = a$.

Example 12.2.3: Writing Equations of Planes Parallel to Coordinate Planes

a. Write an equation of the plane passing through point $(3, 11, 7)$ that is parallel to the yz-plane.
b. Find an equation of the plane passing through points $(6, -2, 9), (0, -2, 4)$, and $(1, -2, -3)$.

Solution

a. When a plane is parallel to the yz-plane, only the y- and z-coordinates may vary. The x-coordinate has the same constant value for all points in this plane, so this plane can be represented by the equation $x = 3$.
b. Each of the points $(6, -2, 9), (0, -2, 4)$, and $(1, -2, -3)$ has the same y-coordinate. This plane can be represented by the equation $y = -2$.

Exercise 12.2.3

Write an equation of the plane passing through point $(1, -6, -4)$ that is parallel to the xy-plane.

Hint

If a plane is parallel to the xy-plane, the z-coordinates of the points in that plane do not vary.

Answer

$$z = -4$$

As we have seen, in $\mathbb{R}^2$ the equation $x = 5$ describes the vertical line passing through point $(5, 0)$. This line is parallel to the y-axis. In a natural extension, the equation $x = 5$ in $\mathbb{R}^3$ describes the plane passing through point $(5, 0, 0)$, which is parallel to the yz-plane. Another natural extension of a familiar equation is found in the equation of a sphere.

Definition: Sphere

A sphere is the set of all points in space equidistant from a fixed point, the center of the sphere (Figure 12.2.12), just as the set of all points in a plane that are equidistant from the center represents a circle. In a sphere, as in a circle, the distance from the center to a point on the sphere is called the *radius*.

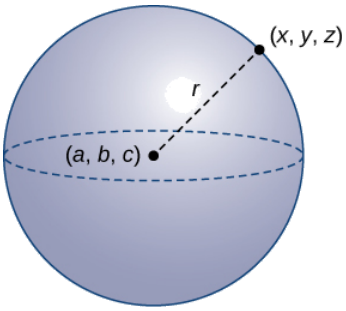


Figure 12.2.12: Each point (x, y, z) on the surface of a sphere is r units away from the center (a, b, c).

The equation of a circle is derived using the distance formula in two dimensions. In the same way, the equation of a sphere is based on the three-dimensional formula for distance.

Standard Equation of a Sphere

The sphere with center (a, b, c) and radius r can be represented by the equation

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2.$$

This equation is known as the *standard equation of a sphere*.

Example 12.2.4: Finding an Equation of a Sphere

Find the standard equation of the sphere with center $(10, 7, 4)$ and point $(-1, 3, -2)$, as shown in Figure 12.2.13.

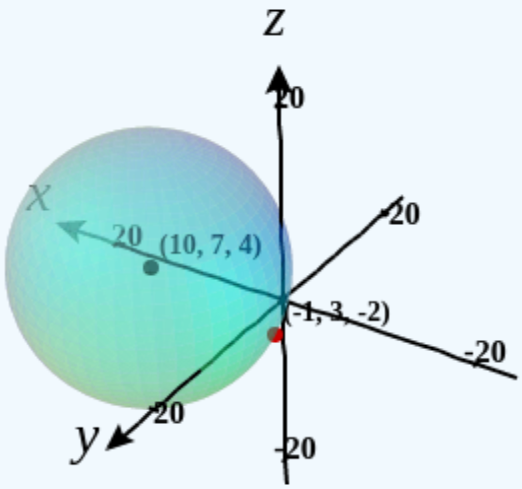


Figure 12.2.13: The sphere centered at $(10, 7, 4)$ containing point $(-1, 3, -2)$.

Solution

Use the distance formula to find the radius r of the sphere:

$$\begin{aligned} r &= \sqrt{(-1-10)^2 + (3-7)^2 + (-2-4)^2} \\ &= \sqrt{(-11)^2 + (-4)^2 + (-6)^2} \\ &= \sqrt{173} \end{aligned}$$

The standard equation of the sphere is

$$(x-10)^2 + (y-7)^2 + (z-4)^2 = 173.$$

? Exercise 12.2.4

Find the standard equation of the sphere with center $(-2, 4, -5)$ containing point $(4, 4, -1)$.

Hint

First use the distance formula to find the radius of the sphere.

Answer

$$(x+2)^2 + (y-4)^2 + (z+5)^2 = 52$$

Example 12.2.5: Finding the Equation of a Sphere

Let $P = (-5, 2, 3)$ and $Q = (3, 4, -1)$, and suppose line segment $\overline{PQ}$ forms the diameter of a sphere (Figure 12.2.14). Find the equation of the sphere.

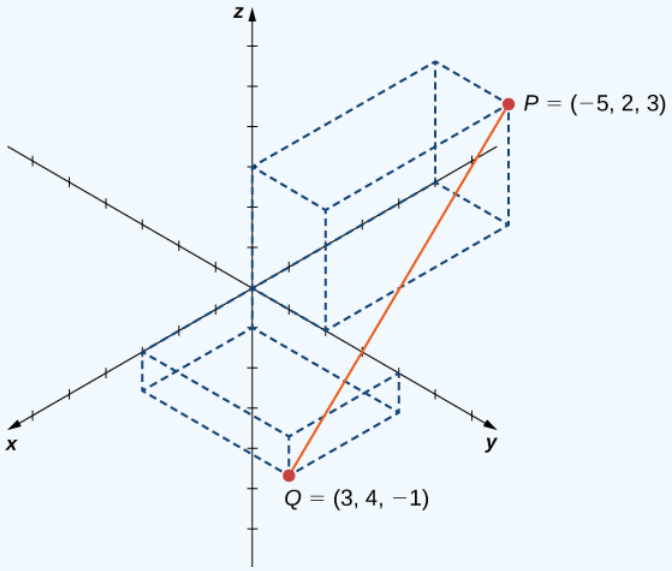


Figure 12.2.14: Line segment $\overline{PQ}$.

Solution:

Since $\overline{PQ}$ is a diameter of the sphere, we know the center of the sphere is the midpoint of $\overline{PQ}$.Then,

$$C = \left(\frac{-5+3}{2}, \frac{2+4}{2}, \frac{3+(-1)}{2}\right) = (-1, 3, 1).$$

Furthermore, we know the radius of the sphere is half the length of the diameter. This gives

$$\begin{aligned} r &= \frac{1}{2}\sqrt{(-5-3)^2 + (2-4)^2 + (3-(-1))^2} \\ &= \frac{1}{2}\sqrt{64+4+16} \\ &= \sqrt{21} \end{aligned}$$

Then, the equation of the sphere is $(x+1)^2 + (y-3)^2 + (z-1)^2 = 21$.

Exercise 12.2.5

Find the equation of the sphere with diameter $\overline{PQ}$, where $P = (2, -1, -3)$ and $Q = (-2, 5, -1)$.

Hint

Find the midpoint of the diameter first.

Answer

$$x^2 + (y-2)^2 + (z+2)^2 = 14$$

Example 12.2.6: Graphing Other Equations in Three Dimensions

Describe the set of points that satisfies $(x-4)(z-2) = 0$, and graph the set.

Solution

We must have either $x - 4 = 0$ or $z - 2 = 0$, so the set of points forms the two planes $x = 4$ and $z = 2$ (Figure 12.2.15).

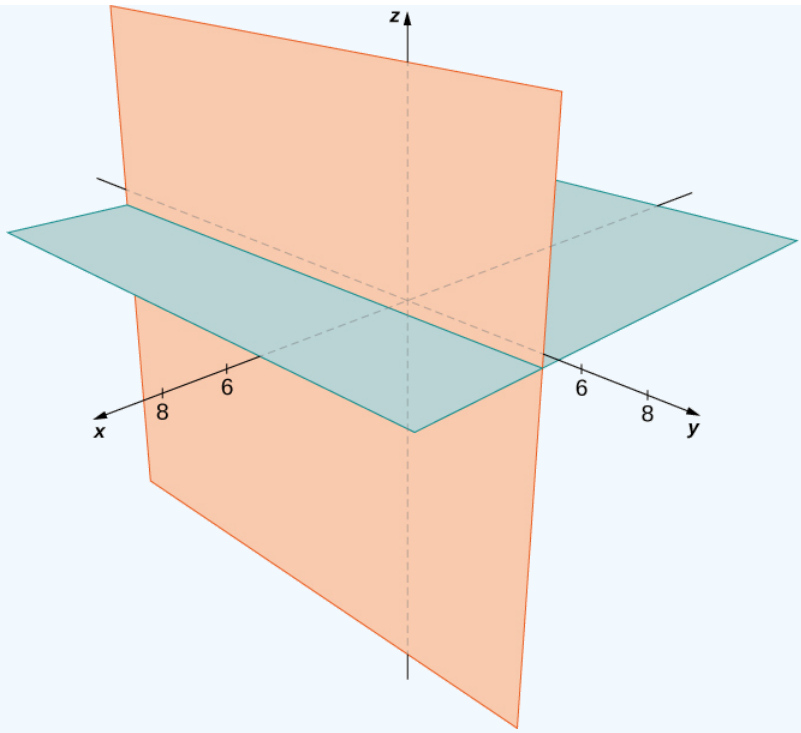


Figure 12.2.15: The set of points satisfying $(x-4)(z-2)=0$ forms the two planes $x=4$ and $z=2$.

Exercise 12.2.6

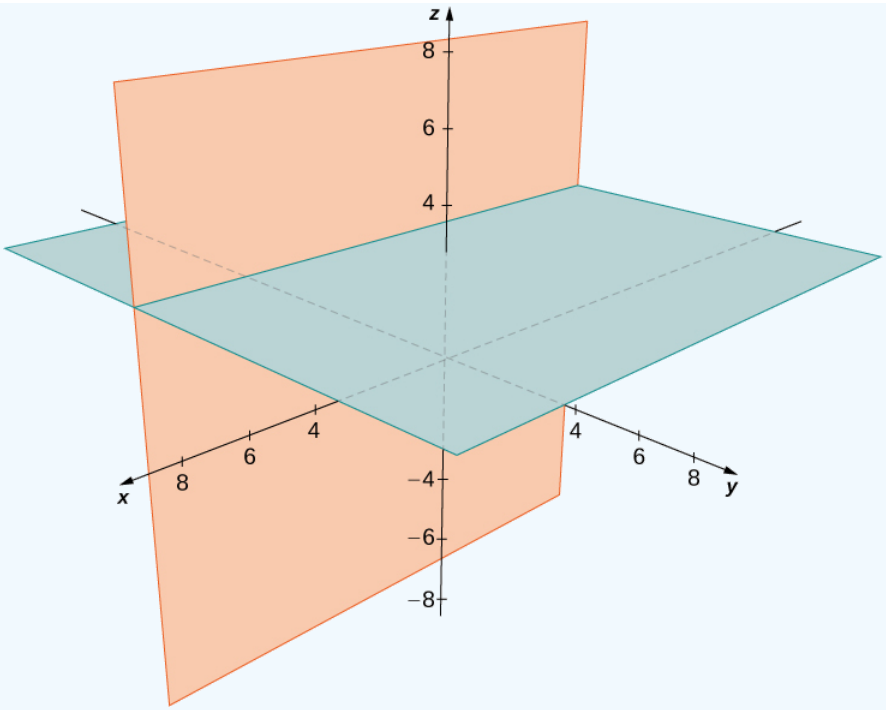
Describe the set of points that satisfies $(y+2)(z-3)=0$, and graph the set.

Hint

One of the factors must be zero.

Answer

The set of points forms the two planes $y=-2$ and $z=3$.



Example 12.2.7: Graphing Other Equations in Three Dimensions

Describe the set of points in three-dimensional space that satisfies $(x-2)^2+(y-1)^2=4$, and graph the set.

Solution

The x- and y-coordinates form a circle in the xy-plane of radius 2, centered at $(2,1)$. Since there is no restriction on the z-coordinate, the three-dimensional result is a circular cylinder of radius 2 centered on the line with $x=2$ and $y=1$. The cylinder extends indefinitely in the z-direction (Figure 12.2.16).

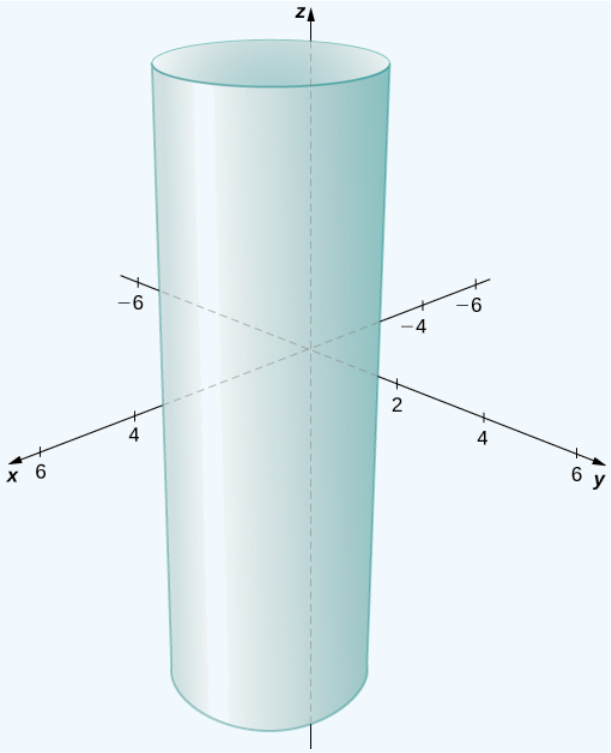


Figure 12.2.16: The set of points satisfying $(x-2)^2+(y-1)^2=4$. This is a cylinder of radius 2 centered on the line with $x=2$ and $y=1$.

? Exercise 12.2.7

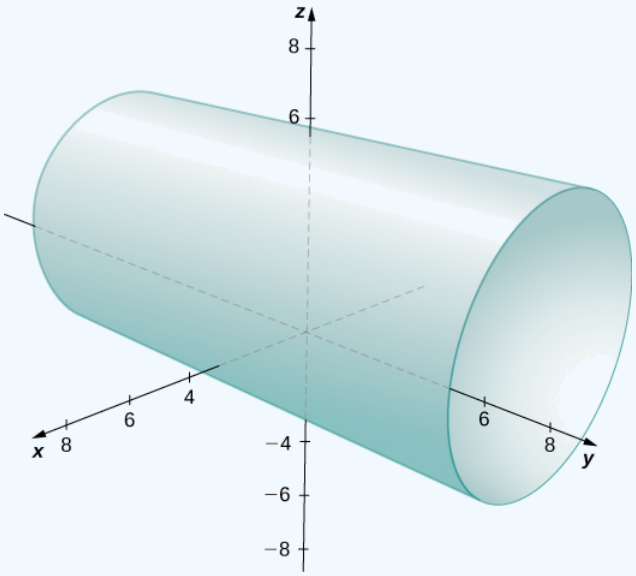
Describe the set of points in three dimensional space that satisfies $x^2+(z-2)^2=16$, and graph the surface.

Hint

Think about what happens if you plot this equation in two dimensions in the xz-plane.

Answer

A cylinder of radius 4 centered on the line with $x=0$ and $z=2$.



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Working with Vectors in $\mathbb{R}^3$

Just like two-dimensional vectors, three-dimensional vectors are quantities with both magnitude and direction, and they are represented by directed line segments (arrows). With a three-dimensional vector, we use a three-dimensional arrow.

Three-dimensional vectors can also be represented in component form. The notation $\vec{\mathbf{v}} = \langle x, y, z \rangle$ is a natural extension of the two-dimensional case, representing a vector with the initial point at the origin, $(0, 0, 0)$, and terminal point (x, y, z). The zero vector is $\vec{\mathbf{0}} = \langle 0, 0, 0 \rangle$. So, for example, the three dimensional vector $\vec{\mathbf{v}} = \langle 2, 4, 1 \rangle$ is represented by a directed line segment from point $(0, 0, 0)$ to point $(2, 4, 1)$ (Figure 12.2.17).

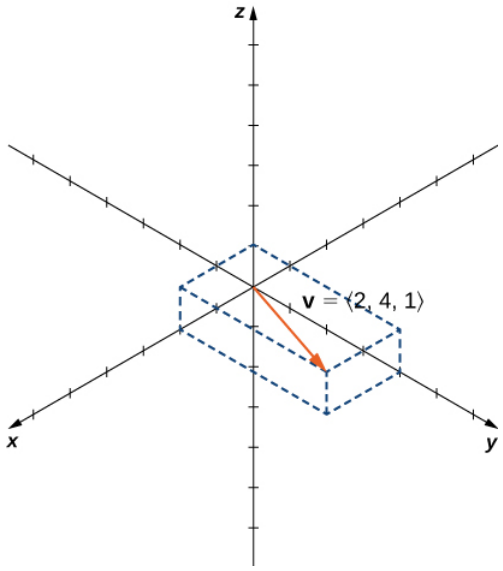


Figure 12.2.17: Vector $\vec{\mathbf{v}} = \langle 2, 4, 1 \rangle$ is represented by a directed line segment from point $(0, 0, 0)$ to point $(2, 4, 1)$.

Vector addition and scalar multiplication are defined analogously to the two-dimensional case. If $\vec{\mathbf{v}} = \langle x_1, y_1, z_1 \rangle$ and $\vec{\mathbf{w}} = \langle x_2, y_2, z_2 \rangle$ are vectors, and k is a scalar, then

$$\vec{\mathbf{v}} + \vec{\mathbf{w}} = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle$$

and

$$k\vec{\mathbf{v}} = \langle kx_1, ky_1, kz_1 \rangle.$$

If $k = -1$, then $k\vec{\mathbf{v}} = (-1)\vec{\mathbf{v}}$ is written as $-\vec{\mathbf{v}}$, and vector subtraction is defined by $\vec{\mathbf{v}} - \vec{\mathbf{w}} = \vec{\mathbf{v}} + (-\vec{\mathbf{w}}) = \vec{\mathbf{v}} + (-1)\vec{\mathbf{w}}$.

The standard unit vectors extend easily into three dimensions as well, $\hat{\mathbf{i}} = \langle 1, 0, 0 \rangle$, $\hat{\mathbf{j}} = \langle 0, 1, 0 \rangle$, and $\hat{\mathbf{k}} = \langle 0, 0, 1 \rangle$, and we use them in the same way we used the standard unit vectors in two dimensions. Thus, we can represent a vector in $\mathbb{R}^3$ in the following ways:

$$\vec{\mathbf{v}} = \langle x, y, z \rangle = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

.

Example 12.2.8: Vector Representations

Let $\overrightarrow{PQ}$ be the vector with initial point $P = (3, 12, 6)$ and terminal point $Q = (-4, -3, 2)$ as shown in Figure 12.2.18. Express $\overrightarrow{PQ}$ in both component form and using standard unit vectors.

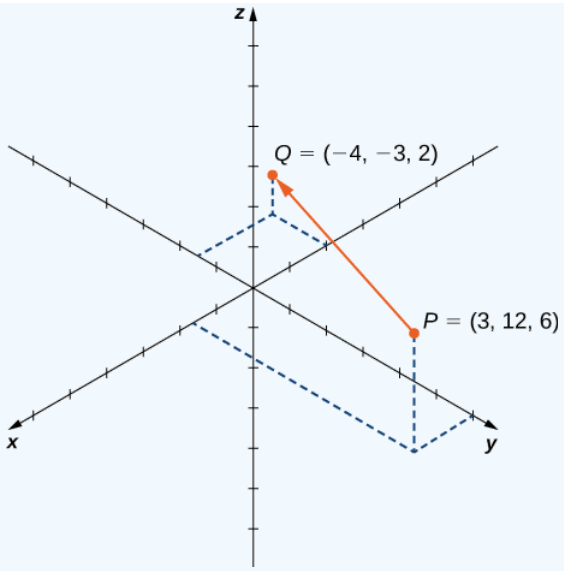


Figure 12.2.18: The vector with initial point $P = (3, 12, 6)$ and terminal point $Q = (-4, -3, 2)$.

Solution

In component form,

$$\begin{aligned}\overrightarrow{PQ} &= \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle \\ &= \langle -4-3, -3-12, 2-6\rangle \\ &= \langle -7, -15, -4\rangle.\end{aligned}$$

In standard unit form,

$$\overrightarrow{PQ} = -7\hat{\mathbf{i}} - 15\hat{\mathbf{j}} - 4\hat{\mathbf{k}}.$$

? Exercise 12.2.8

Let $S = (3, 8, 2)$ and $T = (2, -1, 3)$. Express $\vec{ST}$ in component form and in standard unit form.

Hint

Write $\overrightarrow{ST}$ in component form first. T is the terminal point of $\overrightarrow{ST}$.

Answer

$\overrightarrow{ST} = \langle -1, -9, 1\rangle = -\hat{\mathbf{i}} - 9\hat{\mathbf{j}} + \hat{\mathbf{k}}$

As described earlier, vectors in three dimensions behave in the same way as vectors in a plane. The geometric interpretation of vector addition, for example, is the same in both two- and three-dimensional space (Figure 12.2.19).

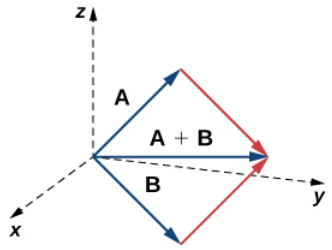


Figure 12.2.19: To add vectors in three dimensions, we follow the same procedures we learned for two dimensions.

We have already seen how some of the algebraic properties of vectors, such as vector addition and scalar multiplication, can be extended to three dimensions. Other properties can be extended in similar fashion. They are summarized here for our reference.

Properties of Vectors in Space

Let $\vec{\mathbf{v}} = \langle x_1, y_1, z_1 \rangle$ and $\vec{\mathbf{w}} = \langle x_2, y_2, z_2 \rangle$ be vectors, and let k be a scalar.

- **Scalar multiplication**:

$$k\vec{\mathbf{v}} = \langle kx_1, ky_1, kz_1 \rangle$$

- **Vector addition**:

$$\vec{\mathbf{v}} + \vec{\mathbf{w}} = \langle x_1, y_1, z_1 \rangle + \langle x_2, y_2, z_2 \rangle = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle$$

- **Vector subtraction**:

$$\vec{\mathbf{v}} - \vec{\mathbf{w}} = \langle x_1, y_1, z_1 \rangle - \langle x_2, y_2, z_2 \rangle = \langle x_1 - x_2, y_1 - y_2, z_1 - z_2 \rangle$$

- **Vector magnitude**:

$$\|\vec{\mathbf{v}}\| = \sqrt{x_1^2 + y_1^2 + z_1^2}$$

- **Unit vector in the direction of $\vec{\mathbf{v}}$**:

$$\frac{1}{\|\vec{\mathbf{v}}\|}\vec{\mathbf{v}} = \frac{1}{\|\vec{\mathbf{v}}\|}\langle x_1, y_1, z_1 \rangle = \langle \frac{x_1}{\|\vec{\mathbf{v}}\|}, \frac{y_1}{\|\vec{\mathbf{v}}\|}, \frac{z_1}{\|\vec{\mathbf{v}}\|} \rangle, \quad \text{if } \vec{\mathbf{v}} \neq \vec{\mathbf{0}}$$

We have seen that vector addition in two dimensions satisfies the commutative, associative, and additive inverse properties. These properties of vector operations are valid for three-dimensional vectors as well. Scalar multiplication of vectors satisfies the distributive property, and the zero vector acts as an additive identity. The proofs to verify these properties in three dimensions are straightforward extensions of the proofs in two dimensions.

Example 12.2.9: Vector Operations in Three Dimensions

Let $\vec{\mathbf{v}} = \langle -2, 9, 5 \rangle$ and $\vec{\mathbf{w}} = \langle 1, -1, 0 \rangle$ (Figure 12.2.20). Find the following vectors.

a. $3\vec{\mathbf{v}} - 2\vec{\mathbf{w}}$
b. $5\|\vec{\mathbf{w}}\|$
c. $\|5\vec{\mathbf{w}}\|$
d. A unit vector in the direction of $\vec{\mathbf{v}}$

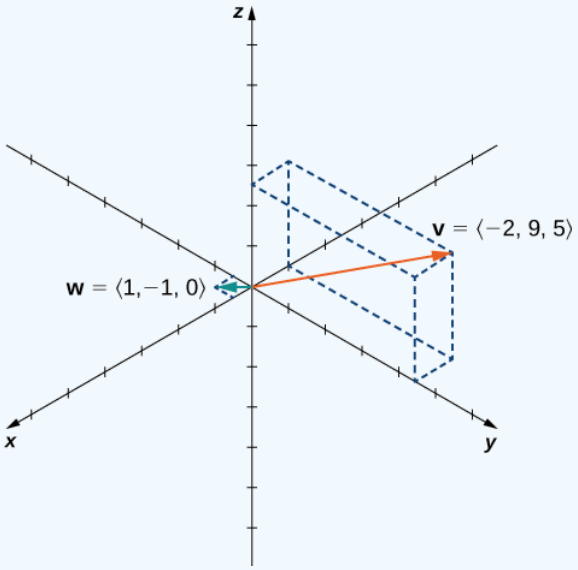


Figure 12.2.20: The vectors $\vec{\mathbf{v}} = \langle -2, 9, 5 \rangle$ and $\vec{\mathbf{w}} = \langle 1, -1, 0 \rangle$.

Solution

a. First, use scalar multiplication of each vector, then subtract:

$$\begin{aligned} 3\vec{\mathbf{v}} - 2\vec{\mathbf{w}} &= 3\langle -2, 9, 5\rangle - 2\langle 1, -1, 0\rangle \\ &= \langle -6, 27, 15\rangle - \langle 2, -2, 0\rangle \\ &= \langle -6-2, 27-(-2), 15-0\rangle \\ &= \langle -8, 29, 15\rangle. \end{aligned}$$

b. Write the equation for the magnitude of the vector, then use scalar multiplication:

$$5\|\vec{\mathbf{w}}\| = 5\sqrt{1^2 + (-1)^2 + 0^2} = 5\sqrt{2}.$$

c. First, use scalar multiplication, then find the magnitude of the new vector. Note that the result is the same as for part b.:

$$\|5\vec{\mathbf{w}}\| = \| \langle 5, -5, 0\rangle \| = \sqrt{5^2 + (-5)^2 + 0^2} = \sqrt{50} = 5\sqrt{2}$$

d. Recall that to find a unit vector in two dimensions, we divide a vector by its magnitude. The procedure is the same in three dimensions:

$$\begin{aligned} \frac{\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|} &= \frac{1}{\|\vec{\mathbf{v}}\|}\langle -2, 9, 5\rangle \\ &= \frac{1}{\sqrt{(-2)^2 + 9^2 + 5^2}}\langle -2, 9, 5\rangle \\ &= \frac{1}{\sqrt{110}}\langle -2, 9, 5\rangle \\ &= \left\langle \frac{-2}{\sqrt{110}}, \frac{9}{\sqrt{110}}, \frac{5}{\sqrt{110}}\right\rangle. \end{aligned}$$

? Exercise 12.2.9:

Let $\vec{\mathbf{v}} = \langle -1, -1, 1\rangle$ and $\vec{\mathbf{w}} = \langle 2, 0, 1\rangle$. Find a unit vector in the direction of $5\vec{\mathbf{v}} + 3\vec{\mathbf{w}}$.

Hint

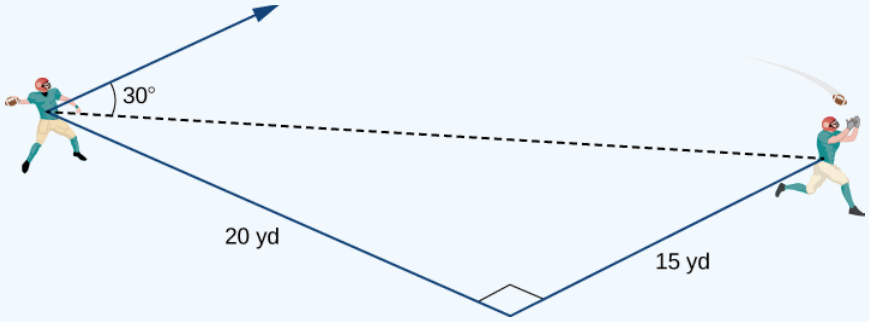
Start by writing $5\vec{\mathbf{v}} + 3\vec{\mathbf{w}}$ in component form.

Answer

$$\left\langle \frac{1}{3\sqrt{10}}, -\frac{5}{3\sqrt{10}}, \frac{8}{3\sqrt{10}}\right\rangle$$

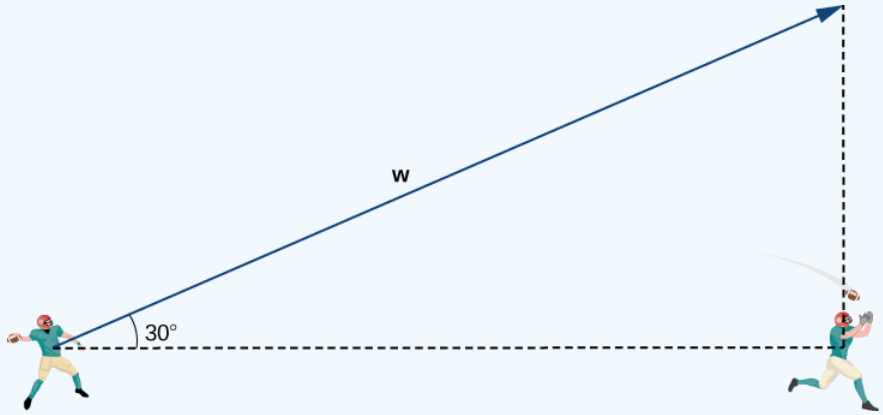
✓ Example 12.2.10: Throwing a Forward Pass

A quarterback is standing on the football field preparing to throw a pass. His receiver is standing 20 yd down the field and 15 yd to the quarterback's left. The quarterback throws the ball at a velocity of 60 mph toward the receiver at an upward angle of $30°$ (see the following figure). Write the initial velocity vector of the ball, $\vec{\mathbf{v}}$, in component form.



Solution

The first thing we want to do is find a vector in the same direction as the velocity vector of the ball. We then scale the vector appropriately so that it has the right magnitude. Consider the vector $\vec{w}$ extending from the quarterback's arm to a point directly above the receiver's head at an angle of $30°$ (see the following figure). This vector would have the same direction as $\vec{v}$, but it may not have the right magnitude.



The receiver is 20 yd down the field and 15 yd to the quarterback's left. Therefore, the straight-line distance from the quarterback to the receiver is

$$\textbf{Dist from QB to receiver} = \sqrt{15^2+20^2} = \sqrt{225+400} = \sqrt{625} = 25 \text{ yd}.$$

We have $\dfrac{25}{\|\vec{w}\|} = \cos 30°$. Then the magnitude of $\vec{w}$ is given by

$$\|\vec{w}\| = \frac{25}{\cos 30°} = \frac{25 \cdot 2}{\sqrt{3}} = \frac{50}{\sqrt{3}} \text{ yd}$$

and the vertical distance from the receiver to the terminal point of $\vec{w}$ is

$$\textbf{Vert dist from receiver to terminal point of } \vec{w} = \|\vec{w}\| \sin 30° = \frac{50}{\sqrt{3}} \cdot \frac{1}{2} = \frac{25}{\sqrt{3}} \text{ yd}.$$

Then $\vec{w} = \langle 20, 15, \dfrac{25}{\sqrt{3}} \rangle$, and has the same direction as $\vec{v}$.

Recall, though, that we calculated the magnitude of $\vec{w}$ to be $\|\vec{w}\| = \dfrac{50}{\sqrt{3}}$ yd, and $\vec{v}$ has magnitude 60 mph. So, we need to multiply vector $\vec{w}$ by an appropriate constant, k. We want to find a value of k so that $\| k\vec{w} \| = 60$ mph*. We have

$$\|k\vec{w}\| = k\|\vec{w}\| = k\frac{50}{\sqrt{3}} \text{ yd},$$

so we want

$$k\left(\frac{50}{\sqrt{3}} \text{ yd}\right) = 60 \text{ mph}$$

$$k = \frac{60\sqrt{3}}{50} \text{ mph / yd}$$

$$k = \frac{6\sqrt{3}}{5} \text{ mph / yd}.$$

Then

$$\vec{v} = k\vec{w} = k\langle 20, 15, \frac{25}{\sqrt{3}} \rangle = \frac{6\sqrt{3}}{5} \langle 20, 15, \frac{25}{\sqrt{3}} \rangle = \langle 24\sqrt{3}, 18\sqrt{3}, 30 \rangle.$$

Let's double-check that $\|\vec{v}\| = 60$ mph. We have

$$\|\vec{v}\| = \sqrt{(24\sqrt{3})^2 + (18\sqrt{3})^2 + (30)^2} = \sqrt{1728 + 972 + 900} = \sqrt{3600} = 60 \text{ mph}.$$

So, we have found the correct components for $\vec{v}$.

Note *

Readers who have been watching the units of measurement may be wondering what exactly is going on at this point: haven't we just mixed yards and miles per hour? We haven't, but the reason is subtle. One way to understand it is to realize that there are really two parallel coordinate systems in this problem: one gives positions down the field, across the field, and up into the air in units of yards; the other gives speeds down the field, across the field, and up into the air in units of miles per hour. The vector $\vec{\mathbf{w}}$ is calculated in the position coordinate system; vector $\vec{\mathbf{v}}$ will be in the speed system. Because corresponding axes in each system are parallel, directions in the two systems are also parallel, so the claim that $\vec{\mathbf{w}}$ and $\vec{\mathbf{v}}$ point in the same direction is correct. The constant k that we're looking for is a conversion factor between the magnitudes of these two vectors, converting from the position system to the speed one in the process. And as seen above, our calculation of k produces the right units for such a conversion, namely miles per hour per yard.

? Exercise 12.2.10

Assume the quarterback and the receiver are in the same place as in the previous example. This time, however, the quarterback throws the ball at velocity of 40 mph and an angle of $45°$. Write the initial velocity vector of the ball, $\vec{\mathbf{v}}$, in component form.

Hint

Follow the process used in the previous example.

Answer

$v = \langle 16\sqrt{2}, 12\sqrt{2}, 20\sqrt{2} \rangle$

Key Concepts

- The three-dimensional coordinate system is built around a set of three axes that intersect at right angles at a single point, the origin. Ordered triples (x, y, z) are used to describe the location of a point in space.
- The distance d between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

- In three dimensions, the equations $x = a,\ y = b$, and $z = c$ describe planes that are parallel to the coordinate planes.
- The standard equation of a sphere with center (a, b, c) and radius r is

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

- In three dimensions, as in two, vectors are commonly expressed in component form, $\vec{\mathbf{v}} = \langle x, y, z \rangle$, or in terms of the standard unit vectors, $\vec{\mathbf{v}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$.
- Properties of vectors in space are a natural extension of the properties for vectors in a plane. Let $\vec{\mathbf{v}} = \langle x_1, y_1, z_1 \rangle$ and $\vec{\mathbf{w}} = \langle x_2, y_2, z_2 \rangle$ be vectors, and let k be a scalar.

  Scalar multiplication:

$$k\vec{\mathbf{v}} = \langle kx_1, ky_1, kz_1 \rangle$$

  Vector addition:

$$\vec{\mathbf{v}} + \vec{\mathbf{w}} = \langle x_1, y_1, z_1 \rangle + \langle x_2, y_2, z_2 \rangle = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle$$

  Vector subtraction:

$$\vec{\mathbf{v}} - \vec{\mathbf{w}} = \langle x_1, y_1, z_1 \rangle - \langle x_2, y_2, z_2 \rangle = \langle x_1 - x_2, y_1 - y_2, z_1 - z_2 \rangle$$

  Vector magnitude:

$$\|\vec{\mathbf{v}}\| = \sqrt{x_1^2 + y_1^2 + z_1^2}$$

Unit vector in the direction of $\vec{\mathbf{v}}$:

$$\frac{\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|} = \frac{1}{\|\vec{\mathbf{v}}\|}\langle x_1, y_1, z_1\rangle = \langle \frac{x_1}{\|\vec{\mathbf{v}}\|}, \frac{y_1}{\|\vec{\mathbf{v}}\|}, \frac{z_1}{\|\vec{\mathbf{v}}\|}\rangle, \ \vec{\mathbf{v}} \neq \vec{\mathbf{0}}$$

Key Equations

Distance between two points in space:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Sphere with center (a, b, c) and radius r:

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

Glossary

coordinate plane
a plane containing two of the three coordinate axes in the three-dimensional coordinate system, named by the axes it contains: the xy-plane, xz-plane, or the yz-plane

right-hand rule
a common way to define the orientation of the three-dimensional coordinate system; when the right hand is curved around the z-axis in such a way that the fingers curl from the positive x-axis to the positive y-axis, the thumb points in the direction of the positive z-axis

octants
the eight regions of space created by the coordinate planes

sphere
the set of all points equidistant from a given point known as the *center*

standard equation of a sphere
$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$ describes a sphere with center (a, b, c) and radius r

three-dimensional rectangular coordinate system
a coordinate system defined by three lines that intersect at right angles; every point in space is described by an ordered triple (x, y, z) that plots its location relative to the defining axes

Contributors

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Example 12.2.10 has been modified by Doug Baldwin and Paul Seeburger to clarify the units of measurement that it uses and how it uses them.

Paul Seeburger also created dynamic versions of Figures 12.2.8, 12.2.9 and 12.2.13 using CalcPlot3D.

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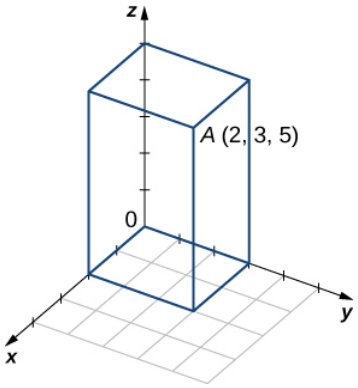
12.2E: Exercises for Section 12.2

1) Consider a rectangular box with one of the vertices at the origin, as shown in the following figure. If point $A(2,3,5)$ is the opposite vertex to the origin, then find

a. the coordinates of the other six vertices of the box and

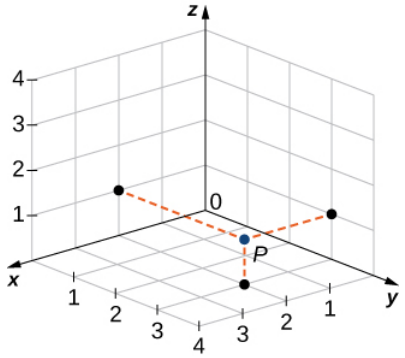
b. the length of the diagonal of the box determined by the vertices O and A.



Answer

a. $(2,0,5), (2,0,0), (2,3,0), (0,3,0), (0,3,5), (0,0,5)$ b. $\sqrt{38}$

2) Find the coordinates of point P and determine its distance to the origin.

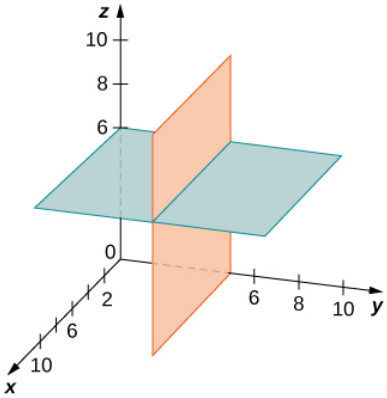


For exercises 3-6, describe and graph the set of points that satisfies the given equation.

3) $(y-5)(z-6)=0$

Answer

A union of two planes: $y=5$ (a plane parallel to the xz-plane) and $z=6$ (a plane parallel to the xy-plane)

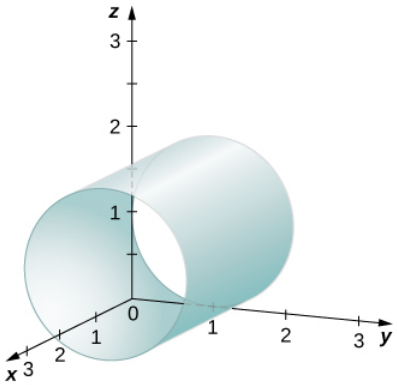


4) $(z-2)(z-5)=0$

5) $(y-1)^2+(z-1)^2=1$

Answer

A cylinder of radius 1 centered on the line $y=1, z=1$



6) $(x-2)^2+(z-5)^2=4$

7) Write the equation of the plane passing through point $(1,1,1)$ that is parallel to the xy-plane.

Answer

$z=1$

8) Write the equation of the plane passing through point $(1,-3,2)$ that is parallel to the xz-plane.

9) Find an equation of the plane passing through points $(1,-3,-2), (0,3,-2)$, and $(1,0,-2)$.

Answer

$z=-2$

10) Find an equation of the plane passing through points $(1,9,2), (1,3,6)$, and $(1,-7,8)$.

For exercises 11-14, find the equation of the sphere in standard form that satisfies the given conditions.

11) Center $C(-1,7,4)$ and radius 4

Answer

$(x+1)^2+(y-7)^2+(z-4)^2=16$

12) Center $C(-4,7,2)$ and radius 6

13) Diameter PQ, where $P(-1,5,7)$ and $Q(-5,2,9)$

Answer

$x+3)^2+(y-3.5)^2+(z-8)^2=\frac{29}{4}$

14) Diameter PQ, where $P(-16,-3,9)$ and $Q(-2,3,5)$

For exercises 15 and 16, find the center and radius of the sphere with an equation in general form that is given.

15) $x^2+y^2+z^2-4z+3=0$

Answer

Center $C(0,0,2)$ and radius 1

16) $x^2+y^2+z^2-6x+8y-10z+25=0$

For exercises 17-20, express vector $\overrightarrow{PQ}$ with the initial point at P and the terminal point at Q

a. in component form and

b. by using standard unit vectors.

17) $P(3,0,2)$ and $Q(-1,-1,4)$

Answer

$a. \overrightarrow{PQ} = \langle -4,-1,2\rangle$
$b. \overrightarrow{PQ} = -4\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

18) $P(0,10,5)$ and $Q(1,1,-3)$

19) $P(-2,5,-8)$ and $M(1,-7,4)$, where M is the midpoint of the line segment $\overline{PQ}$

Answer

$a. \overrightarrow{PQ} = \langle 6,-24,24\rangle$
$b. \overrightarrow{PQ} = 6\hat{\mathbf{i}} - 24\hat{\mathbf{j}} + 24\hat{\mathbf{k}}$

20) $Q(0,7,-6)$ and $M(-1,3,2)$, where M is the midpoint of the line segment $\overline{PQ}$

21) Find terminal point Q of vector $\overrightarrow{PQ} = \langle 7,-1,3\rangle$ with the initial point at $P(-2,3,5)$.

Answer

$Q(5,2,8)$

22) Find initial point P of vector $\overrightarrow{PQ} = \langle -9,1,2\rangle$ with the terminal point at $Q(10,0,-1)$.

For exercises 23-26, use the given vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ to find and express the vectors $\vec{\mathbf{a}} + \vec{\mathbf{b}}$, $4\vec{\mathbf{a}}$, and $-5\vec{\mathbf{a}} + 3\vec{\mathbf{b}}$ in component form.

23) $\vec{\mathbf{a}} = \langle -1,-2,4\rangle, \quad \vec{\mathbf{b}} = \langle -5,6,-7\rangle$

Answer

$\vec{\mathbf{a}} + \vec{\mathbf{b}} = \langle -6,4,-3\rangle, 4\vec{\mathbf{a}} = \langle -4,-8,16\rangle, -5\vec{\mathbf{a}} + 3\vec{\mathbf{b}} = \langle -10,28,-41\rangle$

24) $\vec{\mathbf{a}} = \langle 3,-2,4\rangle, \quad \vec{\mathbf{b}} = \langle -5,6,-9\rangle$

25) $\vec{\mathbf{a}} = -\hat{\mathbf{k}}, \quad \vec{\mathbf{b}} = -\hat{\mathbf{i}}$

Answer

$\vec{\mathbf{a}} + \vec{\mathbf{b}} = \langle -1,0,-1\rangle, 4\vec{\mathbf{a}} = \langle 0,0,-4\rangle, -5\vec{\mathbf{a}} + 3\vec{\mathbf{b}} = \langle -3,0,5\rangle$

26) $\vec{\mathbf{a}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}, \quad \vec{\mathbf{b}} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

For exercises 27-30, vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are given. Find the magnitudes of vectors $\vec{\mathbf{u}} - \vec{\mathbf{v}}$ and $-2\vec{\mathbf{u}}$.

27) $\vec{\mathbf{u}} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}, \quad \vec{\mathbf{v}} = -\hat{\mathbf{i}} + 5\hat{\mathbf{j}} - \hat{\mathbf{k}}$

Answer

$\|\vec{\mathbf{u}} - \vec{\mathbf{v}}\| = \sqrt{38}, \quad \|-2\vec{\mathbf{u}}\| = 2\sqrt{29}$

28) $\vec{\mathbf{u}} = \hat{\mathbf{i}} + \hat{\mathbf{j}}, \quad \vec{\mathbf{v}} = \hat{\mathbf{j}} - \hat{\mathbf{k}}$

29) $\vec{\mathbf{u}} = \langle 2\cos t, -2\sin t, 3\rangle, \quad \vec{\mathbf{v}} = \langle 0,0,3\rangle, \quad$ where t is a real number.

Answer

$\|\vec{\mathbf{u}} - \vec{\mathbf{v}}\| = 2, \quad \|-2\vec{\mathbf{u}}\| = 2\sqrt{13}$

30) $\vec{\mathbf{u}} = \langle 0,1,\sinh t\rangle, \quad \vec{\mathbf{v}} = \langle 1,1,0\rangle, \quad$ where t is a real number.

For exercises 31-36, find the unit vector in the direction of the given vector $\vec{a}$ and express it using standard unit vectors.

31) $\vec{a} = 3\hat{i} - 4\hat{j}$

Answer

$\frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}$

32) $\vec{a} = \langle 4, -3, 6 \rangle$

33) $\vec{a} = \overrightarrow{PQ}$, where $P(-2, 3, 1)$ and $Q(0, -4, 4)$

Answer

$\frac{\sqrt{62}}{31}\hat{i} - \frac{7\sqrt{62}}{62}\hat{j} + \frac{3\sqrt{62}}{62}\hat{k}$

34) $\vec{a} = \overrightarrow{OP}$, where $P(-1, -1, 1)$

35) $\vec{a} = \vec{u} - \vec{v} + \vec{w}$, where $\vec{u} = \hat{i} - \hat{j} - \hat{k}, \quad \vec{v} = 2\hat{i} - \hat{j} + \hat{k}, \quad$ and $\vec{w} = -\hat{i} + \hat{j} + 3\hat{k}$

Answer

$-\frac{\sqrt{6}}{3}\hat{i} + \frac{\sqrt{6}}{6}\hat{j} + \frac{\sqrt{6}}{6}\hat{k}$

36) $\vec{a} = 2\vec{u} + \vec{v} - \vec{w}, \quad$ where $\vec{u} = \hat{i} - \hat{k}, \quad \vec{v} = 2\hat{j} \quad$, and $\vec{w} = \hat{i} - \hat{j}$

37) Determine whether $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are equivalent vectors, where $A(1, 1, 1)$, $B(3, 3, 3)$, $P(1, 4, 5)$, and $Q(3, 6, 7)$.

Answer

Equivalent vectors

38) Determine whether the vectors $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are equivalent, where $A(1, 4, 1)$, $B(-2, 2, 0)$, $P(2, 5, 7)$, and $Q(-3, 2, 1)$.

For exercises 39-42, find vector $\vec{u}$ with a magnitude that is given and satisfies the given conditions.

39) $\vec{v} = \langle 7, -1, 3 \rangle$, $\|\vec{u}\| = 10$, and $\vec{u}$ and $\vec{v}$ have the same direction

Answer

$\vec{u} = \langle \frac{70\sqrt{59}}{59}, -\frac{10\sqrt{59}}{59}, \frac{30\sqrt{59}}{59} \rangle$

40) $\vec{v} = \langle 2, 4, 1 \rangle$, $\|\vec{u}\| = 15$, and $\vec{u}$ and $\vec{v}$ have the same direction

41) $\vec{v} = \langle 2\sin t, 2\cos t, 1 \rangle$, $\|\vec{u}\| = 2$, $\vec{u}$ and $\vec{v}$ have opposite directions for any t, where t is a real number

Answer

$\vec{u} = \langle -\frac{4\sqrt{5}}{5}\sin t, -\frac{4\sqrt{5}}{5}\cos t, -\frac{2\sqrt{5}}{5} \rangle$

42) $\vec{v} = \langle 3\sinh t, 0, 3 \rangle$, $\|\vec{u}\| = 5$, and $\vec{u}$ and $\vec{v}$ have opposite directions for any t, where t is a real number

43) Determine a vector of magnitude 5 in the direction of vector $\overrightarrow{AB}$, where $A(2, 1, 5)$ and $B(3, 4, -7)$.

Answer

$\langle \frac{5\sqrt{154}}{154}, \frac{15\sqrt{154}}{154}, -\frac{30\sqrt{154}}{77} \rangle$

44) Find a vector of magnitude 2 that points in the opposite direction than vector $\overrightarrow{AB}$, where $A(-1, -1, 1)$ and $B(0, 1, 1)$. Express the answer in component form.

45) Consider the points $A(2, \alpha, 0)$, $B(0, 1, \beta)$, and $C(1, 1, \beta)$, where α and β are negative real numbers. Find α and β such that $\|\overrightarrow{OA} - \overrightarrow{OB} + \overrightarrow{OC}\| = \|\overrightarrow{OB}\| = 4$.

Answer

$\alpha = -\sqrt{7}$, $\beta = -\sqrt{15}$

46) Consider points $A(\alpha, 0, 0)$, $B(0, \beta, 0)$, and $C(\alpha, \beta, \beta)$, where α and β are positive real numbers. Find α and β such that $\|\overrightarrow{OA} + \overrightarrow{OB}\| = \sqrt{2}$ and $\|\overrightarrow{OC}\| = \sqrt{3}$.

47) Let $P(x, y, z)$ be a point situated at an equal distance from points $A(1, -1, 0)$ and $B(-1, 2, 1)$. Show that point P lies on the plane of equation $-2x + 3y + z = 2$.

48) Let $P(x, y, z)$ be a point situated at an equal distance from the origin and point $A(4, 1, 2)$. Show that the coordinates of point P satisfy the equation $8x + 2y + 4z = 21$.

49) The points $A, B,$ and C are collinear (in this order) if the relation $\|\overrightarrow{AB}\| + \|\overrightarrow{BC}\| = \|\overrightarrow{AC}\|$ is satisfied. Show that $A(5, 3, -1)$, $B(-5, -3, 1)$, and $C(-15, -9, 3)$ are collinear points.

50) Show that points $A(1, 0, 1)$, $B(0, 1, 1)$, and $C(1, 1, 1)$ are not collinear.

51) [T] A force $\vec{\mathbf{F}}$ of $50\ N$ acts on a particle in the direction of the vector $\overrightarrow{OP}$, where $P(3, 4, 0)$.

a. Express the force as a vector in component form.

b. Find the angle between force $\vec{\mathbf{F}}$ and the positive direction of the x-axis. Express the answer in degrees rounded to the nearest integer.

Answer

$a. \vec{\mathbf{F}} = \langle 30, 40, 0 \rangle; \quad b. 53°$

52) [T] A force $\vec{\mathbf{F}}$ of $40\ N$ acts on a box in the direction of the vector $\overrightarrow{OP}$, where $P(1, 0, 2)$.

a. Express the force as a vector by using standard unit vectors.

b. Find the angle between force $\vec{\mathbf{F}}$ and the positive direction of the x-axis.

53) If $\vec{\mathbf{F}}$ is a force that moves an object from point $P_1(x_1, y_1, z_1)$ to another point $P_2(x_2, y_2, z_2)$, then the displacement vector is defined as $\vec{\mathbf{D}} = (x_2 - x_1)\hat{\mathbf{i}} + (y_2 - y_1)\hat{\mathbf{j}} + (z_2 - z_1)\hat{\mathbf{k}}$. A metal container is lifted 10 m vertically by a constant force $\vec{\mathbf{F}}$. Express the displacement vector $\vec{\mathbf{D}}$ by using standard unit vectors.

Answer

$\vec{\mathbf{D}} = 10\hat{\mathbf{k}}$

54) A box is pulled 4 yd horizontally in the x-direction by a constant force $\vec{\mathbf{F}}$. Find the displacement vector in component form.

55) The sum of the forces acting on an object is called the resultant or net force. An object is said to be in static equilibrium if the resultant force of the forces that act on it is zero. Let $\vec{\mathbf{F}}_1 = \langle 10, 6, 3 \rangle, \vec{\mathbf{F}}_2 = \langle 0, 4, 9 \rangle$, and $\vec{\mathbf{F}}_3 = \langle 10, -3, -9 \rangle$ be three forces acting on a box. Find the force $\vec{\mathbf{F}}_4$ acting on the box such that the box is in static equilibrium. Express the answer in component form.

Answer

$\vec{\mathbf{F}}_4 = \langle -20, -7, -3 \rangle$

56) [T] Let $\vec{\mathbf{F}}_k = \langle 1, k, k^2 \rangle, k = 1, ..., n$ be n forces acting on a particle, with $n \geq 2$.

a. Find the net force $\vec{\mathbf{F}} = \sum_{k=1}^{n} \vec{\mathbf{F}}_k$. Express the answer using standard unit vectors.

b. Use a computer algebra system (CAS) to find n such that $\|\vec{\mathbf{F}}\| < 100$.

57) The force of gravity $\vec{\mathbf{F}}$ acting on an object is given by $\vec{\mathbf{F}} = m\vec{\mathbf{g}}$, where m is the mass of the object (expressed in kilograms) and $\vec{\mathbf{g}}$ is acceleration resulting from gravity, with $\|\vec{\mathbf{g}}\| = 9.8\ N/kg$. A 2-kg disco ball hangs by a chain from the ceiling of a room.

a. Find the force of gravity $\vec{\mathbf{F}}$ acting on the disco ball and find its magnitude.

b. Find the force of tension $\vec{\mathbf{T}}$ in the chain and its magnitude.

Express the answers using standard unit vectors.

Figure 18: (credit: modification of work by Kenneth Lu, Flickr)

Answer

$a. \vec{\mathbf{F}} = -19.6\hat{\mathbf{k}}, \quad \|\vec{\mathbf{F}}\| = 19.6\ N$
$b. \vec{\mathbf{T}} = 19.6\hat{\mathbf{k}}, \quad \|\vec{\mathbf{T}}\| = 19.6\ N$

58) A 5-kg pendant chandelier is designed such that the alabaster bowl is held by four chains of equal length, as shown in the following figure.

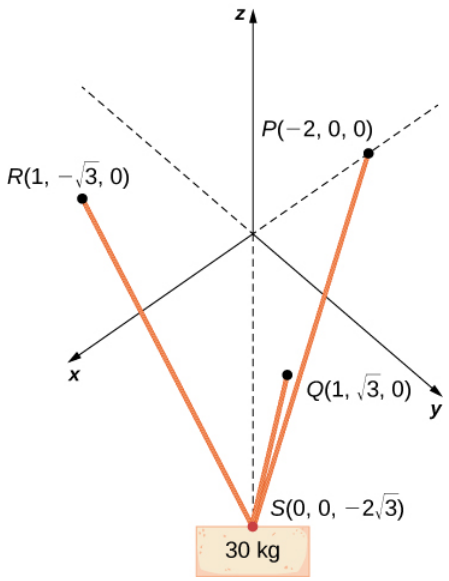
a. Find the magnitude of the force of gravity acting on the chandelier.

b. Find the magnitudes of the forces of tension for each of the four chains (assume chains are essentially vertical).

59) [T] A 30-kg block of cement is suspended by three cables of equal length that are anchored at points $P(-2,0,0), Q(1,\sqrt{3},0)$, and $R(1,-\sqrt{3},0)$. The load is located at $S(0,0,-2\sqrt{3})$, as shown in the following figure. Let $\vec{\mathbf{F}}_1, \vec{\mathbf{F}}_2$, and $\vec{\mathbf{F}}_3$ be the forces of tension resulting from the load in cables RS, QS, and PS, respectively.

a. Find the gravitational force $\vec{\mathbf{F}}$ acting on the block of cement that counterbalances the sum $\vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \vec{\mathbf{F}}_3$ of the forces of tension in the cables.

b. Find forces $\vec{\mathbf{F}}_1, \vec{\mathbf{F}}_2$, and $\vec{\mathbf{F}}_3$. Express the answer in component form.

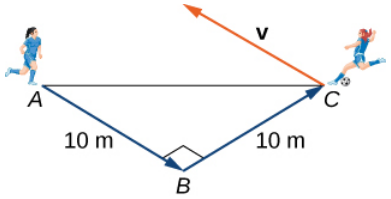


Answer

a. $\vec{\mathbf{F}} = -294\hat{\mathbf{k}}$ N;

b. $\vec{\mathbf{F}}_1 = \langle -\frac{49\sqrt{3}}{3}, 49, -98\rangle, \vec{\mathbf{F}}_2 = \langle -\frac{49\sqrt{3}}{3}, -49, -98\rangle$, and $\vec{\mathbf{F}}_3 = \langle \frac{98\sqrt{3}}{3}, 0, -98\rangle$ (each component is expressed in newtons)

60) Two soccer players are practicing for an upcoming game. One of them runs 10 m from point A to point B. She then turns left at $90°$ and runs 10 m until she reaches point C. Then she kicks the ball with a speed of 10 m/sec at an upward angle of $45°$ to her teammate, who is located at point A. Write the velocity of the ball in component form.



61) Let $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t)\rangle$ be the position vector of a particle at the time $t \in [0, T]$, where $x, y,$ and z are smooth functions on $[0, T]$. The instantaneous velocity of the particle at time t is defined by vector $\vec{\mathbf{v}}(t) = \langle x'(t), y'(t), z'(t)\rangle$, with components that are the derivatives with respect to t, of the functions $x, y,$ and z, respectively. The magnitude $\| \vec{\mathbf{v}}(t) \|$ of the instantaneous velocity vector is called the speed of the particle at time t. Vector $\vec{\mathbf{a}}(t) = \langle x''(t), y''(t), z''(t)\rangle$, with components that are the second derivatives with respect to t, of the functions $x, y,$ and z, respectively, gives the acceleration of the particle at time t. Consider $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t, 2t\rangle$ the position vector of a particle at time $t \in [0, 30]$, where the components of $\vec{\mathbf{r}}$ are expressed in centimeters and time is expressed in seconds.

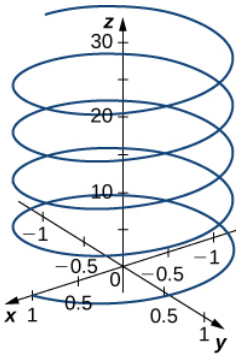
a. Find the instantaneous velocity, speed, and acceleration of the particle after the first second. Round your answer to two decimal places.

b. Use a CAS to visualize the path of the particle—that is, the set of all points of coordinates $(\cos t, \sin t, 2t)$, where $t \in [0, 30]$.

Answer

$a. \vec{\mathbf{v}}(1) = \langle -0.84, 0.54, 2\rangle$ (each component is expressed in centimeters per second); $\| \vec{\mathbf{v}}(1) \| = 2.24$ (expressed in centimeters per second); $\vec{\mathbf{a}}(1) = \langle -0.54, -0.84, 0\rangle$ (each component expressed in centimeters per second squared);

$b.$



62) [T] Let $\vec{\mathbf{r}}(t) = \langle t, 2t^2, 4t^2 \rangle$ be the position vector of a particle at time t (in seconds), where $t \in [0, 10]$ (here the components of $\vec{\mathbf{r}}$ are expressed in centimeters).

a. Find the instantaneous velocity, speed, and acceleration of the particle after the first two seconds. Round your answer to two decimal places.

b. Use a CAS to visualize the path of the particle defined by the points $(t,\ 2t^2,\ 4t^2)$, where $t \in [0,\ 60]$.

Contributors

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12.3: The Dot Product

Learning Objectives

- Calculate the dot product of two given vectors.
- Determine whether two given vectors are perpendicular.
- Find the direction cosines of a given vector.
- Explain what is meant by the vector projection of one vector onto another vector, and describe how to compute it.
- Calculate the work done by a given force.

If we apply a force to an object so that the object moves, we say that work is done by the force. Previously, we looked at a constant force and we assumed the force was applied in the direction of motion of the object. Under those conditions, work can be expressed as the product of the force acting on an object and the distance the object moves. In this chapter, however, we have seen that both force and the motion of an object can be represented by vectors.

In this section, we develop an operation called the dot product, which allows us to calculate work in the case when the force vector and the motion vector have different directions. The dot product essentially tells us how much of the force vector is applied in the direction of the motion vector. The dot product can also help us measure the angle formed by a pair of vectors and the position of a vector relative to the coordinate axes. It even provides a simple test to determine whether two vectors meet at a right angle.

The Dot Product and Its Properties

We have already learned how to add and subtract vectors. In this chapter, we investigate two types of vector multiplication. The first type of vector multiplication is called the dot product, based on the notation we use for it, and it is defined as follows:

Definition: dot product

The *dot product* of vectors $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ is given by the sum of the products of the components

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Note that if u and v are two-dimensional vectors, we calculate the dot product in a similar fashion. Thus, if $\vec{\mathbf{u}} = \langle u_1, u_2 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2 \rangle$, then

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2.$$

When two vectors are combined under addition or subtraction, the result is a vector. When two vectors are combined using the dot product, the result is a scalar. For this reason, the dot product is often called the *scalar product.* It may also be called the *inner product.*

Example 12.3.1: Calculating Dot Products

a. Find the dot product of $\vec{\mathbf{u}} = \langle 3, 5, 2 \rangle$ and $\vec{\mathbf{v}} = \langle -1, 3, 0 \rangle$.
b. Find the scalar product of $\vec{\mathbf{p}} = 10\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$ and $\vec{\mathbf{q}} = -2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 6\hat{\mathbf{k}}$.

Solution:

a. Substitute the vector components into the formula for the dot product:

$$\begin{aligned} \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} &= u_1 v_1 + u_2 v_2 + u_3 v_3 \\ &= 3(-1) + 5(3) + 2(0) \\ &= -3 + 15 + 0 \\ &= 12. \end{aligned}$$

b. The calculation is the same if the vectors are written using standard unit vectors. We still have three components for each vector to substitute into the formula for the dot product:

$$\begin{aligned}\vec{\mathbf{p}} \cdot \vec{\mathbf{q}} &= p_1q_1 + p_2q_2 + p_3q_3 \\ &= 10(-2) + (-4)(1) + (7)(6) \\ &= -20 - 4 + 42 \\ &= 18.\end{aligned}$$

Exercise 12.3.1

Find $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$, where $\vec{\mathbf{u}} = \langle 2, 9, -1\rangle$ and $\vec{\mathbf{v}} = \langle -3, 1, -4\rangle$.

Hint

Multiply corresponding components and then add their products.

Answer

7

Like vector addition and subtraction, the dot product has several algebraic properties. We prove three of these properties and leave the rest as exercises.

Properties of the Dot Product

Let $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ be vectors, and let c be a scalar.

i. *Commutative property*

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \vec{\mathbf{v}} \cdot \vec{\mathbf{u}}$$

ii. *Distributive property*

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} + \vec{\mathbf{u}} \cdot \vec{\mathbf{w}}$$

iii. *Associative property*

$$c(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}) = (c\vec{\mathbf{u}}) \cdot \vec{\mathbf{v}} = \vec{\mathbf{u}} \cdot (c\vec{\mathbf{v}})$$

iv. *Property of magnitude*

$$\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = \|\vec{\mathbf{v}}\|^2$$

Proof

Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3\rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3\rangle$. Then

$$\begin{aligned}\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} &= \langle u_1, u_2, u_3\rangle \cdot \langle v_1, v_2, v_3\rangle \\ &= u_1v_1 + u_2v_2 + u_3v_3 \\ &= v_1u_1 + v_2u_2 + v_3u_3 \\ &= \langle v_1, v_2, v_3\rangle \cdot \langle u_1, u_2, u_3\rangle \\ &= \vec{\mathbf{v}} \cdot \vec{\mathbf{u}}.\end{aligned}$$

The associative property looks like the associative property for real-number multiplication, but pay close attention to the difference between scalar and vector objects:

$$\begin{aligned} c(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}) &= c(u_1v_1 + u_2v_2 + u_3v_3) \\ &= c(u_1v_1) + c(u_2v_2) + c(u_3v_3) \\ &= (cu_1)v_1 + (cu_2)v_2 + (cu_3)v_3 \\ &= \langle cu_1, cu_2, cu_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle \\ &= c\langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle \\ &= (c\vec{\mathbf{u}}) \cdot \vec{\mathbf{v}}. \end{aligned}$$

The proof that $c(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}) = \vec{\mathbf{u}} \cdot (c\vec{\mathbf{v}})$ is similar.

The fourth property shows the relationship between the magnitude of a vector and its dot product with itself:

$$\begin{aligned} \vec{\mathbf{v}} \cdot \vec{\mathbf{v}} &= \langle v_1, v_2, v_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle \\ &= (v_1)^2 + (v_2)^2 + (v_3)^2 \\ &= \left[\sqrt{(v_1)^2 + (v_2)^2 + (v_3)^2}\right]^2 \\ &= \|\vec{\mathbf{v}}\|^2. \end{aligned}$$

□

Note that the definition of the dot product yields $\vec{\mathbf{0}} \cdot \vec{\mathbf{v}} = 0$. By property iv. if $\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = 0$, then $\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

Example 12.3.2: Using Properties of the Dot Product

Let $\vec{\mathbf{a}} = \langle 1, 2, -3 \rangle$, $\vec{\mathbf{b}} = \langle 0, 2, 4 \rangle$, and $\vec{\mathbf{c}} = \langle 5, -1, 3 \rangle$.

Find each of the following products.

a. $(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})\vec{\mathbf{c}}$
b. $\vec{\mathbf{a}} \cdot (2\vec{\mathbf{c}})$
c. $\|\vec{\mathbf{b}}\|^2$

Solution

a. Note that this expression asks for the scalar multiple of $\vec{\mathbf{c}}$ by $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$:

$$\begin{aligned} (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})\vec{\mathbf{c}} &= (\langle 1, 2, -3 \rangle \cdot \langle 0, 2, 4 \rangle)\langle 5, -1, 3 \rangle \\ &= (1(0) + 2(2) + (-3)(4))\langle 5, -1, 3 \rangle \\ &= -8\langle 5, -1, 3 \rangle \\ &= \langle -40, 8, -24 \rangle. \end{aligned}$$

b. This expression is a dot product of vector $\vec{\mathbf{a}}$ and scalar multiple $2\vec{\mathbf{c}}$:

$$\begin{aligned} \vec{\mathbf{a}} \cdot (2\vec{\mathbf{c}}) &= 2(\vec{\mathbf{a}} \cdot \vec{\mathbf{c}}) \\ &= 2(\langle 1, 2, -3 \rangle \cdot \langle 5, -1, 3 \rangle) \\ &= 2(1(5) + 2(-1) + (-3)(3)) \\ &= 2(-6) = -12. \end{aligned}$$

c. Simplifying this expression is a straightforward application of the dot product:

$$\begin{aligned} \|\vec{\mathbf{b}}\|^2 &= \vec{\mathbf{b}} \cdot \vec{\mathbf{b}} \\ &= \langle 0, 2, 4 \rangle \cdot \langle 0, 2, 4 \rangle \\ &= 0^2 + 2^2 + 4^2 \\ &= 0 + 4 + 16 \\ &= 20. \end{aligned}$$

Exercise 12.3.2

Find the following products for $\vec{\mathbf{p}} = \langle 7, 0, 2\rangle$, $\vec{\mathbf{q}} = \langle -2, 2, -2\rangle$, and $\vec{\mathbf{r}} = \langle 0, 2, -3\rangle$.

a. $(\vec{\mathbf{r}} \cdot \vec{\mathbf{p}})\vec{\mathbf{q}}$
b. $\|\vec{\mathbf{p}}\|^2$

Hint

$\vec{\mathbf{r}} \cdot \vec{\mathbf{p}}$ is a scalar.

Answer

a. $(\vec{\mathbf{r}} \cdot \vec{\mathbf{p}})\vec{\mathbf{q}} = \langle 12, -12, 12\rangle$; *b.* $\|\vec{\mathbf{p}}\|^2 = 53$

Using the Dot Product to Find the Angle between Two Vectors

When two nonzero vectors are placed in standard position, whether in two dimensions or three dimensions, they form an angle between them (Figure 12.3.1). The dot product provides a way to find the measure of this angle. This property is a result of the fact that we can express the dot product in terms of the cosine of the angle formed by two vectors.

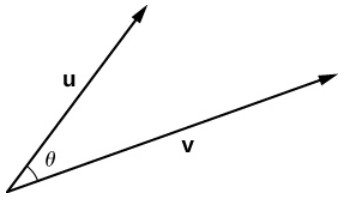


Figure 12.3.1: Let θ be the angle between two nonzero vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ such that $0 \leq \theta \leq \pi$.

Evaluating a Dot Product

The dot product of two vectors is the product of the magnitude of each vector and the cosine of the angle between them:

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\| \cos\theta. \tag{12.3.1}$$

Proof

Place vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ in standard position and consider the vector $\vec{\mathbf{v}} - \vec{\mathbf{u}}$ (Figure 12.3.2). These three vectors form a triangle with side lengths $\|\vec{\mathbf{u}}\|$, $\|\vec{\mathbf{v}}\|$, and $\|\vec{\mathbf{v}} - \vec{\mathbf{u}}\|$.

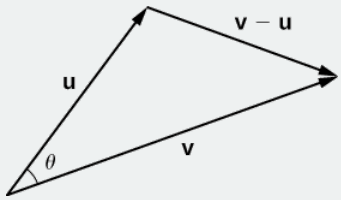


Figure 12.3.2: The lengths of the sides of the triangle are given by the magnitudes of the vectors that form the triangle.

Recall from trigonometry that the law of cosines describes the relationship among the side lengths of the triangle and the angle θ. Applying the law of cosines here gives

$$\|\vec{\mathbf{v}} - \vec{\mathbf{u}}\|^2 = \|\vec{\mathbf{u}}\|^2 + \|\vec{\mathbf{v}}\|^2 - 2\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\| \cos\theta. \tag{12.3.2}$$

The dot product provides a way to rewrite the left side of Equation 12.3.2:

$$\begin{aligned}
\|\vec{\mathbf{v}} - \vec{\mathbf{u}}\|^2 &= (\vec{\mathbf{v}} - \vec{\mathbf{u}}) \cdot (\vec{\mathbf{v}} - \vec{\mathbf{u}}) \\
&= (\vec{\mathbf{v}} - \vec{\mathbf{u}}) \cdot \vec{\mathbf{v}} - (\vec{\mathbf{v}} - \vec{\mathbf{u}}) \cdot \vec{\mathbf{u}} \\
&= \vec{\mathbf{v}} \cdot \vec{\mathbf{v}} - \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} - \vec{\mathbf{v}} \cdot \vec{\mathbf{u}} + \vec{\mathbf{u}} \cdot \vec{\mathbf{u}} \\
&= \vec{\mathbf{v}} \cdot \vec{\mathbf{v}} - \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} - \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} + \vec{\mathbf{u}} \cdot \vec{\mathbf{u}} \\
&= \|\vec{\mathbf{v}}\|^2 - 2\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} + \|\vec{\mathbf{u}}\|^2.
\end{aligned}$$

Substituting into the law of cosines yields

$$\begin{aligned}\|\vec{\mathbf{v}}-\vec{\mathbf{u}}\|^2 &= \|\vec{\mathbf{u}}\|^2+\|\vec{\mathbf{v}}\|^2-2\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|\cos\theta\\ \|\vec{\mathbf{v}}\|^2-2\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}+\|\vec{\mathbf{u}}\|^2 &= \|\vec{\mathbf{u}}\|^2+\|\vec{\mathbf{v}}\|^2-2\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|\cos\theta\\ -2\vec{\mathbf{u}}\cdot\vec{\mathbf{v}} &= -2\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|\cos\theta\\ \vec{\mathbf{u}}\cdot\vec{\mathbf{v}} &= \|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|\cos\theta.\end{aligned}$$

□

We can use the form of the dot product in Equation 12.3.1 to find the measure of the angle between two nonzero vectors by rearranging Equation 12.3.1 to solve for the cosine of the angle:

$$\cos\theta=\frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|}. \tag{12.3.3}$$

Using this equation, we can find the cosine of the angle between two nonzero vectors. Since we are considering the smallest angle between the vectors, we assume $0°\leq\theta\leq 180°$ (or $0\leq\theta\leq\pi$ if we are working in radians). The inverse cosine is unique over this range, so we are then able to determine the measure of the angle θ.

Example 12.3.3: Finding the Angle between Two Vectors

Find the measure of the angle between each pair of vectors.

a. $\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $2\hat{\mathbf{i}}–\hat{\mathbf{j}}–3\hat{\mathbf{k}}$

b. $\langle 2,5,6\rangle$ and $\langle -2,-4,4\rangle$

Solution

a. To find the cosine of the angle formed by the two vectors, substitute the components of the vectors into Equation 12.3.3:

$$\begin{aligned}\cos\theta &= \frac{(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})\cdot(2\hat{\mathbf{i}}-\hat{\mathbf{j}}-3\hat{\mathbf{k}})}{\|\,\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}\,\|\cdot\|\,2\hat{\mathbf{i}}-\hat{\mathbf{j}}-3\hat{\mathbf{k}}\,\|}\\ &= \frac{1(2)+(1)(-1)+(1)(-3)}{\sqrt{1^2+1^2+1^2}\sqrt{2^2+(-1)^2+(-3)^2}}\\ &= \frac{-2}{\sqrt{3}\sqrt{14}}=\frac{-2}{\sqrt{42}}.\end{aligned}$$

Therefore, $\theta=\arccos\dfrac{-2}{\sqrt{42}}$ rad.

b. Start by finding the value of the cosine of the angle between the vectors:

$$\begin{aligned}\cos\theta &= \frac{\langle 2,5,6\rangle\cdot\langle -2,-4,4\rangle}{\|\,\langle 2,5,6\rangle\,\|\cdot\|\,\langle -2,-4,4\rangle\,\|}\\ &= \frac{2(-2)+(5)(-4)+(6)(4)}{\sqrt{2^2+5^2+6^2}\sqrt{(-2)^2+(-4)^2+4^2}}\\ &= \frac{0}{\sqrt{65}\sqrt{36}}=0.\end{aligned}$$

Now, $\cos\theta=0$ and $0\leq\theta\leq\pi$, so $\theta=\pi/2$.

Exercise 12.3.3

Find the measure of the angle, in radians, formed by vectors $\vec{\mathbf{a}}=\langle 1,2,0\rangle$ and $\vec{\mathbf{b}}=\langle 2,4,1\rangle$. Round to the nearest hundredth.

Hint

Use the Equation 12.3.3.

Answer

$\theta \approx 0.22$ rad

The angle between two vectors can be acute $(0 < \cos\theta < 1)$, obtuse $(-1 < \cos\theta < 0)$, or straight $(\cos\theta = -1)$. If $\cos\theta = 1$, then both vectors have the same direction. If $\cos\theta = 0$, then the vectors, when placed in standard position, form a right angle (Figure 12.3.3). We can formalize this result into a theorem regarding orthogonal (perpendicular) vectors.

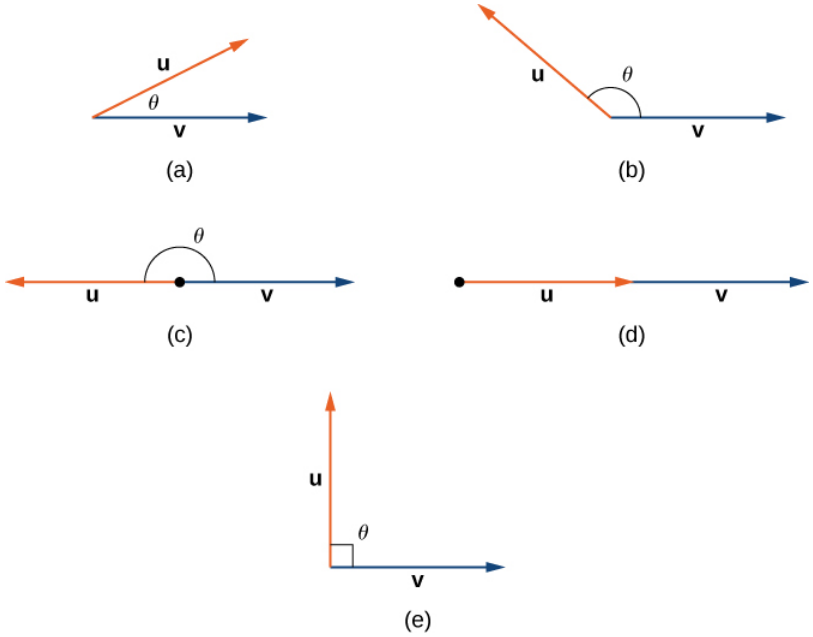


Figure 12.3.3: (a) An acute angle has $0 < \cos\theta < 1$. (b) An obtuse angle has $-1 < \cos\theta < 0$. (c) A straight line has $\cos\theta = -1$. (d) If the vectors have the same direction, $\cos\theta = 1$. (e) If the vectors are orthogonal (perpendicular), $\cos\theta = 0$.

Orthogonal Vectors

The nonzero vectors $\vec{u}$ and $\vec{v}$ are **orthogonal vectors** if and only if $\vec{u} \cdot \vec{v} = 0$.

Proof

Let $\vec{u}$ and $\vec{v}$ be nonzero vectors, and let θ denote the angle between them. First, assume $\vec{u} \cdot \vec{v} = 0$.Then

$$\|\vec{u}\|\|\vec{v}\| \cos\theta = 0.$$

However, $\|\vec{u}\| \neq 0$ and $\|\vec{v}\| \neq 0$, so we must have $\cos\theta = 0$. Hence, $\theta = 90°$, and the vectors are orthogonal.

Now assume $\vec{u}$ and $\vec{v}$ are orthogonal. Then $\theta = 90°$ and we have

$$\begin{aligned} \vec{u} \cdot \vec{v} &= \|\vec{u}\|\|\vec{v}\| \cos\theta \\ &= \|\vec{u}\|\|\vec{v}\| \cos 90° \\ &= \|\vec{u}\|\|\vec{v}\|(0) \\ &= 0. \end{aligned}$$

□

The terms **orthogonal, perpendicular,** and **normal** each indicate that mathematical objects are intersecting at right angles. The use of each term is determined mainly by its context. We say that vectors are orthogonal and lines are perpendicular. The term *normal* is used most often when measuring the angle made with a plane or other surface.

Example 12.3.4: Identifying Orthogonal Vectors

Determine whether $\vec{p} = \langle 1, 0, 5 \rangle$ and $\vec{q} = \langle 10, 3, -2 \rangle$ are orthogonal vectors.

Solution

Using the definition, we need only check the dot product of the vectors:

$$\vec{p} \cdot \vec{q} = 1(10) + (0)(3) + (5)(-2) = 10 + 0 - 10 = 0.$$

Because $\vec{p} \cdot \vec{q} = 0$, the vectors are orthogonal (Figure 12.3.4).

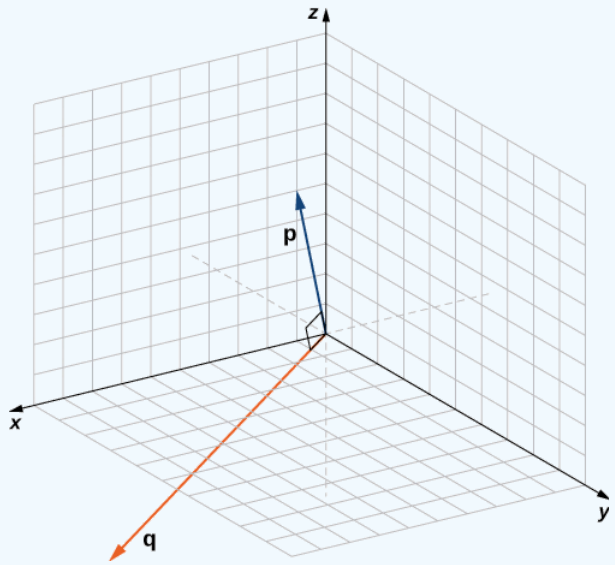


Figure 12.3.4: Vectors $\vec{p}$ and $\vec{q}$ form a right angle when their initial points are aligned.

Exercise 12.3.4

For which value of x is $\vec{p} = \langle 2, 8, -1 \rangle$ orthogonal to $\vec{q} = \langle x, -1, 2 \rangle$?

Hint

Vectors $\vec{p}$ and $\vec{q}$ are orthogonal if and only if $\vec{p} \cdot \vec{q} = 0$.

Answer

$x = 5$

Example 12.3.5: Measuring the Angle Formed by Two Vectors

Let $\vec{v} = \langle 2, 3, 3 \rangle$. Find the measures of the angles formed by the following vectors.

a. $\vec{v}$ and $\hat{i}$
b. $\vec{v}$ and $\hat{j}$
c. $\vec{v}$ and $\hat{k}$

Solution

a.Let α be the angle formed by $\vec{v}$ and $\hat{i}$:

$$\cos\alpha = \frac{\vec{v} \cdot \hat{i}}{\|\vec{v}\| \cdot \|\hat{i}\|} = \frac{\langle 2, 3, 3 \rangle \cdot \langle 1, 0, 0 \rangle}{\sqrt{2^2 + 3^2 + 3^2}\sqrt{1}} = \frac{2}{\sqrt{22}}$$

$$\alpha = \arccos \frac{2}{\sqrt{22}} \approx 1.130 \, \text{rad}.$$

b. Let β represent the angle formed by $\vec{v}$ and $\hat{j}$:

$$\cos\beta = \frac{\vec{v}\cdot\hat{j}}{\|\vec{v}\|\cdot\|\hat{j}\|} = \frac{\langle 2,3,3\rangle\cdot\langle 0,1,0\rangle}{\sqrt{2^2+3^2+3^2}\sqrt{1}} = \frac{3}{\sqrt{22}}$$

$$\beta = \arccos\frac{3}{\sqrt{22}} \approx 0.877\,\text{rad}.$$

c. Let γ represent the angle formed by $\vec{v}$ and $\hat{k}$:

$$\cos\gamma = \frac{\vec{v}\cdot\hat{k}}{\|\vec{v}\|\cdot\|\hat{k}\|} = \frac{\langle 2,3,3\rangle\cdot\langle 0,0,1\rangle}{\sqrt{2^2+3^2+3^2}\sqrt{1}} = \frac{3}{\sqrt{22}}$$

$$\gamma = \arccos\frac{3}{\sqrt{22}} \approx 0.877\,\text{rad}.$$

Exercise 12.3.5

Let $\vec{v} = \langle 3,-5,1\rangle$. Find the measure of the angles formed by each pair of vectors.

a. $\vec{v}$ and $\hat{i}$
b. $\vec{v}$ and $\hat{j}$
c. $\vec{v}$ and $\hat{k}$

Hint

$\hat{i} = \langle 1,0,0\rangle, \hat{j} = \langle 0,1,0\rangle$, and $\hat{k} = \langle 0,0,1\rangle$

Answer

$a.\alpha \approx 1.04$ rad; b. $\beta \approx 2.58$ rad; c. $\gamma \approx 1.40$ rad

The angle a vector makes with each of the coordinate axes, called a direction angle, is very important in practical computations, especially in a field such as engineering. For example, in astronautical engineering, the angle at which a rocket is launched must be determined very precisely. A very small error in the angle can lead to the rocket going hundreds of miles off course. Direction angles are often calculated by using the dot product and the cosines of the angles, called the direction cosines. Therefore, we define both these angles and their cosines.

Definition: direction angles

The angles formed by a nonzero vector and the coordinate axes are called the **direction angles** for the vector (Figure 12.3.5). The cosines for these angles are called the **direction cosines.**

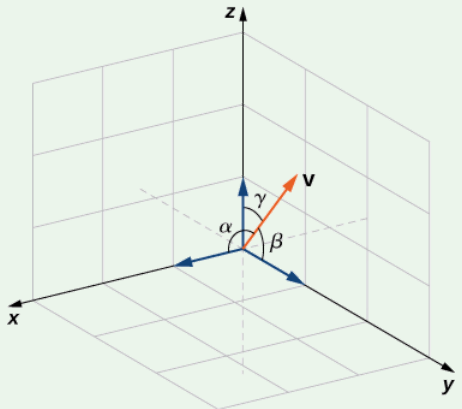


Figure 12.3.5: Angle α is formed by vector $\vec{v}$ and unit vector $\hat{i}$. Angle β is formed by vector $\vec{v}$ and unit vector $\hat{j}$. Angle γ is formed by vector $\vec{v}$ and unit vector $\hat{k}$.

In Example 12.3.5, the direction cosines of $\vec{\mathbf{v}} = \langle 2, 3, 3 \rangle$ are $\cos\alpha = \dfrac{2}{\sqrt{22}}, \cos\beta = \dfrac{3}{\sqrt{22}}$, and $\cos\gamma = \dfrac{3}{\sqrt{22}}$. The direction angles of $\vec{\mathbf{v}}$ are $\alpha = 1.130$ rad, $\beta = 0.877$ rad, and $\gamma = 0.877$ rad.

So far, we have focused mainly on vectors related to force, movement, and position in three-dimensional physical space. However, vectors are often used in more abstract ways. For example, suppose a fruit vendor sells apples, bananas, and oranges. On a given day, he sells 30 apples, 12 bananas, and 18 oranges. He might use a quantity vector, $\vec{\mathbf{q}} = \langle 30, 12, 18 \rangle$, to represent the quantity of fruit he sold that day. Similarly, he might want to use a price vector, $\vec{\mathbf{p}} = \langle 0.50, 0.25, 1 \rangle$, to indicate that he sells his apples for 50¢ each, bananas for 25¢ each, and oranges for $1 apiece. In this example, although we could still graph these vectors, we do not interpret them as literal representations of position in the physical world. We are simply using vectors to keep track of particular pieces of information about apples, bananas, and oranges.

This idea might seem a little strange, but if we simply regard vectors as a way to order and store data, we find they can be quite a powerful tool. Going back to the fruit vendor, let's think about the dot product, $\vec{\mathbf{q}} \cdot \vec{\mathbf{p}}$. We compute it by multiplying the number of apples sold (30) by the price per apple (50¢), the number of bananas sold by the price per banana, and the number of oranges sold by the price per orange. We then add all these values together. So, in this example, the dot product tells us how much money the fruit vendor had in sales on that particular day.

When we use vectors in this more general way, there is no reason to limit the number of components to three. What if the fruit vendor decides to start selling grapefruit? In that case, he would want to use four-dimensional quantity and price vectors to represent the number of apples, bananas, oranges, and grapefruit sold, and their unit prices. As you might expect, to calculate the dot product of four-dimensional vectors, we simply add the products of the components as before, but the sum has four terms instead of three.

Example 12.3.6: Using Vectors in an Economic Context

AAA Party Supply Store sells invitations, party favors, decorations, and food service items such as paper plates and napkins. When AAA buys its inventory, it pays 25¢ per package for invitations and party favors. Decorations cost AAA 50¢ each, and food service items cost 20¢ per package. AAA sells invitations for $2.50 per package and party favors for $1.50 per package. Decorations sell for $4.50 each and food service items for $1.25 per package.

During the month of May, AAA Party Supply Store sells 1258 invitations, 342 party favors, 2426 decorations, and 1354 food service items. Use vectors and dot products to calculate how much money AAA made in sales during the month of May. How much did the store make in profit?

Solution

The cost, price, and quantity vectors are

$$\begin{aligned}\vec{\mathbf{c}} &= \langle 0.25, 0.25, 0.50, 0.20 \rangle \\ \vec{\mathbf{p}} &= \langle 2.50, 1.50, 4.50, 1.25 \rangle \\ \vec{\mathbf{q}} &= \langle 1258, 342, 2426, 1354 \rangle.\end{aligned}$$

AAA sales for the month of May can be calculated using the dot product $\vec{\mathbf{p}} \cdot \vec{\mathbf{q}}$. We have

$$\begin{aligned}\vec{\mathbf{p}} \cdot \vec{\mathbf{q}} &= \langle 2.50, 1.50, 4.50, 1.25 \rangle \cdot \langle 1258, 342, 2426, 1354 \rangle \\ &= 3145 + 513 + 10917 + 1692.5 \\ &= 16267.5.\end{aligned}$$

So, AAA took in $16,267.50 during the month of May. To calculate the profit, we must first calculate how much AAA paid for the items sold. We use the dot product $\vec{\mathbf{c}} \cdot \vec{\mathbf{q}}$ to get

$$\begin{aligned}\vec{\mathbf{c}} \cdot \vec{\mathbf{q}} &= \langle 0.25, 0.25, 0.50, 0.20 \rangle \cdot \langle 1258, 342, 2426, 1354 \rangle \\ &= 314.5 + 85.5 + 1213 + 270.8 \\ &= 1883.8.\end{aligned}$$

So, AAA paid $1,883.80 for the items they sold. Their profit, then, is given by

$$\vec{p} \cdot \vec{q} - \vec{c} \cdot \vec{q} = 16267.5 - 1883.8 = 14383.7.$$

Therefore, AAA Party Supply Store made $14,383.70 in May.

Exercise 12.3.6

On June 1, AAA Party Supply Store decided to increase the price they charge for party favors to $2 per package. They also changed suppliers for their invitations, and are now able to purchase invitations for only 10¢ per package. All their other costs and prices remain the same. If AAA sells 1408 invitations, 147 party favors, 2112 decorations, and 1894 food service items in the month of June, use vectors and dot products to calculate their total sales and profit for June.

Hint

Use four-dimensional vectors for cost, price, and quantity sold.

Answer

Sales = $15,685.50; profit = $14,073.15

Projections

As we have seen, addition combines two vectors to create a resultant vector. But what if we are given a vector and we need to find its component parts? We use vector projections to perform the opposite process; they can break down a vector into its components. The magnitude of a vector projection is a scalar projection. For example, if a child is pulling the handle of a wagon at a 55° angle, we can use projections to determine how much of the force on the handle is actually moving the wagon forward (12.3.6). We return to this example and learn how to solve it after we see how to calculate projections.

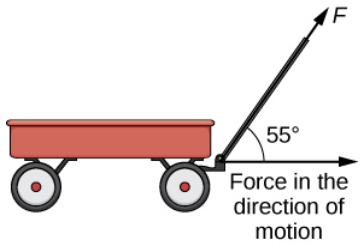


Figure 12.3.6: When a child pulls a wagon, only the horizontal component of the force propels the wagon forward.

Definition: Vector and Projection

The **vector projection** of $\vec{v}$ onto $\vec{u}$ is the vector labeled $\text{proj}_{\vec{u}}\vec{v}$ in Figure 12.3.7. It has the same initial point as $\vec{u}$ and $\vec{v}$ and the same direction as $\vec{u}$, and represents the component of $\vec{v}$ that acts in the direction of $\vec{u}$. If θ represents the angle between $\vec{u}$ and $\vec{v}$, then, by properties of triangles, we know the length of $\text{proj}_{\vec{u}}\vec{v}$ is $\|\text{proj}_{\vec{u}}\vec{v}\| = \|\vec{v}\|\cos\theta$. When expressing $\cos\theta$ in terms of the dot product, this becomes

$$\|\text{proj}_{\vec{u}}\vec{v}\| = \|\vec{v}\|\cos\theta = \|\vec{v}\|\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|}.$$

We now multiply by a unit vector in the direction of $\vec{u}$ to get $\text{proj}_{\vec{u}}\vec{v}$:

$$\text{proj}_{\vec{u}}\vec{v} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|}\left(\frac{1}{\|\vec{u}\|}\vec{u}\right) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|^2}\vec{u}.$$

The length of this vector is also known as the **scalar projection** of $\vec{v}$ onto $\vec{u}$ and is denoted by

$$\|\text{proj}_{\vec{u}}\vec{v}\| = \text{comp}_{\vec{u}}\vec{v} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|}.$$

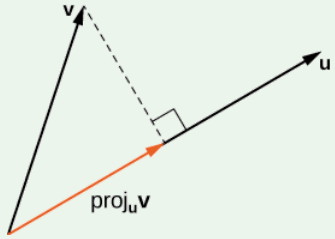


Figure 12.3.7: The projection of $\vec{v}$ onto $\vec{u}$ shows the component of vector $\vec{v}$ in the direction of $\vec{u}$.

Example 12.3.7: Finding Projections

Find the projection of $\vec{v}$ onto $\vec{u}$.

a. $\vec{v} = \langle 3, 5, 1\rangle$ and $\vec{u} = \langle -1, 4, 3\rangle$
b. $\vec{v} = 3\hat{i} - 2\hat{j}$ and $\vec{u} = \hat{i} + 6\hat{j}$

Solution

a. Substitute the components of $\vec{v}$ and $\vec{u}$ into the formula for the projection:

$$\begin{aligned}\text{proj}_{\vec{u}}\vec{v} &= \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|^2}\vec{u}\\ &= \frac{\langle -1,4,3\rangle\cdot\langle 3,5,1\rangle}{\|\langle -1,4,3\rangle\|^2}\langle -1,4,3\rangle\\ &= \frac{-3+20+3}{(-1)^2+4^2+3^2}\langle -1,4,3\rangle\\ &= \frac{20}{26}\langle -1,4,3\rangle\\ &= \langle -\frac{10}{13}, \frac{40}{13}, \frac{30}{13}\rangle.\end{aligned}$$

b. To find the two-dimensional projection, simply adapt the formula to the two-dimensional case:

$$\begin{aligned}\text{proj}_{\vec{u}}\vec{v} &= \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|^2}\vec{u}\\ &= \frac{(\hat{i}+6\hat{j})\cdot(3\hat{i}-2\hat{j})}{\|\hat{i}+6\hat{j}\|^2}(\hat{i}+6\hat{j})\\ &= \frac{1(3)+6(-2)}{1^2+6^2}(\hat{i}+6\hat{j})\\ &= -\frac{9}{37}(\hat{i}+6\hat{j})\\ &= -\frac{9}{37}\hat{i} - \frac{54}{37}\hat{j}.\end{aligned}$$

Sometimes it is useful to decompose vectors—that is, to break a vector apart into a sum. This process is called the **resolution of a vector into components.** Projections allow us to identify two orthogonal vectors having a desired sum. For example, let $\vec{v} = \langle 6, -4\rangle$ and let $\vec{u} = \langle 3, 1\rangle$. We want to decompose the vector $\vec{v}$ into orthogonal components such that one of the component vectors has the same direction as $\vec{u}$.

We first find the component that has the same direction as $\vec{u}$ by projecting $\vec{v}$ onto $\vec{u}$. Let $\vec{p} = \text{proj}_{\vec{u}}\vec{v}$. Then, we have

$$\begin{aligned}\vec{p} &= \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|^2}\vec{u}\\ &= \frac{18-4}{9+1}\vec{u}\\ = \frac{7}{5}\vec{u} = \frac{7}{5}\langle 3,1\rangle &= \langle \frac{21}{5}, \frac{7}{5}\rangle.\end{aligned}$$

Now consider the vector $\vec{\mathbf{q}} = \vec{\mathbf{v}} - \vec{\mathbf{p}}$. We have

$$\begin{aligned}\vec{\mathbf{q}} &= \vec{\mathbf{v}} - \vec{\mathbf{p}} \\ &= \langle 6, -4\rangle - \langle \frac{21}{5}, \frac{7}{5}\rangle \\ &= \langle \frac{9}{5}, -\frac{27}{5}\rangle.\end{aligned}$$

Clearly, by the way we defined $\vec{\mathbf{q}}$, we have $\vec{\mathbf{v}} = \vec{\mathbf{q}} + \vec{\mathbf{p}}$, and

$$\begin{aligned}\vec{\mathbf{q}} \cdot \vec{\mathbf{p}} &= \langle \frac{9}{5}, -\frac{27}{5}\rangle \cdot \langle \frac{21}{5}, \frac{7}{5}\rangle \\ &= \frac{9(21)}{25} + -\frac{27(7)}{25} \\ &= \frac{189}{25} - \frac{189}{25} = 0.\end{aligned}$$

Therefore, $\vec{\mathbf{q}}$ and $\vec{\mathbf{p}}$ are orthogonal.

Example 12.3.8: Resolving Vectors into Components

Express $\vec{\mathbf{v}} = \langle 8, -3, -3\rangle$ as a sum of orthogonal vectors such that one of the vectors has the same direction as $\vec{\mathbf{u}} = \langle 2, 3, 2\rangle$.

Solution

Let $\vec{\mathbf{p}}$ represent the projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$:

$$\begin{aligned}\vec{\mathbf{p}} &= \text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} \\ &= \frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2}\vec{\mathbf{u}} \\ &= \frac{\langle 2, 3, 2\rangle \cdot \langle 8, -3, -3\rangle}{\| \langle 2, 3, 2\rangle \|^2}\langle 2, 3, 2\rangle \\ &= \frac{16 - 9 - 6}{2^2 + 3^2 + 2^2}\langle 2, 3, 2\rangle \\ &= \frac{1}{17}\langle 2, 3, 2\rangle \\ &= \langle \frac{2}{17}, \frac{3}{17}, \frac{2}{17}\rangle.\end{aligned}$$

Then,

$$\begin{aligned}\vec{\mathbf{q}} = \vec{\mathbf{v}} - \vec{\mathbf{p}} &= \langle 8, -3, -3\rangle - \langle \frac{2}{17}, \frac{3}{17}, \frac{2}{17}\rangle \\ &= \langle \frac{134}{17}, -\frac{54}{17}, -\frac{53}{17}\rangle.\end{aligned}$$

To check our work, we can use the dot product to verify that $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ are orthogonal vectors:

$$\begin{aligned}\vec{\mathbf{p}} \cdot \vec{\mathbf{q}} &= \langle \frac{2}{17}, \frac{3}{17}, \frac{2}{17}\rangle \cdot \langle \frac{134}{17}, -\frac{54}{17}, -\frac{53}{17}\rangle \\ &= \frac{268}{289} - \frac{162}{289} - \frac{106}{289} = 0.\end{aligned}$$

Then,

$$\vec{\mathbf{v}} = \vec{\mathbf{p}} + \vec{\mathbf{q}} = \langle \frac{2}{17}, \frac{3}{17}, \frac{2}{17}\rangle + \langle \frac{134}{17}, -\frac{54}{17}, -\frac{53}{17}\rangle.$$

Exercise 12.3.7

Express $\vec{v} = 5\hat{i} - \hat{j}$ as a sum of orthogonal vectors such that one of the vectors has the same direction as $\vec{u} = 4\hat{i} + 2\hat{j}$.

Hint

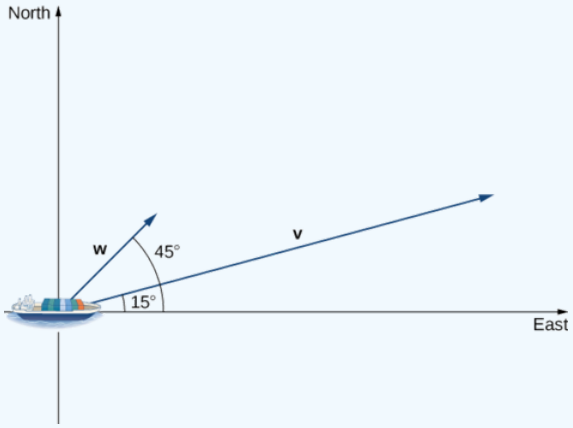
Start by finding the projection of $\vec{v}$ onto $\vec{u}$.

Answer

$\vec{v} = \vec{p} + \vec{q}$, where $\vec{p} = \frac{18}{5}\hat{i} + \frac{9}{5}\hat{j}$ and $\vec{q} = \frac{7}{5}\hat{i} - \frac{14}{5}\hat{j}$

Example 12.3.9: Scalar Projection of Velocity

A container ship leaves port traveling $15°$ north of east. Its engine generates a speed of 20 knots along that path (see the following figure). In addition, the ocean current moves the ship northeast at a speed of 2 knots. Considering both the engine and the current, how fast is the ship moving in the direction $15°$ north of east? Round the answer to two decimal places.



Solution

Let $\vec{v}$ be the velocity vector generated by the engine, and let $\vec{w}$ be the velocity vector of the current. We already know $\|\vec{v}\| = 20$ along the desired route. We just need to add in the scalar projection of $\vec{w}$ onto $\vec{v}$. We get

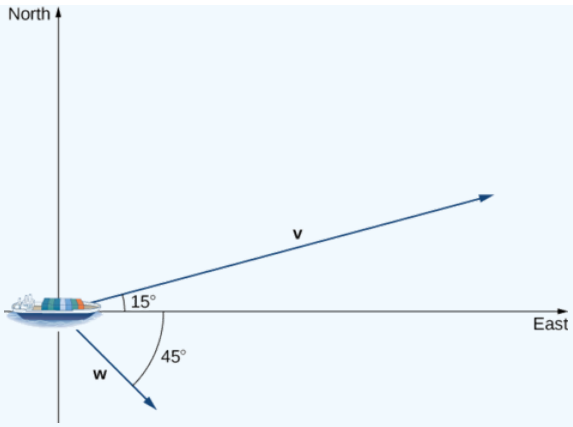
$$\text{comp}_{\vec{v}}\vec{w} = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|}$$

$$= \frac{\|\vec{v}\|\|\vec{w}\|\cos(30°)}{\|\vec{v}\|} = \|\vec{w}\|\cos(30°) = 2\frac{\sqrt{3}}{2} = \sqrt{3} \approx 1.73 \text{ knots}.$$

The ship is moving at 21.73 knots in the direction $15°$ north of east.

Exercise 12.3.8

Repeat the previous example, but assume the ocean current is moving southeast instead of northeast, as shown in the following figure.



Hint

Compute the scalar projection of $\vec{w}$ onto $\vec{v}$.

Answer

21 knots

Work

Now that we understand dot products, we can see how to apply them to real-life situations. The most common application of the dot product of two vectors is in the calculation of work.

From physics, we know that work is done when an object is moved by a force. When the force is constant and applied in the same direction the object moves, then we define the work done as the product of the force and the distance the object travels: $W = Fd$. We saw several examples of this type in earlier chapters. Now imagine the direction of the force is different from the direction of motion, as with the example of a child pulling a wagon. To find the work done, we need to multiply the component of the force that acts in the direction of the motion by the magnitude of the displacement. The dot product allows us to do just that. If we represent an applied force by a vector $\vec{\mathbf{F}}$ and the displacement of an object by a vector $\vec{s}$, then the **work done by the force** is the dot product of $\vec{\mathbf{F}}$ and $\vec{s}$.

Definition: Constant Force

When a constant force is applied to an object so the object moves in a straight line from point P to point Q, the work W done by the force $\vec{\mathbf{F}}$, acting at an angle θ from the line of motion, is given by

$$W = \vec{\mathbf{F}} \cdot \overrightarrow{PQ} = \| \vec{\mathbf{F}} \| \| \overrightarrow{PQ} \| \cos\theta.$$

Let's revisit the problem of the child's wagon introduced earlier. Suppose a child is pulling a wagon with a force having a magnitude of 8 lb on the handle at an angle of 55°. If the child pulls the wagon 50 ft, find the work done by the force (Figure 12.3.8).

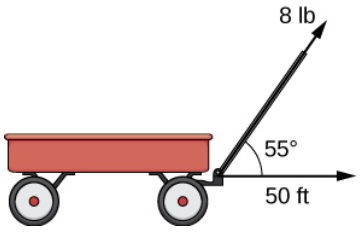


Figure 12.3.8: The horizontal component of the force is the projection of $\vec{\mathbf{F}}$ onto the positive x-axis.

We have

$$W = \| \vec{\mathbf{F}} \| \| \overrightarrow{PQ} \| \cos\theta = 8(50)(\cos(55°)) \approx 229\,\text{ft·lb}.$$

In U.S. standard units, we measure the magnitude of force $\| \vec{\mathbf{F}} \|$ in pounds. The magnitude of the displacement vector $\| \overrightarrow{PQ} \|$ tells us how far the object moved, and it is measured in feet. The customary unit of measure for work, then, is the foot-pound. One foot-pound is the amount of work required to move an object weighing 1 lb a distance of 1 ft straight up. In the metric system, the unit of measure for force is the newton (N), and the unit of measure of magnitude for work is a newton-meter (N·m), or a joule (J).

Example 12.3.10: Calculating Work

A conveyor belt generates a force $\vec{\mathbf{F}} = 5\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$ that moves a suitcase from point $(1, 1, 1)$ to point $(9, 4, 7)$ along a straight line. Find the work done by the conveyor belt. The distance is measured in meters and the force is measured in newtons.

Solution

The displacement vector $\overrightarrow{PQ}$ has initial point $(1, 1, 1)$ and terminal point $(9, 4, 7)$:

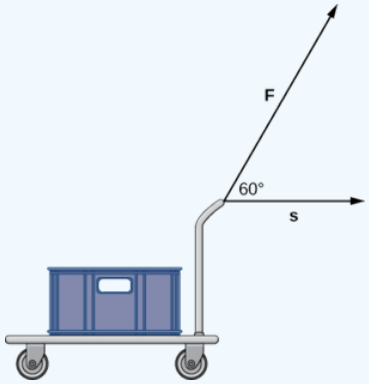
$$\overrightarrow{PQ} = \langle 9 - 1, 4 - 1, 7 - 1 \rangle = \langle 8, 3, 6 \rangle = 8\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}.$$

Work is the dot product of force and displacement:

$$\begin{aligned} W &= \vec{\mathbf{F}} \cdot \overrightarrow{PQ} \\ &= (5\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}) \cdot (8\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}) \\ &= 5(8) + (-3)(3) + 1(6) \\ &= 37\,\text{N·m} \\ &= 37\,\text{J} \end{aligned}$$

Exercise 12.3.9

A constant force of 30 lb is applied at an angle of 60° to pull a handcart 10 ft across the ground. What is the work done by this force?



Hint

Use the definition of work as the dot product of force and distance.

Answer

150 ft-lb

Key Concepts

- The dot product, or scalar product, of two vectors $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ is $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2 + u_3 v_3$.
- The dot product satisfies the following properties:
  - $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \vec{\mathbf{v}} \cdot \vec{\mathbf{u}}$
  - $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} + \vec{\mathbf{u}} \cdot \vec{\mathbf{w}}$
  - $c(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}) = (c\vec{\mathbf{u}}) \cdot \vec{\mathbf{v}} = \vec{\mathbf{u}} \cdot (c\vec{\mathbf{v}})$

- $\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = \|\vec{\mathbf{v}}\|^2$

- The dot product of two vectors can be expressed, alternatively, as $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\| \cos\theta$. This form of the dot product is useful for finding the measure of the angle formed by two vectors.
- Vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are orthogonal if $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 0$.
- The angles formed by a nonzero vector and the coordinate axes are called the *direction angles* for the vector. The cosines of these angles are known as the *direction cosines.*
- The vector projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$ is the vector $\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} = \dfrac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2}\vec{\mathbf{u}}$. The magnitude of this vector is known as the *scalar projection* of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$, given by $\text{comp}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} = \dfrac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|}$.
- Work is done when a force is applied to an object, causing displacement. When the force is represented by the vector $\vec{\mathbf{F}}$ and the displacement is represented by the vector $\vec{\mathbf{s}}$, then the work done W is given by the formula $W = \vec{\mathbf{F}} \cdot \vec{\mathbf{s}} = \| \vec{\mathbf{F}} \| \, \|\vec{\mathbf{s}}\| \cos\theta$.

Key Equations

- **Dot product of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$**

  $$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_1v_1 + u_2v_2 + u_3v_3 = \|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\| \cos\theta$$

- **Cosine of the angle formed by $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$**

  $$\cos\theta = \frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|}$$

- **Vector projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$**

  $$\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} = \frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2}\vec{\mathbf{u}}$$

- **Scalar projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$**

  $$\text{comp}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} = \frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|}$$

- **Work done by a force $\vec{\mathbf{F}}$ to move an object through displacement vector $\overrightarrow{PQ}$**

  $$W = \vec{\mathbf{F}} \cdot \overrightarrow{PQ} = \| \vec{\mathbf{F}} \| \| \overrightarrow{PQ} \| \cos\theta$$

Glossary

direction angles

the angles formed by a nonzero vector and the coordinate axes

direction cosines

the cosines of the angles formed by a nonzero vector and the coordinate axes

dot product or scalar product

$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_1v_1 + u_2v_2 + u_3v_3$ where $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$

scalar projection

the magnitude of the vector projection of a vector

orthogonal vectors

vectors that form a right angle when placed in standard position

vector projection

the component of a vector that follows a given direction

work done by a force

work is generally thought of as the amount of energy it takes to move an object; if we represent an applied force by a vector $\vec{\mathbf{F}}$ and the displacement of an object by a vector $\vec{\mathbf{s}}$, then the work done by the force is the dot product of $\vec{\mathbf{F}}$ and $\vec{\mathbf{s}}$.

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- edited for vector notation by Paul Seeburger

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12.3E: Exercises for Section 12.3

For exercises 1-4, the vectors $\vec{u}$ and $\vec{v}$ are given. Calculate the dot product $\vec{u} \cdot \vec{v}$.

1) $\vec{u} = \langle 3, 0 \rangle, \quad \vec{v} = \langle 2, 2 \rangle$

Answer

$\vec{u} \cdot \vec{v} = 6$

2) $\vec{u} = \langle 3, -4 \rangle, \quad \vec{v} = \langle 4, 3 \rangle$

3) $\vec{u} = \langle 2, 2, -1 \rangle, \quad \vec{v} = \langle -1, 2, 2 \rangle$

Answer

$\vec{u} \cdot \vec{v} = 0$

4) $\vec{u} = \langle 4, 5, -6 \rangle, \quad \vec{v} = \langle 0, -2, -3 \rangle$

For exercises 5-8, the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are given. Determine the vectors $(\vec{a} \cdot \vec{b})\vec{c}$ and $(\vec{a} \cdot \vec{c})\vec{b}$. Express the vectors in component form.

5) $\vec{a} = \langle 2, 0, -3 \rangle, \quad \vec{b} = \langle -4, -7, 1 \rangle, \quad \vec{c} = \langle 1, 1, -1 \rangle$

Answer

$(\vec{a} \cdot \vec{b})\vec{c} = \langle -11, -11, 11 \rangle; (\vec{a} \cdot \vec{c})\vec{b} = \langle -20, -35, 5 \rangle$

6) $\vec{a} = \langle 0, 1, 2 \rangle, \quad \vec{b} = \langle -1, 0, 1 \rangle, \quad \vec{c} = \langle 1, 0, -1 \rangle$

7) $\vec{a} = \hat{i} + \hat{j}, \quad \vec{b} = \hat{i} - \hat{k}, \quad \vec{c} = \hat{i} - 2\hat{k}$

Answer

$(\vec{a} \cdot \vec{b})\vec{c} = \langle 1, 0, -2 \rangle; (\vec{a} \cdot \vec{c})\vec{b} = \langle 1, 0, -1 \rangle$

8) $\vec{a} = \hat{i} - \hat{j} + \hat{k}, \quad \vec{b} = \hat{j} + 3\hat{k}, \quad \vec{c} = -\hat{i} + 2\hat{j} - 4\hat{k}$

For exercises 9-12, two vectors are given.

a. Find the measure of the angle θ between these two vectors. Express the answer in radians rounded to two decimal places, if it is not possible to express it exactly.

b. Is θ an acute angle?

9) [T] $\vec{a} = \langle 3, -1 \rangle, \quad \vec{b} = \langle -4, 0 \rangle$

Answer

$a. \theta = 2.82$ rad; $b. \theta$ is not acute.

10) [T] $\vec{a} = \langle 2, 1 \rangle, \quad \vec{b} = \langle -1, 3 \rangle$

11) $\vec{u} = 3\hat{i}, \quad \vec{v} = 4\hat{i} + 4\hat{j}$

Answer

$a. \theta = \frac{\pi}{4}$ rad; $b. \theta$ is acute.

12) $\vec{u} = 5\hat{i}, \quad \vec{v} = -6\hat{i} + 6\hat{j}$

For exercises 13-18, find the measure of the angle between the three-dimensional vectors $\vec{a}$ and $\vec{b}$. Express the answer in radians rounded to two decimal places, if it is not possible to express it exactly.

13) $\vec{a} = \langle 3, -1, 2 \rangle, \quad \vec{b} = \langle 1, -1, -2 \rangle$

Answer

$\theta = \frac{\pi}{2}$

14) $\vec{\mathbf{a}} = \langle 0, -1, -3 \rangle, \quad \vec{\mathbf{b}} = \langle 2, 3, -1 \rangle$

15) $\vec{\mathbf{a}} = \hat{\mathbf{i}} + \hat{\mathbf{j}}, \quad \vec{\mathbf{b}} = \hat{\mathbf{j}} - \hat{\mathbf{k}}$

Answer

$\theta = \frac{\pi}{3}$

16) $\vec{\mathbf{a}} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}, \quad \vec{\mathbf{b}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$

17) [T] $\vec{\mathbf{a}} = 3\hat{\mathbf{i}} - \hat{\mathbf{j}} - 2\hat{\mathbf{k}}, \quad \vec{\mathbf{b}} = \vec{\mathbf{v}} + \vec{\mathbf{w}}$, where $\vec{\mathbf{v}} = -2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $\vec{\mathbf{w}} = \hat{\mathbf{i}} + 2\hat{\mathbf{k}}$

Answer

$\theta = 2$ rad

18) [T] $\vec{\mathbf{a}} = 3\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}, \quad \vec{\mathbf{b}} = \vec{\mathbf{v}} - \vec{\mathbf{w}}$, where $\vec{\mathbf{v}} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ and $\vec{\mathbf{w}} = 6\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

For exercises 19-22, determine whether the given vectors are orthogonal.

19) $\vec{\mathbf{a}} = \langle x, y \rangle, \quad \vec{\mathbf{b}} = \langle -y, x \rangle$, where x and y are nonzero real numbers

Answer

Orthogonal

20) $\vec{\mathbf{a}} = \langle x, x \rangle, \quad \vec{\mathbf{b}} = \langle -y, y \rangle$, where x and y are nonzero real numbers

21) $\vec{\mathbf{a}} = 3\hat{\mathbf{i}} - \hat{\mathbf{j}} - 2\hat{\mathbf{k}}, \quad \vec{\mathbf{b}} = -2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$

Answer

Not orthogonal

22) $\vec{\mathbf{a}} = \hat{\mathbf{i}} - \hat{\mathbf{j}}, \quad \vec{\mathbf{b}} = 7\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$

23) Find all two-dimensional vectors $\vec{\mathbf{a}}$ orthogonal to vector $\vec{\mathbf{b}} = \langle 3, 4 \rangle$. Express the answer in component form.

Answer

$\vec{\mathbf{a}} = \langle -\frac{4\alpha}{3}, \alpha \rangle$, where $\alpha \neq 0$ is a real number

24) Find all two-dimensional vectors $\vec{\mathbf{a}}$ orthogonal to vector $\vec{\mathbf{b}} = \langle 5, -6 \rangle$. Express the answer by using standard unit vectors.

25) Determine all three-dimensional vectors $\vec{\mathbf{u}}$ orthogonal to vector $\vec{\mathbf{v}} = \langle 1, 1, 0 \rangle$. Express the answer by using standard unit vectors.

Answer

$\vec{\mathbf{u}} = -\alpha\hat{\mathbf{i}} + \alpha\hat{\mathbf{j}} + \beta\hat{\mathbf{k}}$, where α and β are real numbers such that $\alpha^2 + \beta^2 \neq 0$

26) Determine all three-dimensional vectors $\vec{\mathbf{u}}$ orthogonal to vector $\vec{\mathbf{v}} = \hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}}$. Express the answer in component form.

27) Determine the real number α such that vectors $\vec{\mathbf{a}} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ and $\vec{\mathbf{b}} = 9\hat{\mathbf{i}} + \alpha\hat{\mathbf{j}}$ are orthogonal.

Answer

$\alpha = -6$

28) Determine the real number α such that vectors $\vec{\mathbf{a}} = -3\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$ and $\vec{\mathbf{b}} = 2\hat{\mathbf{i}} + \alpha\hat{\mathbf{j}}$ are orthogonal.

29) [T] Consider the points $P(4, 5)$ and $Q(5, -7)$, and note that O represents the origin.

a. Determine vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. Express the answer by using standard unit vectors.

b. Determine the measure of angle O in triangle OPQ. Express the answer in degrees rounded to two decimal places.

Answer

$a. \overrightarrow{OP} = 4\hat{\mathbf{i}} + 5\hat{\mathbf{j}}, \quad \overrightarrow{OQ} = 5\hat{\mathbf{i}} - 7\hat{\mathbf{j}}; \quad b. 105.8°$

30) [T] Consider points $A(1,1), B(2,-7)$, and $C(6,3)$.

a. Determine vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$. Express the answer in component form.

b. Determine the measure of angle B in triangle ABC. Express the answer in degrees rounded to two decimal places.

31) Determine the measure of angle A in triangle ABC, where $A(1,1,8), B(4,-3,-4)$, and $C(-3,1,5)$. Express your answer in degrees rounded to two decimal places.

Answer

$68.33°$

32) Consider points $P(3,7,-2)$ and $Q(1,1,-3)$. Determine the angle between vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. [Remember that O represents the origin.] Express the answer in degrees rounded to two decimal places.

For exercises 33-34, determine which (if any) pairs of the following vectors are orthogonal.

33) $\vec{\mathbf{u}} = \langle 3,7,-2\rangle, \quad \vec{\mathbf{v}} = \langle 5,-3,-3\rangle, \quad \vec{\mathbf{w}} = \langle 0,1,-1\rangle$

Answer

$\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are orthogonal; $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ are orthogonal, but $\vec{\mathbf{u}}$ and $\vec{\mathbf{w}}$ are **not** orthogonal.

34) $\vec{\mathbf{u}} = \hat{\mathbf{i}} - \hat{\mathbf{k}}, \quad \vec{\mathbf{v}} = 5\hat{\mathbf{j}} - 5\hat{\mathbf{k}}, \quad \vec{\mathbf{w}} = 10\hat{\mathbf{j}}$

35) Use vectors to show that a parallelogram with equal diagonals is a **rectangle**.

36) Use vectors to show that the diagonals of a **rhombus** are perpendicular.

37) Show that $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} + \vec{\mathbf{u}} \cdot \vec{\mathbf{w}}$ is true for any vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$.

38) Verify the identity $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} + \vec{\mathbf{u}} \cdot \vec{\mathbf{w}}$ for vectors $\vec{\mathbf{u}} = \langle 1,0,4\rangle, \vec{\mathbf{v}} = \langle -2,3,5\rangle$, and $\vec{\mathbf{w}} = \langle 4,-2,6\rangle$.

For exercises 39-41, determine $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$ using the given information.

39) $\|\vec{\mathbf{u}}\| = 5, \quad \|\vec{\mathbf{v}}\| = 3$, and the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ is $\pi/6$ rad.

Answer

$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = (5)(3)\cos\frac{\pi}{6} = 15 \cdot \frac{\sqrt{3}}{2} = \frac{15\sqrt{3}}{2}$

40) $\|\vec{\mathbf{u}}\| = 20, \quad \|\vec{\mathbf{v}}\| = 15$, and the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ is $5\pi/4$ rad.

41) $\|\vec{\mathbf{u}}\| = 7, \quad \|\vec{\mathbf{v}}\| = 12$, and the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ is $\pi/2$ rad.

Answer

$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = (7)(12)\cos\frac{\pi}{2} = 84 \cdot 0 = 0$

42) Considering the definition of the dot product, $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|\cos\theta$, where θ is the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, what can you say about the angle θ between two nonzero vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ if:

a. $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} > 0$?

b. $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} < 0$?

c. $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 0$?

In exercises 43-45, you are given the vertices of a triangle. Use dot products to determine whether each triangle is *acute*, *obtuse*, or *right*.

43) $(2,3,0), (3,1,-2), (-1,4,5)$

Answer

Label the points P, Q and R. Then determine the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.
Here we see that, if $P = (2,3,0), Q = (3,1,-2)$ and $R = (-1,4,5)$, so we have $\overrightarrow{PQ} =< 1,-2,-2 >$ and $\overrightarrow{PR} =< -3,1,5 >$.

Then $\overrightarrow{PQ} \cdot \overrightarrow{PR} = -15 < 0$, so angle QPR is obtuse. Therefore this triangle must be an ***obtuse triangle***.

44) $(5,1,0),\ (7,1,1),\ (6,3,2)$

45) $(6,1,4),\ (3,2,-1),\ (2,3,1)$

Solution:

Label the points P, Q and R. Then determine the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.
Here we see that, if $P = (6,1,4),\ Q = (3,2,-1)$ and $R = (2,3,1)$, so we have $\overrightarrow{PQ} =< -3,1,-5 >$ and $\overrightarrow{PR} =< -4,2,-3 >$.
Then $\overrightarrow{PQ} \cdot \overrightarrow{PR} = 29 > 0$, so angle QPR is **acute**. This triangle could still be any of the three types, so we need to check another angle.
Considering angle PQR we find $\overrightarrow{QP} =< 3,-1,5 >$ and $\overrightarrow{QR} =< -1,1,2 >$, so $\overrightarrow{QP} \cdot \overrightarrow{QR} = 6 > 0$, showing angle PQR is also **acute**.
We then must check the third angle to determine whether this is a right triangle or an acute triangle.
To check angle PRQ we find $\overrightarrow{RP} =< 4,-2,3 >$ and $\overrightarrow{RQ} =< 1,-1,-2 >$, giving us $\overrightarrow{RP} \cdot \overrightarrow{RQ} = 0$. Angle PRQ is thus a **right** angle.
Therefore this triangle is a ***right triangle***.

46) Which of the following operations are allowed for nonzero vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$, and which are not. Explain your reasoning.

a. $\vec{u} + (\vec{v} \cdot \vec{w})$ b. $(\vec{u} \cdot \vec{v}) \cdot \vec{w}$ c. $(\vec{u} \cdot \vec{v})\,\vec{w}$
d. $(\vec{u} + \vec{v}) \cdot \vec{w}$ e. $(\vec{u} + \vec{v}) \cdot \|\vec{w}\|$ f. $\|\vec{u} + \vec{v}\|\|\vec{w}\|$

Projections

For the exercises 47-50, given the vectors $\vec{u}$ and $\vec{v}$:

a. Find the vector projection $\text{Proj}_{\vec{u}}\vec{v}$ of vector $\vec{v}$ onto vector $\vec{u}$ and the component of $\vec{v}$ that is orthogonal to $\vec{u}$, i.e., $\vec{v}_{\text{perp}}$. Express your answers in component form.

b. Find the scalar projection $\text{comp}_{\vec{u}}\vec{v} = \|\text{Proj}_{\vec{u}}\vec{v}\|$ of vector $\vec{v}$ onto vector $\vec{u}$.

c. Find the vector projection $\text{Proj}_{\vec{v}}\vec{u}$ of vector $\vec{u}$ onto vector $\vec{v}$ and the component of $\vec{u}$ that is orthogonal to $\vec{v}$, i.e., $\vec{u}_{\text{perp}}$. Express your answers in unit vector form.

d. Find the scalar projection $\text{comp}_{\vec{v}}\vec{u}$ of vector $\vec{u}$ onto vector $\vec{v}$.

47) $\vec{u} = 5\hat{i} + 2\hat{j}, \quad \vec{v} = 2\hat{i} + 3\hat{j}$

Answer

a. $\text{Proj}_{\vec{u}}\vec{v} = \langle \frac{80}{29}, \frac{32}{29} \rangle$ and $\vec{v}_{\text{perp}} =< -\frac{22}{29}, \frac{55}{29} >$; b. $\text{comp}_{\vec{u}}\vec{v} = \frac{16}{\sqrt{29}} = \frac{16\sqrt{29}}{29}$;
c. $\text{Proj}_{\vec{v}}\vec{u} = \frac{32}{13}\hat{i} + \frac{48}{13}\hat{j}$ and $\vec{u}_{\text{perp}} = \frac{33}{13}\hat{i} - \frac{22}{13}\hat{j}$; d. $\text{comp}_{\vec{v}}\vec{u} = \frac{16}{\sqrt{13}} = \frac{16\sqrt{13}}{13}$

48) $\vec{u} = \langle -4, 7 \rangle, \quad \vec{v} = \langle 3, 5 \rangle$

49) $\vec{u} = 3\hat{i} + 2\hat{k}, \quad \vec{v} = 2\hat{j} + 4\hat{k}$

Answer

a. $\text{Proj}_{\vec{u}}\vec{v} = \langle \frac{24}{13}, 0, \frac{16}{13} \rangle$ and $\vec{v}_{\text{perp}} =< -\frac{24}{13}, 2, \frac{36}{13} >$; b. $\text{comp}_{\vec{u}}\vec{v} = \frac{8}{\sqrt{13}} = \frac{8\sqrt{13}}{13}$
c. $\text{Proj}_{\vec{v}}\vec{u} = \frac{4}{5}\hat{j} + \frac{8}{5}\hat{k}$ and $\vec{u}_{\text{perp}} = 3\hat{i} - \frac{4}{5}\hat{j} + \frac{2}{5}\hat{k}$; d. $\text{comp}_{\vec{v}}\vec{u} = \frac{\sqrt{80}}{5} = \frac{4\sqrt{5}}{5}$;

50) $\vec{u} = \langle 4, 4, 0 \rangle, \quad \vec{v} = \langle 0, 4, 1 \rangle$

51) Consider the vectors $\vec{u} = 4\hat{i} - 3\hat{j}$ and $\vec{v} = 3\hat{i} + 2\hat{j}$.

a. Find the component form of vector $\text{Proj}_{\vec{u}}\vec{v}$ that represents the projection of $\vec{v}$ onto $\vec{u}$.

b. Write the decomposition $\vec{v} = \vec{w} + \vec{q}$ of vector $\vec{v}$ into the orthogonal components $\vec{w}$ and $\vec{q}$, where $\vec{w}$ is the projection of $\vec{v}$ onto $\vec{u}$ and $\vec{q}$ is the vector component of $\vec{v}$ orthogonal to the direction of $\vec{u}$. That is, $\vec{q} = \vec{v}_{\text{perp}}$.

Answer

a. $\text{Proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} = \langle \frac{24}{25}, -\frac{18}{25} \rangle$; b. $\vec{\mathbf{q}} = \vec{\mathbf{v}}_{\text{perp}} = \langle \frac{51}{25}, \frac{68}{25} \rangle$,
$\vec{\mathbf{v}} = \vec{\mathbf{w}} + \vec{\mathbf{q}} = \text{Proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} + \vec{\mathbf{v}}_{\text{perp}} = \langle \frac{24}{25}, -\frac{18}{25} \rangle + \langle \frac{51}{25}, \frac{68}{25} \rangle$
So we have that, $\vec{\mathbf{v}} = \langle \frac{24}{25}, -\frac{18}{25} \rangle + \langle \frac{51}{25}, \frac{68}{25} \rangle$.

52) Consider vectors $\vec{\mathbf{u}} = 2\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$ and $\vec{\mathbf{v}} = 4\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$.

a. Find the component form of vector $\vec{\mathbf{w}} = \text{Proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}$ that represents the projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$.

b. Write the decomposition $\vec{\mathbf{v}} = \vec{\mathbf{w}} + \vec{\mathbf{q}}$ of vector $\vec{\mathbf{v}}$ into the orthogonal components $\vec{\mathbf{w}}$ and $\vec{\mathbf{q}}$, where $\vec{\mathbf{w}}$ is the projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$ and $\vec{\mathbf{q}}$ is a vector orthogonal to the direction of $\vec{\mathbf{u}}$.

53) A $50,000$-pound truck is parked on a hill with a $5°$ slope above the horizontal (in the positive x-direction). Considering only the force due to gravity, find

a. the component of the weight of the truck that is pulling the truck down the hill (aligned with the road)

b. the component of the weight of the truck that is perpendicular to the road.

c. the magnitude of the force needed to keep the truck from rolling down the hill

d. the magnitude of the force that is perpendicular to the road.

Solution:

Let $\vec{\mathbf{F}}$ be the force due to gravity on the truck (its weight) and let $\vec{\mathbf{r}}$ be the unit vector pointing up the hill to the right.

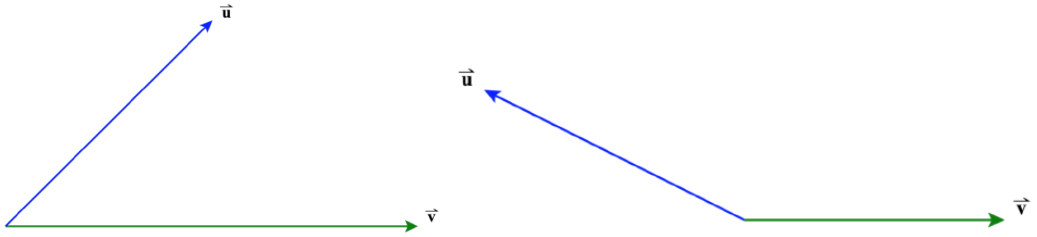
Then
a. the component of the weight of the truck that is pulling the truck down the hill is $\text{Proj}_{\vec{\mathbf{r}}}\vec{\mathbf{F}}$.
Now $\vec{\mathbf{F}} = -50,000\hat{\mathbf{j}}$ and $\vec{\mathbf{r}} = \cos 5°\,\hat{\mathbf{i}} + \sin 5°\,\hat{\mathbf{j}}$ and $\vec{\mathbf{F}} \cdot \vec{\mathbf{r}} = -50,000 \sin 5° \approx -4357.78714$
Note that $\|\vec{\mathbf{r}}\| = 1$.
Then: $\text{Proj}_{\vec{\mathbf{r}}}\vec{\mathbf{F}} = \left(\frac{\vec{\mathbf{F}} \cdot \vec{\mathbf{r}}}{\|\vec{\mathbf{r}}\|^2}\right)\vec{\mathbf{r}} = (-50,000 \sin 5°)\,\vec{\mathbf{r}} = -50000 \sin 5° \cos 5°\,\hat{\mathbf{i}} - 50000 \sin^2 5°\,\hat{\mathbf{j}}$.
Using the identity, $\sin 2\theta = 2\cos\theta \sin\theta$, we can simplify this as
$\text{Proj}_{\vec{\mathbf{r}}}\vec{\mathbf{F}} = -25000 \sin 10°\,\hat{\mathbf{i}} - 50000 \sin^2 5°\,\hat{\mathbf{j}} \approx -4341.2\,\hat{\mathbf{i}} - 379.8\hat{\mathbf{j}}$.

b. the component of the weight of the truck that is perpendicular to the road is $\vec{\mathbf{F}}_{\text{perp}}$.
Since we know that $\vec{\mathbf{F}} = \text{Proj}_{\vec{\mathbf{r}}}\vec{\mathbf{F}} + \vec{\mathbf{F}}_{\text{perp}}$, we have that $\vec{\mathbf{F}}_{\text{perp}} = \vec{\mathbf{F}} - \text{Proj}_{\vec{\mathbf{r}}}\vec{\mathbf{F}}$.
So $\vec{\mathbf{F}}_{\text{perp}} = -50000\hat{\mathbf{j}} - \left(-25000 \sin 10°\,\hat{\mathbf{i}} - 50000 \sin^2 5°\,\hat{\mathbf{j}}\right)$
$= 25000 \sin 10°\,\hat{\mathbf{i}} + (50000 \sin^2 5° - 50000)\,\hat{\mathbf{j}} = 25000 \sin 10°\,\hat{\mathbf{i}} - 50000 \cos^2 5°\,\hat{\mathbf{j}} \approx 4341.2\,\hat{\mathbf{i}} - 49620.2\hat{\mathbf{j}}$.

c. the magnitude of the force needed to keep the truck from rolling down the hill is $\|\text{Proj}_{\vec{\mathbf{r}}}\vec{\mathbf{F}}\| = 50,000 \sin 5°$ lbs ≈ 4357.8 lbs. Note that this force is actually the negative of the projection vector, but its magnitude is the same.

d. the magnitude of the truck's weight that is perpendicular to the road is $\|\vec{\mathbf{F}}_{\text{perp}}\| \approx 49809.7$ lbs.

54) Given the vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ shown in each diagram below, draw in $\text{Proj}_{\vec{\mathbf{v}}}\vec{\mathbf{u}}$ and $\vec{\mathbf{u}}_{\text{perp}}$.



Work

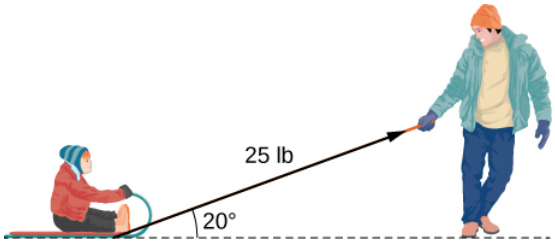
55) Find the work done by force $\vec{\mathbf{F}} = \langle 5, 6, -2 \rangle$ (measured in Newtons) that moves a particle from point $P(3, -1, 0)$ to point $Q(2, 3, 1)$ along a straight line (the distance is measured in meters).

Answer

$17\ \text{N}\cdot\text{m}$

56) [T] A sled is pulled by exerting a force of 100 N on a rope that makes an angle of $25°$ with the horizontal. Find the work done in pulling the sled 40 m. (Round the answer to one decimal place.)

57) [T] A father is pulling his son on a sled at an angle of $20°$ with the horizontal with a force of 25 lb (see the following image). He pulls the sled in a straight path of 50 ft. How much work was done by the man pulling the sled? (Round the answer to the nearest integer.)



Answer

1175 ft·lb

58) [T] A car is towed using a force of 1600 N. The rope used to pull the car makes an angle of 25° with the horizontal. Find the work done in towing the car 2 km. Express the answer in joules ($1\ \text{J} = 1\ \text{N}\cdot\text{m}$) rounded to the nearest integer.

59) [T] A boat sails north aided by a wind blowing in a direction of $N30°E$ with a magnitude of 500 lb. How much work is performed by the wind as the boat moves 100 ft? (Round the answer to two decimal places.)

Answer

$25000\sqrt{3}$ ft-lbs $\approx 43,301.27$ ft-lbs

Solution:

Vector representing the wind: $\vec{\text{w}} = 500\cos 60°\hat{\text{i}} + 500\sin 60°\hat{\text{j}}$
Vector representing the displacement to the north: $\vec{\text{d}} = 100\hat{\text{j}}$
Work done by the wind: $W = \vec{\text{w}} \cdot \vec{\text{d}} = 25000\sqrt{3}$ ft-lbs $\approx 43,301.27$ ft-lbs

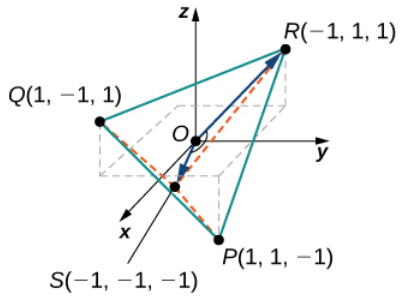
Other Applications of the Dot Product

60) [T] Find the vectors that join the center of a clock to the hours 1:00, 2:00, and 3:00. Assume the clock is circular with a radius of 1 unit.

61) A methane molecule has a carbon atom situated at the origin and four hydrogen atoms located at points $P(1,1,-1), Q(1,-1,1), R(-1,1,1)$, and $S(-1,-1,-1)$ (see figure).

a. Find the distance between the hydrogen atoms located at $P$and R.

b. Find the angle between vectors $\overrightarrow{OS}$ and $\overrightarrow{OR}$ that connect the carbon atom with the hydrogen atoms located at S and R, which is also called the *bond angle*. Express the answer in degrees rounded to two decimal places.



Answer

$a. 2\sqrt{2}; \quad b. 109.47°$

62) Vector $\vec{\mathbf{p}} = \langle 150, 225, 375 \rangle$ represents the price of certain models of bicycles sold by a bicycle shop. Vector $\vec{\mathbf{n}} = \langle 10, 7, 9 \rangle$ represents the number of bicycles sold of each model, respectively. Compute the dot product $\vec{\mathbf{p}} \cdot \vec{\mathbf{n}}$ and state its meaning.

63) [T] Two forces $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$ are represented by vectors with initial points that are at the origin. The first force has a magnitude of 20 lb and passes through the point $P(1, 1, 0)$. The second force has a magnitude of 40 lb and passes through the point $Q(0, 1, 1)$. Let $\vec{\mathbf{F}}$ be the resultant force of forces $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$.

a. Find the magnitude of $\vec{\mathbf{F}}$. (Round the answer to one decimal place.)

b. Find the direction angles of $\vec{\mathbf{F}}$. (Express the answer in degrees rounded to one decimal place.)

Answer

a. $\| \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 \| = 52.9$ lb; b. The direction angles are $\alpha = 74.5°,\ \beta = 36.7°$, and $\gamma = 57.7°$.

64) [T] Consider $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t, 2t \rangle$ the position vector of a particle at time $t \in [0, 30]$, where the components of $\vec{\mathbf{r}}$ are expressed in centimeters and time in seconds. Let $\overrightarrow{OP}$ be the position vector of the particle after 1 sec.

a. Show that all vectors $\overrightarrow{PQ}$, where $Q(x, y, z)$ is an arbitrary point, orthogonal to the instantaneous velocity vector $\vec{\mathbf{v}}(1)$ of the particle after 1 sec, can be expressed as $\overrightarrow{PQ} = \langle x - \cos 1, y - \sin 1, z - 2 \rangle$, where $x \sin 1 - y \cos 1 - 2z + 4 = 0$. The set of point Q describes a plane called the *normal plane* to the path of the particle at point P.

b. Use a CAS to visualize the instantaneous velocity vector and the normal plane at point P along with the path of the particle.

Contributors

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Problems 39-46 and 53 & 54 were created by Paul Seeburger (Monroe Community College).
Solution to Problem 59 also was added by Paul Seeburger.

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12.4: The Cross Product

Learning Objectives

- Calculate the cross product of two given vectors.
- Use determinants to calculate a cross product.
- Find a vector orthogonal to two given vectors.
- Determine areas and volumes by using the cross product.
- Calculate the torque of a given force and position vector.

Imagine a mechanic turning a wrench to tighten a bolt. The mechanic applies a force at the end of the wrench. This creates rotation, or torque, which tightens the bolt. We can use vectors to represent the force applied by the mechanic, and the distance (radius) from the bolt to the end of the wrench. Then, we can represent torque by a vector oriented along the axis of rotation. Note that the torque vector is orthogonal to both the force vector and the radius vector.

In this section, we develop an operation called the **cross product**, which allows us to find a vector orthogonal to two given vectors. Calculating torque is an important application of cross products, and we examine torque in more detail later in the section.

The Cross Product and Its Properties

The dot product is a multiplication of two vectors that results in a scalar. In this section, we introduce a product of two vectors that generates a third vector orthogonal to the first two. Consider how we might find such a vector. Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ be nonzero vectors. We want to find a vector $\vec{\mathbf{w}} = \langle w_1, w_2, w_3 \rangle$ orthogonal to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$—that is, we want to find $\vec{\mathbf{w}}$ such that $\vec{\mathbf{u}} \cdot \vec{\mathbf{w}} = 0$ and $\vec{\mathbf{v}} \cdot \vec{\mathbf{w}} = 0$. Therefore, w_1, w_2, and w_3 must satisfy

$$u_1 w_1 + u_2 w_2 + u_3 w_3 = 0 \tag{12.4.1}$$

$$v_1 w_1 + v_2 w_2 + v_3 w_3 = 0. \tag{12.4.2}$$

If we multiply the top equation by v_3 and the bottom equation by u_3 and subtract, we can eliminate the variable w_3, which gives

$$(u_1 v_3 - v_1 u_3) w_1 + (u_2 v_3 - v_2 u_3) w_2 = 0.$$

If we select

$$w_1 = u_2 v_3 - u_3 v_2$$
$$w_2 = -(u_1 v_3 - u_3 v_1),$$

we get a possible solution vector. Substituting these values back into the original equations (Equations 12.4.1 and 12.4.2) gives

$$w_3 = u_1 v_2 - u_2 v_1.$$

That is, vector

$$\vec{\mathbf{w}} = \langle u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1 \rangle$$

is orthogonal to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, which leads us to define the following operation, called the *cross product*.

Definition: Cross Product

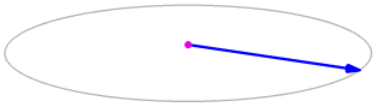
Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$. Then, the *cross product* $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is vector

$$\begin{aligned} \vec{\mathbf{u}} \times \vec{\mathbf{v}} &= (u_2 v_3 - u_3 v_2)\hat{\mathbf{i}} - (u_1 v_3 - u_3 v_1)\hat{\mathbf{j}} + (u_1 v_2 - u_2 v_1)\hat{\mathbf{k}} \\ &= \langle u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1 \rangle. \end{aligned} \tag{12.4.3}$$

From the way we have developed $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$, it should be clear that the cross product is orthogonal to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$. However, it never hurts to check. To show that $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is orthogonal to $\vec{\mathbf{u}}$, we calculate the dot product of $\vec{\mathbf{u}}$ and $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$.

$$\begin{aligned}\vec{\mathbf{u}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) &= \langle u_1, u_2, u_3 \rangle \cdot \langle u_2v_3 - u_3v_2, -u_1v_3 + u_3v_1, u_1v_2 - u_2v_1 \rangle \\ &= u_1(u_2v_3 - u_3v_2) + u_2(-u_1v_3 + u_3v_1) + u_3(u_1v_2 - u_2v_1) \\ &= u_1u_2v_3 - u_1u_3v_2 - u_1u_2v_3 + u_2u_3v_1 + u_1u_3v_2 - u_2u_3v_1 \\ &= (u_1u_2v_3 - u_1u_2v_3) + (-u_1u_3v_2 + u_1u_3v_2) + (u_2u_3v_1 - u_2u_3v_1) \\ &= 0\end{aligned}$$

In a similar manner, we can show that the cross product is also orthogonal to $\vec{\mathbf{v}}$.



The cross product $\vec{\mathrm{a}} \times \vec{\mathrm{b}}$ (vertical, in pink) changes as the angle between the vectors $\vec{\mathrm{a}}$ (blue) and $\vec{\mathrm{b}}$ (red) changes. The cross product (purple) is always perpendicular to both vectors, and has magnitude zero when the vectors are parallel and maximum magnitude $\|\vec{\mathrm{a}}\|\|\vec{\mathrm{b}}\|$ when they are perpendicular. (Public Domain; LucasVB).

Example 12.4.1: Finding a Cross Product

Let $\vec{\mathbf{p}} = \langle -1, 2, 5 \rangle$ and $\vec{\mathbf{q}} = \langle 4, 0, -3 \rangle$ (Figure 12.4.1). Find $\vec{\mathbf{p}} \times \vec{\mathbf{q}}$.

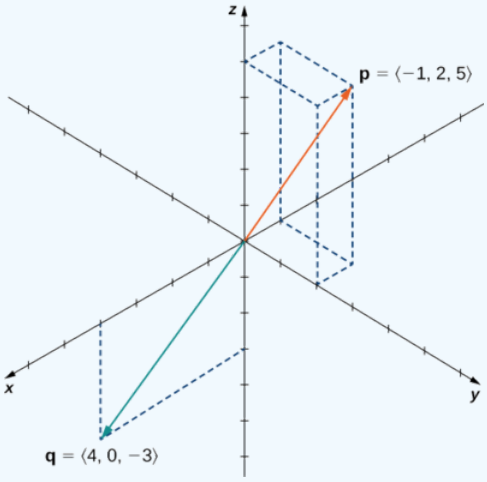


Figure 12.4.1: Finding a cross product to two given vectors.

Solution

Substitute the components of the vectors into Equation 12.4.3:

$$\begin{aligned}\vec{\mathbf{p}} \times \vec{\mathbf{q}} &= \langle -1, 2, 5 \rangle \times \langle 4, 0, -3 \rangle \\ &= \langle p_2q_3 - p_3q_2, -(p_1q_3 - p_3q_1), p_1q_2 - p_2q_1 \rangle \\ &= \langle 2(-3) - 5(0), -(-1)(-3) + 5(4), (-1)(0) - 2(4) \rangle \\ &= \langle -6, 17, -8 \rangle.\end{aligned}$$

Exercise 12.4.1

Find $\vec{\mathbf{p}} \times \vec{\mathbf{q}}$ for $\vec{\mathbf{p}} = \langle 5, 1, 2\rangle$ and $\vec{\mathbf{q}} = \langle -2, 0, 1\rangle$. Express the answer using standard unit vectors.

Hint

Use the formula $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (u_2v_3 - u_3v_2)\hat{\mathbf{i}} - (u_1v_3 - u_3v_1)\hat{\mathbf{j}} + (u_1v_2 - u_2v_1)\hat{\mathbf{k}}$.

Answer

$\vec{\mathbf{p}} \times \vec{\mathbf{q}} = \hat{\mathbf{i}} - 9\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

Although it may not be obvious from Equation 12.4.3, the direction of $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is given by the right-hand rule. If we hold the right hand out with the fingers pointing in the direction of $\vec{\mathbf{u}}$, then curl the fingers toward vector $\vec{\mathbf{v}}$, the thumb points in the direction of the cross product, as shown in Figure 12.4.2.

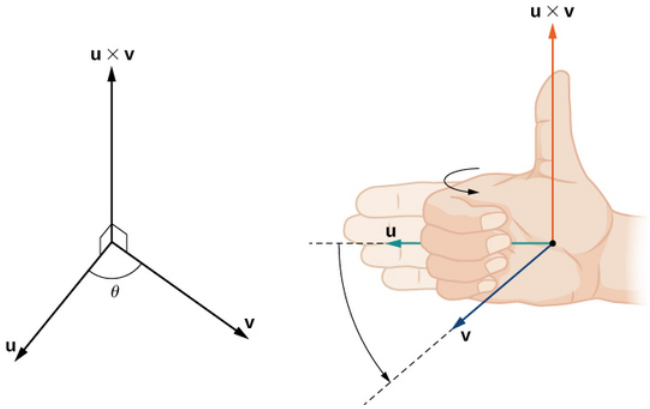


Figure 12.4.2: The direction of $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is determined by the right-hand rule.

Notice what this means for the direction of $\vec{\mathbf{v}} \times \vec{\mathbf{u}}$. If we apply the right-hand rule to $\vec{\mathbf{v}} \times \vec{\mathbf{u}}$, we start with our fingers pointed in the direction of $\vec{\mathbf{v}}$, then curl our fingers toward the vector $\vec{\mathbf{u}}$. In this case, the thumb points in the opposite direction of $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$. (Try it!)

Example 12.4.2: Anticommutativity of the Cross Product

Let $\vec{\mathbf{u}} = \langle 0, 2, 1\rangle$ and $\vec{\mathbf{v}} = \langle 3, -1, 0\rangle$. Calculate $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ and $\vec{\mathbf{v}} \times \vec{\mathbf{u}}$ and graph them.

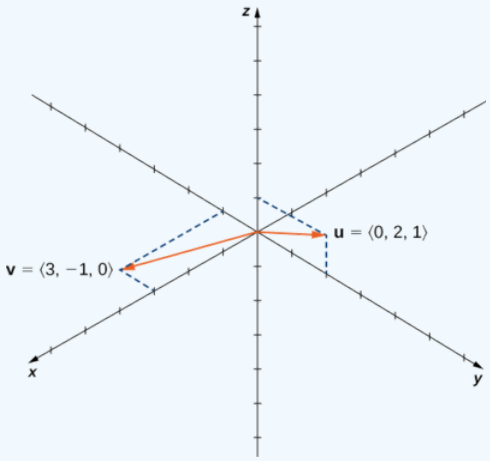


Figure 12.4.3: Are the cross products $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ and $\vec{\mathbf{v}} \times \vec{\mathbf{u}}$ in the same direction?

Solution

We have

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \langle (0 + 1), -(0 - 3), (0 - 6)\rangle = \langle 1, 3, -6\rangle$$

$$\vec{\mathbf{v}} \times \vec{\mathbf{u}} = \langle (-1 - 0), -(3 - 0), (6 - 0)\rangle = \langle -1, -3, 6\rangle.$$

We see that, in this case, $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$ (Figure 12.4.4). We prove this in general later in this section.

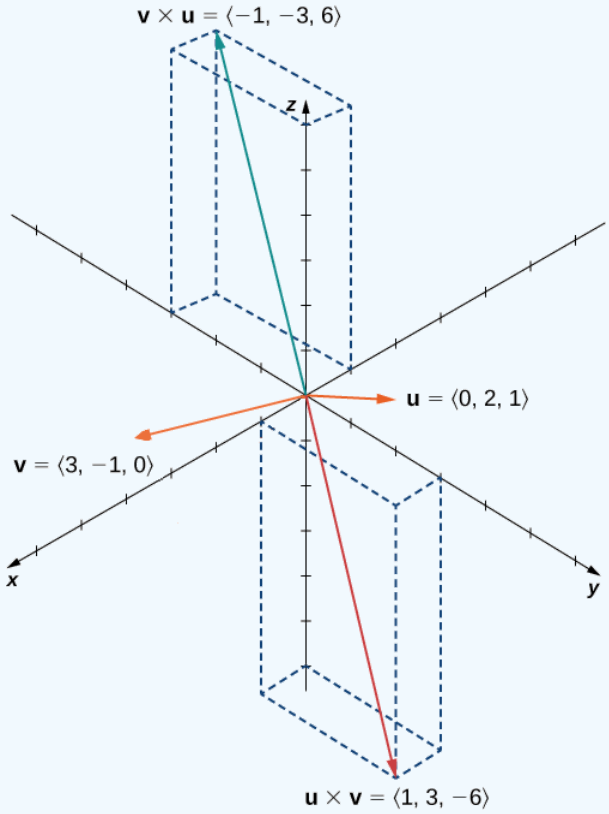


Figure 12.4.4: The cross products $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$ are both orthogonal to $\vec{u}$ and $\vec{v}$, but in opposite directions.

? Exercise 12.4.2

Suppose vectors $\vec{u}$ and $\vec{v}$ lie in the xy-plane (the z-component of each vector is zero). Now suppose the x- and y-components of $\vec{u}$ and the y-component of $\vec{v}$ are all positive, whereas the x-component of $\vec{v}$ is negative. Assuming the coordinate axes are oriented in the usual positions, in which direction does $\vec{u} \times \vec{v}$ point?

Hint

Remember the right-hand rule (Figure 12.4.2).

Answer

Up (the positive z-direction)

The cross products of the standard unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ can be useful for simplifying some calculations, so let's consider these cross products. A straightforward application of the definition shows that

$$\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = \vec{0}.$$

(The cross product of two vectors is a vector, so each of these products results in the zero vector, not the scalar 0.) It's up to you to verify the calculations on your own.

Furthermore, because the cross product of two vectors is orthogonal to each of these vectors, we know that the cross product of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ is parallel to $\hat{\mathbf{k}}$. Similarly, the vector product of $\hat{\mathbf{i}}$ and $\hat{\mathbf{k}}$ is parallel to $\hat{\mathbf{j}}$, and the vector product of $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ is parallel to $\hat{\mathbf{i}}$.

We can use the right-hand rule to determine the direction of each product. Then we have

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$$
$$\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}}$$
$$\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}$$
$$\hat{\mathbf{k}} \times \hat{\mathbf{j}} = -\hat{\mathbf{i}}$$
$$\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$$
$$\hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}.$$

These formulas come in handy later.

Example 12.4.3: Cross Product of Standard Unit Vectors

Find $\hat{\mathbf{i}} \times (\hat{\mathbf{j}} \times \hat{\mathbf{k}})$.

Solution

We know that $\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}$. Therefore, $\hat{\mathbf{i}} \times (\hat{\mathbf{j}} \times \hat{\mathbf{k}}) = \hat{\mathbf{i}} \times \hat{\mathbf{i}} = \vec{0}$.

Exercise 12.4.3

Find $(\hat{\mathbf{i}} \times \hat{\mathbf{j}}) \times (\hat{\mathbf{k}} \times \hat{\mathbf{i}})$.

Hint

Remember the right-hand rule (Figure 12.4.2).

Answer

$-\hat{\mathbf{i}}$

As we have seen, the dot product is often called the *scalar product* because it results in a scalar. The cross product results in a vector, so it is sometimes called the **vector product.** These operations are both versions of vector multiplication, but they have very different properties and applications. Let's explore some properties of the cross product. We prove only a few of them. Proofs of the other properties are left as exercises.

Properties of the Cross Product

Let $\vec{\mathbf{u}}, \vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ be vectors in space, and let c be a scalar.

i. Anticommutative property:

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = -(\vec{\mathbf{v}} \times \vec{\mathbf{u}})$$

ii. Distributive property:

$$\vec{\mathbf{u}} \times (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \vec{\mathbf{u}} \times \vec{\mathbf{v}} + \vec{\mathbf{u}} \times \vec{\mathbf{w}}$$

iii. Multiplication by a constant:

$$c(\vec{\mathbf{u}} \times \vec{\mathbf{v}}) = (c\vec{\mathbf{u}}) \times \vec{\mathbf{v}} = \vec{\mathbf{u}} \times (c\vec{\mathbf{v}})$$

iv. Cross product of the zero vector:

$$\vec{\mathbf{u}} \times \vec{0} = \vec{0} \times \vec{\mathbf{u}} = \vec{0}$$

v. Cross product of a vector with itself:

$$\vec{\mathbf{v}} \times \vec{\mathbf{v}} = \vec{0}$$

vi. Triple scalar product:

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) \cdot \vec{\mathbf{w}}$$

vii. Triple cross product:

$$\vec{\mathbf{u}} \times (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = (\vec{\mathbf{u}} \cdot \vec{\mathbf{w}})\,\vec{\mathbf{v}} - (\vec{\mathbf{u}} \cdot \vec{\mathbf{v}})\,\vec{\mathbf{w}}$$

Proof

For property i, we want to show $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = -(\vec{\mathbf{v}} \times \vec{\mathbf{u}})$. We have

$$\begin{aligned}\vec{\mathbf{u}} \times \vec{\mathbf{v}} &= \langle u_1, u_2, u_3\rangle \times \langle v_1, v_2, v_3\rangle \\ &= \langle u_2v_3 - u_3v_2, -u_1v_3 + u_3v_1, u_1v_2 - u_2v_1\rangle \\ &= -\langle u_3v_2 - u_2v_3, -u_3v_1 + u_1v_3, u_2v_1 - u_1v_2\rangle \\ &= -\langle v_1, v_2, v_3\rangle \times \langle u_1, u_2, u_3\rangle \\ &= -(\vec{\mathbf{v}} \times \vec{\mathbf{u}}).\end{aligned}$$

Unlike most operations we've seen, the cross product is not commutative. This makes sense if we think about the right-hand rule.

For property iv., this follows directly from the definition of the cross product. We have

$$\vec{\mathbf{u}} \times \vec{\mathbf{0}} = \langle u_2(0) - u_3(0), -(u_1(0) - u_3(0)), u_1(0) - u_2(0)\rangle = \langle 0, 0, 0\rangle = \vec{\mathbf{0}}.$$

Then, by property i., $\vec{\mathbf{0}} \times \vec{\mathbf{u}} = \vec{\mathbf{0}}$ as well. Remember that the dot product of a vector and the zero vector is the *scalar* 0, whereas the cross product of a vector with the zero vector is the *vector* $\vec{\mathbf{0}}$.

Property vi. looks like the associative property, but note the change in operations:

$$\begin{aligned}\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) &= u \cdot \langle v_2w_3 - v_3w_2, -v_1w_3 + v_3w_1, v_1w_2 - v_2w_1\rangle \\ &= u_1(v_2w_3 - v_3w_2) + u_2(-v_1w_3 + v_3w_1) + u_3(v_1w_2 - v_2w_1) \\ &= u_1v_2w_3 - u_1v_3w_2 - u_2v_1w_3 + u_2v_3w_1 + u_3v_1w_2 - u_3v_2w_1 \\ &= (u_2v_3 - u_3v_2)w_1 + (u_3v_1 - u_1v_3)w_2 + (u_1v_2 - u_2v_1)w_3 \\ &= \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1\rangle \cdot \langle w_1, w_2, w_3\rangle = (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) \cdot \vec{\mathbf{w}}.\end{aligned}$$

□

Example 12.4.4: Using the Properties of the Cross Product

Use the cross product properties to calculate $(2\hat{\mathbf{i}} \times 3\hat{\mathbf{j}}) \times \hat{\mathbf{j}}$.

Solution

$$\begin{aligned}(2\hat{\mathbf{i}} \times 3\hat{\mathbf{j}}) \times \hat{\mathbf{j}} &= 2(\hat{\mathbf{i}} \times 3\hat{\mathbf{j}}) \times \hat{\mathbf{j}} \\ &= 2(3)(\hat{\mathbf{i}} \times \hat{\mathbf{j}}) \times \hat{\mathbf{j}} \\ &= (6\hat{\mathbf{k}}) \times \hat{\mathbf{j}} \\ &= 6(\hat{\mathbf{k}} \times \hat{\mathbf{j}}) \\ &= 6(-\hat{\mathbf{i}}) = -6\hat{\mathbf{i}}.\end{aligned}$$

Exercise 12.4.4

Use the properties of the cross product to calculate $(\hat{\mathbf{i}} \times \hat{\mathbf{k}}) \times (\hat{\mathbf{k}} \times \hat{\mathbf{j}})$.

Hint

$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = -(\vec{\mathbf{v}} \times \vec{\mathbf{u}})$

Answer

$-\hat{\mathbf{k}}$

So far in this section, we have been concerned with the direction of the vector $\vec{u} \times \vec{v}$, but we have not discussed its magnitude. It turns out there is a simple expression for the magnitude of $\vec{u} \times \vec{v}$ involving the magnitudes of $\vec{u}$ and $\vec{v}$, and the sine of the angle between them.

Magnitude of the Cross Product

Let $\vec{u}$ and $\vec{v}$ be vectors, and let θ be the angle between them. Then, $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \sin\theta$.

Proof

Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ be vectors, and let θ denote the angle between them. Then

$$\begin{aligned}
\|\vec{u} \times \vec{v}\|^2 &= (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2 \\
&= u_2^2v_3^2 - 2u_2u_3v_2v_3 + u_3^2v_2^2 + u_3^2v_1^2 - 2u_1u_3v_1v_3 + u_1^2v_3^2 + u_1^2v_2^2 - 2u_1u_2v_1v_2 + u_2^2v_1^2 \\
&= u_1^2v_1^2 + u_1^2v_2^2 + u_1^2v_3^2 + u_2^2v_1^2 + u_2^2v_2^2 + u_2^2v_3^2 + u_3^2v_1^2 + u_3^2v_2^2 + u_3^2v_3^2 \\
&\quad - (u_1^2v_1^2 + u_2^2v_2^2 + u_3^2v_3^2 + 2u_1u_2v_1v_2 + 2u_1u_3v_1v_3 + 2u_2u_3v_2v_3) \\
&= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2 \\
&= \|\vec{u}\|^2\|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2 \\
&= \|\vec{u}\|^2\|\vec{v}\|^2 - \|\vec{u}\|^2\|\vec{v}\|^2\cos^2\theta \\
&= \|\vec{u}\|^2\|\vec{v}\|^2(1 - \cos^2\theta) \\
&= \|\vec{u}\|^2\|\vec{v}\|^2(\sin^2\theta).
\end{aligned}$$

Taking square roots and noting that $\sqrt{\sin^2\theta} = \sin\theta$ for $0 \le \theta \le 180°$, we have the desired result:

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\|\sin\theta.$$

□

This definition of the cross product allows us to visualize or interpret the product geometrically. It is clear, for example, that the cross product is defined only for vectors in three dimensions, not for vectors in two dimensions. In two dimensions, it is impossible to generate a vector simultaneously orthogonal to two nonparallel vectors.

Example 12.4.5: Calculating the Cross Product

Use "Magnitude of the Cross Product" to find the magnitude of the cross product of $\vec{u} = \langle 0, 4, 0 \rangle$ and $\vec{v} = \langle 0, 0, -3 \rangle$.

Solution

We have

$$\begin{aligned}
\|\vec{u} \times \vec{v}\| &= \|\vec{u}\| \cdot \|\vec{v}\| \cdot \sin\theta \\
&= \sqrt{0^2 + 4^2 + 0^2} \cdot \sqrt{0^2 + 0^2 + (-3)^2} \cdot \sin\frac{\pi}{2} \\
&= 4(3)(1) = 12
\end{aligned}$$

Exercise 12.4.5

Use "Magnitude of the Cross Product" to find the magnitude of $\vec{u} \times \vec{v}$, where $\vec{u} = \langle -8, 0, 0 \rangle$ and $\vec{v} = \langle 0, 2, 0 \rangle$.

Hint

Vectors $\vec{u}$ and $\vec{v}$ are orthogonal.

Answer

16

Determinants and the Cross Product

Using Equation 12.4.3 to find the cross product of two vectors is straightforward, and it presents the cross product in the useful component form. The formula, however, is complicated and difficult to remember. Fortunately, we have an alternative. We can calculate the cross product of two vectors using determinant notation.

A 2×2 determinant is defined by

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - b_1 a_2.$$

For example,

$$\begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix} = 3(1) - 5(-2) = 3 + 10 = 13.$$

A 3×3 determinant is defined in terms of 2×2 determinants as follows:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}. \tag{12.4.4}$$

Equation 12.4.4 is referred to as the expansion of the determinant along the first row. Notice that the multipliers of each of the 2×2 determinants on the right side of this expression are the entries in the first row of the 3×3 determinant. Furthermore, each of the 2×2 determinants contains the entries from the 3×3 determinant that would remain if you crossed out the row and column containing the multiplier. Thus, for the first term on the right, a_1 is the multiplier, and the 2×2 determinant contains the entries that remain if you cross out the first row and first column of the 3×3 determinant. Similarly, for the second term, the multiplier is a_2, and the 2×2 determinant contains the entries that remain if you cross out the first row and second column of the 3×3 determinant. Notice, however, that the coefficient of the second term is negative. The third term can be calculated in similar fashion.

Example 12.4.6: Using Expansion Along the First Row to Compute a 3×3 Determinant

Evaluate the determinant $\begin{vmatrix} 2 & 5 & -1 \\ -1 & 1 & 3 \\ -2 & 3 & 4 \end{vmatrix}$.

Solution

We have

$$\begin{aligned} \begin{vmatrix} 2 & 5 & -1 \\ -1 & 1 & 3 \\ -2 & 3 & 4 \end{vmatrix} &= 2 \begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} - 5 \begin{vmatrix} -1 & 3 \\ -2 & 4 \end{vmatrix} - 1 \begin{vmatrix} -1 & 1 \\ -2 & 3 \end{vmatrix} \\ &= 2(4-9) - 5(-4+6) - 1(-3+2) \\ &= 2(-5) - 5(2) - 1(-1) = -10 - 10 + 1 \\ &= -19 \end{aligned}$$

Exercise 12.4.6

Evaluate the determinant $\begin{vmatrix} 1 & -2 & -1 \\ 3 & 2 & -3 \\ 1 & 5 & 4 \end{vmatrix}$.

Hint

Expand along the first row. Don't forget the second term is negative!

Answer

40

Technically, determinants are defined only in terms of arrays of real numbers. However, the determinant notation provides a useful mnemonic device for the cross product formula.

Rule: Cross Product Calculated by a Determinant

Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ be vectors. Then the cross product $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is given by

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \hat{\mathbf{k}}.$$

Example 12.4.7: Using Determinant Notation to find $\vec{\mathbf{p}} \times \vec{\mathbf{q}}$

Let $\vec{\mathbf{p}} = \langle -1, 2, 5 \rangle$ and $\vec{\mathbf{q}} = \langle 4, 0, -3 \rangle$. Find $\vec{\mathbf{p}} \times \vec{\mathbf{q}}$.

Solution

We set up our determinant by putting the standard unit vectors across the first row, the components of $\vec{\mathbf{u}}$ in the second row, and the components of $\vec{\mathbf{v}}$ in the third row. Then, we have

$$\begin{aligned} \vec{\mathbf{p}} \times \vec{\mathbf{q}} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 2 & 5 \\ 4 & 0 & -3 \end{vmatrix} = \begin{vmatrix} 2 & 5 \\ 0 & -3 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} -1 & 5 \\ 4 & -3 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} -1 & 2 \\ 4 & 0 \end{vmatrix} \hat{\mathbf{k}} \\ &= (-6 - 0)\hat{\mathbf{i}} - (3 - 20)\hat{\mathbf{j}} + (0 - 8)\hat{\mathbf{k}} \\ &= -6\hat{\mathbf{i}} + 17\hat{\mathbf{j}} - 8\hat{\mathbf{k}}. \end{aligned}$$

Notice that this answer confirms the calculation of the cross product in Example 12.4.1.

Exercise 12.4.7

Use determinant notation to find $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$, where $\vec{\mathbf{a}} = \langle 8, 2, 3 \rangle$ and $\vec{\mathbf{b}} = \langle -1, 0, 4 \rangle$.

Hint

Calculate the determinant $\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 8 & 2 & 3 \\ -1 & 0 & 4 \end{vmatrix}$.

Answer

$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = 8\hat{\mathbf{i}} - 35\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

Using the Cross Product

The cross product is very useful for several types of calculations, including finding a vector orthogonal to two given vectors, computing areas of triangles and parallelograms, and even determining the volume of the three-dimensional geometric shape made of parallelograms known as a *parallelepiped*. The following examples illustrate these calculations.

Example 12.4.8: Finding a Unit Vector Orthogonal to Two Given Vectors

Let $\vec{\mathbf{a}} = \langle 5, 2, -1 \rangle$ and $\vec{\mathbf{b}} = \langle 0, -1, 4 \rangle$. Find a unit vector orthogonal to both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

Solution

The cross product $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is orthogonal to both vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$. We can calculate it with a determinant:

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 5 & 2 & -1 \\ 0 & -1 & 4 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 4 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} 5 & -1 \\ 0 & 4 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} 5 & 2 \\ 0 & -1 \end{vmatrix} \hat{\mathbf{k}}$$
$$= (8-1)\hat{\mathbf{i}} - (20-0)\hat{\mathbf{j}} + (-5-0)\hat{\mathbf{k}}$$
$$= 7\hat{\mathbf{i}} - 20\hat{\mathbf{j}} - 5\hat{\mathbf{k}}.$$

Normalize this vector to find a unit vector in the same direction:

$$\|\vec{\mathbf{a}} \times \vec{\mathbf{b}}\| = \sqrt{(7)^2 + (-20)^2 + (-5)^2} = \sqrt{474}.$$

Thus, $\left\langle \frac{7}{\sqrt{474}}, \frac{-20}{\sqrt{474}}, \frac{-5}{\sqrt{474}} \right\rangle$ is a unit vector orthogonal to $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

Simplified, this vector becomes $\left\langle \frac{7\sqrt{474}}{474}, \frac{-10\sqrt{474}}{237}, \frac{-5\sqrt{474}}{474} \right\rangle$.

Exercise 12.4.8

Find a unit vector orthogonal to both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, where $\vec{\mathbf{a}} = \langle 4, 0, 3 \rangle$ and $\vec{\mathbf{b}} = \langle 1, 1, 4 \rangle$.

Hint

Normalize the cross product.

Answer

$\left\langle \frac{-3}{\sqrt{194}}, \frac{-13}{\sqrt{194}}, \frac{4}{\sqrt{194}} \right\rangle$ or, simplified as $\left\langle \frac{-3\sqrt{194}}{194}, \frac{-13\sqrt{194}}{194}, \frac{2\sqrt{194}}{97} \right\rangle$

To use the cross product for calculating areas, we state and prove the following theorem.

Area of a Parallelogram

If we locate vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ such that they form adjacent sides of a parallelogram, then the area of the parallelogram is given by $\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\|$ (Figure 12.4.5).

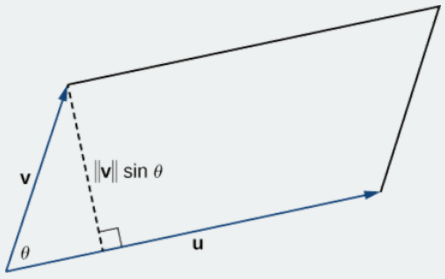


Figure 12.4.5: The parallelogram with adjacent sides $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ has base $\|\vec{\mathbf{u}}\|$ and height $\|\vec{\mathbf{v}}\| \sin\theta$.

Proof

We show that the magnitude of the cross product is equal to the base times height of the parallelogram.

$$\begin{aligned} \text{Area of a parallelogram} &= \text{base} \times \text{height} \\ &= \|\vec{\mathbf{u}}\|(\|\vec{\mathbf{v}}\| \sin\theta) \\ &= \|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\| \end{aligned}$$

□

Example 12.4.9: Finding the Area of a Triangle

Let $P = (1, 0, 0), Q = (0, 1, 0)$, and $R = (0, 0, 1)$ be the vertices of a triangle (Figure 12.4.6). Find its area.

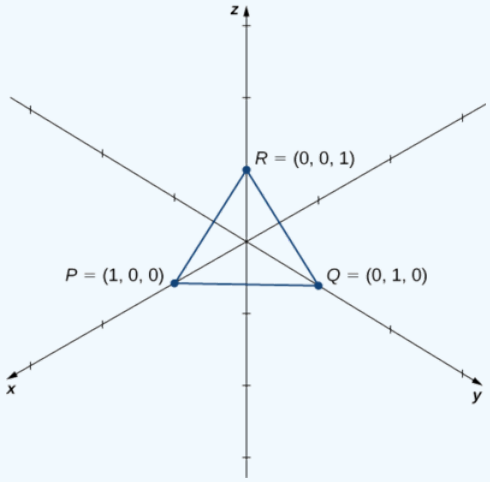


Figure 12.4.6: Finding the area of a triangle by using the cross product.

Solution

We have $\overrightarrow{PQ} = \langle 0-1, 1-0, 0-0\rangle = \langle -1, 1, 0\rangle$ and $\overrightarrow{PR} = \langle 0-1, 0-0, 1-0\rangle = \langle -1, 0, 1\rangle$. The area of the parallelogram with adjacent sides $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is given by $\| \overrightarrow{PQ} \times \overrightarrow{PR} \|$:

$$\begin{aligned} \overrightarrow{PQ} \times \overrightarrow{PR} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} \\ &= (1-0)\hat{\mathbf{i}} - (-1-0)\hat{\mathbf{j}} + (0-(-1))\hat{\mathbf{k}} \\ &= \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}} \end{aligned}$$

$$\begin{aligned} \| \overrightarrow{PQ} \times \overrightarrow{PR} \| &= \| \langle 1, 1, 1\rangle \| \\ &= \sqrt{1^2 + 1^2 + 1^2} \\ &= \sqrt{3}. \end{aligned}$$

The area of ΔPQR is half the area of the parallelogram or $\sqrt{3}/2 \text{ units}^2$.

Exercise 12.4.9

Find the area of the parallelogram $PQRS$ with vertices $P(1, 1, 0)$, $Q(7, 1, 0)$, $R(9, 4, 2)$, and $S(3, 4, 2)$.

Hint

Sketch the parallelogram and identify two vectors that form adjacent sides of the parallelogram.

Answer

$6\sqrt{13} \text{ units}^2$

The Triple Scalar Product

Because the cross product of two vectors is a vector, it is possible to combine the dot product and the cross product. The dot product of a vector with the cross product of two other vectors is called the **triple scalar product** because the result is a scalar.

Definition: Triple Scalar Product

The triple scalar product of vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ is

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}).$$

Calculating a Triple Scalar Product

The triple scalar product of vectors

$$\vec{\mathbf{u}} = u_1\hat{\mathbf{i}} + u_2\hat{\mathbf{j}} + u_3\hat{\mathbf{k}}$$

$$\vec{\mathbf{v}} = v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}} + v_3\hat{\mathbf{k}}$$

and

$$\vec{\mathbf{w}} = w_1\hat{\mathbf{i}} + w_2\hat{\mathbf{j}} + w_3\hat{\mathbf{k}}$$

is the determinant of the 3×3 matrix formed by the components of the vectors:

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}. \tag{12.4.5}$$

Proof

The calculation is straightforward.

$$\begin{aligned} \vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) &= \langle u_1, u_2, u_3 \rangle \cdot \langle v_2w_3 - v_3w_2, -v_1w_3 + v_3w_1, v_1w_2 - v_2w_1 \rangle \\ &= u_1(v_2w_3 - v_3w_2) + u_2(-v_1w_3 + v_3w_1) + u_3(v_1w_2 - v_2w_1) \\ &= u_1(v_2w_3 - v_3w_2) - u_2(v_1w_3 - v_3w_1) + u_3(v_1w_2 - v_2w_1) \\ &= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}. \end{aligned}$$

□

Example 12.4.10: Calculating the Triple Scalar Product

Let $\vec{\mathbf{u}} = \langle 1, 3, 5 \rangle$, $\vec{\mathbf{v}} = \langle 2, -1, 0 \rangle$ and $\vec{\mathbf{w}} = \langle -3, 0, -1 \rangle$. Calculate the triple scalar product $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})$.

Solution

Apply Equation 12.4.5 directly:

$$\begin{aligned} \vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) &= \begin{vmatrix} 1 & 3 & 5 \\ 2 & -1 & 0 \\ -3 & 0 & -1 \end{vmatrix} \\ &= 1\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} - 3\begin{vmatrix} 2 & 0 \\ -3 & -1 \end{vmatrix} + 5\begin{vmatrix} 2 & -1 \\ -3 & 0 \end{vmatrix} \\ &= (1 - 0) - 3(-2 - 0) + 5(0 - 3) \\ &= 1 + 6 - 15 = -8. \end{aligned}$$

Exercise 12.4.10

Calculate the triple scalar product $\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{c}})$, where $\vec{\mathbf{a}} = \langle 2, -4, 1 \rangle$, $\vec{\mathbf{b}} = \langle 0, 3, -1 \rangle$, and $\vec{\mathbf{c}} = \langle 5, -3, 3 \rangle$.

Hint

Place the vectors as the rows of a 3×3 matrix, then calculate the determinant.

Answer

17

When we create a matrix from three vectors, we must be careful about the order in which we list the vectors. If we list them in a matrix in one order and then rearrange the rows, the absolute value of the determinant remains unchanged. However, each time two rows switch places, the determinant changes sign:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = d \qquad \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -d \qquad \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = d \qquad \begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = -d$$

Verifying this fact is straightforward, but rather messy. Let's take a look at this with an example:

$$\begin{aligned} \begin{vmatrix} 1 & 2 & 1 \\ -2 & 0 & 3 \\ 4 & 1 & -1 \end{vmatrix} &= \begin{vmatrix} 0 & 3 \\ 1 & -1 \end{vmatrix} - 2\begin{vmatrix} -2 & 3 \\ 4 & -1 \end{vmatrix} + \begin{vmatrix} -2 & 0 \\ 4 & 1 \end{vmatrix} \\ &= (0-3) - 2(2-12) + (-2-0) \\ &= -3 + 20 - 2 = 15. \end{aligned}$$

Switching the top two rows we have

$$\begin{aligned} \begin{vmatrix} -2 & 0 & 3 \\ 1 & 2 & 1 \\ 4 & 1 & -1 \end{vmatrix} &= -2\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} \\ &= -2(-2-1) + 3(1-8) \\ &= 6 - 21 = -15. \end{aligned}$$

Rearranging vectors in the triple products is equivalent to reordering the rows in the matrix of the determinant. Let $\vec{\mathbf{u}} = u_1\hat{\mathbf{i}} + u_2\hat{\mathbf{j}} + u_3\hat{\mathbf{k}}$, $\vec{\mathbf{v}} = v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}} + v_3\hat{\mathbf{k}}$, and $\vec{\mathbf{w}} = w_1\hat{\mathbf{i}} + w_2\hat{\mathbf{j}} + w_3\hat{\mathbf{k}}$. Applying Calculating a Triple Scalar Product, we have

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

and

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{w}} \times \vec{\mathbf{v}}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

We can obtain the determinant for calculating $\vec{\mathbf{u}} \cdot (\vec{\mathbf{w}} \times \vec{\mathbf{v}})$ by switching the bottom two rows of $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})$. Therefore, $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = -\vec{\mathbf{u}} \cdot (\vec{\mathbf{w}} \times \vec{\mathbf{v}})$.

Following this reasoning and exploring the different ways we can interchange variables in the triple scalar product lead to the following identities:

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = -\vec{\mathbf{u}} \cdot (\vec{\mathbf{w}} \times \vec{\mathbf{v}}) \tag{12.4.6}$$

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = \vec{\mathbf{v}} \cdot (\vec{\mathbf{w}} \times \vec{\mathbf{u}}) = \vec{\mathbf{w}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}}). \tag{12.4.7}$$

Let $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ be two vectors in standard position. If $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are not scalar multiples of each other, then these vectors form adjacent sides of a parallelogram. We saw in Area of a Parallelogram that the area of this parallelogram is $\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\|$. Now suppose we add a third vector $\vec{\mathbf{w}}$ that does not lie in the same plane as $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ but still shares the same initial point. Then these vectors form three edges of a **parallelepiped**, a three-dimensional prism with six faces that are each parallelograms, as shown in Figure 12.4.7. The volume of this prism is the product of the figure's height and the area of its base. The triple scalar product of $\vec{\mathbf{u}}, \vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ provides a simple method for calculating the volume of the parallelepiped defined by these vectors.

Volume of a Parallelepiped

The volume of a parallelepiped with adjacent edges given by the vectors $\vec{u}, \vec{v}$, and $\vec{w}$ is the absolute value of the triple scalar product (Figure 12.4.7):

$$V = |\vec{u} \cdot (\vec{v} \times \vec{w})|.$$

Note that, as the name indicates, the triple scalar product produces a scalar. The volume formula just presented uses the absolute value of a scalar quantity.

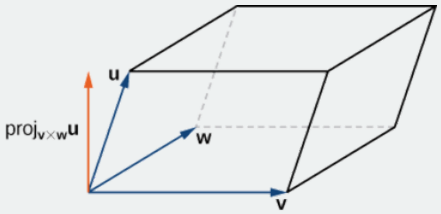


Figure 12.4.7: The height of the parallelepiped is given by $\|\text{proj}_{\vec{v}\times\vec{w}}\vec{u}\|$.

Proof

The area of the base of the parallelepiped is given by $\|\vec{v} \times \vec{w}\|$. The height of the figure is given by $\|\text{proj}_{\vec{v}\times\vec{w}}\vec{u}\|$. The volume of the parallelepiped is the product of the height and the area of the base, so we have

$$\begin{aligned} V &= \| \text{ proj}_{\vec{v}\times\vec{w}}\vec{u} \, \| \, \|\vec{v} \times \vec{w}\| \\ &= \left| \frac{\vec{u} \cdot (\vec{v} \times \vec{w})}{\|\vec{v} \times \vec{w}\|} \right| \|\vec{v} \times \vec{w}\| \\ &= |\vec{u} \cdot (\vec{v} \times \vec{w})|. \end{aligned}$$

□

Example 12.4.11: Calculating the Volume of a Parallelepiped

Let $\vec{u} = \langle -1, -2, 1\rangle, \vec{v} = \langle 4, 3, 2\rangle$, and $\vec{w} = \langle 0, -5, -2\rangle$. Find the volume of the parallelepiped with adjacent edges $\vec{u}, \vec{v}$, and $\vec{w}$ (Figure 12.4.8).

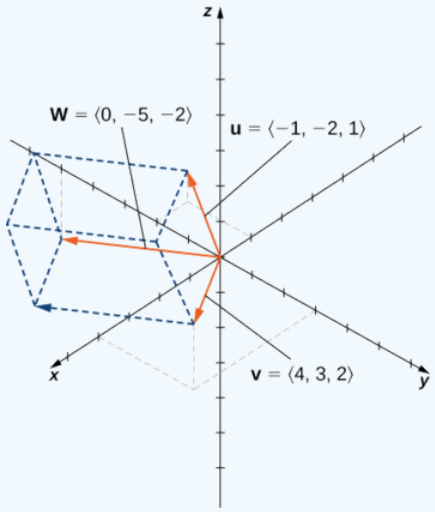


Figure 12.4.8

Solution

We have

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = \begin{vmatrix} -1 & -2 & 1 \\ 4 & 3 & 2 \\ 0 & -5 & -2 \end{vmatrix}$$
$$= (-1)\begin{vmatrix} 3 & 2 \\ -5 & -2 \end{vmatrix} + 2\begin{vmatrix} 4 & 2 \\ 0 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 3 \\ 0 & -5 \end{vmatrix}$$
$$= (-1)(-6+10) + 2(-8-0) + (-20-0)$$
$$= -4 - 16 - 20$$
$$= -40.$$

Thus, the volume of the parallelepiped is $|-40| = 40$ units^3

? Exercise 12.4.11

Find the volume of the parallelepiped formed by the vectors $\vec{\mathbf{a}} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - \hat{\mathbf{k}}$, $\vec{\mathbf{b}} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}}$, and $\vec{\mathbf{c}} = 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$.

Hint

Calculate the triple scalar product by finding a determinant.

Answer

8 units^3

Applications of the Cross Product

The cross product appears in many practical applications in mathematics, physics, and engineering. Let's examine some of these applications here, including the idea of torque, with which we began this section. Other applications show up in later chapters, particularly in our study of vector fields such as gravitational and electromagnetic fields (Introduction to Vector Calculus).

✓ Example 12.4.12: Using the Triple Scalar Product

Use the triple scalar product to show that vectors $\vec{\mathbf{u}} = \langle 2, 0, 5\rangle$, $\vec{\mathbf{v}} = \langle 2, 2, 4\rangle$, and $\vec{\mathbf{w}} = \langle 1, -1, 3\rangle$ are coplanar—that is, show that these vectors lie in the same plane.

Solution

Start by calculating the triple scalar product to find the volume of the parallelepiped defined by $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$:

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = \begin{vmatrix} 2 & 0 & 5 \\ 2 & 2 & 4 \\ 1 & -1 & 3 \end{vmatrix}$$
$$= [2(2)(3) + (0)(4)(1) + 5(2)(-1)] - [5(2)(1) + (2)(4)(-1) + (0)(2)(3)]$$
$$= 2 - 2 = 0.$$

The volume of the parallelepiped is 0 units^3, so one of the dimensions must be zero. Therefore, the three vectors all lie in the same plane.

? Exercise 12.4.12

Are the vectors $\vec{\mathbf{a}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$, $\vec{\mathbf{b}} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$, and $\vec{\mathbf{c}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ coplanar?

Hint

Calculate the triple scalar product.

Answer

No, the triple scalar product is $-4 \neq 0$, so the three vectors form the adjacent edges of a parallelepiped. They are not coplanar.

Example 12.4.13: Finding an Orthogonal Vector

Only a single plane can pass through any set of three noncolinear points. Find a vector orthogonal to the plane containing points $P = (9, -3, -2), Q = (1, 3, 0)$, and $R = (-2, 5, 0)$.

Solution

The plane must contain vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$:

$$\overrightarrow{PQ} = \langle 1 - 9, 3 - (-3), 0 - (-2) \rangle = \langle -8, 6, 2 \rangle$$

$$\overrightarrow{QR} = \langle -2 - 1, 5 - 3, 0 - 0 \rangle = \langle -3, 2, 0 \rangle.$$

The cross product $\overrightarrow{PQ} \times \overrightarrow{QR}$ produces a vector orthogonal to both $\overrightarrow{PQ}$ and $\overrightarrow{QR}$. Therefore, the cross product is orthogonal to the plane that contains these two vectors:

$$\begin{aligned} \overrightarrow{PQ} \times \overrightarrow{QR} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -8 & 6 & 2 \\ -3 & 2 & 0 \end{vmatrix} \\ &= 0\hat{\mathbf{i}} - 6\hat{\mathbf{j}} - 16\hat{\mathbf{k}} - (-18\hat{\mathbf{k}} + 4\hat{\mathbf{i}} + 0\hat{\mathbf{j}}) \\ &= -4\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 2\hat{\mathbf{k}}. \end{aligned}$$

We have seen how to use the triple scalar product and how to find a vector orthogonal to a plane. Now we apply the cross product to real-world situations.

Sometimes a force causes an object to rotate. For example, turning a screwdriver or a wrench creates this kind of rotational effect, called torque.

Definition: Torque

Torque, $\vec{\tau}$ (the Greek letter tau), measures the tendency of a force to produce rotation about an axis of rotation. Let $\vec{\mathbf{r}}$ be a vector with an initial point located on the axis of rotation and with a terminal point located at the point where the force is applied, and let vector $\vec{\mathbf{F}}$ represent the force. Then torque is equal to the cross product of $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$:

$$\vec{\tau} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}.$$

See Figure 12.4.9.

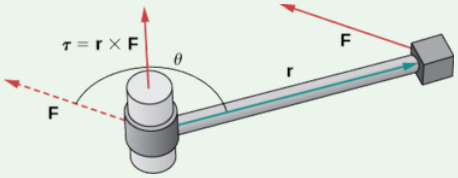


Figure 12.4.9: Torque measures how a force causes an object to rotate.

Think about using a wrench to tighten a bolt. The torque τ applied to the bolt depends on how hard we push the wrench (force) and how far up the handle we apply the force (distance). The torque increases with a greater force on the wrench at a greater distance from the bolt. Common units of torque are the newton-meter or foot-pound. Although torque is dimensionally equivalent to work (it has the same units), the two concepts are distinct. Torque is used specifically in the context of rotation, whereas work typically involves motion along a line.

Example 12.4.14: Evaluating Torque

A bolt is tightened by applying a force of 6 N to a 0.15-m wrench (Figure 12.4.10). The angle between the wrench and the force vector is 40°. Find the magnitude of the torque about the center of the bolt. Round the answer to two decimal places.

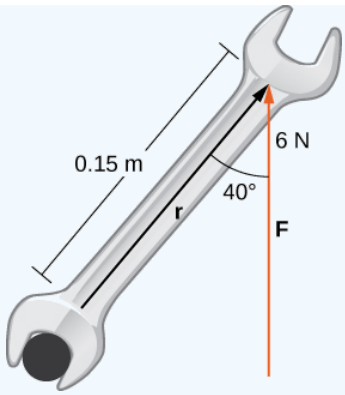


Figure 12.4.10: Torque describes the twisting action of the wrench.

Solution:

Substitute the given information into the equation defining torque:

$$\begin{aligned} \|\vec{\tau}\| &= \|\vec{r} \times \vec{F}\| \\ &= \|\vec{r}\| \, \| \, \vec{F} \, \| \sin\theta \\ &= (0.15\,\text{m})(6\,\text{N}) \sin 40° \\ &\approx 0.58\,\text{N·m}. \end{aligned}$$

Exercise 12.4.14

Calculate the force required to produce 15 N·m torque at an angle of 30° from a 150-cm rod.

Hint

$\|\vec{\tau}\| = 15$ N·m and $\|\vec{r}\| = 1.5$ m

Answer

20 N

Key Concepts

- The cross product $\vec{u} \times \vec{v}$ of two vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is a vector orthogonal to both $\vec{u}$ and $\vec{v}$. Its length is given by $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \sin\theta$, where θ is the angle between $\vec{u}$ and $\vec{v}$. Its direction is given by the right-hand rule.
- The algebraic formula for calculating the cross product of two vectors,

  $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$, is

  $\vec{u} \times \vec{v} = (u_2 v_3 - u_3 v_2)\hat{\mathbf{i}} - (u_1 v_3 - u_3 v_1)\hat{\mathbf{j}} + (u_1 v_2 - u_2 v_1)\hat{\mathbf{k}}$.
- The cross product satisfies the following properties for vectors $\vec{u}, \vec{v}$, and $\vec{w}$, and scalar c:

  $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$

  $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

  $c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$

  $\vec{u} \times \vec{0} = \vec{0} \times \vec{u} = \vec{0}$

  $\vec{v} \times \vec{v} = \vec{0}$

  $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$
- The cross product of vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is the determinant $\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$
- If vectors $\vec{u}$ and $\vec{v}$ form adjacent sides of a parallelogram, then the area of the parallelogram is given by $\|\vec{u} \times \vec{v}\|$.

- The triple scalar product of vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ is $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})$.
- The volume of a parallelepiped with adjacent edges given by vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ is $V = |\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})|$.
- If the triple scalar product of vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ is zero, then the vectors are coplanar. The converse is also true: If the vectors are coplanar, then their triple scalar product is zero.
- The cross product can be used to identify a vector orthogonal to two given vectors or to a plane.
- Torque $\vec{\boldsymbol{\tau}}$ measures the tendency of a force to produce rotation about an axis of rotation. If force $\vec{\mathbf{F}}$ is acting at a distance (displacement) $\vec{\mathbf{r}}$ from the axis, then torque is equal to the cross product of $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$: $\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$.

Key Equations

- **The cross product of two vectors in terms of the unit vectors**

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (u_2 v_3 - u_3 v_2)\hat{\mathbf{i}} - (u_1 v_3 - u_3 v_1)\hat{\mathbf{j}} + (u_1 v_2 - u_2 v_1)\hat{\mathbf{k}}$$

Glossary

cross product

$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (u_2 v_3 - u_3 v_2)\hat{\mathbf{i}} - (u_1 v_3 - u_3 v_1)\hat{\mathbf{j}} + (u_1 v_2 - u_2 v_1)\hat{\mathbf{k}}$, where $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$

determinant

a real number associated with a square matrix

parallelepiped

a three-dimensional prism with six faces that are parallelograms

torque

the effect of a force that causes an object to rotate

triple scalar product

the dot product of a vector with the cross product of two other vectors: $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})$

vector product

the cross product of two vectors

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12.4E: Exercises for Section 12.4

For exercises 1-4, the vectors $\vec{u}$ and $\vec{v}$ are given.

a. Find the cross product $\vec{u} \times \vec{v}$ of the vectors $\vec{u}$ and $\vec{v}$. Express the answer in component form.

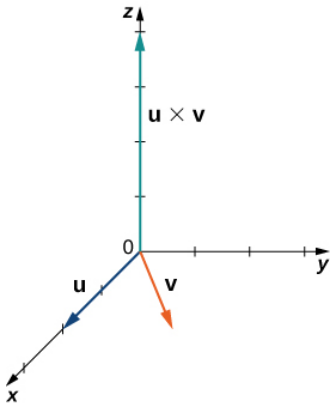
b. Sketch the vectors $\vec{u}$, $\vec{v}$, and $\vec{u} \times \vec{v}$.

1) $\vec{u} = \langle 2, 0, 0 \rangle, \quad \vec{v} = \langle 2, 2, 0 \rangle$

Answer

$a. \vec{u} \times \vec{v} = \langle 0, 0, 4 \rangle;$

$b.$



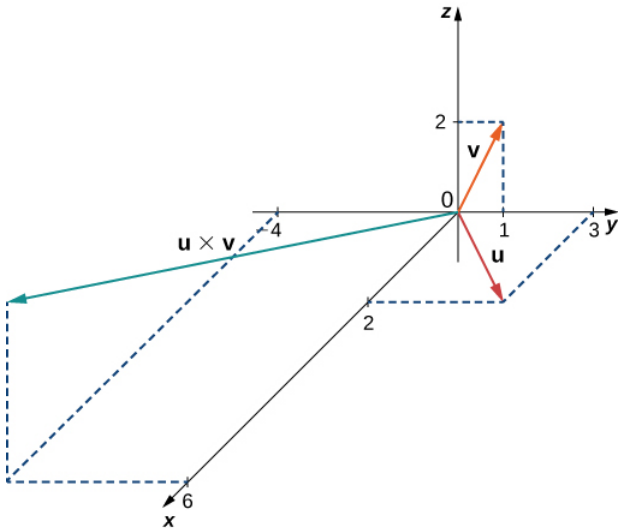
2) $\vec{u} = \langle 3, 2, -1 \rangle, \quad \vec{v} = \langle 1, 1, 0 \rangle$

3) $\vec{u} = 2\hat{i} + 3\hat{j}, \quad \vec{v} = \hat{j} + 2\hat{k}$

Answer

$a. \vec{u} \times \vec{v} = \langle 6, -4, 2 \rangle;$

$b.$



4) $\vec{u} = 2\hat{j} + 3\hat{k}, \quad \vec{v} = 3\hat{i} + \hat{k}$

5) Simplify $(\hat{i} \times \hat{i} - 2\hat{i} \times \hat{j} - 4\hat{i} \times \hat{k} + 3\hat{j} \times \hat{k}) \times \hat{i}$.

Answer

$-2\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$

6) Simplify $\hat{\mathbf{j}} \times (\hat{\mathbf{k}} \times \hat{\mathbf{j}} + 2\hat{\mathbf{j}} \times \hat{\mathbf{i}} - 3\hat{\mathbf{j}} \times \hat{\mathbf{j}} + 5\hat{\mathbf{i}} \times \hat{\mathbf{k}})$.

In exercises 7-10, vectors $\vec{u}$ and $\vec{v}$ are given. Find unit vector $\vec{w}$ in the direction of the cross product vector $\vec{u} \times \vec{v}$. Express your answer using standard unit vectors.

7) $\vec{u} = \langle 3, -1, 2 \rangle, \quad \vec{v} = \langle -2, 0, 1 \rangle$

Answer

$\vec{w} = -\frac{\sqrt{6}}{18}\hat{\mathbf{i}} - \frac{7\sqrt{6}}{18}\hat{\mathbf{j}} - \frac{\sqrt{6}}{9}\hat{\mathbf{k}}$

8) $\vec{u} = \langle 2, 6, 1 \rangle, \quad \vec{v} = \langle 3, 0, 1 \rangle$

9) $\vec{u} = \overrightarrow{AB}, \quad \vec{v} = \overrightarrow{AC}$, where $A(1, 0, 1),\ B(1, -1, 3)$, and $C(0, 0, 5)$

Answer

$\vec{w} = -\frac{4\sqrt{21}}{21}\hat{\mathbf{i}} - \frac{2\sqrt{21}}{21}\hat{\mathbf{j}} - \frac{\sqrt{21}}{21}\hat{\mathbf{k}}$

10) $\vec{u} = \overrightarrow{OP}, \quad \vec{v} = \overrightarrow{PQ}$, where $P(-1, 1, 0)$ and $Q(0, 2, 1)$

11) Determine the real number α such that $\vec{u} \times \vec{v}$ and $\hat{\mathbf{i}}$ are orthogonal, where $\vec{u} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ and $\vec{v} = 4\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \alpha\hat{\mathbf{k}}$.

Answer

$\alpha = 10$

12) Show that $\vec{u} \times \vec{v}$ and $2\hat{\mathbf{i}} - 14\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ cannot be orthogonal for any α real number, where $\vec{u} = \hat{\mathbf{i}} + 7\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\vec{v} = \alpha\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + \hat{\mathbf{k}}$.

13) Show that $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$, where $\vec{u}$ and $\vec{v}$ are nonzero vectors.

14) Show that $\vec{v} \times \vec{u}$ is orthogonal to $(\vec{u} \cdot \vec{v})(\vec{u} + \vec{v}) + \vec{u}$, where $\vec{u}$ and $\vec{v}$ are nonzero vectors.

15) Calculate the determinant $\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & 7 \\ 2 & 0 & 3 \end{vmatrix}$.

Answer

$-3\hat{\mathbf{i}} + 11\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

16) Calculate the determinant $\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 3 & -4 \\ 1 & 6 & -1 \end{vmatrix}$.

For exercises 17-18, the vectors $\vec{u}$ and $\vec{v}$ are given. Use determinant notation to find vector $\vec{w}$ orthogonal to vectors $\vec{u}$ and $\vec{v}$.

17) $\vec{u} = \langle -1, 0, e^t \rangle, \quad \vec{v} = \langle 1, e^{-t}, 0 \rangle$, where t is a real number

Answer

$\vec{w} = \langle -1, e^t, -e^{-t} \rangle$

18) $\vec{u} = \langle 1, 0, x \rangle, \quad \vec{v} = \langle \frac{2}{x}, 1, 0 \rangle$, where x is a nonzero real number

19) Find vector $(\vec{a} - 2\vec{b}) \times \vec{c}$, where $\vec{a} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & -1 & 5 \\ 0 & 1 & 8 \end{vmatrix}$, $\vec{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 1 & 1 \\ 2 & -1 & -2 \end{vmatrix}$, and $\vec{c} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$.

Answer

$-26\hat{\mathbf{i}} + 17\hat{\mathbf{j}} + 9\hat{\mathbf{k}}$

20) Find vector $\vec{c} \times (\vec{a} + 3\vec{b})$, where $\vec{a} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 5 & 0 & 9 \\ 0 & 1 & 0 \end{vmatrix}$, $\vec{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & -1 & 1 \\ 7 & 1 & -1 \end{vmatrix}$, and $\vec{c} = \hat{\mathbf{i}} - \hat{\mathbf{k}}$.

21) [T] Use the cross product $\vec{u} \times \vec{v}$ to find the acute angle between vectors $\vec{u}$ and $\vec{v}$, where $\vec{u} = \hat{i} + 2\hat{j}$ and $\vec{v} = \hat{i} + \hat{k}$. Express the answer in degrees rounded to the nearest integer.

Answer

$72°$

22) [T] Use the cross product $\vec{u} \times \vec{v}$ to find the obtuse angle between vectors $\vec{u}$ and $\vec{v}$, where $\vec{u} = -\hat{i} + 3\hat{j} + \hat{k}$ and $\vec{v} = \hat{i} - 2\hat{j}$. Express the answer in degrees rounded to the nearest integer.

23) Use the sine and cosine of the angle between two nonzero vectors $\vec{u}$ and $\vec{v}$ to prove Lagrange's identity: $\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2\|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2$.

24) Verify Lagrange's identity $\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2\|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2$ for vectors $\vec{u} = -\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{v} = 2\hat{i} - \hat{j}$.

25) Nonzero vectors $\vec{u}$ and $\vec{v}$ are called collinear if there exists a nonzero scalar α such that $\vec{v} = \alpha\vec{u}$. Show that $\vec{u}$ and $\vec{v}$ are collinear if and only if $\vec{u} \times \vec{v} = 0$.

26) Nonzero vectors $\vec{u}$ and $\vec{v}$ are called *collinear* if there exists a nonzero scalar α such that $\vec{v} = \alpha\vec{u}$. Show that vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are collinear, where $A(4, 1, 0)$, $B(6, 5, -2)$, and $C(5, 3, -1)$.

27) Find the area of the parallelogram with adjacent sides $\vec{u} = \langle 3, 2, 0\rangle$ and $\vec{v} = \langle 0, 2, 1\rangle$.

Answer

7

28) Find the area of the parallelogram with adjacent sides $\vec{u} = \hat{i} + \hat{j}$ and $\vec{v} = \hat{i} + \hat{k}$.

29) Consider points $A(3, -1, 2)$, $B(2, 1, 5)$, and $C(1, -2, -2)$.

a. Find the area of parallelogram $ABCD$ with adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

b. Find the area of triangle ABC.

c. Find the distance from point A to line BC.

Answer

a. $5\sqrt{6}$; b. $\frac{5\sqrt{6}}{2}$; c. $\frac{5\sqrt{6}}{\sqrt{59}} = \frac{5\sqrt{354}}{59}$

30) Consider points $A(2, -3, 4)$, $B(0, 1, 2)$, and $C(-1, 2, 0)$.

a. Find the area of parallelogram $ABCD$ with adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

b. Find the area of triangle ABC.

c. Find the distance from point B to line AC.

In exercises 31-32, vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ are given.

a. Find the triple scalar product $\vec{u} \cdot (\vec{v} \times \vec{w})$.

b. Find the volume of the parallelepiped with the adjacent edges $\vec{u}$, $\vec{v}$, and $\vec{w}$.

31) $\vec{u} = \hat{i} + \hat{j}$, $\vec{v} = \hat{j} + \hat{k}$, and $\vec{w} = \hat{i} + \hat{k}$

Answer

$a. 2$; $b. 2$ units^3

32) $\vec{u} = \langle -3, 5, -1\rangle$, $\vec{v} = \langle 0, 2, -2\rangle$, and $\vec{w} = \langle 3, 1, 1\rangle$

33) Calculate the triple scalar products $\vec{v} \cdot (\vec{u} \times \vec{w})$ and $\vec{w} \cdot (\vec{u} \times \vec{v})$, where $\vec{u} = \langle 1, 1, 1\rangle$, $\vec{v} = \langle 7, 6, 9\rangle$, and $\vec{w} = \langle 4, 2, 7\rangle$.

Answer

$\vec{v} \cdot (\vec{u} \times \vec{w}) = -1$, $\vec{w} \cdot (\vec{u} \times \vec{v}) = 1$

34) Calculate the triple scalar products $\vec{w} \cdot (\vec{v} \times \vec{u})$ and $\vec{u} \cdot (\vec{w} \times \vec{v})$, where $\vec{u} = \langle 4, 2, -1\rangle$, $\vec{v} = \langle 2, 5, -3\rangle$, and $\vec{w} = \langle 9, 5, -10\rangle$.

35) Find vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ with a triple scalar product given by the determinant $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 5 \\ 8 & 9 & 2 \end{vmatrix}$. Determine their triple scalar product.

Answer

$\vec{a} = \langle 1, 2, 3 \rangle, \quad \vec{b} = \langle 0, 2, 5 \rangle, \quad \vec{c} = \langle 8, 9, 2 \rangle; \quad \vec{a} \cdot (\vec{b} \times \vec{c}) = -9$

36) The triple scalar product of vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ is given by the determinant $\begin{vmatrix} 0 & -2 & 1 \\ 0 & 1 & 4 \\ 1 & -3 & 7 \end{vmatrix}$. Find vector $\vec{a} - \vec{b} + \vec{c}$.

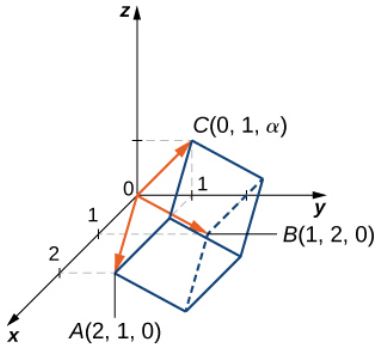
37) Consider the parallelepiped with edges OA, OB, and OC, where $A(2, 1, 0), B(1, 2, 0)$, and $C(0, 1, \alpha)$.

a. Find the real number $\alpha > 0$ such that the volume of the parallelepiped is 3 units^3.

b. For $\alpha = 1$, find the height h from vertex C of the parallelepiped. Sketch the parallelepiped.

Answer

$a.\ \alpha = 1; \quad b.\ h = 1$ unit,



38) Consider points $A(\alpha, 0, 0), B(0, \beta, 0)$, and $C(0, 0, \gamma)$, with α, β, and γ positive real numbers.

a. Determine the volume of the parallelepiped with adjacent sides $\overrightarrow{OA}, \overrightarrow{OB}$, and $\overrightarrow{OC}$.

b. Find the volume of the tetrahedron with vertices O, A, B, and C. (Hint: The volume of the tetrahedron is $1/6$ of the volume of the parallelepiped.)

c. Find the distance from the origin to the plane determined by A, B, and C. Sketch the parallelepiped and tetrahedron.

39) Let u, v, and w be three-dimensional vectors and c be a real number. Prove the following properties of the cross product.

a. $\vec{u} \times \vec{u} = \vec{0}$

b. $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$

c. $c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$

d. $\vec{u} \cdot (\vec{u} \times \vec{v}) = \vec{0}$

40) Show that vectors $\vec{u} = \langle 1, 0, -8 \rangle$, $\vec{v} = \langle 0, 1, 6 \rangle$, and $\vec{w} = \langle -1, 9, 3 \rangle$ satisfy the following properties of the cross product.

a. $\vec{u} \times \vec{u} = \vec{0}$

b. $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$

c. $c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$

d. $\vec{u} \cdot (\vec{u} \times \vec{v}) = \vec{0}$

41) Nonzero vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ are said to be linearly dependent if one of the vectors is a linear combination of the other two. For instance, there exist two nonzero real numbers α and β such that $\vec{w} = \alpha\vec{u} + \beta\vec{v}$. Otherwise, the vectors are called linearly independent. Show that $\vec{u}, \vec{v}$, and $\vec{w}$ could be placed on the same plane if and only if they are linear dependent.

42) Consider vectors $\vec{u} = \langle 1, 4, -7 \rangle$, $\vec{v} = \langle 2, -1, 4 \rangle$, $\vec{w} = \langle 0, -9, 18 \rangle$, and $\vec{p} = \langle 0, -9, 17 \rangle$.

a. Show that $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ can be placed on the same plane by using their triple scalar product

b. Show that $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ can be placed on the same plane by using the definition that there exist two nonzero real numbers α and β such that $w = \alpha u + \beta v$.

c. Show that $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{p}}$ are linearly independent—that is, none of the vectors is a linear combination of the other two.

43) Consider points $A(0,0,2), B(1,0,2), C(1,1,2)$, and $D(0,1,2)$. Are vectors $\overrightarrow{AB}, \overrightarrow{AC}$, and $\overrightarrow{AD}$ linearly dependent (that is, one of the vectors is a linear combination of the other two)?

Answer

Yes, $\overrightarrow{AD} = \alpha\overrightarrow{AB} + \beta\overrightarrow{AC}$, where $\alpha = -1$ and $\beta = 1$.

44) Show that vectors $\hat{\mathbf{i}} + \hat{\mathbf{j}}, \hat{\mathbf{i}} - \hat{\mathbf{j}}$, and $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ are linearly independent—that is, there exist two nonzero real numbers α and β such that $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}} = \alpha(\hat{\mathbf{i}} + \hat{\mathbf{j}}) + \beta(\hat{\mathbf{i}} - \hat{\mathbf{j}})$.

45) Let $\vec{\mathbf{u}} = \langle u_1, u_2 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2 \rangle$ be two-dimensional vectors. The cross product of vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ is not defined. However, if the vectors are regarded as the three-dimensional vectors $\tilde{\vec{\mathbf{u}}} = \langle u_1, u_2, 0 \rangle$ and $\tilde{\vec{\mathbf{v}}} = \langle v_1, v_2, 0 \rangle$, respectively, then, in this case, we can define the cross product of $\tilde{\vec{\mathbf{u}}}$ and $\tilde{\vec{\mathbf{v}}}$. In particular, in determinant notation, the cross product of $\tilde{\vec{\mathbf{u}}}$ and $\tilde{\vec{\mathbf{v}}}$ is given by

$$\tilde{\vec{\mathbf{u}}} \times \tilde{\vec{\mathbf{v}}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_1 & u_2 & 0 \\ v_1 & v_2 & 0 \end{vmatrix}.$$

Use this result to compute $(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) \times (\sin\theta\,\hat{\mathbf{i}} - \cos\theta\,\hat{\mathbf{j}})$, where θ is a real number.

Answer

$-\hat{\mathbf{k}}$

46) Consider points $P(2,1), Q(4,2)$, and $R(1,2)$.

a. Find the area of triangle PQR.

b. Determine the distance from point R to the line passing through P and Q.

47) Determine a vector of magnitude 10 perpendicular to the plane passing through the x-axis and point $P(1,2,4)$.

Answer

$\langle 0, \pm 4\sqrt{5}, 2\sqrt{5} \rangle$

48) Determine a unit vector perpendicular to the plane passing through the z-axis and point $A(3,1,-2)$.

49) Consider $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ two three-dimensional vectors. If the magnitude of the cross product vector $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is k times larger than the magnitude of vector $\vec{\mathbf{u}}$, show that the magnitude of $\vec{\mathbf{v}}$ is greater than or equal to k, where k is a natural number.

50) [T] Assume that the magnitudes of two nonzero vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are known. The function $f(\theta) = \|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\| \sin\theta$ defines the magnitude of the cross product vector $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$, where $\theta \in [0, \pi]$ is the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$.

a. Graph the function f.

b. Find the absolute minimum and maximum of function f. Interpret the results.

c. If $\|\vec{\mathbf{u}}\| = 5$ and $\|\vec{\mathbf{v}}\| = 2$, find the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ if the magnitude of their cross product vector is equal to 9.

51) Find all vectors $\vec{\mathbf{w}} = \langle w_1, w_2, w_3 \rangle$ that satisfy the equation $\langle 1,1,1 \rangle \times \vec{\mathbf{w}} = \langle -1, -1, 2 \rangle$. Hint: You should be able to write all components of $\vec{\mathbf{w}}$ in terms of one of the constants w_1, w_2, or w_3.

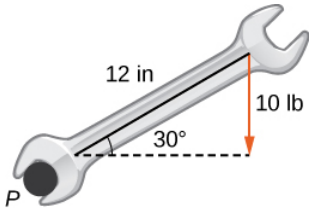
Answer

Writing all components in terms of the constant w_3, one way to represent these vectors is: $\vec{\mathbf{w}} = \langle w_3 - 1, w_3 + 1, w_3 \rangle$, where w_3 is any real number.

Note that we could use any parameter we wish here. We could set $w_3 = a$. Then $\vec{\mathbf{w}} = \langle a - 1, a + 1, a \rangle$ would also represent these vectors.

52) Solve the equation $\vec{\mathbf{w}} \times \langle 1, 0, -1 \rangle = \langle 3, 0, 3 \rangle$, where $\vec{\mathbf{w}} = \langle w_1, w_2, w_3 \rangle$ is a nonzero vector with a magnitude of 3.

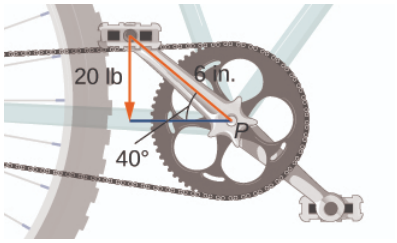
53) [T] A mechanic uses a 12-in. wrench to turn a bolt. The wrench makes a $30°$ angle with the horizontal. If the mechanic applies a vertical force of 10 lb on the wrench handle, what is the magnitude of the torque at point P (see the following figure)? Express the answer in foot-pounds rounded to two decimal places.



Answer

8.66 ft-lb

54) [T] A boy applies the brakes on a bicycle by applying a downward force of 20 lb on the pedal when the 6-in. crank makes a $40°$ angle with the horizontal (see the following figure). Find the torque at point P. Express your answer in foot-pounds rounded to two decimal places.



55) [T] Find the magnitude of the force that needs to be applied to the end of a 20-cm wrench located on the positive direction of the y-axis if the force is applied in the direction $\langle 0, 1, -2 \rangle$ and it produces a 100 N·m torque to the bolt located at the origin.

Answer

$250\sqrt{5}$ N ≈ 559 N

56) [T] What is the magnitude of the force required to be applied to the end of a 1-ft wrench at an angle of $35°$ to produce a torque of 20 N·m?

57) [T] The force vector $\vec{\mathbf{F}}$ acting on a proton with an electric charge of $1.6 \times 10^{-19}\, C$ (in coulombs) moving in a magnetic field $\vec{\mathbf{B}}$ where the velocity vector $\vec{\mathbf{v}}$ is given by $\vec{\mathbf{F}} = 1.6 \times 10^{-19}(\vec{\mathbf{v}} \times \vec{\mathbf{B}})$ (here, $\vec{\mathbf{v}}$ is expressed in meters per second, $\vec{\mathbf{B}}$ is in tesla [T], and $\vec{\mathbf{F}}$ is in newtons [N]). Find the force that acts on a proton that moves in the xy-plane at velocity $\vec{\mathbf{v}} = 10^5\hat{\mathbf{i}} + 10^5\hat{\mathbf{j}}$ (in meters per second) in a magnetic field given by $\vec{\mathbf{B}} = 0.3\hat{\mathbf{j}}$.

Answer

$\vec{\mathbf{F}} = 4.8 \times 10^{-15}\, kN$

58) [T] The force vector $\vec{\mathbf{F}}$ acting on a proton with an electric charge of $1.6 \times 10^{-19}\, C$ moving in a magnetic field $\vec{\mathbf{B}}$ where the velocity vector $\vec{\mathbf{v}}$ is given by $\vec{\mathbf{F}} = 1.6 \times 10^{-19}(\vec{\mathbf{v}} \times \vec{\mathbf{B}})$ (here, $\vec{\mathbf{v}}$ is expressed in meters per second, $\vec{\mathbf{B}}$ in T, and $\vec{\mathbf{F}}$ in N). If the magnitude of force $\vec{\mathbf{F}}$ acting on a proton is $5.9 \times 10^{-17}\, N$ and the proton is moving at the speed of 300 m/sec in magnetic field $\vec{\mathbf{B}}$ of magnitude 2.4 T, find the angle between velocity vector $\vec{\mathbf{v}}$ of the proton and magnetic field $\vec{\mathbf{B}}$. Express the answer in degrees rounded to the nearest integer.

59) [T] Consider $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t, 2t \rangle$ the position vector of a particle at time $t \in [0, 30]$, where the components of $\vec{\mathbf{r}}$ are expressed in centimeters and time in seconds. Let $\overrightarrow{OP}$ be the position vector of the particle after 1 sec.

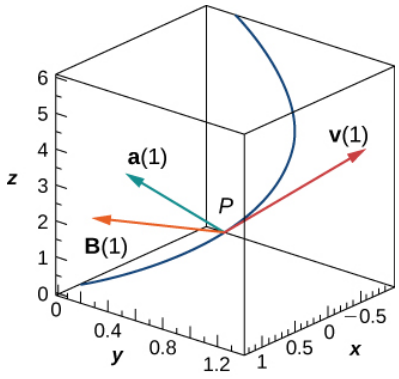
a. Determine unit vector $\vec{\mathbf{B}}(t)$ (called the binormal unit vector) that has the direction of cross product vector $\vec{\mathbf{v}}(t) \times \vec{\mathbf{a}}(t)$, where $\vec{\mathbf{v}}(t)$ and $\vec{\mathbf{a}}(t)$ are the instantaneous velocity vector and, respectively, the acceleration vector of the particle after t seconds.

b. Use a CAS to visualize vectors $\vec{\mathbf{v}}(1)$, $\vec{\mathbf{a}}(1)$, and $\vec{\mathbf{B}}(1)$ as vectors starting at point P along with the path of the particle.

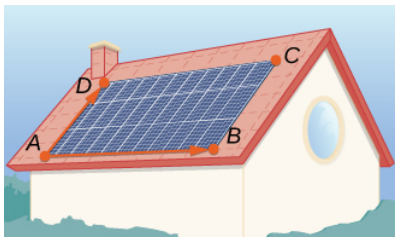
Answer

a. $\vec{\mathbf{B}}(t) = \langle \frac{2\sqrt{5}\sin t}{5}, -\frac{2\sqrt{5}\cos t}{5}, \frac{\sqrt{5}}{5} \rangle$;

b.



60) A solar panel is mounted on the roof of a house. The panel may be regarded as positioned at the points of coordinates (in meters) $A(8,0,0), B(8,18,0), C(0,18,8)$, and $D(0,0,8)$ (see the following figure).



a. Find vector $\vec{\mathbf{n}} = \overrightarrow{AB} \times \overrightarrow{AD}$ perpendicular to the surface of the solar panels. Express the answer using standard unit vectors. Note that the magnitude of this vector should give us the area of rectangle $ABCD$.

b. Assume unit vector $\vec{\mathbf{s}} = \frac{1}{\sqrt{3}}\hat{\mathbf{i}} + \frac{1}{\sqrt{3}}\hat{\mathbf{j}} + \frac{1}{\sqrt{3}}\hat{\mathbf{k}}$ points toward the Sun at a particular time of the day and the flow of solar energy is $\vec{\mathbf{F}} = 900\vec{\mathbf{s}}$ (in watts per square meter $[W/m^2]$). Find the predicted amount of electrical power the panel can produce, which is given by the dot product of vectors $\vec{\mathbf{F}}$ and $\vec{\mathbf{n}}$ (expressed in watts).

c. Determine the angle of elevation of the Sun above the solar panel. Express the answer in degrees rounded to the nearest whole number. (Hint: The angle between vectors $\vec{\mathbf{n}}$ and $\vec{\mathbf{s}}$ and the angle of elevation are complementary.)

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12.5: Equations of Lines and Planes in Space

Learning Objectives

- Write the vector, parametric, and symmetric equations of a line through a given point in a given direction, and a line through two given points.
- Find the distance from a point to a given line.
- Write the vector and scalar equations of a plane through a given point with a given normal.
- Find the distance from a point to a given plane.
- Find the angle between two planes.

By now, we are familiar with writing equations that describe a line in two dimensions. To write an equation for a line, we must know two points on the line, or we must know the direction of the line and at least one point through which the line passes. In two dimensions, we use the concept of slope to describe the orientation, or direction, of a line. In three dimensions, we describe the direction of a line using a vector parallel to the line. In this section, we examine how to use equations to describe lines and planes in space.

Equations for a Line in Space

Let's first explore what it means for two vectors to be parallel. Recall that parallel vectors must have the same or opposite directions. If two nonzero vectors, $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, are parallel, we claim there must be a scalar, k, such that $\vec{\mathbf{u}} = k\vec{\mathbf{v}}$. If $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ have the same direction, simply choose

$$k = \frac{\|\vec{\mathbf{u}}\|}{\|\vec{\mathbf{v}}\|}.$$

If $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ have opposite directions, choose

$$k = -\frac{\|\vec{\mathbf{u}}\|}{\|\vec{\mathbf{v}}\|}.$$

Note that the converse holds as well. If $\vec{\mathbf{u}} = k\vec{\mathbf{v}}$ for some scalar k, then either $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ have the same direction $(k > 0)$ or opposite directions $(k < 0)$, so $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are parallel. Therefore, two nonzero vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are parallel if and only if $\vec{\mathbf{u}} = k\vec{\mathbf{v}}$ for some scalar k. By convention, the zero vector $\vec{\mathbf{0}}$ is considered to be parallel to all vectors.

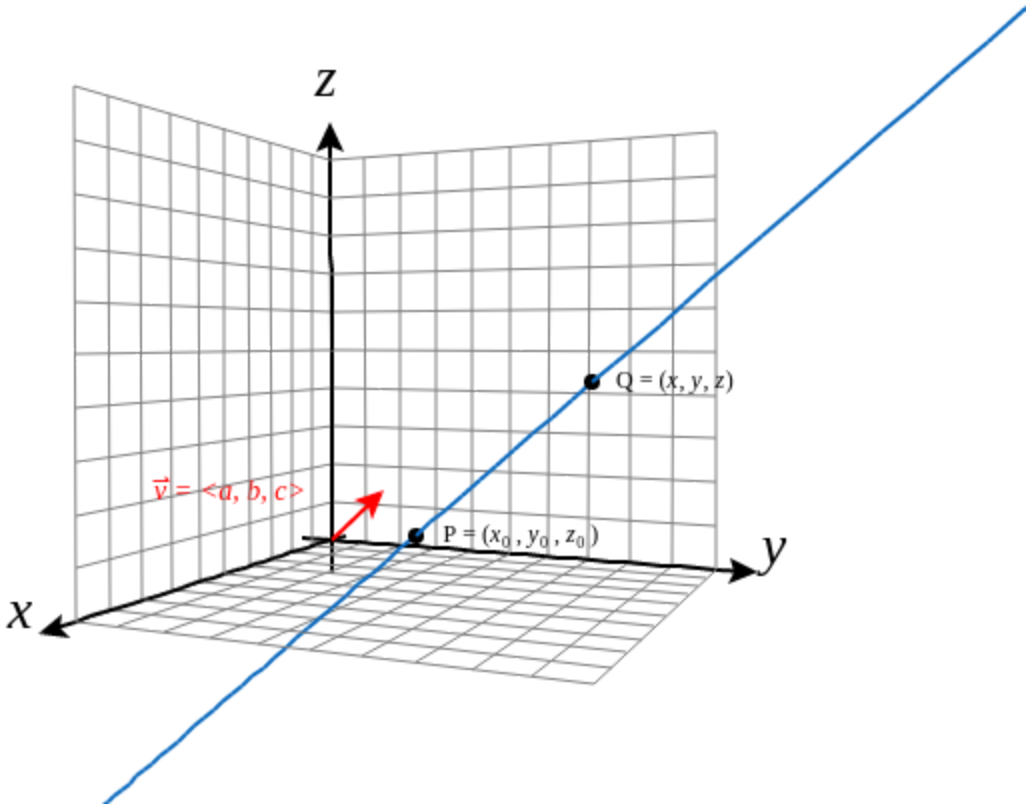


Figure 12.5.1: Vector $\vec{\mathbf{v}}$ is the direction vector for $\overrightarrow{PQ}$.

As in two dimensions, we can describe a line in space using a point on the line and the direction of the line, or a parallel vector, which we call the **direction vector** (Figure 12.5.1). Let L be a line in space passing through point $P(x_0, y_0, z_0)$. Let $\vec{\mathbf{v}} = \langle a, b, c\rangle$ be a vector parallel to L. Then, for any point on line $Q(x, y, z)$, we know that $\overrightarrow{PQ}$ is parallel to $\vec{\mathbf{v}}$. Thus, as we just discussed, there is a scalar, t, such that $\overrightarrow{PQ} = t\vec{\mathbf{v}}$, which gives

$$\begin{aligned} \overrightarrow{PQ} &= t\vec{\mathbf{v}} \\ \langle x - x_0, y - y_0, z - z_0\rangle &= t\langle a, b, c\rangle \\ \langle x - x_0, y - y_0, z - z_0\rangle &= \langle ta, tb, tc\rangle. \end{aligned} \tag{12.5.1}$$

Using vector operations, we can rewrite Equation 12.5.1

$$\begin{aligned} \langle x - x_0, y - y_0, z - z_0\rangle &= \langle ta, tb, tc\rangle \\ \langle x, y, z\rangle - \langle x_0, y_0, z_0\rangle &= t\langle a, b, c\rangle \\ \underbrace{\langle x, y, z\rangle}_{\vec{\mathbf{r}}} &= \underbrace{\langle x_0, y_0, z_0\rangle}_{\vec{\mathbf{r}}_o} + t\underbrace{\langle a, b, c\rangle}_{\vec{\mathbf{v}}}. \end{aligned}$$

Setting $\vec{\mathbf{r}} = \langle x, y, z\rangle$ and $\vec{\mathbf{r}}_0 = \langle x_0, y_0, z_0\rangle$, we now have the **vector equation of a line:**

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_0 + t\vec{\mathbf{v}}. \tag{12.5.2}$$

Equating components, Equation 12.5.2 shows that the following equations are simultaneously true: $x - x_0 = ta, y - y_0 = tb$, and $z - z_0 = tc$. If we solve each of these equations for the component variables x, y, and z, we get a set of equations in which each variable is defined in terms of the parameter t and that, together, describe the line. This set of three equations forms a set of **parametric equations of a line:**

$$x = x_0 + ta$$

$$y = y_0 + tb$$

$$z = z_0 + tc.$$

If we solve each of the equations for t assuming $a, b,$ and c are nonzero, we get a different description of the same line:

$$\frac{x - x_0}{a} = t$$
$$\frac{y - y_0}{b} = t$$
$$\frac{z - z_0}{c} = t.$$

Because each expression equals t, they all have the same value. We can set them equal to each other to create **symmetric equations of a line:**

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

We summarize the results in the following theorem.

Theorem: Parametric and Symmetric Equations of a Line

A line L parallel to vector $\vec{\mathbf{v}} = \langle a, b, c \rangle$ and passing through point $P(x_0, y_0, z_0)$ can be described by the following parametric equations:

$$x = x_0 + ta, y = y_0 + tb,$$

and

$$z = z_0 + tc.$$

If the constants $a, b,$ and c are all nonzero, then L can be described by the symmetric equation of the line:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

The parametric equations of a line are not unique. Using a different parallel vector or a different point on the line leads to a different, equivalent representation. Each set of parametric equations leads to a related set of symmetric equations, so it follows that a symmetric equation of a line is not unique either.

Example 12.5.1: Equations of a Line in Space

Find parametric and symmetric equations of the line passing through points $(1, 4, -2)$ and $(-3, 5, 0)$.

Solution

First, identify a vector parallel to the line:

$$\vec{\mathbf{v}} = \langle -3 - 1, 5 - 4, 0 - (-2) \rangle = \langle -4, 1, 2 \rangle.$$

Use either of the given points on the line to complete the parametric equations:

$$x = 1 - 4t$$
$$y = 4 + t,$$

and

$$z = -2 + 2t.$$

Solve each equation for t to create the symmetric equation of the line:

$$\frac{x-1}{-4} = y-4 = \frac{z+2}{2}.$$

Exercise 12.5.1

Find parametric and symmetric equations of the line passing through points $(1,-3,2)$ and $(5,-2,8)$.

Hint:

Start by finding a vector parallel to the line.

Answer

Possible set of parametric equations: $x = 1+4t,\ y = -3+t,\ z = 2+6t$; related set of symmetric equations:

$$\frac{x-1}{4} = y+3 = \frac{z-2}{6}$$

Sometimes we don't want the equation of a whole line, just a line segment. In this case, we limit the values of our parameter t. For example, let $P(x_0,y_0,z_0)$ and $Q(x_1,y_1,z_1)$ be points on a line, and let $\vec{\mathbf{p}} = \langle x_0,y_0,z_0\rangle$ and $\vec{\mathbf{q}} = \langle x_1,y_1,z_1\rangle$ be the associated position vectors. In addition, let $\vec{\mathbf{r}} = \langle x,y,z\rangle$. We want to find a vector equation for the line segment between P and Q. Using P as our known point on the line, and $\overrightarrow{PQ} = \langle x_1-x_0, y_1-y_0, z_1-z_0\rangle$ as the direction vector equation, Equation 12.5.2 gives

$$\vec{\mathbf{r}} = \vec{\mathbf{p}} + t(\overrightarrow{PQ}). \tag{12.5.3}$$

Equation 12.5.3 can be expanded using properties of vectors:

$$\begin{aligned}
\vec{\mathbf{r}} &= \vec{\mathbf{p}} + t(\overrightarrow{PQ}) \\
&= \langle x_0,y_0,z_0\rangle + t\langle x_1-x_0, y_1-y_0, z_1-z_0\rangle \\
&= \langle x_0,y_0,z_0\rangle + t(\langle x_1,y_1,z_1\rangle - \langle x_0,y_0,z_0\rangle) \\
&= \langle x_0,y_0,z_0\rangle + t\langle x_1,y_1,z_1\rangle - t\langle x_0,y_0,z_0\rangle \\
&= (1-t)\langle x_0,y_0,z_0\rangle + t\langle x_1,y_1,z_1\rangle \\
&= (1-t)\vec{\mathbf{p}} + t\vec{\mathbf{q}}.
\end{aligned}$$

Thus, the vector equation of the line passing through P and Q is

$$\vec{\mathbf{r}} = (1-t)\vec{\mathbf{p}} + t\vec{\mathbf{q}}.$$

Remember that we did not want the equation of the whole line, just the line segment between P and Q. Notice that when $t=0$, we have $\vec{\mathbf{r}} = \vec{\mathbf{p}}$, and when $t=1$, we have $\vec{\mathbf{r}} = \vec{\mathbf{q}}$. Therefore, the vector equation of the line segment between P and Q is

$$\vec{\mathbf{r}} = (1-t)\vec{\mathbf{p}} + t\vec{\mathbf{q}}, 0 \le t \le 1.$$

Going back to Equation 12.5.2, we can also find parametric equations for this line segment. We have

$$\begin{aligned}
\vec{\mathbf{r}} &= \vec{\mathbf{p}} + t(\overrightarrow{PQ}) \\
\langle x,y,z\rangle &= \langle x_0,y_0,z_0\rangle + t\langle x_1-x_0, y_1-y_0, z_1-z_0\rangle \\
&= \langle x_0+t(x_1-x_0), y_0+t(y_1-y_0), z_0+t(z_1-z_0)\rangle.
\end{aligned}$$

Then, the parametric equations are

$$\begin{aligned}
x &= x_0 + t(x_1-x_0) \\
y &= y_0 + t(y_1-y_0) \\
z &= z_0 + t(z_1-z_0), \quad 0 \le t \le 1.
\end{aligned} \tag{12.5.4}$$

Example 12.5.2: Parametric Equations of a Line Segment

Find parametric equations of the line segment between the points $P(2,1,4)$ and $Q(3,-1,3)$.

Solution

Start with the parametric equations for a line (Equations 12.5.4) and work with each component separately:

$$\begin{aligned} x &= x_0 + t(x_1 - x_0) \\ &= 2 + t(3-2) \\ &= 2 + t, \end{aligned}$$

$$\begin{aligned} y &= y_0 + t(y_1 - y_0) \\ &= 1 + t(-1-1) \\ &= 1 - 2t, \end{aligned}$$

and

$$\begin{aligned} z &= z_0 + t(z_1 - z_0) \\ &= 4 + t(3-4) \\ &= 4 - t. \end{aligned}$$

Therefore, the parametric equations for the line segment are

$$\begin{aligned} x &= 2 + t \\ y &= 1 - 2t \\ z &= 4 - t, \quad 0 \le t \le 1. \end{aligned}$$

Exercise 12.5.2

Find parametric equations of the line segment between points $P(-1,3,6)$ and $Q(-8,2,4)$.

Answer

$x = -1 - 7t,\ y = 3 - t,\ z = 6 - 2t, \quad 0 \le t \le 1$

Distance between a Point and a Line

We already know how to calculate the distance between two points in space. We now expand this definition to describe the distance between a point and a line in space. Several real-world contexts exist when it is important to be able to calculate these distances. When building a home, for example, builders must consider "setback" requirements, when structures or fixtures have to be a certain distance from the property line. Air travel offers another example. Airlines are concerned about the distances between populated areas and proposed flight paths.

Let L be a line in the plane and let M be any point not on the line. Then, we define distance d from M to L as the length of line segment $\overline{MP}$, where P is a point on L such that $\overline{MP}$ is perpendicular to L (Figure 12.5.2).

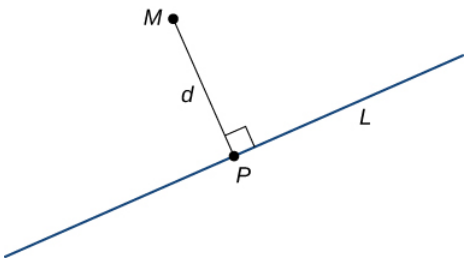


Figure 12.5.2: The distance from point M to line L is the length of $\overline{MP}$.

When we're looking for the distance between a line and a point in space, Figure 12.5.2 still applies. We still define the distance as the length of the perpendicular line segment connecting the point to the line. In space, however, there is no clear way to know

which point on the line creates such a perpendicular line segment, so we select an arbitrary point on the line and use properties of vectors to calculate the distance. Therefore, let P be an arbitrary point on line L and let $\vec{\mathbf{v}}$ be a direction vector for L (Figure 12.5.3).

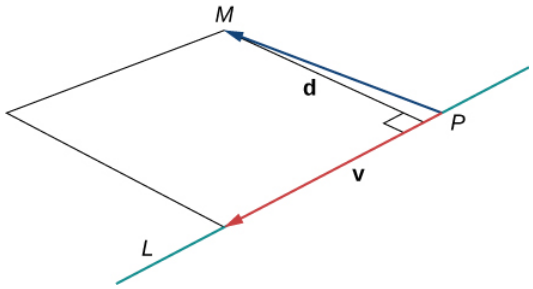


Figure 12.5.3: Vectors $\overrightarrow{PM}$ and $\vec{\mathbf{v}}$ form two sides of a parallelogram with base $\|\vec{\mathbf{v}}\|$ and height d, which is the distance between a line and a point in space.

Vectors $\overrightarrow{PM}$ and $\vec{\mathbf{v}}$ form two sides of a parallelogram with area $\|\overrightarrow{PM} \times \vec{\mathbf{v}}\|$. Using a formula from geometry, the area of this parallelogram can also be calculated as the product of its base and height:

$$\|\overrightarrow{PM} \times \vec{\mathbf{v}}\| = \|\vec{\mathbf{v}}\| d.$$

We can use this formula to find a general formula for the distance between a line in space and any point not on the line.

Distance from a Point to a Line

Let L be a line in space passing through point P with direction vector $\vec{\mathbf{v}}$. If M is any point not on L, then the distance from M to L is

$$d = \frac{\|\overrightarrow{PM} \times \vec{\mathbf{v}}\|}{\|\vec{\mathbf{v}}\|}.$$

Example 12.5.3: Calculating the Distance from a Point to a Line

Find the distance between the point $M = (1, 1, 3)$ and line $\dfrac{x-3}{4} = \dfrac{y+1}{2} = z - 3$.

Solution:

From the symmetric equations of the line, we know that vector $\vec{\mathbf{v}} = \langle 4, 2, 1 \rangle$ is a direction vector for the line. Setting the symmetric equations of the line equal to zero, we see that point $P(3, -1, 3)$ lies on the line. Then,

$$\begin{aligned} \overrightarrow{PM} &= \langle 1-3, 1-(-1), 3-3 \rangle \\ &= \langle -2, 2, 0 \rangle. \end{aligned}$$

To calculate the distance, we need to find $\overrightarrow{PM} \times \vec{\mathbf{v}}$:

$$\begin{aligned} \overrightarrow{PM} \times \vec{\mathbf{v}} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -2 & 2 & 0 \\ 4 & 2 & 1 \end{vmatrix} \\ &= (2-0)\hat{\mathbf{i}} - (-2-0)\hat{\mathbf{j}} + (-4-8)\hat{\mathbf{k}} \\ &= 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 12\hat{\mathbf{k}}. \end{aligned}$$

Therefore, the distance between the point and the line is (Figure 12.5.4)

$$d = \frac{\|\overrightarrow{PM} \times \vec{\mathbf{v}}\|}{\|\vec{\mathbf{v}}\|}$$
$$= \frac{\sqrt{2^2 + 2^2 + 12^2}}{\sqrt{4^2 + 2^2 + 1^2}}$$
$$= \frac{2\sqrt{38}}{\sqrt{21}}$$
$$= \frac{2\sqrt{798}}{21} \text{ units}$$

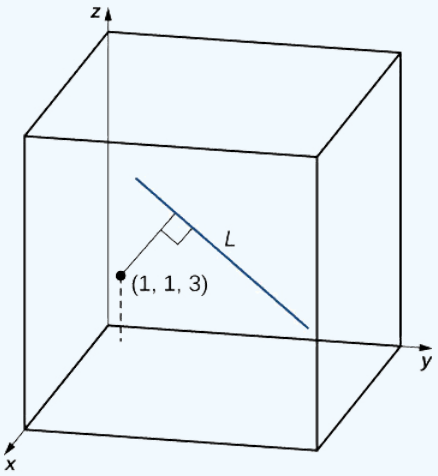


Figure 12.5.4: Point $(1, 1, 3)$ is approximately 2.7 units from the line with symmetric equations $\frac{x-3}{4} = \frac{y+1}{2} = z - 3$.

? Exercise 12.5.3

Find the distance between point $(0, 3, 6)$ and the line with parametric equations $x = 1 - t,\ y = 1 + 2t,\ z = 5 + 3t$.

Hint

Find a vector with initial point $(0, 3, 6)$ and a terminal point on the line, and then find a direction vector for the line.

Answer

$\sqrt{\frac{10}{7}} = \frac{\sqrt{70}}{7}$ units

Relationships between Lines

Given two lines in the two-dimensional plane, the lines are equal, they are parallel but not equal, or they intersect in a single point. In three dimensions, a fourth case is possible. If two lines in space are not parallel, but do not intersect, then the lines are said to be **skew lines** (Figure 12.5.5).

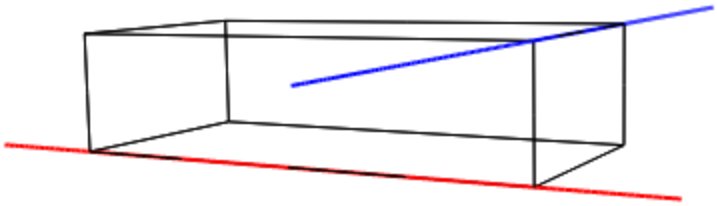


Figure 12.5.5: In three dimensions, it is possible that two lines do not cross, even when they have different directions.

To classify lines as parallel but not equal, equal, intersecting, or skew, we need to know two things: whether the direction vectors are parallel and whether the lines share a point (Figure 12.5.6).

Direction Vectors Are Parallel? \ Lines Share A Common Point?	Yes	No
Yes	Equal	Parallel but not equal
No	Intersecting	Skew

Figure 12.5.6: Determine the relationship between two lines based on whether their direction vectors are parallel and whether they share a point.

Example 12.5.4: Classifying Lines in Space

For each pair of lines, determine whether the lines are equal, parallel but not equal, skew, or intersecting.

a.

- $L_1:\ x = 2s - 1,\ y = s - 1,\ z = s - 4$
- $L_2:\ x = t - 3,\ y = 3t + 8,\ z = 5 - 2t$

b.

- $L_1:\ x = -y = z$
- $L_2:\ \dfrac{x-3}{2} = y = z - 2$

c.

- $L_1:\ x = 6s - 1,\ y = -2s,\ z = 3s + 1$
- $L_2:\ \dfrac{x-4}{6} = \dfrac{y+3}{-2} = \dfrac{z-1}{3}$

Solution

a. Line L_1 has direction vector $\vec{\mathbf{v}}_1 = \langle 2, 1, 1\rangle$; line L_2 has direction vector $\vec{\mathbf{v}}_2 = \langle 1, 3, -2\rangle$. Because the direction vectors are not parallel vectors, the lines are either intersecting or skew. To determine whether the lines intersect, we see if there is a point, (x, y, z), that lies on both lines. To find this point, we use the parametric equations to create a system of equalities:

$$2s - 1 = t - 3;$$

$$s - 1 = 3t + 8;$$

$$s - 4 = 5 - 2t.$$

By the first equation, $t = 2s + 2$. Substituting into the second equation yields

$$s - 1 = 3(2s + 2) + 8$$

$$s - 1 = 6s + 6 + 8$$

$$5s = -15$$

$$s = -3.$$

Substitution into the third equation, however, yields a contradiction:

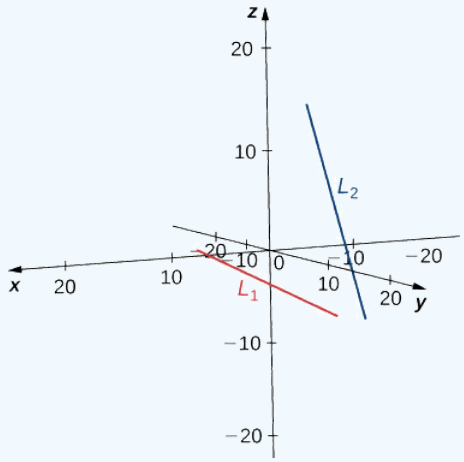
$$s - 4 = 5 - 2(2s + 2)$$

$$s - 4 = 5 - 4s - 4$$

$$5s = 5$$

$$s = 1.$$

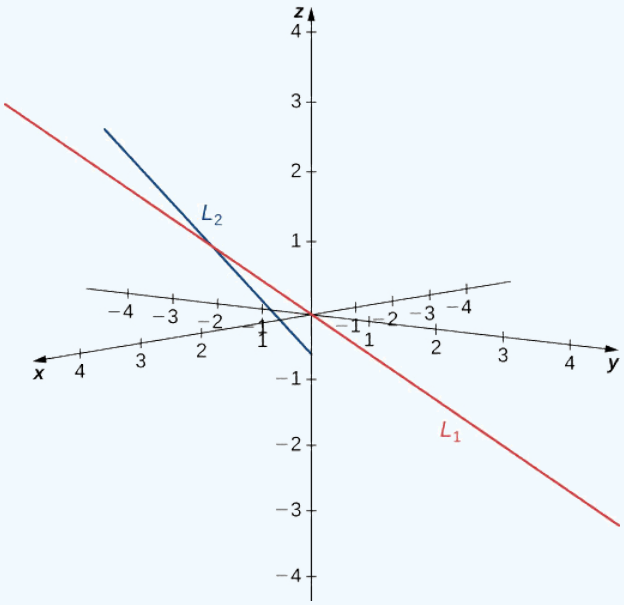
There is no single point that satisfies the parametric equations for L_1 and L_2 simultaneously. These lines do not intersect, so they are skew (see the following figure).



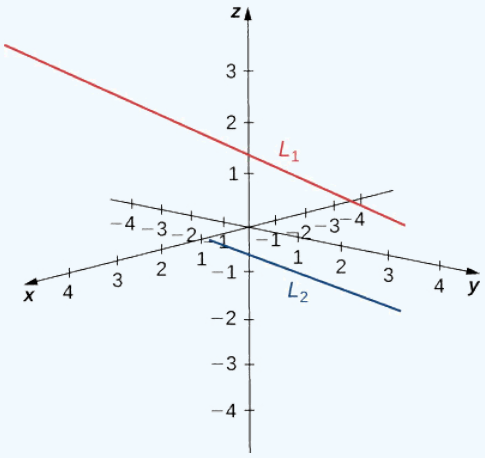
b. Line L_1 has direction vector $\vec{\mathbf{v}}_1 = \langle 1, -1, 1 \rangle$ and passes through the origin, $(0, 0, 0)$. Line L_2 has a different direction vector, $\vec{\mathbf{v}}_2 = \langle 2, 1, 1 \rangle$, so these lines are not parallel or equal. Let r represent the parameter for line L_1 and let s represent the parameter for L_2:

$$\begin{array}{ll} \text{Line } L_1: & \text{Line } L_2: \\ x = r & x = 2s + 3 \\ y = -r & y = s \\ z = r & z = s + 2 \end{array}$$

Solve the system of equations to find $r = 1$ and $s = -1$. If we need to find the point of intersection, we can substitute these parameters into the original equations to get $(1, -1, 1)$ (see the following figure).



c. Lines L_1 and L_2 have equivalent direction vectors: $\vec{\mathbf{v}} = \langle 6, -2, 3 \rangle$. These two lines are parallel (see the following figure).



Exercise 12.5.4

Describe the relationship between the lines with the following parametric equations:

$$x = 1 - 4t,\ y = 3 + t,\ z = 8 - 6t$$

$$x = 2 + 3s,\ y = 2s,\ z = -1 - 3s.$$

Hint

Start by identifying direction vectors for each line. Is one a multiple of the other?

Answer

These lines are skew because their direction vectors are not parallel and there is no point (x, y, z) that lies on both lines.

Equations for a Plane

We know that a line is determined by two points. In other words, for any two distinct points, there is exactly one line that passes through those points, whether in two dimensions or three. Similarly, given any three points that do not all lie on the same line, there is a unique plane that passes through these points. Just as a line is determined by two points, a plane is determined by three.

This may be the simplest way to characterize a plane, but we can use other descriptions as well. For example, given two distinct, intersecting lines, there is exactly one plane containing both lines. A plane is also determined by a line and any point that does not lie on the line. These characterizations arise naturally from the idea that a plane is determined by three points. Perhaps the most surprising characterization of a plane is actually the most useful.

Imagine a pair of orthogonal vectors that share an initial point. Visualize grabbing one of the vectors and twisting it. As you twist, the other vector spins around and sweeps out a plane. Here, we describe that concept mathematically. Let $\vec{\mathbf{n}} = \langle a, b, c\rangle$ be a vector and $P = (x_0, y_0, z_0)$ be a point. Then the set of all points $Q = (x, y, z)$ such that $\overrightarrow{PQ}$ is orthogonal to $\vec{\mathbf{n}}$ forms a plane (Figure 12.5.7). We say that $\vec{\mathbf{n}}$ is a **normal vector**, or perpendicular to the plane. Remember, the dot product of orthogonal vectors is zero. This fact generates the **vector equation of a plane**:

$$\vec{\mathbf{n}} \cdot \overrightarrow{PQ} = 0.$$

Rewriting this equation provides additional ways to describe the plane:

$$\begin{aligned} \vec{\mathbf{n}} \cdot \overrightarrow{PQ} &= 0 \\ \langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle &= 0 \\ a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0. \end{aligned}$$

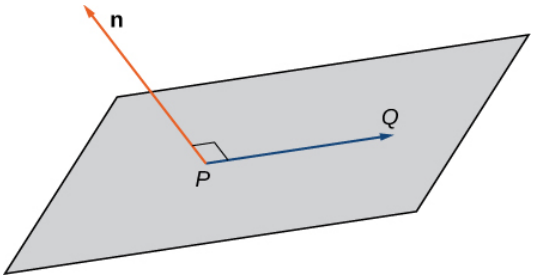


Figure 12.5.7: Given a point P and vector $\vec{\mathbf{n}}$, the set of all points Q with $\overrightarrow{PQ}$ orthogonal to $\vec{\mathbf{n}}$ forms a plane.

Definition: Scalar Equation of a Plane

Given a point P and vector $\vec{\mathbf{n}}$, the set of all points Q satisfying the equation $\vec{\mathbf{n}} \cdot \overrightarrow{PQ} = 0$ forms a plane. The equation

$$\vec{\mathbf{n}} \cdot \overrightarrow{PQ} = 0$$

is known as the *vector equation of a plane*.

The *scalar equation of a plane* (sometimes also called the *standard equation of a plane*) containing point $P = (x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c\rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

This equation can be expressed as $ax + by + cz + d = 0$, where $d = -ax_0 - by_0 - cz_0$. This form of the equation is sometimes called the *general form of the equation of a plane.*

As described earlier in this section, any three points that do not all lie on the same line determine a plane. Given three such points, we can find an equation for the plane containing these points.

Example 12.5.5: Writing an Equation of a Plane Given Three Points in the Plane

Write an equation for the plane containing points $P = (1, 1, -2), Q = (0, 2, 1)$, and $R = (-1, -1, 0)$ in both standard and general forms.

Solution

To write an equation for a plane, we must find a normal vector for the plane. We start by identifying two vectors in the plane:

$$\begin{aligned}\overrightarrow{PQ} &= \langle 0-1, 2-1, 1-(-2)\rangle \\ &= \langle -1, 1, 3\rangle \\ \overrightarrow{QR} &= \langle -1-0, -1-2, 0-1\rangle \\ &= \langle -1, -3, -1\rangle.\end{aligned}$$

The cross product $\overrightarrow{PQ} \times \overrightarrow{QR}$ is orthogonal to both $\overrightarrow{PQ}$ and $\overrightarrow{QR}$, so it is normal to the plane that contains these two vectors:

$$\begin{aligned}\vec{\mathbf{n}} &= \overrightarrow{PQ} \times \overrightarrow{QR} \\ &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 1 & 3 \\ -1 & -3 & -1 \end{vmatrix} \\ &= (-1+9)\hat{\mathbf{i}} - (1+3)\hat{\mathbf{j}} + (3+1)\hat{\mathbf{k}} \\ &= 8\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 4\hat{\mathbf{k}}.\end{aligned}$$

Thus, $\vec{\mathbf{n}} = \langle 8, -4, 4\rangle$, and we can choose any of the three given points to write an equation of the plane:

$$\begin{aligned}8(x-1) - 4(y-1) + 4(z+2) &= 0 \\ 8x - 4y + 4z + 4 &= 0.\end{aligned}$$

The scalar equations of a plane vary depending on the normal vector and point chosen.

Example 12.5.6: Writing an Equation for a Plane Given a Point and a Line

Find an equation of the plane that passes through point $(1, 4, 3)$ and contains the line given by $x = \dfrac{y-1}{2} = z + 1$.

Solution

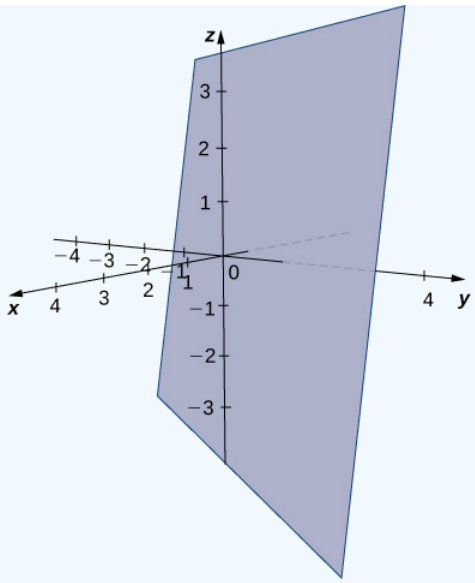
Symmetric equations describe the line that passes through point $(0, 1, -1)$ parallel to vector $\vec{\mathbf{v}}_1 = \langle 1, 2, 1\rangle$ (see the following figure). Use this point and the given point, $(1, 4, 3)$, to identify a second vector parallel to the plane:

$$\vec{\mathbf{v}}_2 = \langle 1-0, 4-1, 3-(-1)\rangle = \langle 1, 3, 4\rangle.$$

Use the cross product of these vectors to identify a normal vector for the plane:

$$\begin{aligned}\vec{\mathbf{n}} &= \vec{\mathbf{v}}_1 \times \vec{\mathbf{v}}_2 \\ &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & 1 \\ 1 & 3 & 4 \end{vmatrix} \\ &= (8-3)\hat{\mathbf{i}} - (4-1)\hat{\mathbf{j}} + (3-2)\hat{\mathbf{k}} \\ &= 5\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}.\end{aligned}$$

The scalar equations for the plane are $5x - 3(y-1) + (z+1) = 0$ and $5x - 3y + z + 4 = 0$.



Exercise 12.5.6

Find an equation of the plane containing the lines L_1 and L_2:

$$L_1 : \ x = -y = z$$

$$L_2 : \ \frac{x-3}{2} = y = z - 2.$$

Hint

Hint: The cross product of the lines' direction vectors gives a normal vector for the plane.

Answer

$$-2(x-1) + (y+1) + 3(z-1) = 0$$

or

$$-2x + y + 3z = 0$$

Now that we can write an equation for a plane, we can use the equation to find the distance d between a point P and the plane. It is defined as the shortest possible distance from P to a point on the plane.

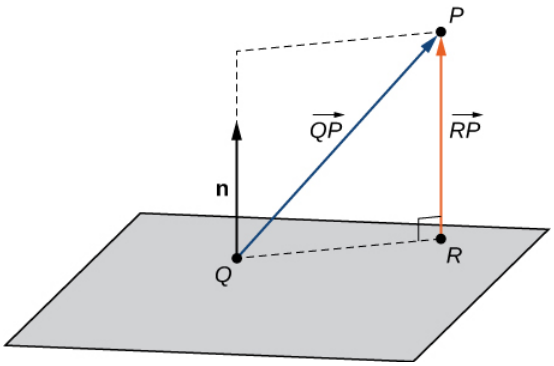


Figure 12.5.8: We want to find the shortest distance from point P to the plane. Let point R be the point in the plane such that, for any other point in the plane Q, $\|\overrightarrow{RP}\| < \|\overrightarrow{QP}\|$.

Just as we find the two-dimensional distance between a point and a line by calculating the length of a line segment perpendicular to the line, we find the three-dimensional distance between a point and a plane by calculating the length of a line segment

perpendicular to the plane. Let R be the point in the plane such that $\overrightarrow{RP}$ is orthogonal to the plane, and let Q be an arbitrary point in the plane. Then the projection of vector $\overrightarrow{QP}$ onto the normal vector describes vector $\overrightarrow{RP}$, as shown in Figure 12.5.8.

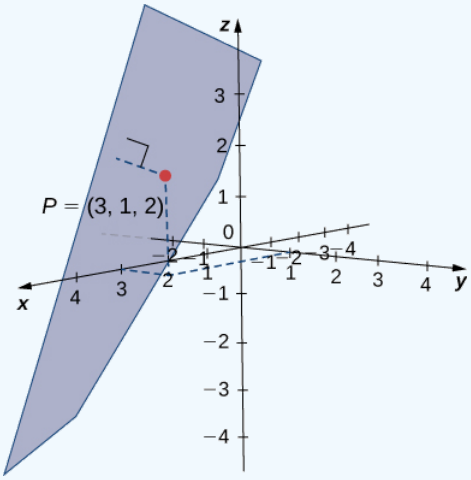
The Distance between a Plane and a Point

Suppose a plane with normal vector $\vec{\mathbf{n}}$ passes through point Q. The distance d from the plane to a point P not in the plane is given by

$$d = \|\text{proj}_{\vec{\mathbf{n}}} \overrightarrow{QP}\| = |\text{ comp}_{\vec{\mathbf{n}}} \overrightarrow{QP}| = \frac{|\overrightarrow{QP} \cdot \vec{\mathbf{n}}|}{\|\vec{\mathbf{n}}\|}. \tag{12.5.5}$$

Example 12.5.7: Distance between a Point and a Plane

Find the distance between point $P = (3, 1, 2)$ and the plane given by $x - 2y + z = 5$ (see the following figure).



Solution

The coefficients of the plane's equation provide a normal vector for the plane: $\vec{\mathbf{n}} = \langle 1, -2, 1\rangle$. To find vector $\overrightarrow{QP}$, we need a point in the plane. Any point will work, so set $y = z = 0$ to see that point $Q = (5, 0, 0)$ lies in the plane. Find the component form of the vector from Q to P:

$$\overrightarrow{QP} = \langle 3 - 5, 1 - 0, 2 - 0\rangle = \langle -2, 1, 2\rangle.$$

Apply the distance formula from Equation 12.5.5:

$$\begin{aligned} d &= \frac{\left|\overrightarrow{QP} \cdot \vec{\mathbf{n}}\right|}{\|\vec{\mathbf{n}}\|} \\ &= \frac{|\langle -2, 1, 2\rangle \cdot \langle 1, -2, 1\rangle|}{\sqrt{1^2 + (-2)^2 + 1^2}} \\ &= \frac{|-2 - 2 + 2|}{\sqrt{6}} \\ &= \frac{2}{\sqrt{6}} = \frac{\sqrt{6}}{3} \text{ units.} \end{aligned}$$

Exercise 12.5.7

Find the distance between point $P = (5, -1, 0)$ and the plane given by $4x + 2y - z = 3$.

Hint

Point $(0, 0, -3)$ lies on the plane.

Answer

$$\frac{15}{\sqrt{21}} = \frac{5\sqrt{21}}{7} \text{ units}$$

Parallel and Intersecting Planes

We have discussed the various possible relationships between two lines in two dimensions and three dimensions. When we describe the relationship between two planes in space, we have only two possibilities: the two distinct planes are parallel or they intersect. When two planes are parallel, their normal vectors are parallel. When two planes intersect, the intersection is a line (Figure 12.5.9).

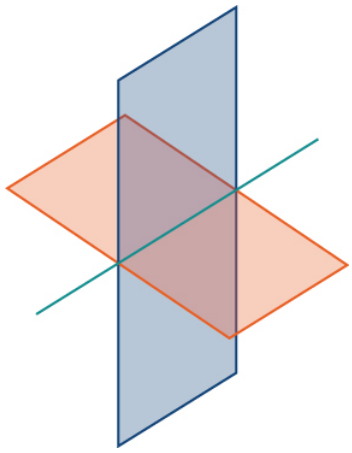
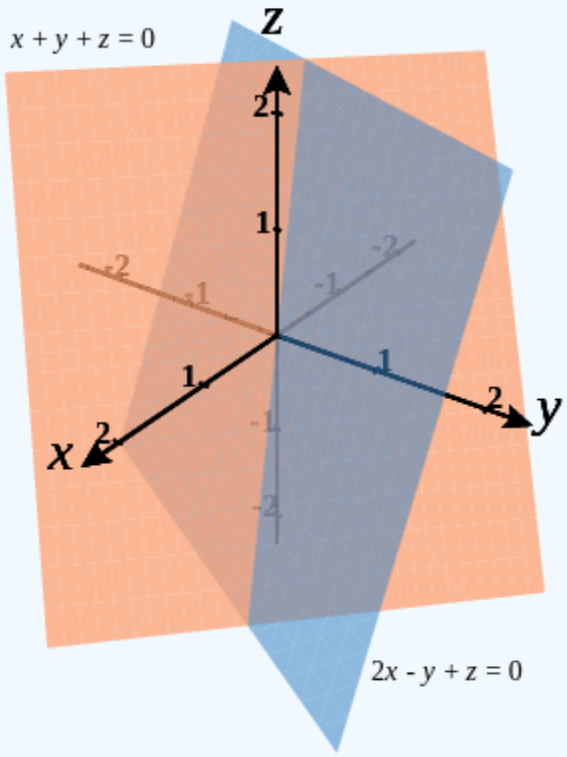


Figure 12.5.9: The intersection of two nonparallel planes is always a line.

We can use the equations of the two planes to find parametric equations for the line of intersection.

Example 12.5.8: Finding the Line of Intersection for Two Planes

Find parametric and symmetric equations for the line formed by the intersection of the planes given by $x + y + z = 0$ and $2x - y + z = 0$ (see the following figure).



Solution

Note that the two planes have nonparallel normals, so the planes intersect. Further, the origin satisfies each equation, so we know the line of intersection passes through the origin. Add the plane equations so we can eliminate one of the variables, in this case, y:

$$x + y + z = 0$$
$$2x - y + z = 0$$
$$\_\_\_\_\_$$
$$3x + 2z = 0.$$

This gives us $x = -\frac{2}{3}z$. We substitute this value into the first equation to express y in terms of z:

$$x + y + z = 0$$
$$-\frac{2}{3}z + y + z = 0$$
$$y + \frac{1}{3}z = 0$$
$$y = -\frac{1}{3}z$$

We now have the first two variables, x and y, in terms of the third variable, z. Now we define z in terms of t. To eliminate the need for fractions, we choose to define the parameter t as $t = -\frac{1}{3}z$. Then, $z = -3t$. Substituting the parametric representation of

z back into the other two equations, we see that the parametric equations for the line of intersection are $x = 2t,\ y = t,\ z = -3t$. The symmetric equations for the line are $\frac{x}{2} = y = \frac{z}{-3}$.

? Exercise 12.5.8

Find parametric equations for the line formed by the intersection of planes $x + y - z = 3$ and $3x - y + 3z = 5$.

Hint

Add the two equations, then express z in terms of x. Then, express y in terms of x.

Answer

$x = t,\ y = 7 - 3t,\ z = 4 - 2t$

In addition to finding the equation of the line of intersection between two planes, we may need to find the angle formed by the intersection of two planes. For example, builders constructing a house need to know the angle where different sections of the roof meet to know whether the roof will look good and drain properly. We can use normal vectors to calculate the angle between the two planes. We can do this because the angle between the normal vectors is the same as the angle between the planes. Figure 12.5.10 shows why this is true.

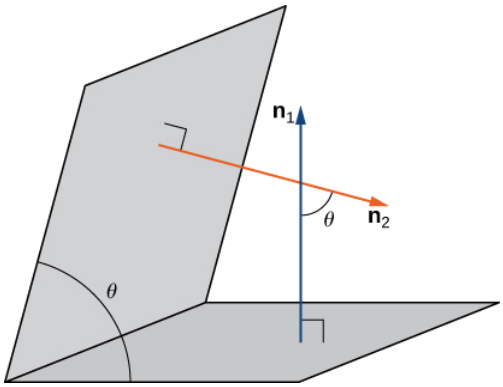


Figure 12.5.10: The angle between two planes has the same measure as the angle between the normal vectors for the planes.

We can find the measure of the angle θ between two intersecting planes by first finding the cosine of the angle, using the following equation:

$$\cos\theta = \frac{|\vec{\mathbf{n}}_1 \cdot \vec{\mathbf{n}}_2|}{\|\vec{\mathbf{n}}_1\|\|\vec{\mathbf{n}}_2\|}.$$

We can then use the angle to determine whether two planes are parallel or orthogonal or if they intersect at some other angle.

✓ Example 12.5.9: Finding the Angle between Two Planes

Determine whether each pair of planes is parallel, orthogonal, or neither. If the planes are intersecting, but not orthogonal, find the measure of the angle between them. Give the answer in radians and round to two decimal places.

a. $x + 2y - z = 8$ and $2x + 4y - 2z = 10$
b. $2x - 3y + 2z = 3$ and $6x + 2y - 3z = 1$
c. $x + y + z = 4$ and $x - 3y + 5z = 1$

Solution:

a. The normal vectors for these planes are $\vec{\mathbf{n}}_1 = \langle 1, 2, -1\rangle$ and $\vec{\mathbf{n}}_2 = \langle 2, 4, -2\rangle$. These two vectors are scalar multiples of each other. The normal vectors are parallel, so the planes are parallel.
b. The normal vectors for these planes are $\vec{\mathbf{n}}_1 = \langle 2, -3, 2\rangle$ and $\vec{\mathbf{n}}_2 = \langle 6, 2, -3\rangle$. Taking the dot product of these vectors, we have

$$\begin{aligned}\vec{\mathbf{n}}_1 \cdot \vec{\mathbf{n}}_2 &= \langle 2, -3, 2\rangle \cdot \langle 6, 2, -3\rangle \\ &= 2(6) - 3(2) + 2(-3) = 0.\end{aligned}$$

The normal vectors are orthogonal, so the corresponding planes are orthogonal as well.

c. The normal vectors for these planes are $\vec{\mathbf{n}}_1 = \langle 1, 1, 1\rangle$ and $\vec{\mathbf{n}}_2 = \langle 1, -3, 5\rangle$:

$$\begin{aligned}\cos\theta &= \frac{|\vec{\mathbf{n}}_1 \cdot \vec{\mathbf{n}}_2|}{\|\vec{\mathbf{n}}_1\|\|\vec{\mathbf{n}}_2\|} \\ &= \frac{|\langle 1, 1, 1\rangle \cdot \langle 1, -3, 5\rangle|}{\sqrt{1^2 + 1^2 + 1^2}\sqrt{1^2 + (-3)^2 + 5^2}} \\ &= \frac{3}{\sqrt{105}}\end{aligned}$$

Then $\theta = \arccos \frac{3}{\sqrt{105}} \approx 1.27$ rad.

Thus the angle between the two planes is about 1.27 rad, or approximately $73°$.

? Exercise 12.5.9

Find the measure of the angle between planes $x + y - z = 3$ and $3x - y + 3z = 5$. Give the answer in radians and round to two decimal places.

Hint

Use the coefficients of the variables in each equation to find a normal vector for each plane.

Answer

1.44 rad

When we find that two planes are parallel, we may need to find the distance between them. To find this distance, we simply select a point in one of the planes. The distance from this point to the other plane is the distance between the planes.

Previously, we introduced the formula for calculating this distance in Equation 12.5.5:

$$d = \frac{\overrightarrow{QP} \cdot \vec{\mathbf{n}}}{\|\vec{\mathbf{n}}\|},$$

where Q is a point on the plane, P is a point not on the plane, and $\vec{n}$ is the normal vector that passes through point Q. Consider the distance from point (x_0, y_0, z_0) to plane $ax + by + cz + k = 0$. Let (x_1, y_1, z_1) be any point in the plane. Substituting into the formula yields

$$\begin{aligned}d &= \frac{|a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|ax_0 + by_0 + cz_0 + k|}{\sqrt{a^2 + b^2 + c^2}}.\end{aligned}$$

We state this result formally in the following theorem.

Distance from a Point to a Plane

Let $P(x_0, y_0, z_0)$ be a point. The distance from P to plane $ax + by + cz + k = 0$ is given by

$$d = \frac{|ax_0 + by_0 + cz_0 + k|}{\sqrt{a^2 + b^2 + c^2}}.$$

Example 12.5.10: Finding the Distance between Parallel Planes

Find the distance between the two parallel planes given by $2x + y - z = 2$ and $2x + y - z = 8$.

Solution

Point $(1, 0, 0)$ lies in the first plane. The desired distance, then, is

$$\begin{aligned} d &= \frac{|ax_0 + by_0 + cz_0 + k|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|2(1) + 1(0) + (-1)(0) + (-8)|}{\sqrt{2^2 + 1^2 + (-1)^2}} \\ &= \frac{6}{\sqrt{6}} = \sqrt{6}\,\text{units} \end{aligned}$$

Exercise 12.5.10:

Find the distance between parallel planes $5x - 2y + z = 6$ and $5x - 2y + z = -3$.

Hint

Set $x = y = 0$ to find a point on the first plane.

Answer

$$\frac{9}{\sqrt{30}} = \frac{3\sqrt{30}}{10}\,\text{units}$$

Distance between Two Skew Lines

Finding the distance from a point to a line or from a line to a plane seems like a pretty abstract procedure. But, if the lines represent pipes in a chemical plant or tubes in an oil refinery or roads at an intersection of highways, confirming that the distance between them meets specifications can be both important and awkward to measure. One way is to model the two pipes as lines, using the techniques in this chapter, and then calculate the distance between them. The calculation involves forming vectors along the directions of the lines and using both the cross product and the dot product.

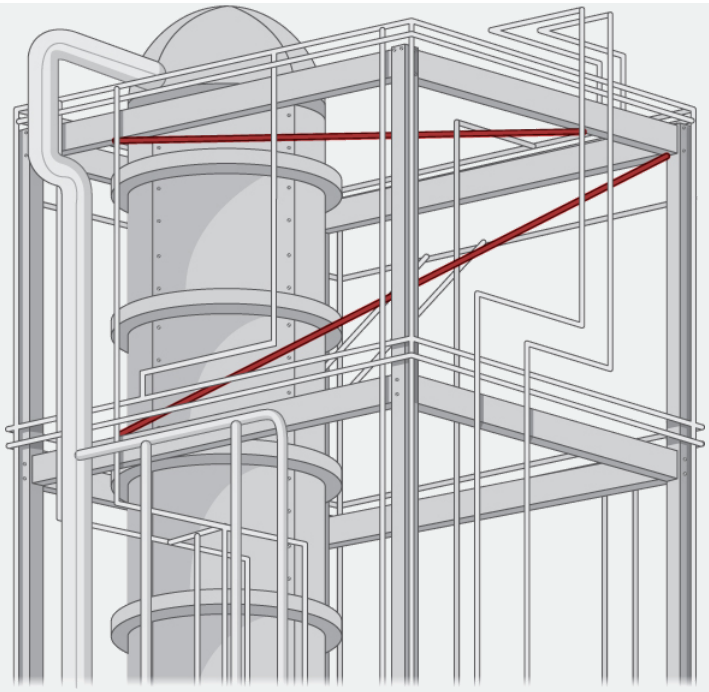


Figure 12.5.11: Industrial pipe installations often feature pipes running in different directions. How can we find the distance between two skew pipes?

The symmetric forms of two lines, L_1 and L_2, are

$$L_1 : \frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$$

$$L_2 : \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}.$$

You are to develop a formula for the distance d between these two lines, in terms of the values $a_1, b_1, c_1; a_2, b_2, c_2; x_1, y_1, z_1;$ and x_2, y_2, z_2. The distance between two lines is usually taken to mean the minimum distance, so this is the length of a line segment or the length of a vector that is perpendicular to both lines and intersects both lines.

1. First, write down two vectors, $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$, that lie along L_1 and L_2, respectively.

2. Find the cross product of these two vectors and call it $\vec{\mathbf{N}}$. This vector is perpendicular to $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$, and hence is perpendicular to both lines.

3. From vector $\vec{\mathbf{N}}$, form a unit vector $\vec{\mathbf{n}}$ in the same direction.

4. Use symmetric equations to find a convenient vector $\vec{\mathbf{v}}_{12}$ that lies between any two points, one on each line. Again, this can be done directly from the symmetric equations.

5. The dot product of two vectors is the magnitude of the projection of one vector onto the other—that is, $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = \|\vec{\mathbf{A}}\|\|\vec{\mathbf{B}}\| \cos\theta$, where θ is the angle between the vectors. Using the dot product, find the projection of vector $\vec{\mathbf{v}}_{12}$ found in step 4 onto unit vector $\vec{\mathbf{n}}$ found in step 3. This projection is perpendicular to both lines, and hence its length must be the perpendicular distance d between them. Note that the value of d may be negative, depending on your choice of vector $\vec{\mathbf{v}}_{12}$ or the order of the cross product, so use absolute value signs around the numerator.

6. Check that your formula gives the correct distance of $|-25|/\sqrt{198} \approx 1.78$ between the following two lines:

$$L_1 : \frac{x-5}{2} = \frac{y-3}{4} = \frac{z-1}{3}$$

$$L_2 : \frac{x-6}{3} = \frac{y-1}{5} = \frac{z}{7}.$$

7. Is your general expression valid when the lines are parallel? If not, why not? (Hint: What do you know about the value of the cross product of two parallel vectors? Where would that result show up in your expression for d?)

8. Demonstrate that your expression for the distance is zero when the lines intersect. Recall that two lines intersect if they are not parallel and they are in the same plane. Hence, consider the direction of $\vec{\mathbf{n}}$ and $\vec{\mathbf{v}}_{12}$. What is the result of their dot product?

9. Consider the following application. Engineers at a refinery have determined they need to install support struts between many of the gas pipes to reduce damaging vibrations. To minimize cost, they plan to install these struts at the closest points between adjacent skewed pipes. Because they have detailed schematics of the structure, they are able to determine the correct lengths of the struts needed, and hence manufacture and distribute them to the installation crews without spending valuable time making measurements.

The rectangular frame structure has the dimensions $4.0 \times 15.0 \times 10.0\,\text{m}$ (height, width, and depth). One sector has a pipe entering the lower corner of the standard frame unit and exiting at the diametrically opposed corner (the one farthest away at the top); call this L_1. A second pipe enters and exits at the two different opposite lower corners; call this L_2 (Figure 12.5.12).

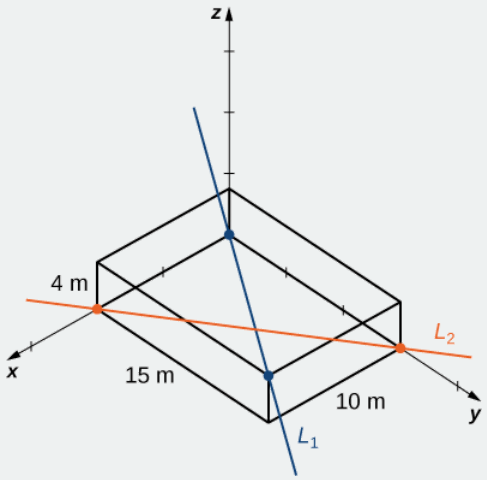


Figure 12.5.12: Two pipes cross through a standard frame unit.

Write down the vectors along the lines representing those pipes, find the cross product between them from which to create the unit vector $\vec{\mathbf{n}}$, define a vector that spans two points on each line, and finally determine the minimum distance between the lines. (Take the origin to be at the lower corner of the first pipe.) Similarly, you may also develop the symmetric equations for each line and substitute directly into your formula.

Key Concepts

- In three dimensions, the direction of a line is described by a direction vector. The vector equation of a line with direction vector $\vec{\mathbf{v}} = \langle a, b, c \rangle$ passing through point $P = (x_0, y_0, z_0)$ is $\vec{\mathbf{r}} = \vec{\mathbf{r}}_0 + t\vec{\mathbf{v}}$, where $\vec{\mathbf{r}}_0 = \langle x_0, y_0, z_0 \rangle$ is the position vector of point P. This equation can be rewritten to form the parametric equations of the line: $x = x_0 + ta, y = y_0 + tb$, and $z = z_0 + tc$. The line can also be described with the symmetric equations $\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$.
- Let L be a line in space passing through point P with direction vector $\vec{\mathbf{v}}$. If Q is any point not on L, then the distance from Q to L is $d = \dfrac{\|\overrightarrow{PQ} \times \vec{\mathbf{v}}\|}{\|\vec{\mathbf{v}}\|}$.
- In three dimensions, two lines may be parallel but not equal, equal, intersecting, or skew.
- Given a point P and vector $\vec{\mathbf{n}}$, the set of all points Q satisfying equation $\vec{\mathbf{n}} \cdot \overrightarrow{PQ} = 0$ forms a plane. Equation $\vec{\mathbf{n}} \cdot \overrightarrow{PQ} = 0$ is known as the *vector equation of a plane*.
- The scalar equation of a plane containing point $P = (x_0, y_0, z_0)$ with normal vector $\vec{\mathbf{n}} = \langle a, b, c \rangle$ is $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$. This equation can be expressed as $ax + by + cz + d = 0$, where $d = -ax_0 - by_0 - cz_0$. This form of the equation is sometimes called the general *form of the equation of a plane*.

- Suppose a plane with normal vector n passes through point Q. The distance D from the plane to point P not in the plane is given by

$$D = \|\text{proj}_{\vec{\mathbf{n}}}\overrightarrow{QP}\| = |\text{ comp}_{\vec{\mathbf{n}}}\vec{QP}| = \frac{|\vec{QP}\cdot\vec{\mathbf{n}}|}{\|\vec{\mathbf{n}}\|}.$$

- The normal vectors of parallel planes are parallel. When two planes intersect, they form a line.
- The measure of the angle θ between two intersecting planes can be found using the equation: $\cos\theta = \frac{|\vec{\mathbf{n}}_1\cdot\vec{\mathbf{n}}_2|}{\|\vec{\mathbf{n}}_1\|\|\vec{\mathbf{n}}_2\|}$, where $\vec{\mathbf{n}}_1$ and $\vec{\mathbf{n}}_2$ are normal vectors to the planes.
- The distance D from point (x_0, y_0, z_0) to plane $ax + by + cz + d = 0$ is given by

$$D = \frac{|a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

.

Key Equations

- **Vector Equation of a Line**

  $\vec{\mathbf{r}} = \vec{\mathbf{r}}_0 + t\vec{\mathbf{v}}$

- **Parametric Equations of a Line**

  $x = x_0 + ta,\ y = y_0 + tb$, and $z = z_0 + tc$

- **Symmetric Equations of a Line**

  $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$

- **Vector Equation of a Plane**

  $\vec{\mathbf{n}}\cdot\overrightarrow{PQ} = 0$

- **Scalar Equation of a Plane**

  $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

- **Distance between a Plane and a Point**

  $d = \|\text{proj}_{\vec{\mathbf{n}}}\overrightarrow{QP}\| = |\text{ comp}_{\vec{\mathbf{n}}}\overrightarrow{QP}| = \frac{|\overrightarrow{QP}\cdot\vec{\mathbf{n}}|}{\|\vec{\mathbf{n}}\|}$

Glossary

direction vector

a vector parallel to a line that is used to describe the direction, or orientation, of the line in space

general form of the equation of a plane

an equation in the form $ax + by + cz + d = 0$, where $\vec{\mathbf{n}} = \langle a, b, c\rangle$ is a normal vector of the plane, $P = (x_0, y_0, z_0)$ is a point on the plane, and $d = -ax_0 - by_0 - cz_0$

normal vector

a vector perpendicular to a plane

parametric equations of a line

the set of equations $x = x_0 + ta, y = y_0 + tb$, and $z = z_0 + tc$ describing the line with direction vector $v = \langle a, b, c\rangle$ passing through point (x_0, y_0, z_0)

scalar equation of a plane

the equation $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ used to describe a plane containing point $P=(x_0,y_0,z_0)$ with normal vector $n=\langle a,b,c\rangle$ or its alternate form $ax+by+cz+d=0$, where $d=-ax_0-by_0-cz_0$

skew lines

two lines that are not parallel but do not intersect

symmetric equations of a line

the equations $\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$ describing the line with direction vector $v=\langle a,b,c\rangle$ passing through point (x_0,y_0,z_0)

vector equation of a line

the equation $\vec{\mathbf{r}}=\vec{\mathbf{r}}_0+t\vec{\mathbf{v}}$ used to describe a line with direction vector $\vec{\mathbf{v}}=\langle a,b,c\rangle$ passing through point $P=(x_0,y_0,z_0)$, where $\vec{\mathbf{r}}_0=\langle x_0,y_0,z_0\rangle$, is the position vector of point P

vector equation of a plane

the equation $\vec{\mathbf{n}}\cdot\overrightarrow{PQ}=0$, where P is a given point in the plane, Q is any point in the plane, and $\vec{\mathbf{n}}$ is a normal vector of the plane

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12.5E: Exercises for Section 12.5

In exercises 1 - 4, points P and Q are given. Let L be the line passing through points P and Q.

a. Find the vector equation of line L.

b. Find parametric equations of line L.

c. Find symmetric equations of line L.

d. Find parametric equations of the line segment determined by P and Q.

1) $P(-3,5,9), \quad Q(4,-7,2)$

Answer

a. $\vec{\mathbf{r}} = \langle -3,5,9\rangle + t\langle 7,-12,-7\rangle, \quad t \in R;$
b. $x = -3+7t, \quad y = 5-12t, \quad z = 9-7t, \quad t \in R;$
c. $\dfrac{x+3}{7} = \dfrac{y-5}{-12} = \dfrac{z-9}{-7};$
d. $x = -3+7t, \quad y = 5-12t, \quad z = 9-7t, \quad 0 \le t \le 1$

2) $P(4,0,5), \quad Q(2,3,1)$

3) $P(-1,0,5), \quad Q(4,0,3)$

Answer

a. $\vec{\mathbf{r}} = \langle -1,0,5\rangle + t\langle 5,0,-2\rangle, \quad t \in R;$
b. $x = -1+5t, y = 0, z = 5-2t, \quad t \in R;$
c. $\dfrac{x+1}{5} = \dfrac{z-5}{-2}, \quad y = 0;$
d. $x = -1+5t, \quad y = 0, \quad z = 5-2t, \quad t \in [0,1]$

4) $P(7,-2,6), \quad Q(-3,0,6)$

For exercises 5 - 8, point P and vector $\vec{\mathbf{v}}$ are given. Let L be the line passing through point P with direction $\vec{\mathbf{v}}$.

a. Find parametric equations of line L.

b. Find symmetric equations of line L.

c. Find the intersection of the line with the xy-plane.

5) $P(1,-2,3), \ \vec{\mathbf{v}} = \langle 1,2,3\rangle$

Answer

a. $x = 1+t, \quad y = -2+2t, \quad z = 3+3t, \quad t \in R;$
b. $\dfrac{x-1}{1} = \dfrac{y+2}{2} = \dfrac{z-3}{3};$
c. $(0,-4,0)$

6) $P(3,1,5), \ \vec{\mathbf{v}} = \langle 1,1,1\rangle$

7) $P(3,1,5), \ \vec{\mathbf{v}} = \overrightarrow{QR}$, where $Q(2,2,3)$ and $R(3,2,3)$

Answer

a. $x = 3+t, \quad y = 1, \quad z = 5, \quad t \in R;$
b. $y = 1, \quad z = 5;$
c. The line does not intersect the xy-plane.

8) $P(2,3,0), \ \vec{\mathbf{v}} = \overrightarrow{QR}$, where $Q(0,4,5)$ and $R(0,4,6)$

For exercises 9 and 10, line L is given.

a. Find a point P that belongs to the line and a direction vector $\vec{v}$ of the line. Express $\vec{v}$ in component form.

b. Find the distance from the origin to line L.

9) $x = 1 + t, \quad y = 3 + t, \quad z = 5 + 4t, \quad t \in R$

Answer

a. A possible point and direction vector are $P(1, 3, 5)$ and $\vec{v} = \langle 1, 1, 4 \rangle$, but these answers are not unique.
b. $\sqrt{3}$ units

10) $-x = y + 1, \quad z = 2$

11) Find the distance between point $A(-3, 1, 1)$ and the line of symmetric equations

$x = -y = -z.$

Answer

$\frac{2\sqrt{2}}{\sqrt{3}} = \frac{2\sqrt{6}}{3}$ units

12) Find the distance between point $A(4, 2, 5)$ and the line of parametric equations

$x = -1 - t, \ y = -t, \ z = 2, \ t \in R.$

For exercises 13 - 14, lines L_1 and L_2 are given.

a. Verify whether lines L_1 and L_2 are parallel.

b. If the lines L_1 and L_2 are parallel, then find the distance between them.

13) $L_1 : x = 1 + t, \quad y = t, \quad z = 2 + t, \quad t \in R$ and $L_2 : x - 3 = y - 1 = z - 3$

Answer

a. Parallel;
b. $\frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$ units

14) $L_1 : x = 2, \quad y = 1, \quad z = t, \quad t \in R$ and $L_2 : x = 1, \quad y = 1, \quad z = 2 - 3t, \quad t \in R$

15) Show that the line passing through points $P(3, 1, 0)$ and $Q(1, 4, -3)$ is perpendicular to the line with equation $x = 3t, \quad y = -32 + 8t, \quad z = -9 + 6t, \quad t \in R.$

Answer

$\overrightarrow{PQ} = \langle -2, 3, -3 \rangle$ is the direction vector of the line through points P and Q, and the direction vector of the line defined by the parametric equations above is $\vec{v} = \langle 3, 8, 6 \rangle$.
Since $\vec{v} \cdot \overrightarrow{PQ} = -6 + 24 - 18 = 0$, the two direction vectors are orthogonal.
Now all we need to show is that the two lines intersect.
The line through points $P(3, 1, 0)$ and $Q(1, 4, -3)$ has parametric equations: $x = 3 - 2u$, $y = 1 + 3u$, and $z = -3u$.
Setting the x- and z-coordinates of the two lines equal, we obtain the system of equations:

$$3t = 3 - 2u \quad \text{and} \quad -9 + 6t = -3u$$

Solving this system using substitution gives us, $u = -3$ and $t = 3$. Plugging these values of t and u back into the parametric equations of these two lines gives us the intersection point with coordinates $(9, -8, 9)$ on both lines.
Therefore the lines intersect and the line through points P and Q with direction vector $\overrightarrow{PQ}$ is perpendicular to the other line.

16) Are the lines of equations $x = -2 + 2t, \quad y = -6, \quad z = 2 + 6t, \quad t \in R$ and $x = -1 + t, \quad y = 1 + t, \quad z = t, \quad t \in R$, perpendicular to each other?

17) Find the point of intersection of the lines of equations $x = -2y = 3z$ and $x = -5 - t, \quad y = -1 + t, \quad z = t - 11, \quad t \in R.$

Answer

$(-12, 6, -4)$

18) Find the intersection point of the x-axis with the line of parametric equations $x = 10 + t, \quad y = 2 - 2t, \quad z = -3 + 3t, \quad t \in R.$

For exercises 19 - 22, lines L_1 and L_2 are given. Determine whether the lines are *equal, parallel but not equal, skew,* or *intersecting*.

19) $L_1 : x = y - 1 = -z$ and $L_2 : x - 2 = -y = \dfrac{z}{2}$

Answer

The lines are skew.

20) $L_1 : x = 2t, \quad y = 0, \quad z = 3, \quad t \in R$ and $L_2 : x = 0, \quad y = 8 + s, \quad z = 7 + s, \quad s \in R$

21) $L_1 : x = -1 + 2t, \quad y = 1 + 3t, \quad z = 7t, \quad t \in R$ and $L_2 : x - 1 = \frac{2}{3}(y - 4) = \frac{2}{7}z - 2$

Answer

The lines are equal.

22) $L_1 : 3x = y + 1 = 2z$ and $L_2 : x = 6 + 2t, \quad y = 17 + 6t, \quad z = 9 + 3t, \quad t \in R$

23) Consider line L of symmetric equations $x - 2 = -y = \dfrac{z}{2}$ and point $A(1, 1, 1)$.

a. Find parametric equations for a line parallel to L that passes through point A.

b. Find symmetric equations of a line skew to L and that passes through point A.

c. Find symmetric equations of a line that intersects L and passes through point A.

Answer

a. $x = 1 + t, \quad y = 1 - t, \quad z = 1 + 2t, \quad t \in R$

b. For instance, the line passing through A with direction vector $j : x = 1, z = 1$

c. For instance, the line passing through A and point $(2, 0, 0)$ that belongs to L is a line that intersects; $L : \frac{x-1}{-1} = y - 1 = z - 1$

24) Consider line L of parametric equations $x = t, \quad y = 2t, \quad z = 3, \quad t \in R.$

a. Find parametric equations for a line parallel to L that passes through the origin.

b. Find parametric equations of a line skew to L that passes through the origin.

c. Find symmetric equations of a line that intersects L and passes through the origin.

For exercises 25 - 28, point P and vector $\vec{\mathrm{n}}$ are given.

a. Find the scalar equation of the plane that passes through P and has normal vector $\vec{\mathrm{n}}$.

b. Find the general form of the equation of the plane that passes through P and has normal vector $\vec{\mathrm{n}}$.

25) $P(0, 0, 0), \quad \vec{\mathrm{n}} = 3\hat{\mathrm{i}} - 2\hat{\mathrm{j}} + 4\hat{\mathrm{k}}$

Answer

a. $3x - 2y + 4z = 0$

b. $3x - 2y + 4z = 0$

26) $P(3, 2, 2), \quad \vec{\mathrm{n}} = 2\hat{\mathrm{i}} + 3\hat{\mathrm{j}} - \hat{\mathrm{k}}$

27) $P(1, 2, 3), \quad \vec{\mathrm{n}} = \langle 1, 2, 3 \rangle$

Answer

a. $(x-1)+2(y-2)+3(z-3)=0$

b. $x+2y+3z-14=0$

28) $P(0,0,0), \quad \vec{\mathbf{n}}=\langle -3,2,-1\rangle$

For exercises 29 - 32, the equation of a plane is given.

a. Find normal vector $\vec{\mathbf{n}}$ to the plane. Express $\vec{\mathbf{n}}$ using standard unit vectors.

b. Find the intersections of the plane with each of the coordinate axes (its intercepts).

c. Sketch the plane.

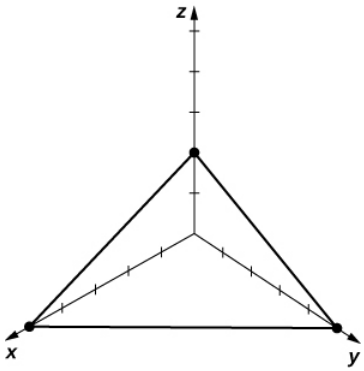
29) [T] $4x+5y+10z-20=0$

Answer

a. $\vec{\mathbf{n}}=4\hat{\mathbf{i}}+5\hat{\mathbf{j}}+10\hat{\mathbf{k}}$

b. $(5,0,0)$, $(0,4,0)$, and $(0,0,2)$

c.



30) $3x+4y-12=0$

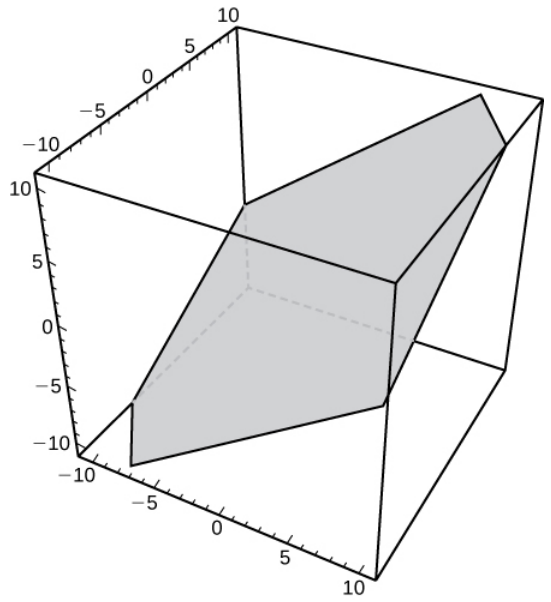
31) $3x-2y+4z=0$

Answer

a. $\vec{\mathbf{n}}=3\hat{\mathbf{i}}-2\hat{\mathbf{j}}+4\hat{\mathbf{k}}$

b. $(0,0,0)$

c.



32) $x + z = 0$

33) Given point $P(1, 2, 3)$ and vector $\vec{\mathbf{n}} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$, find point Q on the x-axis such that $\overrightarrow{PQ}$ and $\vec{\mathbf{n}}$ are orthogonal.

Answer

$(3, 0, 0)$

34) Show there is no plane perpendicular to $\vec{\mathbf{n}} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$ that passes through points $P(1, 2, 3)$ and $Q(2, 3, 4)$.

35) Find parametric equations of the line passing through point $P(-2, 1, 3)$ that is perpendicular to the plane of equation $2x - 3y + z = 7$.

Answer

$x = -2 + 2t, \quad y = 1 - 3t, \quad z = 3 + t, \quad t \in R$

36) Find symmetric equations of the line passing through point $P(2, 5, 4)$ that is perpendicular to the plane of equation $2x + 3y - 5z = 0$.

37) Show that line $\dfrac{x-1}{2} = \dfrac{y+1}{3} = \dfrac{z-2}{4}$ is parallel to plane $x - 2y + z = 6$.

38) Find the real number α such that the line of parametric equations $x = t, \quad y = 2 - t, \quad z = 3 + t, \quad t \in R$ is parallel to the plane of equation $\alpha x + 5y + z - 10 = 0$.

For exercises 39 - 42, the equations of two planes are given.

a. Determine whether the planes are *parallel*, *orthogonal*, or *neither*.

b. If the planes are neither parallel nor orthogonal, then find the measure of the angle between the planes. Express the answer in degrees rounded to the nearest integer.

c. If the planes intersect, find the line of intersection of the planes, providing the parametric equations of this line.

39) [T] $x + y + z = 0, \quad 2x - y + z - 7 = 0$

Answer

a. The planes are neither parallel nor orthogonal.

b. $62°$

c. $x = -1 + 2t$

$y = -4 + t$
$z = 5 - 3t$

40) $5x - 3y + z = 4, \quad x + 4y + 7z = 1$

41) $x - 5y - z = 1, \quad 5x - 25y - 5z = -3$

Answer

a. The planes are parallel.

42) [T] $x - 3y + 6z = 4, \quad 5x + y - z = 4$

For exercises 43 - 46, determine whether the given line intersects with the given plane. If they do intersect, state the point of intersection.

43) Plane: $2x + y - z = 11$ Line: $x = 1 + t, \quad y = 3 - 2t, \quad z = 2 + 4t$

Answer

They intersect at point $(-1, 7, -6)$.

44) Plane: $-x + 2y + z = 2$ Line: $x = 1 + 2t, \quad y = -2 + t, \quad z = 5 - 3t$

Answer

They intersect at point $\left(-\frac{1}{3}, -\frac{8}{3}, 7\right)$.

45) Plane: $x - 3y + 2z = 4$ Line: $x = 2 - t, \quad y = t, \quad z = 4 + 2t$

Answer

The line does not intersect with this plane.

46) Plane: $x - 3y + 2z = 10$ Line: $x = 2 - t, \quad y = t, \quad z = 4 + 2t$

Answer

The line is actually fully contained in this plane, so every point on the line is on the plane. For example, when $t = 0$ we have the point, $(2, 0, 4)$.

47) Show that the lines of equations $x = t, \quad y = 1 + t, \quad z = 2 + t, \quad t \in R$, and $\frac{x}{2} = \frac{y - 1}{3} = z - 3$ are skew, and find the distance between them.

Answer

$\frac{1}{\sqrt{6}} = \frac{\sqrt{6}}{6}$ units

48) Show that the lines of equations $x = -1 + t, \quad y = -2 + t, \quad z = 3t, \quad t \in R$, and $x = 5 + s, \quad y = -8 + 2s, \quad z = 7s, \quad s \in R$ are skew, and find the distance between them.

49) Consider point $C(-3, 2, 4)$ and the plane of equation $2x + 4y - 3z = 8$.

a. Find the radius of the sphere with center C tangent to the given plane.

b. Find point P of tangency.

Answer

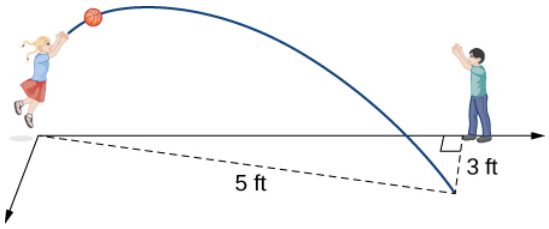
a. $r = \frac{18}{\sqrt{29}} = \frac{18\sqrt{29}}{29}$
b. $P\left(-\frac{51}{29}, \frac{130}{29}, \frac{62}{29}\right)$

50) Consider the plane of equation $x - y - z - 8 = 0$.

a. Find the equation of the sphere with center C at the origin that is tangent to the given plane.

b. Find parametric equations of the line passing through the origin and the point of tangency.

51) Two children are playing with a ball. The girl throws the ball to the boy. The ball travels in the air, curves 3 ft to the right, and falls 5 ft away from the girl (see the following figure). If the plane that contains the trajectory of the ball is perpendicular to the ground, find its equation.



Answer

$4x - 3y = 0$

52) [T] John allocates d dollars to consume monthly three goods of prices a, b, and c. In this context, the budget equation is defined as $ax + by + cz = d$, where $x \geq 0,\ y \geq 0$, and $z \geq 0$ represent the number of items bought from each of the goods. The budget set is given by $\{(x, y, z) \mid ax + by + cz \leq d,\ x \geq 0,\ y \geq 0,\ z \geq 0\}$, and the budget plane is the part of the plane of equation $ax + by + cz = d$ for which $x \geq 0,\ y \geq 0$, and $z \geq 0$. Consider $a = \$8,\ b = \$5,\ c = \$10$, and $d = \$500$.

a. Use a CAS to graph the budget set and budget plane.

b. For $z = 25$, find the new budget equation and graph the budget set in the same system of coordinates.

53) [T] Consider $\vec{\mathbf{r}}(t) = \langle \sin t, \cos t, 2t \rangle$ the position vector of a particle at time $t \in [0, 3]$, where the components of $\vec{\mathbf{r}}$ are expressed in centimeters and time is measured in seconds. Let $\overrightarrow{OP}$ be the position vector of the particle after 1 sec.

a. Determine the velocity vector $\vec{\mathbf{v}}(1)$ of the particle after 1 sec.

b. Find the scalar equation of the plane that is perpendicular to $v(1)$ and passes through point P. This plane is called the normal plane to the path of the particle at point P.

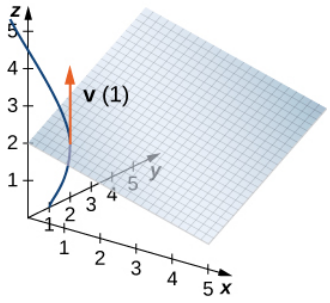
c. Use a CAS to visualize the path of the particle along with the velocity vector and normal plane at point P.

Answer

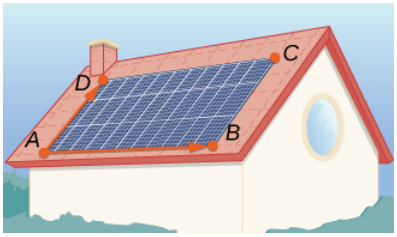
a. $\vec{\mathbf{v}}(1) = \langle \cos 1, -\sin 1, 2 \rangle$

b. $(\cos 1)(x - \sin 1) - (\sin 1)(y - \cos 1) + 2(z - 2) = 0$

c.



54) [T] A solar panel is mounted on the roof of a house. The panel may be regarded as positioned at the points of coordinates (in meters) $A(8, 0, 0),\ B(8, 18, 0),\ C(0, 18, 8)$, and $D(0, 0, 8)$ (see the following figure).



a. Find the general form of the equation of the plane that contains the solar panel by using points A, B, and C, and show that its normal vector is equivalent to $\overrightarrow{AB} \times \overrightarrow{AD}$.

b. Find parametric equations of line L_1 that passes through the center of the solar panel and has direction vector $\vec{s} = \frac{1}{\sqrt{3}}\hat{\mathbf{i}} + \frac{1}{\sqrt{3}}\hat{\mathbf{j}} + \frac{1}{\sqrt{3}}\hat{\mathbf{k}}$, which points toward the position of the Sun at a particular time of day.

c. Find symmetric equations of line L_2 that passes through the center of the solar panel and is perpendicular to it.

d. Determine the angle of elevation of the Sun above the solar panel by using the angle between lines L_1 and L_2.

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Problems 15 and 43 - 46 created by Paul Seeburger

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12.6: Quadric Surfaces

Learning Objectives

- Identify a cylinder as a type of three-dimensional surface.
- Recognize the main features of ellipsoids, paraboloids, and hyperboloids.
- Use traces to draw the intersections of quadric surfaces with the coordinate planes.

We have been exploring vectors and vector operations in three-dimensional space, and we have developed equations to describe lines, planes, and spheres. In this section, we use our knowledge of planes and spheres, which are examples of three-dimensional figures called *surfaces*, to explore a variety of other surfaces that can be graphed in a three-dimensional coordinate system.

Identifying Cylinders

The first surface we'll examine is the cylinder. Although most people immediately think of a hollow pipe or a soda straw when they hear the word cylinder, here we use the broad mathematical meaning of the term. As we have seen, cylindrical surfaces don't have to be circular. A rectangular heating duct is a cylinder, as is a rolled-up yoga mat, the cross-section of which is a spiral shape.

In the two-dimensional coordinate plane, the equation $x^2 + y^2 = 9$ describes a circle centered at the origin with radius 3. In three-dimensional space, this same equation represents a surface. Imagine copies of a circle stacked on top of each other centered on the z-axis (Figure $12.6.1$), forming a hollow tube. We can then construct a cylinder from the set of lines parallel to the z-axis passing through the circle $x^2 + y^2 = 9$ in the xy-plane, as shown in the figure. In this way, any curve in one of the coordinate planes can be extended to become a surface.

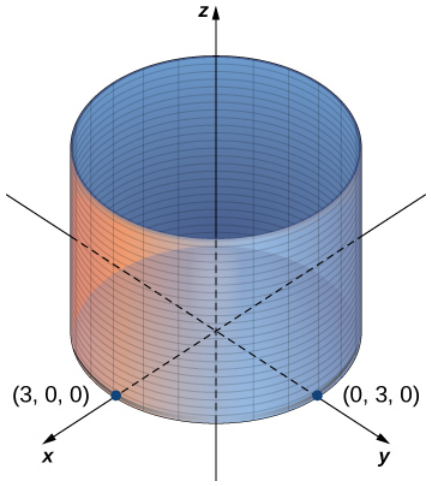


Figure $12.6.1$: In three-dimensional space, the graph of equation $x^2 + y^2 = 9$ is a cylinder with radius 3 centered on the z-axis. It continues indefinitely in the positive and negative directions.

Definition: cylinders and rulings

A set of lines parallel to a given line passing through a given curve is known as a cylindrical surface, or *cylinder*. The parallel lines are called rulings.

From this definition, we can see that we still have a cylinder in three-dimensional space, even if the curve is not a circle. Any curve can form a cylinder, and the rulings that compose the cylinder may be parallel to any given line (Figure $12.6.2$).

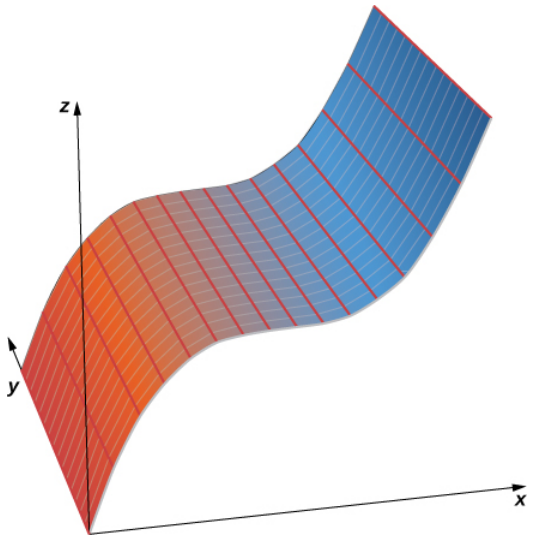


Figure 12.6.2: In three-dimensional space, the graph of equation $z = x^3$ is a cylinder, or a cylindrical surface with rulings parallel to the y-axis.

Example 12.6.1: Graphing Cylindrical Surfaces

Sketch the graphs of the following cylindrical surfaces.

a. $x^2 + z^2 = 25$
b. $z = 2x^2 - y$
c. $y = \sin x$

Solution

a. The variable y can take on any value without limit. Therefore, the lines ruling this surface are parallel to the y-axis. The intersection of this surface with the xz-plane forms a circle centered at the origin with radius 5 (see Figure 12.6.3).

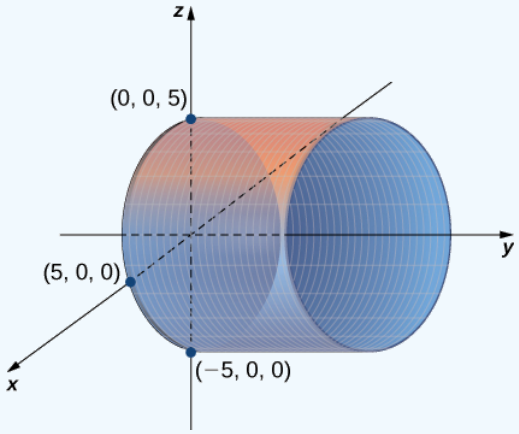


Figure 12.6.3: The graph of equation $x^2 + z^2 = 25$ is a cylinder with radius 5 centered on the y-axis.

b. In this case, the equation contains all three variables —x, y, and z— so none of the variables can vary arbitrarily. The easiest way to visualize this surface is to use a computer graphing utility (Figure 12.6.4).

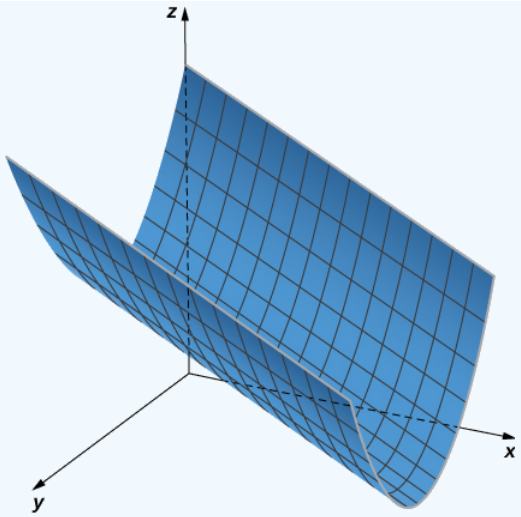


Figure 12.6.4

c. In this equation, the variable z can take on any value without limit. Therefore, the lines composing this surface are parallel to the z-axis. The intersection of this surface with the xy-plane outlines curve $y = \sin x$ (Figure 12.6.5).

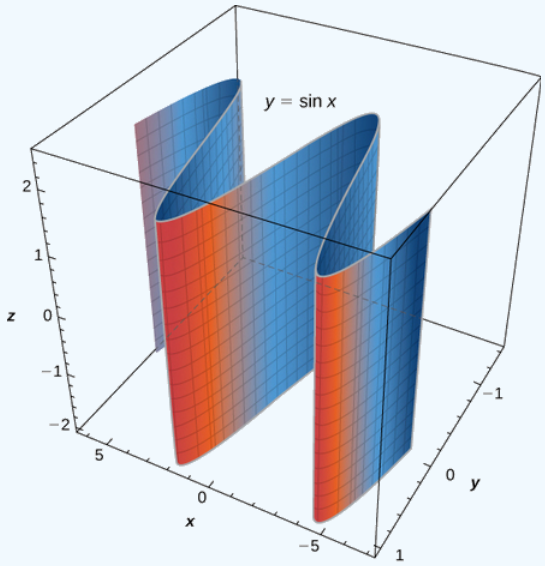


Figure 12.6.5: The graph of equation $y = \sin x$ is formed by a set of lines parallel to the z-axis passing through curve $y = \sin x$ in the xy-plane.

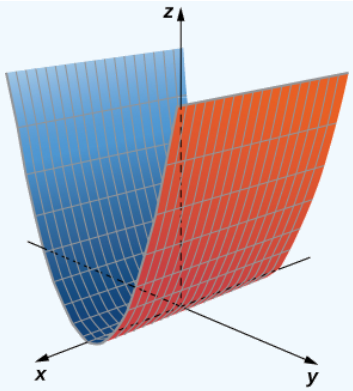
Exercise 12.6.1:

Sketch or use a graphing tool to view the graph of the cylindrical surface defined by equation $z = y^2$.

Hint

The variable x can take on any value without limit.

Answer



When sketching surfaces, we have seen that it is useful to sketch the intersection of the surface with a plane parallel to one of the coordinate planes. These curves are called traces. We can see them in the plot of the cylinder in Figure 12.6.6.

Definition: traces

The *traces* of a surface are the cross-sections created when the surface intersects a plane parallel to one of the coordinate planes.

Traces are useful in sketching cylindrical surfaces. For a cylinder in three dimensions, though, only one set of traces is useful. Notice, in Figure 12.6.6, that the trace of the graph of $z = \sin x$ in the xz-plane is useful in constructing the graph. The trace in the xy-plane, though, is just a series of parallel lines, and the trace in the yz-plane is simply one line.

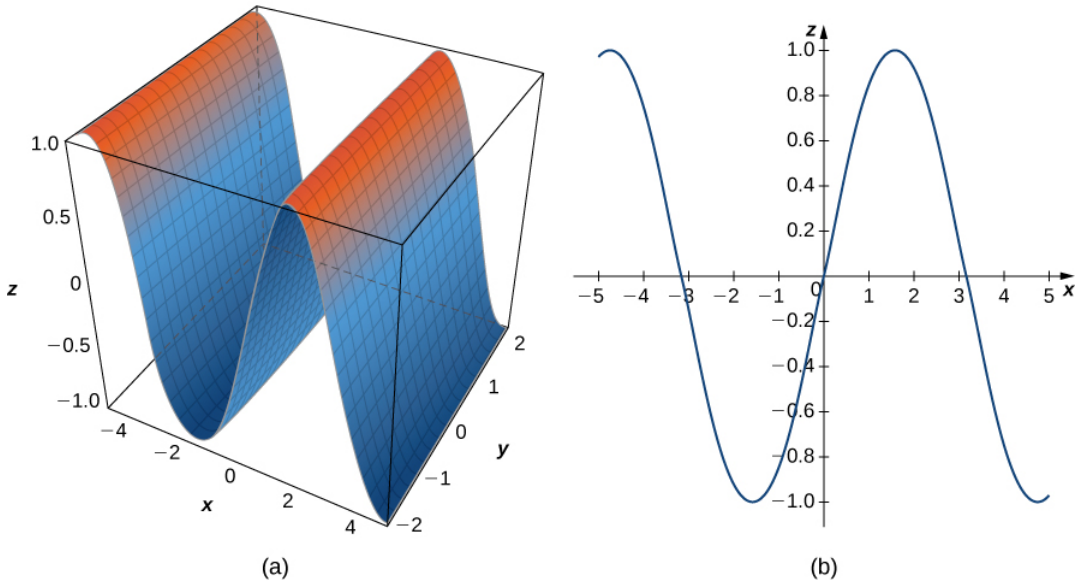


Figure 12.6.6: (a) This is one view of the graph of equation $z = \sin x$. (b) To find the trace of the graph in the xz-plane, set $y = 0$. The trace is simply a two-dimensional sine wave.

Cylindrical surfaces are formed by a set of parallel lines. Not all surfaces in three dimensions are constructed so simply, however. We now explore more complex surfaces, and traces are an important tool in this investigation.

Quadric Surfaces

We have learned about surfaces in three dimensions described by first-order equations; these are planes. Some other common types of surfaces can be described by second-order equations. We can view these surfaces as three-dimensional extensions of the conic sections we discussed earlier: the ellipse, the parabola, and the hyperbola. We call these graphs quadric surfaces

Definition: Quadric surfaces and conic sections

Quadric surfaces are the graphs of equations that can be expressed in the form

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Jz + K = 0.$$

When a quadric surface intersects a coordinate plane, the trace is a conic section.

An *ellipsoid* is a surface described by an equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Set $x = 0$ to see the trace of the ellipsoid in the yz-plane. To see the traces in the xy- and xz-planes, set $z = 0$ and $y = 0$, respectively. Notice that, if $a = b$, the trace in the xy-plane is a circle. Similarly, if $a = c$, the trace in the xz-plane is a circle and, if $b = c$, then the trace in the yz-plane is a circle. A sphere, then, is an ellipsoid with $a = b = c$.

Example 12.6.2: Sketching an Ellipsoid

Sketch the ellipsoid

$$\frac{x^2}{2^2} + \frac{y^2}{3^2} + \frac{z^2}{5^2} = 1.$$

Solution

Start by sketching the traces. To find the trace in the xy-plane, set $z = 0$: $\frac{x^2}{2^2} + \frac{y^2}{3^2} = 1$ (Figure 12.6.7). To find the other traces, first set $y = 0$ and then set $x = 0$.

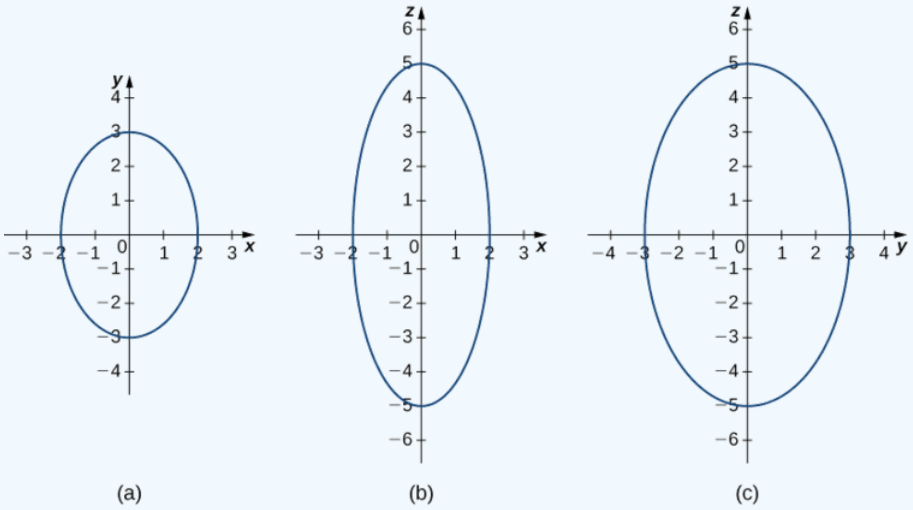
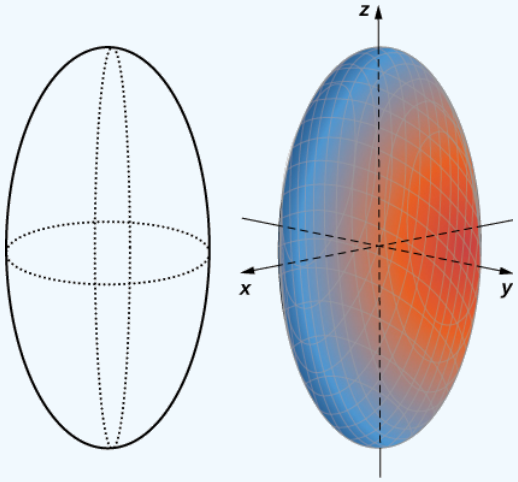


Figure 12.6.7: (a) This graph represents the trace of equation $\frac{x^2}{2^2} + \frac{y^2}{3^2} + \frac{z^2}{5^2} = 1$ in the xy-plane, when we set $z = 0$. (b) When we set $y = 0$, we get the trace of the ellipsoid in the xz-plane, which is an ellipse. (c) When we set $x = 0$, we get the trace of the ellipsoid in the yz-plane, which is also an ellipse.

Now that we know what traces of this solid look like, we can sketch the surface in three dimensions (Figure 12.6.8).



(a) (b)

Figure 12.6.8: (a) The traces provide a framework for the surface. (b) The center of this ellipsoid is the origin.

The trace of an ellipsoid is an ellipse in each of the coordinate planes. However, this does not have to be the case for all quadric surfaces. Many quadric surfaces have traces that are different kinds of conic sections, and this is usually indicated by the name of the surface. For example, if a surface can be described by an equation of the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c}$$

then we call that surface an **elliptic paraboloid**. The trace in the xy-plane is an ellipse, but the traces in the xz-plane and yz-plane are parabolas (Figure 12.6.9). Other elliptic paraboloids can have other orientations simply by interchanging the variables to give us a different variable in the linear term of the equation $\frac{x^2}{a^2}+\frac{z^2}{c^2}=\frac{y}{b}$ or $\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{x}{a}$.

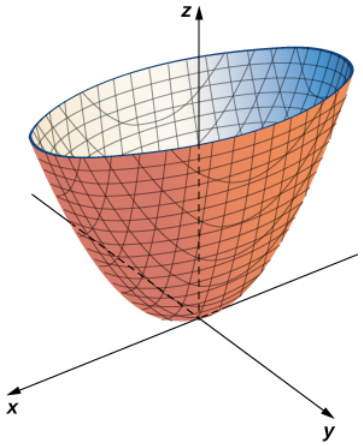


Figure 12.6.9: This quadric surface is called an elliptic paraboloid.

Example 12.6.3: Identifying Traces of Quadric Surfaces

Describe the traces of the elliptic paraboloid $x^2+\frac{y^2}{2^2}=\frac{z}{5}$.

Solution

To find the trace in the xy-plane, set $z=0: x^2+\frac{y^2}{2^2}=0$. The trace in the plane $z=0$ is simply one point, the origin. Since a single point does not tell us what the shape is, we can move up the z-axis to an arbitrary plane to find the shape of other traces of the figure.

The trace in plane $z = 5$ is the graph of equation $x^2 + \frac{y^2}{2^2} = 1$, which is an ellipse. In the xz-plane, the equation becomes $z = 5x^2$. The trace is a parabola in this plane and in any plane with the equation $y = b$.

In planes parallel to the yz-plane, the traces are also parabolas, as we can see in Figure 12.6.10.

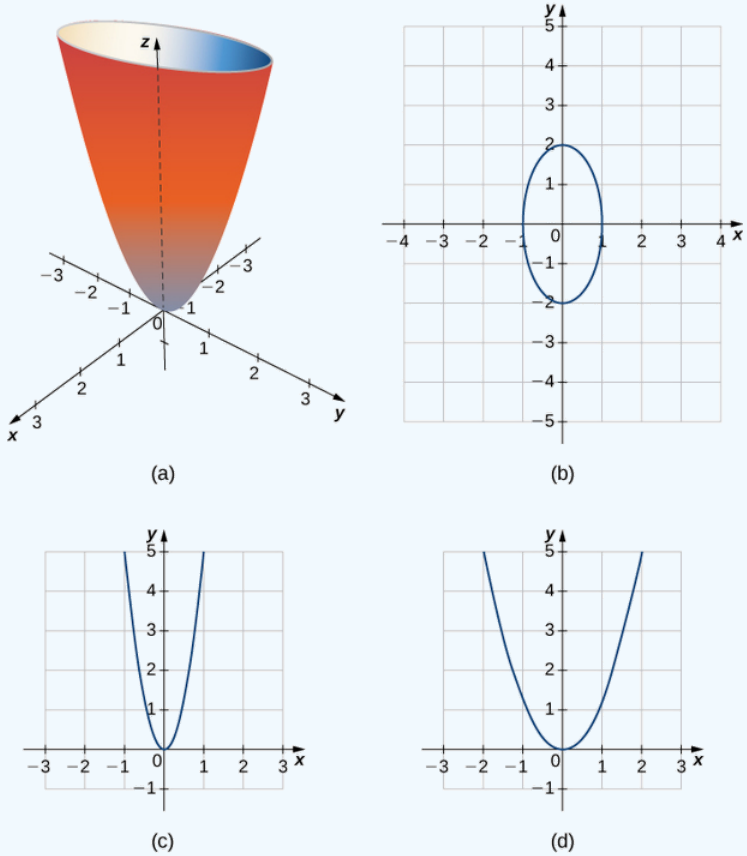


Figure 12.6.10: (a) The paraboloid $x^2 + \frac{y^2}{2^2} = \frac{z}{5}$. (b) The trace in plane $z = 5$. (c) The trace in the xz-plane. (d) The trace in the yz-plane.

Exercise 12.6.2:

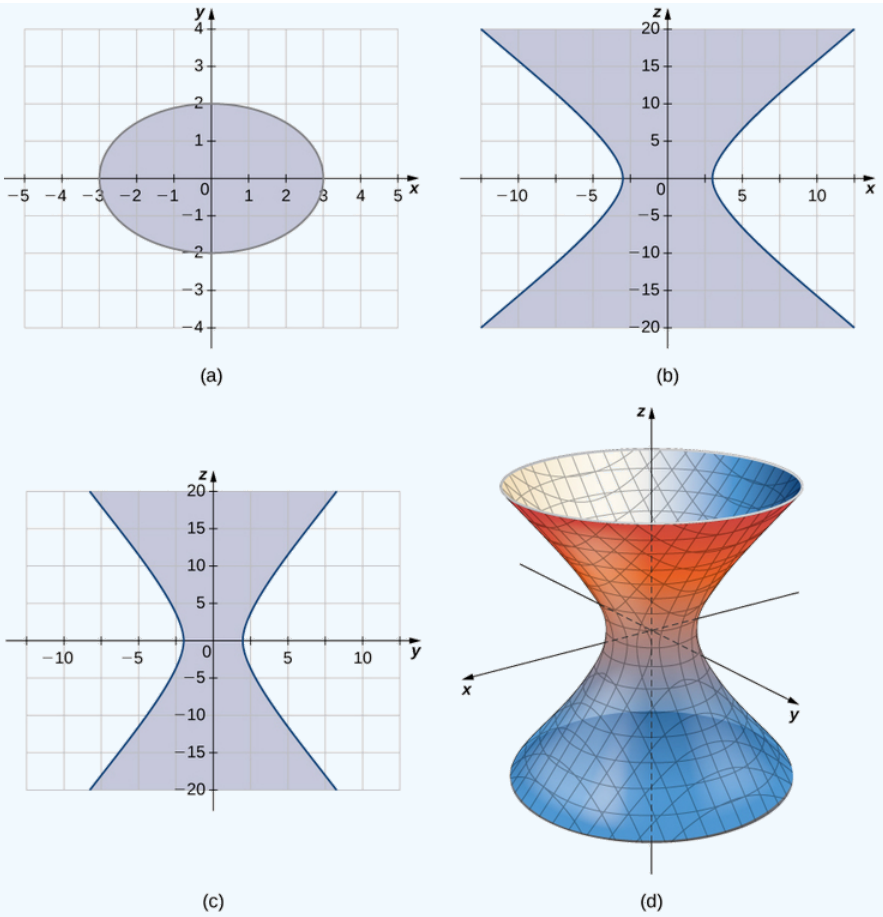
A hyperboloid of one sheet is any surface that can be described with an equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$. Describe the traces of the hyperboloid of one sheet given by equation $\frac{x^2}{3^2} + \frac{y^2}{2^2} - \frac{z^2}{5^2} = 1$.

Hint

To find the traces in the coordinate planes, set each variable to zero individually.

Answer

The traces parallel to the xy-plane are ellipses and the traces parallel to the xz- and yz-planes are hyperbolas. Specifically, the trace in the xy-plane is ellipse $\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$, the trace in the xz-plane is hyperbola $\frac{x^2}{3^2} - \frac{z^2}{5^2} = 1$, and the trace in the yz-plane is hyperbola $\frac{y^2}{2^2} - \frac{z^2}{5^2} = 1$ (see the following figure).



Hyperboloids of one sheet have some fascinating properties. For example, they can be constructed using straight lines, such as in the sculpture in Figure 12.6.11a. In fact, cooling towers for nuclear power plants are often constructed in the shape of a hyperboloid. The builders are able to use straight steel beams in the construction, which makes the towers very strong while using relatively little material (Figure 12.6.11b).

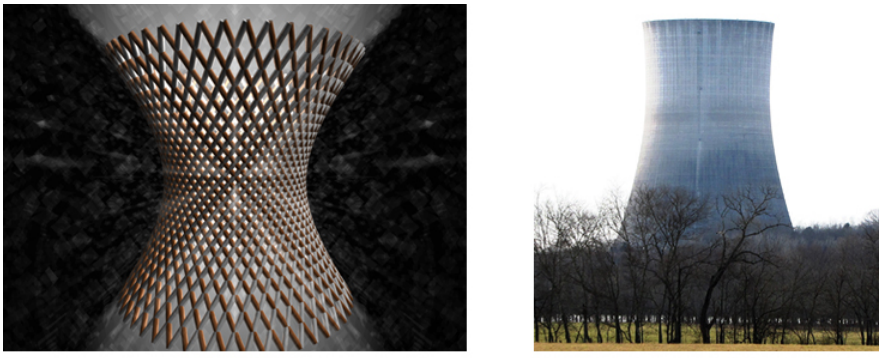


Figure 12.6.11: (a) A sculpture in the shape of a hyperboloid can be constructed of straight lines. (b) Cooling towers for nuclear power plants are often built in the shape of a hyperboloid.

Example 12.6.4: Chapter Opener: Finding the Focus of a Parabolic Reflector

Energy hitting the surface of a parabolic reflector is concentrated at the focal point of the reflector (Figure 12.6.12). If the surface of a parabolic reflector is described by equation $\frac{x^2}{100}+\frac{y^2}{100}=\frac{z}{4}$, where is the focal point of the reflector?

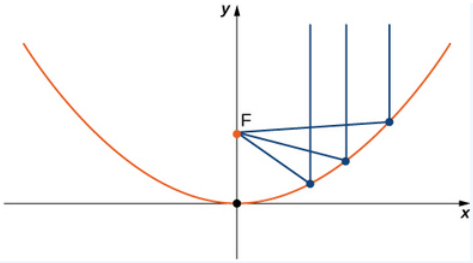


Figure 12.6.12: Energy reflects off of the parabolic reflector and is collected at the focal point. (credit: modification of CGP Grey, Wikimedia Commons)

Solution

Since z is the first-power variable, the axis of the reflector corresponds to the z-axis. The coefficients of x^2 and y^2 are equal, so the cross-section of the paraboloid perpendicular to the z-axis is a circle. We can consider a trace in the xz-plane or the yz-plane; the result is the same. Setting $y = 0$, the trace is a parabola opening up along the z-axis, with standard equation $x^2 = 4pz$, where p is the focal length of the parabola. In this case, this equation becomes $x^2 = 100 \cdot \frac{z}{4} = 4pz$ or $25 = 4p$. So p is 6.25 m, which tells us that the focus of the paraboloid is 6.25 m up the axis from the vertex. Because the vertex of this surface is the origin, the focal point is $(0, 0, 6.25)$.

Seventeen standard quadric surfaces can be derived from the general equation

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Jz + K = 0.$$

The following figures summarize the most important ones.

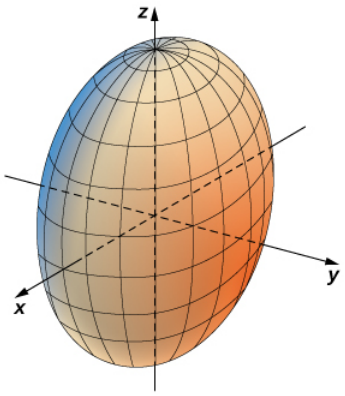
Characteristics of Common Quadric Surfaces

Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Traces
In plane $z = p$: an ellipse
In plane $y = q$: an ellipse
In plane $x = r$: an ellipse

If $a = b = c$, then this surface is a sphere.

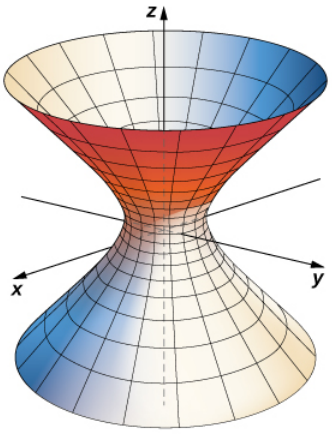


Hyperboloid of One Sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Traces
In plane $z = p$: an ellipse
In plane $y = q$: a hyperbola
In plane $x = r$: a hyperbola

In the equation for this surface, two of the variables have positive coefficients and one has a negative coefficient. The axis of the surface corresponds to the variable with the negative coefficient.



Hyperboloid of Two Sheets

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Traces
In plane $z = p$: an ellipse or the empty set (no trace)
In plane $y = q$: a hyperbola
In plane $x = r$: a hyperbola

In the equation for this surface, two of the variables have negative coefficients and one has a positive coefficient. The axis of the surface corresponds to the variable with a positive coefficient. The surface does not intersect the coordinate plane perpendicular to the axis.

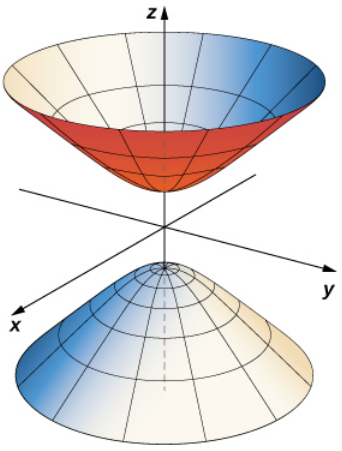


Figure 12.6.13: Characteristics of Common Quadric Surfaces: Ellipsoid, Hyperboloid of One Sheet, Hyperboloid of Two Sheets.

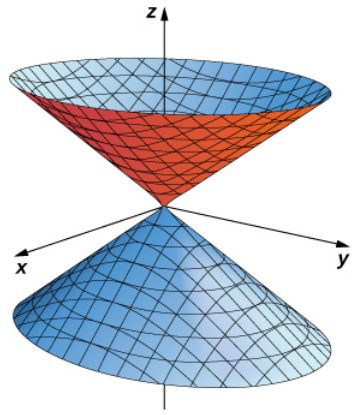
Characteristics of Common Quadric Surfaces

Elliptic Cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

Traces
In plane $z = p$: an ellipse
In plane $y = q$: a hyperbola
In plane $x = r$: a hyperbola
In the xz – plane: a pair of lines that intersect at the origin
In the yz – plane: a pair of lines that intersect at the origin

The axis of the surface corresponds to the variable with a negative coefficient. The traces in the coordinate planes parallel to the axis are intersecting lines.

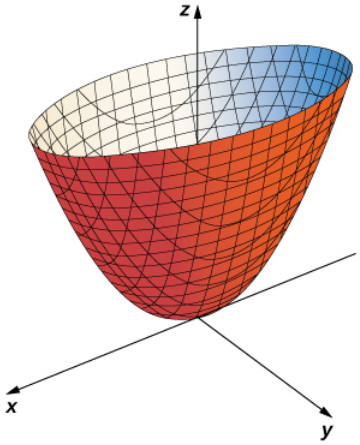


Elliptic Paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Traces
In plane $z = p$: an ellipse
In plane $y = q$: a parabola
In plane $x = r$: a parabola

The axis of the surface corresponds to the linear variable.



Hyperbolic Paraboloid

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

Traces
In plane $z = p$: a hyperbola
In plane $y = q$: a parabola
In plane $x = r$: a parabola

The axis of the surface corresponds to the linear variable.

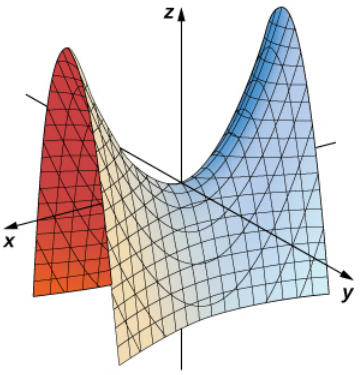


Figure 12.6.14: Characteristics of Common Quadric Surfaces: Elliptic Cone, Elliptic Paraboloid, Hyperbolic Paraboloid.

Example 12.6.5: Identifying Equations of Quadric Surfaces

Identify the surfaces represented by the given equations.

a. $16x^2 + 9y^2 + 16z^2 = 144$
b. $9x^2 - 18x + 4y^2 + 16y - 36z + 25 = 0$

Solution

a. The $x, y,$ and z terms are all squared, and are all positive, so this is probably an ellipsoid. However, let's put the equation into the standard form for an ellipsoid just to be sure. We have

$$16x^2 + 9y^2 + 16z^2 = 144.$$

Dividing through by 144 gives

$$\frac{x^2}{9}+\frac{y^2}{16}+\frac{z^2}{9}=1.$$

So, this is, in fact, an ellipsoid, centered at the origin.

b. We first notice that the z term is raised only to the first power, so this is either an elliptic paraboloid or a hyperbolic paraboloid. We also note there are x terms and y terms that are not squared, so this quadric surface is not centered at the origin. We need to complete the square to put this equation in one of the standard forms. We have

$$\begin{aligned}
9x^2-18x+4y^2+16y-36z+25&=0\\
9x^2-18x+4y^2+16y+25&=36z\\
9(x^2-2x)+4(y^2+4y)+25&=36z\\
9(x^2-2x+1-1)+4(y^2+4y+4-4)+25&=36z\\
9(x-1)^2-9+4(y+2)^2-16+25&=36z\\
9(x-1)^2+4(y+2)^2&=36z\\
\frac{(x-1)^2}{4}+\frac{(y+2)^2}{9}&=z.
\end{aligned}$$

This is an elliptic paraboloid centered at $(1,-2,0)$.

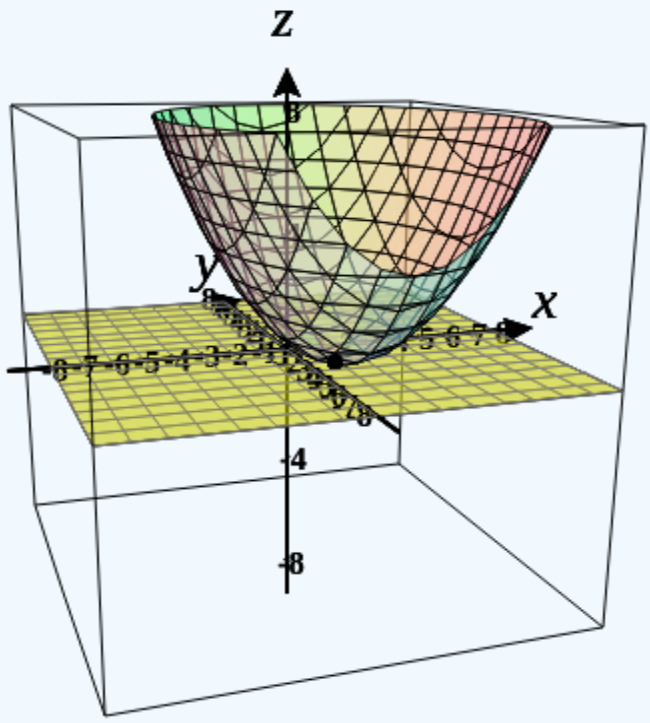


Figure 12.6.15: Plot of $z=\frac{(x-1)^2}{4}+\frac{(y+2)^2}{9}$ in CalcPlot3D.

Exercise 12.6.3

Identify the surface represented by equation $9x^2 + y^2 - z^2 + 2z - 10 = 0$.

Hint

Look at the signs and powers of the x, y, and z terms

Answer

Hyperboloid of one sheet, centered at $(0, 0, 1)$.

Key Concepts

- A set of lines parallel to a given line passing through a given curve is called a *cylinder*, or a cylindrical surface. The parallel lines are called *rulings*.
- The intersection of a three-dimensional surface and a plane is called a trace. To find the trace in the xy-, yz-, or xz-planes, set $z = 0$, $x = 0$, or $y = 0$, respectively.
- Quadric surfaces are three-dimensional surfaces with traces composed of conic sections. Every quadric surface can be expressed with an equation of the form

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Jz + K = 0.$$

- To sketch the graph of a quadric surface, start by sketching the traces to understand the framework of the surface.
- Important quadric surfaces are summarized in Figures 12.6.13 and 12.6.14.

Glossary

cylinder

a set of lines parallel to a given line passing through a given curve

ellipsoid

a three-dimensional surface described by an equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$; all traces of this surface are ellipses

elliptic cone

a three-dimensional surface described by an equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$; traces of this surface include ellipses and intersecting lines

elliptic paraboloid

a three-dimensional surface described by an equation of the form $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$; traces of this surface include ellipses and parabolas

hyperboloid of one sheet

a three-dimensional surface described by an equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$; traces of this surface include ellipses and hyperbolas

hyperboloid of two sheets

a three-dimensional surface described by an equation of the form $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$; traces of this surface include ellipses and hyperbolas

quadric surfaces

surfaces in three dimensions having the property that the traces of the surface are conic sections (ellipses, hyperbolas, and parabolas)

rulings

parallel lines that make up a cylindrical surface

trace

the intersection of a three-dimensional surface with a coordinate plane

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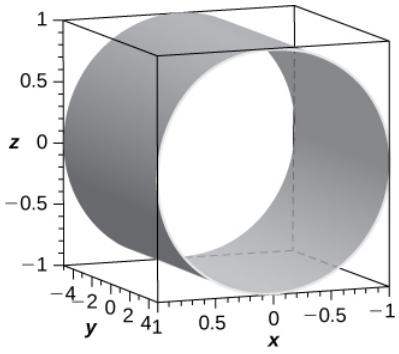
12.6E: Exercises for Section 12.6

For exercises 1 - 6, sketch and describe the cylindrical surface of the given equation.

1) [T] $x^2 + z^2 = 1$

Answer

The surface is a cylinder with the rulings parallel to the y-axis.

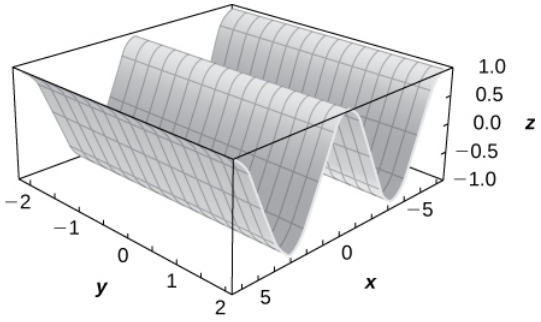


2) [T] $x^2 + y^2 = 9$

3) [T] $z = \cos\left(\frac{\pi}{2} + x\right)$

Answer

The surface is a cylinder with rulings parallel to the y-axis.



4) [T] $z = e^x$

5) [T] $z = 9 - y^2$

Answer

The surface is a cylinder with rulings parallel to the x-axis.

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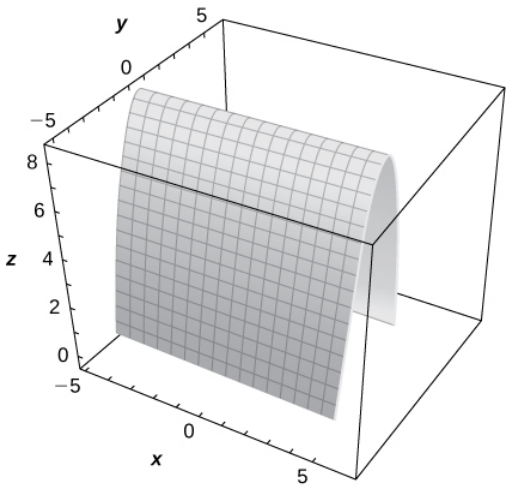


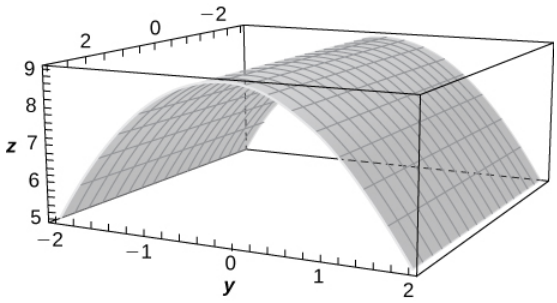
6) [T] $z = \ln x$

For exercises 7 - 10, the graph of a quadric surface is given.

a. Specify the name of the quadric surface.

b. Determine the axis of symmetry of the quadric surface.

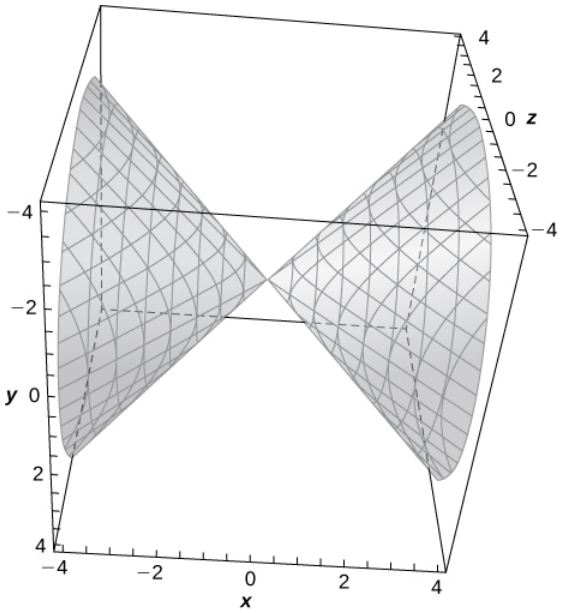
7)



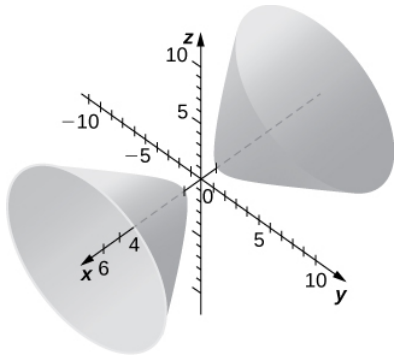
Answer

a. Cylinder; b. The x-axis

8)



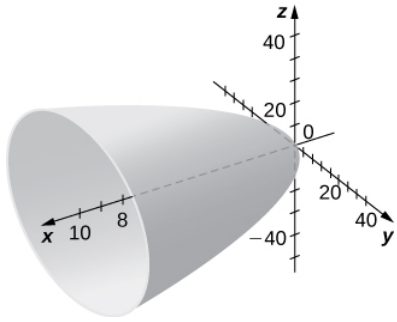
9)



Answer

a. Hyperboloid of two sheets; b. The x-axis

10)



For exercises 11 - 16, match the given quadric surface with its corresponding equation in standard form.

a. $\frac{x^2}{4}+\frac{y^2}{9}-\frac{z^2}{12}=1$

b. $\frac{x^2}{4}-\frac{y^2}{9}-\frac{z^2}{12}=1$

c. $\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{12}=1$

d. $z = 4x^2 + 3y^2$

e. $z = 4x^2 - y^2$

f. $4x^2 + y^2 - z^2 = 0$

11) Hyperboloid of two sheets

Answer

b.

12) Ellipsoid

13) Elliptic paraboloid

Answer

d.

14) Hyperbolic paraboloid

15) Hyperboloid of one sheet

Answer

a.

16) Elliptic cone

For exercises 17 - 28, rewrite the given equation of the quadric surface in standard form. Identify the surface.

17) $-x^2 + 36y^2 + 36z^2 = 9$

Answer

$-\frac{x^2}{9} + \frac{y^2}{\frac{1}{4}} + \frac{z^2}{\frac{1}{4}} = 1$, hyperboloid of one sheet with the x-axis as its axis of symmetry

18) $-4x^2 + 25y^2 + z^2 = 100$

19) $-3x^2 + 5y^2 - z^2 = 10$

Answer

$-\frac{x^2}{\frac{10}{3}} + \frac{y^2}{2} - \frac{z^2}{10} = 1$, hyperboloid of two sheets with the y-axis as its axis of symmetry

20) $3x^2 - y^2 - 6z^2 = 18$

21) $5y = x^2 - z^2$

Answer

$y = -\frac{z^2}{5} + \frac{x^2}{5}$, hyperbolic paraboloid with the y-axis as its axis of symmetry

22) $8x^2 - 5y^2 - 10z = 0$

23) $x^2 + 5y^2 + 3z^2 - 15 = 0$

Answer

$\frac{x^2}{15} + \frac{y^2}{3} + \frac{z^2}{5} = 1$, ellipsoid

24) $63x^2 + 7y^2 + 9z^2 - 63 = 0$

25) $x^2 + 5y^2 - 8z^2 = 0$

Answer

$\frac{x^2}{40} + \frac{y^2}{8} - \frac{z^2}{5} = 0$, elliptic cone with the z-axis as its axis of symmetry

26) $5x^2 - 4y^2 + 20z^2 = 0$

27) $6x = 3y^2 + 2z^2$

Answer

$x = \frac{y^2}{2} + \frac{z^2}{3}$, elliptic paraboloid with the x-axis as its axis of symmetry

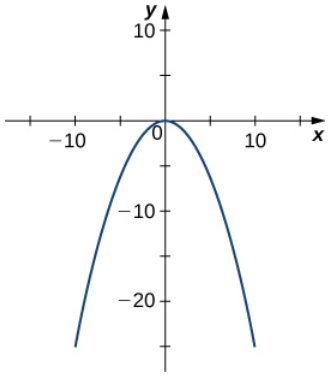
28) $49y = x^2 + 7z^2$

For exercises 29 - 34, find the trace of the given quadric surface in the specified plane of coordinates and sketch it.

29) [T] $x^2 + z^2 + 4y = 0, \quad z = 0$

Answer

Parabola $y = -\frac{x^2}{4}$,

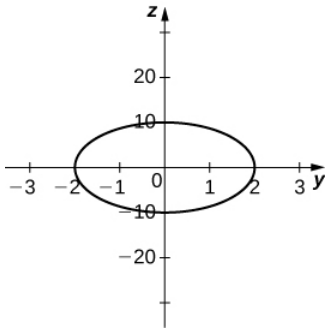


30) [T] $x^2 + z^2 + 4y = 0, \quad x = 0$

31) [T] $-4x^2 + 25y^2 + z^2 = 100, \quad x = 0$

Answer

Ellipse $\frac{y^2}{4} + \frac{z^2}{100} = 1$,

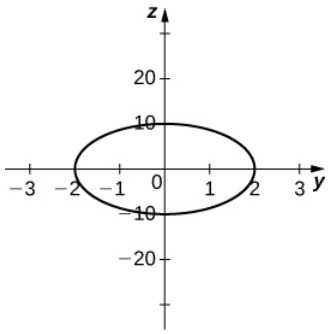


32) [T] $-4x^2 + 25y^2 + z^2 = 100, \quad y = 0$

33) [T] $x^2 + \frac{y^2}{4} + \frac{z^2}{100} = 1, \quad x = 0$

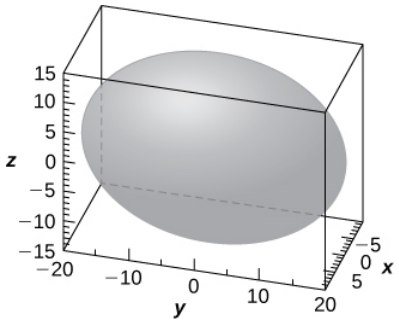
Answer

Ellipse $\frac{y^2}{4} + \frac{z^2}{100} = 1$,



34) [T] $x^2 - y - z^2 = 1, \quad y = 0$

35) Use the graph of the given quadric surface to answer the questions.



a. Specify the name of the quadric surface.

b. Which of the equations—$16x^2 + 9y^2 + 36z^2 = 3600,\ 9x^2 + 36y^2 + 16z^2 = 3600$, or $36x^2 + 9y^2 + 16z^2 = 3600$ —corresponds to the graph?

c. Use b. to write the equation of the quadric surface in standard form.

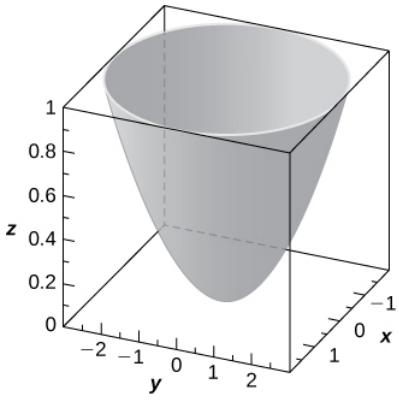
Answer

a. Ellipsoid

b. The third equation

c. $\frac{x^2}{100} + \frac{y^2}{400} + \frac{z^2}{225} = 1$

36) Use the graph of the given quadric surface to answer the questions.



a. Specify the name of the quadric surface.

b. Which of the equations—$36z = 9x^2 + y^2,\ 9x^2 + 4y^2 = 36z$, or $-36z = -81x^2 + 4y^2$ —corresponds to the graph above?

c. Use b. to write the equation of the quadric surface in standard form.

For exercises 37 - 42, the equation of a quadric surface is given.

a. Use the method of completing the square to write the equation in standard form.

b. Identify the surface.

37) $x^2+2z^2+6x-8z+1=0$

Answer

a. $\dfrac{(x+3)^2}{16}+\dfrac{(z-2)^2}{8}=1$

b. Cylinder centered at $(-3,2)$ with rulings parallel to the y-axis

38) $4x^2-y^2+z^2-8x+2y+2z+3=0$

39) $x^2+4y^2-4z^2-6x-16y-16z+5=0$

Answer

a. $\dfrac{(x-3)^2}{4}+(y-2)^2-(z+2)^2=1$

b. Hyperboloid of one sheet centered at $(3,2,-2)$, with the z-axis as its axis of symmetry

40) $x^2+z^2-4y+4=0$

41) $x^2+\dfrac{y^2}{4}-\dfrac{z^2}{3}+6x+9=0$

Answer

a. $(x+3)^2+\dfrac{y^2}{4}-\dfrac{z^2}{3}=0$

b. Elliptic cone centered at $(-3,0,0)$, with the z-axis as its axis of symmetry

42) $x^2-y^2+z^2-12z+2x+37=0$

43) Write the standard form of the equation of the ellipsoid centered at the origin that passes through points $A(2,0,0)$, $B(0,0,1)$, and $C\left(12,\sqrt{11},\frac{1}{2}\right)$.

Answer

$\dfrac{x^2}{4}+\dfrac{y^2}{16}+z^2=1$

44) Write the standard form of the equation of the ellipsoid centered at point $P(1,1,0)$ that passes through points $A(6,1,0)$, $B(4,2,0)$ and $C(1,2,1)$.

45) Determine the intersection points of elliptic cone $x^2-y^2-z^2=0$ with the line of symmetric equations $\dfrac{x-1}{2}=\dfrac{y+1}{3}=z$.

Answer

$(1,-1,0)$ and $\left(\frac{13}{3},4,\frac{5}{3}\right)$

46) Determine the intersection points of parabolic hyperboloid $z=3x^2-2y^2$ with the line of parametric equations $x=3t$, $y=2t$, $z=19t$, where $t\in R$.

47) Find the equation of the quadric surface with points $P(x,y,z)$ that are equidistant from point $Q(0,-1,0)$ and plane of equation $y=1$. Identify the surface.

Answer

$x^2+z^2+4y=0$, elliptic paraboloid

48) Find the equation of the quadric surface with points $P(x,y,z)$ that are equidistant from point $Q(0,2,0)$ and plane of equation $y=-2$. Identify the surface.

49) If the surface of a parabolic reflector is described by equation $400z=x^2+y^2$, find the focal point of the reflector.

Answer

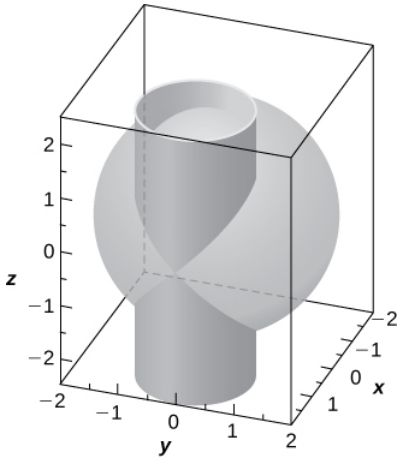
$(0, 0, 100)$

50) Consider the parabolic reflector described by equation $z = 20x^2 + 20y^2$. Find its focal point.

51) Show that quadric surface $x^2 + y^2 + z^2 + 2xy + 2xz + 2yz + x + y + z = 0$ reduces to two parallel planes.

52) Show that quadric surface $x^2 + y^2 + z^2 - 2xy - 2xz + 2yz - 1 = 0$ reduces to two parallel planes passing.

53) [T] The intersection between cylinder $(x-1)^2 + y^2 = 1$ and sphere $x^2 + y^2 + z^2 = 4$ is called a *Viviani curve.*



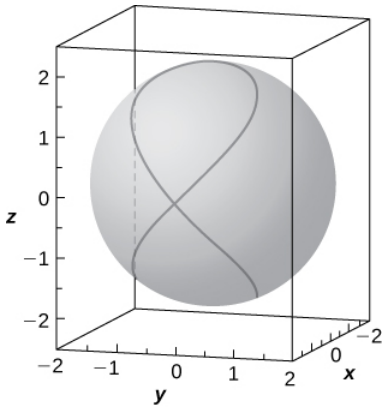
a. Solve the system consisting of the equations of the surfaces to find the equation of the intersection curve. (Hint: Find x and y in terms of z.)

b. Use a computer algebra system (CAS) or CalcPlot3D to visualize the intersection curve on sphere $x^2 + y^2 + z^2 = 4$.

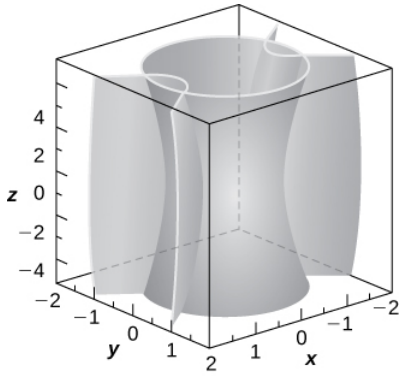
Answer

a. $x = 2 - \dfrac{z^2}{2}, \quad y = \pm\dfrac{z}{2}\sqrt{4 - z^2}$, where $z \in [-2, 2]$;

b.



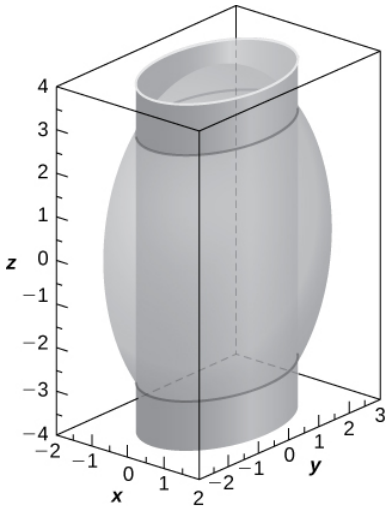
54) Hyperboloid of one sheet $25x^2 + 25y^2 - z^2 = 25$ and elliptic cone $-25x^2 + 75y^2 + z^2 = 0$ are represented in the following figure along with their intersection curves. Identify the intersection curves and find their equations (Hint: Find y from the system consisting of the equations of the surfaces.)



55) [T] Use a CAS or CalcPlot3D to create the intersection between cylinder $9x^2 + 4y^2 = 18$ and ellipsoid $36x^2 + 16y^2 + 9z^2 = 144$, and find the equations of the intersection curves.

Answer

two ellipses of equations $\frac{x^2}{2} + \frac{y^2}{\frac{9}{2}} = 1$ in planes $z = \pm 2\sqrt{2}$



56) [T] A spheroid is an ellipsoid with two equal semiaxes. For instance, the equation of a spheroid with the z-axis as its axis of symmetry is given by $\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1$, where a and c are positive real numbers. The spheroid is called oblate if $c < a$, and prolate for $c > a$.

a. The eye cornea is approximated as a prolate spheroid with an axis that is the eye, where $a = 8.7mm$ and $c = 9.6mm$.Write the equation of the spheroid that models the cornea and sketch the surface.

b. Give two examples of objects with prolate spheroid shapes.

57) [T] In cartography, Earth is approximated by an oblate spheroid rather than a sphere. The radii at the equator and poles are approximately 3963mi and 3950mi, respectively.

a. Write the equation in standard form of the ellipsoid that represents the shape of Earth. Assume the center of Earth is at the origin and that the trace formed by plane $z = 0$ corresponds to the equator.

b. Sketch the graph.

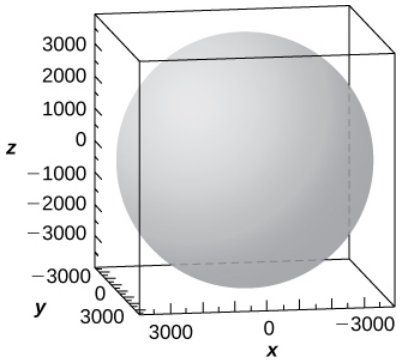
c. Find the equation of the intersection curve of the surface with plane $z = 1000$ that is parallel to the xy-plane. The intersection curve is called a *parallel.*

d. Find the equation of the intersection curve of the surface with plane $x + y = 0$ that passes through the z-axis. The intersection curve is called a meridian.

Answer

a. $\frac{x^2}{3963^2}+\frac{y^2}{3963^2}+\frac{z^2}{3950^2}=1$

b.



c. The intersection curve is the ellipse of equation $\frac{x^2}{3963^2}+\frac{y^2}{3963^2}=\frac{(2950)(4950)}{3950^2}$, and the intersection is an ellipse.

d. The intersection curve is the ellipse of equation $\frac{2y^2}{3963^2}+\frac{z^2}{3950^2}=1$.

58) [T] A set of buzzing stunt magnets (or “rattlesnake eggs”) includes two sparkling, polished, superstrong spheroid-shaped magnets well-known for children’s entertainment. Each magnet is 1.625 in. long and 0.5 in. wide at the middle. While tossing them into the air, they create a buzzing sound as they attract each other.

a. Write the equation of the prolate spheroid centered at the origin that describes the shape of one of the magnets.

b. Write the equations of the prolate spheroids that model the shape of the buzzing stunt magnets. Use a CAS or CalcPlot3D to create the graphs.

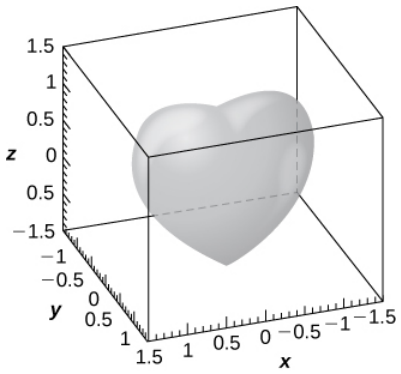
59) [T] A heart-shaped surface is given by equation $(x^2+\frac{9}{4}y^2+z^2-1)^3-x^2z^3-\frac{9}{80}y^2z^3=0$.

a. Use a CAS or CalcPlot3D to graph the surface that models this shape.

b. Determine and sketch the trace of the heart-shaped surface on the xz-plane.

Answer

a.



b. The intersection curve is $(x^2+z^2-1)^3-x^2z^3=0$.

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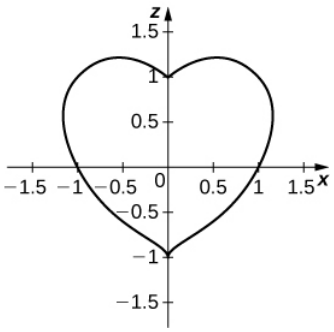
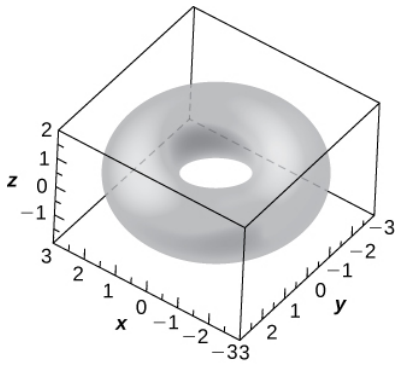


60) [T] The ring torus symmetric about the z-axis is a special type of surface in topology and its equation is given by $(x^2+y^2+z^2+R^2-r^2)^2 = 4R^2(x^2+y^2)$, where $R > r > 0$. The numbers R and r are called are the major and minor radii, respectively, of the surface. The following figure shows a ring torus for which $R = 2$ and $r = 1$.



a. Write the equation of the ring torus with $R = 2$ and $r = 1$, and use a CAS or CalcPlot3D to graph the surface. Compare the graph with the figure given.

b. Determine the equation and sketch the trace of the ring torus from a. on the xy-plane.

c. Give two examples of objects with ring torus shapes.

Contributors

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12.7: Cylindrical and Spherical Coordinates

Learning Objectives

- Convert from cylindrical to rectangular coordinates.
- Convert from rectangular to cylindrical coordinates.
- Convert from spherical to rectangular coordinates.
- Convert from rectangular to spherical coordinates.

The Cartesian coordinate system provides a straightforward way to describe the location of points in space. Some surfaces, however, can be difficult to model with equations based on the Cartesian system. This is a familiar problem; recall that in two dimensions, polar coordinates often provide a useful alternative system for describing the location of a point in the plane, particularly in cases involving circles. In this section, we look at two different ways of describing the location of points in space, both of them based on extensions of polar coordinates. As the name suggests, cylindrical coordinates are useful for dealing with problems involving cylinders, such as calculating the volume of a round water tank or the amount of oil flowing through a pipe. Similarly, spherical coordinates are useful for dealing with problems involving spheres, such as finding the volume of domed structures.

Cylindrical Coordinates

When we expanded the traditional Cartesian coordinate system from two dimensions to three, we simply added a new axis to model the third dimension. Starting with polar coordinates, we can follow this same process to create a new three-dimensional coordinate system, called the cylindrical coordinate system. In this way, cylindrical coordinates provide a natural extension of polar coordinates to three dimensions.

Definition: The Cylindrical Coordinate System

In the cylindrical coordinate system**,** a point in space (Figure 12.7.1) is represented by the ordered triple (r, θ, z), where

- (r, θ) are the polar coordinates of the point's projection in the xy-plane
- z is the usual z-**coordinate** in the Cartesian coordinate system

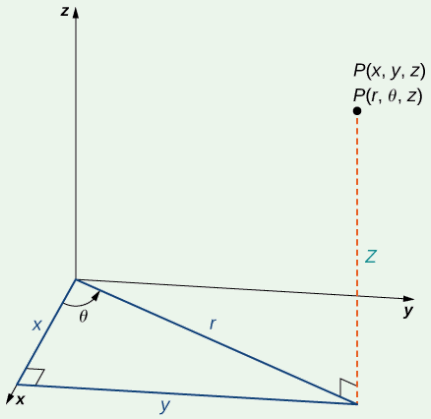


Figure 12.7.1: The right triangle lies in the xy-plane. The length of the hypotenuse is r and θ is the measure of the angle formed by the positive x-axis and the hypotenuse. The z-coordinate describes the location of the point above or below the xy-plane.

In the xy-plane, the right triangle shown in Figure 12.7.1 provides the key to transformation between cylindrical and Cartesian, or rectangular, coordinates.

Conversion between Cylindrical and Cartesian Coordinates

The rectangular coordinates (x, y, z) and the cylindrical coordinates (r, θ, z) of a point are related as follows:

These equations are used to convert from cylindrical coordinates to rectangular coordinates.

- $x = r\cos\theta$

- $y = r \sin \theta$
- $z = z$

These equations are used to convert from rectangular coordinates to cylindrical coordinates

1. $r^2 = x^2 + y^2$
2. $\tan \theta = \dfrac{y}{x}$
3. $z = z$

As when we discussed conversion from rectangular coordinates to polar coordinates in two dimensions, it should be noted that the equation $\tan \theta = \dfrac{y}{x}$ has an infinite number of solutions. However, if we restrict θ to values between 0 and 2π, then we can find a unique solution based on the quadrant of the xy-plane in which original point (x, y, z) is located. Note that if $x = 0$, then the value of θ is either $\dfrac{\pi}{2}, \dfrac{3\pi}{2}$, or 0, depending on the value of y.

Notice that these equations are derived from properties of right triangles. To make this easy to see, consider point P in the xy-plane with rectangular coordinates $(x, y, 0)$ and with cylindrical coordinates $(r, \theta, 0)$, as shown in Figure 12.7.2.

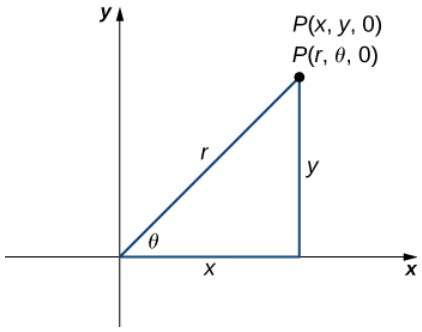


Figure 12.7.2: The Pythagorean theorem provides equation $r^2 = x^2 + y^2$. Right-triangle relationships tell us that $x = r \cos \theta, y = r \sin \theta$, and $\tan \theta = y/x$.

Let's consider the differences between rectangular and cylindrical coordinates by looking at the surfaces generated when each of the coordinates is held constant. If c is a constant, then in rectangular coordinates, surfaces of the form $x = c, y = c$, or $z = c$ are all planes. Planes of these forms are parallel to the yz-plane, the xz-plane, and the xy-plane, respectively. When we convert to cylindrical coordinates, the z-coordinate does not change. Therefore, in cylindrical coordinates, surfaces of the form $z = c$ are planes parallel to the xy-plane. Now, let's think about surfaces of the form $r = c$. The points on these surfaces are at a fixed distance from the z-axis. In other words, these surfaces are vertical circular cylinders. Last, what about $\theta = c$? The points on a surface of the form $\theta = c$ are at a fixed angle from the x-axis, which gives us a half-plane that starts at the z-axis (Figures 12.7.3 and 12.7.4).

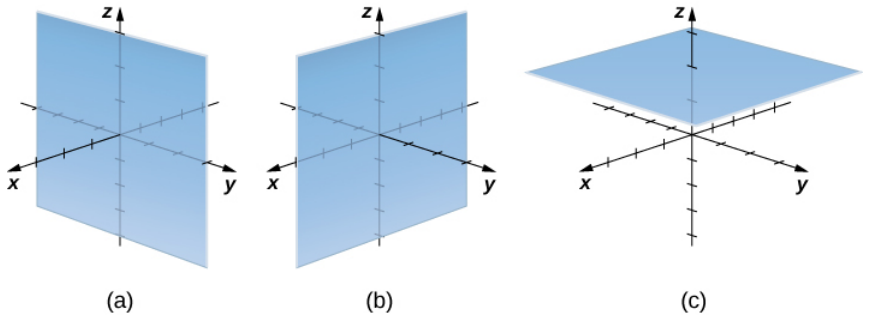


Figure 12.7.3: In rectangular coordinates, (a) surfaces of the form $x = c$ are planes parallel to the yz-plane, (b) surfaces of the form $y = c$ are planes parallel to the xz-plane, and (c) surfaces of the form $z = c$ are planes parallel to the xy-plane.

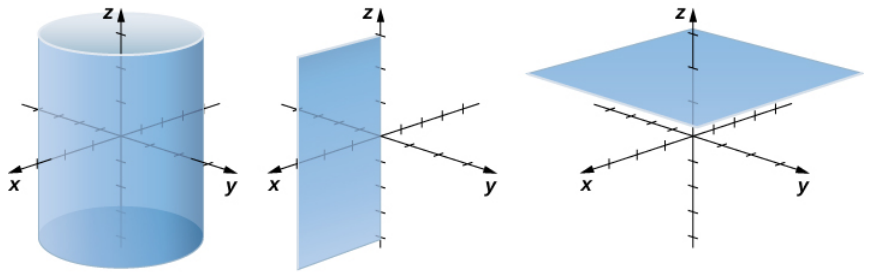


Figure 12.7.4: In cylindrical coordinates, (a) surfaces of the form $r = c$ are vertical cylinders of radius r, (b) surfaces of the form $\theta = c$ are half-planes at angle θ from the x-axis, and (c) surfaces of the form $z = c$ are planes parallel to the xy-plane.

Example 12.7.1: Converting from Cylindrical to Rectangular Coordinates

Plot the point with cylindrical coordinates $(4, \frac{2\pi}{3}, -2)$ and express its location in rectangular coordinates.

Solution

Conversion from cylindrical to rectangular coordinates requires a simple application of the equations listed in Conversion between Cylindrical and Cartesian Coordinates:

$$x = r\cos\theta = 4\cos\frac{2\pi}{3} = -2$$
$$y = r\sin\theta = 4\sin\frac{2\pi}{3} = 2\sqrt{3}.$$
$$z = -2$$

The point with cylindrical coordinates $(4, \frac{2\pi}{3}, -2)$ has rectangular coordinates $(-2, 2\sqrt{3}, -2)$ (Figure 12.7.5).

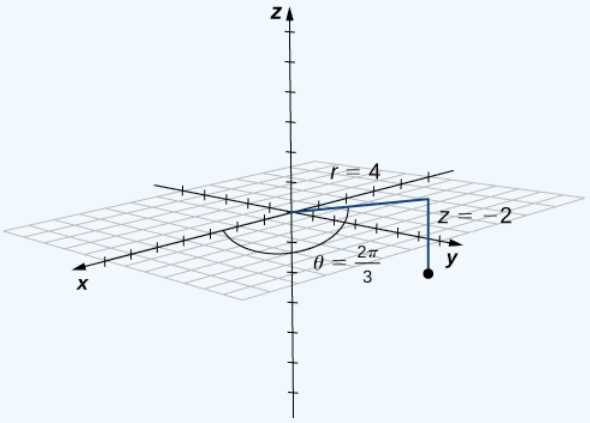


Figure 12.7.5: The projection of the point in the xy-plane is 4 units from the origin. The line from the origin to the point's projection forms an angle of $\frac{2\pi}{3}$ with the positive x-axis. The point lies 2 units below the xy-plane.

Exercise 12.7.1

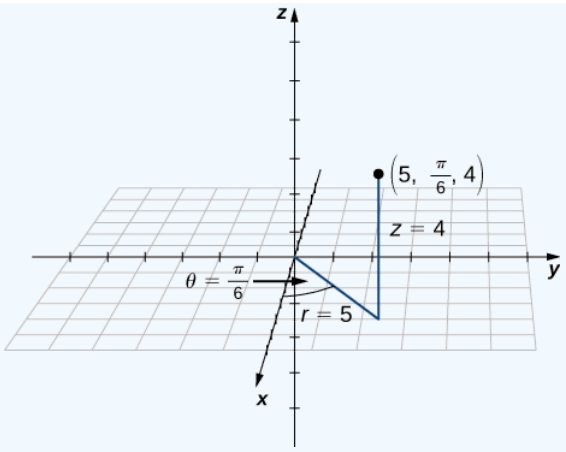
Point R has cylindrical coordinates $(5, \frac{\pi}{6}, 4)$. Plot R and describe its location in space using rectangular, or Cartesian, coordinates.

Hint

The first two components match the polar coordinates of the point in the xy-plane.

Answer

The rectangular coordinates of the point are $(\frac{5\sqrt{3}}{2}, \frac{5}{2}, 4)$.



If this process seems familiar, it is with good reason. This is exactly the same process that we followed in Introduction to Parametric Equations and Polar Coordinates to convert from polar coordinates to two-dimensional rectangular coordinates.

Example 12.7.2: Converting from Rectangular to Cylindrical Coordinates

Convert the rectangular coordinates $(1, -3, 5)$ to cylindrical coordinates.

Solution

Use the second set of equations from Conversion between Cylindrical and Cartesian Coordinates to translate from rectangular to cylindrical coordinates:

$$\begin{aligned} r^2 &= x^2 + y^2 \\ r &= \pm\sqrt{1^2 + (-3)^2} \\ &= \pm\sqrt{10}. \end{aligned}$$

We choose the positive square root, so $r = \sqrt{10}$.Now, we apply the formula to find θ. In this case, y is negative and x is positive, which means we must select the value of θ between $\frac{3\pi}{2}$ and 2π:

$$\begin{aligned} \tan\theta &= \frac{y}{x} & &= \frac{-3}{1} \\ \theta &= \arctan(-3) & &\approx 5.03\,\text{rad}. \end{aligned}$$

In this case, the z-coordinates are the same in both rectangular and cylindrical coordinates:

$$z = 5.$$

The point with rectangular coordinates $(1, -3, 5)$ has cylindrical coordinates approximately equal to $(\sqrt{10}, 5.03, 5)$.

Exercise 12.7.2

Convert point $(-8, 8, -7)$ from Cartesian coordinates to cylindrical coordinates.

Hint

$r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$

Answer

$(8\sqrt{2}, \frac{3\pi}{4}, -7)$

The use of cylindrical coordinates is common in fields such as physics. Physicists studying electrical charges and the capacitors used to store these charges have discovered that these systems sometimes have a cylindrical symmetry. These systems have

complicated modeling equations in the Cartesian coordinate system, which make them difficult to describe and analyze. The equations can often be expressed in more simple terms using cylindrical coordinates. For example, the cylinder described by equation $x^2 + y^2 = 25$ in the Cartesian system can be represented by cylindrical equation $r = 5$.

Example 12.7.3: Identifying Surfaces in the Cylindrical Coordinate System

Describe the surfaces with the given cylindrical equations.

a. $\theta = \dfrac{\pi}{4}$
b. $r^2 + z^2 = 9$
c. $z = r$

Solution

a. When the angle θ is held constant while r and z are allowed to vary, the result is a half-plane (Figure 12.7.6).

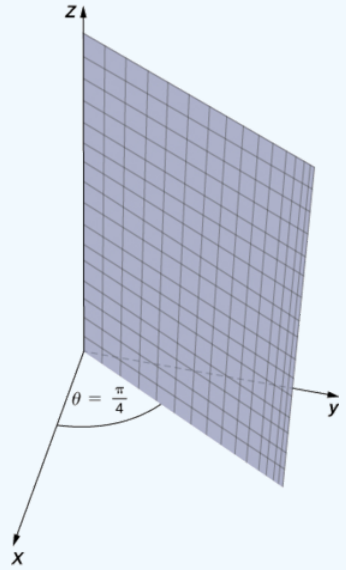


Figure 12.7.6: In polar coordinates, the equation $\theta = \pi/4$ describes the ray extending diagonally through the first quadrant. In three dimensions, this same equation describes a half-plane.

b. Substitute $r^2 = x^2 + y^2$ into equation $r^2 + z^2 = 9$ to express the rectangular form of the equation: $x^2 + y^2 + z^2 = 9$. This equation describes a sphere centered at the origin with radius 3 (Figure 12.7.7).

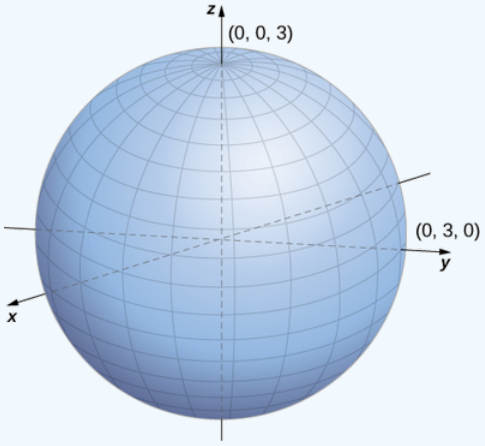


Figure 12.7.7: The sphere centered at the origin with radius 3 can be described by the cylindrical equation $r^2 + z^2 = 9$.

c. To describe the surface defined by equation $z = r$, is it useful to examine traces parallel to the xy-plane. For example, the trace in plane $z = 1$ is circle $r = 1$, the trace in plane $z = 3$ is circle $r = 3$, and so on. Each trace is a circle. As the value of z increases, the radius of the circle also increases. The resulting surface is a cone (Figure 12.7.8).

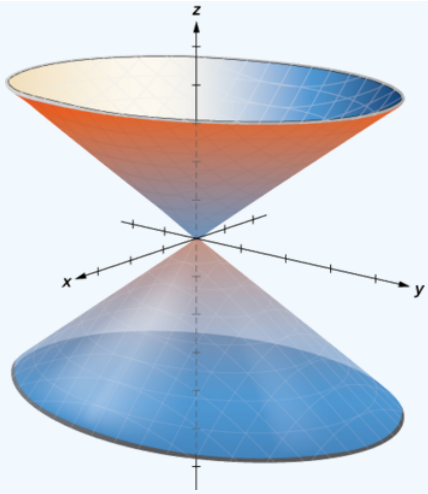


Figure 12.7.8: The traces in planes parallel to the xy-plane are circles. The radius of the circles increases as z increases.

? Exercise 12.7.3

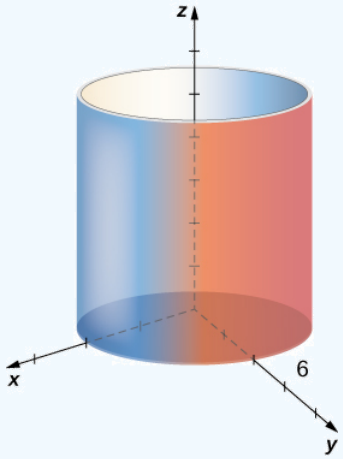
Describe the surface with cylindrical equation $r = 6$.

Hint

The θ and z components of points on the surface can take any value.

Answer

This surface is a cylinder with radius 6.



Spherical Coordinates

In the Cartesian coordinate system, the location of a point in space is described using an ordered triple in which each coordinate represents a distance. In the cylindrical coordinate system, the location of a point in space is described using two distances (r and z) and an angle measure (θ). In the spherical coordinate system, we again use an ordered triple to describe the location of a point in space. In this case, the triple describes one distance and two angles. Spherical coordinates make it simple to describe a sphere, just as cylindrical coordinates make it easy to describe a cylinder. Grid lines for spherical coordinates are based on angle measures, like those for polar coordinates.

Definition: spherical coordinate system

In the *spherical coordinate system*, a point P in space (Figure 12.7.9) is represented by the ordered triple (ρ, θ, φ) where

- ρ (the Greek letter rho) is the distance between P and the origin $(\rho \neq 0)$;
- θ is the same angle used to describe the location in cylindrical coordinates;
- φ (the Greek letter phi) is the angle formed by the positive z-axis and line segment $\overline{OP}$, where O is the origin and $0 \leq \varphi \leq \pi$.

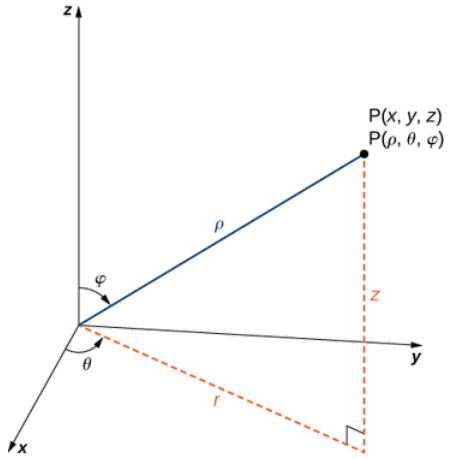


Figure 12.7.9: The relationship among spherical, rectangular, and cylindrical coordinates.

By convention, the origin is represented as $(0, 0, 0)$ in spherical coordinates.

HOW TO: Converting among Spherical, Cylindrical, and Rectangular Coordinates

Rectangular coordinates (x, y, z), cylindrical coordinates (r, θ, z), and spherical coordinates (ρ, θ, φ) of a point are related as follows:

Convert from spherical coordinates to rectangular coordinates

These equations are used to convert from spherical coordinates to rectangular coordinates.

- $x = \rho \sin\varphi \cos\theta$
- $y = \rho \sin\varphi \sin\theta$
- $z = \rho \cos\varphi$

Convert from rectangular coordinates to spherical coordinates

These equations are used to convert from rectangular coordinates to spherical coordinates.

- $\rho^2 = x^2 + y^2 + z^2$
- $\tan\theta = \dfrac{y}{x}$
- $\varphi = \arccos\left(\dfrac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$.

Convert from spherical coordinates to cylindrical coordinates

These equations are used to convert from spherical coordinates to cylindrical coordinates.

- $r = \rho \sin\varphi$
- $\theta = \theta$
- $z = \rho \cos\varphi$

Convert from cylindrical coordinates to spherical coordinates

These equations are used to convert from cylindrical coordinates to spherical coordinates.

- $\rho = \sqrt{r^2 + z^2}$
- $\theta = \theta$

- $\varphi = \arccos\left(\dfrac{z}{\sqrt{r^2+z^2}}\right)$

The formulas to convert from spherical coordinates to rectangular coordinates may seem complex, but they are straightforward applications of trigonometry. Looking at Figure 12.7.10, it is easy to see that $r = \rho \sin\varphi$. Then, looking at the triangle in the xy-plane with r as its hypotenuse, we have $x = r\cos\theta = \rho\sin\varphi\cos\theta$. The derivation of the formula for y is similar. Figure 12.7.10 also shows that $\rho^2 = r^2 + z^2 = x^2 + y^2 + z^2$ and $z = \rho\cos\varphi$. Solving this last equation for φ and then substituting $\rho = \sqrt{r^2+z^2}$ (from the first equation) yields $\varphi = \arccos\left(\dfrac{z}{\sqrt{r^2+z^2}}\right)$. Also, note that, as before, we must be careful when using the formula $\tan\theta = \dfrac{y}{x}$ to choose the correct value of θ.

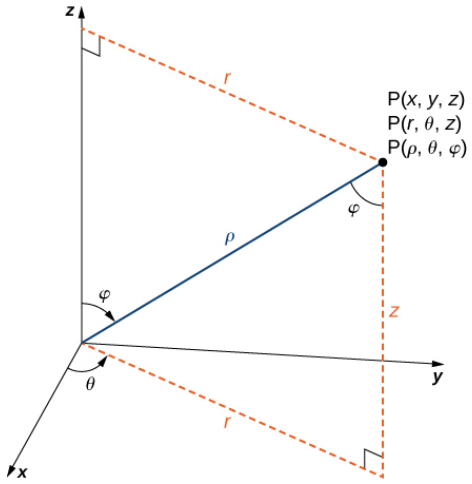


Figure 12.7.10: The equations that convert from one system to another are derived from right-triangle relationships.

As we did with cylindrical coordinates, let's consider the surfaces that are generated when each of the coordinates is held constant. Let c be a constant, and consider surfaces of the form $\rho = c$. Points on these surfaces are at a fixed distance from the origin and form a sphere. The coordinate θ in the spherical coordinate system is the same as in the cylindrical coordinate system, so surfaces of the form $\theta = c$ are half-planes, as before. Last, consider surfaces of the form $\varphi = c$. The points on these surfaces are at a fixed angle from the z-axis and form a half-cone (Figure 12.7.11).

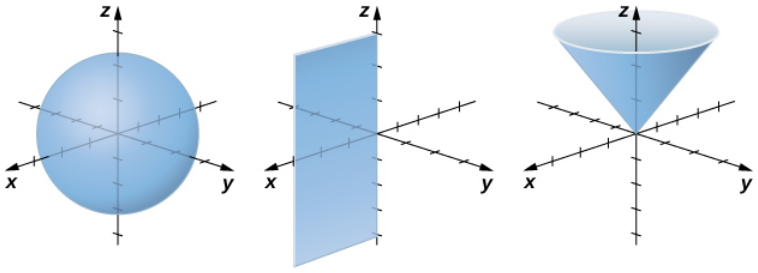


Figure 12.7.11: In spherical coordinates, surfaces of the form $\rho = c$ are spheres of radius ρ (a), surfaces of the form $\theta = c$ are half-planes at an angle θ from the x-axis (b), and surfaces of the form $\phi = c$ are half-cones at an angle ϕ from the z-axis (c).

Example 12.7.4: Converting from Spherical Coordinates

Plot the point with spherical coordinates $(8, \dfrac{\pi}{3}, \dfrac{\pi}{6})$ and express its location in both rectangular and cylindrical coordinates.

Solution

Use the equations in Converting among Spherical, Cylindrical, and Rectangular Coordinates to translate between spherical and cylindrical coordinates (Figure 12.7.12):

$$
\begin{aligned}
x &= \rho \sin\varphi \cos\theta \\
&= 8 \sin\left(\frac{\pi}{6}\right) \cos\left(\frac{\pi}{3}\right) \\
&= 8\left(\frac{1}{2}\right)\frac{1}{2} \\
&= 2 \\
y &= \rho \sin\varphi \sin\theta \\
&= 8 \sin\left(\frac{\pi}{6}\right) \sin\left(\frac{\pi}{3}\right) \\
&= 8\left(\frac{1}{2}\right)\frac{\sqrt{3}}{2} \\
&= 2\sqrt{3} \\
z &= \rho \cos\varphi \\
&= 8 \cos\left(\frac{\pi}{6}\right) \\
&= 8\left(\frac{\sqrt{3}}{2}\right) \\
&= 4\sqrt{3}
\end{aligned}
$$

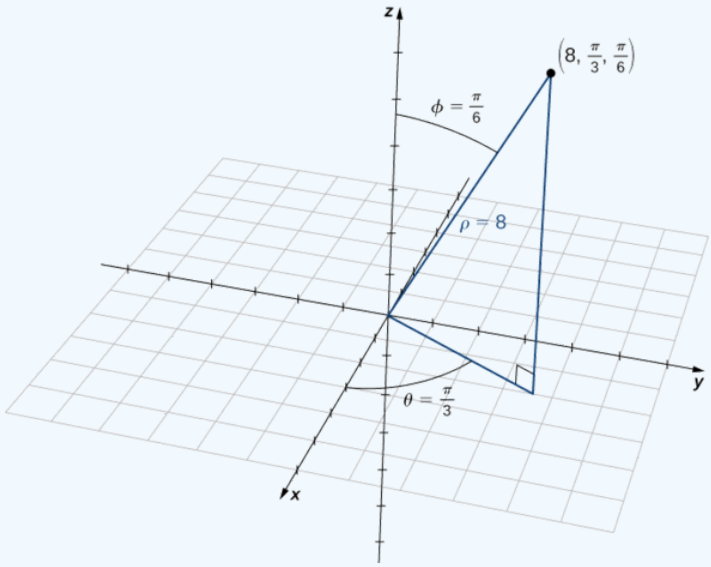


Figure 12.7.12: The projection of the point in the xy-plane is 4 units from the origin. The line from the origin to the point's projection forms an angle of $\pi/3$ with the positive x-axis. The point lies $4\sqrt{3}$ units above the xy-plane.

The point with spherical coordinates $(8, \frac{\pi}{3}, \frac{\pi}{6})$ has rectangular coordinates $(2, 2\sqrt{3}, 4\sqrt{3})$.

Finding the values in cylindrical coordinates is equally straightforward:

$$
\begin{aligned}
r &= \rho \sin\varphi \\
&= 8 \sin\frac{\pi}{6} \\
&= 4 \\
\theta &= \theta \\
z &= \rho \cos\varphi \\
&= 8 \cos\frac{\pi}{6} \\
&= 4\sqrt{3}.
\end{aligned}
$$

Thus, cylindrical coordinates for the point are $(4, \frac{\pi}{3}, 4\sqrt{3})$.

Exercise 12.7.4

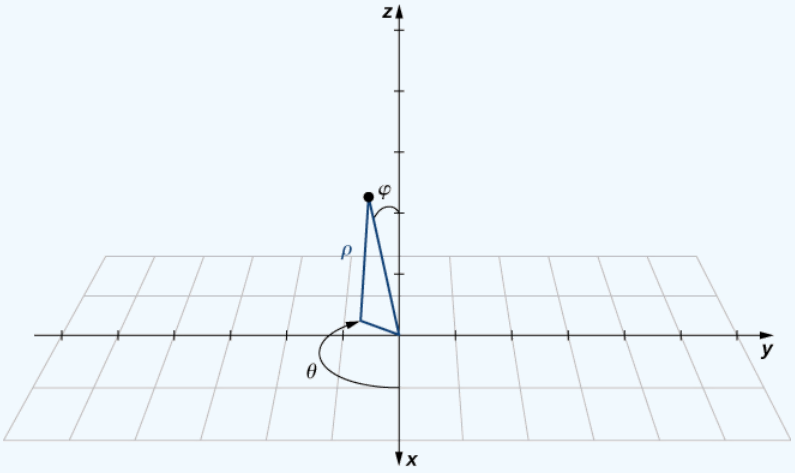
Plot the point with spherical coordinates $(2, -\frac{5\pi}{6}, \frac{\pi}{6})$ and describe its location in both rectangular and cylindrical coordinates.

Hint

Converting the coordinates first may help to find the location of the point in space more easily.

Answer

Cartesian: $(-\frac{\sqrt{3}}{2}, -\frac{1}{2}, \sqrt{3})$, cylindrical: $(1, -\frac{5\pi}{6}, \sqrt{3})$



Example 12.7.5: Converting from Rectangular Coordinates

Convert the rectangular coordinates $(-1, 1, \sqrt{6})$ to both spherical and cylindrical coordinates.

Solution

Start by converting from rectangular to spherical coordinates:

$$\rho^2 = x^2 + y^2 + z^2 = (-1)^2 + 1^2 + (\sqrt{6})^2 = 8$$

$$\tan\theta = \frac{1}{-1}$$

$$\rho = 2\sqrt{2} \text{ and } \theta = \arctan(-1) = \frac{3\pi}{4}.$$

Because $(x, y) = (-1, 1)$, then the correct choice for θ is $\frac{3\pi}{4}$.

There are actually two ways to identify φ. We can use the equation $\varphi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$. A more simple approach, however, is to use equation $z = \rho\cos\varphi$. We know that $z = \sqrt{6}$ and $\rho = 2\sqrt{2}$, so

$$\sqrt{6} = 2\sqrt{2}\cos\varphi, \text{ so } \cos\varphi = \frac{\sqrt{6}}{2\sqrt{2}} = \frac{\sqrt{3}}{2}$$

and therefore $\varphi = \frac{\pi}{6}$. The spherical coordinates of the point are $(2\sqrt{2}, \frac{3\pi}{4}, \frac{\pi}{6})$.

To find the cylindrical coordinates for the point, we need only find r:

$$r = \rho\sin\varphi = 2\sqrt{2}\sin(\frac{\pi}{6}) = \sqrt{2}.$$

The cylindrical coordinates for the point are $(\sqrt{2}, \frac{3\pi}{4}, \sqrt{6})$.

Example 12.7.6: Identifying Surfaces in the Spherical Coordinate System

Describe the surfaces with the given spherical equations.

a. $\theta = \frac{\pi}{3}$
b. $\varphi = \frac{5\pi}{6}$
c. $\rho = 6$
d. $\rho = \sin\theta \sin\varphi$

Solution

a. The variable θ represents the measure of the same angle in both the cylindrical and spherical coordinate systems. Points with coordinates $(\rho, \frac{\pi}{3}, \varphi)$ lie on the plane that forms angle $\theta = \frac{\pi}{3}$ with the positive x-axis. Because $\rho > 0$, the surface described by equation $\theta = \frac{\pi}{3}$ is the half-plane shown in Figure 12.7.13.

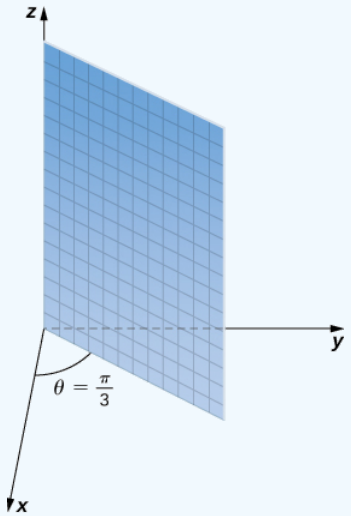


Figure 12.7.13: The surface described by equation $\theta = \frac{\pi}{3}$ is a half-plane.

b. Equation $\varphi = \frac{5\pi}{6}$ describes all points in the spherical coordinate system that lie on a line from the origin forming an angle measuring $\frac{5\pi}{6}$ rad with the positive z-axis. These points form a half-cone (Figure 12.7.14). Because there is only one value for φ that is measured from the positive z-axis, we do not get the full cone (with two pieces).

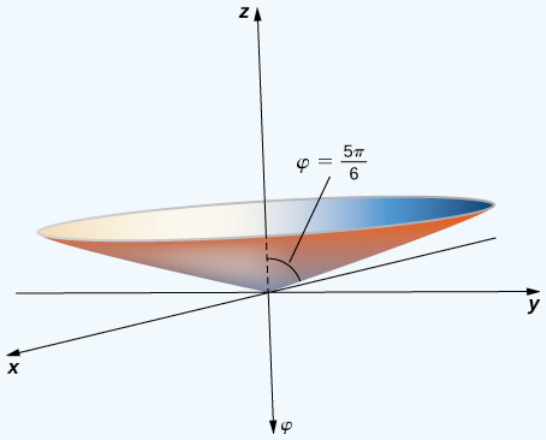


Figure 12.7.14: The equation $\varphi = \frac{5\pi}{6}$ describes a cone.

To find the equation in rectangular coordinates, use equation $\varphi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$.

$$\frac{5\pi}{6} = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$$\cos\frac{5\pi}{6} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$-\frac{\sqrt{3}}{2} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{3}{4} = \frac{z^2}{x^2+y^2+z^2}$$

$$\frac{3x^2}{4} + \frac{3y^2}{4} + \frac{3z^2}{4} = z^2$$

$$\frac{3x^2}{4} + \frac{3y^2}{4} - \frac{z^2}{4} = 0.$$

This is the equation of a cone centered on the z-axis.

c. Equation $\rho = 6$ describes the set of all points 6 units away from the origin—a sphere with radius 6 (Figure 12.7.15).

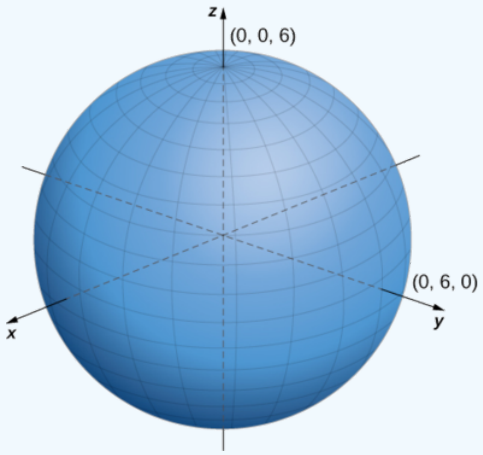


Figure 12.7.15: Equation $\rho = 6$ describes a sphere with radius 6.

d. To identify this surface, convert the equation from spherical to rectangular coordinates, using equations $y = \rho\sin\varphi\sin\theta$ and $\rho^2 = x^2+y^2+z^2$:

$\rho = \sin\theta\sin\varphi$

$\rho^2 = \rho\sin\theta\sin\varphi$ Multiply both sides of the equation by ρ.

$x^2+y^2+z^2 = y$ Substitute rectangular variables using the equations above.

$x^2+y^2-y+z^2 = 0$ Subtract y from both sides of the equation.

$x^2+y^2-y+\frac{1}{4}+z^2 = \frac{1}{4}$ Complete the square.

$x^2+\left(y-\frac{1}{2}\right)^2+z^2 = \frac{1}{4}$. Rewrite the middle terms as a perfect square.

The equation describes a sphere centered at point $(0, \frac{1}{2}, 0)$ with radius $\frac{1}{2}$.

Exercise 12.7.5

Describe the surfaces defined by the following equations.

a. $\rho = 13$
b. $\theta = \frac{2\pi}{3}$
c. $\varphi = \frac{\pi}{4}$

Hint

Think about what each component represents and what it means to hold that component constant.

Answer a

This is the set of all points 13 units from the origin. This set forms a sphere with radius 13.

Answer b

This set of points forms a half plane. The angle between the half plane and the positive x-axis is $\theta = \dfrac{2\pi}{3}$.

Answer c

Let P be a point on this surface. The position vector of this point forms an angle of $\varphi = \dfrac{\pi}{4}$ with the positive z-axis, which means that points closer to the origin are closer to the axis. These points form a half-cone.

Spherical coordinates are useful in analyzing systems that have some degree of symmetry about a point, such as the volume of the space inside a domed stadium or wind speeds in a planet's atmosphere. A sphere that has Cartesian equation $x^2 + y^2 + z^2 = c^2$ has the simple equation $\rho = c$ in spherical coordinates.

In geography, latitude and longitude are used to describe locations on Earth's surface, as shown in Figure $12.7.16$. Although the shape of Earth is not a perfect sphere, we use spherical coordinates to communicate the locations of points on Earth. Let's assume Earth has the shape of a sphere with radius 4000 mi. We express angle measures in degrees rather than radians because latitude and longitude are measured in degrees.

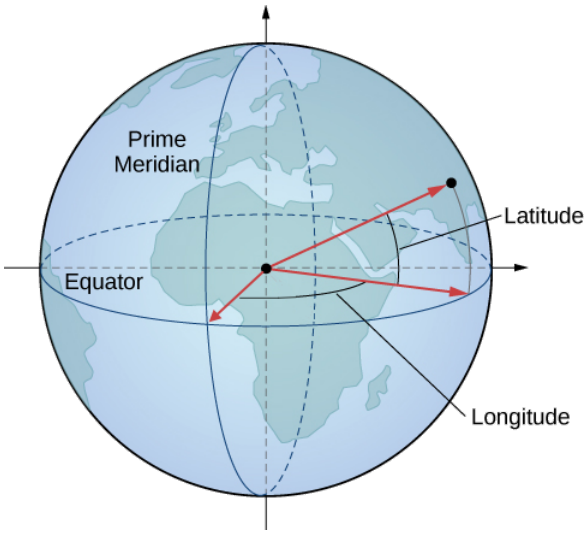


Figure $12.7.16$: In the latitude–longitude system, angles describe the location of a point on Earth relative to the equator and the prime meridian.

Let the center of Earth be the center of the sphere, with the ray from the center through the North Pole representing the positive z-axis. The prime meridian represents the trace of the surface as it intersects the xz-plane. The equator is the trace of the sphere intersecting the xy-plane.

Example $12.7.7$: Converting Latitude and Longitude to Spherical Coordinates

The latitude of Columbus, Ohio, is $40°$ N and the longitude is $83°$ W, which means that Columbus is $40°$ north of the equator. Imagine a ray from the center of Earth through Columbus and a ray from the center of Earth through the equator directly south of Columbus. The measure of the angle formed by the rays is $40°$. In the same way, measuring from the prime meridian, Columbus lies $83°$ to the west. Express the location of Columbus in spherical coordinates.

Solution

The radius of Earth is 4000mi, so $\rho = 4000$. The intersection of the prime meridian and the equator lies on the positive x-axis. Movement to the west is then described with negative angle measures, which shows that $\theta = -83°$, Because Columbus lies $40°$ north of the equator, it lies $50°$ south of the North Pole, so $\varphi = 50°$. In spherical coordinates, Columbus lies at point $(4000, -83°, 50°)$.

Exercise 12.7.6

Sydney, Australia is at $34°S$ and $151°E$. Express Sydney's location in spherical coordinates.

Hint

Because Sydney lies south of the equator, we need to add $90°$ to find the angle measured from the positive z-axis.

Answer

$(4000, 151°, 124°)$

Cylindrical and spherical coordinates give us the flexibility to select a coordinate system appropriate to the problem at hand. A thoughtful choice of coordinate system can make a problem much easier to solve, whereas a poor choice can lead to unnecessarily complex calculations. In the following example, we examine several different problems and discuss how to select the best coordinate system for each one.

Example 12.7.8: Choosing the Best Coordinate System

In each of the following situations, we determine which coordinate system is most appropriate and describe how we would orient the coordinate axes. There could be more than one right answer for how the axes should be oriented, but we select an orientation that makes sense in the context of the problem. Note: There is not enough information to set up or solve these problems; we simply select the coordinate system (Figure 12.7.17).

a. Find the center of gravity of a bowling ball.
b. Determine the velocity of a submarine subjected to an ocean current.
c. Calculate the pressure in a conical water tank.
d. Find the volume of oil flowing through a pipeline.
e. Determine the amount of leather required to make a football.

(a) (b) (c)

(d) (e)

Figure 12.7.17: (credit: (a) modification of work by scl hua, Wikimedia, (b) modification of work by DVIDSHUB, Flickr, (c) modification of work by Michael Malak, Wikimedia, (d) modification of work by Sean Mack, Wikimedia, (e) modification of work by Elvert Barnes, Flickr)

Solution

a. Clearly, a bowling ball is a sphere, so spherical coordinates would probably work best here. The origin should be located at the physical center of the ball. There is no obvious choice for how the x-, y- and z-axes should be oriented. Bowling balls normally have a weight block in the center. One possible choice is to align the z-axis with the axis of symmetry of the weight block.

b. A submarine generally moves in a straight line. There is no rotational or spherical symmetry that applies in this situation, so rectangular coordinates are a good choice. The z-axis should probably point upward. The x- and y-axes could be aligned to point east and north, respectively. The origin should be some convenient physical location, such as the starting position of the submarine or the location of a particular port.

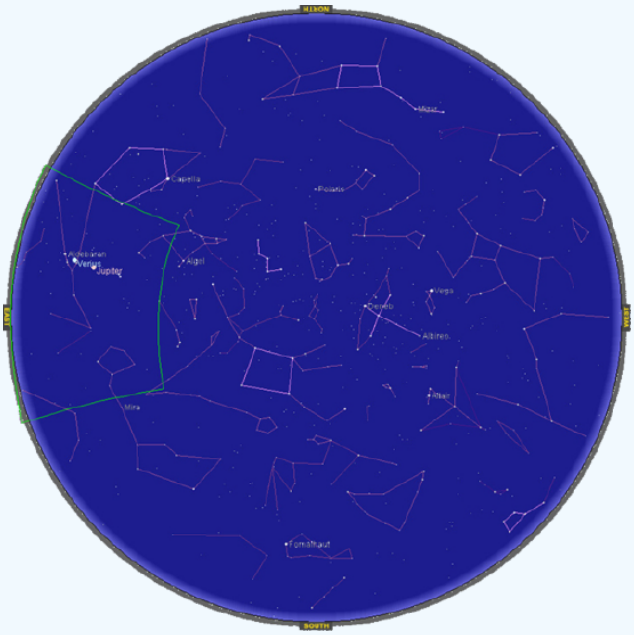
c. A cone has several kinds of symmetry. In cylindrical coordinates, a cone can be represented by equation $z = kr$, where k is a constant. In spherical coordinates, we have seen that surfaces of the form $\varphi = c$ are half-cones. Last, in rectangular coordinates, elliptic cones are quadric surfaces and can be represented by equations of the form $z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$. In this case, we could choose any of the three. However, the equation for the surface is more complicated in rectangular coordinates than in the other two systems, so we might want to avoid that choice. In addition, we are talking about a water tank, and the depth of the water might come into play at some point in our calculations, so it might be nice to have a component that represents height and depth directly. Based on this reasoning, cylindrical coordinates might be the best choice. Choose the z-axis to align with the axis of the cone. The orientation of the other two axes is arbitrary. The origin should be the bottom point of the cone.

d. A pipeline is a cylinder, so cylindrical coordinates would be best the best choice. In this case, however, we would likely choose to orient our z-axis with the center axis of the pipeline. The x-axis could be chosen to point straight downward or to some other logical direction. The origin should be chosen based on the problem statement. Note that this puts the z-axis in a horizontal orientation, which is a little different from what we usually do. It may make sense to choose an unusual orientation for the axes if it makes sense for the problem.

e. A football has rotational symmetry about a central axis, so cylindrical coordinates would work best. The z-axis should align with the axis of the ball. The origin could be the center of the ball or perhaps one of the ends. The position of the x-axis is arbitrary.

Exercise 12.7.7

Which coordinate system is most appropriate for creating a star map, as viewed from Earth (see the following figure)?



How should we orient the coordinate axes?

Hint

What kinds of symmetry are present in this situation?

Answer

Spherical coordinates with the origin located at the center of the earth, the z-axis aligned with the North Pole, and the x-axis aligned with the prime meridian

Key Concepts

- In the cylindrical coordinate system, a point in space is represented by the ordered triple (r, θ, z), where (r, θ) represents the polar coordinates of the point's projection in the xy-plane and z represents the point's projection onto the z-axis.
- To convert a point from cylindrical coordinates to Cartesian coordinates, use equations $x = r\cos\theta, y = r\sin\theta$, and $z = z$.
- To convert a point from Cartesian coordinates to cylindrical coordinates, use equations $r^2 = x^2 + y^2, \tan\theta = \dfrac{y}{x}$, and $z = z$.
- In the spherical coordinate system, a point P in space is represented by the ordered triple (ρ, θ, φ), where ρ is the distance between P and the origin $(\rho \neq 0), \theta$ is the same angle used to describe the location in cylindrical coordinates, and φ is the angle formed by the positive z-axis and line segment $\overline{OP}$, where O is the origin and $0 \leq \varphi \leq \pi$.
- To convert a point from spherical coordinates to Cartesian coordinates, use equations $x = \rho\sin\varphi\cos\theta, y = \rho\sin\varphi\sin\theta$, and $z = \rho\cos\varphi$.
- To convert a point from Cartesian coordinates to spherical coordinates, use equations $\rho^2 = x^2 + y^2 + z^2, \tan\theta = \dfrac{y}{x}$, and $\varphi = \arccos\left(\dfrac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$.
- To convert a point from spherical coordinates to cylindrical coordinates, use equations $r = \rho\sin\varphi, \theta = \theta$, and $z = \rho\cos\varphi$.
- To convert a point from cylindrical coordinates to spherical coordinates, use equations $\rho = \sqrt{r^2 + z^2}, \theta = \theta$, and $\varphi = \arccos\left(\dfrac{z}{\sqrt{r^2 + z^2}}\right)$.

Glossary

cylindrical coordinate system

a way to describe a location in space with an ordered triple (r, θ, z), where (r, θ) represents the polar coordinates of the point's projection in the xy-plane, and z represents the point's projection onto the z-axis

spherical coordinate system

a way to describe a location in space with an ordered triple (ρ, θ, φ), where ρ is the distance between P and the origin $(\rho \neq 0)$, θ is the same angle used to describe the location in cylindrical coordinates, and φ is the angle formed by the positive z-axis and line segment $\bar{OP}$, where O is the origin and $0 \leq \varphi \leq \pi$

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- Paul Seeburger edited the LaTeX on the page

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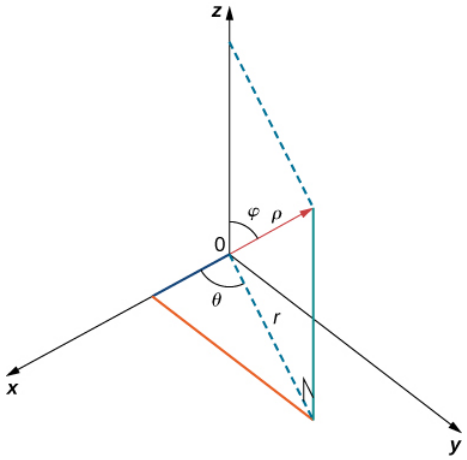
12.7E: Exercises for Section 12.7

Use the following figure as an aid in identifying the relationship between the rectangular, cylindrical, and spherical coordinate systems.



For exercises 1 - 4, the cylindrical coordinates (r, θ, z) of a point are given. Find the rectangular coordinates (x, y, z) of the point.

1) $\left(4, \frac{\pi}{6}, 3\right)$

Answer

$(2\sqrt{3}, 2, 3)$

2) $\left(3, \frac{\pi}{3}, 5\right)$

3) $\left(4, \frac{7\pi}{6}, 3\right)$

Answer

$(-2\sqrt{3}, -2, 3)$

4) $(2, \pi, -4)$

For exercises 5 - 8, the rectangular coordinates (x, y, z) of a point are given. Find the cylindrical coordinates (r, θ, z) of the point.

5) $(1, \sqrt{3}, 2)$

Answer

$\left(2, \frac{\pi}{3}, 2\right)$

6) $(1, 1, 5)$

7) $(3, -3, 7)$

Answer

$\left(3\sqrt{2}, -\frac{\pi}{4}, 7\right)$

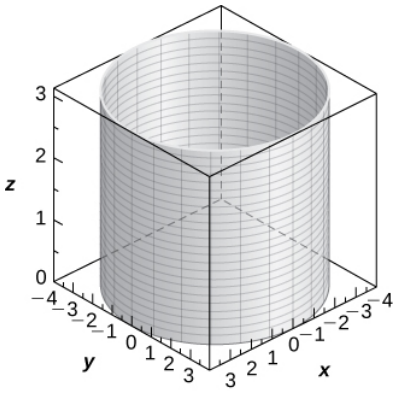
8) $(-2\sqrt{2}, 2\sqrt{2}, 4)$

For exercises 9 - 16, the equation of a surface in cylindrical coordinates is given. Find the equation of the surface in rectangular coordinates. Identify and graph the surface.

9) [T] $r = 4$

Answer

A cylinder of equation $x^2 + y^2 = 16$, with its center at the origin and rulings parallel to the z-axis,

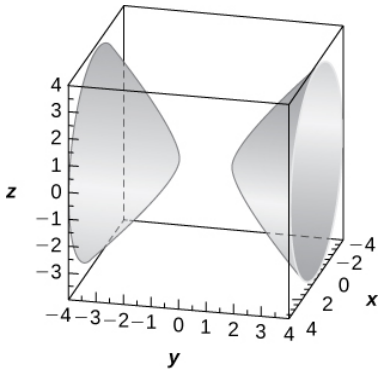


10) [T] $z = r^2 \cos^2 \theta$

11) [T] $r^2 \cos(2\theta) + z^2 + 1 = 0$

Answer

Hyperboloid of two sheets of equation $-x^2 + y^2 - z^2 = 1$, with the y-axis as the axis of symmetry,

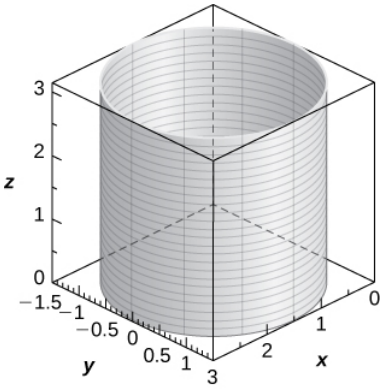


12) [T] $r = 3 \sin \theta$

13) [T] $r = 2 \cos \theta$

Answer

Cylinder of equation $x^2 - 2x + y^2 = 0$, with a center at $(1, 0, 0)$ and radius 1, with rulings parallel to the z-axis,

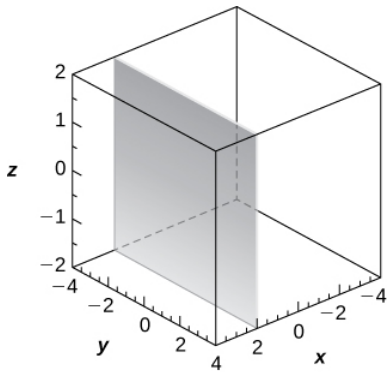


14) [T] $r^2 + z^2 = 5$

15) [T] $r = 2 \sec \theta$

Answer

Plane of equation $x = 2$,



16) [T] $r = 3\csc\theta$

For exercises 17 - 22, the equation of a surface in rectangular coordinates is given. Find the equation of the surface in cylindrical coordinates.

17) $z = 3$

Answer

$z = 3$

18) $x = 6$

19) $x^2 + y^2 + z^2 = 9$

Answer

$r^2 + z^2 = 9$

20) $y = 2x^2$

21) $x^2 + y^2 - 16x = 0$

Answer

$r = 16\cos\theta, \quad r = 0$

22) $x^2 + y^2 - 3\sqrt{x^2 + y^2} + 2 = 0$

For exercises 23 - 26, the spherical coordinates (ρ, θ, φ) of a point are given. Find the rectangular coordinates (x, y, z) of the point.

23) $(3, 0, \pi)$

Answer

$(0, 0, -3)$

24) $\left(1, \frac{\pi}{6}, \frac{\pi}{6}\right)$

25) $\left(12, -\frac{\pi}{4}, \frac{\pi}{4}\right)$

Answer

$(6, -6, \sqrt{2})$

26) $\left(3, \frac{\pi}{4}, \frac{\pi}{6}\right)$

For exercises 27 - 30, the rectangular coordinates (x, y, z) of a point are given. Find the spherical coordinates (ρ, θ, φ) of the point. Express the measure of the angles in degrees rounded to the nearest integer.

27) $(4, 0, 0)$

Answer

$(4, 0, 90°)$

28) $(-1, 2, 1)$

29) $(0, 3, 0)$

Answer

$(3, 90°, 90°)$

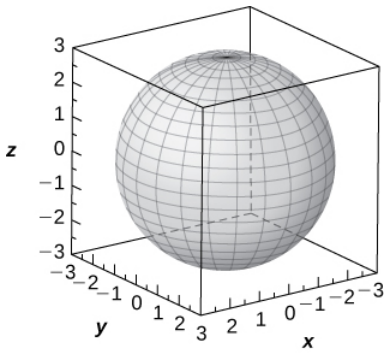
30) $(-2, 2\sqrt{3}, 4)$

For exercises 31 - 36, the equation of a surface in spherical coordinates is given. Find the equation of the surface in rectangular coordinates. Identify and graph the surface.

31) [T] $\rho = 3$

Answer

Sphere of equation $x^2 + y^2 + z^2 = 9$ centered at the origin with radius 3,

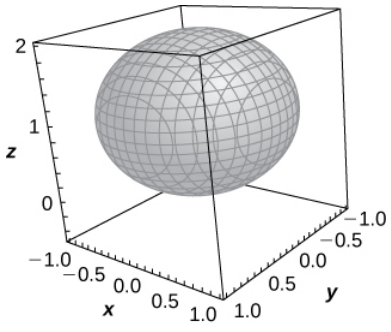


32) [T] $\varphi = \frac{\pi}{3}$

33) [T] $\rho = 2\cos\varphi$

Answer

Sphere of equation $x^2 + y^2 + (z-1)^2 = 1$ centered at $(0, 0, 1)$ with radius 1,

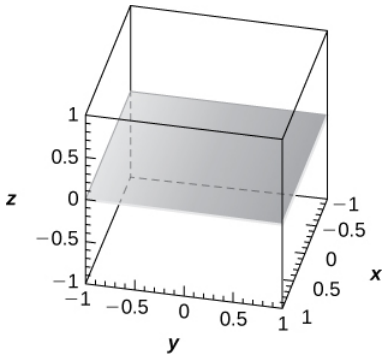


34) [T] $\rho = 4\csc\varphi$

35) [T] $\varphi = \frac{\pi}{2}$

Answer

The xy-plane of equation $z = 0$,



36) [T] $\rho = 6 \csc \varphi \sec \theta$

For exercises 37 - 40, the equation of a surface in rectangular coordinates is given. Find the equation of the surface in spherical coordinates. Identify the surface.

37) $x^2 + y^2 - 3z^2 = 0, \quad z \neq 0$

Answer

$\varphi = \frac{\pi}{3}$ or $\varphi = \frac{2\pi}{3}$; Elliptic cone

38) $x^2 + y^2 + z^2 - 4z = 0$

39) $z = 6$

Answer

$\rho \cos \varphi = 6$; Plane at $z = 6$

40) $x^2 + y^2 = 9$

For exercises 41 - 44, the cylindrical coordinates of a point are given. Find its associated spherical coordinates, with the measure of the angle φ in radians rounded to four decimal places.

41) [T] $(1, \frac{\pi}{4}, 3)$

Answer

$\left(\sqrt{10}, \frac{\pi}{4}, 0.3218\right)$

42) [T] $(5, \pi, 12)$

43) $(3, \frac{\pi}{2}, 3)$

Answer

$(3\sqrt{2}, \frac{\pi}{2}, \frac{\pi}{4})$

44) $(3, -\frac{\pi}{6}, 3)$

For exercises 45 - 48, the spherical coordinates of a point are given. Find its associated cylindrical coordinates.

45) $(2, -\frac{\pi}{4}, \frac{\pi}{2})$

Answer

$(2, -\frac{\pi}{4}, 0)$

46) $(4, \frac{\pi}{4}, \frac{\pi}{6})$

47) $(8, \frac{\pi}{3}, \frac{\pi}{2})$

Answer

$(8, \frac{\pi}{3}, 0)$

48) $\left(9, -\frac{\pi}{6}, \frac{\pi}{3}\right)$

For exercises 49 - 52, find the most suitable system of coordinates to describe the solids.

49) The solid situated in the first octant with a vertex at the origin and enclosed by a cube of edge length a, where $a > 0$

Answer

Cartesian system, $\{(x, y, z) \mid 0 \le x \le a,\ 0 \le y \le a,\ 0 \le z \le a\}$

50) A spherical shell determined by the region between two concentric spheres centered at the origin, of radii of a and b, respectively, where $b > a > 0$

51) A solid inside sphere $x^2 + y^2 + z^2 = 9$ and outside cylinder $\left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{4}$

Answer

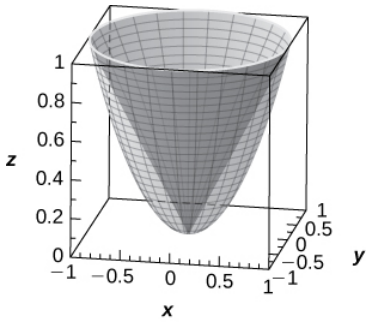
Cylindrical system, $\{(r, \theta, z) \mid r^2 + z^2 \le 9,\ r \ge 3\cos\theta,\ 0 \le \theta \le 2\pi\}$

52) A cylindrical shell of height 10 determined by the region between two cylinders with the same center, parallel rulings, and radii of 2 and 5, respectively

53) [T] Use a CAS or CalcPlot3D to graph in cylindrical coordinates the region between elliptic paraboloid $z = x^2 + y^2$ and cone $x^2 + y^2 - z^2 = 0$.

Answer

The region is described by the set of points $\{(r, \theta, z) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ r^2 \le z \le r\}$.



54) [T] Use a CAS or CalcPlot3D to graph in spherical coordinates the "ice cream-cone region" situated above the xy-plane between sphere $x^2 + y^2 + z^2 = 4$ and elliptical cone $x^2 + y^2 - z^2 = 0$.

55) Washington, DC, is located at $39°$ N and $77°$ W (see the following figure). Assume the radius of Earth is 4000 mi. Express the location of Washington, DC, in spherical coordinates.

Answer

$(4000, -77°, 51°)$

56) San Francisco is located at $37.78°N$ and $122.42°W$. Assume the radius of Earth is 4000mi. Express the location of San Francisco in spherical coordinates.

57) Find the latitude and longitude of Rio de Janeiro if its spherical coordinates are $(4000, -43.17°, 102.91°)$.

Answer

$43.17°W, 22.91°S$

58) Find the latitude and longitude of Berlin if its spherical coordinates are $(4000, 13.38°, 37.48°)$.

59) [T] Consider the torus of equation $\left(x^2+y^2+z^2+R^2-r^2\right)^2 = 4R^2(x^2+y^2)$, where $R \geq r > 0$.

a. Write the equation of the torus in spherical coordinates.

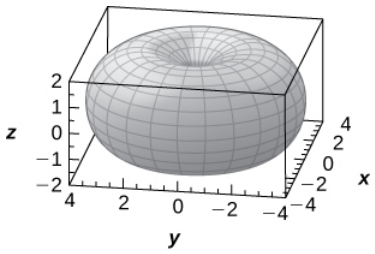
b. If $R = r$, the surface is called a horn torus. Show that the equation of a horn torus in spherical coordinates is $\rho = 2R\sin\varphi$.

c. Use a CAS or CalcPlot3D to graph the horn torus with $R = r = 2$ in spherical coordinates.

Answer

a. $\rho = 0, \quad \rho + R^2 - r^2 - 2R\sin\varphi = 0$

c.



60) [T] The "bumpy sphere" with an equation in spherical coordinates is $\rho = a + b\cos(m\theta)\sin(n\varphi)$, with $\theta \in [0, 2\pi]$ and $\varphi \in [0, \pi]$, where a and b are positive numbers and m and n are positive integers, may be used in applied mathematics to model tumor growth.

a. Show that the "bumpy sphere" is contained inside a sphere of equation $\rho = a + b$. Find the values of θ and φ at which the two surfaces intersect.

b. Use a CAS or CalcPlot3D to graph the surface for $a = 14, b = 2, m = 4$, and $n = 6$ along with sphere $\rho = a + b$.

c. Find the equation of the intersection curve of the surface at b. with the cone $\varphi = \frac{\pi}{12}$. Graph the intersection curve in the plane of intersection.

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Links to CalcPlot3D added by Paul Seeburger (Monroe Community College).

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12.R: Chapter 12 Review Exercises

For exercises 1 - 4, determine whether the statement is *true* or *false*. Justify the answer with a proof or a counterexample.

1) For vectors $\vec{a}$ and $\vec{b}$ and any given scalar c, $c(\vec{a} \cdot \vec{b}) = (c\vec{a}) \cdot \vec{b}$.

Answer

True; See proof in Section 11.3

2) For vectors $\vec{a}$ and $\vec{b}$ and any given scalar c, $c(\vec{a} \times \vec{b}) = (c\vec{a}) \times \vec{b}$.

3) The symmetric equation for the line of intersection between two planes $x + y + z = 2$ and $x + 2y - 4z = 5$ is given by $-\frac{x-1}{6} = \frac{y-1}{5} = z$.

Answer

False, Converting the symmetric equations above to the parametric equations of the line, we get:
$x = 1 - 6t$
$y = 1 + 5t$
$z = t$

If this line lies on each plane, we should get an identity (like 5 = 5) when we substitute each expression of t in the each plane's equation.

Substituting into the first plane equation, we get: $(1 - 6t) + (1 + 5t) + t = 2$ ✓

Thus we know that this line does lie on the first plane.

But when we substitute into the second plane equation, we obtain: $(1 - 6t) + 2(1 + 5t) - 4(t) = 1 - 6t + 2 + 10t - 4t = 3 \neq 5$

Since we do not get an identity, we know this line is **not** on the second plane and can therefore not be the line of intersection of the two planes.

4) If $\vec{a} \cdot \vec{b} = 0$, then $\vec{a}$ is perpendicular to $\vec{b}$.

Answer

False, since $\vec{a}$ or $\vec{b}$ could also be the zero vector.

For exercises 5 and 6, use the given vectors to find the quantities.

5) $\vec{a} = 9\hat{i} - 2\hat{j}, \quad \vec{b} = -3\hat{i} + \hat{k}$

a. $3\vec{a} + \vec{b}$

b. $\|\vec{a}\|$

c. $\vec{a} \times \|\vec{b} \times \vec{a}\|$

d. $\|\vec{b} \times \vec{a}\|$

Answer

a. $\langle 24, -6, 1 \rangle$
b. $\sqrt{85}$
c. Can't cross a vector with a scalar
d. 11

6) $\vec{a} = 2\hat{i} + \hat{j} - 9\hat{k}, \quad \vec{b} = -\hat{i} + 2\hat{k}, \quad \vec{c} = 4\hat{i} - 2\hat{j} + \hat{k}$

a. $2\vec{a} - \vec{b}$

b. $\|\vec{\mathbf{b}} \times \vec{\mathbf{c}}\|$

c. $\vec{\mathbf{b}} \times \left(\vec{\mathbf{b}} \times \vec{\mathbf{c}}\right)$

d. $\vec{\mathbf{c}} \times \|\vec{\mathbf{b}} \times \vec{\mathbf{a}}\|$

e. $\text{Proj}_{\vec{\mathbf{a}}} \vec{\mathbf{b}}$

7) Find the values of a such that vectors $\langle 2, 4, a\rangle$ and $\langle 0, -1, a\rangle$ are orthogonal.

Answer

$a = \pm 2$

For exercises 8 and 9, find the unit vectors.

8) Find the unit vector that has the same direction as vector $\vec{\mathbf{v}}$ that begins at $(0, -3)$ and ends at $(4, 10)$.

9) Find the unit vector that has the same direction as vector $\vec{\mathbf{v}}$ that begins at $(1, 4, 10)$ and ends at $(3, 0, 4)$.

Answer

$\langle \frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}}\rangle = \langle \frac{\sqrt{14}}{14}, -\frac{\sqrt{14}}{7}, -\frac{3\sqrt{14}}{14}\rangle$

For exercises 10 and 11, find the area or volume of the given shapes.

10) The parallelogram spanned by vectors $\vec{\mathbf{a}} = \langle 1, 13\rangle$ and $\vec{\mathbf{b}} = \langle 3, 21\rangle$

11) The parallelepiped formed by $\vec{\mathbf{a}} = \langle 1, 4, 1\rangle$ and $\vec{\mathbf{b}} = \langle 3, 6, 2\rangle$, and $\vec{\mathbf{c}} = \langle -2, 1, -5\rangle$

Answer

27 units^2

For exercises 12 and 13, find the parametric equations and the vector equation of the line with the given properties.

12) The line that passes through point $(2, -3, 7)$ that is parallel to vector $\langle 1, 3, -2\rangle$

13) The line that passes through points $(1, 3, 5)$ and $(-2, 6, -3)$

Answer

$x = 1 - 3t, y = 3 + 3t, z = 5 - 8t, \quad \vec{\mathbf{r}}(t) = (1 - 3t)\hat{\mathbf{i}} + 3(1 + t)\hat{\mathbf{j}} + (5 - 8t)\hat{\mathbf{k}}$

For exercises 14 and 15, find the equation of the plane with the given properties.

14) The plane that passes through point $(4, 7, -1)$ and has normal vector $\vec{\mathbf{n}} = \langle 3, 4, 2\rangle$

15) The plane that passes through points $(0, 1, 5), (2, -1, 6)$, and $(3, 2, 5)$.

Answer

$-x + 3y + 8z = 43$

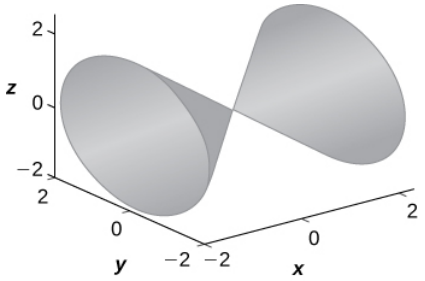
For exercises 16 and 17, find the traces for the surfaces in planes $x = k$, $y = k$, and $z = k$. Then, describe and draw the surfaces.

16) $9x^2 + 4y^2 - 16y + 36z^2 = 20$

17) $x^2 = y^2 + z^2$

Answer

$x = k$ trace: $k^2 = y^2 + z^2$ is a circle, $y = k$ trace: $x^2 - z^2 = k^2$ is a hyperbola (or a pair of lines if $k = 0$), $z = k$ trace: $x^2 - y^2 = k^2$ is a hyperbola (or a pair of lines if $k = 0$). The surface is a cone.



For exercises 18 and 19, write the given equation in cylindrical coordinates and spherical coordinates.

18) $x^2 + y^2 + z^2 = 144$

19) $z = x^2 + y^2 - 1$

Answer

Cylindrical: $z = r^2 - 1$, spherical: $\cos\varphi = \rho \sin^2 \varphi - \frac{1}{\rho}$

For exercises 20 and 21, convert the given equations from cylindrical or spherical coordinates to rectangular coordinates. Identify the given surface.

20) $\rho^2(\sin^2(\varphi) - \cos^2(\varphi)) = 1$

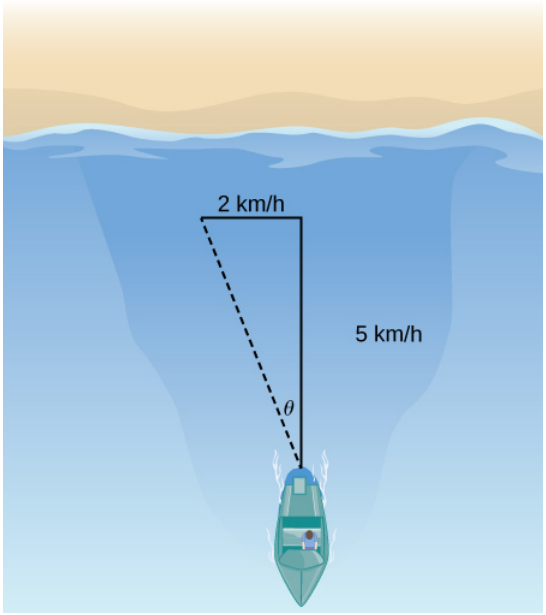
21) $r^2 - 2r\cos(\theta) + z^2 = 1$

Answer

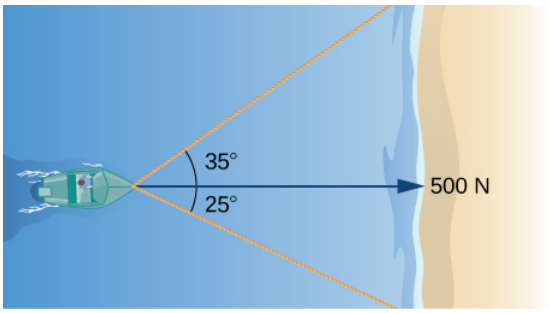
$x^2 - 2x + y^2 + z^2 = 1$, sphere

For exercises 22 and 23, consider a small boat crossing a river.

22) If the boat velocity is 5 km/h due north in still water and the water has a current of 2 km/h due west (see the following figure), what is the velocity of the boat relative to shore? What is the angle θ that the boat is actually traveling?



23) When the boat reaches the shore, two ropes are thrown to people to help pull the boat ashore. One rope is at an angle of $25°$ and the other is at $35°$. If the boat must be pulled straight and at a force of 500 N, find the magnitude of force for each rope (see the following figure).



Answer

331 N and 244 N

24) An airplane is flying in the direction of 52° east of north with a speed of 450 mph. A strong wind has a bearing 33° east of north with a speed of 50 mph. What is the resultant ground speed and bearing of the airplane?

25) Calculate the work done by moving a particle from position $(1, 2, 0)$ to $(8, 4, 5)$ along a straight line with a force $\vec{\mathbf{F}} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$.

Answer

15 J

In problems 26 and 27, consider your unsuccessful attempt to take the tire off your car using a wrench to loosen the bolts. Assume the wrench is 0.3 m long and you are able to apply a 200-N force.

26) Because your tire is flat, you are only able to apply your force at a $60°$ angle. What is the torque at the center of the bolt? Assume this force is not enough to loosen the bolt.

27) Someone lends you a tire jack and you are now able to apply a 200-N force at an $80°$ angle. Is your resulting torque going to be more or less? What is the new resulting torque at the center of the bolt? Assume this force is not enough to loosen the bolt.

Answer

More, 59.09 J

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CHAPTER OVERVIEW

13: Vector-Valued Functions

A vector-valued function, also referred to as a vector function, is a mathematical function of one or more variables whose range is a set of multidimensional vectors or infinite-dimensional vectors. The input of a vector-valued function could be a scalar or a vector. Vector-valued functions provide a useful method for studying various curves both in the plane and in three-dimensional space. We can apply this concept to calculate the velocity, acceleration, arc length, and curvature of an object's trajectory. In this chapter, we examine these methods and show how they are used.

13.0: Prelude to Vector-Valued Functions
13.1: Vector-Valued Functions and Space Curves
13.1E: Exercises for Section 13.1
13.2: Calculus of Vector-Valued Functions
13.2E: Exercises for Section 13.2
13.3: Arc Length and Curvature
13.3E: Exercises for Section 13.3
13.4: Motion in Space
13.4E: Exercises for Section 13.4
13.R: Chapter 13 Review Exercises

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13.0: Prelude to Vector-Valued Functions

In 1705, using Sir Isaac Newton's new laws of motion, the astronomer Edmond Halley made a prediction. He stated that comets that had appeared in 1531, 1607, and 1682 were actually the same comet and that it would reappear in 1758. Halley was proved to be correct, although he did not live to see it. However, the comet was later named in his honor. Halley's Comet follows an elliptical path through the solar system, with the Sun appearing at one focus of the ellipse. This motion is predicted by Johannes Kepler's first law of planetary motion, which we mentioned briefly previously. Kepler's third law of planetary motion can be used with the calculus of vector-valued functions to find the average distance of Halley's Comet from the Sun.

Figure 13.0.1: Halley's Comet appeared in view of Earth in 1986 and will appear again in 2061.

Vector-valued functions provide a useful method for studying various curves both in the plane and in three-dimensional space. We can apply this concept to calculate the velocity, acceleration, arc length, and curvature of an object's trajectory. In this chapter, we examine these methods and show how they are used.

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13.1: Vector-Valued Functions and Space Curves

Learning Objectives

- Write the general equation of a vector-valued function in component form and unit-vector form.
- Recognize parametric equations for a space curve.
- Describe the shape of a helix and write its equation.
- Define the limit of a vector-valued function.

Our study of vector-valued functions combines ideas from our earlier examination of single-variable calculus with our description of vectors in three dimensions from the preceding chapter. In this section, we extend concepts from earlier chapters and also examine new ideas concerning curves in three-dimensional space. These definitions and theorems support the presentation of material in the rest of this chapter and also in the remaining chapters of the text.

Definition of a Vector-Valued Function

Our first step in studying the calculus of vector-valued functions is to define what exactly a vector-valued function is. We can then look at graphs of vector-valued functions and see how they define curves in both two and three dimensions.

Definition: Vector-valued Functions

A vector-valued function is a function of the form

$$\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} \quad \text{or} \quad \vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}},$$

where the component functions f, g, and h, are real-valued functions of the parameter t. Vector-valued functions are also written in the form

$$\vec{\mathbf{r}}(t) = \langle f(t),\, g(t)\rangle \quad \text{or} \quad \vec{\mathbf{r}}(t) = \langle f(t),\, g(t),\, h(t)\rangle.$$

In both cases, the first form of the function defines a two-dimensional vector-valued function; the second form describes a three-dimensional vector-valued function.

The parameter t can lie between two real numbers: $a \le t \le b$. Another possibility is that the value of t might take on all real numbers. Last, the component functions themselves may have domain restrictions that enforce restrictions on the value of t. We often use t as a parameter because t can represent time.

Example 13.1.1: Evaluating Vector-Valued Functions and Determining Domains

For each of the following vector-valued functions, evaluate $\vec{\mathbf{r}}(0)$, $\vec{\mathbf{r}}(\frac{\pi}{2})$, and $\vec{\mathbf{r}}(\frac{2\pi}{3})$. Do any of these functions have domain restrictions?

a. $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 3\sin t\,\hat{\mathbf{j}}$
b. $\vec{\mathbf{r}}(t) = 3\tan t\,\hat{\mathbf{i}} + 4\sec t\,\hat{\mathbf{j}} + 5t\,\hat{\mathbf{k}}$

Solution

a. To calculate each of the function values, substitute the appropriate value of t into the function:

$$\begin{aligned}
\vec{\mathbf{r}}(0) &= 4\cos(0)\hat{\mathbf{i}} + 3\sin(0)\hat{\mathbf{j}} \\
&= 4\hat{\mathbf{i}} + 0\hat{\mathbf{j}} = 4\hat{\mathbf{i}} \\
\vec{\mathbf{r}}\left(\frac{\pi}{2}\right) &= 4\cos\left(\frac{\pi}{2}\right)\hat{\mathbf{i}} + 3\sin\left(\frac{\pi}{2}\right)\hat{\mathbf{j}} \\
&= 0\hat{\mathbf{i}} + 3\hat{\mathbf{j}} = 3\hat{\mathbf{j}} \\
\vec{\mathbf{r}}\left(\frac{2\pi}{3}\right) &= 4\cos\left(\frac{2\pi}{3}\right)\hat{\mathbf{i}} + 3\sin\left(\frac{2\pi}{3}\right)\hat{\mathbf{j}} \\
&= 4\left(-\tfrac{1}{2}\right)\hat{\mathbf{i}} + 3\left(\tfrac{\sqrt{3}}{2}\right)\hat{\mathbf{j}} = -2\hat{\mathbf{i}} + \tfrac{3\sqrt{3}}{2}\hat{\mathbf{j}}
\end{aligned}$$

To determine whether this function has any domain restrictions, consider the component functions separately. The first component function is $f(t) = 4\cos t$ and the second component function is $g(t) = 3\sin t$. Neither of these functions has a domain restriction, so the domain of $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 3\sin t\,\hat{\mathbf{j}}$ is all real numbers.

b. To calculate each of the function values, substitute the appropriate value of t into the function:

$$\begin{aligned}
\vec{\mathbf{r}}(0) &= 3\tan(0)\hat{\mathbf{i}} + 4\sec(0)\hat{\mathbf{j}} + 5(0)\hat{\mathbf{k}} \\
&= 0\hat{\mathbf{i}} + 4j + 0\hat{\mathbf{k}} = 4\hat{\mathbf{j}} \\
\vec{\mathbf{r}}\left(\frac{\pi}{2}\right) &= 3\tan\left(\frac{\pi}{2}\right)\hat{\mathbf{i}} + 4\sec\left(\frac{\pi}{2}\right)\hat{\mathbf{j}} + 5\left(\frac{\pi}{2}\right)\hat{\mathbf{k}}, \text{ which does not exist} \\
\vec{\mathbf{r}}\left(\frac{2\pi}{3}\right) &= 3\tan\left(\frac{2\pi}{3}\right)\hat{\mathbf{i}} + 4\sec\left(\frac{2\pi}{3}\right)\hat{\mathbf{j}} + 5\left(\frac{2\pi}{3}\right)\hat{\mathbf{k}} \\
&= 3(-\sqrt{3})\hat{\mathbf{i}} + 4(-2)\hat{\mathbf{j}} + \frac{10\pi}{3}\hat{\mathbf{k}} \\
&= (-3\sqrt{3})\hat{\mathbf{i}} - 8\hat{\mathbf{j}} + \frac{10\pi}{3}\hat{\mathbf{k}}
\end{aligned}$$

To determine whether this function has any domain restrictions, consider the component functions separately. The first component function is $f(t) = 3\tan t$, the second component function is $g(t) = 4\sec t$, and the third component function is $h(t) = 5t$. The first two functions are not defined for odd multiples of $\frac{\pi}{2}$, so the function is not defined for odd multiples of $\frac{\pi}{2}$. Therefore,

$$\mathrm{D}_{\vec{\mathbf{r}}} = \left\{t \,|\, t \neq \frac{(2n+1)\pi}{2}\right\},$$

where n is any integer.

Exercise 13.1.1

For the vector-valued function $\vec{\mathbf{r}}(t) = (t^2 - 3t)\,\hat{\mathbf{i}} + (4t + 1)\,\hat{\mathbf{j}}$, evaluate $\vec{\mathbf{r}}(0)$, $\vec{\mathbf{r}}(1)$, and $\vec{\mathbf{r}}(-4)$. Does this function have any domain restrictions?

Hint

Substitute the appropriate values of t into the function.

Answer

$\vec{\mathbf{r}}(0) = \hat{\mathbf{j}}$, $\vec{\mathbf{r}}(1) = -2\hat{\mathbf{i}} + 5\hat{\mathbf{j}}$, $\vec{\mathbf{r}}(-4) = 28\hat{\mathbf{i}} - 15\hat{\mathbf{j}}$

The domain of $\vec{\mathbf{r}}(t) = (t^2 - 3t)\hat{\mathbf{i}} + (4t + 1)\hat{\mathbf{j}}$ is all real numbers.

Example 13.1.1 illustrates an important concept. The domain of a vector-valued function consists of real numbers. The domain can be all real numbers or a subset of the real numbers. The range of a vector-valued function consists of vectors. Each real number in the domain of a vector-valued function is mapped to either a two- or a three-dimensional vector.

Graphing Vector-Valued Functions

Recall that a plane vector consists of two quantities: direction and magnitude. Given any point in the plane (the *initial point*), if we move in a specific direction for a specific distance, we arrive at a second point. This represents the *terminal point* of the vector. We calculate the components of the vector by subtracting the coordinates of the initial point from the coordinates of the terminal point.

A vector is considered to be in *standard position* if the initial point is located at the origin. When graphing a vector-valued function, we typically graph the vectors in the domain of the function in standard position, because doing so guarantees the uniqueness of the graph. This convention applies to the graphs of three-dimensional vector-valued functions as well. The graph of a vector-valued function of the form

$$\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}$$

consists of the set of all points $(f(t),\, g(t))$, and the path it traces is called a **plane curve**. The graph of a vector-valued function of the form

$$\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}$$

consists of the set of all points $(f(t),\ g(t),\ h(t))$, and the path it traces is called a **space curve**. Any representation of a plane curve or space curve using a vector-valued function is called a **vector parameterization** of the curve.

Each plane curve and space curve has an **orientation**, indicated by arrows drawn in on the curve, that shows the direction of motion along the curve as the value of the parameter t increases.

Example 13.1.2 : Graphing a Vector-Valued Function

Create a graph of each of the following vector-valued functions:

a. The plane curve represented by $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 3\sin t\,\hat{\mathbf{j}},\ 0 \le t \le 2\pi$
b. The plane curve represented by $\vec{\mathbf{r}}(t) = 4\cos(t^3)\,\hat{\mathbf{i}} + 3\sin(t^3)\,\hat{\mathbf{j}},\ 0 \le t \le \sqrt[3]{2\pi}$
c. The space curve represented by $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 4\sin t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}},\ 0 \le t \le 4\pi$

Solution

a. As with any graph, we start with a table of values. We then graph each of the vectors in the second column of the table in standard position and connect the terminal points of each vector to form a curve (Figure 13.1.1). This curve turns out to be an ellipse centered at the origin.

Table 13.1.1: Table of Values for $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 3\sin t\,\hat{\mathbf{j}},\ 0 \le t \le 2\pi$

t	$\vec{\mathbf{r}}(t)$	t	$\vec{\mathbf{r}}(t)$
0	$4\hat{\mathbf{i}}$	π	$-4\hat{\mathbf{i}}$
$\frac{\pi}{4}$	$2\sqrt{2}\hat{\mathbf{i}} + \frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$	$\frac{5\pi}{4}$	$-2\sqrt{2}\hat{\mathbf{i}} - \frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$
$\frac{\pi}{2}$	$3\hat{\mathbf{j}}$	$\frac{3\pi}{2}$	$-3\hat{\mathbf{j}}$
$\frac{3\pi}{4}$	$-2\sqrt{2}\hat{\mathbf{i}} + \frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$	$\frac{7\pi}{4}$	$2\sqrt{2}\hat{\mathbf{i}} - \frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$
2π	$4\hat{\mathbf{i}}$		

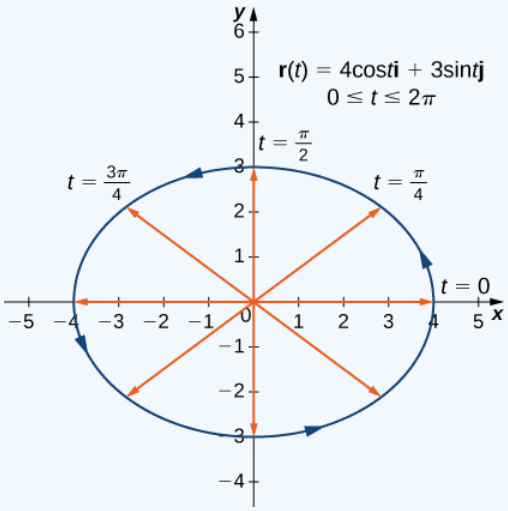


Figure 13.1.1: The graph of the first vector-valued function is an ellipse.

b. The table of values for $\vec{\mathbf{r}}(t) = 4\cos(t^3)\,\hat{\mathbf{i}} + 3\sin(t^3)\,\hat{\mathbf{j}},\ 0 \le t \le \sqrt[3]{2\pi}$ is as follows:

Table of Values for $\vec{\mathbf{r}}(t) = 4\cos(t^3)\,\hat{\mathbf{i}} + 3\sin(t^3)\,\hat{\mathbf{j}},\ 0 \le t \le \sqrt[3]{2\pi}$

t	$\vec{\mathbf{r}}(t)$	t	$\vec{\mathbf{r}}(t)$
0	$4\hat{\mathbf{i}}$	$\sqrt[3]{\pi}$	$-4\hat{\mathbf{i}}$

t	$\vec{\mathbf{r}}(t)$	t	$\vec{\mathbf{r}}(t)$
$\sqrt[3]{\frac{\pi}{4}}$	$2\sqrt{2}\hat{\mathbf{i}}+\frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$	$\sqrt[3]{\frac{5\pi}{4}}$	$-2\sqrt{2}\hat{\mathbf{i}}-\frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$
$\sqrt[3]{\frac{\pi}{2}}$	$3\hat{\mathbf{j}}$	$\sqrt[3]{\frac{3\pi}{2}}$	$-3\hat{\mathbf{j}}$
$\sqrt[3]{\frac{3\pi}{4}}$	$-2\sqrt{2}\hat{\mathbf{i}}+\frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$	$\sqrt[3]{\frac{7\pi}{4}}$	$2\sqrt{2}\hat{\mathbf{i}}-\frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$
$\sqrt[3]{2\pi}$	$4\hat{\mathbf{i}}$		

The graph of this curve is also an ellipse centered at the origin.

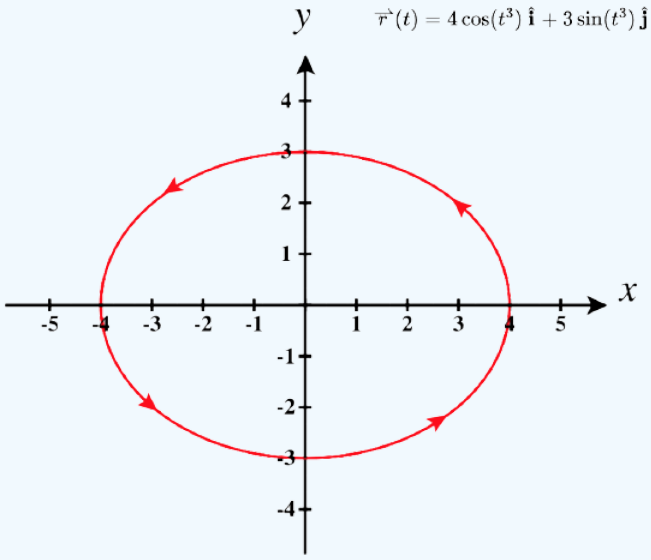


Figure 13.1.2: The graph of the second vector-valued function is also an ellipse.

c. We go through the same procedure for a three-dimensional vector function.

Table of Values for $\mathrm{r(t)} = 4\cos \mathrm{t}\hat{\mathbf{i}} + 4\sin \mathrm{t}\hat{\mathbf{j}} + \mathrm{t}\hat{\mathbf{k}}$, $0 \le \mathrm{t} \le 4\pi$

t	$\vec{\mathbf{r}}(t)$	t	$\vec{\mathbf{r}}(t)$
0	$4\hat{\mathbf{i}}$	π	$-4\hat{\mathbf{i}}+\pi\hat{\mathbf{k}}$
$\frac{\pi}{4}$	$2\sqrt{2}\hat{\mathbf{i}}+2\sqrt{2}\hat{\mathbf{j}}+\frac{\pi}{4}\hat{\mathbf{k}}$	$\frac{5\pi}{4}$	$-2\sqrt{2}\hat{\mathbf{i}}-2\sqrt{2}\hat{\mathbf{j}}+\frac{5\pi}{4}\hat{\mathbf{k}}$
$\frac{\pi}{2}$	$4\hat{\mathbf{j}}+\frac{\pi}{2}\hat{\mathbf{k}}$	$\frac{3\pi}{2}$	$-4\hat{\mathbf{j}}+\frac{3\pi}{2}\hat{\mathbf{k}}$
$\frac{3\pi}{4}$	$-2\sqrt{2}\hat{\mathbf{i}}+2\sqrt{2}\hat{\mathbf{j}}+\frac{3\pi}{4}\hat{\mathbf{k}}$	$\frac{7\pi}{4}$	$2\sqrt{2}\hat{\mathbf{i}}-2\sqrt{2}\hat{\mathbf{j}}+\frac{7\pi}{4}\hat{\mathbf{k}}$
2π	$4\hat{\mathbf{j}}+2\pi\hat{\mathbf{k}}$		

The values then repeat themselves, except for the fact that the coefficient of $\hat{\mathbf{k}}$ is always increasing (13.1.3). This curve is called a helix. Notice that if the $\hat{\mathbf{k}}$ component is eliminated, then the function becomes $\vec{\mathbf{r}}(t) = 4\cos t\hat{\mathbf{i}} + 4\sin t\hat{\mathbf{j}}$, which is a circle of radius 4 centered at the origin.

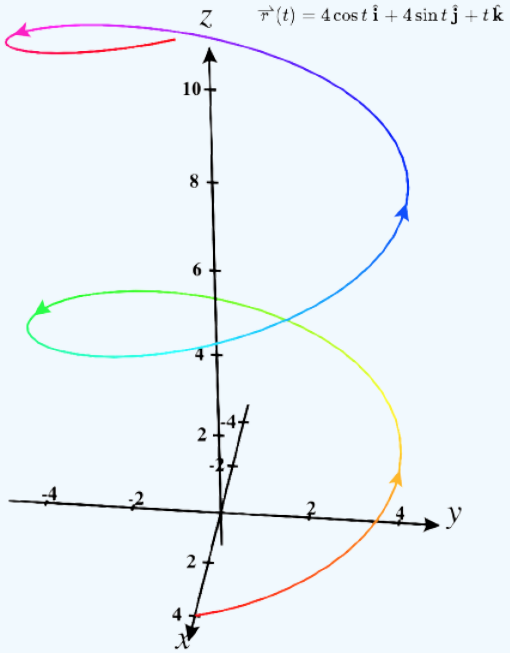


Figure 13.1.3: The graph of the third vector-valued function is a helix.

You may notice that the graphs in parts a. and b. are identical. This happens because the function describing the curve in part b. is a so-called reparameterization of the function describing the curve in part a. In fact, any curve has an infinite number of reparameterizations; for example, we can replace t with $2t$ in any of the three previous curves without changing the shape of the curve. The interval over which t is defined may change, but that is all. We return to this idea later in this chapter when we study arc-length parameterization. As mentioned, the name of the shape of the curve of the graph in 13.1.3 is a *helix*. The curve resembles a spring, with a circular cross-section looking down along the z-axis. It is possible for a helix to be elliptical in cross-section as well. For example, the vector-valued function $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 3\sin t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$ describes an elliptical helix. The projection of this helix into the xy-plane is an ellipse. Last, the arrows in the graph of this helix indicate the orientation of the curve as t progresses from 0 to 4π.

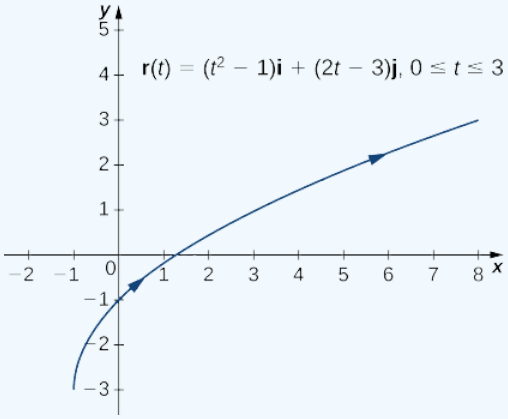
Exercise 13.1.2

Create a graph of the vector-valued function $\vec{\mathbf{r}}(t) = (t^2 - 1)\hat{\mathbf{i}} + (2t - 3)\hat{\mathbf{j}}$, $0 \le t \le 3$.

Hint

Start by making a table of values, then graph the vectors for each value of t.

Answer



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At this point, you may notice a similarity between vector-valued functions and parameterized curves. Indeed, given a vector-valued function $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}$ we can define $x = f(t)$ and $y = g(t)$. If a restriction exists on the values of t (for example, t is restricted to the interval $[a, b]$ for some constants $a < b$, then this restriction is enforced on the parameter. The graph of the parameterized function would then agree with the graph of the vector-valued function, except that the vector-valued graph would represent vectors rather than points. Since we can parameterize a curve defined by a function $y = f(x)$, it is also possible to represent an arbitrary plane curve by a vector-valued function.

Limits and Continuity of a Vector-Valued Function

We now take a look at the limit of a vector-valued function. This is important to understand to study the calculus of vector-valued functions.

Definition: limit of a vector-valued function

A vector-valued function $\vec{\mathbf{r}}$ approaches the limit $\vec{\mathbf{L}}$ as t approaches a, written

$$\lim_{t\to a} \vec{\mathbf{r}}(t) = \vec{\mathbf{L}},$$

provided

$$\lim_{t\to a} \left\| \vec{\mathbf{r}}(t) - \vec{\mathbf{L}} \right\| = 0.$$

This is a rigorous definition of the limit of a vector-valued function. In practice, we use the following theorem:

Theorem: Limit of a vector-valued function

Let f, g, and h be functions of t. Then the limit of the vector-valued function $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}}$ as t approaches a is given by

$$\lim_{t\to a} \vec{\mathbf{r}}(t) = [\lim_{t\to a} f(t)]\hat{\mathbf{i}} + [\lim_{t\to a} g(t)]\hat{\mathbf{j}}, \tag{13.1.1}$$

provided the limits $\lim_{t\to a} f(t)$ and $\lim_{t\to a} g(t)$ exist.

Similarly, the limit of the vector-valued function $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$ as t approaches a is given by

$$\lim_{t\to a} \vec{\mathbf{r}}(t) = [\lim_{t\to a} f(t)]\hat{\mathbf{i}} + [\lim_{t\to a} g(t)]\hat{\mathbf{j}} + [\lim_{t\to a} h(t)]\hat{\mathbf{k}}, \tag{13.1.2}$$

provided the limits $\lim_{t\to a} f(t)$, $\lim_{t\to a} g(t)$ and $\lim_{t\to a} h(t)$ exist.

In the following example, we show how to calculate the limit of a vector-valued function.

Example 13.1.3: Evaluating the Limit of a Vector-Valued Function

For each of the following vector-valued functions, calculate $\lim_{t\to 3} \vec{\mathbf{r}}(t)$ for

a. $\vec{\mathbf{r}}(t) = (t^2 - 3t + 4)\hat{\mathbf{i}} + (4t + 3)\hat{\mathbf{j}}$
b. $\vec{\mathbf{r}}(t) = \frac{2t-4}{t+1}\hat{\mathbf{i}} + \frac{t}{t^2+1}\hat{\mathbf{j}} + (4t - 3)\hat{\mathbf{k}}$

Solution

a. Use Equation 13.1.1 and substitute the value $t = 3$ into the two component expressions:

$$\begin{aligned}\lim_{t\to 3} \vec{\mathbf{r}}(t) &= \lim_{t\to 3} \left[(t^2 - 3t + 4)\hat{\mathbf{i}} + (4t + 3)\hat{\mathbf{j}}\right] \\ &= \left[\lim_{t\to 3}(t^2 - 3t + 4)\right]\hat{\mathbf{i}} + \left[\lim_{t\to 3}(4t + 3)\right]\hat{\mathbf{j}} \\ &= 4\hat{\mathbf{i}} + 15\hat{\mathbf{j}}\end{aligned}$$

b. Use Equation 13.1.2 and substitute the value $t = 3$ into the three component expressions:

$$\begin{aligned}\lim_{t\to 3}\vec{\mathbf{r}}(t) &= \lim_{t\to 3}\left(\frac{2t-4}{t+1}\hat{\mathbf{i}}+\frac{t}{t^2+1}\hat{\mathbf{j}}+(4t-3)\hat{\mathbf{k}}\right)\\ &= \left[\lim_{t\to 3}\left(\frac{2t-4}{t+1}\right)\right]\hat{\mathbf{i}}+\left[\lim_{t\to 3}\left(\frac{t}{t^2+1}\right)\right]\hat{\mathbf{j}}+\left[\lim_{t\to 3}(4t-3)\right]\hat{\mathbf{k}}\\ &= \tfrac{1}{2}\hat{\mathbf{i}}+\tfrac{3}{10}\hat{\mathbf{j}}+9\hat{\mathbf{k}}\end{aligned}$$

Exercise 13.1.3

Calculate $\lim_{t\to 2}\vec{\mathbf{r}}(t)$ for the function $\vec{\mathbf{r}}(t)=\sqrt{t^2+3t-1}\,\hat{\mathbf{i}}-(4t-3)\hat{\mathbf{j}}-\sin\frac{(t+1)\pi}{2}\hat{\mathbf{k}}$

Hint

Use Equation 13.1.2 from the preceding theorem.

Answer

$$\lim_{t\to 2}\vec{\mathbf{r}}(t)=3\hat{\mathbf{i}}-5\hat{\mathbf{j}}+\hat{\mathbf{k}}$$

Now that we know how to calculate the limit of a vector-valued function, we can define **continuity at a point** for such a function.

Definitions

Let f, g, and h be functions of t. Then, the vector-valued function $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}$ is **continuous at point** $t=a$ if the following three conditions hold:

1. $\vec{\mathbf{r}}(a)$ exists
2. $\lim_{t\to a}\vec{\mathbf{r}}(t)$ exists
3. $\lim_{t\to a}\vec{\mathbf{r}}(t)=\vec{\mathbf{r}}(a)$

Similarly, the vector-valued function $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}+h(t)\hat{\mathbf{k}}$ is **continuous at point** $t=a$ if the following three conditions hold:

1. $\vec{\mathbf{r}}(a)$ exists
2. $\lim_{t\to a}\vec{\mathbf{r}}(t)$ exists
3. $\lim_{t\to a}\vec{\mathbf{r}}(t)=\vec{\mathbf{r}}(a)$

Summary

- A vector-valued function is a function of the form $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}$ or $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}+h(t)\hat{\mathbf{k}}$, where the component functions f, g, and h are real-valued functions of the parameter t.
- The graph of a vector-valued function of the form $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}$ is called a *plane curve*. The graph of a vector-valued function of the form $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}+h(t)\hat{\mathbf{k}}$ is called a *space curve*.
- It is possible to represent an arbitrary plane curve by a vector-valued function.
- To calculate the limit of a vector-valued function, calculate the limits of the component functions separately.

Key Equations

- **Vector-valued function**
  $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}$ or $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}+h(t)\hat{\mathbf{k}}$,or $\vec{\mathbf{r}}(t)=\langle f(t),g(t)\rangle$ or $\vec{\mathbf{r}}(t)=\langle f(t),g(t),h(t)\rangle$
- **Limit of a vector-valued function**
  $\lim_{t\to a}\vec{\mathbf{r}}(t)=[\lim_{t\to a}f(t)]\hat{\mathbf{i}}+[\lim_{t\to a}g(t)]\hat{\mathbf{j}}$ or $\lim_{t\to a}\vec{\mathbf{r}}(t)=[\lim_{t\to a}f(t)]\hat{\mathbf{i}}+[\lim_{t\to a}g(t)]\hat{\mathbf{j}}+[\lim_{t\to a}h(t)]\hat{\mathbf{k}}$

Glossary

component functions

the component functions of the vector-valued function $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}$ are $f(t)$ and $g(t)$, and the component functions of the vector-valued function $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}+h(t)\hat{\mathbf{k}}$ are $f(t)$, $g(t)$ and $h(t)$

helix

a three-dimensional curve in the shape of a spiral

limit of a vector-valued function

a vector-valued function $\vec{\mathbf{r}}(t)$ has a limit $\vec{\mathbf{L}}$ as t approaches a if $\lim t \to a \left|\vec{\mathbf{r}}(t) - \vec{\mathbf{L}}\right| = 0$

plane curve

the set of ordered pairs $(f(t), g(t))$ together with their defining parametric equations $x = f(t)$ and $y = g(t)$

reparameterization

an alternative parameterization of a given vector-valued function

space curve

the set of ordered triples $(f(t), g(t), h(t))$ together with their defining parametric equations $x = f(t)$, $y = g(t)$ and $z = h(t)$

vector parameterization

any representation of a plane or space curve using a vector-valued function

vector-valued function

a function of the form $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}}$ or $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$,where the component functions f, g, and h are real-valued functions of the parameter t.

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- Edited by Paul Seeburger (Monroe Community College)

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13.1E: EXERCISES FOR SECTION 13.1

INTRODUCTION TO VECTOR-VALUED FUNCTIONS

1) Give the component functions $x = f(t)$ and $y = g(t)$ for the vector-valued function $\vec{\mathbf{r}}(t) = 3\sec t\,\hat{\mathbf{i}} + 2\tan t\,\hat{\mathbf{j}}$.

2) Given $\vec{\mathbf{r}}(t) = 3\sec t\hat{\mathbf{i}} + 2\tan t\hat{\mathbf{j}}$, find the following values (if possible).

a. $\vec{\mathbf{r}}\left(\frac{\pi}{4}\right)$
b. $\vec{\mathbf{r}}(\pi)$
c. $\vec{\mathbf{r}}\left(\frac{\pi}{2}\right)$

3) Sketch the curve of the vector-valued function $\vec{\mathbf{r}}(t) = 3\sec t\hat{\mathbf{i}} + 2\tan t\hat{\mathbf{j}}$ and give the orientation of the curve. Sketch asymptotes as a guide to the graph.

LIMITS OF VECTOR-VALUED FUNCTIONS

4) Evaluate $\lim\limits_{t\to 0}\left(e^t\hat{\mathbf{i}} + \frac{\sin t}{t}\hat{\mathbf{j}} + e^{-t}\hat{\mathbf{k}}\right)$

5) Given the vector-valued function $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t\rangle$ find the following values:

a. $\lim\limits_{t\to\frac{\pi}{4}} \vec{\mathbf{r}}(t)$
b. $\vec{\mathbf{r}}\left(\frac{\pi}{3}\right)$
c. Is $\vec{\mathbf{r}}(t)$ continuous at $t = \frac{\pi}{3}$?
d. Graph $\vec{\mathbf{r}}(t)$.

6) Given the vector-valued function $\vec{\mathbf{r}}(t) = \langle t, t^2 + 1\rangle$, find the following values:

a. $\lim\limits_{t\to -3} \vec{\mathbf{r}}(t)$
b. $\vec{\mathbf{r}}(-3)$
c. Is $\vec{\mathbf{r}}(t)$ continuous at $x = -3$?
d. $\vec{\mathbf{r}}(t+2) - \vec{\mathbf{r}}(t)$

7) Let $\vec{\mathbf{r}}(t) = e^t\hat{\mathbf{i}} + \sin t\hat{\mathbf{j}} + \ln t\hat{\mathbf{k}}$. Find the following values:

a. $\vec{\mathbf{r}}\left(\frac{\pi}{4}\right)$
b. $\lim\limits_{t\to\frac{\pi}{4}} \vec{\mathbf{r}}(t)$
c. Is $\vec{\mathbf{r}}(t)$ continuous at $t = \frac{\pi}{4}$?

For exercises 8 - 13, find the limit of the following vector-valued functions at the indicated value of t.

8) $\lim\limits_{t\to 4}\langle\sqrt{t-3}, \frac{\sqrt{t}-2}{t-4}, \tan\left(\frac{\pi}{t}\right)\rangle$

9) $\lim\limits_{t\to\frac{\pi}{2}} \vec{\mathbf{r}}(t)$ for $\vec{\mathbf{r}}(t) = e^t\hat{\mathbf{i}} + \sin t\hat{\mathbf{j}} + \ln t\hat{\mathbf{k}}$

10) $\lim\limits_{t\to\infty}\langle e^{-2t}, \frac{2t+3}{3t-1}, \arctan(2t)\rangle$

11) $\lim\limits_{t\to e^2}\langle t\ln(t), \frac{\ln t}{t^2}, \sqrt{\ln(t^2)}\rangle$

12) $\lim\limits_{t\to\frac{\pi}{6}}\langle\cos 2t, \sin 2t, 1\rangle$

13) $\lim\limits_{t\to\infty} \vec{\mathbf{r}}(t)$ for $\vec{\mathbf{r}}(t) = 2e^{-t}\mathbf{i} + e^{-t}\hat{\mathbf{j}} + \ln(t-1)\hat{\mathbf{k}}$

DOMAIN OF A VECTOR-VALUED FUNCTION

For problems 14 - 17, find the domain of the vector-valued functions.

14) Domain: $\vec{\mathbf{r}}(t) = \langle t^2, t, \sin t\rangle$

15) Domain: $\vec{\mathbf{r}}(t) = \langle t^2, \tan t, \ln t\rangle$

16) Domain: $\vec{\mathbf{r}}(t) = \langle t^2, \sqrt{t-3}, \frac{3}{2t+1}\rangle$

17) Domain: $\vec{\mathbf{r}}(t) = \langle \csc(t), \frac{1}{\sqrt{t-3}}, \ln(t-2)\rangle$

18) a. Find the domain of $\vec{\mathbf{r}}(t) = 2e^{-t}\hat{\mathbf{i}} + e^{-t}\hat{\mathbf{j}} + \ln(t-1)\hat{\mathbf{k}}$.

b. For what values of t is $\vec{\mathbf{r}}(t) = 2e^{-t}\hat{\mathbf{i}} + e^{-t}\hat{\mathbf{j}} + \ln(t-1)\hat{\mathbf{k}}$ continuous?

19) Domain: $\vec{\mathbf{r}}(t) = (\arccos t)\,\hat{\mathbf{i}} + \sqrt{2t-1}\,\hat{\mathbf{j}} + \ln(t)\,\hat{\mathbf{k}}$

VISUALIZING VECTOR-VALUED FUNCTIONS

20) Describe the curve defined by the vector-valued function $\vec{\mathbf{r}}(t) = (1+t)\hat{\mathbf{i}} + (2+5t)\hat{\mathbf{j}} + (-1+6t)\hat{\mathbf{k}}$.

21) Let $\vec{\mathbf{r}}(t) = \langle \cos t, t, \sin t \rangle$ and use it to answer the following questions.

a. For what values of t is $\vec{\mathbf{r}}(t)$ continuous?
b. Sketch the graph of $\vec{\mathbf{r}}(t)$.

22) Produce a careful sketch of the graph of $\vec{\mathbf{r}}(t) = t^2\hat{\mathbf{i}} + t\hat{\mathbf{j}}$.

In questions 23 - 25, use a graphing utility to sketch each of the vector-valued functions:

23) **[T]** $\vec{\mathbf{r}}(t) = 2\cos^2 t\hat{\mathbf{i}} + (2-\sqrt{t})\hat{\mathbf{j}}$

24) **[T]** $\vec{\mathbf{r}}(t) = \langle e^{\cos(3t)}, e^{-\sin(t)} \rangle$

25) **[T]** $\vec{\mathbf{r}}(t) = \langle 2 - \sin(2t), 3 + 2\cos t \rangle$

FINDING EQUATIONS IN x AND y FOR THE PATH TRACED OUT BY VECTOR-VALUED FUNCTIONS

For questions 26-33, eliminate the parameter t, write the equation in Cartesian coordinates, then sketch the graph of the vector-valued functions.

26) $\vec{\mathbf{r}}(t) = 2t\hat{\mathbf{i}} + t^2\hat{\mathbf{j}}$
(*Hint:* Let $x = 2t$ and $y = t^2$. Solve the first equation for t in terms of x and substitute this result into the second equation.)

27) $\vec{\mathbf{r}}(t) = t^3\hat{\mathbf{i}} + 2t\hat{\mathbf{j}}$

28) $\vec{\mathbf{r}}(t) = \sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}}$

29) $\vec{\mathbf{r}}(t) = 3\cos t\,\hat{\mathbf{i}} + 3\sin t\,\hat{\mathbf{j}}$

30) $\vec{\mathbf{r}}(t) = \langle \sin t, 4\cos t \rangle$

31) $\vec{\mathbf{r}}(t) = 2\sin t\,\hat{\mathbf{i}} - 3\cos t\,\hat{\mathbf{j}}$

32) $\vec{\mathbf{r}}(t) = \tan t\,\hat{\mathbf{i}} - 2\sec t\,\hat{\mathbf{j}}$

33) $\vec{\mathbf{r}}(t) = 3\sec t\,\hat{\mathbf{i}} + 4\tan t\,\hat{\mathbf{j}}$

FINDING A VECTOR-VALUED FUNCTION TO TRACE OUT THE GRAPH OF AN EQUATION IN x AND y

For questions 34 - 40, find a vector-valued function that traces out the given curve in the indicated direction.

34) $4x^2 + 9y^2 = 36$; clockwise and counterclockwise

35) $y = x^2$; from left to right

36) The line through P and Q where P is $(1, 4, -2)$ and Q is $(3, 9, 6)$

37) The circle, $x^2 + y^2 = 36$, oriented clockwise, with position $(-6, 0)$ at time $t = 0$.

38) The ellipse, $x^2 + \dfrac{y^2}{36} = 1$, oriented counterclockwise

39) The hyperbola, $\dfrac{y^2}{36} - x^2 = 1$, top piece is oriented from left-to-right

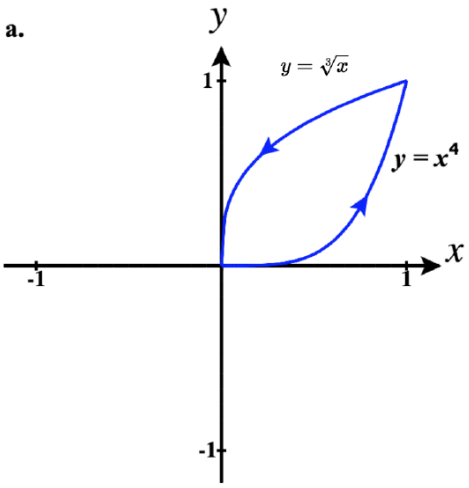
40) The hyperbola, $\dfrac{x^2}{49} - \dfrac{y^2}{64} = 1$, right piece is oriented from bottom-to-top

PARAMETERIZING A PIECEWISE PATH

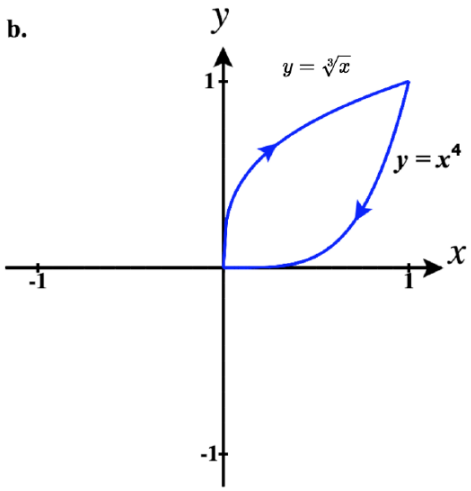
For questions 41 - 44, provide a parameterization for each piecewise path. Try to write a parameterization that starts with $t = 0$ and progresses on through values of t as you move from one piece to another.

41)

a.

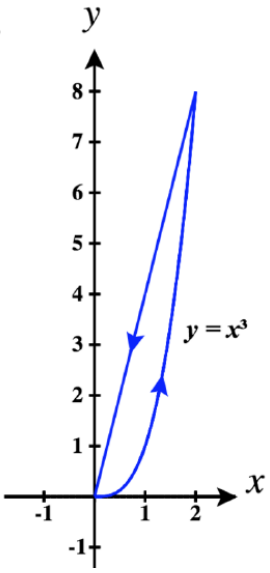


b.

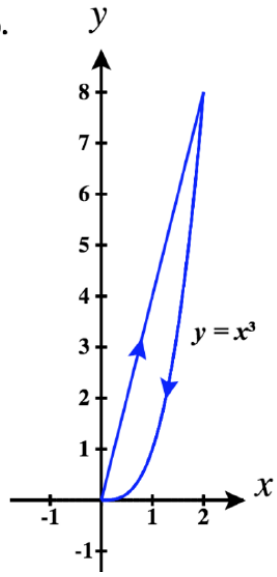


42)

a.

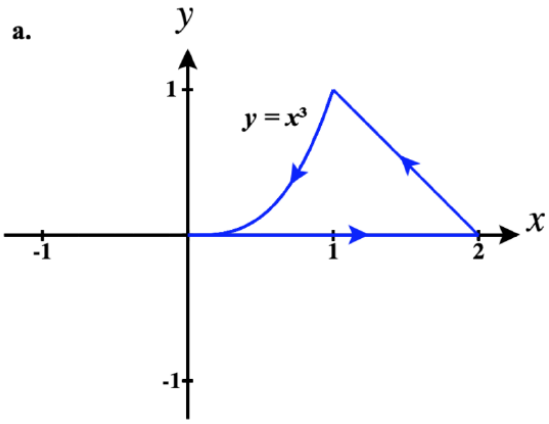


b.

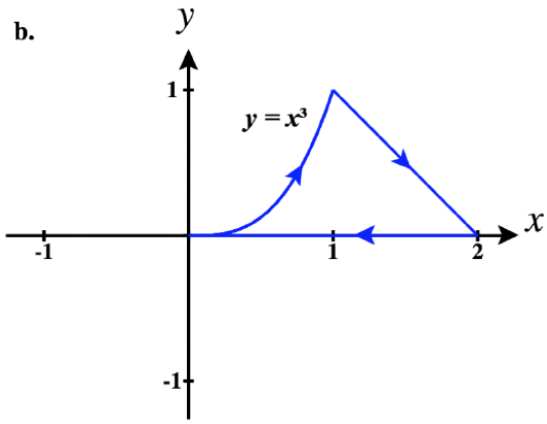


43)

a.



b.



44)

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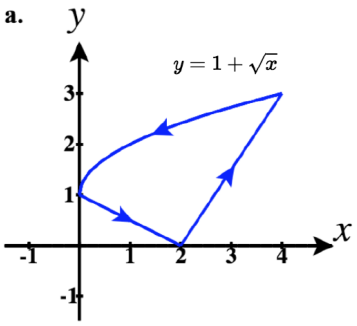
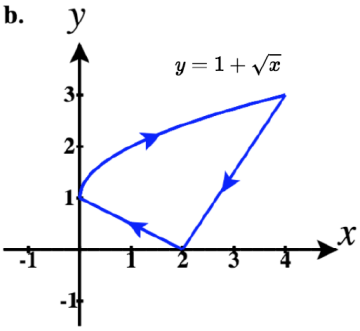
a.



b.



ADDITIONAL VECTOR-VALUED FUNCTION QUESTIONS

For questions 45 - 48, consider the curve described by the vector-valued function $\vec{\mathbf{r}}(t) = (50e^{-t}\cos t)\hat{\mathbf{i}} + (50e^{-t}\sin t)\hat{\mathbf{j}} + (5 - 5e^{-t})\hat{\mathbf{k}}$.

45) What is the initial point of the path corresponding to $\vec{\mathbf{r}}(0)$?

46) What is $\lim_{t\to\infty} \vec{\mathbf{r}}(t)$?

47) **[T]** Use technology to sketch the curve.

48) Eliminate the parameter t to show that $z = 5 - \frac{r}{10}$ where $r^2 = x^2 + y^2$.

49) **[T]** Let $\vec{\mathbf{r}}(t) = \cos t\hat{\mathbf{i}} + \sin t\hat{\mathbf{j}} + 0.3\sin(2t)\hat{\mathbf{k}}$. Use technology to graph the curve (called the *roller-coaster curve*) over the interval $[0, 2\pi)$. Choose at least two views to determine the peaks and valleys.

50) **[T]** Use the result of the preceding problem to construct an equation of a roller coaster with a steep drop from the peak and steep incline from the "valley." Then, use technology to graph the equation.

51) Use the results of the preceding two problems to construct an equation of a path of a roller coaster with more than two turning points (peaks and valleys).

52) Complete the following investigation.

a. Graph the curve $\vec{\mathbf{r}}(t) = (4 + \cos(18t))\cos(t)\hat{\mathbf{i}} + (4 + \cos(18t)\sin(t))\hat{\mathbf{j}} + 0.3\sin(18t)\hat{\mathbf{k}}$ using two viewing angles of your choice to see the overall shape of the curve.
b. Does the curve resemble a "slinky"?
c. What changes to the equation should be made to increase the number of coils of the slinky?

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13.2: Calculus of Vector-Valued Functions

Learning Objectives

- Write an expression for the derivative of a vector-valued function.
- Find the tangent vector at a point for a given position vector.
- Find the unit tangent vector at a point for a given position vector and explain its significance.
- Calculate the definite integral of a vector-valued function.

To study the calculus of vector-valued functions, we follow a similar path to the one we took in studying real-valued functions. First, we define the derivative, then we examine applications of the derivative, then we move on to defining integrals. However, we will find some interesting new ideas along the way as a result of the vector nature of these functions and the properties of space curves.

Derivatives of Vector-Valued Functions

Now that we have seen what a vector-valued function is and how to take its limit, the next step is to learn how to differentiate a vector-valued function. The definition of the derivative of a vector-valued function is nearly identical to the definition of a real-valued function of one variable. However, because the range of a vector-valued function consists of vectors, the same is true for the range of the derivative of a vector-valued function.

Definition: Derivative of Vector-Valued Functions

The derivative of a vector-valued function $\vec{\mathbf{r}}(t)$ is

$$\vec{\mathbf{r}}'(t) = \lim_{\Delta t \to 0} \frac{\vec{\mathbf{r}}(t+\Delta t) - \vec{\mathbf{r}}(t)}{\Delta t} \tag{13.2.1}$$

provided the limit exists. If $\vec{\mathbf{r}}'(t)$ exists, then $\vec{\mathbf{r}}(t)$ is differentiable at t. If $\vec{\mathbf{r}}'(t)$ exists for all t in an open interval (a, b) then $\vec{\mathbf{r}}(t)$ is differentiable over the interval (a, b). For the function to be differentiable over the closed interval $[a, b]$, the following two limits must exist as well:

$$\vec{\mathbf{r}}'(a) = \lim_{\Delta t \to 0^+} \frac{\vec{\mathbf{r}}(a+\Delta t) - \vec{\mathbf{r}}(a)}{\Delta t}$$

and

$$\vec{\mathbf{r}}'(b) = \lim_{\Delta t \to 0^-} \frac{\vec{\mathbf{r}}(b+\Delta t) - \vec{\mathbf{r}}(b)}{\Delta t}$$

Many of the rules for calculating derivatives of real-valued functions can be applied to calculating the derivatives of vector-valued functions as well. Recall that the derivative of a real-valued function can be interpreted as the slope of a tangent line or the instantaneous rate of change of the function. The derivative of a vector-valued function can be understood to be an instantaneous rate of change as well; for example, when the function represents the position of an object at a given point in time, the derivative represents its velocity at that same point in time.

We now demonstrate taking the derivative of a vector-valued function.

Example 13.2.1: Finding the Derivative of a Vector-Valued Function

Use the definition to calculate the derivative of the function

$$\vec{\mathbf{r}}(t) = (3t+4)\,\hat{\mathbf{i}} + (t^2 - 4t + 3)\,\hat{\mathbf{j}}.$$

Solution

Let's use Equation 13.2.1:

$$
\begin{aligned}
\vec{\mathbf{r}}'(t) &= \lim_{\Delta t \to 0} \frac{\vec{\mathbf{r}}(t+\Delta t) - \vec{\mathbf{r}}(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{[(3(t+\Delta t)+4)\,\hat{\mathbf{i}} + ((t+\Delta t)^2 - 4(t+\Delta t)+3)\,\hat{\mathbf{j}}] - [(3t+4)\,\hat{\mathbf{i}} + (t^2-4t+3)\,\hat{\mathbf{j}}]}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{(3t+3\Delta t+4)\,\hat{\mathbf{i}} - (3t+4)\,\hat{\mathbf{i}} + (t^2+2t\Delta t+(\Delta t)^2-4t-4\Delta t+3)\,\hat{\mathbf{j}} - (t^2-4t+3)\,\hat{\mathbf{j}}}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{(3\Delta t)\,\hat{\mathbf{i}} + (2t\Delta t + (\Delta t)^2 - 4\Delta t)\,\hat{\mathbf{j}}}{\Delta t} \\
&= \lim_{\Delta t \to 0} (3\,\hat{\mathbf{i}} + (2t + \Delta t - 4)\,\hat{\mathbf{j}}) \\
&= 3\,\hat{\mathbf{i}} + (2t-4)\,\hat{\mathbf{j}}
\end{aligned}
$$

? Exercise 13.2.1

Use the definition to calculate the derivative of the function $\vec{\mathbf{r}}(t) = (2t^2+3)\,\hat{\mathbf{i}} + (5t-6)\,\hat{\mathbf{j}}$.

Hint

Use Equation 13.2.1.

Answer

$$\vec{\mathbf{r}}'(t) = 4t\,\hat{\mathbf{i}} + 5\,\hat{\mathbf{j}}$$

Notice that in the calculations in Example 13.2.1, we could also obtain the answer by first calculating the derivative of each component function, then putting these derivatives back into the vector-valued function. This is always true for calculating the derivative of a vector-valued function, whether it is in two or three dimensions. We state this in the following theorem. The proof of this theorem follows directly from the definitions of the limit of a vector-valued function and the derivative of a vector-valued function.

Theorem 13.2.1: Differentiation of Vector-Valued Functions

Let f, g, and h be differentiable functions of t.

1. If $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}$ then
$$\vec{\mathbf{r}}'(t) = f'(t)\,\hat{\mathbf{i}} + g'(t)\,\hat{\mathbf{j}}.$$
2. If $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}$ then
$$\vec{\mathbf{r}}'(t) = f'(t)\,\hat{\mathbf{i}} + g'(t)\,\hat{\mathbf{j}} + h'(t)\,\hat{\mathbf{k}}.$$

✓ Example 13.2.2: Calculating the Derivative of Vector-Valued Functions

Use Theorem 13.2.1 to calculate the derivative of each of the following functions.

a. $\vec{\mathbf{r}}(t) = (6t+8)\,\hat{\mathbf{i}} + (4t^2+2t-3)\,\hat{\mathbf{j}}$
b. $\vec{\mathbf{r}}(t) = 3\cos t\,\hat{\mathbf{i}} + 4\sin t\,\hat{\mathbf{j}}$
c. $\vec{\mathbf{r}}(t) = e^t \sin t\,\hat{\mathbf{i}} + e^t \cos t\,\hat{\mathbf{j}} - e^{2t}\,\hat{\mathbf{k}}$

Solution

We use Theorem 13.2.1 and what we know about differentiating functions of one variable.

a. The first component of
$$\vec{\mathbf{r}}(t) = (6t+8)\,\hat{\mathbf{i}} + (4t^2+2t-3)\,\hat{\mathbf{j}}$$

is $f(t) = 6t + 8$. The second component is $g(t) = 4t^2 + 2t - 3$. We have $f'(t) = 6$ and $g'(t) = 8t + 2$, so the Theorem 13.2.1 gives $\vec{\mathbf{r}}\,'(t) = 6\,\hat{\mathbf{i}} + (8t + 2)\,\hat{\mathbf{j}}$.

b. The first component is $f(t) = 3\cos t$ and the second component is $g(t) = 4\sin t$. We have $f'(t) = -3\sin t$ and $g'(t) = 4\cos t$, so we obtain $\vec{\mathbf{r}}\,'(t) = -3\sin t\,\hat{\mathbf{i}} + 4\cos t\,\hat{\mathbf{j}}$.

c. The first component of $\vec{\mathbf{r}}(t) = e^t \sin t\,\hat{\mathbf{i}} + e^t \cos t\,\hat{\mathbf{j}} - e^{2t}\,\hat{\mathbf{k}}$ is $f(t) = e^t \sin t$, the second component is $g(t) = e^t \cos t$, and the third component is $h(t) = -e^{2t}$. We have $f'(t) = e^t(\sin t + \cos t)$, $g'(t) = e^t(\cos t - \sin t)$, and $h'(t) = -2e^{2t}$, so the theorem gives $\vec{\mathbf{r}}\,'(t) = e^t(\sin t + \cos t)\,\hat{\mathbf{i}} + e^t(\cos t - \sin t)\,\hat{\mathbf{j}} - 2e^{2t}\,\hat{\mathbf{k}}$.

Exercise 13.2.2

Calculate the derivative of the function

$$\vec{\mathbf{r}}(t) = (t \ln t)\,\hat{\mathbf{i}} + (5e^t)\,\hat{\mathbf{j}} + (\cos t - \sin t)\,\hat{\mathbf{k}}.$$

Hint

Identify the component functions and use Theorem 13.2.1.

Answer

$$\vec{\mathbf{r}}\,'(t) = (1 + \ln t)\,\hat{\mathbf{i}} + 5e^t\,\hat{\mathbf{j}} - (\sin t + \cos t)\,\hat{\mathbf{k}}$$

We can extend to vector-valued functions the properties of the derivative that we presented previously. In particular, the constant multiple rule, the sum and difference rules, the product rule, and the chain rule all extend to vector-valued functions. However, in the case of the product rule, there are actually three extensions:

1. for a real-valued function multiplied by a vector-valued function,
2. for the dot product of two vector-valued functions, and
3. for the cross product of two vector-valued functions.

Theorem: Properties of the Derivative of Vector-Valued Functions

Let $\vec{\mathbf{r}}$ and $\vec{\mathbf{u}}$ be differentiable vector-valued functions of t, let f be a differentiable real-valued function of t, and let c be a scalar.

i.	$\frac{d}{dt}[c\vec{\mathbf{r}}(t)]$	$=$	$c\vec{\mathbf{r}}\,'(t)$	Scalar multiple
ii.	$\frac{d}{dt}[\vec{\mathbf{r}}(t) \pm \vec{\mathbf{u}}(t)]$	$=$	$\vec{\mathbf{r}}\,'(t) \pm \vec{\mathbf{u}}\,'(t)$	Sum and difference
iii.	$\frac{d}{dt}[f(t)\vec{\mathbf{u}}(t)]$	$=$	$f'(t)\vec{\mathbf{u}}(t) + f(t)\vec{\mathbf{u}}\,'(t)$	Scalar product
iv.	$\frac{d}{dt}[\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{u}}(t)]$	$=$	$\vec{\mathbf{r}}\,'(t) \cdot \vec{\mathbf{u}}(t) + \vec{\mathbf{r}}(t) \cdot \vec{\mathbf{u}}\,'(t)$	Dot product
v.	$\frac{d}{dt}[\vec{\mathbf{r}}(t) \times \vec{\mathbf{u}}(t)]$	$=$	$\vec{\mathbf{r}}\,'(t) \times \vec{\mathbf{u}}(t) + \vec{\mathbf{r}}(t) \times \vec{\mathbf{u}}\,'(t)$	Cross product
vi.	$\frac{d}{dt}[\vec{\mathbf{r}}(f(t))]$	$=$	$\vec{\mathbf{r}}\,'(f(t)) \cdot f'(t)$	Chain rule
vii.	If $\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}(t)$	$=$	c, then $\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}\,'(t) = 0$.	

Proof

The proofs of the first two properties follow directly from the definition of the derivative of a vector-valued function. The third property can be derived from the first two properties, along with the product rule. Let $\vec{\mathbf{u}}(t) = g(t)\,\hat{\mathbf{i}} + h(t)\,\hat{\mathbf{j}}$. Then

$$\begin{aligned}\frac{d}{dt}[f(t)\vec{\mathbf{u}}(t)] &= \frac{d}{dt}[f(t)(g(t)\,\hat{\mathbf{i}} + h(t)\,\hat{\mathbf{j}})] \\ &= \frac{d}{dt}[f(t)g(t)\,\hat{\mathbf{i}} + f(t)h(t)\,\hat{\mathbf{j}}] \\ &= \frac{d}{dt}[f(t)g(t)]\,\hat{\mathbf{i}} + \frac{d}{dt}[f(t)h(t)]\,\hat{\mathbf{j}} \\ &= (f'(t)g(t) + f(t)g'(t))\,\hat{\mathbf{i}} + (f'(t)h(t) + f(t)h'(t))\,\hat{\mathbf{j}} \\ &= f'(t)\vec{\mathbf{u}}(t) + f(t)\vec{\mathbf{u}}'(t).\end{aligned}$$

To prove property iv. let $\vec{\mathbf{r}}(t) = f_1(t)\,\hat{\mathbf{i}} + g_1(t)\,\hat{\mathbf{j}}$ and $\vec{\mathbf{u}}(t) = f_2(t)\,\hat{\mathbf{i}} + g_2(t)\,\hat{\mathbf{j}}$. Then

$$\begin{aligned}\frac{d}{dt}[\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{u}}(t)] &= \frac{d}{dt}[f_1(t)f_2(t) + g_1(t)g_2(t)] \\ &= f_1{}'(t)f_2(t) + f_1(t)f_2{}'(t) + g_1{}'(t)g_2(t) + g_1(t)g_2{}'(t) = f_1{}'(t)f_2(t) + g_1 \\ &\quad {}'(t)g_2(t) + f_1(t)f_2{}'(t) + g_1(t)g_2{}'(t) \\ &= (f_1{}'\,\hat{\mathbf{i}} + g_1{}'\,\hat{\mathbf{j}})\cdot(f_2\,\hat{\mathbf{i}} + g_2\,\hat{\mathbf{j}}) + (f_1\,\hat{\mathbf{i}} + g_1\,\hat{\mathbf{j}})\cdot(f_2{}'\,\hat{\mathbf{i}} + g_2{}'\,\hat{\mathbf{j}}) \\ &= \vec{\mathbf{r}}'(t)\cdot\vec{\mathbf{u}}(t) + \vec{\mathbf{r}}(t)\cdot\vec{\mathbf{u}}'(t).\end{aligned}$$

The proof of property v. is similar to that of property iv. Property vi. can be proved using the chain rule. Last, property vii. follows from property iv:

$$\begin{aligned}\frac{d}{dt}[\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}(t)] &= \frac{d}{dt}[c] \\ \vec{\mathbf{r}}'(t)\cdot\vec{\mathbf{r}}(t) + \vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}'(t) &= 0 \\ 2\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}'(t) &= 0 \\ \vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}'(t) &= 0\end{aligned}$$

Now for some examples using these properties.

Example 13.2.3: Using the Properties of Derivatives of Vector-Valued Functions

Given the vector-valued functions

$$\vec{\mathbf{r}}(t) = (6t+8)\,\hat{\mathbf{i}} + (4t^2 + 2t - 3)\,\hat{\mathbf{j}} + 5t\,\hat{\mathbf{k}}$$

and

$$\vec{\mathbf{u}}(t) = (t^2 - 3)\,\hat{\mathbf{i}} + (2t+4)\,\hat{\mathbf{j}} + (t^3 - 3t)\,\hat{\mathbf{k}},$$

calculate each of the following derivatives using the properties of the derivative of vector-valued functions.

a. $\frac{d}{dt}[\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{u}}(t)]$
b. $\frac{d}{dt}[\vec{\mathbf{u}}(t)\times\vec{\mathbf{u}}'(t)]$

Solution

We have $\vec{\mathbf{r}}'(t) = 6\,\hat{\mathbf{i}} + (8t+2)\,\hat{\mathbf{j}} + 5\,\hat{\mathbf{k}}$ and $\vec{\mathbf{u}}'(t) = 2t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + (3t^2 - 3)\,\hat{\mathbf{k}}$. Therefore, according to **property iv**:

1.
$$\begin{aligned}\frac{d}{dt}[\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{u}}(t)] &= \vec{\mathbf{r}}'(t)\cdot\vec{\mathbf{u}}(t) + \vec{\mathbf{r}}(t)\cdot\vec{\mathbf{u}}'(t) \\ &= (6\,\hat{\mathbf{i}} + (8t+2)\,\hat{\mathbf{j}} + 5\,\hat{\mathbf{k}})\cdot((t^2-3)\,\hat{\mathbf{i}} + (2t+4)\,\hat{\mathbf{j}} + (t^3-3t)\,\hat{\mathbf{k}}) \\ &\quad +((6t+8)\,\hat{\mathbf{i}} + (4t^2+2t-3)\,\hat{\mathbf{j}} + 5t\,\hat{\mathbf{k}})\cdot(2t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + (3t^2-3)\,\hat{\mathbf{k}}) \\ &= 6(t^2-3) + (8t+2)(2t+4) + 5(t^3-3t) \\ &\quad +2t(6t+8) + 2(4t^2+2t-3) + 5t(3t^2-3) \\ &= 20t^3 + 42t^2 + 26t - 16.\end{aligned}$$

2. First, we need to adapt **property v** for this problem:

$$\frac{d}{dt}[\vec{\mathbf{u}}(t) \times \vec{\mathbf{u}}'(t)] = \vec{\mathbf{u}}'(t) \times \vec{\mathbf{u}}'(t) + \vec{\mathbf{u}}(t) \times \vec{\mathbf{u}}''(t).$$

Recall that the cross product of any vector with itself is zero. Furthermore, $\vec{\mathbf{u}}''(t)$ represents the second derivative of $\vec{\mathbf{u}}(t)$:

$$\vec{\mathbf{u}}''(t) = \frac{d}{dt}[\vec{\mathbf{u}}'(t)] = \frac{d}{dt}[2t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + (3t^2 - 3)\,\hat{\mathbf{k}}] = 2\,\hat{\mathbf{i}} + 6t\,\hat{\mathbf{k}}.$$

Therefore,

$$\begin{aligned}\frac{d}{dt}[\vec{\mathbf{u}}(t) \times \vec{\mathbf{u}}'(t)] &= 0 + ((t^2 - 3)\,\hat{\mathbf{i}} + (2t + 4)\,\hat{\mathbf{j}} + (t^3 - 3t)\,\hat{\mathbf{k}}) \times (2\,\hat{\mathbf{i}} + 6t\,\hat{\mathbf{k}}) \\ &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ t^2 - 3 & 2t + 4 & t^3 - 3t \\ 2 & 0 & 6t \end{vmatrix} \\ &= 6t(2t + 4)\,\hat{\mathbf{i}} - (6t(t^2 - 3) - 2(t^3 - 3t))\,\hat{\mathbf{j}} - 2(2t + 4)\,\hat{\mathbf{k}} \\ &= (12t^2 + 24t)\,\hat{\mathbf{i}} + (12t - 4t^3)\,\hat{\mathbf{j}} - (4t + 8)\,\hat{\mathbf{k}}.\end{aligned}$$

? Exercise 13.2.3

Calculate $\frac{d}{dt}[\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}'(t)]$ and $\frac{d}{dt}[\vec{\mathbf{u}}(t) \times \vec{\mathbf{r}}(t)]$ for the vector-valued functions:

- $\vec{\mathbf{r}}(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} - e^{2t}\,\hat{\mathbf{k}}$
- $\vec{\mathbf{u}}(t) = t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} + \cos t\,\hat{\mathbf{k}}$,

Hint

Follow the same steps as in Example 13.2.3.

Answer

$\frac{d}{dt}[\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}'(t)] = 8e^{4t}$

$\frac{d}{dt}[\vec{\mathbf{u}}(t) \times \vec{\mathbf{r}}(t)] = -(e^{2t}(\cos t + 2\sin t) + \cos 2t)\,\hat{\mathbf{i}} + (e^{2t}(2t + 1) - \sin 2t)\,\hat{\mathbf{j}} + (t\cos t + \sin t - \cos 2t)\,\hat{\mathbf{k}}$

Tangent Vectors and Unit Tangent Vectors

Recall that the derivative at a point can be interpreted as the slope of the tangent line to the graph at that point. In the case of a vector-valued function, the derivative provides a tangent vector to the curve represented by the function. Consider the vector-valued function

$$\vec{\mathbf{r}}(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} \tag{13.2.2}$$

The derivative of this function is

$$\vec{\mathbf{r}}'(t) = -\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}}$$

If we substitute the value $t = \pi/6$ into both functions we get

$$\vec{\mathbf{r}}\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}\,\hat{\mathbf{i}} + \frac{1}{2}\,\hat{\mathbf{j}}$$

and

$$\vec{\mathbf{r}}'\left(\frac{\pi}{6}\right) = -\frac{1}{2}\,\hat{\mathbf{i}} + \frac{\sqrt{3}}{2}\,\hat{\mathbf{j}}.$$

The graph of this function appears in Figure 13.2.1, along with the vectors $\vec{\mathbf{r}}\left(\frac{\pi}{6}\right)$ and $\vec{\mathbf{r}}'\left(\frac{\pi}{6}\right)$.

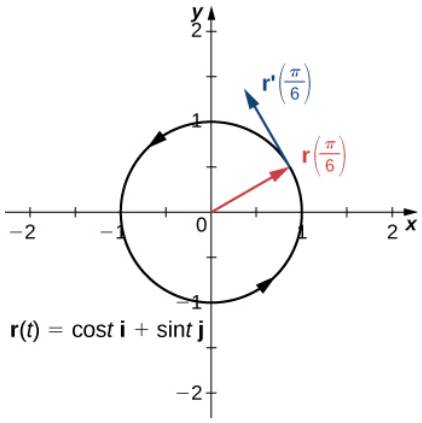

Figure 13.2.1: The tangent line at a point is calculated from the derivative of the vector-valued function $\vec{\mathbf{r}}(t)$.

Notice that the vector $\vec{\mathbf{r}}'\left(\frac{\pi}{6}\right)$ is tangent to the circle at the point corresponding to $t = \frac{\pi}{6}$. This is an example of a tangent vector to the plane curve defined by Equation 13.2.2.

Definition: principal unit tangent vector

Let C be a curve defined by a vector-valued function $\vec{\mathbf{r}}$, and assume that $\vec{\mathbf{r}}'(t)$ exists when $\mathrm{t} = \mathrm{t}_0$ A tangent vector $\vec{\mathbf{r}}$ at $t = t_0$ is any vector such that, when the tail of the vector is placed at point $\vec{\mathbf{r}}(t_0)$ on the graph, vector $\vec{\mathbf{r}}$ is tangent to curve C. Vector $\vec{\mathbf{r}}\,'(t_0)$ is an example of a tangent vector at point $t = t_0$. Furthermore, assume that $\vec{\mathbf{r}}'(t) \neq 0$. The principal unit tangent vector at t is defined to be

$$\vec{\mathbf{T}}(t) = \frac{\vec{\mathbf{r}}'(t)}{\|\vec{\mathbf{r}}'(t)\|},$$

provided $\|\vec{\mathbf{r}}'(t)\| \neq 0$.

The unit tangent vector is exactly what it sounds like: a unit vector that is tangent to the curve. To calculate a unit tangent vector, first find the derivative $\vec{\mathbf{r}}\,'(t)$. Second, calculate the magnitude of the derivative. The third step is to divide the derivative by its magnitude.

Example 13.2.4: Finding a Unit Tangent Vector

Find the unit tangent vector for each of the following vector-valued functions:

a. $\vec{\mathbf{r}}(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}$
b. $\vec{\mathbf{u}}(t) = (3t^2 + 2t)\,\hat{\mathbf{i}} + (2 - 4t^3)\,\hat{\mathbf{j}} + (6t + 5)\,\hat{\mathbf{k}}$

Solution

a.

$$\begin{aligned}
&\text{First step:} && \vec{\mathbf{r}}'(t) &&= -\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}} \\
&\text{Second step:} && \|\vec{\mathbf{r}}'(t)\| &&= \sqrt{(-\sin t)^2 + (\cos t)^2} = 1 \\
&\text{Third step:} && \vec{\mathbf{T}}(t) &&= \frac{\vec{\mathbf{r}}'(t)}{\|\vec{\mathbf{r}}'(t)\|} = \frac{-\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}}}{1} = -\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}}
\end{aligned}$$

b.

$$\begin{aligned}
&\text{First step:} && \vec{\mathbf{r}}'(t) &&= (6t + 2)\,\hat{\mathbf{i}} - 12t^2\,\hat{\mathbf{j}} + 6\,\hat{\mathbf{k}} \\
&\text{Second step:} && \|\vec{\mathbf{r}}'(t)\| &&= \sqrt{(6t + 2)^2 + (-12t^2)^2 + 6^2} \\
& && &&= \sqrt{144t^4 + 36t^2 + 24t + 40} \\
& && &&= 2\sqrt{36t^4 + 9t^2 + 6t + 10} \\
&\text{Third step:} && \vec{\mathbf{T}}(t) &&= \frac{\vec{\mathbf{r}}'(t)}{\|\vec{\mathbf{r}}'(t)\|} = \frac{(6t + 2)\,\hat{\mathbf{i}} - 12t^2\,\hat{\mathbf{j}} + 6\,\hat{\mathbf{k}}}{2\sqrt{36t^4 + 9t^2 + 6t + 10}} \\
& && &&= \frac{3t + 1}{\sqrt{36t^4 + 9t^2 + 6t + 10}}\,\hat{\mathbf{i}} - \frac{6t^2}{\sqrt{36t^4 + 9t^2 + 6t + 10}}\,\hat{\mathbf{j}} + \frac{3}{\sqrt{36t^4 + 9t^2 + 6t + 10}}\,\hat{\mathbf{k}}
\end{aligned}$$

Exercise 13.2.4

Find the unit tangent vector for the vector-valued function

$$\vec{\mathbf{r}}(t) = (t^2 - 3)\,\hat{\mathbf{i}} + (2t + 1)\,\hat{\mathbf{j}} + (t - 2)\,\hat{\mathbf{k}}.$$

Hint

Follow the same steps as in Example 13.2.4.

Answer

$$\vec{\mathbf{T}}(t) = \frac{2t}{\sqrt{4t^2 + 5}}\,\hat{\mathbf{i}} + \frac{2}{\sqrt{4t^2 + 5}}\,\hat{\mathbf{j}} + \frac{1}{\sqrt{4t^2 + 5}}\,\hat{\mathbf{k}}$$

Integrals of Vector-Valued Functions

We introduced antiderivatives of real-valued functions in Antiderivatives and definite integrals of real-valued functions in The Definite Integral. Each of these concepts can be extended to vector-valued functions. Also, just as we can calculate the derivative of a vector-valued function by differentiating the component functions separately, we can calculate the antiderivative in the same manner. Furthermore, the Fundamental Theorem of Calculus applies to vector-valued functions as well.

The antiderivative of a vector-valued function appears in applications. For example, if a vector-valued function represents the velocity of an object at time *t*, then its antiderivative represents position. Or, if the function represents the acceleration of the object at a given time, then the antiderivative represents its velocity.

Definition: Definite and Indefinite Integrals of Vector-Valued Functions

Let f, g, and h be integrable real-valued functions over the closed interval $[a, b]$.

1. The indefinite integral of a vector-valued function $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}$ is

$$\int [f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}]\,dt = \left[\int f(t)\,dt\right]\hat{\mathbf{i}} + \left[\int g(t)\,dt\right]\hat{\mathbf{j}}.$$

   The definite integral of a vector-valued function is

$$\int_a^b [f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}]\,dt = \left[\int_a^b f(t)\,dt\right]\hat{\mathbf{i}} + \left[\int_a^b g(t)\,dt\right]\hat{\mathbf{j}}.$$

2. The indefinite integral of a vector-valued function $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}$ is

$$\int [f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}]\,dt = \left[\int f(t)\,dt\right]\hat{\mathbf{i}} + \left[\int g(t)\,dt\right]\hat{\mathbf{j}} + \left[\int h(t)\,dt\right]\hat{\mathbf{k}}.$$

   The definite integral of the vector-valued function is

$$\int_a^b [f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}]\,dt = \left[\int_a^b f(t)\,dt\right]\hat{\mathbf{i}} + \left[\int_a^b g(t)\,dt\right]\hat{\mathbf{j}} + \left[\int_a^b h(t)\,dt\right]\hat{\mathbf{k}}.$$

Since the indefinite integral of a vector-valued function involves indefinite integrals of the component functions, each of these component integrals contains an integration constant. They can all be different. For example, in the two-dimensional case, we can have

$$\int f(t)\,dt = F(t) + C_1 \text{ and } \int g(t)\,dt = G(t) + C_2,$$

where F and G are antiderivatives of f and g, respectively. Then

$$\begin{aligned}\int [f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}]\,dt &= \left[\int f(t)\,dt\right]\hat{\mathbf{i}} + \left[\int g(t)\,dt\right]\hat{\mathbf{j}} \\ &= (F(t)+C_1)\,\hat{\mathbf{i}} + (G(t)+C_2)\,\hat{\mathbf{j}} \\ &= F(t)\,\hat{\mathbf{i}} + G(t)\,\hat{\mathbf{j}} + C_1\,\hat{\mathbf{i}} + C_2\,\hat{\mathbf{j}} \\ &= F(t)\,\hat{\mathbf{i}} + G(t)\,\hat{\mathbf{j}} + \vec{\mathbf{C}}\end{aligned}$$

where $\vec{\mathbf{C}} = C_1\,\hat{\mathbf{i}} + C_2\,\hat{\mathbf{j}}$. Therefore, the *integration constants* becomes a *constant vector*.

Example 13.2.5: Integrating Vector-Valued Functions

Calculate each of the following integrals:

a. $\displaystyle\int [(3t^2+2t)\,\hat{\mathbf{i}} + (3t-6)\,\hat{\mathbf{j}} + (6t^3+5t^2-4)\,\hat{\mathbf{k}}]\,dt$

b. $\displaystyle\int [\langle t, t^2, t^3\rangle \times \langle t^3, t^2, t\rangle]\,dt$

c. $\displaystyle\int_0^{\frac{\pi}{3}} [\sin 2t\,\hat{\mathbf{i}} + \tan t\,\hat{\mathbf{j}} + e^{-2t}\,\hat{\mathbf{k}}]\,dt$

Solution

a. We use the first part of the definition of the integral of a space curve:

$$\begin{aligned}\int [(3t^2+2t)\,\hat{\mathbf{i}} + (3t-6)\,\hat{\mathbf{j}} + (6t^3+5t^2-4)\,\hat{\mathbf{k}}]\,dt &= \left[\int 3t^2+2t\,dt\right]\hat{\mathbf{i}} + \left[\int 3t-6\,dt\right]\hat{\mathbf{j}} \\ &\quad + \left[\int 6t^3+5t^2-4\,dt\right]\hat{\mathbf{k}} \\ &= (t^3+t^2)\,\hat{\mathbf{i}} + \left(\frac{3}{2}t^2-6t\right)\hat{\mathbf{j}} \\ &\quad + \left(\frac{3}{2}t^4+\frac{5}{3}t^3-4t\right)\hat{\mathbf{k}} + \vec{\mathbf{C}}.\end{aligned}$$

b.

c. First calculate $\langle t, t^2, t^3\rangle \times \langle t^3, t^2, t\rangle$:

$$\begin{aligned}\langle t, t^2, t^3\rangle \times \langle t^3, t^2, t\rangle &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ t & t^2 & t^3 \\ t^3 & t^2 & t\end{vmatrix} \\ &= (t^2(t) - t^3(t^2))\,\hat{\mathbf{i}} - (t^2 - t^3(t^3))\,\hat{\mathbf{j}} + (t(t^2) - t^2(t^3))\,\hat{\mathbf{k}} \\ &= (t^3 - t^5)\,\hat{\mathbf{i}} + (t^6 - t^2)\,\hat{\mathbf{j}} + (t^3 - t^5)\,\hat{\mathbf{k}}.\end{aligned}$$

Next, substitute this back into the integral and integrate:

$$\begin{aligned}\int [\langle t, t^2, t^3\rangle \times \langle t^3, t^2, t\rangle]\,dt &= \int (t^3 - t^5)\,\hat{\mathbf{i}} + (t^6 - t^2)\,\hat{\mathbf{j}} + (t^3 - t^5)\,\hat{\mathbf{k}}\,dt \\ &= \left(\frac{t^4}{4} - \frac{t^6}{6}\right)\hat{\mathbf{i}} + \left(\frac{t^7}{7} - \frac{t^3}{3}\right)\hat{\mathbf{j}} + \left(\frac{t^4}{4} - \frac{t^6}{6}\right)\hat{\mathbf{k}} + \vec{\mathbf{C}}.\end{aligned}$$

d. Use the second part of the definition of the integral of a space curve:

$$\begin{aligned}\int_0^{\frac{\pi}{3}} [\sin 2t\,\hat{\mathbf{i}} + \tan t\,\hat{\mathbf{j}} + e^{-2t}\,\hat{\mathbf{k}}]\,dt &= \left[\int_0^{\frac{\pi}{3}} \sin 2t\,dt\right]\hat{\mathbf{i}} + \left[\int_0^{\frac{\pi}{3}} \tan t\,dt\right]\hat{\mathbf{j}} + \left[\int_0^{\frac{\pi}{3}} e^{-2t}\,dt\right]\hat{\mathbf{k}} \\ &= \left(-\tfrac{1}{2}\cos 2t\right)\Big|_0^{\pi/3}\hat{\mathbf{i}} - \left(\ln|\cos t|\right)\Big|_0^{\pi/3}\hat{\mathbf{j}} - \left(\tfrac{1}{2}e^{-2t}\right)\Big|_0^{\pi/3}\hat{\mathbf{k}} \\ &= \left(-\tfrac{1}{2}\cos\tfrac{2\pi}{3} + \tfrac{1}{2}\cos 0\right)\hat{\mathbf{i}} - \left(\ln\left(\cos\tfrac{\pi}{3}\right) - \ln(\cos 0)\right)\hat{\mathbf{j}} \\ &\quad - \left(\tfrac{1}{2}e^{-2\pi/3} - \tfrac{1}{2}e^{-2(0)}\right)\hat{\mathbf{k}} \\ &= \left(\tfrac{1}{4} + \tfrac{1}{2}\right)\hat{\mathbf{i}} - (-\ln 2)\,\hat{\mathbf{j}} - \left(\tfrac{1}{2}e^{-2\pi/3} - \tfrac{1}{2}\right)\hat{\mathbf{k}} \\ &= \tfrac{3}{4}\,\hat{\mathbf{i}} + (\ln 2)\,\hat{\mathbf{j}} + \left(\tfrac{1}{2} - \tfrac{1}{2}e^{-2\pi/3}\right)\hat{\mathbf{k}}.\end{aligned}$$

Exercise 13.2.5

Calculate the following integral:

$$\int_1^3 [(2t+4)\,\hat{\mathbf{i}} + (3t^2 - 4t)\,\hat{\mathbf{j}}]\,dt$$

Hint

Use the definition of the definite integral of a plane curve.

Answer

$$\int_1^3 [(2t+4)\,\hat{\mathbf{i}} + (3t^2 - 4t)\,\hat{\mathbf{j}}]\,dt = 16\,\hat{\mathbf{i}} + 10\,\hat{\mathbf{j}}$$

Summary

- To calculate the derivative of a vector-valued function, calculate the derivatives of the component functions, then put them back into a new vector-valued function.
- Many of the properties of differentiation of scalar functions also apply to vector-valued functions.
- The derivative of a vector-valued function $\vec{\mathbf{r}}(t)$ is also a tangent vector to the curve. The unit tangent vector $\vec{\mathbf{T}}(t)$ is calculated by dividing the derivative of a vector-valued function by its magnitude.
- The antiderivative of a vector-valued function is found by finding the antiderivatives of the component functions, then putting them back together in a vector-valued function.
- The definite integral of a vector-valued function is found by finding the definite integrals of the component functions, then putting them back together in a vector-valued function.

Key Equations

- **Derivative of a vector-valued function**

$$\vec{\mathbf{r}}'(t) = \lim_{\Delta t \to 0} \frac{\vec{\mathbf{r}}(t+\Delta t) - \vec{\mathbf{r}}(t)}{\Delta t}$$

- **Principal unit tangent vector**

$$\vec{\mathbf{T}}(t) = \frac{\vec{\mathbf{r}}'(t)}{\|\vec{\mathbf{r}}'(t)\|}$$

- **Indefinite integral of a vector-valued function**

$$\int [f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}]\,dt = \left[\int f(t)\,dt\right]\hat{\mathbf{i}} + \left[\int g(t)\,dt\right]\hat{\mathbf{j}} + \left[\int h(t)\,dt\right]\hat{\mathbf{k}}$$

- **Definite integral of a vector-valued function**

$$\int_a^b [f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}]\,dt = \left[\int_a^b f(t)\,dt\right]\hat{\mathbf{i}} + \left[\int_a^b g(t)\,dt\right]\hat{\mathbf{j}} + \left[\int_a^b h(t)\,dt\right]\hat{\mathbf{k}}$$

Glossary

definite integral of a vector-valued function

the vector obtained by calculating the definite integral of each of the component functions of a given vector-valued function, then using the results as the components of the resulting function

derivative of a vector-valued function

the derivative of a vector-valued function $\vec{\mathbf{r}}(t)$ is $\vec{\mathbf{r}}'(t) = \lim_{\Delta t \to 0} \frac{\vec{\mathbf{r}}(t+\Delta t) - \vec{\mathbf{r}}(t)}{\Delta t}$, provided the limit exists

indefinite integral of a vector-valued function

a vector-valued function with a derivative that is equal to a given vector-valued function

principal unit tangent vector

a unit vector tangent to a curve C

tangent vector

to $\vec{\mathbf{r}}(t)$ at $t = t_0$ any vector $\vec{\mathbf{v}}$ such that, when the tail of the vector is placed at point $\vec{\mathbf{r}}(t_0)$ on the graph, vector $\vec{\mathbf{v}}$ is tangent to curve C

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13.2E: EXERCISES FOR SECTION 13.2

TAKING DERIVATIVES OF VECTOR-VALUED FUNCTIONS

In questions 1 - 10, compute the derivative of each vector-valued function.

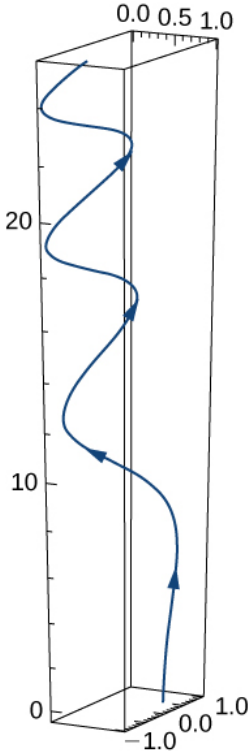
1) $\vec{\mathbf{r}}(t) = t^3\,\hat{\mathbf{i}} + 3t^2\,\hat{\mathbf{j}} + \frac{t^3}{6}\,\hat{\mathbf{k}}$

Answer

$\vec{\mathbf{r}}'(t) = 3t^2\,\hat{\mathbf{i}} + 6t\,\hat{\mathbf{j}} + \frac{1}{2}t^2\,\hat{\mathbf{k}}$

2) $\vec{\mathbf{r}}(t) = \sin(t)\,\hat{\mathbf{i}} + \cos(t)\,\hat{\mathbf{j}} + e^t\,\hat{\mathbf{k}}$

3) $\vec{\mathbf{r}}(t) = e^{-t}\,\hat{\mathbf{i}} + \sin(3t)\,\hat{\mathbf{j}} + 10\sqrt{t}\,\hat{\mathbf{k}}$. A sketch of the graph is shown here. Notice the varying periodic nature of the graph.



Answer

$\vec{\mathbf{r}}'(t) = -e^{-t}\,\hat{\mathbf{i}} + 3\cos(3t)\,\hat{\mathbf{j}} + \frac{5}{\sqrt{t}}\,\hat{\mathbf{k}}$

4) $\vec{\mathbf{r}}(t) = e^t\,\hat{\mathbf{i}} + 2e^t\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$

5) $\vec{\mathbf{r}}(t) = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$

Answer

$\vec{\mathbf{r}}'(t) = \langle 0, 0, 0\rangle = \vec{\mathbf{0}}$

6) $\vec{\mathbf{r}}(t) = te^t\,\hat{\mathbf{i}} + t\ln(t)\,\hat{\mathbf{j}} + \sin(3t)\,\hat{\mathbf{k}}$

7) $\vec{\mathbf{r}}(t) = \left\langle \frac{1}{t+1},\ \arctan(t),\ \ln t^3 \right\rangle$

Answer

$\vec{\mathbf{r}}'(t) = \langle \frac{-1}{(t+1)^2}, \frac{1}{1+t^2}, \frac{3}{t}\rangle$

8) $\vec{\mathbf{r}}(t) = \langle \tan 2t,\ \sec 2t,\ \sin^2 t\rangle$

9) $\vec{\mathbf{r}}(t) = \langle 3,\ 4\sin(3t),\ t\cos(t)\rangle$

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Answer

$\vec{\mathbf{r}}'(t) = \langle 0, 12\cos(3t), \cos t - t\sin t\rangle$

10) $\vec{\mathbf{r}}(t) = t^2\,\hat{\mathbf{i}} + te^{-2t}\,\hat{\mathbf{j}} - 5e^{-4t}\,\hat{\mathbf{k}}$

11) a. Describe and sketch the curve represented by the vector-valued function $\vec{\mathbf{r}}(t) = \langle 6t, 6t - t^2\rangle$.

b. Locate the highest point on the curve $\vec{\mathbf{r}}(t) = \langle 6t, 6t - t^2\rangle$ and give the value of the function at this point.

Answer

b. $\vec{\mathbf{r}}(t) = \langle 18, 9\rangle$ at $t = 3$

12) Find the parametric equations of the tangent line to the curve $\vec{\mathbf{r}}(t) = \langle t, t^2, t\rangle$ at $t = 2$.

13) Find the parametric equations of the tangent line to the curve $\vec{\mathbf{r}}(t) = \langle e^t, e^{-t}, 0\rangle$ at $t = 0$.

Answer

$x = 1 + t, \quad y = 1 - t, \quad z = 0$

14) Compute the first, second, and third derivatives of $\vec{\mathbf{r}}(t) = 3t\,\hat{\mathbf{i}} + 6\ln(t)\,\hat{\mathbf{j}} + 5e^{-3t}\,\hat{\mathbf{k}}$.

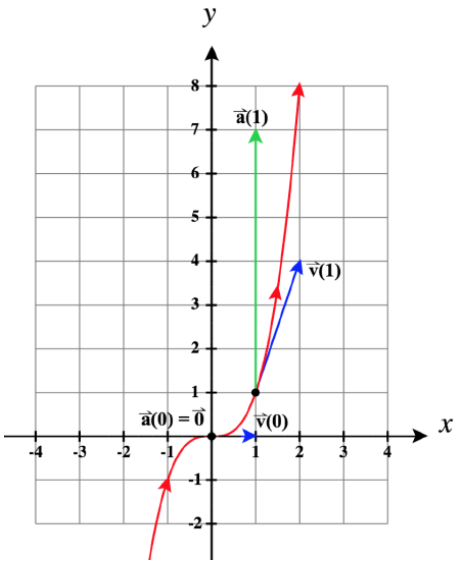
DESCRIBING MOTION WITH VECTOR-VALUED FUNCTIONS

In questions 15 - 17, find the velocity and acceleration at the given times, plot the graph of the position function, and draw in the velocity and acceleration vectors at the corresponding locations on the curve.

15) $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + t^3\,\hat{\mathbf{j}}$, at $t = 0$ and at $t = 1$

Answer

	$t = 0:$	$t = 1:$
$\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + t^3\,\hat{\mathbf{j}},$	$\vec{\mathbf{r}}(0) = \vec{\mathbf{0}},$	$\vec{\mathbf{r}}(1) = \hat{\mathbf{i}} + \hat{\mathbf{j}}$
$\vec{\mathbf{v}}(t) = \hat{\mathbf{i}} + 3t^2\,\hat{\mathbf{j}},$	$\vec{\mathbf{v}}(0) = \hat{\mathbf{i}},$	$\vec{\mathbf{v}}(1) = \hat{\mathbf{i}} + 3\hat{\mathbf{j}}$
$\vec{\mathbf{a}}(t) = 6t\,\hat{\mathbf{j}},$	$\vec{\mathbf{a}}(0) = \vec{\mathbf{0}},$	$\vec{\mathbf{a}}(1) = 6\,\hat{\mathbf{j}}$

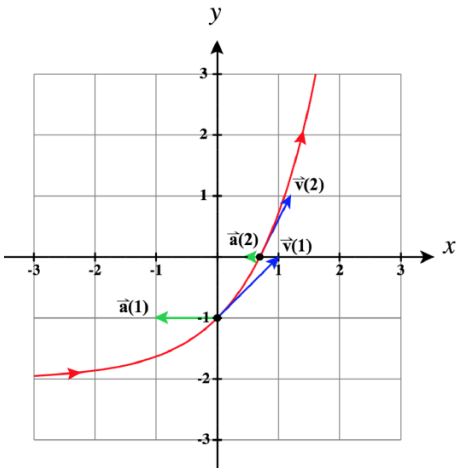


16) $\vec{\mathbf{r}}(t) = \cos t\,\hat{\mathbf{i}} + \sin 3t\,\hat{\mathbf{j}}$, at $t = 0$, at $t = \frac{\pi}{4}$, and at $t = \frac{\pi}{2}$

17) $\vec{\mathbf{r}}(t) = \ln t\,\hat{\mathbf{i}} + (t-2)\,\hat{\mathbf{j}}$, at $t = 1$ and at $t = 2$

Answer

	$t=1:$	$t=2:$
$\vec{\mathbf{r}}(t)=(\ln t)\,\hat{\mathbf{i}}+(t-2)\,\hat{\mathbf{j}},$	$\vec{\mathbf{r}}(1)=-\hat{\mathbf{j}},$	$\vec{\mathbf{r}}(2)=(\ln 2)\,\hat{\mathbf{i}}$
$\vec{\mathbf{v}}(t)=\frac{1}{t}\,\hat{\mathbf{i}}+\hat{\mathbf{j}},$	$\vec{\mathbf{v}}(1)=\hat{\mathbf{i}}+\hat{\mathbf{j}},$	$\vec{\mathbf{v}}(2)=\frac{1}{2}\hat{\mathbf{i}}+\hat{\mathbf{j}}$
$\vec{\mathbf{a}}(t)=-\frac{1}{t^2}\,\hat{\mathbf{i}},$	$\vec{\mathbf{a}}(1)=-\hat{\mathbf{i}},$	$\vec{\mathbf{a}}(2)=-\frac{1}{4}\,\hat{\mathbf{i}}$



In questions 18 - 24, find the velocity, speed, and acceleration of a particle with the given position function. Remember that the speed is the magnitude of the velocity represented by $\|\vec{\mathbf{v}}(t)\|$ or $\|\vec{\mathbf{r}}'(t)\|$.

18) $\vec{\mathbf{r}}(t)=e^{2t}\,\hat{\mathbf{i}}+\sin t\,\hat{\mathbf{j}}$

19) $\vec{\mathbf{r}}(t)=\cos t^3\,\hat{\mathbf{i}}+\sin t^3\,\hat{\mathbf{j}}$

Answer

$\vec{\mathbf{v}}(t)=-3t^2\sin t^3\,\hat{\mathbf{i}}+3t^2\cos t^3\,\hat{\mathbf{j}},\quad \text{speed}(t)=\|\vec{\mathbf{v}}(t)\|=3t^2,\quad \vec{\mathbf{a}}(t)=\left(-6t\sin t^3-9t^4\cos t^3\right)\hat{\mathbf{i}}+\left(6t\cos t^3-9t^4\sin t^3\right)\hat{\mathbf{j}}$

20) $\vec{\mathbf{r}}(t)=\langle e^t,e^{-t},0\rangle$

21) $\vec{\mathbf{r}}(t)=\langle t+\cos t,t-\sin t\rangle$

Answer

$\vec{\mathbf{v}}(t)=\langle 1-\sin t,1-\cos t\rangle,\quad \text{speed}(t)=\|\vec{\mathbf{v}}(t)\|=\sqrt{3-2(\sin t+\cos t)},\quad \vec{\mathbf{a}}(t)=\langle -\cos t,\sin t\rangle$

22) $\vec{\mathbf{r}}(t)=\dfrac{2t-1}{2t+1}\,\hat{\mathbf{i}}+\ln(1-4t^2)\,\hat{\mathbf{j}}$

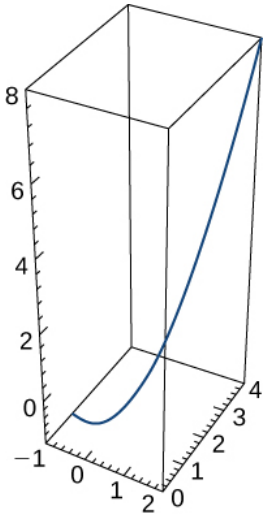
23) $\vec{\mathbf{r}}(t)=\cos 3t\,\hat{\mathbf{i}}+\sin 3t\,\hat{\mathbf{j}}+0.5t\,\hat{\mathbf{k}}$

Answer

$\vec{\mathbf{v}}(t)=-3\sin 3t\,\hat{\mathbf{i}}+3\cos 3t\,\hat{\mathbf{j}}+0.5\,\hat{\mathbf{k}},\quad \text{speed}(t)=\|\vec{\mathbf{v}}(t)\|=\sqrt{9.25}\ \text{units/sec},\quad \vec{\mathbf{a}}(t)=-9\cos 3t\,\hat{\mathbf{i}}-9\sin 3t\,\hat{\mathbf{j}}$

24) $\vec{\mathbf{r}}(t)=e^{-t}\,\hat{\mathbf{i}}+(\ln t)\,\hat{\mathbf{j}}+(\sin 7t)\,\hat{\mathbf{k}}$

25) Consider the position vector for a particle to be $\vec{\mathbf{r}}(t)=t\,\hat{\mathbf{i}}+t^2\,\hat{\mathbf{j}}+t^3\,\hat{\mathbf{k}}$. The graph is shown here:



a. Find the velocity vector at any time.

b. Find the speed of the particle at time $t = 2$ sec.

c. Find the acceleration at time $t = 2$ sec.

Answer

a. $\vec{\mathbf{v}}(t) = \hat{\mathbf{i}} + 2t\hat{\mathbf{j}} + 3t^2\hat{\mathbf{k}}$

b. $\sqrt{161}$ units/sec

Note that $\text{speed}(t) = \|\vec{\mathbf{v}}(t)\| = \sqrt{1^2 + (2t)^2 + (3t^2)^2} = \sqrt{1 + 4t^2 + 9t^4}$.

Hence, $\text{speed}(2) = \sqrt{1 + 16 + 9(16)} = \sqrt{161}$ units/sec.

c. Since $\vec{\mathbf{a}}(t) = 2\hat{\mathbf{j}} + 6t\hat{\mathbf{k}}$, $\vec{\mathbf{a}}(2) = 2\hat{\mathbf{j}} + 12\hat{\mathbf{k}}$

26) A particle travels along the path of an ellipse with the equation $\vec{\mathbf{r}}(t) = \cos t\,\hat{\mathbf{i}} + 2\sin t\,\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$. Find the following:

a. Velocity of the particle

Answer

$\vec{\mathbf{v}}(t) = \langle -\sin t, 2\cos t, 0\rangle$

b. Speed of the particle at $t = \frac{\pi}{4}$

c. Acceleration of the particle at $t = \frac{\pi}{4}$

Answer

$\vec{\mathbf{a}}(t) = \langle -\frac{\sqrt{2}}{2}, -\sqrt{2}, 0\rangle$

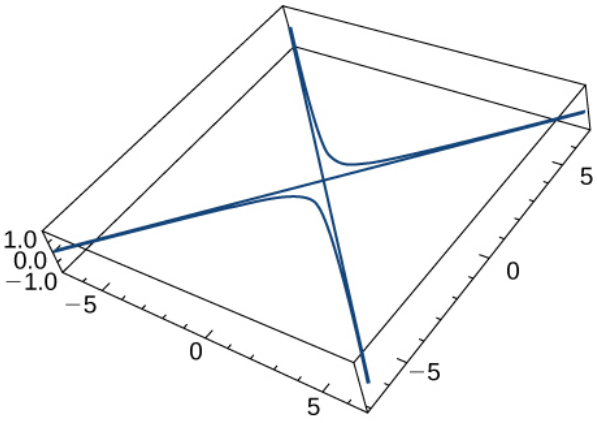
27) Show that if the speed of a particle traveling along a curve represented by a vector-valued function is constant, then the velocity function is always perpendicular to the acceleration function.

Answer

$$\begin{aligned} \|\vec{\mathbf{v}}(t)\| &= k \\ \vec{\mathbf{v}}(t)\cdot\vec{\mathbf{v}}(t) &= k^2 \\ \frac{d}{dt}\left(\vec{\mathbf{v}}(t)\cdot\vec{\mathbf{v}}(t)\right) &= \frac{d}{dt}\left(k^2\right) = 0 \\ \vec{\mathbf{v}}(t)\cdot\vec{\mathbf{v}}'(t) + \vec{\mathbf{v}}'(t)\cdot\vec{\mathbf{v}}(t) &= 0 \\ 2\vec{\mathbf{v}}(t)\cdot\vec{\mathbf{v}}'(t) &= 0 \\ \vec{\mathbf{v}}(t)\cdot\vec{\mathbf{v}}'(t) &= 0 \end{aligned}$$

The last statement implies that the velocity and acceleration are perpendicular or orthogonal.

28) Given the vector-valued function $\vec{\mathbf{r}}(t) = \langle \tan t, \sec t, 0\rangle$ (graph is shown here), find the following:



a. Velocity

Answer

$\vec{\mathbf{v}}(t) = \langle \sec^2 t,\ \sec t \tan t \rangle$

b. Speed

Answer

$\|\vec{\mathbf{v}}(t)\| = \sqrt{\sec^4 t + \sec^2 t \tan^2 t} = \sqrt{(\sec^2 t)(\sec^2 t + \tan^2 t)}$

c. Acceleration

Answer

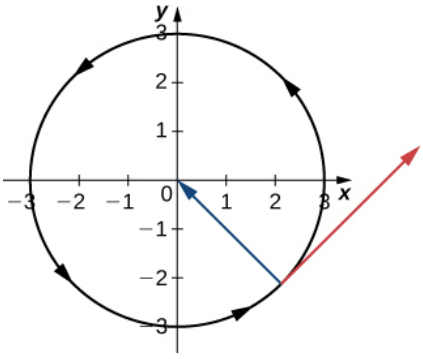
$\vec{\mathbf{a}}(t) = \langle 2\sec^2 t \tan t,\ \sec t \tan^2 t + \sec^3 t \rangle$

29) Find the minimum speed of a particle traveling along the curve $\vec{\mathbf{r}}(t) = \langle t + \cos t, t - \sin t \rangle$, where $t \in [0, 2\pi)$. Then also find its maximum speed on this interval.

Answer

Min. speed is $\sqrt{3 - 2\sqrt{2}} \approx 0.41421$ when $t = \frac{\pi}{4}$.
Max. speed is $\sqrt{3 + 2\sqrt{2}} \approx 2.41421$ when $t = \frac{5\pi}{4}$.

For questions 30 - 31, consider a particle that moves on a circular path of radius b according to the function $\vec{\mathbf{r}}(t) = b\cos(\omega t)\,\hat{\mathbf{i}} + b\sin(\omega t)\,\hat{\mathbf{j}}$, where ω is the angular velocity, $\frac{d\theta}{dt}$.



30) Show that the speed of the particle is proportional to the angular velocity.

31) Find the velocity function and show that $\vec{\mathbf{v}}(t)$ is always orthogonal to $\vec{\mathbf{r}}(t)$.

Answer

$\vec{\mathbf{r}}'(t) = -b\omega\sin(\omega t)\,\hat{\mathbf{i}} + b\omega\cos(\omega t)\,\hat{\mathbf{j}}$. To show orthogonality, note that $\vec{\mathbf{r}}'(t) \cdot \vec{\mathbf{r}}(t) = 0$.

SMOOTHNESS OF VECTOR-VALUED FUNCTIONS

For questions 32 - 40,

a. Determine any values of t at which $\vec{\mathbf{r}}$ is not smooth.

b. Determine the open intervals on which $\vec{\mathbf{r}}$ is smooth.

c. Graph the vector-valued function and describe its behavior at the points where it is not smooth.

32) $\vec{\mathbf{r}}(t) = \langle 3t, 5t^2 - 1 \rangle$

33) $\vec{\mathbf{r}}(t) = t^3\,\hat{\mathbf{i}} + 5t^2\,\hat{\mathbf{j}}$

Answer

a. $\vec{\mathbf{r}}$ is not smooth at $t = 0$, since $\vec{\mathbf{r}}'(0) = \vec{\mathbf{0}}$.
b. $\vec{\mathbf{r}}$ is smooth on the open intervals $(-\infty, 0)$ and $(0, \infty)$.
c. There is a cusp when $t = 0$.

34) $\vec{\mathbf{r}}(t) = \langle 5,\ 2\sin(t),\ \cos(t) \rangle$

35) $\vec{\mathbf{r}}(t) = \langle t^3 - 3t^2, 7 \rangle$

Answer

a. $\vec{\mathbf{r}}$ is not smooth at $t = 0$ and $t = 2$, since $\vec{\mathbf{r}}'(0) = \vec{\mathbf{0}}$ and $\vec{\mathbf{r}}'(2) = \vec{\mathbf{0}}$.
b. $\vec{\mathbf{r}}$ is smooth on the open intervals $(-\infty, 0)$, $(0, 2)$, and $(2, \infty)$.
c. The motion on the curve reverses along the same path at both $t = 0$ and $t = 2$.

36) $\vec{\mathbf{r}}(t) = t^2\,\hat{\mathbf{i}} + t^3\,\hat{\mathbf{j}} - 5e^{-4t}\,\hat{\mathbf{k}}$

37) $\vec{\mathbf{r}}(t) = \left\langle \ln(t^2 + 4t + 5),\ \dfrac{t^3}{3} - 4t,\ 5 \right\rangle$

Answer

a. $\vec{\mathbf{r}}$ is not smooth at $t = -2$, since $\vec{\mathbf{r}}'(-2) = \vec{\mathbf{0}}$.
Since the domain of $\vec{\mathbf{r}}$ is $(-\infty, \infty)$, this is all we have to remove.
b. $\vec{\mathbf{r}}$ is smooth on the open intervals $(-\infty, -2)$ and $(-2, \infty)$.
c. There is a cusp when $t = -2$.

38) $\vec{\mathbf{r}}(t) = (5\cos t - \cos 5t)\,\hat{\mathbf{i}} + (5\sin t - \sin 5t)\,\hat{\mathbf{j}}$, for $0 \le t \le 2\pi$

39) $\vec{\mathbf{r}}(t) = \sqrt{t^3 + 9t^2}\,\hat{\mathbf{i}} + (t^2 + 12t)\,\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$

Answer

a. The domain of $\vec{\mathbf{r}}$ is $[-9, \infty)$.
And $\vec{\mathbf{r}}$ is not smooth at $t = -6$, since $\vec{\mathbf{r}}'(-6) = \vec{\mathbf{0}}$.
The domain of $\vec{\mathbf{r}}'$ is $(-9, \infty)$, since $\vec{\mathbf{r}}'$ is undefined at $t = -9$.
b. $\vec{\mathbf{r}}$ is smooth on the open intervals $(-9, -6)$ and $(-6, \infty)$.
c. There is a cusp when $t = -6$.

40) $\vec{\mathbf{r}}(t) = \cos^3 t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}$, for $0 \le t \le 2\pi$

Answer

a. The domain of $\vec{\mathbf{r}}$ is $(-\infty, \infty)$.
$\vec{\mathbf{r}}'(t) = -3(\cos^2 t)(\sin t)\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}}$. It's domain is also $(-\infty, \infty)$.
But note that both components have a factor of $\cos t$, so both components will be 0 when $\cos t = 0$.
Therefore, $\vec{\mathbf{r}}$ is not smooth at $t = \frac{\pi}{2}$ and at $t = \frac{3\pi}{2}$, since $\vec{\mathbf{r}}'\left(\frac{\pi}{2}\right) = \vec{\mathbf{0}}$ and $\vec{\mathbf{r}}'\left(\frac{3\pi}{2}\right) = \vec{\mathbf{0}}$. Note then that $\vec{\mathbf{r}}$ is not smooth for any odd multiple of $\frac{\pi}{2}$, that is for $t = \frac{(2n+1)\pi}{2}$, for any integer value n.
b. $\vec{\mathbf{r}}$ is smooth on the open intervals $\left(\dfrac{(2n-1)\pi}{2}, \dfrac{(2n+1)\pi}{2}\right)$, for any integer value n.
c. There is a cusp when $t = \dfrac{(2n+1)\pi}{2}$, for any integer value n.

PROPERTIES OF THE DERIVATIVE

For questions, 41 - 43, evaluate each expression given that $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}} - t^4\,\hat{\mathbf{k}}$ and $\vec{\mathbf{s}}(t) = \sin(t)\,\hat{\mathbf{i}} + e^t\,\hat{\mathbf{j}} + \cos(t)\,\hat{\mathbf{k}}$

-

41) $\frac{d}{dt}\left[\vec{\mathbf{r}}(t^2)\right]$

Answer

$\frac{d}{dt}\left[\vec{\mathbf{r}}(t^2)\right] = \langle 2t, 4t^3, -8t^7\rangle$

42) $\frac{d}{dt}\left[t^2 \cdot \vec{\mathbf{s}}(t)\right]$

43) $\frac{d}{dt}\left[\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{s}}(t)\right]$

Answer

$\frac{d}{dt}\left[\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{s}}(t)\right] = \sin t + 2te^t - 4t^3\cos t + t\cos t + t^2e^t + t^4\sin t$

44) Find $\vec{\mathbf{r}}'(t) \cdot \vec{\mathbf{r}}''(t)$ *for* $\vec{\mathbf{r}}(t) = -3t^5\,\hat{\mathbf{i}} + 5t\,\hat{\mathbf{j}} + 2t^2\,\hat{\mathbf{k}}$.

Answer

$\vec{\mathbf{r}}'(t) \cdot \vec{\mathbf{r}}''(t) = 900t^7 + 16t$

45) Given $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + 3t\,\hat{\mathbf{j}} + t^2\,\hat{\mathbf{k}}$ and $\vec{\mathbf{u}}(t) = 4t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}} + t^3\,\hat{\mathbf{k}}$, find $\frac{d}{dt}(\vec{\mathbf{r}}(t) \times \vec{\mathbf{u}}(t))$.

46) Evaluate $\frac{d}{dt}\left[\vec{\mathbf{u}}(t) \times \vec{\mathbf{u}}'(t)\right]$ given $\vec{\mathbf{u}}(t) = t^2\,\hat{\mathbf{i}} - 2t\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$.

Answer

$\frac{d}{dt}\left[\vec{\mathbf{u}}(t) \times \vec{\mathbf{u}}'(t)\right] = 0\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + 4t\,\hat{\mathbf{k}}$

47) Given $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + 2\sin t\,\hat{\mathbf{j}} + 2\cos t\,\hat{\mathbf{k}}$ and $\vec{\mathbf{u}}(t) = \frac{1}{t}\,\hat{\mathbf{i}} + 2\sin t\,\hat{\mathbf{j}} + 2\cos t\,\hat{\mathbf{k}}$, find the following:

a. $\vec{\mathbf{r}}(t) \times \vec{\mathbf{u}}(t)$

Answer

$\vec{\mathbf{r}}(t) \times \vec{\mathbf{u}}(t) = \langle 0,\ 2(\cos t)\left(\frac{1}{t} - t\right),\ 2(\sin t)\left(t - \frac{1}{t}\right)\rangle$

b. $\frac{d}{dt}\left(\vec{\mathbf{r}}(t) \times \vec{\mathbf{u}}(t)\right)$

Answer

$\frac{d}{dt}\left(\vec{\mathbf{r}}(t) \times \vec{\mathbf{u}}(t)\right) = \langle 0,\ 2(\sin t)\left(t - \frac{1}{t}\right) - 2(\cos t)\left(1 + \frac{1}{t^2}\right),\ 2(\sin t)\left(1 + \frac{1}{t^2}\right) + 2(\cos t)\left(t - \frac{1}{t}\right)\rangle$

c. Now, use the product rule for the derivative of the cross product of two vectors and show this result is the same as the answer for the preceding problem.

UNIT TANGENT VECTORS

For questions 48 - 51, find a *unit tangent vector* at the indicated value of t.

48) $\vec{\mathbf{r}}(t) = 3t^3\,\hat{\mathbf{i}} + 2t^2\,\hat{\mathbf{j}} + \frac{1}{t}\,\hat{\mathbf{k}};\quad t = 1$

49) $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + \sin(2t)\,\hat{\mathbf{j}} + \cos(3t)\,\hat{\mathbf{k}};\quad t = \frac{\pi}{3}$

Answer

$\vec{\mathbf{r}}'\left(\frac{\pi}{3}\right) = \langle 1, -1, 0\rangle$ is a tangent vector, so a unit tangent vector would be:

$\frac{1}{\sqrt{2}}\langle 1, -1, 0\rangle \quad = \quad \langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0\rangle$

50) $\vec{\mathbf{r}}(t) = \cos(2t)\,\hat{\mathbf{i}} + 2\sin t\,\hat{\mathbf{j}} + t^2\,\hat{\mathbf{k}};\quad t = \frac{\pi}{2}$

51) $\vec{\mathbf{r}}(t) = 3e^t\,\hat{\mathbf{i}} + 2e^{-3t}\,\hat{\mathbf{j}} + 4e^{2t}\,\hat{\mathbf{k}};\quad t = \ln(2)$

Answer

$\vec{\mathbf{r}}'(\ln(2)) = \langle 6, -\frac{3}{4}, 32\rangle$ is a tangent vector, so a unit tangent vector would be:

$$\frac{1}{\sqrt{1060.5625}}\left\langle 6, -\frac{3}{4}, 32\right\rangle \quad = \quad \left\langle \frac{24\sqrt{16969}}{16969}, -\frac{12\sqrt{16969}}{67876}, \frac{128\sqrt{16969}}{16969}\right\rangle$$

For questions 52 - 58, find the unit tangent vector $\vec{\mathbf{T}}(t)$ for the following parameterized curves.

52) $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + 3t\,\hat{\mathbf{j}} + t^2\,\hat{\mathbf{k}}$

53) $\vec{\mathbf{r}}(t) = 6\,\hat{\mathbf{i}} + \cos(3t)\,\hat{\mathbf{j}} + 3\sin(4t)\,\hat{\mathbf{k}}, \quad 0 \le t < 2\pi$

Answer

$$\vec{\mathbf{T}}(t) = \frac{1}{\sqrt{9\sin^2(3t) + 144\cos^2(4t)}}\langle 0, -3\sin(3t), 12\cos(4t)\rangle$$

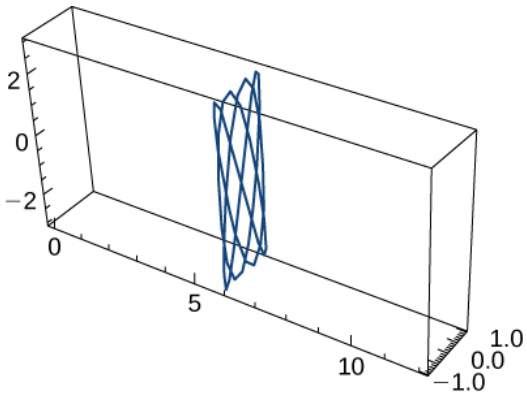
54) $\vec{\mathbf{r}}(t) = \langle t\cos t, t\sin t\rangle$

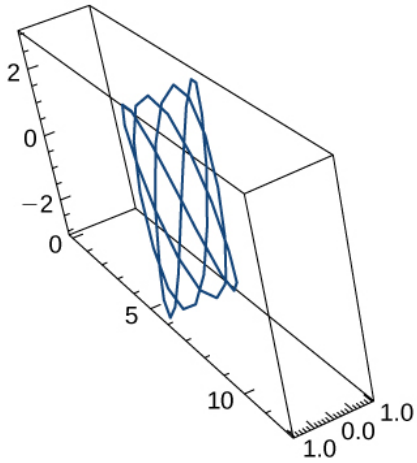
55) $\vec{\mathbf{r}}(t) = \langle t+1, 2t+1, 2t+2\rangle$

Answer

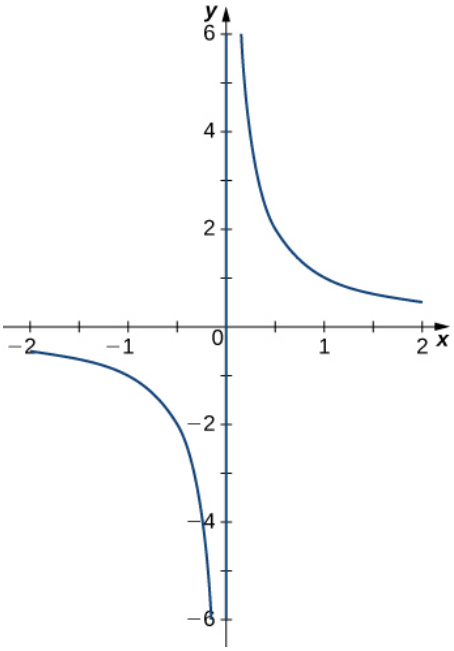
$\vec{\mathbf{T}}(t) = \frac{1}{3}\langle 1, 2, 2\rangle$

56) $\vec{\mathbf{r}}(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} + \sin t\,\hat{\mathbf{k}}, \quad 0 \le t < 2\pi$. Two views of this curve are presented here:





57) $\vec{\mathbf{r}}(t) = \left\langle t, \frac{1}{t}\right\rangle$. The graph is shown here:



Answer

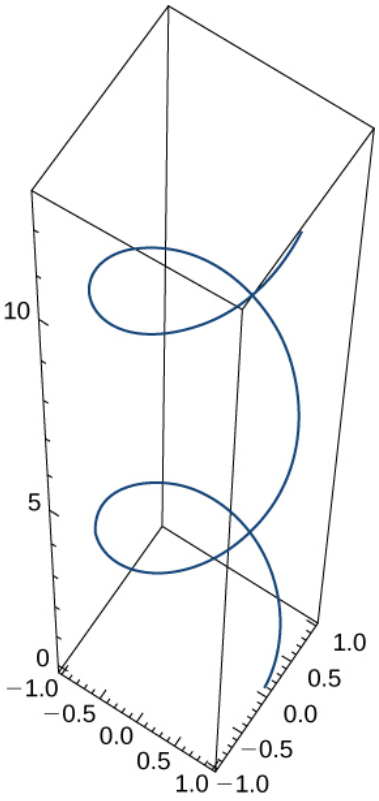
$$\vec{\mathbf{T}}(t) = \left\langle \frac{t^2}{\sqrt{t^4+1}}, \frac{-1}{\sqrt{t^4+1}} \right\rangle$$

58) $\vec{\mathbf{r}}(t) = 3\cos(4t)\,\hat{\mathbf{i}} + 3\sin(4t)\,\hat{\mathbf{j}} + 5t\,\hat{\mathbf{k}}, \quad 1 \le t \le 2$

Answer

$$\vec{\mathbf{T}}(t) = -\tfrac{12}{13}\sin(4t)\,\hat{\mathbf{i}} + \tfrac{12}{13}\cos(4t)\,\hat{\mathbf{j}} + \tfrac{5}{13}\,\hat{\mathbf{k}}$$

59) A particle travels along the path of a helix with the equation $\vec{\mathbf{r}}(t) = \cos(t)\,\hat{\mathbf{i}} + \sin(t)\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$. See the graph presented here:



Find the following:

a. Velocity of the particle at any time

Answer

$\vec{\mathbf{v}}(t) = \langle -\sin t, \cos t, 1 \rangle$

b. Speed of the particle at any time

c. Acceleration of the particle at any time

Answer

$\vec{\mathbf{a}}(t) = -\cos t\,\hat{\mathbf{i}} - \sin t\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}$

d. Find the unit tangent vector for the helix.

INTEGRATION OF VECTOR-VALUED FUNCTIONS

Evaluate the following integrals:

60) $\int \left(e^t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} + \frac{1}{2t-1}\,\hat{\mathbf{k}}\right) dt$

61) $\int_0^1 \vec{\mathbf{r}}(t)\,dt$, where $\vec{\mathbf{r}}(t) = \left\langle \sqrt[3]{t}, \frac{1}{t+1}, e^{-t} \right\rangle$

Answer

$\frac{3}{4}\,\hat{\mathbf{i}} + \ln(2)\,\hat{\mathbf{j}} + (1 - \frac{1}{e})\,\hat{\mathbf{k}}$

62) Evaluate $\int_0^3 \|t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}}\| dt$.

Answer

$\frac{1}{3}(10^{\frac{3}{2}} - 1)$

63) The acceleration function, initial velocity, and initial position of a particle are

$$\vec{\mathbf{a}}(t) = -5\cos t\,\hat{\mathbf{i}} - 5\sin t\,\hat{\mathbf{j}},$$
$$\vec{\mathbf{v}}(0) = 9\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}}, \quad \text{and}$$
$$\vec{\mathbf{r}}(0) = 5\,\hat{\mathbf{i}}$$

Find $\vec{\mathbf{v}}(t)$ and $\vec{\mathbf{r}}(t)$.

Answer

$\vec{\mathbf{v}}(t) = (9 - 5\sin t)\,\hat{\mathbf{i}} + (-3 + 5\cos t)\,\hat{\mathbf{j}}$
$\vec{\mathbf{r}}(t) = (9t + 5\cos t)\,\hat{\mathbf{i}} + (-3t + 5\sin t)\,\hat{\mathbf{j}}$

64) Find the antiderivative of $\vec{\mathbf{r}}'(t) = \cos(2t)\,\hat{\mathbf{i}} - 2\sin t\,\hat{\mathbf{j}} + \frac{1}{1+t^2}\,\hat{\mathbf{k}}$ that satisfies the initial condition $\vec{\mathbf{r}}(0) = 3\,\hat{\mathbf{i}} - 2\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$.

65) An object starts from rest at point $P(1, 2, 0)$ and moves with an acceleration of $\vec{\mathbf{a}}(t) = \hat{\mathbf{j}} + 2\,\hat{\mathbf{k}}$, where $\|\vec{\mathbf{a}}(t)\|$ is measured in feet per second per second. Find the location of the object after $t = 2$ sec.

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13.3: Arc Length and Curvature

Learning Objectives

- Determine the length of a particle's path in space by using the arc-length function.
- Explain the meaning of the curvature of a curve in space and state its formula.
- Describe the meaning of the normal and binormal vectors of a curve in space.

In this section, we study formulas related to curves in both two and three dimensions, and see how they are related to various properties of the same curve. For example, suppose a vector-valued function describes the motion of a particle in space. We would like to determine how far the particle has traveled over a given time interval, which can be described by the arc length of the path it follows. Or, suppose that the vector-valued function describes a road we are building and we want to determine how sharply the road curves at a given point. This is described by the curvature of the function at that point. We explore each of these concepts in this section.

Arc Length for Vector Functions

We have seen how a vector-valued function describes a curve in either two or three dimensions. Recall that the formula for the arc length of a curve defined by the parametric functions $x = x(t), y = y(t), t_1 \leq t \leq t_2$ is given by

$$s = \int_{t_1}^{t_2} \sqrt{(x'(t))^2 + (y'(t))^2}dt.$$

In a similar fashion, if we define a smooth curve using a vector-valued function $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}$, where $a \leq t \leq b$, the arc length is given by the formula

$$s = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}dt.$$

In three dimensions, if the vector-valued function is described by $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}$ over the same interval $a \leq t \leq b$, the arc length is given by

$$s = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}dt.$$

Theorem: Arc-Length Formulas for Plane and Space curves

Plane curve: Given a smooth curve C defined by the function $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}$, where t lies within the interval $[a, b]$, the arc length of C over the interval is

$$s = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}dt \tag{13.3.1}$$

$$= \int_a^b \|\vec{\mathbf{r}}'(t)\|dt. \tag{13.3.2}$$

Space curve: Given a smooth curve C defined by the function $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}$, where t lies within the interval $[a, b]$, the arc length of C over the interval is

$$s = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}dt \tag{13.3.3}$$

$$= \int_a^b \|\vec{\mathbf{r}}'(t)\|dt. \tag{13.3.4}$$

The two formulas are very similar; they differ only in the fact that a space curve has three component functions instead of two. Note that the formulas are defined for smooth curves: curves where the vector-valued function $\vec{\mathbf{r}}(t)$ is differentiable with a non-zero derivative. The smoothness condition guarantees that the curve has no cusps (or corners) that could make the formula problematic.

Example 13.3.1: Finding the Arc Length

Calculate the arc length for each of the following vector-valued functions:

a. $\vec{\mathbf{r}}(t) = (3t-2)\,\hat{\mathbf{i}} + (4t+5)\,\hat{\mathbf{j}}, \quad 1 \le t \le 5$

b. $\vec{\mathbf{r}}(t) = \langle t\cos t, t\sin t, 2t\rangle, 0 \le t \le 2\pi$

Solution

a. Using Equation 13.3.2, $\vec{\mathbf{r}}'(t) = 3\,\hat{\mathbf{i}} + 4\,\hat{\mathbf{j}}$, so

$$\begin{aligned} s &= \int_a^b \|\vec{\mathbf{r}}'(t)\|dt \\ &= \int_1^5 \sqrt{3^2+4^2}dt \\ &= \int_1^5 5dt = 5t\big|_1^5 = 20. \end{aligned}$$

b. Using Equation 13.3.4, $\vec{\mathbf{r}}'(t) = \langle \cos t - t\sin t, \sin t + t\cos t, 2\rangle$, so

$$\begin{aligned} s &= \int_a^b \| \vec{\mathbf{r}}'(t) \| dt \\ &= \int_0^{2\pi} \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 2^2}dt \\ &= \int_0^{2\pi} \sqrt{(\cos^2 t - 2t\sin t\cos t + t^2\sin^2 t) + (\sin^2 t + 2t\sin t\cos t + t^2\cos^2 t) + 4}dt \\ &= \int_0^{2\pi} \sqrt{\cos^2 t + \sin^2 t + t^2(\cos^2 t + \sin^2 t) + 4}dt \\ &= \int_0^{2\pi} \sqrt{t^2+5}dt \end{aligned}$$

Here we can use a table integration formula

$$\int \sqrt{u^2+a^2}du = \frac{u}{2}\sqrt{u^2+a^2} + \frac{a^2}{2}\ln\left|u + \sqrt{u^2+a^2}\right| + C,$$

so we obtain

$$\begin{aligned} \int_0^{2\pi} \sqrt{t^2+5}dt &= \frac{1}{2}\left(t\sqrt{t^2+5} + 5\ln\left|t+\sqrt{t^2+5}\right|\right)_0^{2\pi} \\ &= \frac{1}{2}\left(2\pi\sqrt{4\pi^2+5} + 5\ln\left(2\pi + \sqrt{4\pi^2+5}\right)\right) - \frac{5}{2}\ln\sqrt{5} \\ &\approx 25.343 \text{ units}. \end{aligned}$$

Exercise 13.3.1

Calculate the arc length of the parameterized curve

$$\vec{\mathbf{r}}(t) = \langle 2t^2+1, 2t^2-1, t^3\rangle, \quad 0 \le t \le 3.$$

Hint

Use Equation 13.3.4.

Answer

$\vec{\mathbf{r}}'(t) = \langle 4t, 4t, 3t^2\rangle$, so $s = \frac{1}{27}(113^{3/2} - 32^{3/2}) \approx 37.785$ units

We now return to the helix introduced earlier in this chapter. A vector-valued function that describes a helix can be written in the form

$$\vec{\mathbf{r}}(t) = R\cos\left(\frac{2\pi Nt}{h}\right)\hat{\mathbf{i}} + R\sin\left(\frac{2\pi Nt}{h}\right)\hat{\mathbf{j}} + t\hat{\mathbf{k}}, 0 \le t \le h,$$

where R represents the radius of the helix, h represents the height (distance between two consecutive turns), and the helix completes N turns. Let's derive a formula for the arc length of this helix using Equation 13.3.4. First of all,

$$\vec{\mathbf{r}}\,'(t) = -\frac{2\pi NR}{h}\sin\left(\frac{2\pi Nt}{h}\right)\hat{\mathbf{i}} + \frac{2\pi NR}{h}\cos\left(\frac{2\pi Nt}{h}\right)\hat{\mathbf{j}} + \hat{\mathbf{k}}.$$

Therefore,

$$\begin{aligned}
s &= \int_a^b \|\vec{\mathbf{r}}\,'(t)\|dt \\
&= \int_0^h \sqrt{\left(-\frac{2\pi NR}{h}\sin\left(\frac{2\pi Nt}{h}\right)\right)^2 + \left(\frac{2\pi NR}{h}\cos\left(\frac{2\pi Nt}{h}\right)\right)^2 + 1^2}dt \\
&= \int_0^h \sqrt{\frac{4\pi^2N^2R^2}{h^2}\left(\sin^2\left(\frac{2\pi Nt}{h}\right) + \cos^2\left(\frac{2\pi Nt}{h}\right)\right) + 1}dt \\
&= \int_0^h \sqrt{\frac{4\pi^2N^2R^2}{h^2} + 1}dt \\
&= \left[t\sqrt{\frac{4\pi^2N^2R^2}{h^2} + 1}\right]_0^h \\
&= h\sqrt{\frac{4\pi^2N^2R^2 + h^2}{h^2}} \\
&= \sqrt{4\pi^2N^2R^2 + h^2}.
\end{aligned}$$

This gives a formula for the length of a wire needed to form a helix with N turns that has radius R and height h.

Arc-Length Parameterization

We now have a formula for the arc length of a curve defined by a vector-valued function. Let's take this one step further and examine what an **arc-length function** is.

If a vector-valued function represents the position of a particle in space as a function of time, then the arc-length function measures how far that particle travels as a function of time. The formula for the arc-length function follows directly from the formula for arc length:

$$s = \int_a^t \sqrt{(f'(u))^2 + (g'(u))^2 + (h'(u))^2}du. \tag{13.3.5}$$

If the curve is in two dimensions, then only two terms appear under the square root inside the integral. The reason for using the independent variable u is to distinguish between time and the variable of integration. Since $s(t)$ measures distance traveled as a function of time, $s'(t)$ measures the speed of the particle at any given time. Since we have a formula for $s(t)$ in Equation 13.3.5, we can differentiate both sides of the equation:

$$\begin{aligned}
s'(t) &= \frac{d}{dt}\left[\int_a^t \sqrt{(f'(u))^2 + (g'(u))^2 + (h'(u))^2}du\right] \\
&= \frac{d}{dt}\left[\int_a^t \|\vec{\mathbf{r}}\,'(u)\|du\right] \\
&= \|\vec{\mathbf{r}}\,'(t)\|.
\end{aligned}$$

If we assume that $\vec{\mathbf{r}}(t)$ defines a smooth curve, then the arc length is always increasing, so $s'(t) > 0$ for $t > a$. Last, if $\vec{\mathbf{r}}(t)$ is a curve on which $\|\vec{\mathbf{r}}\,'(t)\| = 1$ for all t, then

$$s(t) = \int_a^t \|\vec{\mathbf{r}}\,'(u)\|\,du = \int_a^t 1\,du = t - a,$$

which means that t represents the arc length as long as $a = 0$.

Theorem: Arc-Length Function

Let $\vec{\mathbf{r}}(t)$ describe a smooth curve for $t \geq a$. Then the arc-length function is given by

$$s(t) = \int_a^t \|\vec{\mathbf{r}}'(u)\| \, du$$

Furthermore,

$$\frac{ds}{dt} = \|\vec{\mathbf{r}}'(t)\| > 0.$$

If $\|\vec{\mathbf{r}}'(t)\| = 1$ for all $t \geq a$, then the parameter t represents the arc length from the starting point at $t = a$.

A useful application of this theorem is to find an alternative parameterization of a given curve, called an **arc-length parameterization**. Recall that any vector-valued function can be reparameterized via a change of variables. For example, if we have a function $\vec{\mathbf{r}}(t) = \langle 3\cos t, 3\sin t \rangle, 0 \leq t \leq 2\pi$ that parameterizes a circle of radius 3, we can change the parameter from t to $4t$, obtaining a new parameterization $\vec{\mathbf{r}}(t) = \langle 3\cos 4t, 3\sin 4t \rangle$. The new parameterization still defines a circle of radius 3, but now we need only use the values $0 \leq t \leq \pi/2$ to traverse the circle once.

Suppose that we find the arc-length function $s(t)$ and are able to solve this function for t as a function of s. We can then reparameterize the original function $\vec{\mathbf{r}}(t)$ by substituting the expression for t back into $\vec{\mathbf{r}}(t)$. The vector-valued function is now written in terms of the parameter s. Since the variable s represents the arc length, we call this an *arc-length parameterization* of the original function $\vec{\mathbf{r}}(t)$. One advantage of finding the arc-length parameterization is that the distance traveled along the curve starting from $s = 0$ is now equal to the parameter s. The arc-length parameterization also appears in the context of curvature (which we examine later in this section) and line integrals.

Example 13.3.2: Finding an Arc-Length Parameterization

Find the arc-length parameterization for each of the following curves:

a. $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 4\sin t\,\hat{\mathbf{j}}, \quad t \geq 0$
b. $\vec{\mathbf{r}}(t) = \langle t + 3, 2t - 4, 2t \rangle, \quad t \geq 3$

Solution

a. First we find the arc-length function using Equation 13.3.5:

$$\begin{aligned} s(t) &= \int_a^t \|\vec{\mathbf{r}}'(u)\| \, du \\ &= \int_0^t \|\langle -4\sin u, 4\cos u \rangle\| \, du \\ &= \int_0^t \sqrt{(-4\sin u)^2 + (4\cos u)^2} \, du \\ &= \int_0^t \sqrt{16\sin^2 u + 16\cos^2 u} \, du \\ &= \int_0^t 4 \, du = 4t, \end{aligned}$$

which gives the relationship between the arc length s and the parameter t as $s = 4t$; so, $t = s/4$. Next we replace the variable t in the original function $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 4\sin t\,\hat{\mathbf{j}}$ with the expression $s/4$ to obtain

$$\vec{\mathbf{r}}(s) = 4\cos\left(\frac{s}{4}\right)\hat{\mathbf{i}} + 4\sin\left(\frac{s}{4}\right)\hat{\mathbf{j}}.$$

This is the arc-length parameterization of $\vec{\mathbf{r}}(t)$. Since the original restriction on t was given by $t \geq 0$, the restriction on s becomes $s/4 \geq 0$, or $s \geq 0$.

b. The arc-length function is given by Equation 13.3.5:

$$\begin{aligned} s(t) &= \int_a^t \|\vec{\mathbf{r}}'(u)\|\, du \\ &= \int_3^t \|\langle 1,2,2\rangle\|\, du \\ &= \int_3^t \sqrt{1^2+2^2+2^2}\, du \\ &= \int_3^t 3\, du \\ &= 3t-9. \end{aligned}$$

Therefore, the relationship between the arc length s and the parameter t is $s = 3t - 9$, so $t = \frac{s}{3} + 3$. Substituting this into the original function $\vec{\mathbf{r}}(t) = \langle t+3, 2t-4, 2t\rangle$ yields

$$\vec{\mathbf{r}}(s) = \langle\left(\frac{s}{3}+3\right)+3,\, 2\left(\frac{s}{3}+3\right)-4,\, 2\left(\frac{s}{3}+3\right)\rangle = \langle\frac{s}{3}+6, \frac{2s}{3}+2, \frac{2s}{3}+6\rangle.$$

This is an arc-length parameterization of $\vec{\mathbf{r}}(t)$. The original restriction on the parameter t was $t \geq 3$, so the restriction on s is $(s/3)+3 \geq 3$, or $s \geq 0$.

? Exercise 13.3.2

Find the arc-length function for the helix

$$\vec{\mathbf{r}}(t) = \langle 3\cos t, 3\sin t, 4t\rangle, \quad t \geq 0.$$

Then, use the relationship between the arc length and the parameter t to find an arc-length parameterization of $\vec{\mathbf{r}}(t)$.

Hint

Start by finding the arc-length function.

Answer

$s = 5t$, or $t = s/5$. Substituting this into $\vec{\mathbf{r}}(t) = \langle 3\cos t, 3\sin t, 4t\rangle$ gives

$$\vec{\mathbf{r}}(s) = \langle 3\cos\left(\frac{s}{5}\right), 3\sin\left(\frac{s}{5}\right), \frac{4s}{5}\rangle, \quad s \geq 0$$

Curvature

An important topic related to arc length is curvature. The concept of curvature provides a way to measure how sharply a smooth curve turns. A circle has constant curvature. The smaller the radius of the circle, the greater the curvature.

Think of driving down a road. Suppose the road lies on an arc of a large circle. In this case you would barely have to turn the wheel to stay on the road. Now suppose the radius is smaller. In this case you would need to turn more sharply to stay on the road. In the case of a curve other than a circle, it is often useful first to inscribe a circle to the curve at a given point so that it is tangent to the curve at that point and "hugs" the curve as closely as possible in a neighborhood of the point (Figure 13.3.1). The curvature of the graph at that point is then defined to be the same as the curvature of the inscribed circle.

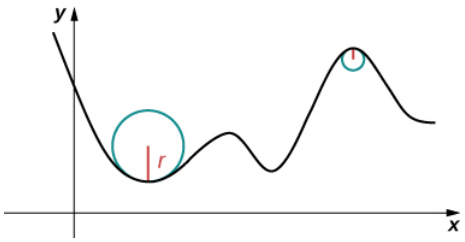


Figure 13.3.1: The graph represents the curvature of a function $y = f(x)$. The sharper the turn in the graph, the greater the curvature, and the smaller the radius of the inscribed circle.

Definition: Curvature

Let C be a smooth curve in the plane or in space given by $\vec{\mathbf{r}}(s)$, where s is the arc-length parameter. The curvature κ at s is

$$\kappa = \left\| \frac{d\vec{\mathbf{T}}}{ds} \right\| = \|\vec{\mathbf{T}}'(s)\|.$$

Visit this video for more information about the curvature of a space curve.

The formula in the definition of curvature is not very useful in terms of calculation. In particular, recall that $\vec{\mathbf{T}}(t)$ represents the unit tangent vector to a given vector-valued function $\vec{\mathbf{r}}(t)$, and the formula for $\vec{\mathbf{T}}(t)$ is

$$\vec{\mathbf{T}}(t) = \frac{\vec{\mathbf{r}}\,'(t)}{\| \vec{\mathbf{r}}\,'(t) \|}.$$

To use the formula for curvature, it is first necessary to express $\vec{\mathbf{r}}(t)$ in terms of the arc-length parameter s, then find the unit tangent vector $\vec{\mathbf{T}}(s)$ for the function $\vec{\mathbf{r}}(s)$, then take the derivative of $\vec{\mathbf{T}}(s)$ with respect to s. This is a tedious process. Fortunately, there are equivalent formulas for curvature.

Theorem: Alternate Formulas of Curvature

If C is a smooth curve given by $\vec{\mathbf{r}}(t)$, then the curvature κ of C at t is given by

$$\kappa = \frac{\|\vec{\mathbf{T}}'(t)\|}{\|\vec{\mathbf{r}}\,'(t)\|}. \tag{13.3.6}$$

If C is a three-dimensional curve, then the curvature can be given by the formula

$$\kappa = \frac{\|\vec{\mathbf{r}}\,'(t) \times \vec{\mathbf{r}}\,''(t)\|}{\|\vec{\mathbf{r}}\,'(t)\|^3}. \tag{13.3.7}$$

If C is the graph of a function $y = f(x)$ and both y' and y'' exist, then the curvature κ at point (x, y) is given by

$$\kappa = \frac{|y''|}{[1 + (y')^2]^{3/2}}. \tag{13.3.8}$$

Proof

The first formula follows directly from the chain rule:

$$\frac{d\vec{\mathbf{T}}}{dt} = \frac{d\vec{\mathbf{T}}}{ds}\frac{ds}{dt},$$

where s is the arc length along the curve C. Dividing both sides by ds/dt, and taking the magnitude of both sides gives

$$\left\| \frac{d\vec{\mathbf{T}}}{ds} \right\| = \left\| \frac{\vec{\mathbf{T}}'(t)}{\frac{ds}{dt}} \right\|.$$

Since $ds/dt = \|\vec{\mathbf{r}}\,'(t)\|$, this gives the formula for the curvature κ of a curve C in terms of any parameterization of C:

$$\kappa = \frac{\|\vec{\mathbf{T}}'(t)\|}{\|\vec{\mathbf{r}}\,'(t)\|}.$$

In the case of a three-dimensional curve, we start with the formulas $\vec{\mathbf{T}}(t) = (\vec{\mathbf{r}}\,'(t))/\|\vec{\mathbf{r}}\,'(t)\|$ and $ds/dt = \|\vec{\mathbf{r}}\,'(t)\|$. Therefore, $\vec{\mathbf{r}}\,'(t) = (ds/dt)\vec{\mathbf{T}}(t)$. We can take the derivative of this function using the scalar product formula:

$$\vec{\mathbf{r}}\,''(t) = \frac{d^2s}{dt^2}\vec{\mathbf{T}}(t) + \frac{ds}{dt}\vec{\mathbf{T}}'(t).$$

Using these last two equations we get

$$\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t) = \frac{ds}{dt}\vec{\mathbf{T}}(t) \times \left(\frac{d^2s}{dt^2}\vec{\mathbf{T}}(t) + \frac{ds}{dt}\vec{\mathbf{T}}'(t)\right)$$
$$= \frac{ds}{dt}\frac{d^2s}{dt^2}\vec{\mathbf{T}}(t) \times \vec{\mathbf{T}}(t) + (\frac{ds}{dt})^2\vec{\mathbf{T}}(t) \times \vec{\mathbf{T}}'(t).$$

Since $\vec{\mathbf{T}}(t) \times \vec{\mathbf{T}}(t) = 0$, this reduces to

$$\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t) = \left(\frac{ds}{dt}\right)^2 \vec{\mathbf{T}}(t) \times \vec{\mathbf{T}}'(t).$$

Since $\vec{\mathbf{T}}'$ is parallel to $\vec{\mathbf{N}}$, and $\vec{\mathbf{T}}$ is orthogonal to $\vec{\mathbf{N}}$, it follows that $\vec{\mathbf{T}}$ and $\vec{\mathbf{T}}'$ are orthogonal. This means that $\|\vec{\mathbf{T}} \times \vec{\mathbf{T}}'\| = \|\vec{\mathbf{T}}\|\|\vec{\mathbf{T}}'\| \sin(\pi/2) = \|\vec{\mathbf{T}}'\|$, so

$$\|\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t)\| = \left(\frac{ds}{dt}\right)^2 \|\vec{\mathbf{T}}'(t)\|.$$

Now we solve this equation for $\|\vec{\mathbf{T}}'(t)\|$ and use the fact that $ds/dt = \|\vec{\mathbf{r}}'(t)\|$:

$$\|\vec{\mathbf{T}}'(t)\| = \frac{\|\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t)\|}{\|\vec{\mathbf{r}}'(t)\|^2}.$$

Then, we divide both sides by $\|\vec{\mathbf{r}}'(t)\|$. This gives

$$\kappa = \frac{\|\vec{\mathbf{T}}'(t)\|}{\|\vec{\mathbf{r}}'(t)\|} = \frac{\|\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t)\|}{\|\vec{\mathbf{r}}'(t)\|^3}.$$

This proves 13.3.7. To prove 13.3.8, we start with the assumption that curve C is defined by the function $y = f(x)$. Then, we can define $\vec{\mathbf{r}}(t) = x\,\hat{\mathbf{i}} + f(x)\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}$. Using the previous formula for curvature:

$$\vec{\mathbf{r}}'(t) = \hat{\mathbf{i}} + f'(x)\,\hat{\mathbf{j}}$$
$$\vec{\mathbf{r}}''(t) = f''(x)\,\hat{\mathbf{j}}$$
$$\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & f'(x) & 0 \\ 0 & f''(x) & 0 \end{vmatrix} = f''(x)\,\hat{\mathbf{k}}.$$

Therefore,

$$\kappa = \frac{\|\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t)\|}{\|\vec{\mathbf{r}}'(t)\|^3} = \frac{|f''(x)|}{(1 + [f'(x)]^2)^{3/2}}$$

Example 13.3.3: Finding Curvature

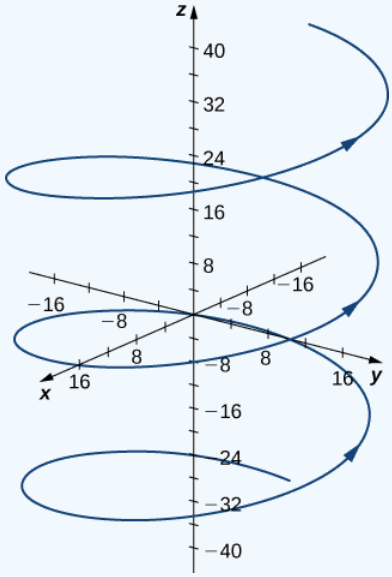
Find the curvature for each of the following curves at the given point:

a. $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 4\sin t\,\hat{\mathbf{j}} + 3t\,\hat{\mathbf{k}}, \quad t = \dfrac{4\pi}{3}$

b. $\mathrm{f(x)} = \sqrt{4\mathrm{x} - \mathrm{x}^2}, \mathrm{x} = 2$

Solution

a. This function describes a helix.



The curvature of the helix at $t = (4\pi)/3$ can be found by using 13.3.6. First, calculate $\vec{\mathbf{T}}(t)$:

$$\begin{aligned}\vec{\mathbf{T}}(t) &= \frac{\vec{\mathbf{r}}\,'(t)}{\|\vec{\mathbf{r}}\,'(t)\|} \\ &= \frac{\langle -4\sin t, 4\cos t, 3\rangle}{\sqrt{(-4\sin t)^2 + (4\cos t)^2 + 3^2}} \\ &= \langle -\frac{4}{5}\sin t, \frac{4}{5}\cos t, \frac{3}{5}\rangle.\end{aligned}$$

Next, calculate $\vec{\mathbf{T}}\,'(t)$:

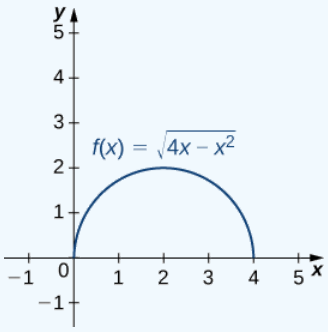
$$\vec{\mathbf{T}}\,'(t) = \langle -\frac{4}{5}\cos t, -\frac{4}{5}\sin t, 0\rangle.$$

Last, apply 13.3.6 :

$$\begin{aligned}\kappa = \frac{\|\vec{\mathbf{T}}\,'(t)\|}{\|\vec{\mathbf{r}}\,'(t)\|} &= \frac{\|\langle -\frac{4}{5}\cos t, -\frac{4}{5}\sin t, 0\rangle\|}{\|\langle -4\sin t, 4\cos t, 3\rangle\|} \\ &= \frac{\sqrt{(-\frac{4}{5}\cos t)^2 + (-\frac{4}{5}\sin t)^2 + 0^2}}{\sqrt{(-4\sin t)^2 + (4\cos t)^2 + 3^2}} \\ &= \frac{4/5}{5} = \frac{4}{25}.\end{aligned}$$

The curvature of this helix is constant at all points on the helix.

2. This function describes a semicircle.



To find the curvature of this graph, we must use 13.3.8. First, we calculate y' and y'':

$$\begin{aligned} y &= \sqrt{4x - x^2} = (4x - x^2)^{1/2} \\ y' &= \frac{1}{2}(4x - x^2)^{-1/2}(4 - 2x) = (2 - x)(4x - x^2)^{-1/2} \\ y'' &= -(4x - x^2)^{-1/2} + (2 - x)(-\frac{1}{2})(4x - x^2)^{-3/2}(4 - 2x) \\ &= -\frac{4x - x^2}{(4x - x^2)^{3/2}} - \frac{(2 - x)^2}{(4x - x^2)^{3/2}} \\ &= \frac{x^2 - 4x - (4 - 4x + x^2)}{(4x - x^2)^{3/2}} \\ &= -\frac{4}{(4x - x^2)^{3/2}}. \end{aligned}$$

Then, we apply 13.3.8:

$$\begin{aligned} \kappa &= \frac{|y''|}{[1 + (y')^2]^{3/2}} \\ &= \frac{\left| -\frac{4}{(4x - x^2)^{3/2}} \right|}{\left[1 + ((2 - x)(4x - x^2)^{-1/2})^2\right]^{3/2}} = \frac{\left|\frac{4}{(4x - x^2)^{3/2}}\right|}{\left[1 + \frac{(2 - x)^2}{4x - x^2}\right]^{3/2}} \\ &= \frac{\left|\frac{4}{(4x - x^2)^{3/2}}\right|}{\left[\frac{4x - x^2 + x^2 - 4x + 4}{4x - x^2}\right]^{3/2}} = \left|\frac{4}{(4x - x^2)^{3/2}}\right| \cdot \frac{(4x - x^2)^{3/2}}{8} \\ &= \frac{1}{2}. \end{aligned}$$

The curvature of this circle is equal to the reciprocal of its radius. There is a minor issue with the absolute value in 13.3.8 ; however, a closer look at the calculation reveals that the denominator is positive for any value of x.

? Exercise 13.3.3

Find the curvature of the curve defined by the function

$$y = 3x^2 - 2x + 4$$

at the point $x = 2$.

Hint

Use 13.3.8.

Answer

$\kappa = \frac{6}{101^{3/2}} \approx 0.0059$

The Normal and Binormal Vectors

We have seen that the derivative $\vec{\mathbf{r}}\,'(t)$ of a vector-valued function is a tangent vector to the curve defined by $\vec{\mathbf{r}}(t)$, and the unit tangent vector $\vec{\mathbf{T}}(t)$ can be calculated by dividing $\vec{\mathbf{r}}\,'(t)$ by its magnitude. When studying motion in three dimensions, two other vectors are useful in describing the motion of a particle along a path in space: the principal unit normal vector and the **binormal vector**.

Definition: Binormal Vectors

Let C be a three-dimensional **smooth** curve represented by $\vec{\mathbf{r}}$ over an open interval I. If $\vec{\mathbf{T}}'(t) \neq \vec{\mathbf{0}}$, then the principal unit normal vector at t is defined to be

$$\vec{\mathbf{N}}(t) = \frac{\vec{\mathbf{T}}'(t)}{\|\vec{\mathbf{T}}'(t)\|}. \tag{13.3.9}$$

The binormal vector at t is defined as

$$\vec{\mathbf{B}}(t) = \vec{\mathbf{T}}(t) \times \vec{\mathbf{N}}(t), \tag{13.3.10}$$

where $\vec{\mathbf{T}}(t)$ is the unit tangent vector.

Note that, by definition, the binormal vector is orthogonal to both the unit tangent vector and the normal vector. Furthermore, $\vec{\mathbf{B}}(t)$ is always a unit vector. This can be shown using the formula for the magnitude of a cross product.

$$\|\vec{\mathbf{B}}(t)\| = \|\vec{\mathbf{T}}(t) \times \vec{\mathbf{N}}(t)\| = \|\vec{\mathbf{T}}(t)\|\|\vec{\mathbf{N}}(t)\| \sin\theta,$$

where θ is the angle between $\vec{\mathbf{T}}(t)$ and $\vec{\mathbf{N}}(t)$. Since $\vec{\mathbf{N}}(t)$ is the derivative of a unit vector, property (vii) of the derivative of a vector-valued function tells us that $\vec{\mathbf{T}}(t)$ and $\vec{\mathbf{N}}(t)$ are orthogonal to each other, so $\theta = \pi/2$. Furthermore, they are both unit vectors, so their magnitude is 1. Therefore, $\|\vec{\mathbf{T}}(t)\|\|\vec{\mathbf{N}}(t)\| \sin\theta = (1)(1)\sin(\pi/2) = 1$ and $\vec{\mathbf{B}}(t)$ is a unit vector.

The principal unit normal vector can be challenging to calculate because the unit tangent vector involves a quotient, and this quotient often has a square root in the denominator. In the three-dimensional case, finding the cross product of the unit tangent vector and the unit normal vector can be even more cumbersome. Fortunately, we have alternative formulas for finding these two vectors, and they are presented in Motion in Space.

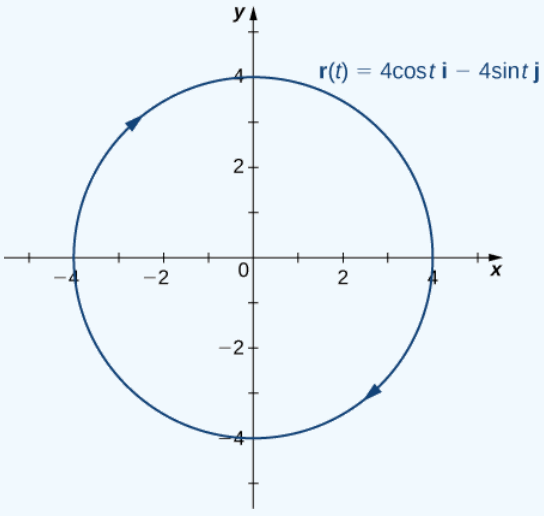
Example 13.3.4: Finding the Principal Unit Normal Vector and Binormal Vector

For each of the following vector-valued functions, find the principal unit normal vector. Then, if possible, find the binormal vector.

1. $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} - 4\sin t\,\hat{\mathbf{j}}$
2. $\vec{\mathbf{r}}(t) = (6t+2)\,\hat{\mathbf{i}} + 5t^2\,\hat{\mathbf{j}} - 8t\,\hat{\mathbf{k}}$

Solution

1. This function describes a circle.



To find the principal unit normal vector, we first must find the unit tangent vector $\vec{\mathbf{T}}(t)$:

$$\begin{aligned}\vec{\mathbf{T}}(t) &= \frac{\vec{\mathbf{r}}'(t)}{\|\vec{\mathbf{r}}'(t)\|} \\ &= \frac{-4\sin t\,\hat{\mathbf{i}} - 4\cos t\,\hat{\mathbf{j}}}{\sqrt{(-4\sin t)^2 + (-4\cos t)^2}} \\ &= \frac{-4\sin t\,\hat{\mathbf{i}} - 4\cos t\,\hat{\mathbf{j}}}{\sqrt{16\sin^2 t + 16\cos^2 t}} \\ &= \frac{-4\sin t\,\hat{\mathbf{i}} - 4\cos t\,\hat{\mathbf{j}}}{\sqrt{16(\sin^2 t + \cos^2 t)}} \\ &= \frac{-4\sin t\,\hat{\mathbf{i}} - 4\cos t\,\hat{\mathbf{j}}}{4} \\ &= -\sin t\,\hat{\mathbf{i}} - \cos t\,\hat{\mathbf{j}}.\end{aligned}$$

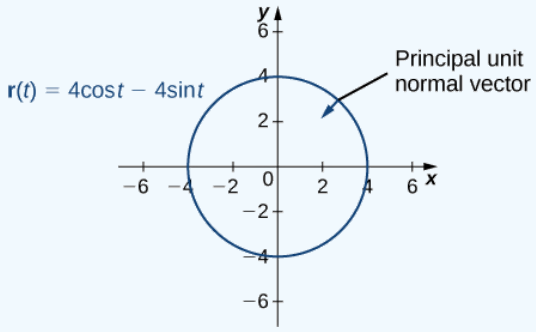
Next, we use 13.3.9 :

$$\begin{aligned}\vec{\mathbf{N}}(t) &= \frac{\vec{\mathbf{T}}'(t)}{\|\vec{\mathbf{T}}'(t)\|} \\ &= \frac{-\cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}}{\sqrt{(-\cos t)^2 + (\sin t)^2}} \\ &= \frac{-\cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}}{\sqrt{\cos^2 t + \sin^2 t}} \\ &= -\cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}.\end{aligned}$$

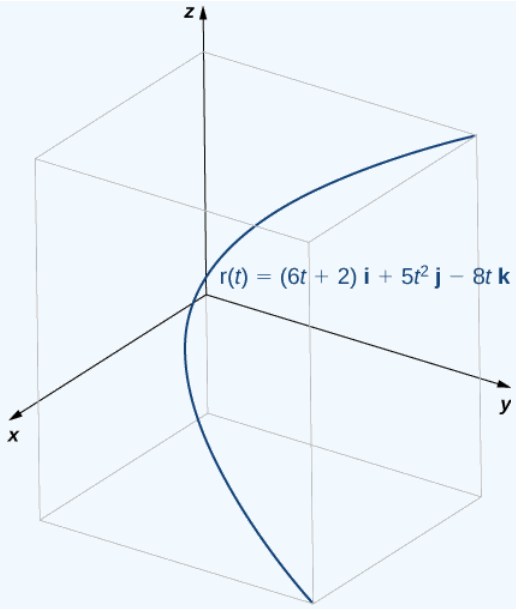
Notice that the unit tangent vector and the principal unit normal vector are orthogonal to each other for all values of t:

$$\begin{aligned}\vec{\mathbf{T}}(t) \cdot \vec{\mathbf{N}}(t) &= \langle -\sin t, -\cos t\rangle \cdot \langle -\cos t, \sin t\rangle \\ &= \sin t\cos t - \cos t\sin t \\ &= 0.\end{aligned}$$

Furthermore, the principal unit normal vector points toward the center of the circle from every point on the circle. Since $\vec{\mathbf{r}}(t)$ defines a curve in two dimensions, we cannot calculate the binormal vector.



2. This function looks like this:



To find the principal unit normal vector, we first find the unit tangent vector $\vec{\mathbf{T}}(t)$:

$$\begin{aligned}\vec{\mathbf{T}}(t) &= \frac{\vec{\mathbf{r}}'(t)}{\|\vec{\mathbf{r}}'(t)\|} \\ &= \frac{6\,\hat{\mathbf{i}} + 10t\,\hat{\mathbf{j}} - 8\,\hat{\mathbf{k}}}{\sqrt{6^2 + (10t)^2 + (-8)^2}} \\ &= \frac{6\,\hat{\mathbf{i}} + 10t\,\hat{\mathbf{j}} - 8\,\hat{\mathbf{k}}}{\sqrt{36 + 100t^2 + 64}} \\ &= \frac{6\,\hat{\mathbf{i}} + 10t\,\hat{\mathbf{j}} - 8\,\hat{\mathbf{k}}}{\sqrt{100(t^2+1)}} \\ &= \frac{3\,\hat{\mathbf{i}} - 5t\,\hat{\mathbf{j}} - 4\,\hat{\mathbf{k}}}{5\sqrt{t^2+1}} \\ &= \frac{3}{5}(t^2+1)^{-1/2}\,\hat{\mathbf{i}} - t(t^2+1)^{-1/2}\,\hat{\mathbf{j}} - \frac{4}{5}(t^2+1)^{-1/2}\,\hat{\mathbf{k}}.\end{aligned}$$

Next, we calculate $\vec{\mathbf{T}}'(t)$ and $\|\vec{\mathbf{T}}'(t)\|$:

$$\begin{aligned}\vec{\mathbf{T}}'(t) &= \frac{3}{5}(-\frac{1}{2})(t^2+1)^{-3/2}(2t)\,\hat{\mathbf{i}} - ((t^2+1)^{-1/2} - t(\frac{1}{2})(t^2+1)^{-3/2}(2t))\,\hat{\mathbf{j}} \\ &\quad - \frac{4}{5}(-\frac{1}{2})(t^2+1)^{-3/2}(2t)\,\hat{\mathbf{k}} \\ &= -\frac{3t}{5(t^2+1)^{3/2}}\,\hat{\mathbf{i}} - \frac{1}{(t^2+1)^{3/2}}\,\hat{\mathbf{j}} + \frac{4t}{5(t^2+1)^{3/2}}\,\hat{\mathbf{k}} \\ \|\vec{\mathbf{T}}'(t)\| &= \sqrt{\Big(-\frac{3t}{5(t^2+1)^{3/2}}\Big)^2 + \Big(-\frac{1}{(t^2+1)^{3/2}}\Big)^2 + \Big(\frac{4t}{5(t^2+1)^{3/2}}\Big)^2} \\ &= \sqrt{\frac{9t^2}{25(t^2+1)^3} + \frac{1}{(t^2+1)^3} + \frac{16t^2}{25(t^2+1)^3}} \\ &= \sqrt{\frac{25t^2+25}{25(t^2+1)^3}} \\ &= \sqrt{\frac{1}{(t^2+1)^2}} \\ &= \frac{1}{t^2+1}.\end{aligned}$$

Therefore, according to 13.3.9 :

$$\begin{aligned}\vec{\mathbf{N}}(t) &= \frac{\vec{\mathbf{T}}'(t)}{\|\vec{\mathbf{T}}'(t)\|} \\ &= \left(-\frac{3t}{5(t^2+1)^{3/2}}\,\hat{\mathbf{i}} - \frac{1}{(t^2+1)^{3/2}}\,\hat{\mathbf{j}} + \frac{4t}{5(t^2+1)^{3/2}}\,\hat{\mathbf{k}}\right)(t^2+1) \\ &= -\frac{3t}{5(t^2+1)^{1/2}}\,\hat{\mathbf{i}} - \frac{5}{5(t^2+1)^{1/2}}\,\hat{\mathbf{j}} + \frac{4t}{5(t^2+1)^{1/2}}\,\hat{\mathbf{k}} \\ &= -\frac{3t\,\hat{\mathbf{i}} + 5\,\hat{\mathbf{j}} - 4t\,\hat{\mathbf{k}}}{5\sqrt{t^2+1}}.\end{aligned}$$

Once again, the unit tangent vector and the principal unit normal vector are orthogonal to each other for all values of t:

$$\begin{aligned}\vec{\mathbf{T}}(t)\cdot\vec{\mathbf{N}}(t) &= \left(\frac{3\,\hat{\mathbf{i}} - 5t\,\hat{\mathbf{j}} - 4\,\hat{\mathbf{k}}}{5\sqrt{t^2+1}}\right)\cdot\left(-\frac{3t\,\hat{\mathbf{i}} + 5\,\hat{\mathbf{j}} - 4t\,\hat{\mathbf{k}}}{5\sqrt{t^2+1}}\right) \\ &= \frac{3(-3t) - 5t(-5) - 4(4t)}{25(t^2+1)} \\ &= \frac{-9t + 25t - 16t}{25(t^2+1)} \\ &= 0.\end{aligned}$$

Last, since $\vec{\mathbf{r}}(t)$ represents a three-dimensional curve, we can calculate the binormal vector using 13.3.10 :

$$\begin{aligned}\vec{\mathbf{B}}(t) &= \vec{\mathbf{T}}(t)\times\vec{\mathbf{N}}(t) \\ &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{3}{5\sqrt{t^2+1}} & -\dfrac{5t}{5\sqrt{t^2+1}} & -\dfrac{4}{5\sqrt{t^2+1}} \\ -\dfrac{3t}{5\sqrt{t^2+1}} & -\dfrac{5}{5\sqrt{t^2+1}} & \dfrac{4t}{5\sqrt{t^2+1}} \end{vmatrix} \\ &= \left(\left(-\frac{5t}{5\sqrt{t^2+1}}\right)\left(\frac{4t}{5\sqrt{t^2+1}}\right) - \left(-\frac{4}{5\sqrt{t^2+1}}\right)\left(-\frac{5}{5\sqrt{t^2+1}}\right)\right)\hat{\mathbf{i}} \\ &- \left(\left(\frac{3}{5\sqrt{t^2+1}}\right)\left(\frac{4t}{5\sqrt{t^2+1}}\right) - \left(-\frac{4}{5\sqrt{t^2+1}}\right)\left(-\frac{3t}{5\sqrt{t^2+1}}\right)\right)\hat{\mathbf{j}} \\ &+ \left(\left(\frac{3}{5\sqrt{t^2+1}}\right)\left(-\frac{5}{5\sqrt{t^2+1}}\right) - \left(-\frac{5t}{5\sqrt{t^2+1}}\right)\left(-\frac{3t}{5\sqrt{t^2+1}}\right)\right)\hat{\mathbf{k}} \\ &= \left(\frac{-20t^2-20}{25(t^2+1)}\right)\hat{\mathbf{i}} + \left(\frac{-15-15t^2}{25(t^2+1)}\right)\hat{\mathbf{k}} \\ &= -20\left(\frac{t^2+1}{25(t^2+1)}\right)\hat{\mathbf{i}} - 15\left(\frac{t^2+1}{25(t^2+1)}\right)\hat{\mathbf{k}} \\ &= -\frac{4}{5}\,\hat{\mathbf{i}} - \frac{3}{5}\,\hat{\mathbf{k}}.\end{aligned}$$

Exercise 13.3.4

Find the unit normal vector for the vector-valued function $\vec{\mathbf{r}}(t) = (t^2-3t)\,\hat{\mathbf{i}} + (4t+1)\,\hat{\mathbf{j}}$ and evaluate it at $t=2$.

Hint

First, find $\vec{\mathbf{T}}(t)$, then use 13.3.9.

Answer

$\vec{\mathbf{N}}(2) = \dfrac{\sqrt{2}}{2}(\hat{\mathbf{i}} - \hat{\mathbf{j}})$

For any smooth curve in three dimensions that is defined by a vector-valued function, we now have formulas for the unit tangent vector $\vec{\mathbf{T}}$, the unit normal vector $\vec{\mathbf{N}}$, and the binormal vector $\vec{\mathbf{B}}$. The unit normal vector and the binormal vector form a plane that is perpendicular to the curve at any point on the curve, called the normal plane. In addition, these three vectors form a frame of reference in three-dimensional space called the *Frenet frame of reference* (also called the **TNB** frame) (Figure 13.3.2). Last, the plane determined by the vectors $\vec{\mathbf{T}}$ and $\vec{\mathbf{N}}$ forms the osculating plane of C at any point P on the curve.

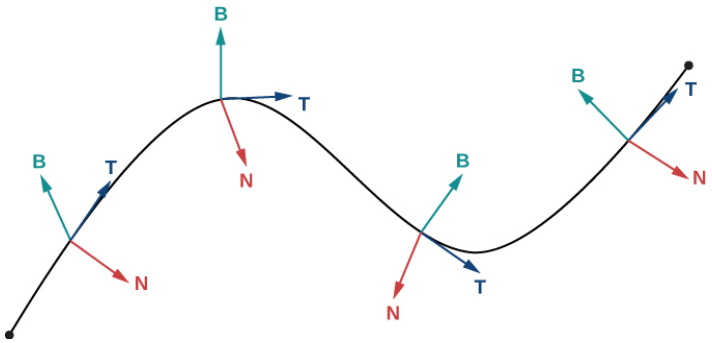


Figure 13.3.2: This figure depicts a Frenet frame of reference. At every point P *on a three-dimensional curve, the unit tangent, unit normal, and binormal vectors form a three-dimensional frame of reference.*

Suppose we form a circle in the osculating plane of C at point P on the curve. Assume that the circle has the same curvature as the curve does at point P and let the circle have radius r. Then, the curvature of the circle is given by $\frac{1}{r}$. We call r the radius of curvature of the curve, and it is equal to the reciprocal of the curvature. If this circle lies on the concave side of the curve and is tangent to the curve at point P, then this circle is called the *osculating circle* of C at P, as shown in Figure 13.3.3.

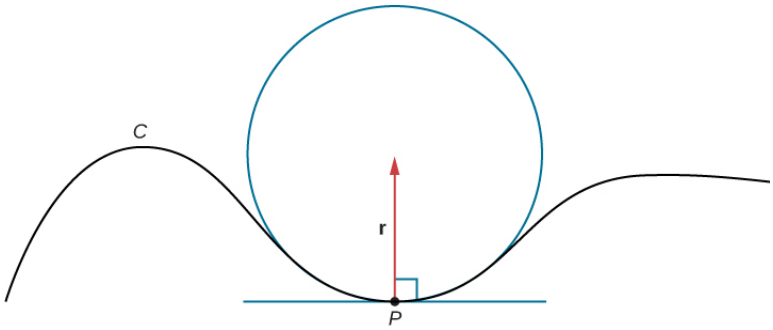


Figure 13.3.3: In this osculating circle, the circle is tangent to curve C at point P *and shares the same curvature.*

For more information on osculating circles, see this demonstration on curvature and torsion, this article on osculating circles, and this discussion of Serret formulas.

To find the equation of an osculating circle in two dimensions, we need find only the center and radius of the circle.

Example 13.3.5: Finding the Equation of an Osculating Circle

Find the equation of the osculating circle of the curve defined by the function $y = x^3 - 3x + 1$ at $x = 1$.

Solution

Figure 13.3.4 shows the graph of $y = x^3 - 3x + 1$.

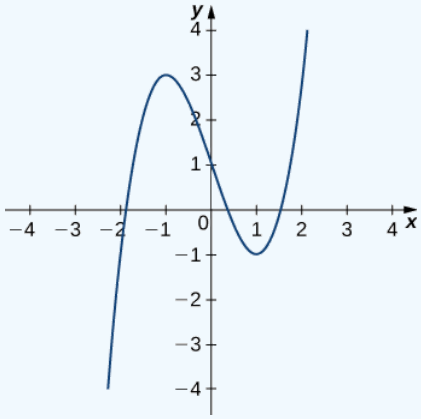


Figure 13.3.4: We want to find the osculating circle of this graph at the point where $x = 1$.

First, let's calculate the curvature at $x = 1$:

$$\kappa = \frac{|f''(x)|}{\left(1 + [f'(x)]^2\right)^{3/2}} = \frac{|6x|}{(1 + [3x^2 - 3]^2)^{3/2}}.$$

This gives $\kappa = 6$. Therefore, the radius of the osculating circle is given by $R = \frac{1}{\kappa} = \frac{1}{6}$. Next, we then calculate the coordinates of the center of the circle. When $x = 1$, the slope of the tangent line is zero. Therefore, the center of the osculating circle is directly above the point on the graph with coordinates $(1, -1)$. The center is located at $(1, -\frac{5}{6})$. The formula for a circle with radius r and center (h, k) is given by $(x - h)^2 + (y - k)^2 = r^2$. Therefore, the equation of the osculating circle is $(x - 1)^2 + (y + \frac{5}{6})^2 = \frac{1}{36}$. The graph and its osculating circle appears in the following graph.

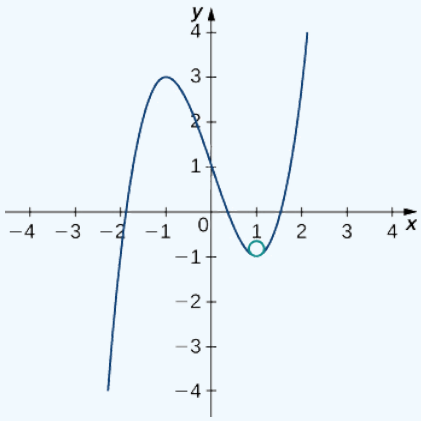


Figure 13.3.5: The osculating circle has radius $R = \frac{1}{6}$.

? Exercise 13.3.5

Find the equation of the osculating circle of the curve defined by the vector-valued function $y = 2x^2 - 4x + 5$ at $x = 1$.

Hint

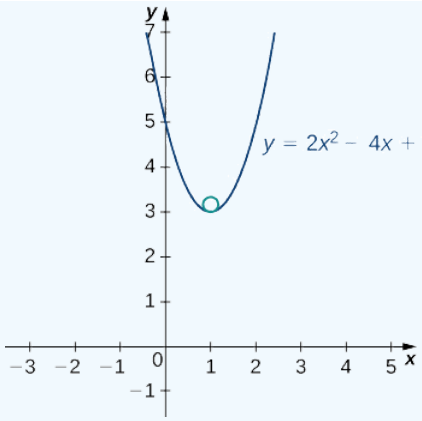
Use 13.3.8 to find the curvature of the graph, then draw a graph of the function around $x = 1$ to help visualize the circle in relation to the graph.

Answer

$\kappa = \frac{4}{[1+(4x-4)^2]^{3/2}}$

At the point $x = 1$, the curvature is equal to 4. Therefore, the radius of the osculating circle is $\frac{1}{4}$.

A graph of this function appears next:



The vertex of this parabola is located at the point $(1, 3)$. Furthermore, the center of the osculating circle is directly above the vertex. Therefore, the coordinates of the center are $(1, \frac{13}{4})$. The equation of the osculating circle is $(x-1)^2 + (y - \frac{13}{4})^2 = \frac{1}{16}$.

Key Concepts

- The arc-length function for a vector-valued function is calculated using the integral formula $s(t) = \int_a^b \|\vec{\mathbf{r}}'(t)\|\, dt$. This formula is valid in both two and three dimensions.
- The curvature of a curve at a point in either two or three dimensions is defined to be the curvature of the inscribed circle at that point. The arc-length parameterization is used in the definition of curvature.
- There are several different formulas for curvature. The curvature of a circle is equal to the reciprocal of its radius.
- The principal unit normal vector at t is defined to be

$$\vec{\mathbf{N}}(t) = \frac{\vec{\mathbf{T}}'(t)}{\|\vec{\mathbf{T}}'(t)\|}.$$

- The binormal vector at t is defined as $\vec{\mathbf{B}}(t) = \vec{\mathbf{T}}(t) \times \vec{\mathbf{N}}(t)$, where $\vec{\mathbf{T}}(t)$ is the unit tangent vector.
- The Frenet frame of reference is formed by the unit tangent vector, the principal unit normal vector, and the binormal vector.
- The osculating circle is tangent to a curve at a point and has the same curvature as the tangent curve at that point.

Key Equations

- **Arc length of space curve**
  $s = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt = \int_a^b \|\vec{\mathbf{r}}'(t)\|\, dt$
- **Arc-length function**
  $s(t) = \int_a^t \sqrt{f'(u))^2 + (g'(u))^2 + (h'(u))^2}\, du \text{ or } s(t) = \int_a^t \|\vec{\mathbf{r}}'(u)\|\, du$
- $\kappa = \frac{\|\vec{\mathbf{T}}'(t)\|}{\|\vec{\mathbf{r}}'(t)\|}$ or $\kappa = \frac{\|\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t)\|}{\|\vec{\mathbf{r}}'(t)\|^3}$ or $\kappa = \frac{|y''|}{[1+(y')^2]^{3/2}}$
- **Principal unit normal vector**
  $\vec{\mathbf{N}}(t) = \frac{\vec{\mathbf{T}}'(t)}{\|\vec{\mathbf{T}}'(t)\|}$
- **Binormal vector**
  $\vec{\mathbf{B}}(t) = \vec{\mathbf{T}}(t) \times \vec{\mathbf{N}}(t)$

Glossary

arc-length function

a function $s(t)$ that describes the arc length of curve C as a function of t

arc-length parameterization

a reparameterization of a vector-valued function in which the parameter is equal to the arc length

binormal vector

a unit vector orthogonal to the unit tangent vector and the unit normal vector

curvature

the derivative of the unit tangent vector with respect to the arc-length parameter

Frenet frame of reference

(TNB frame) a frame of reference in three-dimensional space formed by the unit tangent vector, the unit normal vector, and the binormal vector

normal plane

a plane that is perpendicular to a curve at any point on the curve

osculating circle

a circle that is tangent to a curve C at a point P and that shares the same curvature

osculating plane

the plane determined by the unit tangent and the unit normal vector

principal unit normal vector

a vector orthogonal to the unit tangent vector, given by the formula $\frac{\vec{\mathbf{T}}'(t)}{\|\vec{\mathbf{T}}'(t)\|}$

radius of curvature

the reciprocal of the curvature

smooth

curves where the vector-valued function $\vec{\mathbf{r}}(t)$ is differentiable with a non-zero derivative

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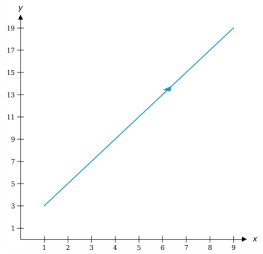
13.3E: Exercises for Section 13.3

Determining Arc Length

In questions 1 - 5, find the arc length of the curve on the given interval.

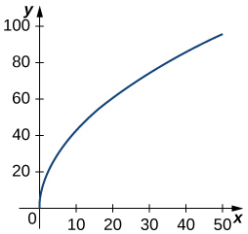
1) $\vec{\mathbf{r}}(t) = t^2\,\hat{\mathbf{i}} + (2t^2+1)\,\hat{\mathbf{j}}, \quad 1 \le t \le 3$. This portion of the graph is shown here:



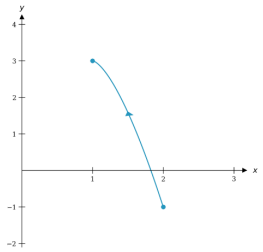
Answer

$8\sqrt{5}$ units ≈ 17.9 units

2) $\vec{\mathbf{r}}(t) = t^2\,\hat{\mathbf{i}} + 14t\,\hat{\mathbf{j}}, \quad 0 \le t \le 7$. This portion of the graph is shown here:



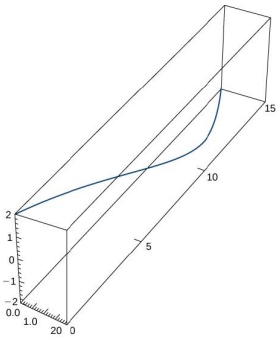
3) $\vec{\mathbf{r}}(t) = \langle t^2+1, 4t^3+3\rangle, \quad -1 \le t \le 0$. This portion of the graph is shown here:



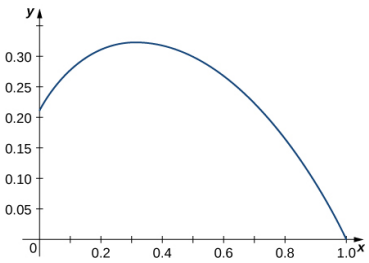
Answer

$\frac{1}{54}(37^{3/2}-1)$ units ≈ 4.15 units

4) $\vec{\mathbf{r}}(t) = \langle 2\sin t, 5t, 2\cos t\rangle, \quad 0 \le t \le \pi$. This portion of the graph is shown here:



5) $\vec{\mathbf{r}}(t) = \langle e^{-t\cos t}, e^{-t\sin t}\rangle$ over the interval $[0, \frac{\pi}{2}]$. Here is the portion of the graph on the indicated interval:



6) Set up an integral to represent the arc length from $t = 0$ to $t = 2$ along the curve traced out by $\vec{\mathbf{r}}(t) = \langle t, t^4\rangle$. Then use technology to approximate this length to the nearest thousandth of a unit.

7) Find the length of one turn of the helix given by $\vec{\mathbf{r}}(t) = \frac{1}{2}\cos t\,\hat{\mathbf{i}} + \frac{1}{2}\sin t\,\hat{\mathbf{j}} + \frac{\sqrt{3}}{2}\,t\,\hat{\mathbf{k}}$.

Answer

Length $= 2\pi$ units

8) Find the arc length of the vector-valued function $\vec{\mathbf{r}}(t) = -t\,\hat{\mathbf{i}} + 4t\hat{\mathbf{j}} + 3t\,\hat{\mathbf{k}}$ over $[0, 1]$.

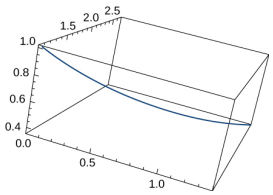
9) A particle travels in a circle with the equation of motion $\vec{\mathbf{r}}(t) = 3\cos t\,\hat{\mathbf{i}} + 3\sin t\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}$. Find the distance traveled around the circle by the particle.

Answer

6π units

10) Set up an integral to find the circumference of the ellipse with the equation $\vec{\mathbf{r}}(t) = \cos t\,\hat{\mathbf{i}} + 2\sin t\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}$.

11) Find the length of the curve $\vec{\mathbf{r}}(t) = \langle\sqrt{2}t, e^t, e^{-t}\rangle$ over the interval $0 \le t \le 1$. The graph is shown here:



Answer

$(e - \frac{1}{e})$ units ≈ 2.35 units

12) Find the length of the curve $\vec{\mathbf{r}}(t) = \langle 2\sin t,\ 5t,\ 2\cos t\rangle$ for $t \in [-10, 10]$.

Unit Tangent Vectors and Unit Normal Vectors

13) The position function for a particle is $\vec{\mathbf{r}}(t) = a\cos(\omega t)\,\hat{\mathbf{i}} + b\sin(\omega t)\,\hat{\mathbf{j}}$. Find the unit tangent vector and the unit normal vector at $t = 0$.

Solution:

$$\vec{\mathbf{r}}'(t) = -a\omega \sin(\omega t)\,\hat{\mathbf{i}} + b\omega \cos(\omega t)\,\hat{\mathbf{j}}$$

$$\|\vec{\mathbf{r}}'(t)\| = \sqrt{a^2\omega^2 \sin^2(\omega t) + b^2\omega^2 \cos^2(\omega t)}$$

$$\vec{\mathbf{T}}(t) = \frac{\vec{\mathbf{r}}'(t)}{\|\vec{\mathbf{r}}'(t)\|} = \frac{-a\omega \sin(\omega t)\,\hat{\mathbf{i}} + b\omega \cos(\omega t)\,\hat{\mathbf{j}}}{\sqrt{a^2\omega^2 \sin^2(\omega t) + b^2\omega^2 \cos^2(\omega t)}}$$

$$\vec{\mathbf{T}}(0) = \frac{b\omega\,\hat{\mathbf{j}}}{\sqrt{(b\omega)^2}} = \frac{b\omega\,\hat{\mathbf{j}}}{|b\omega|}$$

If $b\omega > 0,\ \vec{\mathbf{T}}(0) = \hat{\mathbf{j}}$, and if $b\omega < 0,\ T(0) = -\hat{\mathbf{j}}$

Answer

If $b\omega > 0,\ \vec{\mathbf{T}}(0) = \hat{\mathbf{j}}$, and if $b\omega < 0,\ \vec{\mathbf{T}}(0) = -\hat{\mathbf{j}}$

If $a > 0,\ \vec{\mathbf{N}}(0) = -\hat{\mathbf{i}}$, and if $a < 0,\ \vec{\mathbf{N}}(0) = \hat{\mathbf{i}}$

14) Given $\vec{\mathbf{r}}(t) = a\cos(\omega t)\,\hat{\mathbf{i}} + b\sin(\omega t)\,\hat{\mathbf{j}}$, find the binormal vector $\vec{\mathbf{B}}(0)$.

15) Given $\vec{\mathbf{r}}(t) = \langle 2e^t, e^t\cos t, e^t \sin t\rangle$, determine the unit tangent vector $\vec{\mathbf{T}}(t)$.

Answer

$$\vec{\mathbf{T}}(t) = \left\langle \frac{2}{\sqrt{6}}, \frac{\cos t - \sin t}{\sqrt{6}}, \frac{\cos t + \sin t}{\sqrt{6}} \right\rangle$$

$$= \left\langle \frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}(\cos t - \sin t), \frac{\sqrt{6}}{6}(\cos t + \sin t) \right\rangle$$

16) Given $\vec{\mathbf{r}}(t) = \langle 2e^t,\ e^t\cos t,\ e^t \sin t\rangle$, find the unit tangent vector $\vec{\mathbf{T}}(t)$ evaluated at $t = 0$, $\vec{\mathbf{T}}(0)$.

17) Given $\vec{\mathbf{r}}(t) = \langle 2e^t,\ e^t\cos t,\ e^t \sin t\rangle$, determine the unit normal vector $\vec{\mathbf{N}}(t)$.

Answer

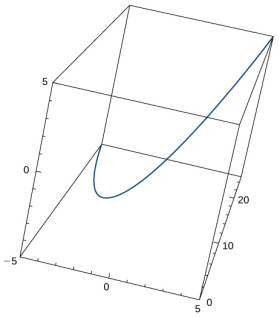
$$\vec{\mathbf{N}}(t) = \langle 0,\ -\tfrac{\sqrt{2}}{2}(\sin t + \cos t),\ \tfrac{\sqrt{2}}{2}(\cos t - \sin t)\rangle$$

18) Given $\vec{\mathbf{r}}(t) = \langle 2e^t,\ e^t\cos t,\ e^t \sin t\rangle$, find the unit normal vector $\vec{\mathbf{N}}(t)$ evaluated at $t = 0$, $\vec{\mathbf{N}}(0)$.

Answer

$$\vec{\mathbf{N}}(0) = \langle 0,\ -\tfrac{\sqrt{2}}{2},\ \tfrac{\sqrt{2}}{2}\rangle$$

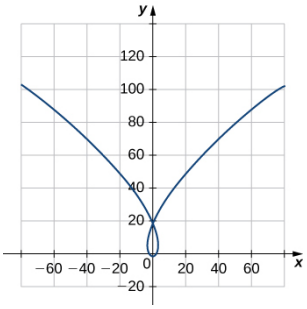
19) Given $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$, find the unit tangent vector $\vec{\mathbf{T}}(t)$. The graph is shown here:



Answer

$$\vec{\mathbf{T}}(t) = \frac{1}{\sqrt{4t^2+2}} < 1, 2t, 1 >$$

20) Find the unit tangent vector $\vec{\mathbf{T}}(t)$ and unit normal vector $\vec{\mathbf{N}}(t)$ at $t = 0$ for the plane curve $\vec{\mathbf{r}}(t) = \langle t^3 - 4t, 5t^2 - 2\rangle$. The graph is shown here:



21) Find the unit tangent vector $\vec{\mathbf{T}}(t)$ for $\vec{\mathbf{r}}(t) = 3t\,\hat{\mathbf{i}} + 5t^2\,\hat{\mathbf{j}} + 2t\,\hat{\mathbf{k}}$.

Answer

$$\vec{\mathbf{T}}(t) = \frac{1}{\sqrt{100t^2 + 13}}(3\,\hat{\mathbf{i}} + 10t\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}})$$

22) Find the principal normal vector to the curve $\vec{\mathbf{r}}(t) = \langle 6\cos t, 6\sin t\rangle$ at the point determined by $t = \frac{\pi}{3}$.

23) Find $\vec{\mathbf{T}}(t)$ for the curve $\vec{\mathbf{r}}(t) = (t^3 - 4t)\,\hat{\mathbf{i}} + (5t^2 - 2)\,\hat{\mathbf{j}}$.

Answer

$$\vec{\mathbf{T}}(t) = \frac{1}{\sqrt{9t^4 + 76t^2 + 16}}\left((3t^2 - 4)\,\hat{\mathbf{i}} + 10t\,\hat{\mathbf{j}}\right)$$

24) Find $\vec{\mathbf{N}}(t)$ for the curve $\vec{\mathbf{r}}(t) = (t^3 - 4t)\,\hat{\mathbf{i}} + (5t^2 - 2)\,\hat{\mathbf{j}}$.

25) Find the unit tangent vector $\vec{\mathbf{T}}(t)$ for $\vec{\mathbf{r}}(t) = \langle 2\sin t,\ 5t,\ 2\cos t\rangle$.

Answer

$$\vec{\mathbf{T}}(t) = \langle \tfrac{2\sqrt{29}}{29}\cos t,\ \tfrac{5\sqrt{29}}{29},\ -\tfrac{2\sqrt{29}}{29}\sin t\rangle$$

26) Find the unit normal vector $\vec{\mathbf{N}}(t)$ for $\vec{\mathbf{r}}(t) = \langle 2\sin t,\ 5t,\ 2\cos t\rangle$.

Answer

$$\vec{\mathbf{N}}(t) = \langle -\sin t,\ 0,\ -\cos t\rangle$$

Arc Length Parameterizations

27) Find the arc-length function $\vec{\mathbf{s}}(t)$ for the line segment given by $\vec{\mathbf{r}}(t) = \langle 3 - 3t,\ 4t\rangle$. Then write the arc-length parameterization of r with s as the parameter.

Answer

Arc-length function: $s(t) = 5t$; The arc-length parameterization of $\vec{\mathbf{r}}(t)$: $\vec{\mathbf{r}}(s) = \left(3 - \frac{3s}{5}\right)\,\hat{\mathbf{i}} + \frac{4s}{5}\,\hat{\mathbf{j}}$

28) Parameterize the helix $\vec{\mathbf{r}}(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$ using the arc-length parameter s, from $t = 0$.

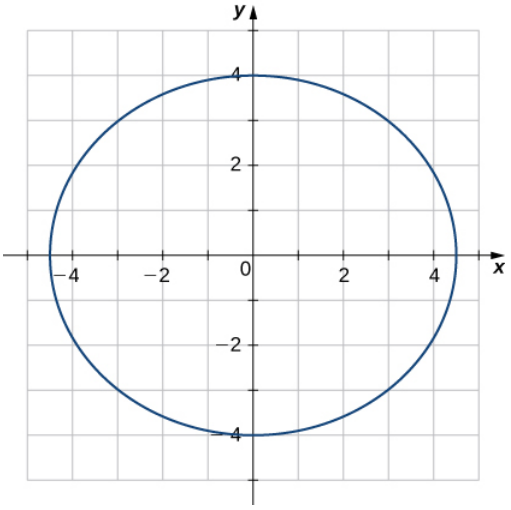
29) Parameterize the curve using the arc-length parameter s, at the point at which $t = 0$ for $\vec{\mathbf{r}}(t) = e^t \sin t\,\hat{\mathbf{i}} + e^t \cos t\,\hat{\mathbf{j}}$

Answer

$$\vec{\mathbf{r}}(s) = \left(1 + \frac{s}{\sqrt{2}}\right)\sin\left(\ln\left(1 + \frac{s}{\sqrt{2}}\right)\right)\,\hat{\mathbf{i}} + \left(1 + \frac{s}{\sqrt{2}}\right)\cos\left(\ln\left(1 + \frac{s}{\sqrt{2}}\right)\right)\,\hat{\mathbf{j}}$$

Curvature and the Osculating Circle

30) Find the curvature of the curve $\vec{\mathbf{r}}(t) = 5\cos t\,\hat{\mathbf{i}} + 4\sin t\,\hat{\mathbf{j}}$ at $t = \pi/3$. (*Note:* The graph is an ellipse.)



31) Find the x-coordinate at which the curvature of the curve $y = 1/x$ is a maximum value.

Answer

The maximum value of the curvature occurs at $x = 1$.

32) Find the curvature of the curve $\vec{\mathbf{r}}(t) = 5\cos t\,\hat{\mathbf{i}} + 5\sin t\,\hat{\mathbf{j}}$. Does the curvature depend upon the parameter t?

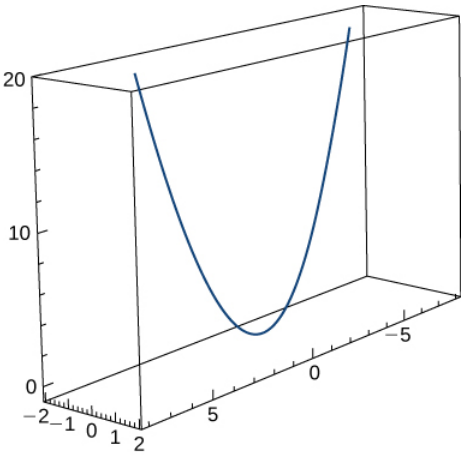
33) Find the curvature κ for the curve $y = x - \frac{1}{4}x^2$ at the point $x = 2$.

Answer

$\frac{1}{2}$

34) Find the curvature κ for the curve $y = \frac{1}{3}x^3$ at the point $x = 1$.

35) Find the curvature κ of the curve $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + 6t^2\,\hat{\mathbf{j}} + 4t\,\hat{\mathbf{k}}$. The graph is shown here:



Answer

$$\kappa \approx \frac{49.477}{(17 + 144t^2)^{3/2}}$$

36) Find the curvature of $\vec{\mathbf{r}}(t) = \langle 2\sin t, 5t, 2\cos t\rangle$.

37) Find the curvature of $\vec{\mathbf{r}}(t) = \sqrt{2}t\,\hat{\mathbf{i}} + e^t\,\hat{\mathbf{j}} + e^{-t}\,\hat{\mathbf{k}}$ at point $P(0, 1, 1)$.

Answer

$\frac{1}{2\sqrt{2}}$

38) At what point does the curve $y = e^x$ have maximum curvature?

39) What happens to the curvature as $x \to \infty$ for the curve $y = e^x$?

Answer

The curvature approaches zero.

40) Find the point of maximum curvature on the curve $y = \ln x$.

41) Find the equations of the normal plane and the osculating plane of the curve $\vec{\mathbf{r}}(t) = \langle 2\sin(3t), t, 2\cos(3t)\rangle$ at point $(0, \pi, -2)$.

Answer

$y = 6x + \pi$ and $x + 6y = 6\pi$

42) Find equations of the osculating circles of the ellipse $4y^2 + 9x^2 = 36$ at the points $(2, 0)$ and $(0, 3)$.

43) Find the equation for the osculating plane at point $t = \pi/4$ on the curve $\vec{\mathbf{r}}(t) = \cos(2t)\,\hat{\mathbf{i}} + \sin(2t)\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$.

Answer

$x + 2z = \frac{\pi}{2}$

44) Find the radius of curvature of $6y = x^3$ at the point $(2, \frac{4}{3})$.

45) Find the curvature at each point (x, y) on the hyperbola $\vec{\mathbf{r}}(t) = \langle a\cosh(t), b\sinh(t)\rangle$.

Answer

$$\frac{a^4b^4}{(b^4x^2 + a^4y^2)^{3/2}}$$

46) Calculate the curvature of the circular helix $\vec{\mathbf{r}}(t) = r\sin(t)\,\hat{\mathbf{i}} + r\cos(t)\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$.

47) Find the radius of curvature of $y = \ln(x + 1)$ at point $(2, \ln 3)$.

Answer

$\frac{10\sqrt{10}}{3}$

48) Find the radius of curvature of the hyperbola $xy = 1$ at point $(1, 1)$.

A particle moves along the plane curve C described by $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}}$. Use this parameterization to answer questions 49 - 51.

49) Find the length of the curve over the interval $[0, 2]$.

Answer

$\frac{1}{4}\left[4\sqrt{17} + \ln\left(4 + \sqrt{17}\right)\right]$ units ≈ 4.64678 units

50) Find the curvature of the plane curve at $t = 0, 1, 2$.

51) Describe the curvature as t increases from $t = 0$ to $t = 2$.

Answer

The curvature is decreasing over this interval.

The surface of a large cup is formed by revolving the graph of the function $y = 0.25x^{1.6}$ from $x = 0$ to $x = 5$ about the y-axis (measured in centimeters).

52) **[T]** Use technology to graph the surface.

53) Find the curvature κ of the generating curve as a function of x.

Answer

$$\kappa = \frac{30}{x^{2/5}\left(25+4x^{6/5}\right)^{3/2}}$$

Note that initially your answer may be:

$$\frac{6}{25x^{2/5}\left(1+\frac{4}{25}x^{6/5}\right)^{3/2}}$$

We can simplify it as follows:

$$\begin{aligned}\frac{6}{25x^{2/5}\left(1+\frac{4}{25}x^{6/5}\right)^{3/2}} &= \frac{6}{25x^{2/5}\left[\frac{1}{25}\left(25+4x^{6/5}\right)\right]^{3/2}} \\ &= \frac{6}{25x^{2/5}\left(\frac{1}{25}\right)^{3/2}\left[25+4x^{6/5}\right]^{3/2}} \\ &= \frac{6}{\frac{25}{125}x^{2/5}\left[25+4x^{6/5}\right]^{3/2}} \\ &= \frac{30}{x^{2/5}\left(25+4x^{6/5}\right)^{3/2}}\end{aligned}$$

54) **[T]** Use technology to graph the curvature function.

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13.4: Motion in Space

Learning Objectives

- Describe the velocity and acceleration vectors of a particle moving in space.
- Explain the tangential and normal components of acceleration.
- State Kepler's laws of planetary motion.

We have now seen how to describe curves in the plane and in space, and how to determine their properties, such as arc length and curvature. All of this leads to the main goal of this chapter, which is the description of motion along plane curves and space curves. We now have all the tools we need; in this section, we put these ideas together and look at how to use them.

Motion Vectors in the Plane and in Space

Our starting point is using vector-valued functions to represent the position of an object as a function of time. All of the following material can be applied either to curves in the plane or to space curves. For example, when we look at the orbit of the planets, the curves defining these orbits all lie in a plane because they are elliptical. However, a particle traveling along a helix moves on a curve in three dimensions.

Definition: Speed, Velocity, and Acceleration

Let $\vec{\mathbf{r}}(t)$ be a twice-differentiable vector-valued function of the parameter t that represents the position of an object as a function of time.

The velocity vector $\vec{\mathbf{v}}(t)$ of the object is given by

$$\text{Velocity } = \vec{\mathbf{v}}(t) = \vec{\mathbf{r}}\,'(t). \tag{13.4.1}$$

The acceleration vector $\vec{\mathbf{a}}(t)$ is defined to be

$$\text{Acceleration } = \vec{\mathbf{a}}(t) = \vec{\mathbf{v}}\,'(t) = \vec{\mathbf{r}}\,''(t). \tag{13.4.2}$$

The *speed* is defined to be

$$\text{Speed } = v(t) = \|\vec{\mathbf{v}}(t)\| = \|\vec{\mathbf{r}}\,'(t)\| = \frac{ds}{dt}. \tag{13.4.3}$$

Since $\vec{\mathbf{r}}(t)$ can be in either two or three dimensions, these vector-valued functions can have either two or three components. In two dimensions, we define $\vec{\mathbf{r}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}}$ and in three dimensions $\vec{\mathbf{r}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$. Then the velocity, acceleration, and speed can be written as shown in the following table.

Table 13.4.1: Formulas for Position, Velocity, Acceleration, and Speed

Quantity	Two Dimensions	Three Dimensions
Position	$\vec{\mathbf{r}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}}$	$\vec{\mathbf{r}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$
Velocity	$\vec{\mathbf{v}}(t) = x\,'(t)\hat{\mathbf{i}} + y\,'(t)\hat{\mathbf{j}}$	$\vec{\mathbf{v}}(t) = x\,'(t)\hat{\mathbf{i}} + y\,'(t)\hat{\mathbf{j}} + z\,'(t)\hat{\mathbf{k}}$
Acceleration	$\vec{\mathbf{a}}(t) = x\,''(t)\hat{\mathbf{i}} + y\,''(t)\hat{\mathbf{j}}$	$\vec{\mathbf{a}}(t) = x\,''(t)\hat{\mathbf{i}} + y\,''(t)\hat{\mathbf{j}} + z\,''(t)\hat{\mathbf{k}}$
Speed	$\|\vec{\mathbf{v}}(t)\| = \sqrt{(x\,'(t))^2 + (y\,'(t))^2}$	$\|\vec{\mathbf{v}}(t)\| = \sqrt{(x\,'(t))^2 + (y\,'(t))^2 + (z\,'(t))^2}$

Example 13.4.1: Studying Motion Along a Parabola

A particle moves in a parabolic path defined by the vector-valued function $\vec{\mathbf{r}}(t) = t^2\hat{\mathbf{i}} + \sqrt{5 - t^2}\hat{\mathbf{j}}$, where t measures time in seconds.

1. Find the velocity, acceleration, and speed as functions of time.

2. Sketch the curve along with the velocity vector at time $t = 1$.

Solution

1. We use Equations 13.4.1, 13.4.2, and 13.4.3:

$$\begin{aligned}
\vec{\mathbf{v}}(t) = \vec{\mathbf{r}}\,'(t) &= 2t\hat{\mathbf{i}} - \frac{t}{\sqrt{5-t^2}}\hat{\mathbf{j}} \\
\vec{\mathbf{a}}(t) = \vec{\mathbf{v}}\,'(t) &= 2\hat{\mathbf{i}} - 5(5-t^2)^{-\frac{3}{2}}\hat{\mathbf{j}} \\
||\vec{\mathbf{v}}(t)|| &= ||\vec{\mathbf{r}}\,'(t)|| \\
&= \sqrt{(2t)^2 + \left(-\frac{t}{\sqrt{5-t^2}}\right)^2} \\
&= \sqrt{4t^2 + \frac{t^2}{5-t^2}} \\
&= \sqrt{\frac{21t^2 - 4t^4}{5-t^2}}.
\end{aligned}$$

2. The graph of $\vec{\mathbf{r}}(t) = t^2\hat{\mathbf{i}} + \sqrt{5-t^2}\hat{\mathbf{j}}$ is a portion of a parabola (Figure 13.4.1).

When $t = 1$, $\vec{\mathbf{r}}(1) = (1)^2\hat{\mathbf{i}} + \sqrt{5-(1)^2}\hat{\mathbf{j}} = \hat{\mathbf{i}} + \sqrt{4}\hat{\mathbf{j}} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}}$.

Thus the particle would be located at the point $(1, 2)$ when $t = 1$.

The velocity vector at $t = 1$ is

$$\begin{aligned}
\vec{\mathbf{v}}(1) = \vec{\mathbf{r}}\,'(1) &= 2(1)\hat{\mathbf{i}} - \frac{1}{\sqrt{5-1^2}}\hat{\mathbf{j}} \\
&= 2\hat{\mathbf{i}} - \frac{1}{2}\hat{\mathbf{j}}
\end{aligned}$$

and the acceleration vector at $t = 1$ is

$$\vec{\mathbf{a}}(1) = \vec{\mathbf{v}}\,'(1) = 2\hat{\mathbf{i}} - 5(5-1^2)^{-3/2}\hat{\mathbf{j}} = 2\hat{\mathbf{i}} - \frac{5}{8}\hat{\mathbf{j}}.$$

Notice that the velocity vector is tangent to the path, as is always the case.

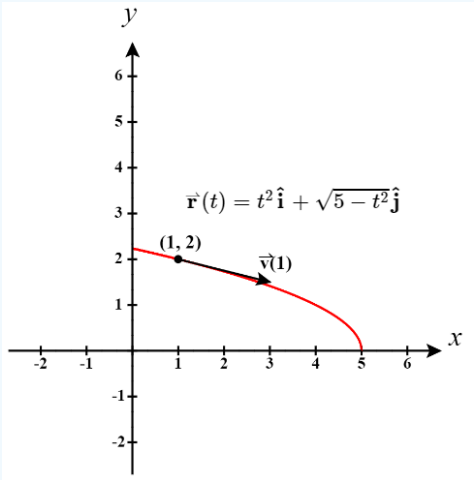

Figure 13.4.1: This graph depicts the velocity vector at time $t = 1$ for a particle moving in a parabolic path.

Exercise 13.4.1

A particle moves in a path defined by the vector-valued function $\vec{\mathbf{r}}(t) = (t^2 - 3t)\,\hat{\mathbf{i}} + (2t - 4)\,\hat{\mathbf{j}} + (t + 2)\,\hat{\mathbf{k}}$, where t measures time in seconds and where distance is measured in feet. Find the velocity, acceleration, and speed as functions of time.

Hint

Use Equations 13.4.1, 13.4.2, and 13.4.3.

Answer

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{r}}\,'(t) = (2t - 3)\,\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$$

$$\vec{\mathbf{a}}(t) = \vec{\mathbf{v}}\,'(t) = 2\,\hat{\mathbf{i}}$$

$$||\vec{\mathbf{r}}\,'(t)|| = \sqrt{(2t-3)^2 + 2^2 + 1^2} = \sqrt{4t^2 - 12t + 14}$$

The units for velocity and speed are feet per second, and the units for acceleration are feet per second squared.

To gain a better understanding of the velocity and acceleration vectors, imagine you are driving along a curvy road. If you do not turn the steering wheel, you would continue in a straight line and run off the road. The speed at which you are traveling when you run off the road, coupled with the direction, gives a vector representing your velocity, as illustrated in Figure 13.4.2.

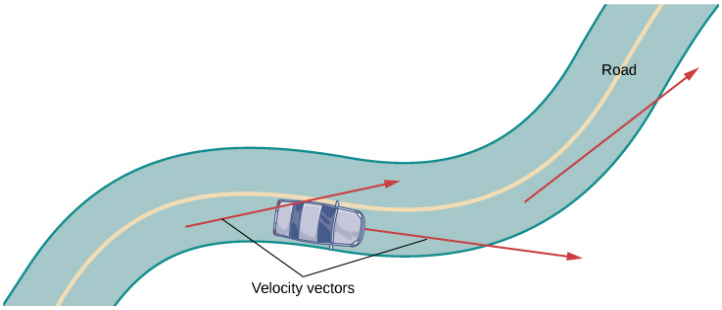

Figure 13.4.2: At each point along a road traveled by a car, the velocity vector of the car is tangent to the path traveled by the car.

However, the fact that you must turn the steering wheel to stay on the road indicates that your velocity is always changing (even if your speed is not) because your *direction* is constantly changing to keep you on the road. As you turn to the right, your acceleration vector also points to the right. As you turn to the left, your acceleration vector points to the left. This indicates that your velocity and acceleration vectors are constantly changing, regardless of whether your actual speed varies (Figure 13.4.3).

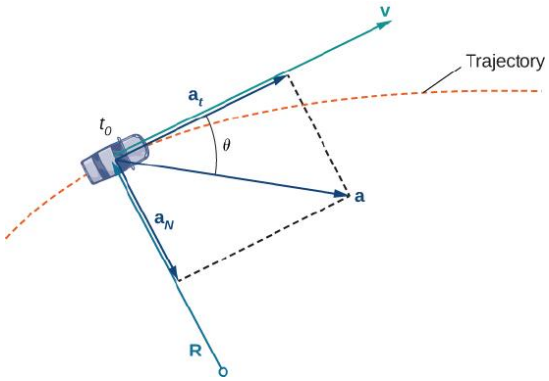

Figure 13.4.3: The dashed line represents the trajectory of an object (a car, for example). The acceleration vector points toward the inside of the turn at all times.

Components of the Acceleration Vector

We can combine some of the concepts discussed in Arc Length and Curvature with the acceleration vector to gain a deeper understanding of how this vector relates to motion in the plane and in space. Recall that the unit tangent vector $\vec{\mathbf{T}}$ and the unit normal vector $\vec{\mathbf{N}}$ form an osculating plane at any point P on the curve defined by a vector-valued function $\vec{\mathbf{r}}(t)$. The following

theorem shows that the acceleration vector $\vec{\mathbf{a}}(t)$ lies in the osculating plane and can be written as a linear combination of the unit tangent and the unit normal vectors.

Theorem 13.4.1: The Plane of the Acceleration Vector

The acceleration vector $\vec{\mathbf{a}}(t)$ of an object moving along a curve traced out by a twice-differentiable function $\vec{\mathbf{r}}(t)$ lies in the plane formed by the unit tangent vector $\vec{\mathbf{T}}(t)$ and the principal unit normal vector $\vec{\mathbf{N}}(t)$ to C. Furthermore,

$$\vec{\mathbf{a}}(t) = v'(t)\vec{\mathbf{T}}(t) + [v(t)]^2\kappa\vec{\mathbf{N}}(t)$$

Here, $v(t) = \|\vec{\mathbf{v}}(t)\|$ is the speed of the object and κ is the curvature of C traced out by $\vec{\mathbf{r}}(t)$.

Proof

Because $\vec{\mathbf{v}}(t) = \vec{\mathbf{r}}\,'(t)$ and $\vec{\mathbf{T}}(t) = \dfrac{\vec{\mathbf{r}}\,'(t)}{||\vec{\mathbf{r}}\,'(t)||}$, we have $\vec{\mathbf{v}}(t) = ||\vec{\mathbf{r}}\,'(t)||\vec{\mathbf{T}}(t) = v(t)\vec{\mathbf{T}}(t)$.

Now we differentiate this equation:

$$\vec{\mathbf{a}}(t) = \vec{\mathbf{v}}\,'(t) = \frac{d}{dt}\left(v(t)\vec{\mathbf{T}}(t)\right) = v'(t)\vec{\mathbf{T}}(t) + v(t)\vec{\mathbf{T}}\,'(t)$$

Since $\vec{\mathbf{N}}(t) = \dfrac{\vec{\mathbf{T}}\,'(t)}{||\vec{\mathbf{T}}\,'(t)||}$, we know $\vec{\mathbf{T}}\,'(t) = ||\vec{\mathbf{T}}\,'(t)||\vec{\mathbf{N}}(t)$, so

$$\vec{\mathbf{a}}(t) = v'(t)\vec{\mathbf{T}}(t) + v(t)||\vec{\mathbf{T}}\,'(t)||\vec{\mathbf{N}}(t).$$

A formula for curvature is $\kappa = \dfrac{||\vec{\mathbf{T}}\,'(t)||}{||\vec{\mathbf{r}}\,'(t)||}$, so $\vec{\mathbf{T}}\,'(t) = \kappa||\vec{\mathbf{r}}\,'(t)|| = \kappa v(t)$.

This gives $\vec{\mathbf{a}}(t) = v'(t)\vec{\mathbf{T}}(t) + \kappa(v(t))^2\vec{\mathbf{N}}(t)$.

□

The coefficients of $\vec{\mathbf{T}}(t)$ and $\vec{\mathbf{N}}(t)$ are referred to as the **tangential component of acceleration** and the **normal component of acceleration**, respectively. We write $a_{\vec{\mathbf{T}}}$ to denote the tangential component and $a_{\vec{\mathbf{N}}}$ to denote the normal component.

Theorem 13.4.2: Tangential and Normal Components of Acceleration

Let $\vec{\mathbf{r}}(t)$ be a vector-valued function that denotes the position of an object as a function of time. Then $\vec{\mathbf{a}}(t) = \vec{\mathbf{r}}\,''(t)$ is the acceleration vector. The tangential and normal components of acceleration $a_{\vec{\mathbf{T}}}$ and $a_{\vec{\mathbf{N}}}$ are given by the formulas

$$a_{\vec{\mathbf{T}}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{T}} = \frac{\vec{\mathbf{v}} \cdot \vec{\mathbf{a}}}{||\vec{\mathbf{v}}||} \tag{13.4.4}$$

and

$$a_{\vec{\mathbf{N}}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{N}} = \frac{||\vec{\mathbf{v}} \times \vec{\mathbf{a}}||}{||\vec{\mathbf{v}}||} = \sqrt{||\vec{\mathbf{a}}||^2 - \left(a_{\vec{\mathbf{T}}}\right)^2}. \tag{13.4.5}$$

These components are related by the formula

$$\vec{\mathbf{a}}(t) = a_{\vec{\mathbf{T}}}\vec{\mathbf{T}}(t) + a_{\vec{\mathbf{N}}}\vec{\mathbf{N}}(t). \tag{13.4.6}$$

Here $\vec{\mathbf{T}}(t)$ is the unit tangent vector to the curve defined by $\vec{\mathbf{r}}(t)$, and $\vec{\mathbf{N}}(t)$ is the unit normal vector to the curve defined by $\vec{\mathbf{r}}(t)$.

The normal component of acceleration is also called the *centripetal component of acceleration* or sometimes the *radial component of acceleration*. To understand centripetal acceleration, suppose you are traveling in a car on a circular track at a constant speed. Then, as we saw earlier, the acceleration vector points toward the center of the track at all times. As a rider in the car, you feel a pull toward the *outside* of the track because you are constantly turning. This sensation acts in the opposite direction of centripetal

acceleration. The same holds true for non-circular paths. The reason is that your body tends to travel in a straight line and resists the force resulting from acceleration that push it toward the side. Note that at point B in Figure 13.4.4 the acceleration vector is pointing backward. This is because the car is decelerating as it goes into the curve.

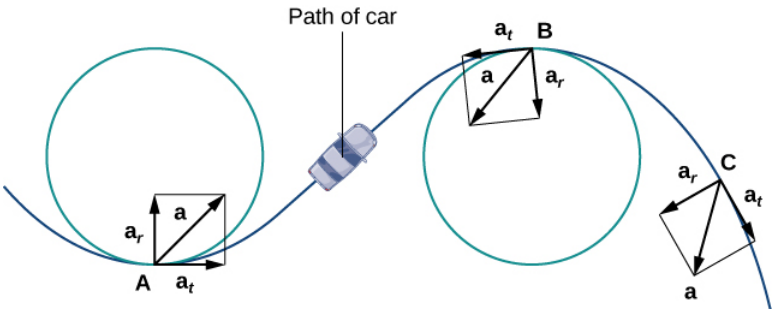

Figure 13.4.4: The tangential and normal components of acceleration can be used to describe the acceleration vector.

The tangential and normal unit vectors at any given point on the curve provide a frame of reference at that point. The tangential and normal components of acceleration are the projections of the acceleration vector onto $\vec{\mathbf{T}}$ and $\vec{\mathbf{N}}$, respectively.

Example 13.4.2: Finding Components of Acceleration

A particle moves in a path defined by the vector-valued function $\vec{\mathbf{r}}(t) = t^2\,\hat{\mathbf{i}} + (2t-3)\,\hat{\mathbf{j}} + (3t^2-3t)\,\hat{\mathbf{k}}$, where t measures time in seconds and distance is measured in feet.

a. Find $a_{\vec{\mathbf{T}}}$ and $a_{\vec{\mathbf{N}}}$ as functions of t.
b. Find $a_{\vec{\mathbf{T}}}$ and $a_{\vec{\mathbf{N}}}$ at time $t = 2$.

Solution

a. Let's start deriving the velocity and acceleration functions:

$$\begin{aligned}\vec{\mathbf{v}}(t) &= \vec{\mathbf{r}}\,'(t)\\ &= 2t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + (6t-3)\,\hat{\mathbf{k}}\\ \vec{\mathbf{a}}(t) &= \vec{\mathbf{v}}\,'(t)\\ &= 2\,\hat{\mathbf{i}} + 6\,\hat{\mathbf{k}}\end{aligned}$$

Now we apply Equation 13.4.4:

$$\begin{aligned}a_{\vec{\mathbf{T}}} &= \frac{\vec{\mathbf{v}}\cdot\vec{\mathbf{a}}}{||\vec{\mathbf{v}}||}\\ &= \frac{(2t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + (6t-3)\,\hat{\mathbf{k}})\cdot(2\,\hat{\mathbf{i}} + 6\,\hat{\mathbf{k}})}{||2t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + (6t-3)\,\hat{\mathbf{k}}||}\\ &= \frac{4t + 6(6t-3)}{\sqrt{(2t)^2 + 2^2 + (6t-3)^2}}\\ &= \frac{40t-18}{40t^2-36t+13}\end{aligned}$$

Now we can apply Equation 13.4.5:

$$
\begin{aligned}
a_{\vec{\mathbf{N}}} &= \sqrt{||\vec{\mathbf{a}}||^2 - \left(a_{\vec{\mathbf{T}}}\right)^2} \\
&= \sqrt{||2\,\hat{\mathbf{i}} + 6\,\hat{\mathbf{k}}||^2 - \left(\frac{40t-18}{\sqrt{40t^2-36t+13}}\right)^2} \\
&= \sqrt{4 + 36 - \frac{(40t-18)^2}{40t^2-36t+13}} \\
&= \sqrt{\frac{40(40t^2-36t+13) - (1600t^2-1440t+324)}{40t^2-36t+13}} \\
&= \sqrt{\frac{196}{40t^2-36t+13}} \\
&= \frac{14}{\sqrt{40t^2-36t+13}}
\end{aligned}
$$

b. We must evaluate each of the answers from part a at $t = 2$:

$$
\begin{aligned}
a_{\vec{\mathbf{T}}}(2) &= \frac{40(2)-18}{\sqrt{40(2)^2-36(2)+13}} \\
&= \frac{80-18}{\sqrt{160-72+13}} \\
&= \frac{62}{\sqrt{101}} \\
a_{\vec{\mathbf{N}}}(2) &= \frac{14}{\sqrt{40(2)^2-36(2)+13}} \\
&= \frac{14}{\sqrt{160-72+13}} = \frac{14}{\sqrt{101}}.
\end{aligned}
$$

The units of acceleration are feet per second squared, as are the units of the normal and tangential components of acceleration.

Exercise 13.4.2

An object moves in a path defined by the vector-valued function $\vec{\mathbf{r}}(t) = 4t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}}$, where t measures time in seconds.

a. Find $a_{\vec{\mathbf{T}}}$ and $a_{\vec{\mathbf{N}}}$ as functions of t.
b. Find $a_{\vec{\mathbf{T}}}$ and $a_{\vec{\mathbf{N}}}$ at time $t = -3$.

Hint

Use Equations 13.4.4 and 13.4.5

Answer

a.

$$
\begin{aligned}
a_{\vec{\mathbf{T}}} = \frac{\vec{\mathbf{v}}(t)\cdot\vec{\mathbf{a}}(t)}{||\vec{\mathbf{v}}(t)||} &= \frac{\vec{\mathbf{r}}\,'(t)\cdot\vec{\mathbf{r}}\,''(t)}{||\vec{\mathbf{r}}\,'(t)||} \\
&= \frac{(4\,\hat{\mathbf{i}} + 2t\,\hat{\mathbf{j}})\cdot(2\,\hat{\mathbf{j}})}{||4\,\hat{\mathbf{i}} + 2t\,\hat{\mathbf{j}}||} \\
&= \frac{4t}{\sqrt{4^2+(2t)^2}} \\
&= \frac{2t}{\sqrt{2+t^2}}
\end{aligned}
$$

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$$a_{\vec{\mathbf{N}}} = \sqrt{||\vec{\mathbf{a}}||^2 - a_{\vec{\mathbf{T}}}^2}$$
$$= \sqrt{||2\hat{\mathbf{j}}||^2 - \left(\frac{2t}{\sqrt{2+t^2}}\right)^2}$$
$$= \sqrt{4 - \frac{4t^2}{2+t^2}}$$

b.

$$a_{\vec{\mathbf{T}}}(-3) = \frac{2(-3)}{\sqrt{2+(-3)^2}}$$
$$= \frac{-6}{\sqrt{11}}$$

$$a_{\vec{\mathbf{N}}}(-3) = \sqrt{4 - \frac{4(-3)^2}{2+(-3)^2}}$$
$$= \sqrt{4 - \frac{36}{11}}$$
$$= \sqrt{\frac{8}{11}}$$
$$= \frac{2\sqrt{2}}{\sqrt{11}}$$

Projectile Motion

Now let's look at an application of vector functions. In particular, let's consider the effect of gravity on the motion of an object as it travels through the air, and how it determines the resulting trajectory of that object. In the following, we ignore the effect of air resistance. This situation, with an object moving with an initial velocity but with no forces acting on it other than gravity, is known as projectile motion. It describes the motion of objects from golf balls to baseballs, and from arrows to cannonballs.

First we need to choose a coordinate system. If we are standing at the origin of this coordinate system, then we choose the positive y-axis to be up, the negative y-axis to be down, and the positive x-axis to be forward (i.e., away from the thrower of the object). The effect of gravity is in a downward direction, so Newton's second law tells us that the force on the object resulting from gravity is equal to the mass of the object times the acceleration resulting from gravity, or $\vec{\mathbf{F}}_g = m\vec{\mathbf{a}}$, where $\vec{\mathbf{F}}_g$ represents the force from gravity and $\vec{\mathbf{a}} = -g\,\hat{\mathbf{j}}$ represents the acceleration resulting from gravity at Earth's surface. The value of g in the English system of measurement is approximately 32 ft/sec^2 and it is approximately 9.8 m/sec^2 in the metric system. This is the only force acting on the object. Since gravity acts in a downward direction, we can write the force resulting from gravity in the form $\vec{\mathbf{F}}_g = -mg\,\hat{\mathbf{j}}$, as shown in Figure 13.4.5.

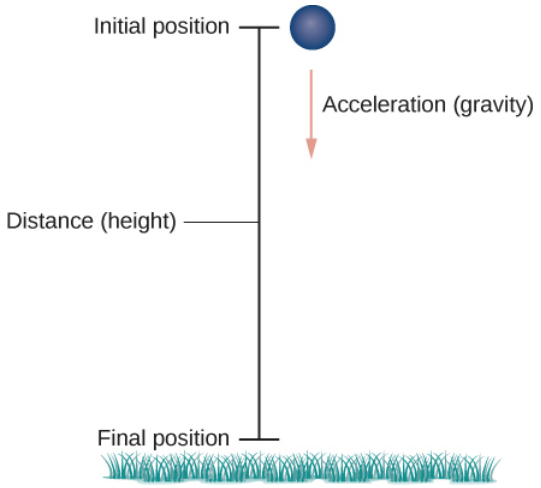


Figure 13.4.5: An object is falling under the influence of gravity.

Newton's second law also tells us that $F = m\vec{\mathbf{a}}$, where $\vec{\mathbf{a}}$ represents the acceleration vector of the object. This force must be equal to the force of gravity at all times, so we therefore know that

$$\begin{aligned} \vec{\mathbf{F}} &= \vec{\mathbf{F}}_g \\ m\vec{\mathbf{a}} &= -mg\hat{\mathbf{j}} \\ \vec{\mathbf{a}} &= -g\hat{\mathbf{j}}. \end{aligned}$$

Now we use the fact that the acceleration vector is the first derivative of the velocity vector. Therefore, we can rewrite the last equation in the form

$$\vec{\mathbf{v}}\,'(t) = -g\hat{\mathbf{j}}$$

By taking the antiderivative of each side of this equation we obtain

$$\vec{\mathbf{v}}(t) = \int -g\hat{\mathbf{j}}\, dt = -gt\hat{\mathbf{j}} + \vec{\mathbf{C}}_1$$

for some constant vector $\vec{\mathbf{C}}_1$. To determine the value of this vector, we can use the velocity of the object at a fixed time, say at time $t = 0$. We call this velocity the *initial velocity*: $\vec{\mathbf{v}}(0) = \vec{\mathbf{v}}_0$. Therefore, $\vec{\mathbf{v}}(0) = -g(0)\,\hat{\mathbf{j}} + \vec{\mathbf{C}}_1 = \vec{\mathbf{v}}_0$ and $\vec{\mathbf{C}}_1 = \vec{\mathbf{v}}_0$. This gives the velocity vector as $\vec{\mathbf{v}}(t) = -gt\hat{\mathbf{j}} + \vec{\mathbf{v}}_0$.

Next we use the fact that velocity $\vec{\mathbf{v}}(t)$ is the derivative of position $\vec{\mathbf{s}}(t)$. This gives the equation

$$\vec{\mathbf{s}}\,'(t) = -gt\hat{\mathbf{j}} + \vec{\mathbf{v}}_0.$$

Taking the antiderivative of both sides of this equation leads to

$$\begin{aligned} \vec{\mathbf{s}}(t) &= \int -gt\hat{\mathbf{j}} + \vec{\mathbf{v}}_0\, dt \\ &= -\frac{1}{2}gt^2\hat{\mathbf{j}} + \vec{\mathbf{v}}_0 t + \vec{\mathbf{C}}_2 \end{aligned}$$

with another unknown constant vector $\vec{\mathbf{C}}_2$. To determine the value of $\vec{\mathbf{C}}_2$, we can use the position of the object at a given time, say at time $t = 0$. We call this position the *initial position*: $\vec{\mathbf{s}}(0) = \vec{\mathbf{s}}_0$. Therefore, $\vec{\mathbf{s}}(0) = -(1/2)g(0)^2\,\hat{\mathbf{j}} + \vec{\mathbf{v}}_0(0) + \vec{\mathbf{C}}_2 = \vec{\mathbf{s}}_0$. This gives the position of the object at any time as

$$\vec{\mathbf{s}}(t) = -\frac{1}{2}gt^2\hat{\mathbf{j}} + \vec{\mathbf{v}}_0 t + \vec{\mathbf{s}}_0.$$

Let's take a closer look at the initial velocity and initial position. In particular, suppose the object is thrown upward from the origin at an angle θ to the horizontal, with initial speed $\vec{\mathbf{v}}_0$. How can we modify the previous result to reflect this scenario? First, we can assume it is thrown from the origin. If not, then we can move the origin to the point from where it is thrown. Therefore, $\vec{\mathbf{s}}_0 = \vec{\mathbf{0}}$, as shown in Figure 13.4.6.

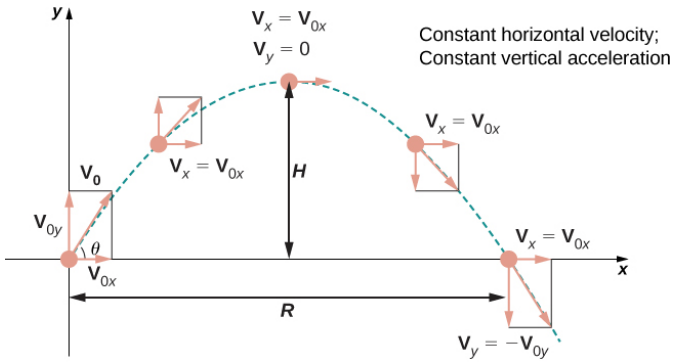


Figure 13.4.6: Projectile motion when the object is thrown upward at an angle θ. The horizontal motion is at constant velocity and the vertical motion is at constant acceleration.

We can rewrite the initial velocity vector in the form $\vec{\mathbf{v}}_0 = v_0\cos\theta\,\hat{\mathbf{i}} + v_0\sin\theta\,\hat{\mathbf{j}}$. Then the equation for the position function $\vec{\mathbf{s}}(t)$ becomes

$$\begin{aligned}\vec{\mathbf{s}}(t) &= -\frac{1}{2}gt^2\,\hat{\mathbf{j}} + v_0 t\cos\theta\,\hat{\mathbf{i}} + v_0 t\sin\theta\,\hat{\mathbf{j}} \\ &= v_0 t\cos\theta\,\hat{\mathbf{i}} + v_0 t\sin\theta\,\hat{\mathbf{j}} - \frac{1}{2}gt^2\,\hat{\mathbf{j}} \\ &= v_0 t\cos\theta\,\hat{\mathbf{i}} + \left(v_0 t\sin\theta - \frac{1}{2}gt^2\right)\hat{\mathbf{j}}.\end{aligned}$$

The coefficient of $\hat{\mathbf{i}}$ represents the horizontal component of $\vec{\mathbf{s}}(t)$ and is the horizontal distance of the object from the origin at time t. The maximum value of the horizontal distance (measured at the same initial and final altitude) is called the range R. The coefficient of $\hat{\mathbf{j}}$ represents the vertical component of $\vec{\mathbf{s}}(t)$ and is the altitude of the object at time t. The maximum value of the vertical distance is the height H.

Example 13.4.3: Motion of a Cannonball

During an Independence Day celebration, a cannonball is fired from a cannon on a cliff toward the water. The cannon is aimed at an angle of 30° above horizontal and the initial speed of the cannonball is 600 ft/sec. The cliff is 100 ft above the water (Figure 13.4.7).

a. Find the maximum height of the cannonball.
b. How long will it take for the cannonball to splash into the sea?
c. How far out to sea will the cannonball hit the water?

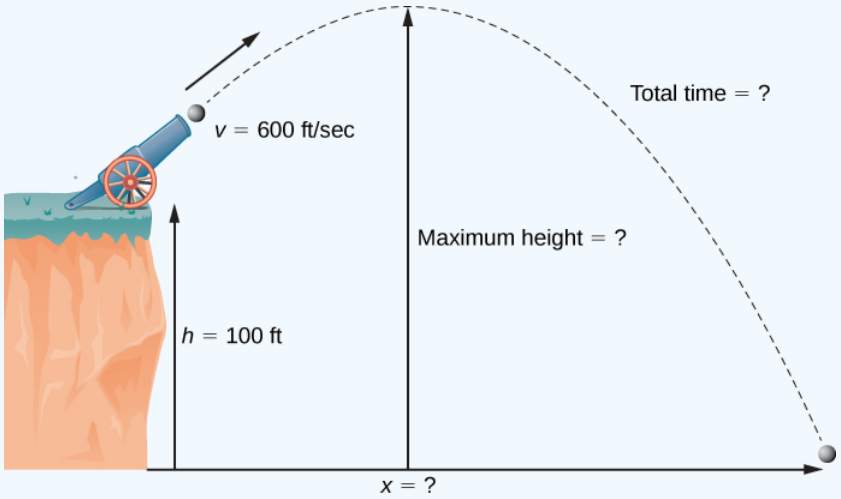


Figure 13.4.7: The flight of a cannonball (ignoring air resistance) is projectile motion.

Solution

We use the equation

$$\vec{\mathbf{s}}(t) = v_0 t\cos\theta\,\hat{\mathbf{i}} + \left(v_0 t\sin\theta - \frac{1}{2}gt^2\right)\hat{\mathbf{j}}$$

with $\theta = 30^\circ$, $g = 32\frac{\text{ft}}{\text{sec}^2}$, and $v_0 = 600\frac{\text{ft}}{\text{sec}}$. Then the position equation becomes

$$\begin{aligned}\vec{\mathbf{s}}(t) &= 600t(\cos 30^\circ)\,\hat{\mathbf{i}} + \left(600t\sin 30^\circ - \frac{1}{2}(32)t^2\right)\hat{\mathbf{j}} \\ &= 300t\sqrt{3}\,\hat{\mathbf{i}} + \left(300t - 16t^2\right)\hat{\mathbf{j}}\end{aligned}$$

a. The cannonball reaches its maximum height when the vertical component of its velocity is zero, because the cannonball is neither rising nor falling at that point. The velocity vector is

$$\begin{aligned}\vec{\mathbf{v}}(t) &= \vec{\mathbf{s}}\,'(t) \\ &= 300\sqrt{3}\,\hat{\mathbf{i}} + (300 - 32t)\,\hat{\mathbf{j}}\end{aligned}$$

Therefore, the vertical component of velocity is given by the expression $300 - 32t$. Setting this expression equal to zero and solving for t gives $t = 9.375$ sec. The height of the cannonball at this time is given by the vertical component of the position vector, evaluated at $t = 9.375$.

$$\begin{aligned}\vec{\mathbf{s}}(9.375) &= 300(9.375)\sqrt{3}\,\hat{\mathbf{i}} + (300(9.375) - 16(9.375)^2)\,\hat{\mathbf{j}} \\ &= 4871.39\,\hat{\mathbf{i}} + 1406.25\,\hat{\mathbf{j}}\end{aligned}$$

Therefore, the maximum height of the cannonball is 1406.39 ft above the cannon, or 1506.39 ft above sea level.

b. When the cannonball lands in the water, it is 100 ft below the cannon. Therefore, the vertical component of the position vector is equal to -100. Setting the vertical component of $\vec{\mathbf{s}}(t)$ equal to -100 and solving, we obtain

$$\begin{aligned}300t - 16t^2 &= -100 \\ 16t^2 - 300t - 100 &= 0 \\ 4t^2 - 75 - 25 &= 0 \\ t &= \frac{75 \pm \sqrt{(-75)^2 - 4(4)(-25)}}{2(4)} \\ &= \frac{75 \pm \sqrt{6025}}{8} \\ &= \frac{75 \pm 5\sqrt{241}}{8}\end{aligned}$$

The positive value of t that solves this equation is approximately 19.08. Therefore, the cannonball hits the water after approximately 19.08 sec.

c. To find the distance out to sea, we simply substitute the answer from part (b) into $\vec{\mathbf{s}}(t)$:

$$\begin{aligned}\vec{\mathbf{s}}(19.08) &= 300(19.08)\sqrt{3}\,\hat{\mathbf{i}} + \left(300(19.08) - 16(19.08)^2\right)\,\hat{\mathbf{j}} \\ &= 9914.26\,\hat{\mathbf{i}} - 100.7424\,\hat{\mathbf{j}}\end{aligned}$$

Therefore, the ball hits the water about 9914.26 ft away from the base of the cliff. Notice that the vertical component of the position vector is very close to −100, which tells us that the ball just hit the water. Note that 9914.26 feet is not the true range of the cannon since the cannonball lands in the ocean at a location below the cannon. The range of the cannon would be determined by finding how far out the cannonball is when its height is 100 ft above the water (the same as the altitude of the cannon).

? Exercise 13.4.3

An archer fires an arrow at an angle of 40° above the horizontal with an initial speed of 98 m/sec. The height of the archer is 171.5 cm. Find the horizontal distance the arrow travels before it hits the ground.

Hint

The equation for the position vector needs to account for the height of the archer in meters.

Answer

967.15 m

One final question remains: In general, what is the maximum distance a projectile can travel, given its initial speed? To determine this distance, we assume the projectile is fired from ground level and we wish it to return to ground level. In other words, we want to determine an equation for the range. In this case, the equation of projectile motion is

$$\vec{\mathbf{s}} = v_0 t\cos\theta\,\hat{\mathbf{i}} + \left(v_0 t\sin\theta - \frac{1}{2}gt^2\right)\,\hat{\mathbf{j}}.$$

Setting the second component equal to zero and solving for t yields

$$v_0 t \sin\theta - \frac{1}{2} g t^2 = 0$$
$$t\left(v_0 \sin\theta - \frac{1}{2} g t\right) = 0$$

Therefore, either $t = 0$ or $t = \dfrac{2v_0 \sin\theta}{g}$. We are interested in the second value of t, so we substitute this into $\vec{\mathbf{s}}(t)$, which gives

$$\begin{aligned}\vec{\mathbf{s}}\left(\frac{2v_0 \sin\theta}{g}\right) &= v_0\left(\frac{2v_0 \sin\theta}{g}\right)\cos\theta\,\hat{\mathbf{i}} + \left(v_0\left(\frac{2v_0 \sin\theta}{g}\right)\sin\theta - \frac{1}{2}g\left(\frac{2v_0 \sin\theta}{g}\right)^2\right)\hat{\mathbf{j}} \\ &= \left(\frac{2v_0^2 \sin\theta\cos\theta}{g}\right)\hat{\mathbf{i}} \\ &= \frac{v_0^2 \sin 2\theta}{g}\,\hat{\mathbf{i}}.\end{aligned}$$

Thus, the expression for the range of a projectile fired at an angle θ is

$$R = \frac{v_0^2 \sin 2\theta}{g}\,\hat{\mathbf{i}}.$$

The only variable in this expression is θ. To maximize the distance traveled, take the derivative of the coefficient of $\hat{\mathbf{i}}$ with respect to θ and set it equal to zero:

$$\begin{aligned}\frac{d}{d\theta}\left(\frac{v_0^2 \sin 2\theta}{g}\right) &= 0 \\ \frac{2v_0^2 \cos 2\theta}{g} &= 0 \\ \theta &= 45^\circ\end{aligned}$$

This value of θ) is the smallest positive value that makes the derivative equal to zero. Therefore, in the absence of air resistance, the best angle to fire a projectile (to maximize the range) is at a 45° angle. The distance it travels is given by

$$\vec{\mathbf{s}}\left(\frac{2v_0 \sin 45^\circ}{g}\right) = \frac{v_0^2 \sin 90^\circ}{g}\,\hat{\mathbf{i}} = \frac{v_0^2}{g}\,\hat{\mathbf{i}}$$

Therefore, the range for an angle of 45° is $\dfrac{v_0^2}{g}$ units.

Kepler's Laws

During the early 1600s, Johannes Kepler was able to use the amazingly accurate data from his mentor Tycho Brahe to formulate his three laws of planetary motion, now known as Kepler's laws of planetary motion. These laws also apply to other objects in the solar system in orbit around the Sun, such as comets (e.g., Halley's comet) and asteroids. Variations of these laws apply to satellites in orbit around Earth.

Theorem 13.4.2: Kepler's Laws of Planetary Motion

1. The path of any planet about the Sun is elliptical in shape, with the center of the Sun located at one focus of the ellipse (the law of ellipses).
2. A line drawn from the center of the Sun to the center of a planet sweeps out equal areas in equal time intervals (the law of equal areas) (Figure 13.4.8).
3. The ratio of the squares of the periods of any two planets is equal to the ratio of the cubes of the lengths of their semimajor orbital axes (the Law of Harmonies).

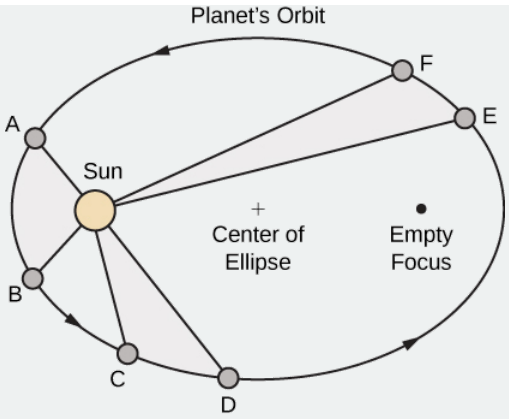


Figure 13.4.8: Kepler's first and second laws are pictured here. The Sun is located at a focus of the elliptical orbit of any planet. Furthermore, the shaded areas are all equal, assuming that the amount of time measured as the planet moves is the same for each region.

Kepler's third law is especially useful when using appropriate units. In particular, *1 astronomical unit* is defined to be the average distance from Earth to the Sun, and is now recognized to be 149,597,870,700 m or, approximately 93,000,000 mi. We therefore write 1 A.U. = 93,000,000 mi. Since the time it takes for Earth to orbit the Sun is 1 year, we use Earth years for units of time. Then, substituting 1 year for the period of Earth and 1 A.U. for the average distance to the Sun, Kepler's third law can be written as

$$T_p^2 = D_p^3$$

for any planet in the solar system, where T_P is the period of that planet measured in Earth years and D_P is the average distance from that planet to the Sun measured in astronomical units. Therefore, if we know the average distance from a planet to the Sun (in astronomical units), we can then calculate the length of its year (in Earth years), and vice versa.

Kepler's laws were formulated based on observations from Brahe; however, they were not proved formally until Sir Isaac Newton was able to apply calculus. Furthermore, Newton was able to generalize Kepler's third law to other orbital systems, such as a moon orbiting around a planet. Kepler's original third law only applies to objects orbiting the Sun.

Proof

Let's now prove Kepler's first law using the calculus of vector-valued functions. First we need a coordinate system. Let's place the Sun at the origin of the coordinate system and let the vector-valued function $\vec{\mathbf{r}}(t)$ represent the location of a planet as a function of time. Newton proved Kepler's law using his second law of motion and his law of universal gravitation. Newton's second law of motion can be written as $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$, where $\vec{\mathbf{F}}$ represents the net force acting on the planet. His law of universal gravitation can be written in the form $\vec{\mathbf{F}} = -\dfrac{GmM}{||\vec{\mathbf{r}}||^2} \cdot \dfrac{\vec{\mathbf{r}}}{||\vec{\mathbf{r}}||}$, which indicates that the force resulting from the gravitational attraction of the Sun points back toward the Sun, and has magnitude $\dfrac{GmM}{||\vec{\mathbf{r}}||^2}$ (Figure 13.4.9).

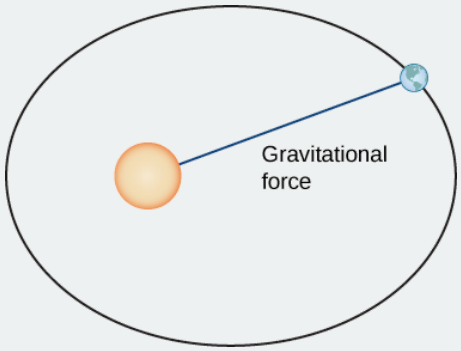


Figure 13.4.9: The gravitational force between Earth and the Sun is equal to the mass of the earth times its acceleration.

Setting these two forces equal to each other, and using the fact that $\vec{\mathbf{a}}(t) = \vec{\mathbf{v}}\,'(t)$, we obtain

$$m\vec{\mathbf{v}}\,'(t) = -\frac{GmM}{\|\vec{\mathbf{r}}\|^2}\cdot\frac{\vec{\mathbf{r}}}{\|\vec{\mathbf{r}}\|},$$

which can be rewritten as

$$\frac{d\vec{\mathbf{v}}}{dt} = -\frac{GM}{||\vec{\mathbf{r}}||^3}\vec{\mathbf{r}}.$$

This equation shows that the vectors $d\vec{\mathbf{v}}/dt$ and $\vec{\mathbf{r}}$ are parallel to each other, so $d\vec{\mathbf{v}}/dt \times \vec{\mathbf{r}} = \vec{\mathbf{0}}$. Next, let's differentiate $\vec{\mathbf{r}} \times \vec{\mathbf{v}}$ with respect to time:

$$\frac{d}{dt}(\vec{\mathbf{r}} \times \vec{\mathbf{v}}) = \frac{d\vec{\mathbf{r}}}{dt} \times \vec{\mathbf{v}} + \vec{\mathbf{r}} \times \frac{d\vec{\mathbf{v}}}{dt} = \vec{\mathbf{v}} \times \vec{\mathbf{v}} + \vec{\mathbf{0}} = \vec{\mathbf{0}}. \tag{13.4.7}$$

This proves that $\vec{\mathbf{r}} \times \vec{\mathbf{v}}$ is a constant vector, which we call $\vec{\mathbf{C}}$. Since $\vec{\mathbf{r}}$ and $\vec{\mathbf{v}}$ are both perpendicular to $\vec{\mathbf{C}}$ for all values of t, they must lie in a plane perpendicular to $\vec{\mathbf{C}}$. Therefore, the motion of the planet lies in a plane.

Next we calculate the expression $d\vec{\mathbf{v}}/dt \times \vec{\mathbf{C}}$:

$$\frac{d\vec{\mathbf{v}}}{dt} \times \vec{\mathbf{C}} = -\frac{GM}{||\vec{\mathbf{r}}||^3}\vec{\mathbf{r}} \times (\vec{\mathbf{r}} \times \vec{\mathbf{v}}) = -\frac{GM}{||\vec{\mathbf{r}}||^3}[(\vec{\mathbf{r}} \cdot \vec{\mathbf{v}})\vec{\mathbf{r}} - (\vec{\mathbf{r}} \cdot \vec{\mathbf{r}})\vec{\mathbf{v}}]. \tag{13.4.8}$$

The last equality in Equation 13.4.8 is from the triple cross product formula (see the cross product section in Introduction to Vectors in Space). We need an expression for $\vec{\mathbf{r}} \cdot \vec{\mathbf{v}}$. To calculate this, we differentiate $\vec{\mathbf{r}} \cdot \vec{\mathbf{r}}$ with respect to time:

$$\frac{d}{dt}(\vec{\mathbf{r}} \cdot \vec{\mathbf{r}}) = \frac{d\vec{\mathbf{r}}}{dt} \cdot \vec{\mathbf{r}} + \vec{\mathbf{r}} \cdot \frac{d\vec{\mathbf{r}}}{dt} = 2\vec{\mathbf{r}} \cdot \frac{d\vec{\mathbf{r}}}{dt} = 2\vec{\mathbf{r}} \cdot \vec{\mathbf{v}}. \tag{13.4.9}$$

Since $\vec{\mathbf{r}} \cdot \vec{\mathbf{r}} = ||\vec{\mathbf{r}}||^2$, we also have

$$\frac{d}{dt}(\vec{\mathbf{r}} \cdot \vec{\mathbf{r}}) = \frac{d}{dt}||\vec{\mathbf{r}}||^2 = 2||\vec{\mathbf{r}}||\frac{d}{dt}||\vec{\mathbf{r}}||. \tag{13.4.10}$$

Combining Equation 13.4.9 and Equation 13.4.10, we get

$$2\vec{\mathbf{r}} \cdot \vec{\mathbf{v}} = 2||\vec{\mathbf{r}}||\frac{d}{dt}||\vec{\mathbf{r}}||$$
$$\vec{\mathbf{r}} \cdot \vec{\mathbf{v}} = ||\vec{\mathbf{r}}||\frac{d}{dt}||\vec{\mathbf{r}}||.$$

Substituting this into Equation 13.4.8 gives us

$$\begin{aligned}\frac{d\vec{\mathbf{v}}}{dt} \times \vec{\mathbf{C}} &= -\frac{GM}{||\vec{\mathbf{r}}||^3}[(\vec{\mathbf{r}} \cdot \vec{\mathbf{v}})\vec{\mathbf{r}} - (\vec{\mathbf{r}} \cdot \vec{\mathbf{r}})\vec{\mathbf{v}}] \\ &= -\frac{GM}{||\vec{\mathbf{r}}||^3}\left[||\vec{\mathbf{r}}||\left(\frac{d}{dt}||\vec{\mathbf{r}}||\right)\vec{\mathbf{r}} - ||\vec{\mathbf{r}}||^2\vec{\mathbf{v}}\right] \\ &= -GM\left[\frac{1}{||\vec{\mathbf{r}}||^2}\left(\frac{d}{dt}||\vec{\mathbf{r}}||\right)\vec{\mathbf{r}} - \frac{1}{||\vec{\mathbf{r}}||}\vec{\mathbf{v}}\right] \\ &= GM\left[\frac{\vec{\mathbf{v}}}{||\vec{\mathbf{r}}||} - \frac{\vec{\mathbf{r}}}{||\vec{\mathbf{r}}||^2}\left(\frac{d}{dt}||\vec{\mathbf{r}}||\right)\right].\end{aligned} \tag{13.4.11}$$

However,

$$\begin{aligned}\frac{d}{dt}\frac{\vec{\mathbf{r}}}{||\vec{\mathbf{r}}||} &= \frac{\frac{d}{dt}(\vec{\mathbf{r}})||\vec{\mathbf{r}}|| - \vec{\mathbf{r}}\frac{d}{dt}||\vec{\mathbf{r}}||}{||\vec{\mathbf{r}}||^2} \\ &= \frac{\frac{d\vec{\mathbf{r}}}{dt}}{||\vec{\mathbf{r}}||} - \frac{\vec{\mathbf{r}}}{||\vec{\mathbf{r}}||^2}\frac{d}{dt}||\vec{\mathbf{r}}|| \\ &= \frac{\vec{\mathbf{v}}}{||\vec{\mathbf{r}}||} - \frac{\vec{\mathbf{r}}}{||\vec{\mathbf{r}}||^2}\frac{d}{dt}||\vec{\mathbf{r}}||.\end{aligned}$$

Therefore, Equation 13.4.11 becomes

$$\frac{d\vec{\mathbf{v}}}{dt} \times \vec{\mathbf{C}} = GM\left(\frac{d}{dt}\frac{\vec{\mathbf{r}}}{||\vec{\mathbf{r}}||}\right).$$

Since $\vec{\mathbf{C}}$ is a constant vector, we can integrate both sides and obtain

$$\vec{\mathbf{v}} \times \vec{\mathbf{C}} = GM\frac{\vec{\mathbf{r}}}{||\vec{\mathbf{r}}||} + \vec{\mathbf{D}},$$

where $\vec{\mathbf{D}}$ is a constant vector. Our goal is to solve for $||\vec{\mathbf{r}}||$. Let's start by calculating $\vec{\mathbf{r}} \cdot \left(\vec{\mathbf{v}} \times \vec{\mathbf{C}}\right)$:

$$\vec{\mathbf{r}} \cdot \left(\vec{\mathbf{v}} \times \vec{\mathbf{C}}\right) = GM\frac{||\vec{\mathbf{r}}||^2}{||\vec{\mathbf{r}}||} + \vec{\mathbf{r}} \cdot \vec{\mathbf{D}} = GM||\vec{\mathbf{r}}|| + \vec{\mathbf{r}} \cdot \vec{\mathbf{D}}.$$

However, $\vec{\mathbf{r}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{C}}) = (\vec{\mathbf{r}} \times \vec{\mathbf{v}}) \cdot \vec{\mathbf{C}}$, so

$$(\vec{\mathbf{r}} \times \vec{\mathbf{v}}) \cdot \vec{\mathbf{C}} = GM||\vec{\mathbf{r}}|| + \vec{\mathbf{r}} \cdot \vec{\mathbf{D}}.$$

Since $\vec{\mathbf{r}} \times \vec{\mathbf{v}} = \vec{\mathbf{C}}$, we have

$$||\vec{\mathbf{C}}||^2 = GM||\vec{\mathbf{r}}|| + \vec{\mathbf{r}} \cdot \vec{\mathbf{D}}.$$

Note that $\vec{\mathbf{r}} \cdot \vec{\mathbf{D}} = ||\vec{\mathbf{r}}||||\vec{\mathbf{D}}|| \cos\theta$, where θ is the angle between $\vec{\mathbf{r}}$ and $\vec{\mathbf{D}}$. Therefore,

$$||\vec{\mathbf{C}}||^2 = GM||\vec{\mathbf{r}}|| + ||\vec{\mathbf{r}}||||\vec{\mathbf{D}}|| \cos\theta$$

Solving for $||\vec{\mathbf{r}}||$,

$$||\vec{\mathbf{r}}|| = \frac{||\vec{\mathbf{C}}||^2}{GM + ||\vec{\mathbf{D}}|| \cos\theta} = \frac{||\vec{\mathbf{C}}||^2}{GM}\left(\frac{1}{1 + e\cos\theta}\right).$$

where $e = ||\vec{\mathbf{D}}||/GM$. This is the polar equation of a conic with a focus at the origin, which we set up to be the Sun. It is a hyperbola if $e > 1$, a parabola if $e = 1$, or an ellipse if $e < 1$. Since planets have closed orbits, the only possibility is an ellipse. However, at this point it should be mentioned that hyperbolic comets do exist. These are objects that are merely passing through the solar system at speeds too great to be trapped into orbit around the Sun. As they pass close enough to the Sun, the gravitational field of the Sun deflects the trajectory enough so the path becomes hyperbolic.

□

Kepler's third law of planetary motion can be modified to the case of one object in orbit around an object other than the Sun, such as the Moon around the Earth. In this case, Kepler's third law becomes

$$P^2 = \frac{4\pi^2 a^3}{G(m+M)}, \tag{13.4.12}$$

where *m* is the mass of the Moon and *M* is the mass of Earth, *a* represents the length of the major axis of the elliptical orbit, and *P* represents the period.

Example 13.4.4: Using Kepler's Third Law for Nonheliocentric Orbits

Given that the mass of the Moon is 7.35×10^{22} kg, the mass of Earth is 5.97×10^{24} kg, $G = 6.67 \times 10^{-11}\text{m/kg} \cdot \text{sec}^2$, and the period of the moon is 27.3 days, let's find the length of the major axis of the orbit of the Moon around Earth.

Solution

It is important to be consistent with units. Since the universal gravitational constant contains seconds in the units, we need to use seconds for the period of the Moon as well:

$$27.3 \text{ days} \times \frac{24 \text{ hr}}{1 \text{ day}} \times \frac{3600 \text{ esc}}{1 \text{ hour}} = 2,358,720 \text{ sec}$$

Substitute all the data into Equation 13.4.12 and solve for a:

$$
\begin{aligned}
(2,358,720sec)^2 &= \frac{4\pi^2 a^3}{\left(6.67\times10^{-11}\frac{m}{\text{kg}\times\text{sec}^2}\right)(7.35\times10^{22}\text{kg}+5.97\times10^{24}\text{kg})} \\
5.563\times10^{12} &= \frac{4\pi^2 a^3}{(6.67\times10^{-11}\text{m}^3)(6.04\times10^{24})} \\
(5.563\times10^{12})(6.67\times10^{-11}\text{m}^3)(6.04\times10^{24}) &= 4\pi^2 a^3 \\
a^3 &= \frac{2.241\times10^{27}}{4\pi^2}\text{m}^3 \\
a &= 3.84\times10^{8}\text{m} \\
&\approx 384,000\,\text{km}.
\end{aligned}
$$

Analysis

According to solarsystem.nasa.gov, the actual average distance from the Moon to Earth is 384,400 km. This is calculated using reflectors left on the Moon by Apollo astronauts back in the 1960s.

Exercise 13.4.4

Titan is the largest moon of Saturn. The mass of Titan is approximately $1.35\times10^{23}kg$. The mass of Saturn is approximately 5.68×10^{26} kg. Titan takes approximately 16 days to orbit Saturn. Use this information, along with the universal gravitation constant $G=6.67\times10^{-11}\text{m/kg}\cdot\text{sec}^2$ to estimate the distance from Titan to Saturn.

Hint

Make sure your units agree, then use Equation 13.4.12.

Answer

$$a\approx1.224\times10^{9}\text{m}=1,224,000\text{km}$$

Example 13.4.5: Halley's Comet

We now return to the chapter opener, which discusses the motion of Halley's comet around the Sun. Kepler's first law states that Halley's comet follows an elliptical path around the Sun, with the Sun as one focus of the ellipse. The period of Halley's comet is approximately 76.1 years, depending on how closely it passes by Jupiter and Saturn as it passes through the outer solar system. Let's use $T=76.1$ years. What is the average distance of Halley's comet from the Sun?

Solution

Using the equation $T^2=D^3$ with $T=76.1$, we obtain $D^3=5791.21$, so $D\approx17.96$ A.U. This comes out to approximately 1.67×10^9 mi.

A natural question to ask is: What are the maximum (aphelion) and minimum (perihelion) distances from Halley's Comet to the Sun? The eccentricity of the orbit of Halley's Comet is 0.967 (Source: http://nssdc.gsfc.nasa.gov/planetary...cometfact.html). Recall that the formula for the eccentricity of an ellipse is $e=c/a$, where a is the length of the semimajor axis and c is the distance from the center to either focus. Therefore, $0.967=c/17.96$ and $c\approx17.37$ A.U. Subtracting this from a gives the perihelion distance $p=a-c=17.96-17.37=0.59$ A.U. According to the

National Space Science Data Center (Source: http://nssdc.gsfc.nasa.gov/planetary...cometfact.html), the perihelion distance for Halley's comet is 0.587 A.U. To calculate the aphelion distance, we add

$$P = a + c = 17.96 + 17.37 = 35.33 \text{ A.U.}$$

This is approximately 3.3×10^9 mi. The average distance from Pluto to the Sun is 39.5 A.U. (Source: http://www.oarval.org/furthest.htm), so it would appear that Halley's Comet stays just within the orbit of Pluto.

NAVIGATING A BANKED TURN

How fast can a racecar travel through a circular turn without skidding and hitting the wall? The answer could depend on several factors:

- The weight of the car;
- The friction between the tires and the road;
- The radius of the circle;
- The "steepness" of the turn.

In this project we investigate this question for NASCAR racecars at the Bristol Motor Speedway in Tennessee. Before considering this track in particular, we use vector functions to develop the mathematics and physics necessary for answering questions such as this.

A car of mass m moves with constant angular speed ω around a circular curve of radius R (Figure 13.4.9). The curve is banked at an angle θ. If the height of the car off the ground is h, then the position of the car at time t is given by the function $\vec{\mathbf{r}}(t) =< R\cos(\omega t), R\sin(\omega t), h >$.

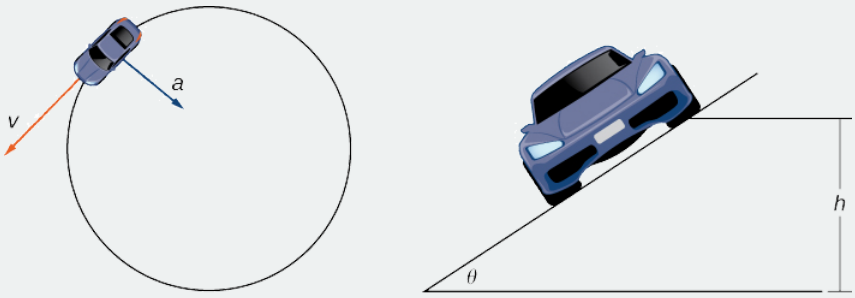


Figure 13.4.9: Views of a race car moving around a track.

1. Find the velocity function $\vec{\mathbf{v}}(t)$ of the car. Show that $\vec{\mathbf{v}}$ is tangent to the circular curve. This means that, without a force to keep the car on the curve, the car will shoot off of it.
2. Show that the speed of the car is ωR. Use this to show that $(2\pi 4)/\|\vec{\mathbf{v}}\| = (2\pi)/\omega$.
3. Find the acceleration $\vec{\mathbf{a}}$. Show that this vector points toward the center of the circle and that $\|\vec{\mathbf{a}}\| = R\omega^2$.
4. The force required to produce this circular motion is called the *centripetal force*, and it is denoted $\vec{\mathbf{F}}_{cent}$. This force points toward the center of the circle (not toward the ground). Show that $\|\vec{\mathbf{F}}_{cent}\| = (m|\vec{\mathbf{v}}|^2)/R$.

As the car moves around the curve, three forces act on it: gravity, the force exerted by the road (this force is perpendicular to the ground), and the friction force (Figure 13.4.10). Because describing the frictional force generated by the tires and the road is complex, we use a standard approximation for the frictional force. Assume that $\vec{\mathbf{f}} = \mu\vec{\mathbf{N}}$ for some positive constant μ. The constant μ is called the *coefficient of friction*.

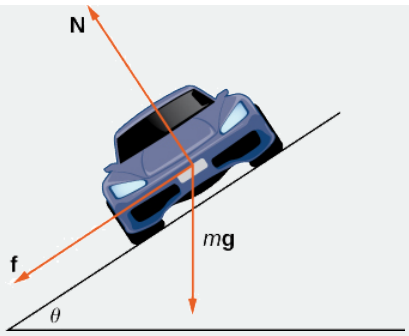


Figure 13.4.10: The car has three forces acting on it: gravity (denoted by $m\vec{\mathbf{g}}$), *the friction force* $\vec{\mathbf{f}}$, *and the force exerted by the road* $\vec{\mathbf{N}}$.

Let v_{max} denote the maximum speed the car can attain through the curve without skidding. In other words, v_{max} is the fastest speed at which the car can navigate the turn. When the car is traveling at this speed, the magnitude of the centripetal force is

$$\|\vec{\mathbf{F}}_{cent}\| = \frac{m(v_{max})^2}{R}.$$

The next three questions deal with developing a formula that relates the speed v_{max} to the banking angle θ.

5. Show that $\vec{\mathbf{N}}\cos\theta = m\vec{\mathbf{g}} + \vec{\mathbf{f}}\sin\theta$. Conclude that $\vec{\mathbf{N}} = (m\vec{\mathbf{g}})/(\cos\theta - \mu\sin\theta)$.
6. The centripetal force is the sum of the forces in the horizontal direction, since the centripetal force points toward the center of the circular curve. Show that

   $$\vec{\mathbf{F}}_{cent} = \vec{\mathbf{N}}\sin\theta + \vec{\mathbf{f}}\cos\theta.$$

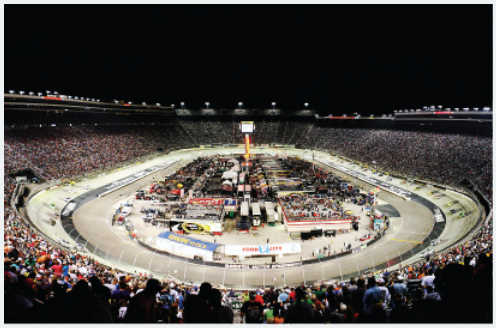
   Conclude that

   $$\vec{\mathbf{F}}_{cent} = \frac{\sin\theta + \mu\cos\theta}{cos\theta - \mu\sin\theta} m\vec{\mathbf{g}}.$$

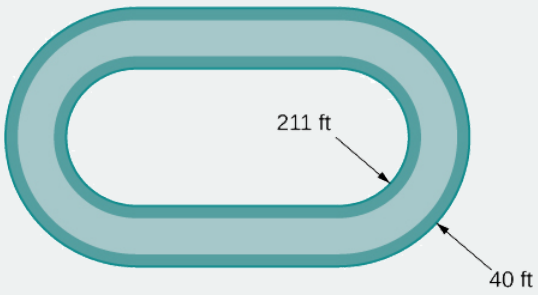
7. Show that $(v_{\max})^2 = ((\sin\theta + \mu\ cos\theta)/(\cos\theta - \mu\sin\theta))gR$. Conclude that the maximum speed does not actually depend on the mass of the car.

   Now that we have a formula relating the maximum speed of the car and the banking angle, we are in a position to answer the questions like the one posed at the beginning of the project.

   The Bristol Motor Speedway is a NASCAR short track in Bristol, Tennessee. The track has the approximate shape shown in Figure 13.4.11. Each end of the track is approximately semicircular, so when cars make turns they are traveling along an approximately circular curve. If a car takes the inside track and speeds along the bottom of turn 1, the car travels along a semicircle of radius approximately 211 ft with a banking angle of 24°. If the car decides to take the outside track and speeds along the top of turn 1, then the car travels along a semicircle with a banking angle of 28°. (The track has variable angle banking.)



(a)



(b)

Figure 13.4.11: At the Bristol Motor Speedway, Bristol, Tennessee (a), the turns have an inner radius of about 211 ft and a width of 40 ft (b). (credit: part (a) photo by Raniel Diaz, Flickr)

The coefficient of friction for a normal tire in dry conditions is approximately 0.7. Therefore, we assume the coefficient for a NASCAR tire in dry conditions is approximately 0.98.

Before answering the following questions, note that it is easier to do computations in terms of feet and seconds, and then convert the answers to miles per hour as a final step.

8. In dry conditions, how fast can the car travel through the bottom of the turn without skidding?
9. In dry conditions, how fast can the car travel through the top of the turn without skidding?
10. In wet conditions, the coefficient of friction can become as low as 0.1. If this is the case, how fast can the car travel through the bottom of the turn without skidding?
11. Suppose the measured speed of a car going along the outside edge of the turn is 105 mph. Estimate the coefficient of friction for the car's tires.

Key Concepts

- If $\vec{\mathbf{r}}(t)$ represents the position of an object at time t, then $\vec{\mathbf{r}}\,'(t)$ represents the velocity and $\vec{\mathbf{r}}\,''(t)$ represents the acceleration of the object at time t. The magnitude of the velocity vector is speed.
- The acceleration vector always points toward the concave side of the curve defined by $\vec{\mathbf{r}}(t)$. The tangential and normal components of acceleration $a_{\vec{\mathbf{T}}}$ and $a_{\vec{\mathbf{N}}}$ are the projections of the acceleration vector onto the unit tangent and unit normal vectors to the curve.
- Kepler's three laws of planetary motion describe the motion of objects in orbit around the Sun. His third law can be modified to describe motion of objects in orbit around other celestial objects as well.
- Newton was able to use his law of universal gravitation in conjunction with his second law of motion and calculus to prove Kepler's three laws.

Key Equations

- **Velocity**

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{r}}\,'(t)$$

- **Acceleration**

$$\vec{\mathbf{a}}(t) = \vec{\mathbf{v}}\,'(t) = \vec{\mathbf{r}}\,''(t)$$

- **Speed**

$$v(t) = ||\vec{\mathbf{v}}(t)|| = ||\vec{\mathbf{r}}\,'(t)|| = \frac{ds}{dt}$$

- **Tangential component of acceleration**

$$a_{\vec{\mathbf{T}}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{T}} = \frac{\vec{\mathbf{v}} \cdot \vec{\mathbf{a}}}{||\vec{\mathbf{v}}||}$$

- **Normal component of acceleration**

$$a_{\vec{\mathbf{N}}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{N}} = \frac{||\vec{\mathbf{v}} \times \vec{\mathbf{a}}||}{||\vec{\mathbf{v}}||} = \sqrt{||\vec{\mathbf{a}}||^2 - \left(a_{\vec{\mathbf{T}}}\right)^2}$$

Glossary

acceleration vector
the second derivative of the position vector

Kepler's laws of planetary motion
three laws governing the motion of planets, asteroids, and comets in orbit around the Sun

normal component of acceleration

the coefficient of the unit normal vector $\vec{\mathbf{N}}$ when the acceleration vector is written as a linear combination of $\vec{\mathbf{T}}$ and $\vec{\mathbf{N}}$

projectile motion
motion of an object with an initial velocity but no force acting on it other than gravity

tangential component of acceleration
the coefficient of the unit tangent vector $\vec{\mathbf{T}}$ when the acceleration vector is written as a linear combination of $\vec{\mathbf{T}}$ and $\vec{\mathbf{N}}$

velocity vector
the derivative of the position vector

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- Edited by Paul Seeburger
  Paul Seeburger added finding point $(1, 2)$ when $t = 1$ in Example 13.4.1.
  He also created Figure 13.4.1.

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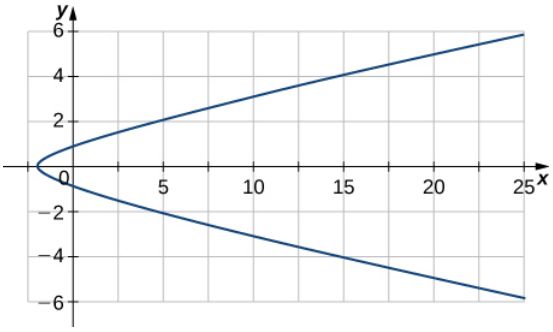
13.4E: Exercises for Section 13.4

1) Given $\vec{\mathbf{r}}(t) = (3t^2 - 2)\,\hat{\mathbf{i}} + (2t - \sin t)\,\hat{\mathbf{j}}$,

a. find the velocity of a particle moving along this curve.

b. find the acceleration of a particle moving along this curve.



Answer

a. $\vec{\mathbf{v}}(t) = 6t\,\hat{\mathbf{i}} + (2 - \cos t)\,\hat{\mathbf{i}}$
b. $\vec{\mathbf{a}}(t) = 6\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{i}}$

In questions 2 - 5, given the position function, find the velocity, acceleration, and speed in terms of the parameter t.

2) $\vec{\mathbf{r}}(t) = e^{-t}\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}} + \tan t\,\hat{\mathbf{k}}$

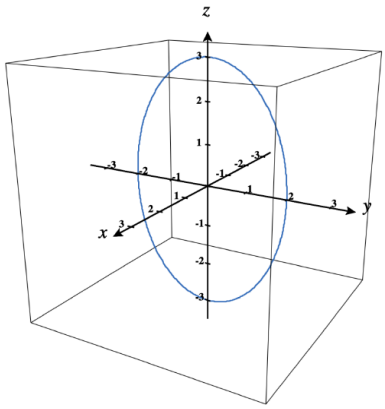
3) $\vec{\mathbf{r}}(t) = \langle 3\cos t,\ 3\sin t,\ t^2 \rangle$

Answer

$\vec{\mathbf{v}}(t) = -3\sin t\,\hat{\mathbf{i}} + 3\cos t\,\hat{\mathbf{j}} + 2t\,\hat{\mathbf{k}}$
$\vec{\mathbf{a}}(t) = -3\cos t\,\hat{\mathbf{i}} - 3\sin t\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}}$
$\text{Speed}(t) = \|\vec{\mathbf{v}}(t)\| = \sqrt{9 + 4t^2}$

4) $\vec{\mathbf{r}}(t) = t^5\,\hat{\mathbf{i}} + (3t^2 + 2t - 5)\,\hat{\mathbf{j}} + (3t - 1)\,\hat{\mathbf{k}}$

5) $\vec{\mathbf{r}}(t) = 2\cos t\,\hat{\mathbf{j}} + 3\sin t\,\hat{\mathbf{k}}$. The graph is shown here:



Answer

$\vec{\mathbf{v}}(t) = -2\sin t\,\hat{\mathbf{j}} + 3\cos t\,\hat{\mathbf{k}}$
$\vec{\mathbf{a}}(t) = -2\cos t\,\hat{\mathbf{j}} - 3\sin t\,\hat{\mathbf{k}}$
$\text{Speed}(t) = \|\vec{\mathbf{v}}(t)\| = \sqrt{4\sin^2 t + 9\cos^2 t} = \sqrt{4 + 5\cos^2 t}$

In questions 6 - 8, find the velocity, acceleration, and speed of a particle with the given position function.

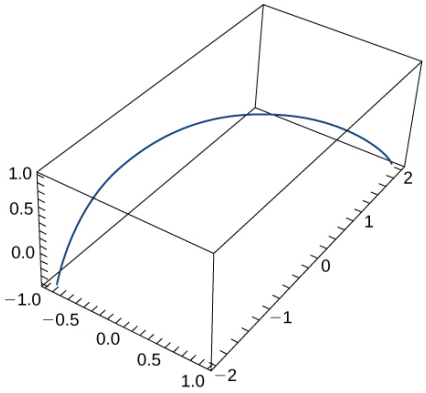
6) $\vec{\mathbf{r}}(t) = \langle t^2 - 1, t \rangle$

7) $\vec{\mathbf{r}}(t) = \langle e^t, e^{-t} \rangle$

Answer

$\vec{\mathbf{v}}(t) = \langle e^t, -e^{-t} \rangle,$
$\vec{\mathbf{a}}(t) = \langle e^t, e^{-t} \rangle,$
$\|\vec{\mathbf{v}}(t)\| = \sqrt{e^{2t} + e^{-2t}}$

8) $\vec{\mathbf{r}}(t) = \langle \sin t, t, \cos t \rangle$. The graph is shown here:



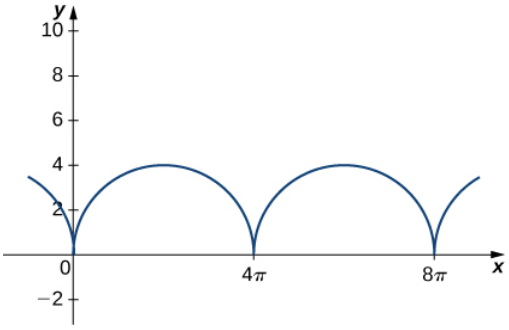
9) The position function of an object is given by $\vec{\mathbf{r}}(t) = \langle t^2, 5t, t^2 - 16t \rangle$. At what time is the speed a minimum?

Answer

$t = 4$

10) Let $\vec{\mathbf{r}}(t) = r\cosh(\omega t)\,\hat{\mathbf{i}} + r\sinh(\omega t)\,\hat{\mathbf{j}}$. Find the velocity and acceleration vectors and show that the acceleration is proportional to $\vec{\mathbf{r}}(t)$.

11) Consider the motion of a point on the circumference of a rolling circle. As the circle rolls, it generates the cycloid $\vec{\mathbf{r}}(t) = (\omega t - \sin(\omega t))\,\hat{\mathbf{i}} + (1 - \cos(\omega t))\,\hat{\mathbf{j}}$, where ω is the angular velocity of the circle and b is the radius of the circle:

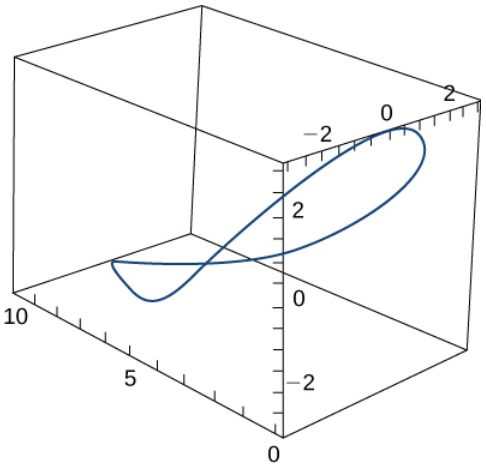


Find the equations for the velocity, acceleration, and speed of the particle at any time.

Answer

$\vec{\mathbf{v}}(t) = (\omega - \omega\cos(\omega t))\,\hat{\mathbf{i}} + (\omega\sin(\omega t))\,\hat{\mathbf{j}}$
$\vec{\mathbf{a}}(t) = (\omega^2\sin(\omega t))\,\hat{\mathbf{i}} + (\omega^2\cos(\omega t))\,\hat{\mathbf{j}}$
$\text{speed}(t) = \sqrt{(\omega - \omega\cos(\omega t))^2 + (\omega\sin(\omega t))^2}$
$= \sqrt{\omega^2 - 2\omega^2\cos(\omega t) + \omega^2\cos^2(\omega t) + \omega^2\sin^2(\omega t)}$
$= \sqrt{2\omega^2(1 - \cos(\omega t))}$

12) A person on a hang glider is spiraling upward as a result of the rapidly rising air on a path having position vector $\vec{\mathbf{r}}(t) = (3\cos t)\,\hat{\mathbf{i}} + (3\sin t)\,\hat{\mathbf{j}} + t^2\,\hat{\mathbf{k}}$. The path is similar to that of a helix, although it is not a helix. The graph is shown here:



Find the following quantities:

a. The velocity and acceleration vectors

b. The glider's speed at any time

Answer

$\|\vec{\mathbf{v}}(t)\| = \sqrt{9 + 4t^2}$

c. The times, if any, at which the glider's acceleration is orthogonal to its velocity

13) Given that $\vec{\mathbf{r}}(t) = \langle e^{-5t} \sin t,\ e^{-5t} \cos t,\ 4e^{-5t} \rangle$ is the position vector of a moving particle, find the following quantities:

a. The velocity of the particle

Answer

$\vec{\mathbf{v}}(t) = \langle e^{-5t}(\cos t - 5 \sin t),\ -e^{-5t}(\sin t + 5 \cos t),\ -20e^{-5t} \rangle$

b. The speed of the particle

c. The acceleration of the particle

Answer

$\vec{\mathbf{a}}(t) = \langle e^{-5t}(-\sin t - 5 \cos t) - 5e^{-5t}(\cos t - 5 \sin t),\ -e^{-5t}(\cos t - 5 \sin t) + 5e^{-5t}(\sin t + 5 \cos t),\ 100e^{-5t} \rangle$

14) Find the maximum speed of a point on the circumference of an automobile tire of radius 1 ft when the automobile is traveling at 55 mph.

15) Find the position vector-valued function $\vec{\mathbf{r}}(t)$, given that $\vec{\mathbf{a}}(t) = \hat{\mathbf{i}} + e^t \hat{\mathbf{j}}$, $\vec{\mathbf{v}}(0) = 2\hat{\mathbf{j}}$, and $\vec{\mathbf{r}}(0) = 2\hat{\mathbf{i}}$.

16) Find $\vec{\mathbf{r}}(t)$ given that $\vec{\mathbf{a}}(t) = -32\hat{\mathbf{j}}$, $\vec{\mathbf{v}}(0) = 600\sqrt{3}\,\hat{\mathbf{i}} + 600\,\hat{\mathbf{j}}$, and $\vec{\mathbf{r}}(0) = \vec{\mathbf{0}}$.

17) The acceleration of an object is given by $\vec{\mathbf{a}}(t) = t\hat{\mathbf{j}} + t\hat{\mathbf{k}}$. The velocity at $t = 1$ sec is $\vec{\mathbf{v}}(1) = 5\hat{\mathbf{j}}$ and the position of the object at $t = 1$ sec is $\vec{\mathbf{r}}(1) = 0\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$. Find the object's position at any time.

Answer

$\vec{\mathbf{r}}(t) = 0\hat{\mathbf{i}} + \left(\frac{1}{6}t^3 + 4.5t - \frac{14}{3}\right)\hat{\mathbf{j}} + \left(\frac{1}{6}t^3 - \frac{1}{2}t + \frac{1}{3}\right)\hat{\mathbf{k}}$

Projectile Motion

18) A projectile is shot in the air from ground level with an initial velocity of 500 m/sec at an angle of 60° with the horizontal.

a. At what time does the projectile reach maximum height?

Answer

44.185 sec

b. What is the approximate maximum height of the projectile?

c. At what time is the maximum range of the projectile attained?

Answer

$t = 88.37$ sec

d. What is the maximum range?

e. What is the total flight time of the projectile?

Answer

$t = 88.37$ sec

19) A projectile is fired at a height of 1.5 m above the ground with an initial velocity of 100 m/sec and at an angle of 30° above the horizontal. Use this information to answer the following questions:

a. Determine the maximum height of the projectile.

b. Determine the range of the projectile.

Answer

The range is approximately 886.29 m.

20) A golf ball is hit in a horizontal direction off the top edge of a building that is 100 ft tall. How fast must the ball be launched to land 450 ft away?

21) A projectile is fired from ground level at an angle of 8° with the horizontal. The projectile is to have a range of 50 m. Find the minimum velocity (speed) necessary to achieve this range.

Answer

$v = 42.16$ m/sec

22) Prove that an object moving in a straight line at a constant speed has an acceleration of zero.

Finding Components of Acceleration & Kepler's Laws

23) Find the tangential and normal components of acceleration for $\vec{\mathbf{r}}(t) = t^2\,\hat{\mathbf{i}} + 2t\,\hat{\mathbf{j}}$ when $t = 1$.

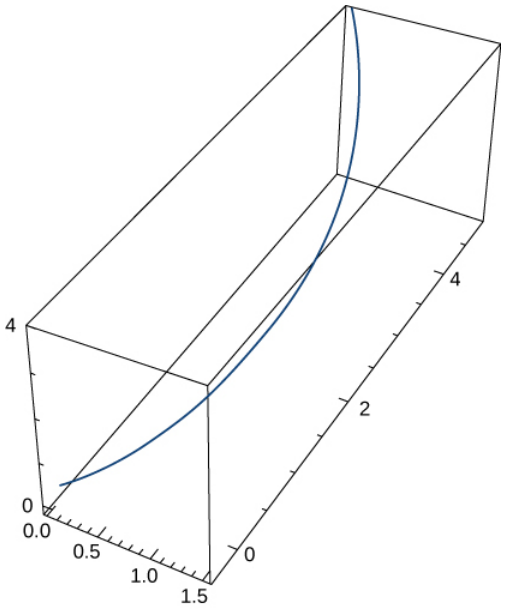
Answer

$a_{\vec{\mathbf{T}}} = \sqrt{2}, \quad a_{\vec{\mathbf{N}}} = \sqrt{2}$

In questions 24 - 30, find the tangential and normal components of acceleration.

24) $\vec{\mathbf{r}}(t) = \langle \cos(2t),\, \sin(2t),\, 1 \rangle$

25) $\vec{\mathbf{r}}(t) = \langle e^t \cos t,\, e^t \sin t,\, e^t \rangle$. The graph is shown here:



Answer

$$a_{\vec{\mathbf{T}}} = \sqrt{3}e^t, \quad a_{\vec{\mathbf{N}}} = \sqrt{2}e^t$$

26) $\vec{\mathbf{r}}(t) = \langle \frac{2}{3}(1+t)^{3/2}, \frac{2}{3}(1-t)^{3/2}, \sqrt{2}t \rangle$

27) $\vec{\mathbf{r}}(t) = \left\langle 2t, t^2, \frac{t^3}{3} \right\rangle$

Answer

$$a_{\vec{\mathbf{T}}} = 2t, \quad a_{\vec{\mathbf{N}}} = 2$$

28) $\vec{\mathbf{r}}(t) = t^2\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}} + t^3\,\hat{\mathbf{k}}$

29) $\vec{\mathbf{r}}(t) = \langle 6t, 3t^2, 2t^3 \rangle$

Answer

$$a_{\vec{\mathbf{T}}} = \frac{6t + 12t^3}{\sqrt{1 + t^2 + t^4}}, \quad a_{\vec{\mathbf{N}}} = 6\sqrt{\frac{1 + 4t^2 + t^4}{1 + t^2 + t^4}}$$

30) $\vec{\mathbf{r}}(t) = 3\cos(2\pi t)\,\hat{\mathbf{i}} + 3\sin(2\pi t)\,\hat{\mathbf{j}}$

Answer

$$a_{\vec{\mathbf{T}}} = 0, \quad a_{\vec{\mathbf{N}}} = 12\pi^2$$

31) Find the tangential and normal components of acceleration for $\vec{\mathbf{r}}(t) = a\cos(\omega t)\,\hat{\mathbf{i}} + b\sin(\omega t)\,\hat{\mathbf{j}}$ at $t = 0$.

Answer

$$a_{\vec{\mathbf{T}}} = 0, \quad a_{\vec{\mathbf{N}}} = a\omega^2$$

32) Suppose that the position function for an object in three dimensions is given by the equation $\vec{\mathbf{r}}(t) = t\cos(t)\,\hat{\mathbf{i}} + t\sin(t)\,\hat{\mathbf{j}} + 3t\,\hat{\mathbf{k}}$.

a. Show that the particle moves on a circular cone.

b. Find the angle between the velocity and acceleration vectors when $t = 1.5$.

c. Find the tangential and normal components of acceleration when $t = 1.5$.

Answer

c. $a_{\vec{\mathbf{T}}} = 0.43 \text{ m/sec}^2, \quad a_{\vec{\mathbf{N}}} = 2.46 \text{ m/sec}^2$

33) The force on a particle is given by $\vec{\mathbf{f}}(t) = (\cos t)\,\hat{\mathbf{i}} + (\sin t)\,\hat{\mathbf{j}}$. The particle is located at point $(c, 0)$ at $t = 0$. The initial velocity of the particle is given by $\vec{\mathbf{v}}(0) = v_0\,\hat{\mathbf{j}}$. Find the path of the particle of mass m. (Recall, $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$.)

Answer

$$\vec{\mathbf{r}}(t) = \left(\frac{-\cos t}{m} + c + \tfrac{1}{m}\right)\hat{\mathbf{i}} + \left(\frac{-\sin t}{m} + \left(v_0 + \tfrac{1}{m}\right)t\right)\hat{\mathbf{j}}$$

34) An automobile that weighs 2700 lb makes a turn on a flat road while traveling at 56 ft/sec. If the radius of the turn is 70 ft, what is the required frictional force to keep the car from skidding?

35) Using Kepler's laws, it can be shown that $v_0 = \sqrt{\dfrac{2GM}{r_0}}$ is the minimum speed needed when $\theta = 0$ so that an object will escape from the pull of a central force resulting from mass M. Use this result to find the minimum speed when $\theta = 0$ for a space capsule to escape from the gravitational pull of Earth if the probe is at an altitude of 300 km above Earth's surface.

Answer

10.94 km/sec

36) Find the time in years it takes the dwarf planet Pluto to make one orbit about the Sun given that $a = 39.5$ A.U.

Contributors

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13.R: Chapter 13 Review Exercises

***True or False*? Justify your answer with a proof or a counterexample.**

1. A parametric equation that passes through points P and Q can be given by $\vec{\mathbf{r}}(t) = \langle t^2,\ 3t + 1,\ t - 2\rangle$, where $P(1, 4, -1)$ and $Q(16, 11, 2)$.

2. $\dfrac{d}{dt}\left[\vec{\mathbf{u}}(t) \times \vec{\mathbf{u}}(t)\right] = 2\vec{\mathbf{u}}'(t) \times \vec{\mathbf{u}}(t)$

Answer

False, $\dfrac{d}{dt}\left[\vec{\mathbf{u}}(t) \times \vec{\mathbf{u}}(t)\right] = \vec{\mathbf{0}}$.

3. The curvature of a circle of radius r is constant everywhere. Furthermore, the curvature is equal to $1/r$.

4. The speed of a particle with a position function $\vec{\mathbf{r}}(t)$ is $\dfrac{\vec{\mathbf{r}}'(t)}{\|\vec{\mathbf{r}}'(t)\|}$.

Answer

False, it is $\|\vec{\mathbf{r}}'(t)\|$

Find the domains of the vector-valued functions.

5. $\vec{\mathbf{r}}(t) = \langle \sin(t),\ \ln(t),\ \sqrt{t}\rangle$

6. $\vec{\mathbf{r}}(t) = \left\langle e^t,\ \dfrac{1}{\sqrt{4 - t}},\ \sec t \right\rangle$

Answer

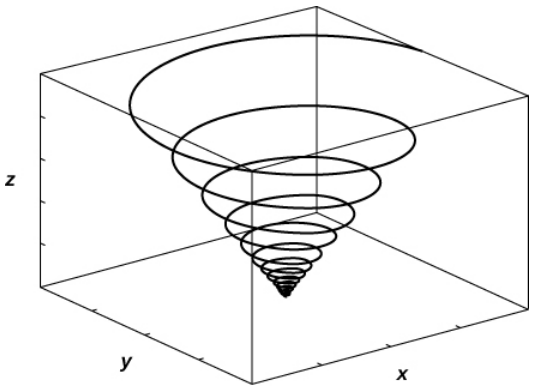
$t < 4,\ t \neq \dfrac{n\pi}{2}$

Sketch the curves for the following vector equations. Use a calculator if needed.

7. **[T]** $\vec{\mathbf{r}}(t) = \langle t^2,\ t^3\rangle$

8. **[T]** $\vec{\mathbf{r}}(t) = \langle \sin(20t)e^{-t},\ \cos(20t)e^{-t},\ e^{-t}\rangle$

Answer



Find a vector function that describes the following curves.

9. Intersection of the cylinder $x^2 + y^2 = 4$ with the plane $x + z = 6$

10. Intersection of the cone $z = \sqrt{x^2 + y^2}$ and plane $z = y - 4$

Answer

$\vec{\mathbf{r}}(t) = \left\langle t,\ 2 - \frac{t^2}{8},\ -2 - \frac{t^2}{8}\right\rangle$

Find the derivatives of $\vec{\mathbf{u}}(t)$, $\vec{\mathbf{u}}'(t)$, $\vec{\mathbf{u}}'(t) \times \vec{\mathbf{u}}(t)$, $\vec{\mathbf{u}}(t) \times \vec{\mathbf{u}}'(t)$, and $\vec{\mathbf{u}}(t) \cdot \vec{\mathbf{u}}'(t)$. Find the unit tangent vector.

11. $\vec{\mathbf{u}}(t) = \langle e^t,\, e^{-t} \rangle$

12. $\vec{\mathbf{u}}(t) = \langle t^2,\, 2t+6,\, 4t^5 - 12 \rangle$

Answer

$\vec{\mathbf{u}}'(t) = \langle 2t,\, 2,\, 20t^4 \rangle$,
$\vec{\mathbf{u}}''(t) = \langle 2,\, 0,\, 80t^3 \rangle$,
$\frac{d}{dt}\left[\vec{\mathbf{u}}'(t) \times \vec{\mathbf{u}}(t)\right] = \langle -480t^3 - 160t^4,\, 24 + 75t^2,\, 12 + 4t \rangle$,
$\frac{d}{dt}\left[\vec{\mathbf{u}}(t) \times \vec{\mathbf{u}}'(t)\right] = \langle 480t^3 + 160t^4,\, -24 - 75t^2,\, -12 - 4t \rangle$,
$\frac{d}{dt}\left[\vec{\mathbf{u}}(t) \cdot \vec{\mathbf{u}}'(t)\right] = 720t^8 - 9600t^3 + 6t^2 + 4$,
unit tangent vector: $\vec{\mathbf{T}}(t) = \dfrac{2t}{\sqrt{400t^8 + 4t^2 + 4}}\,\hat{\mathbf{i}} + \dfrac{2}{\sqrt{400t^8 + 4t^2 + 4}}\,\hat{\mathbf{j}} + \dfrac{20t^4}{\sqrt{400t^8 + 4t^2 + 4}}\,\hat{\mathbf{k}}$

Evaluate the following integrals.

13. $\displaystyle\int \left(\tan(t)\sec(t)\,\hat{\mathbf{i}} - te^{3t}\,\hat{\mathbf{j}}\right)\,dt$

14. $\displaystyle\int_1^4 \vec{\mathbf{u}}(t)\,dt$, with $\vec{\mathbf{u}}(t) = \left\langle \frac{\ln t}{t},\, \frac{1}{\sqrt{t}},\, \sin\left(\frac{t\pi}{4}\right) \right\rangle$

Answer

$\dfrac{\ln(4^2)}{2}\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + \dfrac{2(2+\sqrt{2})}{\pi}\,\hat{\mathbf{k}}$

Find the length for the following curves.

15. $\vec{\mathbf{r}}(t) = \langle 3t,\, 4\cos t,\, 4\sin t \rangle$ for $1 \le t \le 4$

16. $\vec{\mathbf{r}}(t) = 2\,\hat{\mathbf{i}} + t\,\hat{\mathbf{j}} + 3t^2\,\hat{\mathbf{k}}$ for $0 \le t \le 1$

Answer

$\dfrac{\sqrt{37}}{2} + \frac{1}{12}\sinh^{-1} 6$

Reparameterize the following functions with respect to their arc length measured from $t = 0$ in direction of increasing t.

17. $\vec{\mathbf{r}}(t) = 2t\,\hat{\mathbf{i}} + (4t-5)\,\hat{\mathbf{j}} + (1-3t)\,\hat{\mathbf{k}}$

18. $\vec{\mathbf{r}}(t) = \cos(2t)\,\hat{\mathbf{i}} + 8t\,\hat{\mathbf{j}} - \sin(2t)\,\hat{\mathbf{k}}$

Answer

$\vec{\mathbf{r}}(t(s)) = \cos\left(\frac{2s}{\sqrt{65}}\right)\,\hat{\mathbf{i}} + \frac{8s}{\sqrt{65}}\,\hat{\mathbf{j}} - \sin\left(\frac{2s}{\sqrt{65}}\right)\,\hat{\mathbf{k}}$

Find the curvature for the following vector functions.

19. $\vec{\mathbf{r}}(t) = (2\sin t)\,\hat{\mathbf{i}} - 4t\,\hat{\mathbf{j}} + (2\cos t)\,\hat{\mathbf{k}}$

20. $\vec{\mathbf{r}}(t) = \sqrt{2}e^t\,\hat{\mathbf{i}} + \sqrt{2}e^{-t}\,\hat{\mathbf{j}} + 2t\,\hat{\mathbf{k}}$

Answer

$\dfrac{e^{2t}}{\left(e^{2t}+1\right)^2}$

21. Find the unit tangent vector, the unit normal vector, and the binormal vector for $\vec{\mathbf{r}}(t) = 2\cos t\,\hat{\mathbf{i}} + 3t\,\hat{\mathbf{j}} + 2sint\,\hat{\mathbf{k}}$.

22. Find the tangential and normal acceleration components with the position vector $\vec{\mathbf{r}}(t) = \langle \cos t,\, \sin t,\, e^t \rangle$.

Answer

$a_T = \dfrac{e^{2t}}{1+e^{2t}}$,

$$a_N = \frac{\sqrt{2e^{2t} + 4e^{2t}\sin t \cos t + 1}}{1 + e^{2t}}$$

23. A Ferris wheel car is moving at a constant speed v and has a constant radius r. Find the tangential and normal acceleration of the Ferris wheel car.

24. The position of a particle is given by $\vec{\mathbf{r}}(t) = \langle t^2, \ln t, \sin(\pi t)\rangle$, where t is measured in seconds and r is measured in meters. Find the velocity, acceleration, and speed functions. What are the position, velocity, speed, and acceleration of the particle at 1 sec?

Answer

$\vec{\mathbf{v}}(t) = \left\langle 2t, \frac{1}{t}, \pi\cos(\pi t)\right\rangle$ m/sec,
$\vec{\mathbf{a}}(t) = \left\langle 2, -\frac{1}{t^2}, -\pi^2\sin(\pi t)\right\rangle$ m/sec^2,
$\text{speed}(t) = \sqrt{4t^2 + \frac{1}{t^2} + \pi^2\cos^2(\pi t)}$ m/sec;
At $t = 1$, $\vec{\mathbf{r}}(1) = \langle 1, 0, 0\rangle$ m, $\vec{\mathbf{v}}(1) = \langle 2, -1, \pi\rangle$ m/sec, $\vec{\mathbf{a}}(1) = \langle 2, -1, 0\rangle$ m/sec^2, and $\text{speed}(1) = \sqrt{5 + \pi^2}$ m/sec

The following problems consider launching a cannonball out of a cannon. The cannonball is shot out of the cannon with an angle θ and initial velocity $\vec{\mathbf{v}}_0$. The only force acting on the cannonball is gravity, so we begin with a constant acceleration $\vec{\mathbf{a}}(t) = -g\,\hat{\mathbf{j}}$.

25. Find the velocity vector function $\vec{\mathbf{v}}(t)$.

26. Find the position vector $\vec{\mathbf{r}}(t)$ and the parametric representation for the position.

Answer

$\vec{\mathbf{r}}(t) = \vec{\mathbf{v}}_0 t - \frac{gt^2}{2}\,\hat{\mathbf{j}}$,
$\vec{\mathbf{r}}(t) = \langle v_0(\cos\theta)t,\, v_0(\sin\theta)t, -\frac{gt^2}{2}\rangle$ where $v_0 = \|\vec{\mathbf{v}}_0\|$.

27. At what angle do you need to fire the cannonball for the horizontal distance to be greatest? What is the total distance it would travel?

Contributors

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CHAPTER OVERVIEW

14: Differentiation of Functions of Several Variables

When dealing with a function of more than one independent variable, several questions naturally arise. For example, how do we calculate limits of functions of more than one variable? The definition of derivative we used before involved a limit. Does the new definition of derivative involve limits as well? Do the rules of differentiation apply in this context? Can we find relative extrema of functions using derivatives? All these questions are answered in this chapter.

14.0: Prelude to Differentiation of Functions of Several Variables
14.1: Functions of Several Variables
14.1E: Exercises for Section 14.1
14.2: Limits and Continuity
14.2E: Exercises for Section 14.2
14.3: Partial Derivatives
14.3E: Exercises for Section 14.3
14.4: Tangent Planes and Linear Approximations
14.4E: Exercises for Section 14.4
14.5: The Chain Rule for Multivariable Functions
14.5E: Exercises for Section 14.5
14.6: Directional Derivatives and the Gradient
14.6E: Exercises for Section 14.6
14.7: Maxima/Minima Problems
14.7E: Exercises for Section 14.7
14.8: Lagrange Multipliers
14.8E: Exercises for Section 14.8
14.R: Chapter 14 Review Exercises

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14.0: Prelude to Differentiation of Functions of Several Variables

In Introduction to Applications of Derivatives, we studied how to determine the maximum and minimum of a function of one variable over a closed interval. This function might represent the temperature over a given time interval, the position of a car as a function of time, or the altitude of a jet plane as it travels from New York to San Francisco. In each of these examples, the function has one independent variable.

Figure 14.0.1: Americans use (and lose) millions of golf balls a year, which keeps golf ball manufacturers in business. In this chapter, we study a profit model and learn methods for calculating optimal production levels for a typical golf ball manufacturing company. (credit: modification of work by oatsy40, Flickr)

Suppose, however, that we have a quantity that depends on more than one variable. For example, temperature can depend on location and the time of day, or a company's profit model might depend on the number of units sold and the amount of money spent on advertising. In this chapter, we look at a company that produces golf balls. We develop a profit model and, under various restrictions, we find that the optimal level of production and advertising dollars spent determines the maximum possible profit. Depending on the nature of the restrictions, both the method of solution and the solution itself changes.

When dealing with a function of more than one independent variable, several questions naturally arise. For example, how do we calculate limits of functions of more than one variable? The definition of derivative we used before involved a limit. Does the new definition of derivative involve limits as well? Do the rules of differentiation apply in this context? Can we find relative extrema of functions using derivatives? All these questions are answered in this chapter.

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14.1: Functions of Several Variables

Learning Objectives

- Recognize a function of two variables and identify its domain and range.
- Sketch a graph of a function of two variables.
- Sketch several traces or level curves of a function of two variables.
- Recognize a function of three or more variables and identify its level surfaces.

Our first step is to explain what a function of more than one variable is, starting with functions of two independent variables. This step includes identifying the domain and range of such functions and learning how to graph them. We also examine ways to relate the graphs of functions in three dimensions to graphs of more familiar planar functions.

Functions of Two Variables

The definition of a function of two variables is very similar to the definition for a function of one variable. The main difference is that, instead of mapping values of one variable to values of another variable, we map ordered pairs of variables to another variable.

Definition: Function of Two Variables

A *function of two variables* $z = f(x, y)$ maps each ordered pair (x, y) in a subset D of the real plane R^2 to a unique real number z. The set D is called the domain of the function. The range of f is the set of all real numbers z that has at least one ordered pair $(x, y) \in D$ such that $f(x, y) = z$ as shown in Figure 14.1.1.

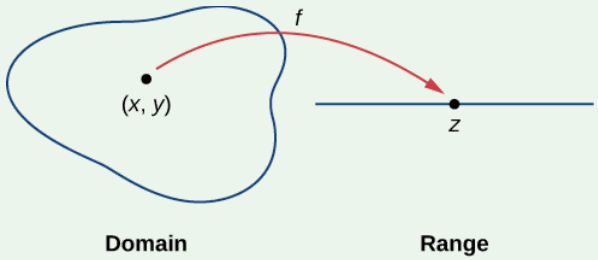


Figure 14.1.1: The domain of a function of two variables consists of ordered pairs (x, y).

Determining the domain of a function of two variables involves taking into account any domain restrictions that may exist. Let's take a look.

Example 14.1.1: Domains and Ranges for Functions of Two Variables

Find the domain and range of each of the following functions:

a. $f(x, y) = 3x + 5y + 2$
b. $g(x, y) = \sqrt{9 - x^2 - y^2}$

Solution

a. This is an example of a linear function in two variables. There are no values or combinations of x and y that cause $f(x, y)$ to be undefined, so the domain of f is R^2. To determine the range, first pick a value for z. We need to find a solution to the equation $f(x, y) = z$, or $3x - 5y + 2 = z$. One such solution can be obtained by first setting $y = 0$, which yields the equation $3x + 2 = z$. The solution to this equation is $x = \frac{z-2}{3}$, which gives the ordered pair $\left(\frac{z-2}{3}, 0\right)$ as a solution to the equation $f(x, y) = z$ for any value of z. Therefore, the range of the function is all real numbers, or R.

b. For the function $g(x, y)$ to have a real value, the quantity under the square root must be nonnegative:

$$9 - x^2 - y^2 \geq 0.$$

This inequality can be written in the form

$$x^2 + y^2 \leq 9.$$

Therefore, the domain of $g(x, y)$ is $\{(x, y) \in R^2 \mid x^2 + y^2 \leq 9\}$. The graph of this set of points can be described as a disk of radius 3 centered at the origin. The domain includes the boundary circle as shown in the following graph.

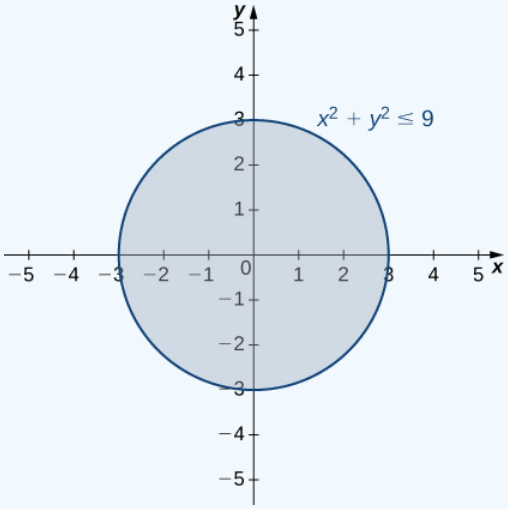


Figure 14.1.2: The domain of the function $g(x, y) = \sqrt{9 - x^2 - y^2}$ is a closed disk of radius 3.

To determine the range of $g(x, y) = \sqrt{9 - x^2 - y^2}$ we start with a point (x_0, y_0) on the boundary of the domain, which is defined by the relation $x^2 + y^2 = 9$. It follows that $x_0^2 + y_0^2 = 9$ and

$$\begin{aligned} g(x_0, y_0) &= \sqrt{9 - x_0^2 - y_0^2} \\ &= \sqrt{9 - (x_0^2 + y_0^2)} \\ &= \sqrt{9 - 9} \\ &= 0. \end{aligned}$$

If $x_0^2 + y_0^2 = 0$ (in other words, $x_0 = y_0 = 0$), then

$$\begin{aligned} g(x_0, y_0) &= \sqrt{9 - x_0^2 - y_0^2} \\ &= \sqrt{9 - (x_0^2 + y_0^2)} \\ &= \sqrt{9 - 0} = 3. \end{aligned}$$

This is the maximum value of the function. Given any value c between 0 and 3, we can find an entire set of points inside the domain of g such that $g(x, y) = c$:

$$\begin{aligned} \sqrt{9 - x^2 - y^2} &= c \\ 9 - x^2 - y^2 &= c^2 \\ x^2 + y^2 &= 9 - c^2. \end{aligned}$$

Since $9 - c^2 > 0$, this describes a circle of radius $\sqrt{9 - c^2}$ centered at the origin. Any point on this circle satisfies the equation $g(x, y) = c$. Therefore, the range of this function can be written in interval notation as $[0, 3]$.

? Exercise 14.1.1

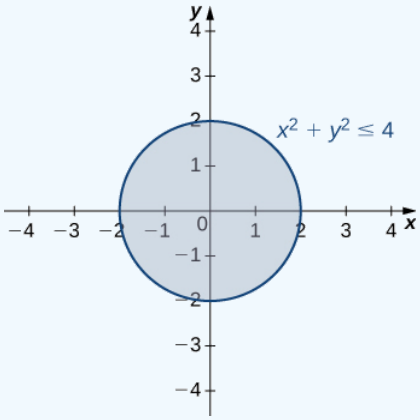
Find the domain and range of the function $f(x, y) = \sqrt{36 - 9x^2 - 9y^2}$.

Hint

Determine the set of ordered pairs that do not make the radicand negative.

Solution

The domain is $\{(x,y)|x^2+y^2 \leq 4\}$ the shaded circle defined by the inequality $x^2+y^2 \leq 4$, which has a circle of radius 2 as its boundary. The range is $[0,6]$.



Graphing Functions of Two Variables

Suppose we wish to graph the function $z=f(x,y)$. This function has two independent variables (x and y) and one dependent variable (z). When graphing a function $y=f(x)$ of one variable, we use the Cartesian plane. We are able to graph any ordered pair (x,y) in the plane, and every point in the plane has an ordered pair (x,y) associated with it. With a function of two variables, each ordered pair (x,y) in the domain of the function is mapped to a real number z. Therefore, the graph of the function f consists of ordered triples (x,y,z). The graph of a function $z=f(x,y)$ of two variables is called a surface.

To understand more completely the concept of plotting a set of ordered triples to obtain a surface in three-dimensional space, imagine the (x,y) coordinate system laying flat. Then, every point in the domain of the function f has a unique z-value associated with it. If z is positive, then the graphed point is located above the xy-plane, if z is negative, then the graphed point is located below the xy-plane. The set of all the graphed points becomes the two-dimensional surface that is the graph of the function f.

Example 14.1.2: Graphing Functions of Two Variables

Create a graph of each of the following functions:

a. $g(x,y)=\sqrt{9-x^2-y^2}$
b. $f(x,y)=x^2+y^2$

Solution

a. In Example 14.1.2, we determined that the domain of $g(x,y)=\sqrt{9-x^2-y^2}$ is $\{(x,y) \in R^2 \mid x^2+y^2 \leq 9\}$ and the range is $\{z \in R^2 \mid 0 \leq z \leq 3\}$. When $x^2+y^2=9$ we have $g(x,y)=0$. Therefore any point on the circle of radius 3 centered at the origin in the xy-plane maps to $z=0$ in R^3. If $x^2+y^2=8$, then $g(x,y)=1$, so any point on the circle of radius $2\sqrt{2}$ centered at the origin in the xy-plane maps to $z=1$ in R^3. As x^2+y^2 gets closer to zero, the value of z approaches 3. When $x^2+y^2=0$, then $g(x,y)=3$. This is the origin in the xy-plane If x^2+y^2 is equal to any other value between 0 and 9, then $g(x,y)$ equals some other constant between 0 and 3. The surface described by this function is a hemisphere centered at the origin with radius 3 as shown in the following graph.

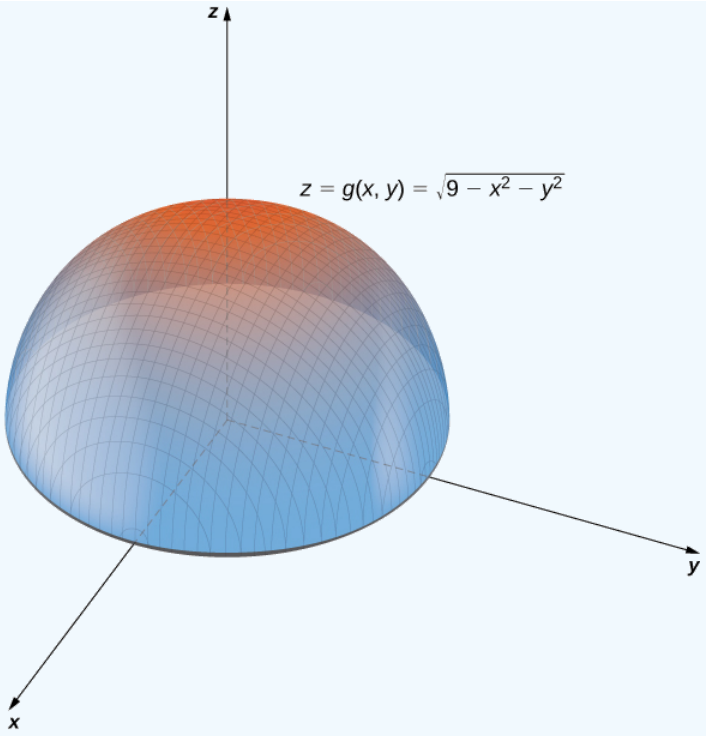


Figure 14.1.3: Graph of the hemisphere represented by the given function of two variables.

b. This function also contains the expression $x^2 + y^2$. Setting this expression equal to various values starting at zero, we obtain circles of increasing radius. The minimum value of $f(x, y) = x^2 + y^2$ is zero (attained when $x = y = 0$.. When $x = 0$, the function becomes $z = y^2$, and when $y = 0$, then the function becomes $z = x^2$. These are cross-sections of the graph, and are parabolas. Recall from Introduction to Vectors in Space that the name of the graph of $f(x, y) = x^2 + y^2$ is a **paraboloid**. The graph of f appears in the following graph.

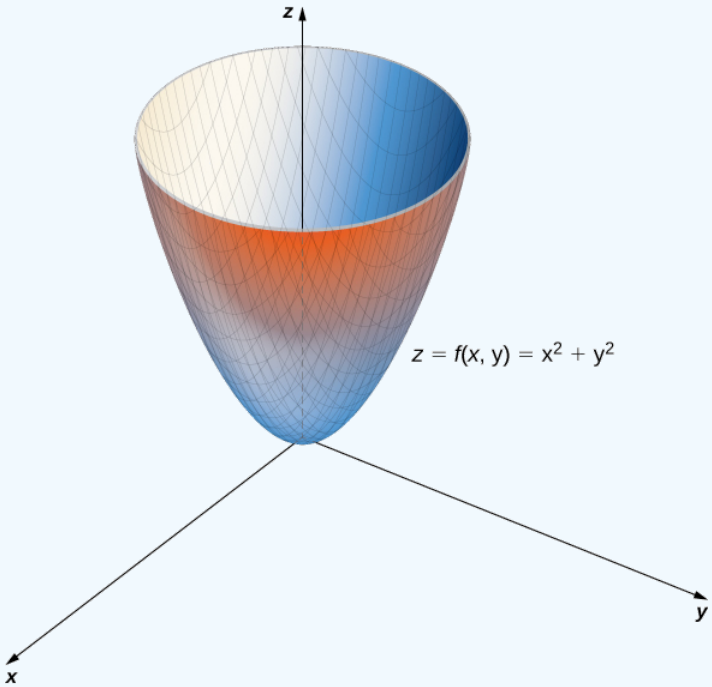


Figure 14.1.4: A paraboloid is the graph of the given function of two variables.

Example 14.1.3: Nuts and Bolts

A profit function for a hardware manufacturer is given by

$$f(x,y) = 16 - (x-3)^2 - (y-2)^2,$$

where x is the number of nuts sold per month (measured in thousands) and y represents the number of bolts sold per month (measured in thousands). Profit is measured in thousands of dollars. Sketch a graph of this function.

Solution

This function is a polynomial function in two variables. The domain of f consists of (x,y) coordinate pairs that yield a nonnegative profit:

$$16 - (x-3)^2 - (y-2)^2 \geq 0$$
$$(x-3)^2 + (y-2)^2 \leq 16.$$

This is a disk of radius 4 centered at $(3,2)$. A further restriction is that both x and y must be nonnegative. When $x = 3$ and $y = 2, f(x,y) = 16$. Note that it is possible for either value to be a noninteger; for example, it is possible to sell 2.5 thousand nuts in a month. The domain, therefore, contains thousands of points, so we can consider all points within the disk. For any $z < 16$, we can solve the equation $f(x,y) = 16$:

$$16 - (x-3)^2 - (y-2)^2 = z$$
$$(x-3)^2 + (y-2)^2 = 16 - z.$$

Since $z < 16$, we know that $16 - z > 0$, so the previous equation describes a circle with radius $\sqrt{16-z}$ centered at the point $(3,2)$. Therefore. the range of $f(x,y)$ is $\{z \in \mathbb{R} | z \leq 16\}$. The graph of $f(x,y)$ is also a paraboloid, and this paraboloid points downward as shown.

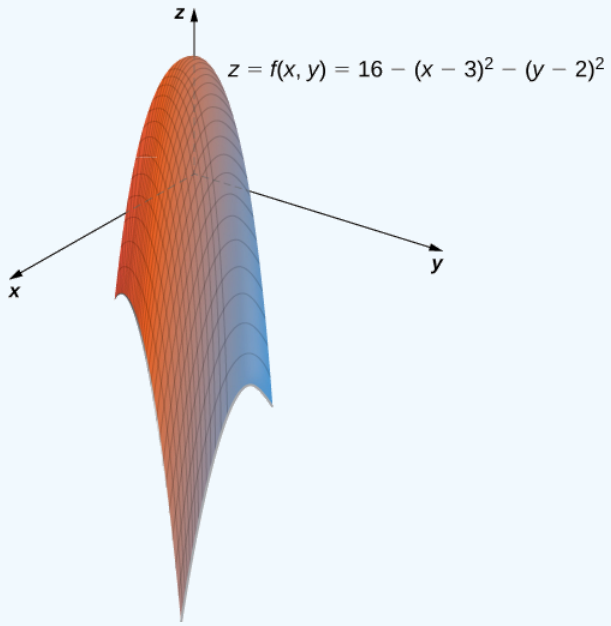
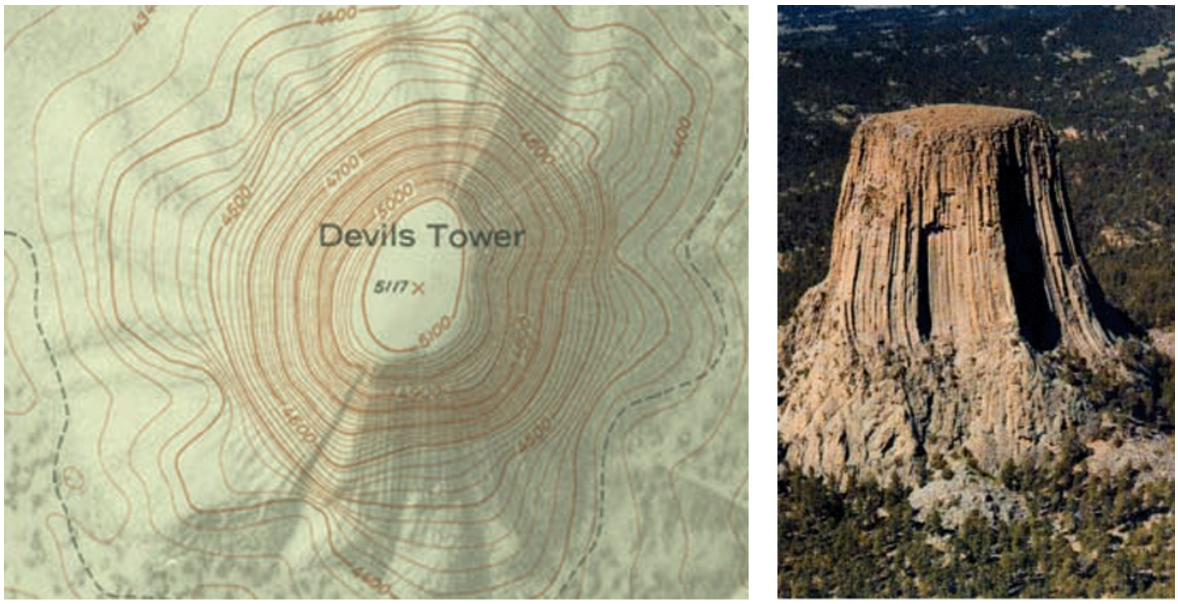


Figure 14.1.5: The graph of the given function of two variables is also a paraboloid.

Level Curves

If hikers walk along rugged trails, they might use a topographical map that shows how steeply the trails change. A *topographical map* contains curved lines called contour lines. Each contour line corresponds to the points on the map that have equal elevation (Figure 14.1.6). A level curve of a function of two variables $f(x,y)$ is completely analogous to a contour line on a topographical map.



(a) (b)

Figure 14.1.6: (a) A topographical map of Devil's Tower, Wyoming. Lines that are close together indicate very steep terrain. (b) A perspective photo of Devil's Tower shows just how steep its sides are. Notice the top of the tower has the same shape as the center of the topographical map.

Definition: Level Curves

Given a function $f(x, y)$ and a number c in the range of f, a level curve of a function of two variables for the value c is defined to be the set of points satisfying the equation $f(x, y) = c$.

Returning to the function $g(x, y) = \sqrt{9 - x^2 - y^2}$, we can determine the level curves of this function. The range of g is the closed interval $[0, 3]$. First, we choose any number in this closed interval—say, $c = 2$. The level curve corresponding to $c = 2$ is described by the equation

$$\sqrt{9 - x^2 - y^2} = 2.$$

To simplify, square both sides of this equation:

$$9 - x^2 - y^2 = 4.$$

Now, multiply both sides of the equation by -1 and add 9 to each side:

$$x^2 + y^2 = 5.$$

This equation describes a circle centered at the origin with radius $\sqrt{5}$. Using values of c between 0 and 3 yields other circles also centered at the origin. If $c = 3$, then the circle has radius 0, so it consists solely of the origin. Figure 14.1.7 is a graph of the level curves of this function corresponding to $c = 0, 1, 2$, and 3. Note that in the previous derivation it may be possible that we introduced extra solutions by squaring both sides. This is not the case here because the range of the square root function is nonnegative.

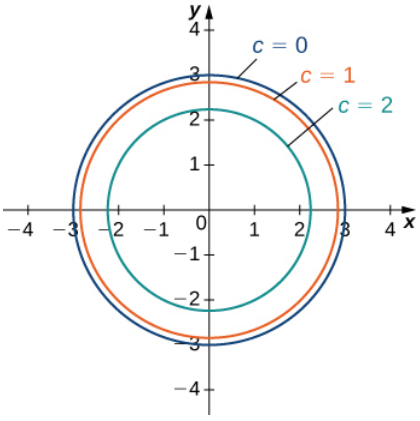


Figure 14.1.7: Level curves of the function $g(x,y)=\sqrt{9-x^2-y^2}$, using $c=0,1,2$, and $3(c=3$ corresponds to the origin).

A graph of the various level curves of a function is called a *contour map*.

Example 14.1.4: Making a Contour Map

Given the function $f(x,y)=\sqrt{8+8x-4y-4x^2-y^2}$, find the level curve corresponding to $c=0$. Then create a contour map for this function. What are the domain and range of f?

Solution

To find the level curve for $c=0$, we set $f(x,y)=0$ and solve. This gives

$$0=\sqrt{8+8x-4y-4x^2-y^2}.$$

We then square both sides and multiply both sides of the equation by -1:

$$4x^2+y^2-8x+4y-8=0.$$

Now, we rearrange the terms, putting the x terms together and the y terms together, and add 8 to each side:

$$4x^2-8x+y^2+4y=8.$$

Next, we group the pairs of terms containing the same variable in parentheses, and factor 4 from the first pair:

$$4(x^2-2x)+(y^2+4y)=8.$$

Then we complete the square in each pair of parentheses and add the correct value to the right-hand side:

$$4(x^2-2x+1)+(y^2+4y+4)=8+4(1)+4.$$

Next, we factor the left-hand side and simplify the right-hand side:

$$4(x-1)^2+(y+2)^2=16.$$

Last, we divide both sides by 16 :

$$\frac{(x-1)^2}{4}+\frac{(y+2)^2}{16}=1. \tag{14.1.1}$$

This equation describes an ellipse centered at $(1,-2)$. The graph of this ellipse appears in the following graph.

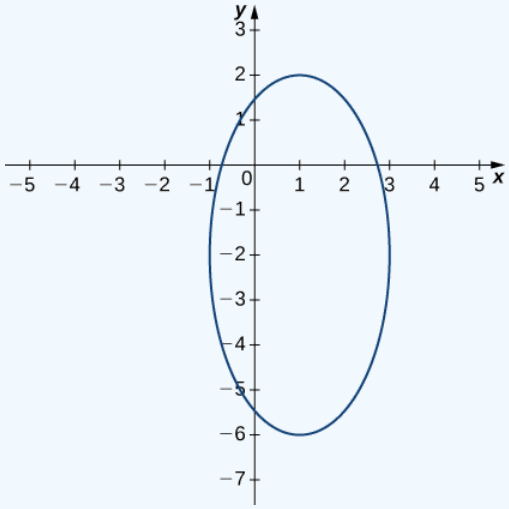


Figure 14.1.8: Level curve of the function $f(x,y)=\sqrt{8+8x-4y-4x^2-y^2}$ corresponding to $c=0$

We can repeat the same derivation for values of c less than 4. Then, Equation 14.1.1 becomes

$$\frac{4(x-1)^2}{16-c^2}+\frac{(y+2)^2}{16-c^2}=1$$

for an arbitrary value of c. Figure 14.1.9 shows a contour map for $f(x,y)$ using the values $c=0,1,2,$ and 3. When $c=4$, the level curve is the point $(-1,2)$.

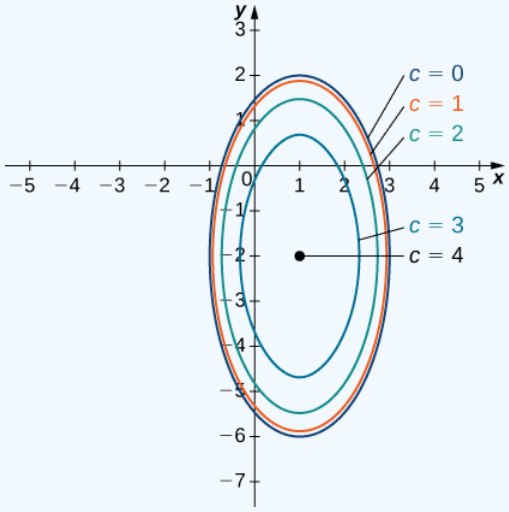


Figure 14.1.9: Contour map for the function $f(x,y)=\sqrt{8+8x-4y-4x^2-y^2}$ using the values $c=0,1,2,3,$ and 4.

Finding the Domain & Range

Since this is a square root function, the radicand must not be negative. So we have

$$8+8x-4y-4x^2-y^2\geq 0$$

Recognizing that the boundary of the domain is an ellipse, we repeat the steps we showed above to obtain

$$\frac{(x-1)^2}{4}+\frac{(y+2)^2}{16}\leq 1$$

So the domain of f can be written: $\left\{(x,y)\mid \frac{(x-1)^2}{4}+\frac{(y+2)^2}{16}\leq 1\right\}$.

To find the range of f, we need to consider the possible outputs of this square root function. We know the output cannot be negative, so we need to next check if its output is ever 0. From the work we completed above to find the level curve for $c=0$, we know the value of f is 0 for any point on that level curve (on the ellipse, $\frac{(x-1)^2}{4}+\frac{(y+2)^2}{16}=1$). So we know the lower bound of the range of this function is 0.

To determine the upper bound for the range of the function in this problem, it's easier if we first complete the square under the radical.

$$\begin{aligned} f(x,y) &= \sqrt{8+8x-4y-4x^2-y^2} \\ &= \sqrt{8-4(x^2-2x\quad)-(y^2+4y\quad)} \\ &= \sqrt{8-4(x^2-2x+1-1)-(y^2+4y+4-4)} \\ &= \sqrt{8-4(x^2-2x+1)+4-(y^2+4y+4)+4} \\ &= \sqrt{16-4(x-1)^2-(y+2)^2} \end{aligned}$$

Now that we have f in this form, we can see how large the radicand can be. Since we are subtracting two perfect squares from 16, we know that the value of the radicand cannot be greater than 16. At the point $(1,-2)$, we can see the radicand will be 16 (since we will be subtracting 0 from 16 at this point. This gives us the maximum value of f, that is $f(1,-2)=\sqrt{16}=4$.

So the range of this function is $[0,4]$.

Exercise 14.1.2

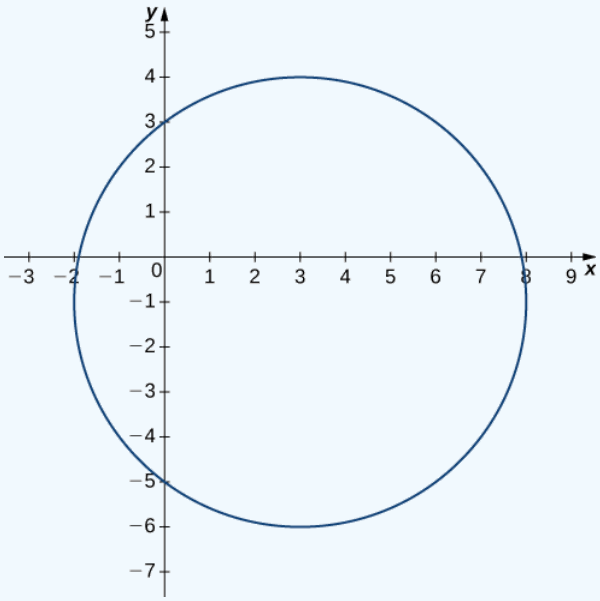
Find and graph the level curve of the function $g(x,y)=x^2+y^2-6x+2y$ corresponding to $c=15$.

Hint

First, set $g(x,y)=15$ and then complete the square.

Solution

The equation of the level curve can be written as $(x-3)^2+(y+1)^2=25$, which is a circle with radius 5 centered at $(3,-1)$.



Another useful tool for understanding the **graph of a function of two variables** is called a vertical trace. Level curves are always graphed in the xy-plane, but as their name implies, vertical traces are graphed in the xz- or yz-planes.

Definition: Vertical Traces

Consider a function $z=f(x,y)$ with domain $D\subseteq\mathbb{R}^2$. A *vertical trace* of the function can be either the set of points that solves the equation $f(a,y)=z$ for a given constant $x=a$ or $f(x,b)=z$ for a given constant $y=b$.

https://math.libretexts.org/@go/page/2600

Example 14.1.5: Finding Vertical Traces

Find vertical traces for the function $f(x,y) = \sin x \cos y$ corresponding to $x = -\frac{\pi}{4}, 0$, and $\frac{\pi}{4}$, and $y = -\frac{\pi}{4}, 0$, and $\frac{\pi}{4}$.

Solution

First set $x = -\frac{\pi}{4}$ in the equation $z = \sin x \cos y$:

$$z = \sin\left(-\frac{\pi}{4}\right)\cos y = -\frac{\sqrt{2}\cos y}{2} \approx -0.7071 \cos y.$$

This describes a cosine graph in the plane $x = -\frac{\pi}{4}$. The other values of z appear in the following table.

Vertical Traces Parallel to the xz-Plane for the Function $f(x,y) = \sin x \cos y$

c	Vertical Trace for $x = c$
$-\frac{\pi}{4}$	$z = -\frac{\sqrt{2}\cos y}{2}$
0	$z = 0$
$\frac{\pi}{4}$	$z = \frac{\sqrt{2}\cos y}{2}$

In a similar fashion, we can substitute the y-values in the equation $f(x,y)$ to obtain the traces in the yz-plane, as listed in the following table.

Vertical Traces Parallel to the yz-Plane for the Function $f(x,y) = \sin x \cos y$

d	Vertical Trace for $y = d$
$\frac{\pi}{4}$	$z = \frac{\sqrt{2}\sin x}{2}$
0	$z = \sin x$
$-\frac{\pi}{4}$	$z = \frac{\sqrt{2}\sin x}{2}$

The three traces in the xz-plane are cosine functions; the three traces in the yz-plane are sine functions. These curves appear in the intersections of the surface with the planes $x = -\frac{\pi}{4}, x = 0, x = \frac{\pi}{4}$ and $y = -\frac{\pi}{4}, y = 0, y = \frac{\pi}{4}$ as shown in the following figure.

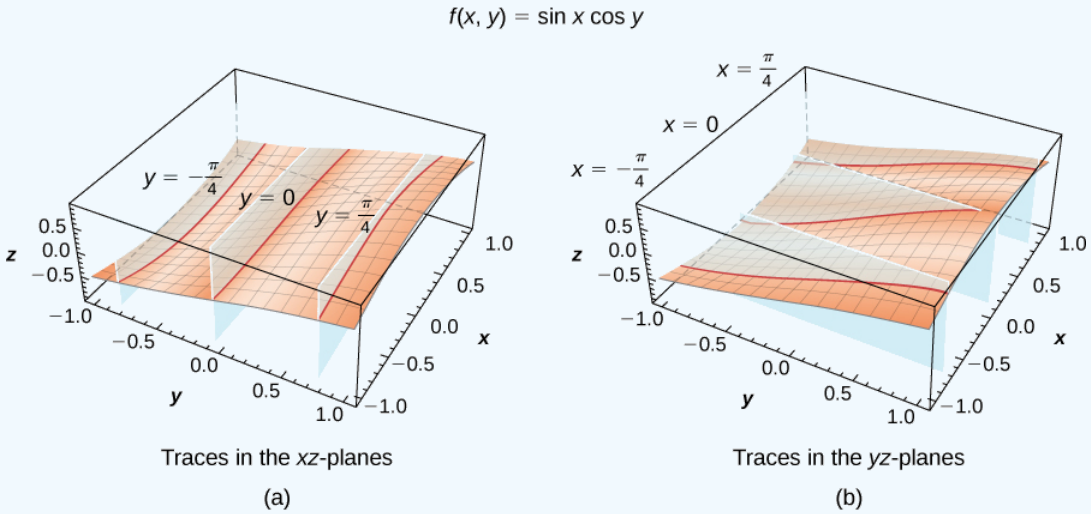


Figure 14.1.10: Vertical traces of the function $f(x,y)$ are cosine curves in the $xz-planes$ (a) and sine curves in the $yz-planes$ (b).

Exercise 14.1.3

Determine the equation of the vertical trace of the function $g(x,y) = -x^2 - y^2 + 2x + 4y - 1$ corresponding to $y = 3$, and describe its graph.

Hint

Set $y = 3$ in the equation $z = -x^2 - y^2 + 2x + 4y - 1$ and complete the square.

Solution

$z = 3 - (x-1)^2$. This function describes a parabola opening downward in the plane $y = 3$.

Functions of two variables can produce some striking-looking surfaces. Figure 14.1.11 shows two examples.

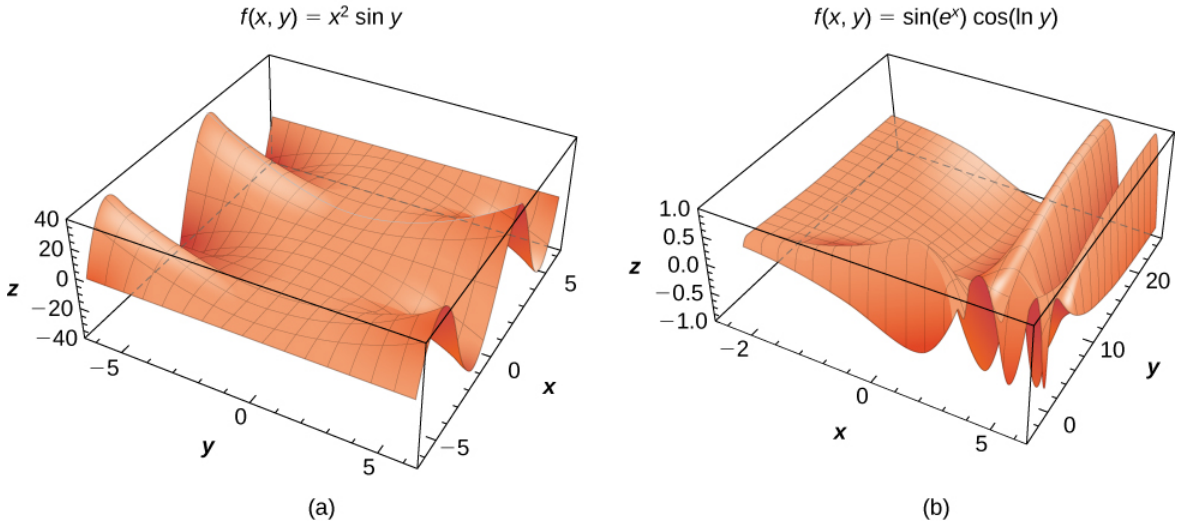


Figure 14.1.11: Examples of surfaces representing functions of two variables: (a) a combination of a power function and a sine function and (b) a combination of trigonometric, exponential, and logarithmic functions.

Functions of More Than Two Variables

So far, we have examined only functions of two variables. However, it is useful to take a brief look at functions of more than two variables. Two such examples are

$$\underbrace{f(x,y,z) = x^2 - 2xy + y^2 + 3yz - z^2 + 4x - 2y + 3x - 6}_{\text{a polynomial in three variables}}$$

and

$$g(x,y,t) = (x^2 - 4xy + y^2)\sin t - (3x + 5y)\cos t.$$

In the first function, (x,y,z) represents a point in space, and the function f maps each point in space to a fourth quantity, such as temperature or wind speed. In the second function, (x,y) can represent a point in the plane, and t can represent time. The function might map a point in the plane to a third quantity (for example, pressure) at a given time t. The method for finding the domain of a function of more than two variables is analogous to the method for functions of one or two variables.

Example 14.1.6: Domains for Functions of Three Variables

Find the domain of each of the following functions:

a. $f(x,y,z) = \dfrac{3x - 4y + 2z}{\sqrt{9 - x^2 - y^2 - z^2}}$

b. $g(x,y,t) = \dfrac{\sqrt{2t - 4}}{x^2 - y^2}$

Solution:

a. For the function $f(x,y,z)=\dfrac{3x-4y+2z}{\sqrt{9-x^2-y^2-z^2}}$ to be defined (and be a real value), two conditions must hold:

1. The denominator **cannot** be zero.
2. The radicand **cannot** be negative.

Combining these conditions leads to the inequality

$$9-x^2-y^2-z^2>0.$$

Moving the variables to the other side and reversing the inequality gives the domain as

$$domain(f)=\{(x,y,z)\in R^3 \mid x^2+y^2+z^2<9\},$$

which describes a ball of radius 3 centered at the origin. (Note: The surface of the ball is not included in this domain.)

b. For the function $g(x,y,t)=\dfrac{\sqrt{2t-4}}{x^2-y^2}$ to be defined (and be a real value), two conditions must hold:

1. The radicand cannot be negative.
2. The denominator cannot be zero.

Since the radicand cannot be negative, this implies $2t-4\geq 0$, and therefore that $t\geq 2$. Since the denominator cannot be zero, $x^2-y^2\neq 0$, or $x^2\neq y^2$, Which can be rewritten as $y=\pm x$, which are the equations of two lines passing through the origin. Therefore, the domain of g is

$$domain(g)=\{(x,y,t)|y\neq\pm x, t\geq 2\}.$$

Exercise 14.1.4

Find the domain of the function $h(x,y,t)=(3t-6)\sqrt{y-4x^2+4}$.

Hint

Check for values that make radicands negative or denominators equal to zero.

Solution

$$domain(h)=\{(x,y,t)\in\mathbb{R}^3 \mid y\geq 4x^2-4\}$$

Functions of two variables have level curves, which are shown as curves in the $xy-plane$. However, when the function has three variables, the curves become surfaces, so we can define level surfaces for functions of three variables.

Definition: Level Surface of a Function of Three Variables

Given a function $f(x,y,z)$ and a number c in the range of f, a *level surface of a function of three variables* is defined to be the set of points satisfying the equation $f(x,y,z)=c$.

Example 14.1.7: Finding a Level Surface

Find the level surface for the function $f(x,y,z)=4x^2+9y^2-z^2$ corresponding to $c=1$.

Solution

The level surface is defined by the equation $4x^2+9y^2-z^2=1$. This equation describes a hyperboloid of one sheet as shown in Figure 14.1.12.

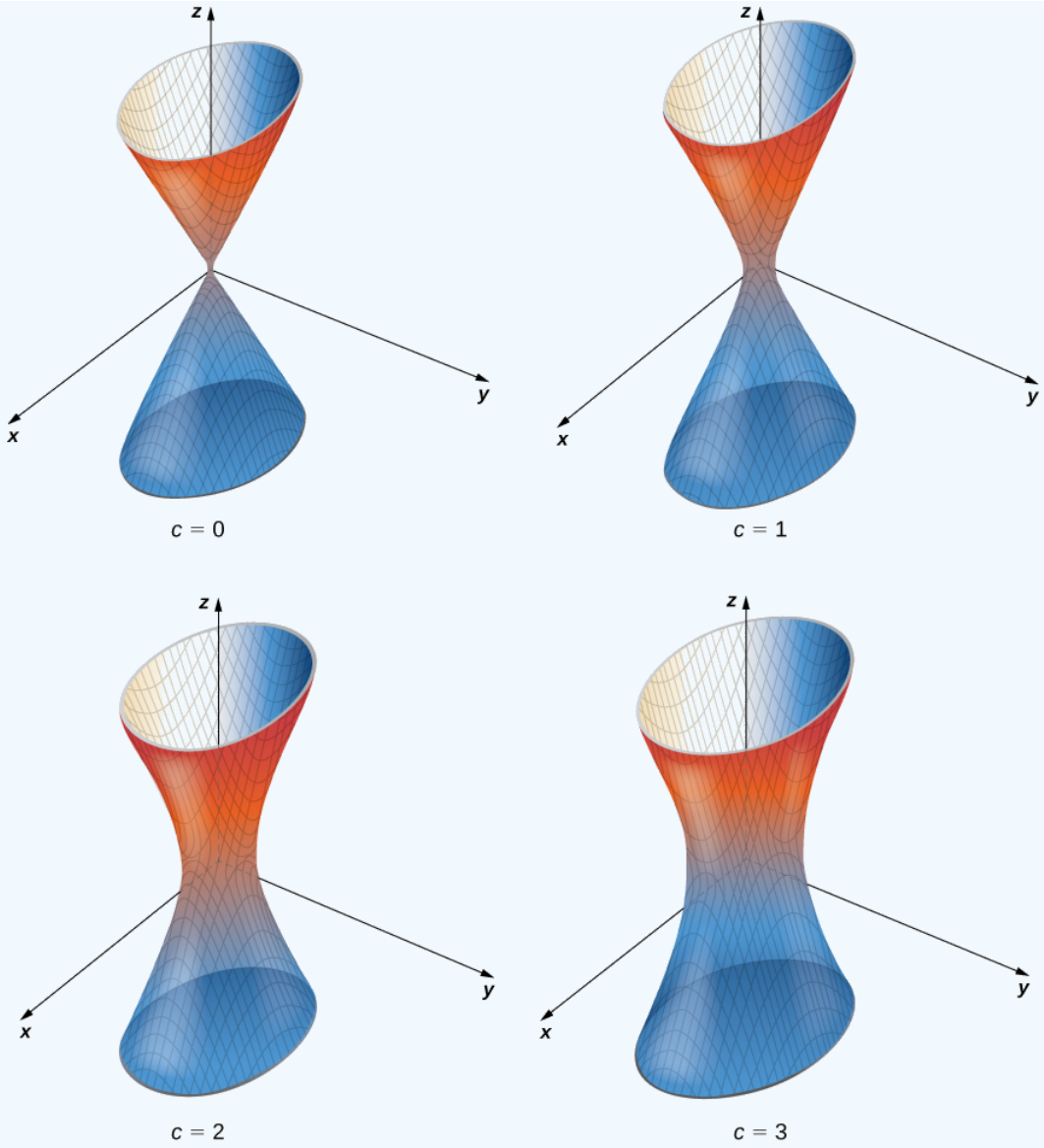


Figure 14.1.12: A hyperboloid of one sheet with some of its level surfaces.

? Exercise 14.1.5

Find the equation of the level surface of the function

$$g(x, y, z) = x^2 + y^2 + z^2 - 2x + 4y - 6z$$

corresponding to $c = 2$, and describe the surface, if possible.

Hint

Set $g(x, y, z) = c$ and complete the square.

Solution

$(x-1)^2 + (y+2)^2 + (z-3)^2 = 16$ describes a sphere of radius 4 centered at the point $(1, -2, 3)$.

Summary

- The graph of a function of two variables is a surface in $\mathbb{R}^3$ and can be studied using level curves and vertical traces.
- A set of level curves is called a contour map.

Key Equations

- **Vertical trace**

  $f(a, y) = z$ for $x = a$ or $f(x, b) = z$ for $y = b$

- **Level surface of a function of three variables**

  $f(x, y, z) = c$

Glossary

contour map
a plot of the various level curves of a given function $f(x, y)$

function of two variables
a function $z = f(x, y)$ that maps each ordered pair (x, y) in a subset D of R^2 to a unique real number z

graph of a function of two variables
a set of ordered triples (x, y, z) that satisfies the equation $z = f(x, y)$ plotted in three-dimensional Cartesian space

level curve of a function of two variables
the set of points satisfying the equation $f(x, y) = c$ for some real number c in the range of f

level surface of a function of three variables
the set of points satisfying the equation $f(x, y, z) = c$ for some real number c in the range of f

surface
the graph of a function of two variables, $z = f(x, y)$

vertical trace
the set of ordered triples (c, y, z) that solves the equation $f(c, y) = z$ for a given constant $x = c$ or the set of ordered triples (x, d, z) that solves the equation $f(x, d) = z$ for a given constant $y = d$

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14.1E: EXERCISES FOR SECTION 14.1

For the following exercises, evaluate each function at the indicated values.

1) $W(x,y) = 4x^2 + y^2$. Find $W(2,-1), W(-3,6)$.

Answer

$W(2,-1) = 17, \quad W(-3,6) = 72$

2) $W(x,y) = 4x^2 + y^2$. Find $W(2+h, 3+h)$.

3) The volume of a right circular cylinder is calculated by a function of two variables, $V(x,y) = \pi x^2 y$, where x is the radius of the right circular cylinder and y represents the height of the cylinder. Evaluate $V(2,5)$ and explain what this means.

Answer

$V(2,5) = 20\pi \text{ units}^3$ This is the volume when the radius is 2 and the height is 5.

4) An oxygen tank is constructed of a right cylinder of height y and radius x with two hemispheres of radius x mounted on the top and bottom of the cylinder. Express the volume of the cylinder as a function of two variables, x and y, find $V(10,2)$, and explain what this means.

For exercises 5 - 10, find the domain and range of the given function. State the domain in set-builder notation and the range in interval notation.

5) $V(x,y) = 4x^2 + y^2$

Answer

Domain: $\{(x,y) \mid x \in \mathbb{R}, y \in \mathbb{R}\}$ That is, all points in the xy-plane

Range: $[0, \infty)$

6) $f(x,y) = \sqrt{x^2 + y^2 - 4}$

Answer

Domain: $\{(x,y) \mid x^2 + y^2 \geq 4\}$

Range: $[0, \infty)$

7) $f(x,y) = 4\ln(y^2 - x)$

Answer

Domain: $\{(x,y) \mid x < y^2\}$

Range: $(-\infty, \infty)$

8) $g(x,y) = \sqrt{16 - 4x^2 - y^2}$

Answer

Domain: $\{(x,y) \mid \frac{x^2}{4} + \frac{y^2}{16} \leq 1\}$

Range: $[0,4]$

9) $z = \arccos(y - x)$

Answer

Domain: $\{(x,y) \mid x - 1 \leq y \leq x + 1\}$ That is, all points between the graphs of $y = x - 1$ and $y = x + 1$.

Range: $[0, \pi]$

10) $f(x,y) = \frac{y+2}{x^2}$

Answer

Domain: $\{(x,y) \mid x \neq 0\}$

Range: $(-\infty, \infty)$

Find the range of the functions.

11) $g(x,y)=\sqrt{16-4x^2-y^2}$

Answer

$\{z \mid 0 \le z \le 4\}$ or in interval notation: $[0,4]$

12) $V(x,y)=4x^2+y^2$

13) $z=y^2-x^2$

Answer

The set $\mathbb{R}$

In exercises 14 - 29, find the level curves of each function at the indicated values of c to visualize the given function. Sketch a contour plot for those exercises where you are asked for more than 3 values of c.

14) $z(x,y)=y^2-x^2, \quad c=1$

15) $z(x,y)=y^2-x^2, \quad c=4$

Answer

$y^2-x^2=4$, a hyperbola

16) $g(x,y)=x^2+y^2; \quad c=0,1,2,3,4,9$

17) $g(x,y)=4-x-y; \quad c=0,1,2,3,4$

Answer

Level curves are lines with $y=-x+(4-c)$.

For each value of c these are:

$c=0: \ y=-x+4,$

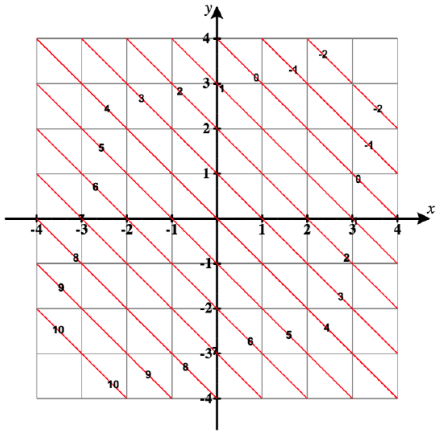
$c=1: \ y=-x+3,$

$c=2: \ y=-x+2,$

$c=3: \ y=-x+1,$

$c=4: \ y=-x.$

The contour plot consists of a series of parallel lines.



18) $f(x,y)=xy; c=1; \quad c=-1$

19) $h(x,y)=2x-y; \quad c=-2,0,2$

Answer

$2x-y=0, 2x-y=-2, 2x-y=2$; three lines

20) $f(x,y)=x^2-y; \quad c=1,2$

21) $g(x,y)=\dfrac{x}{x+y}; c=-1,0,1,2$

Answer

Level curves are lines with the form $y=x\left(\dfrac{1-c}{c}\right)$. At $c=0$, we solve it directly from the equation $\dfrac{x}{x+y}=0$ to get $x=0$.

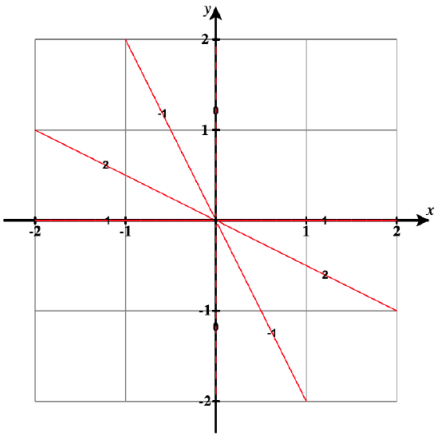
For each value of c these are:

$c=-1: y=-2x$,

$c=0: x=0,$ with $y\neq 0$,

$c=1: y=0,$ with $x\neq 0$,

$c=2: y=-\frac{1}{2}x$.



22) $g(x,y)=x^3-y; \quad c=-1,0,2$

23) $g(x,y)=e^{xy}; \quad c=\frac{1}{2},3$

Answer

The level curves have the form, $y=\dfrac{\ln c}{x}$.

For each value of c these are:

$c=\frac{1}{2}: y=\dfrac{\ln\frac{1}{2}}{x}$ that can be rewritten as, $y=-\dfrac{\ln 2}{x}$

$c=3: y=\dfrac{\ln 3}{x}$.

24) $f(x,y)=x^2; \quad c=4,9$

25) $f(x,y)=xy-x; \quad c=-2,0,2$

Answer

Level curves have the form: $y=\dfrac{c}{x}+1$.

Here $y=\dfrac{-2}{x}+1, \quad y=1, \quad y=\dfrac{2}{x}+1$ or $xy-x=-2$, $xy-x=0$, $xy-x=2$

26) $h(x,y)=\ln(x^2+y^2); \quad c=-1,0,1$

27) $g(x,y)=\ln\left(\dfrac{y}{x^2}\right); \quad c=-2,0,2$

Answer

The level curves have the form, $y=e^c x^2$.

For each value of c these are:

$c=-2: y=e^{-2}x^2$,

$c=0: y=x^2$,

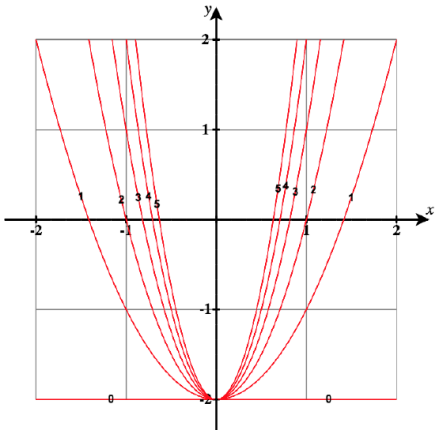
$c=2: y=e^2x^2$.

28) $z=f(x,y)=\sqrt{x^2+y^2}, \quad c=3$

29) $f(x,y)=\dfrac{y+2}{x^2}$, $c=$ any constant

Answer

The level curves are parabolas of the form $y=cx^2-2,$ **with** $x\neq 0$.



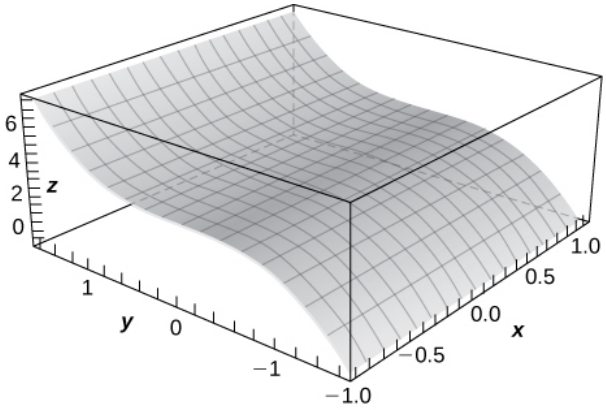
In exercises 30-32, find the vertical traces of the functions at the indicated values of x and y, and plot the traces.

30) $z=4-x-y, \quad x=2$

31) $f(x,y)=3x+y^3, \quad x=1$

Answer

$z=3+y^3$, a curve in the zy-**plane** with rulings parallel to the x-axis



32) $z=\cos\sqrt{x^2+y^2}, \quad x=1$

In exercises 33 - 38, find the domain and range of each function.

33) $z=\sqrt{100-4x^2-25y^2}$

Answer

Domain: $\left\{(x,y)\,|\,\dfrac{x^2}{25}+\dfrac{y^2}{4}\leq 1\right\}$

Range: $[0,10]$

34) $z=\ln(x-y^2)$

35) $f(x,y,z)=\dfrac{1}{\sqrt{36-4x^2-9y^2-z^2}}$

Answer

Domain: $\left\{(x,y,z)\,|\,\dfrac{x^2}{9}+\dfrac{y^2}{4}+\dfrac{z^2}{36}<1\right\}$

Range: $\left[\frac{1}{6},\infty\right)$

36) $f(x,y,z) = \sqrt{49 - x^2 - y^2 - z^2}$

37) $f(x,y,z) = \sqrt[3]{16 - x^2 - y^2 - z^2}$

Answer

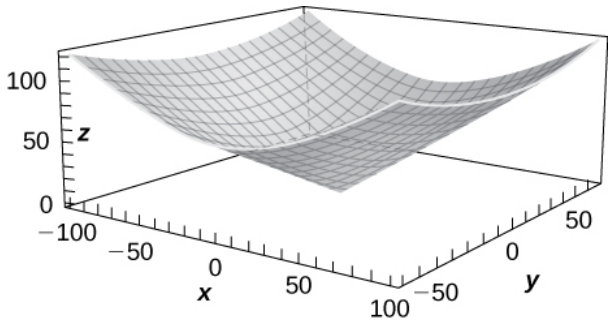
Domain: All points in xyz-space
Range: $\left(-\infty, \sqrt[3]{16}\right]$

38) $f(x,y) = \cos\sqrt{x^2 + y^2}$

In exercises 39 - 40, plot a graph of the function.

39) $z = f(x,y) = \sqrt{x^2 + y^2}$

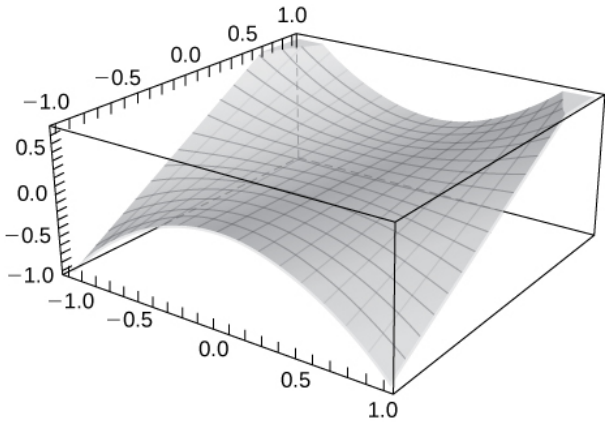
Answer



40) $z = x^2 + y^2$

41) Use technology to graph $z = x^2 y$.

Answer

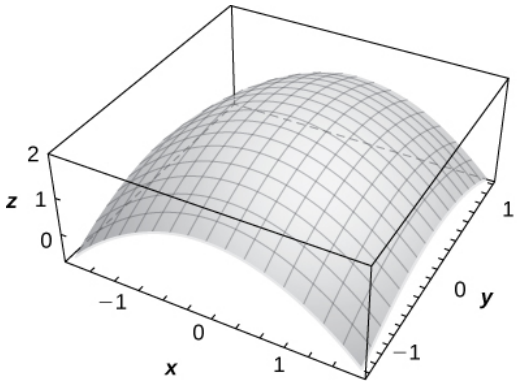


In exercises 42 - 46, sketch the function by finding its level curves. Verify the graph using technology, such as CalcPlot3D.

42) $f(x,y) = \sqrt{4 - x^2 - y^2}$

43) $f(x,y) = 2 - \sqrt{x^2 + y^2}$

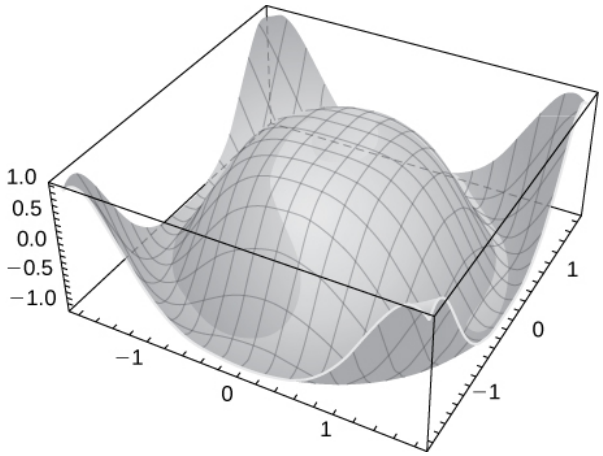
Answer



44) $z = 1 + e^{-x^2-y^2}$

45) $z = \cos\sqrt{x^2+y^2}$

Answer



46) $z = y^2 - x^2$

47) Describe the contour lines for several values of c for $z = x^2 + y^2 - 2x - 2y$.

Answer

The contour lines are concentric circles centered at the point, $(1,1)$.
You can see this by completing the square after setting this function equal to c.
That is, we write $x^2 - 2x + 1 + y^2 - 2y + 1 = c + 2$ which can be rewritten as, $(x-1)^2 + (y-1)^2 = c + 2$.
This gives us circles centered at the point, $(1,1)$, each with a radius of $\sqrt{c+2}$.

In exercises, 48 - 52, find the level surface for the given value of c for each function of three variables and describe it.

48) $w(x,y,z) = x - 2y + z, \quad c = 4$

49) $w(x,y,z) = x^2 + y^2 + z^2, \quad c = 9$

Answer

$x^2 + y^2 + z^2 = 9$, a sphere of radius 3

50) $w(x,y,z) = x^2 + y^2 - z^2, \quad c = -4$

51) $w(x,y,z) = x^2 + y^2 - z^2, \quad c = 4$

Answer

$x^2 + y^2 - z^2 = 4$, a hyperboloid of one sheet

52) $w(x,y,z) = 9x^2 - 4y^2 + 36z^2, \quad c = 0$

In exercises 53 - 55, find an equation of the level curve of f that contains the point P.

53) $f(x,y) = 1 - 4x^2 - y^2, \quad P(0,1)$

Answer

$4x^2 + y^2 = 1,$

54) $g(x,y) = y^2 \arctan x, \quad P(1,2)$

55) $g(x,y) = e^{xy}(x^2 + y^2), \quad P(1,0)$

Answer

$1 = e^{xy}(x^2 + y^2)$

56) The strength E of an electric field at point (x,y,z) resulting from an infinitely long charged wire lying along the y-axis is given by $E(x,y,z) = k/\sqrt{x^2 + y^2}$, where k is a positive constant. For simplicity, let $k = 1$ and find the equations of the level surfaces for $E = 10$ and $E = 100$.

57) A thin plate made of iron is located in the xy-plane The temperature T in degrees Celsius at a point $P(x,y)$ is inversely proportional to the square of its distance from the origin. Express T as a function of x and y.

Answer

$T(x,y) = \dfrac{k}{x^2 + y^2}$

58) Refer to the preceding problem. Using the temperature function found there, determine the proportionality constant if the temperature at point $P(1,2)$ is $50°C$. Use this constant to determine the temperature at point $Q(3,4)$.

59) Refer to the preceding problem. Find the level curves for $T = 40°C$ and $T = 100°C$, and describe what the level curves represent.

Answer

$x^2 + y^2 = \dfrac{k}{40}, \quad x^2 + y^2 = \dfrac{k}{100}$. The level curves represent circles of radii $\sqrt{10k}/20$ and $\sqrt{k}/10$

CONTRIBUTORS

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.
- Paul Seeburger (Monroe Community College) added the contour plots to answers for problems 17, 21 and 29.

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14.2: Limits and Continuity

Learning Objectives

- Calculate the limit of a function of two variables.
- Learn how a function of two variables can approach different values at a boundary point, depending on the path of approach.
- State the conditions for continuity of a function of two variables.
- Verify the continuity of a function of two variables at a point.
- Calculate the limit of a function of three or more variables and verify the continuity of the function at a point.

We have now examined functions of more than one variable and seen how to graph them. In this section, we see how to take the limit of a function of more than one variable, and what it means for a function of more than one variable to be continuous at a point in its domain. It turns out these concepts have aspects that just don't occur with functions of one variable.

Limit of a Function of Two Variables

Recall from Section 2.5 that the definition of a limit of a function of one variable:

Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a. Let L be a real number. Then

$$\lim_{x \to a} f(x) = L$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$, such that if $0 < |x - a| < \delta$ for all x in the domain of f, then

$$|f(x) - L| < \varepsilon.$$

Before we can adapt this definition to define a limit of a function of two variables, we first need to see how to extend the idea of an open interval in one variable to an open interval in two variables.

Definition: δ Disks

Consider a point $(a, b) \in \mathbb{R}^2$. A δ **disk** centered at point (a, b) is defined to be an open disk of radius δ centered at point (a, b) — that is,

$$\{(x, y) \in \mathbb{R}^2 \mid (x - a)^2 + (y - b)^2 < \delta^2\}$$

as shown in Figure 14.2.1.

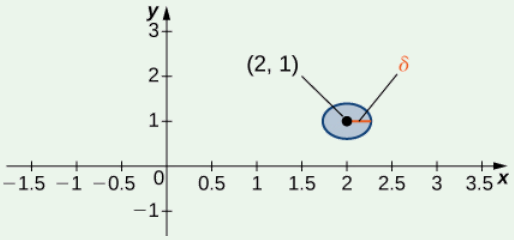

Figure 14.2.1: A δ disk centered around the point $(2, 1)$.

The idea of a δ disk appears in the definition of the limit of a function of two variables. If δ is small, then all the points (x, y) in the δ disk are close to (a, b). This is completely analogous to x being close to a in the definition of a limit of a function of one variable. In one dimension, we express this restriction as

$$a - \delta < x < a + \delta.$$

In more than one dimension, we use a δ disk.

Definition: limit of a function of two variables

Let f be a function of two variables, x and y. The limit of $f(x,y)$ as (x,y) approaches (a,b) is L, written

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for each $\varepsilon > 0$ there exists a small enough $\delta > 0$ such that for all points (x,y) in a δ disk around (a,b), except possibly for (a,b) itself, the value of $f(x,y)$ is no more than ε away from L (Figure 14.2.2).

Using symbols, we write the following: For any $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(x,y) - L| < \varepsilon$$

whenever

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta.$$

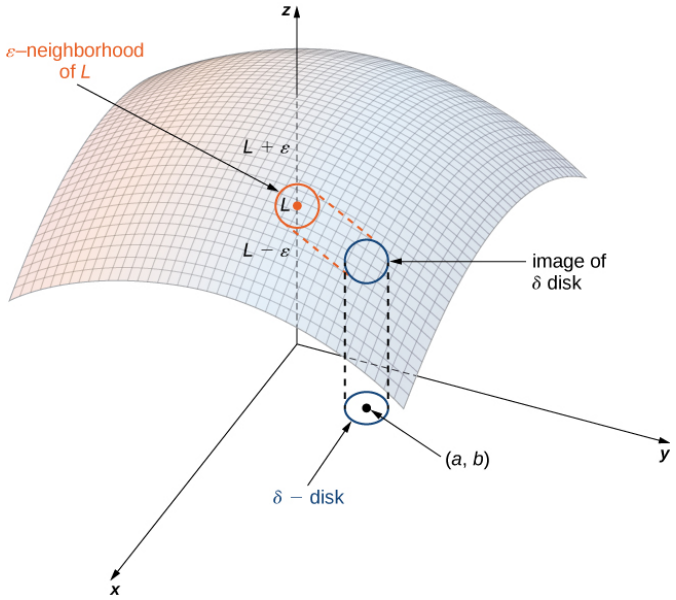

Figure 14.2.2: The limit of a function involving two variables requires that $f(x,y)$ be within ε of L whenever (x,y) is within δ of (a,b). The smaller the value of ε, the smaller the value of δ.

Proving that a limit exists using the definition of a **limit of a function of two variables** can be challenging. Instead, we use the following theorem, which gives us shortcuts to finding limits. The formulas in this theorem are an extension of the formulas in the limit laws theorem in the section titled The Limit Laws in single-variable calculus.

Theorem 14.2.1: Limit Laws for Functions of Two Variables

Let $f(x,y)$ and $g(x,y)$ be defined for all $(x,y) \neq (a,b)$ in a neighborhood around (a,b), and assume the neighborhood is contained completely inside the domain of f. Assume that L and M are real numbers such that

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

and

$$\lim_{(x,y)\to(a,b)} g(x,y) = M,$$

and let c be a constant. Then each of the following statements holds:

Constant Law:

$$\lim_{(x,y)\to(a,b)} c = c$$

Identity Laws:

$$\lim_{(x,y)\to(a,b)} x = a$$

$$\lim_{(x,y)\to(a,b)} y = b$$

Sum Law:

$$\lim_{(x,y)\to(a,b)} (f(x,y) + g(x,y)) = L + M$$

Difference Law:

$$\lim_{(x,y)\to(a,b)} (f(x,y) - g(x,y)) = L - M$$

Constant Multiple Law:

$$\lim_{(x,y)\to(a,b)} (cf(x,y)) = cL$$

Product Law:

$$\lim_{(x,y)\to(a,b)} (f(x,y)g(x,y)) = LM$$

Quotient Law:

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M} \text{ for } M \neq 0$$

Power Law:

$$\lim_{(x,y)\to(a,b)} (f(x,y))^n = L^n$$

for any positive integer n.

Root Law:

$$\lim_{(x,y)\to(a,b)} \sqrt[n]{f(x,y)} = \sqrt[n]{L}$$

for all L if n is odd and positive, and for $L \geq 0$ if n is even and positive.

The proofs of these properties are similar to those for the limits of functions of one variable. We can apply these laws to finding limits of various functions.

Example 14.2.1: Finding the Limit of a Function of Two Variables

Find each of the following limits:

a. $\lim\limits_{(x,y)\to(2,-1)} (x^2 - 2xy + 3y^2 - 4x + 3y - 6)$

b. $\lim\limits_{(x,y)\to(2,-1)} \dfrac{2x + 3y}{4x - 3y}$

Solution

a. First use the sum and difference laws to separate the terms:

$$\begin{aligned}\lim_{(x,y)\to(2,-1)} (x^2 - 2xy + 3y^2 - 4x + 3y - 6) \\ = \left(\lim_{(x,y)\to(2,-1)} x^2\right) - \left(\lim_{(x,y)\to(2,-1)} 2xy\right) + \left(\lim_{(x,y)\to(2,-1)} 3y^2\right) - \left(\lim_{(x,y)\to(2,-1)} 4x\right) \\ + \left(\lim_{(x,y)\to(2,-1)} 3y\right) - \left(\lim_{(x,y)\to(2,-1)} 6\right).\end{aligned}$$

Next, use the constant multiple law on the second, third, fourth, and fifth limits:

$$= (\lim_{(x,y)\to(2,-1)} x^2) - 2(\lim_{(x,y)\to(2,-1)} xy) + 3(\lim_{(x,y)\to(2,-1)} y^2) - 4(\lim_{(x,y)\to(2,-1)} x)$$
$$+3(\lim_{(x,y)\to(2,-1)} y) - \lim_{(x,y)\to(2,-1)} 6.$$

Now, use the power law on the first and third limits, and the product law on the second limit:

$$\left(\lim_{(x,y)\to(2,-1)} x\right)^2 - 2\left(\lim_{(x,y)\to(2,-1)} x\right)\left(\lim_{(x,y)\to(2,-1)} y\right) + 3\left(\lim_{(x,y)\to(2,-1)} y\right)^2$$
$$-4\left(\lim_{(x,y)\to(2,-1)} x\right) + 3\left(\lim_{(x,y)\to(2,-1)} y\right) - \lim_{(x,y)\to(2,-1)} 6.$$

Last, use the identity laws on the first six limits and the constant law on the last limit:

$$\lim_{(x,y)\to(2,-1)} (x^2 - 2xy + 3y^2 - 4x + 3y - 6) = (2)^2 - 2(2)(-1) + 3(-1)^2 - 4(2) + 3(-1) - 6$$
$$= -6.$$

b. Before applying the quotient law, we need to verify that the limit of the denominator is nonzero. Using the difference law, constant multiple law, and identity law,

$$\lim_{(x,y)\to(2,-1)} (4x - 3y) = \lim_{(x,y)\to(2,-1)} 4x - \lim_{(x,y)\to(2,-1)} 3y$$
$$= 4(\lim_{(x,y)\to(2,-1)} x) - 3(\lim_{(x,y)\to(2,-1)} y)$$
$$= 4(2) - 3(-1) = 11.$$

Since the limit of the denominator is nonzero, the quotient law applies. We now calculate the limit of the numerator using the difference law, constant multiple law, and identity law:

$$\lim_{(x,y)\to(2,-1)} (2x + 3y) = \lim_{(x,y)\to(2,-1)} 2x + \lim_{(x,y)\to(2,-1)} 3y$$
$$= 2(\lim_{(x,y)\to(2,-1)} x) + 3(\lim_{(x,y)\to(2,-1)} y)$$
$$= 2(2) + 3(-1) = 1.$$

Therefore, according to the quotient law we have

$$\lim_{(x,y)\to(2,-1)} \frac{2x + 3y}{4x - 3y} = \frac{\lim_{(x,y)\to(2,-1)} (2x + 3y)}{\lim_{(x,y)\to(2,-1)} (4x - 3y)}$$
$$= \frac{1}{11}.$$

? Exercise 14.2.1:

Evaluate the following limit:

$$\lim_{(x,y)\to(5,-2)} \sqrt[3]{\frac{x^2 - y}{y^2 + x - 1}}.$$

Hint

Use the limit laws.

Answer

$$\lim_{(x,y)\to(5,-2)} \sqrt[3]{\frac{x^2 - y}{y^2 + x - 1}} = \frac{3}{2}$$

Since we are taking the limit of a function of two variables, the point (a, b) is in $\mathbb{R}^2$, and it is possible to approach this point from an infinite number of directions. Sometimes when calculating a limit, the answer varies depending on the path taken toward (a, b). If

this is the case, then the limit fails to exist. In other words, the limit must be unique, regardless of path taken.

Example 14.2.2: Limits That Fail to Exist

Show that neither of the following limits exist:

a. $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy}{3x^2+y^2}$

b. $\lim\limits_{(x,y)\to(0,0)} \dfrac{4xy^2}{x^2+3y^4}$

Solution

a. The domain of the function $f(x,y)=\dfrac{2xy}{3x^2+y^2}$ consists of all points in the xy-plane except for the point $(0,0)$ (Figure 14.2.3). To show that the limit does not exist as (x,y) approaches $(0,0)$, we note that it is impossible to satisfy the definition of a limit of a function of two variables because of the fact that the function takes different values along different lines passing through point $(0,0)$. First, consider the line $y=0$ in the xy-plane. Substituting $y=0$ into $f(x,y)$ gives

$$f(x,0)=\frac{2x(0)}{3x^2+0^2}=0$$

for any value of x. Therefore the value of f remains constant for any point on the x-axis, and as y approaches zero, the function remains fixed at zero.

Next, consider the line $y=x$. Substituting $y=x$ into $f(x,y)$ gives

$$f(x,x)=\frac{2x(x)}{3x^2+x^2}=\frac{2x^2}{4x^2}=\tfrac{1}{2}.$$

This is true for any point on the line $y=x$. If we let x approach zero while staying on this line, the value of the function remains fixed at $\frac{1}{2}$, regardless of how small x is.

Choose a value for ε that is less than $1/2$—say, $1/4$. Then, no matter how small a δ disk we draw around $(0,0)$, the values of $f(x,y)$ for points inside that δ disk will include both 0 and $\frac{1}{2}$. Therefore, the definition of limit at a point is never satisfied and the limit fails to exist.

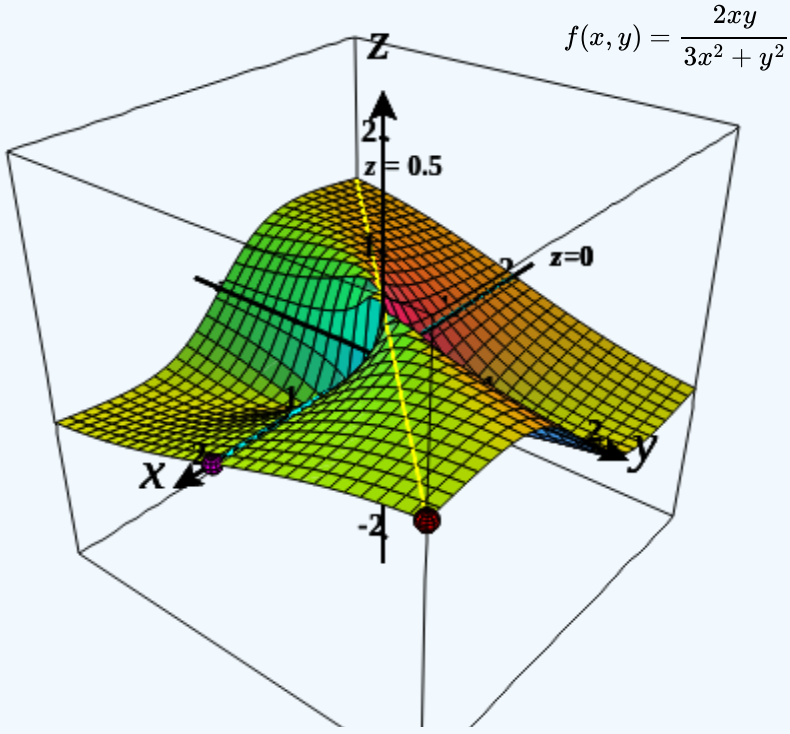

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Figure 14.2.3: Graph of the function $f(x,y)=\dfrac{2xy}{3x^2+y^2}$. Along the line $y=0$, the function is equal to zero; along the line $y=x$, the function is equal to $\frac{1}{2}$.

b. In a similar fashion to a., we can approach the origin along any straight line passing through the origin. If we try the x-axis (i.e., $y=0$), then the function remains fixed at zero. The same is true for the y-axis. Suppose we approach the origin along a straight line of slope k. The equation of this line is $y=kx$. Then the limit becomes

$$\begin{aligned}\lim_{(x,y)\to(0,0)}\frac{4xy^2}{x^2+3y^4} &= \lim_{(x,y)\to(0,0)}\frac{4x(kx)^2}{x^2+3(kx)^4}\\ &= \lim_{(x,y)\to(0,0)}\frac{4k^2x^3}{x^2+3k^4x^4}\\ &= \lim_{(x,y)\to(0,0)}\frac{4k^2x}{1+3k^4x^2}\\ &= \frac{\lim\limits_{(x,y)\to(0,0)}(4k^2x)}{\lim\limits_{(x,y)\to(0,0)}(1+3k^4x^2)}\\ &= 0.\end{aligned}$$

regardless of the value of k. It would seem that the limit is equal to zero. What if we chose a curve passing through the origin instead? For example, we can consider the parabola given by the equation $x=y^2$. Substituting y^2 in place of x in $f(x,y)$ gives

$$\begin{aligned}\lim_{(x,y)\to(0,0)}\frac{4xy^2}{x^2+3y^4} &= \lim_{(x,y)\to(0,0)}\frac{4(y^2)y^2}{(y^2)^2+3y^4}\\ &= \lim_{(x,y)\to(0,0)}\frac{4y^4}{y^4+3y^4}\\ &= \lim_{(x,y)\to(0,0)}1\\ &= 1.\end{aligned}$$

By the same logic in part a, it is impossible to find a δ disk around the origin that satisfies the definition of the limit for any value of $\varepsilon<1$. Therefore,

$$\lim_{(x,y)\to(0,0)}\frac{4xy^2}{x^2+3y^4}$$

does **not** exist.

? Exercise 14.2.2:

Show that

$$\lim_{(x,y)\to(2,1)}\frac{(x-2)(y-1)}{(x-2)^2+(y-1)^2}$$

does not exist.

Hint

Pick a line with slope k passing through point $(2,1)$.

Answer

If $y=k(x-2)+1$, then $\lim_{(x,y)\to(2,1)}\dfrac{(x-2)(y-1)}{(x-2)^2+(y-1)^2}=\dfrac{k}{1+k^2}$. Since the answer depends on k, the limit fails to exist.

Interior Points and Boundary Points

To study continuity and differentiability of a function of two or more variables, we first need to learn some new terminology.

Definition: interior and boundary points

Let S be a subset of $\mathbb{R}^2$ (Figure 14.2.4).

1. A point P_0 is called an *interior point* of S if there is a δ disk centered around P_0 contained completely in S.
2. A point P_0 is called a *boundary point* of S if every δ disk centered around P_0 contains points both inside and outside S.

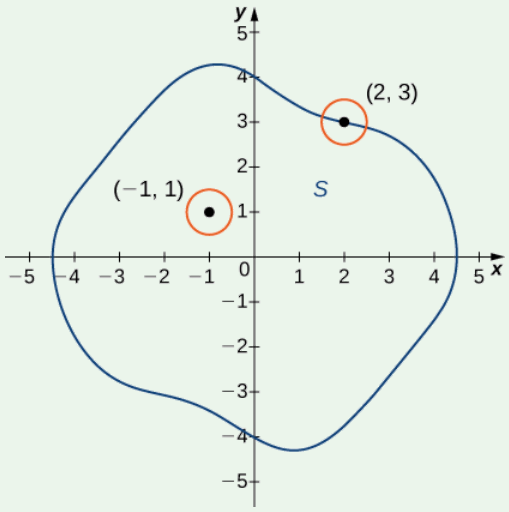

Figure 14.2.4: In the set S shown, $(-1,1)$ is an interior point and $(2,3)$ is a boundary point.

Definition: Open and closed sets

Let S be a subset of $\mathbb{R}^2$ (Figure 14.2.4).

1. S is called an *open set* if every point of S is an interior point.
2. S is called a *closed set* if it contains all its boundary points.

An example of an open set is a δ disk. If we include the boundary of the disk, then it becomes a closed set. A set that contains some, but not all, of its boundary points is neither open nor closed. For example if we include half the boundary of a δ disk but not the other half, then the set is neither open nor closed.

Definition: connected sets and Regions

Let S be a subset of $\mathbb{R}^2$ (Figure 14.2.4).

1. An open set S is a *connected set* if it cannot be represented as the union of two or more disjoint, nonempty open subsets.
2. A set S is a *region* if it is open, connected, and nonempty.

The definition of a limit of a function of two variables requires the δ disk to be contained inside the domain of the function. However, if we wish to find the limit of a function at a boundary point of the domain, the δ **disk** is not contained inside the domain. By definition, some of the points of the δ **disk** are inside the domain and some are outside. Therefore, we need only consider points that are inside both the δ disk and the domain of the function. This leads to the definition of the limit of a function at a boundary point.

Definition

Let f be a function of two variables, x and y, and suppose (a,b) is on the boundary of the domain of f. Then, the limit of $f(x,y)$ as (x,y) approaches (a,b) is L, written

$$\lim_{(x,y)\to(a,b)} f(x,y) = L,$$

if for any $\varepsilon > 0$, there exists a number $\delta > 0$ such that for any point (x,y) inside the domain of f and within a suitably small distance positive δ of (a,b), the value of $f(x,y)$ is no more than ε away from L (Figure 14.2.2). Using symbols, we can write:

For any $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(x,y) - L| < \varepsilon \text{ whenever } 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta.$$

Example 14.2.3: Limit of a Function at a Boundary Point

Prove

$$\lim_{(x,y)\to(4,3)} \sqrt{25 - x^2 - y^2} = 0.$$

Solution

The domain of the function $f(x,y) = \sqrt{25 - x^2 - y^2}$ is $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 25\}$, which is a circle of radius 5 centered at the origin, along with its interior as shown in Figure 14.2.5.

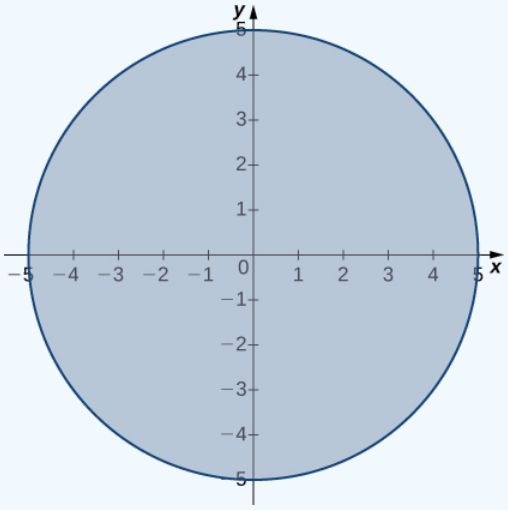

Figure 14.2.5: Domain of the function $f(x,y) = \sqrt{25 - x^2 - y^2}$.

We can use the limit laws, which apply to limits at the boundary of domains as well as interior points:

$$\begin{aligned}\lim_{(x,y)\to(4,3)} \sqrt{25 - x^2 - y^2} &= \sqrt{\lim_{(x,y)\to(4,3)} (25 - x^2 - y^2)} \\ &= \sqrt{\lim_{(x,y)\to(4,3)} 25 - \lim_{(x,y)\to(4,3)} x^2 - \lim_{(x,y)\to(4,3)} y^2} \\ &= \sqrt{25 - 4^2 - 3^2} \\ &= 0\end{aligned}$$

See the following graph.

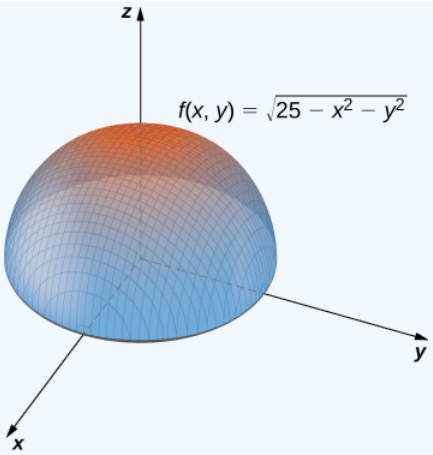

Figure 14.2.6: Graph of the function $f(x,y)=\sqrt{25-x^2-y^2}$.

Exercise 14.2.3

Evaluate the following limit:

$$\lim_{(x,y)\to(5,-2)} \sqrt{29-x^2-y^2}.$$

Hint

Determine the domain of $f(x,y)=\sqrt{29-x^2-y^2}$.

Answer

$$\lim_{(x,y)\to(5,-2)} \sqrt{29-x^2-y^2}=\sqrt{29-5^2-(-2)^2}=\sqrt{29-25-4}=\sqrt{0}=0$$

Continuity of Functions of Two Variables

In Continuity, we defined the continuity of a function of one variable and saw how it relied on the limit of a function of one variable. In particular, three conditions are necessary for $f(x)$ to be continuous at point $x=a$

1. $f(a)$ exists.
2. $\lim\limits_{x\to a} f(x)$ exists.
3. $\lim\limits_{x\to a} f(x) = f(a)$.

These three conditions are necessary for continuity of a function of two variables as well.

Definition: continuous Functions

A function $f(x,y)$ is continuous at a point (a,b) in its domain if the following conditions are satisfied:

1. $f(a,b)$ exists.
2. $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists.
3. $\lim\limits_{(x,y)\to(a,b)} f(x,y)=f(a,b)$.

Example 14.2.4: Demonstrating Continuity for a Function of Two Variables

Show that the function

$$f(x,y)=\frac{3x+2y}{x+y+1}$$

is continuous at point $(5,-3)$.

Solution

There are three conditions to be satisfied, per the definition of continuity. In this example, $a = 5$ and $b = -3$.

1. $f(a, b)$ exists. This is true because the domain of the function f consists of those ordered pairs for which the denominator is nonzero (i.e., $x + y + 1 \neq 0$). Point $(5, -3)$ satisfies this condition. Furthermore,

$$f(a, b) = f(5, -3) = \frac{3(5) + 2(-3)}{5 + (-3) + 1} = \frac{15 - 6}{2 + 1} = 3.$$

2. $\lim\limits_{(x,y)\to(a,b)} f(x, y)$ exists. This is also true:

$$\begin{aligned} \lim_{(x,y)\to(a,b)} f(x, y) &= \lim_{(x,y)\to(5,-3)} \frac{3x + 2y}{x + y + 1} \\ &= \frac{\lim\limits_{(x,y)\to(5,-3)} (3x + 2y)}{\lim\limits_{(x,y)\to(5,-3)} (x + y + 1)} \\ &= \frac{15 - 6}{5 - 3 + 1} \\ &= 3. \end{aligned}$$

3. $\lim\limits_{(x,y)\to(a,b)} f(x, y) = f(a, b)$. This is true because we have just shown that both sides of this equation equal three.

? Exercise 14.2.4

Show that the function

$$f(x, y) = \sqrt{26 - 2x^2 - y^2}$$

is continuous at point $(2, -3)$.

Hint

Use the three-part definition of continuity.

Answer

1. The domain of f contains the ordered pair $(2, -3)$ because $f(a, b) = f(2, -3) = \sqrt{16 - 2(2)^2 - (-3)^2} = 3$
2. $\lim\limits_{(x,y)\to(a,b)} f(x, y) = 3$
3. $\lim\limits_{(x,y)\to(a,b)} f(x, y) = f(a, b) = 3$

Continuity of a function of any number of variables can also be defined in terms of delta and epsilon. A function of two variables is continuous at a point (x_0, y_0) in its domain if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that, whenever $\sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$ it is true, $|f(x, y) - f(a, b)| < \varepsilon$. This definition can be combined with the formal definition (that is, the *epsilon–delta definition*) of continuity of a function of one variable to prove the following theorems:

Theorem 14.2.2: The Sum of Continuous Functions is Continuous

If $f(x, y)$ is continuous at (x_0, y_0), and $g(x, y)$ is continuous at (x_0, y_0), then $f(x, y) + g(x, y)$ is continuous at (x_0, y_0).

Theorem 14.2.3: The Product of Continuous Functions is Continuous

If $g(x)$ is continuous at x_0 and $h(y)$ is continuous at y_0, then $f(x, y) = g(x)h(y)$ is continuous at (x_0, y_0).

Theorem 14.2.4: The Composition of Continuous Functions is Continuous

Let g be a function of two variables from a domain $D \subseteq \mathbb{R}^2$ to a range $R \subseteq R$. Suppose g is continuous at some point $(x_0, y_0) \in D$ and define $z_0 = g(x_0, y_0)$. Let f be a function that maps R to R such that z_0 is in the domain of f. Last, assume f is continuous at z_0. Then $f \circ g$ is continuous at (x_0, y_0) as shown in Figure 14.2.7.

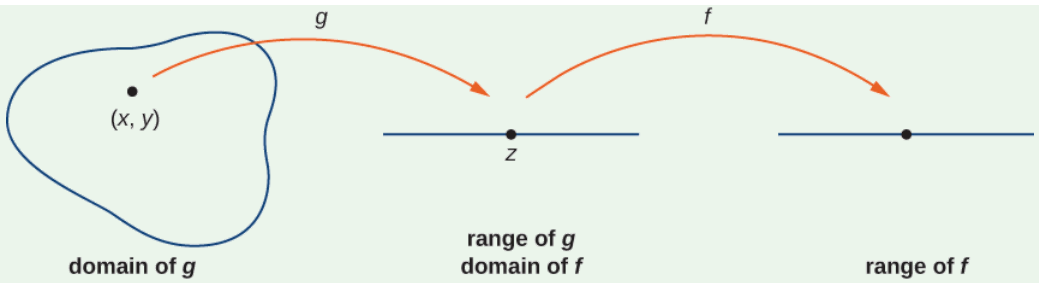

Figure 14.2.7: The composition of two continuous functions is continuous.

Let's now use the previous theorems to show continuity of functions in the following examples.

Example 14.2.5: More Examples of Continuity of a Function of Two Variables

Show that the functions $f(x,y) = 4x^3y^2$ and $g(x,y) = \cos(4x^3y^2)$ are continuous everywhere.

Solution

The polynomials $g(x) = 4x^3$ and $h(y) = y^2$ are continuous at every real number, and therefore by the product of continuous functions theorem, $f(x,y) = 4x^3y^2$ is continuous at every point (x,y) in the xy-plane. Since $f(x,y) = 4x^3y^2$ is continuous at every point (x,y) in the xy-plane and $g(x) = \cos x$ is continuous at every real number x, the continuity of the composition of functions tells us that $g(x,y) = \cos(4x^3y^2)$ is continuous at every point (x,y) in the xy-plane.

Exercise 14.2.5

Show that the functions $f(x,y) = 2x^2y^3 + 3$ and $g(x,y) = (2x^2y^3 + 3)^4$ are continuous everywhere.

Hint

Use the continuity of the sum, product, and composition of two functions.

Answer

The polynomials $g(x) = 2x^2$ and $h(y) = y^3$ are continuous at every real number; therefore, by the product of continuous functions theorem, $f(x,y) = 2x^2y^3$ is continuous at every point (x,y) in the xy-plane. Furthermore, any constant function is continuous everywhere, so $g(x,y) = 3$ is continuous at every point (x,y) in the xy-plane. Therefore, $f(x,y) = 2x^2y^3 + 3$ is continuous at every point (x,y) in the xy-plane. Last, $h(x) = x^4$ is continuous at every real number x, so by the continuity of composite functions theorem $g(x,y) = (2x^2y^3 + 3)^4$ is continuous at every point (x,y) in the xy-plane.

Functions of Three or More Variables

The limit of a function of three or more variables occurs readily in applications. For example, suppose we have a function $f(x,y,z)$ that gives the temperature at a physical location (x,y,z) in three dimensions. Or perhaps a function $g(x,y,z,t)$ can indicate air pressure at a location (x,y,z) at time t. How can we take a limit at a point in $\mathbb{R}^3$? What does it mean to be continuous at a point in four dimensions?

The answers to these questions rely on extending the concept of a δ disk into more than two dimensions. Then, the ideas of the limit of a function of three or more variables and the continuity of a function of three or more variables are very similar to the definitions given earlier for a function of two variables.

Definition: δ-balls

Let (x_0,y_0,z_0) be a point in $\mathbb{R}^3$. Then, a δ-ball in three dimensions consists of all points in $\mathbb{R}^3$ lying at a distance of less than δ from (x_0,y_0,z_0) —that is,

$$\{(x,y,z) \in \mathbb{R}^3 \mid \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} < \delta\}.$$

To define a δ-ball in higher dimensions, add additional terms under the radical to correspond to each additional dimension. For example, given a point $P = (w_0, x_0, y_0, z_0)$ in $\mathbb{R}^4$, a δ ball around P can be described by

$$\{(w, x, y, z) \in \mathbb{R}^4 \mid \sqrt{(w - w_0)^2 + (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} < \delta\}.$$

To show that a limit of a function of three variables exists at a point (x_0, y_0, z_0), it suffices to show that for any point in a δ ball centered at (x_0, y_0, z_0), the value of the function at that point is arbitrarily close to a fixed value (the limit value). All the limit laws for functions of two variables hold for functions of more than two variables as well.

Example 14.2.6: Finding the Limit of a Function of Three Variables

Find

$$\lim_{(x,y,z)\to(4,1,-3)} \frac{x^2y - 3z}{2x + 5y - z}.$$

Solution

Before we can apply the quotient law, we need to verify that the limit of the denominator is nonzero. Using the difference law, the identity law, and the constant law,

$$\begin{aligned}\lim_{(x,y,z)\to(4,1,-3)} (2x + 5y - z) &= 2(\lim_{(x,y,z)\to(4,1,-3)} x) + 5(\lim_{(x,y,z)\to(4,1,-3)} y) - (\lim_{(x,y,z)\to(4,1,-3)} z) \\ &= 2(4) + 5(1) - (-3) \\ &= 16.\end{aligned}$$

Since this is nonzero, we next find the limit of the numerator. Using the product law, power law, difference law, constant multiple law, and identity law,

$$\begin{aligned}\lim_{(x,y,z)\to(4,1,-3)} (x^2y - 3z) &= (\lim_{(x,y,z)\to(4,1,-3)} x)^2(\lim_{(x,y,z)\to(4,1,-3)} y) - 3\lim_{(x,y,z)\to(4,1,-3)} z \\ &= (4^2)(1) - 3(-3) \\ &= 16 + 9 \\ &= 25\end{aligned}$$

Last, applying the quotient law:

$$\lim_{(x,y,z)\to(4,1,-3)} \frac{x^2y - 3z}{2x + 5y - z} = \frac{\lim\limits_{(x,y,z)\to(4,1,-3)} (x^2y - 3z)}{\lim\limits_{(x,y,z)\to(4,1,-3)} (2x + 5y - z)} = \frac{25}{16}$$

Exercise 14.2.6

Find

$$\lim_{(x,y,z)\to(4,-1,3)} \sqrt{13 - x^2 - 2y^2 + z^2}$$

Hint

Use the limit laws and the continuity of the composition of functions.

Answer

$$\lim_{(x,y,z)\to(4,-1,3)} \sqrt{13 - x^2 - 2y^2 + z^2} = 2$$

Key Concepts

- To study limits and continuity for functions of two variables, we use a δ disk centered around a given point.
- A function of several variables has a limit if for any point in a δ ball centered at a point P, the value of the function at that point is arbitrarily close to a fixed value (the limit value).

- The limit laws established for a function of one variable have natural extensions to functions of more than one variable.
- A function of two variables is continuous at a point if the limit exists at that point, the function exists at that point, and the limit and function are equal at that point.

Glossary

boundary point

a point P_0 of R is a boundary point if every δ disk centered around P_0 contains points both inside and outside R

closed set

a set S that contains all its boundary points

connected set

an open set S that cannot be represented as the union of two or more disjoint, nonempty open subsets

δ disk

an open disk of radius δ centered at point (a, b)

δ ball

all points in $\mathbb{R}^3$ lying at a distance of less than δ from (x_0, y_0, z_0)

interior point

a point P_0 of $\mathbb{R}$ is a boundary point if there is a δ disk centered around P_0 contained completely in $\mathbb{R}$

open set

a set S that contains none of its boundary points

region

an open, connected, nonempty subset of $\mathbb{R}^2$

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14.2E: EXERCISES FOR SECTION 14.2

1) Use the limit laws for functions of two variables to evaluate each limit below, given that $\lim\limits_{(x,y)\to(a,b)} f(x,y)=5$ and $\lim\limits_{(x,y)\to(a,b)} g(x,y)=2$.

a. $\lim\limits_{(x,y)\to(a,b)} [f(x,y)+g(x,y)]$

b. $\lim\limits_{(x,y)\to(a,b)} [f(x,y)g(x,y)]$

c. $\lim\limits_{(x,y)\to(a,b)} \left[\dfrac{7f(x,y)}{g(x,y)}\right]$

d. $\lim\limits_{(x,y)\to(a,b)} \left[\dfrac{2f(x,y)-4g(x,y)}{f(x,y)-g(x,y)}\right]$

Answer

a. $\lim\limits_{(x,y)\to(a,b)} [f(x,y)+g(x,y)] = \lim\limits_{(x,y)\to(a,b)} f(x,y) + \lim\limits_{(x,y)\to(a,b)} g(x,y) = 5+2=7$

b. $\lim\limits_{(x,y)\to(a,b)} [f(x,y)g(x,y)] = \left(\lim\limits_{(x,y)\to(a,b)} f(x,y)\right)\left(\lim\limits_{(x,y)\to(a,b)} g(x,y)\right) = 5(2)=10$

c. $\lim\limits_{(x,y)\to(a,b)} \left[\dfrac{7f(x,y)}{g(x,y)}\right] = \dfrac{7\left(\lim\limits_{(x,y)\to(a,b)} f(x,y)\right)}{\lim\limits_{(x,y)\to(a,b)} g(x,y)} = \dfrac{7(5)}{2} = 17.5$

d. $\lim\limits_{(x,y)\to(a,b)} \left[\dfrac{2f(x,y)-4g(x,y)}{f(x,y)-g(x,y)}\right] = \dfrac{2\left(\lim\limits_{(x,y)\to(a,b)} f(x,y)\right)-4\left(\lim\limits_{(x,y)\to(a,b)} g(x,y)\right)}{\lim\limits_{(x,y)\to(a,b)} f(x,y)-\lim\limits_{(x,y)\to(a,b)} g(x,y)} = \dfrac{2(5)-4(2)}{5-2} = \dfrac{2}{3}$

In exercises 2 - 4, find the limit of the function.

2) $\lim\limits_{(x,y)\to(1,2)} x$

3) $\lim\limits_{(x,y)\to(1,2)} \dfrac{5x^2y}{x^2+y^2}$

Answer

$\lim\limits_{(x,y)\to(1,2)} \dfrac{5x^2y}{x^2+y^2} = 2$

4) Show that the limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{5x^2y}{x^2+y^2}$ exists and is the same along the paths: y-axis and x-axis, and along $y=x$.

In exercises 5 - 19, evaluate the limits at the indicated values of x and y. If the limit does not exist, state this and explain why the limit does not exist.

5) $\lim\limits_{(x,y)\to(0,0)} \dfrac{4x^2+10y^2+4}{4x^2-10y^2+6}$

Answer

$\lim\limits_{(x,y)\to(0,0)} \dfrac{4x^2+10y^2+4}{4x^2-10y^2+6} = \dfrac{2}{3}$

6) $\lim\limits_{(x,y)\to(11,13)} \sqrt{\dfrac{1}{xy}}$

7) $\lim\limits_{(x,y)\to(0,1)} \dfrac{y^2\sin x}{x}$

Answer

$\lim\limits_{(x,y)\to(0,1)} \dfrac{y^2\sin x}{x} = 1$

8) $\lim\limits_{(x,y)\to(0,0)} \sin\left(\dfrac{x^8+y^7}{x-y+10}\right)$

9) $\lim\limits_{(x,y)\to(\pi/4,1)} \dfrac{y\tan x}{y+1}$

Answer

$\lim\limits_{(x,y)\to(\pi/4,1)} \dfrac{y\tan x}{y+1} = \dfrac{1}{2}$

10) $\lim\limits_{(x,y)\to(0,\pi/4)} \dfrac{\sec x + 2}{3x - \tan y}$

11) $\lim\limits_{(x,y)\to(2,5)} \left(\dfrac{1}{x} - \dfrac{5}{y}\right)$

Answer

$$\lim_{(x,y)\to(2,5)} \left(\frac{1}{x} - \frac{5}{y}\right) = -\frac{1}{2}$$

12) $\lim\limits_{(x,y)\to(4,4)} x \ln y$

13) $\lim\limits_{(x,y)\to(4,4)} e^{-x^2-y^2}$

Answer

$$\lim_{(x,y)\to(4,4)} e^{-x^2-y^2} = e^{-32}$$

14) $\lim\limits_{(x,y)\to(0,0)} \sqrt{9 - x^2 - y^2}$

15) $\lim\limits_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y)$

Answer

$$\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y) = 11$$

16) $\lim\limits_{(x,y)\to(\pi,\pi)} x \sin\left(\dfrac{x+y}{4}\right)$

17) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy+1}{x^2+y^2+1}$

Answer

$$\lim_{(x,y)\to(0,0)} \frac{xy+1}{x^2+y^2+1} = 1$$

18) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$

19) $\lim\limits_{(x,y)\to(0,0)} \ln(x^2+y^2)$

Answer

The limit does not exist because when x and y both approach zero, the function approaches $\ln 0$, which is undefined (approaches negative infinity).

In exercises 20 - 21, complete the statement.

20) A point (x_0, y_0) in a plane region R is an interior point of R if _____.

21) A point (x_0, y_0) in a plane region R is called a boundary point of R if _____.

Answer

Every open disk centered at (x_0, y_0) contains points inside R and outside R.

In exercises 22 - 25, use algebraic techniques to evaluate the limit.

22) $\lim\limits_{(x,y)\to(2,1)} \dfrac{x-y-1}{\sqrt{x-y}-1}$

23) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4-4y^4}{x^2+2y^2}$

Answer

$$\lim_{(x,y)\to(0,0)} \frac{x^4-4y^4}{x^2+2y^2} = 0$$

24) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3-y^3}{x-y}$

25) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-xy}{\sqrt{x}-\sqrt{y}}$

Answer

$$\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} = 0$$

In exercises 26 - 27, evaluate the limits of the functions of three variables.

26) $\lim_{(x,y,z)\to(1,2,3)} \frac{xz^2-y^2z}{xyz-1}$

27) $\lim_{(x,y,z)\to(0,0,0)} \frac{x^2-y^2-z^2}{x^2+y^2-z^2}$

Answer

The limit does not exist.

In exercises 28 - 31, evaluate the limit of the function by determining the value the function approaches along the indicated paths. If the limit does not exist, explain why not.

28) $\lim_{(x,y)\to(0,0)} \frac{xy+y^3}{x^2+y^2}$

a. Along the x-axis $(y=0)$

b. Along the y-axis $(x=0)$

c. Along the path $y=2x$

29) Evaluate $\lim_{(x,y)\to(0,0)} \frac{xy+y^3}{x^2+y^2}$ using the results of previous problem.

Answer

The limit does not exist. The function approaches two different values along different paths.

30) $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$

a. Along the x-axis $(y=0)$

b. Along the y-axis $(x=0)$

c. Along the path $y=x^2$

31) Evaluate $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ using the results of previous problem.

Answer

The limit does not exist because the function approaches two different values along the paths.

In exercises 32 - 35, discuss the continuity of each function. Find the largest region in the xy-plane in which each function is continuous.

32) $f(x,y)=\sin(xy)$

33) $f(x,y)=\ln(x+y)$

Answer

The function f is continuous in the region $y>-x$.

34) $f(x,y)=e^{3xy}$

35) $f(x,y)=\frac{1}{xy}$

Answer

The function f is continuous at all points in the xy-plane except at points on the x- and y-axes.

In exercises 36 - 38, determine the region in which the function is continuous. Explain your answer.

36) $f(x,y)=\frac{x^2y}{x^2+y^2}$

37) $f(x,y)=\begin{cases}\frac{x^2y}{x^2+y^2} & if(x,y)\neq(0,0)\\ 0 & if(x,y)=(0,0)\end{cases}$

Hint:

Show that the function approaches different values along two different paths.

Answer

The function is continuous at $(0,0)$ since the limit of the function at $(0,0)$ is 0, the same value of $f(0,0)$.

38) $f(x,y)=\dfrac{\sin(x^2+y^2)}{x^2+y^2}$

39) Determine whether $g(x,y)=\dfrac{x^2-y^2}{x^2+y^2}$ is continuous at $(0,0)$.

Answer

The function is discontinuous at $(0,0)$. The limit at $(0,0)$ fails to exist and $g(0,0)$ does not exist.

40) Create a plot using graphing software to determine where the limit does not exist. Determine the region of the coordinate plane in which $f(x,y)=\dfrac{1}{x^2-y}$ is continuous.

41) Determine the region of the xy-plane in which the composite function $g(x,y)=\arctan(\frac{xy^2}{x+y})$ is continuous. Use technology to support your conclusion.

Answer

Since the function $\arctan x$ is continuous over $(-\infty,\infty)$, $g(x,y)=\arctan(\frac{xy^2}{x+y})$ is continuous where $z=\dfrac{xy^2}{x+y}$ is continuous. The inner function z is continuous on all points of the xy-plane except where $y=-x$. Thus, $g(x,y)=\arctan(\frac{xy^2}{x+y})$ is continuous on all points of the coordinate plane except at points at which $y=-x$.

42) Determine the region of the xy-plane in which $f(x,y)=\ln(x^2+y^2-1)$ is continuous. Use technology to support your conclusion. (Hint: Choose the range of values for x and y carefully!)

43) At what points in space is $g(x,y,z)=x^2+y^2-2z^2$ continuous?

Answer

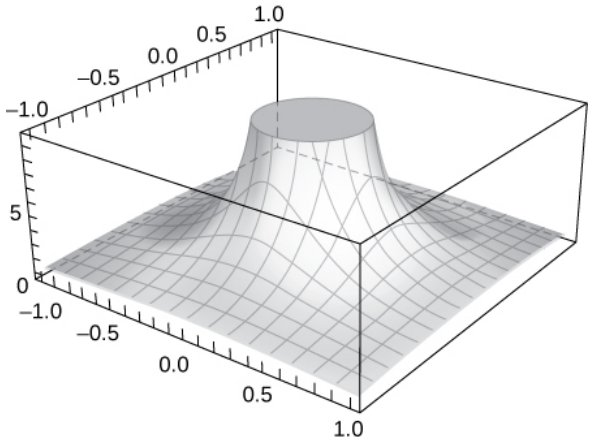
All points $P(x,y,z)$ in space

44) At what points in space is $g(x,y,z)=\dfrac{1}{x^2+z^2-1}$ continuous?

45) Show that $\lim\limits_{(x,y)\to(0,0)}\dfrac{1}{x^2+y^2}$ does not exist at $(0,0)$ by plotting the graph of the function.

Answer

The graph increases without bound as x and y both approach zero.



46) [T] Evaluate $\lim\limits_{(x,y)\to(0,0)}\dfrac{-xy^2}{x^2+y^4}$ by plotting the function using a CAS. Determine analytically the limit along the path $x=y^2$.

47) [T]

a. Use a CAS to draw a contour map of $z=\sqrt{9-x^2-y^2}$.

b. What is the name of the geometric shape of the level curves?

c. Give the general equation of the level curves.

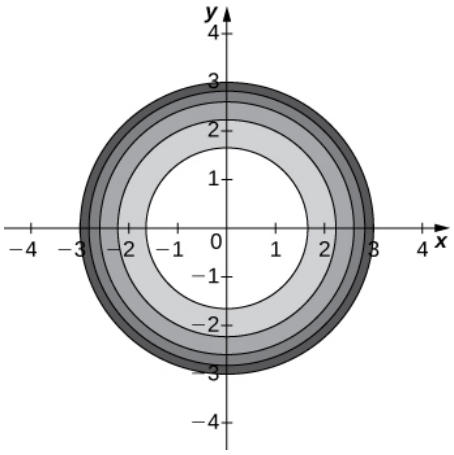
d. What is the maximum value of z?

e. What is the domain of the function?

f. What is the range of the function?

Answer

a.



b. The level curves are circles centered at $(0,0)$ with radius $9-c$.

c. $x^2+y^2=9-c$

d. $z=3$

e. $\{(x,y)\in R^2 \mid x^2+y^2\le 9\}$

f. $\{z|0\le z\le 3\}$

48) True or False: If we evaluate $\lim\limits_{(x,y)\to(0,0)} f(x)$ along several paths and each time the limit is 1, we can conclude that $\lim\limits_{(x,y)\to(0,0)} f(x)=1$.

49) Use polar coordinates to find $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}$. You can also find the limit using L'Hôpital's rule.

Answer

$$\lim_{(x,y)\to(0,0)} \frac{\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}=1$$

50) Use polar coordinates to find $\lim\limits_{(x,y)\to(0,0)} \cos(x^2+y^2)$.

51) Discuss the continuity of $f(g(x,y))$ where $f(t)=1/t$ and $g(x,y)=2x-5y$.

Answer

$f(g(x,y))$ is continuous at all points (x,y) that are not on the line $2x-5y=0$.

52) Given $f(x,y)=x^2-4y$, find $\lim\limits_{h\to 0} \dfrac{f(x+h,y)-f(x,y)}{h}$.

53) Given $f(x,y)=x^2-4y$, find $\lim\limits_{h\to 0} \dfrac{f(1+h,y)-f(1,y)}{h}$.

Answer

$$\lim_{h\to 0} \frac{f(1+h,y)-f(1,y)}{h}=2$$

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14.3: Partial Derivatives

Learning Objectives

- Calculate the partial derivatives of a function of two variables.
- Calculate the partial derivatives of a function of more than two variables.
- Determine the higher-order derivatives of a function of two variables.
- Explain the meaning of a partial differential equation and give an example.

Now that we have examined limits and continuity of functions of two variables, we can proceed to study derivatives. Finding derivatives of functions of two variables is the key concept in this chapter, with as many applications in mathematics, science, and engineering as differentiation of single-variable functions. However, we have already seen that limits and continuity of multivariable functions have new issues and require new terminology and ideas to deal with them. This carries over into differentiation as well.

Derivatives of a Function of Two Variables

When studying derivatives of functions of one variable, we found that one interpretation of the derivative is an instantaneous rate of change of y as a function of x. Leibniz notation for the derivative is dy/dx, which implies that y is the dependent variable and x is the independent variable. For a function $z = f(x, y)$ of two variables, x and y are the independent variables and z is the dependent variable. This raises two questions right away: How do we adapt Leibniz notation for functions of two variables? Also, what is an interpretation of the derivative? The answer lies in partial derivatives.

Definition: Partial Derivatives

Let $f(x, y)$ be a function of two variables. Then the *partial derivative* of f with respect to x, written as $\partial f/\partial x$, or f_x, is defined as

$$\frac{\partial f}{\partial x} = f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h} \tag{14.3.1}$$

The partial derivative of f with respect to y, written as $\partial f/\partial y$, or f_y, is defined as

$$\frac{\partial f}{\partial y} = f_y(x, y) = \lim_{k \to 0} \frac{f(x, y + k) - f(x, y)}{k}. \tag{14.3.2}$$

This definition shows two differences already. First, the notation changes, in the sense that we still use a version of Leibniz notation, but the d in the original notation is replaced with the symbol ∂. (This rounded "d" is usually called "partial," so $\partial f/\partial x$ is spoken as the "partial of f with respect to x.") This is the first hint that we are dealing with partial derivatives. Second, we now have two different derivatives we can take, since there are two different independent variables. Depending on which variable we choose, we can come up with different partial derivatives altogether, and often do.

Example 14.3.1: Calculating Partial Derivatives from the Definition

Use the definition of the partial derivative as a limit to calculate $\partial f/\partial x$ and $\partial f/\partial y$ for the function

$$f(x, y) = x^2 - 3xy + 2y^2 - 4x + 5y - 12.$$

Solution

First, calculate $f(x + h, y)$.

$$\begin{aligned} f(x + h, y) &= (x + h)^2 - 3(x + h)y + 2y^2 - 4(x + h) + 5y - 12 \\ &= x^2 + 2xh + h^2 - 3xy - 3hy + 2y^2 - 4x - 4h + 5y - 12. \end{aligned}$$

Next, substitute this into Equation 14.3.1 and simplify:

$$\begin{aligned}\frac{\partial f}{\partial x} &= \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}\\ &= \lim_{h\to 0}\frac{(x^2+2xh+h^2-3xy-3hy+2y^2-4x-4h+5y-12)-(x^2-3xy+2y^2-4x+5y-12)}{h}\\ &= \lim_{h\to 0}\frac{x^2+2xh+h^2-3xy-3hy+2y^2-4x-4h+5y-12-x^2+3xy-2y^2+4x-5y+12}{h}\\ &= \lim_{h\to 0}\frac{2xh+h^2-3hy-4h}{h}\\ &= \lim_{h\to 0}\frac{h(2x+h-3y-4)}{h}\\ &= \lim_{h\to 0}(2x+h-3y-4)\\ &= 2x-3y-4.\end{aligned}$$

To calculate $\frac{\partial f}{\partial y}$, first calculate $f(x,y+h)$:

$$\begin{aligned}f(x,y+h) &= x^2-3x(y+h)+2(y+h)^2-4x+5(y+h)-12\\ &= x^2-3xy-3xh+2y^2+4yh+2h^2-4x+5y+5h-12.\end{aligned}$$

Next, substitute this into Equation 14.3.2 and simplify:

$$\begin{aligned}\frac{\partial f}{\partial y} &= \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}\\ &= \lim_{h\to 0}\frac{(x^2-3xy-3xh+2y^2+4yh+2h^2-4x+5y+5h-12)-(x^2-3xy+2y^2-4x+5y-12)}{h}\\ &= \lim_{h\to 0}\frac{x^2-3xy-3xh+2y^2+4yh+2h^2-4x+5y+5h-12-x^2+3xy-2y^2+4x-5y+12}{h}\\ &= \lim_{h\to 0}\frac{-3xh+4yh+2h^2+5h}{h}\\ &= \lim_{h\to 0}\frac{h(-3x+4y+2h+5)}{h}\\ &= \lim_{h\to 0}(-3x+4y+2h+5)\\ &= -3x+4y+5\end{aligned}$$

? Exercise 14.3.1

Use the definition of the partial derivative as a limit to calculate $\partial f/\partial x$ and $\partial f/\partial y$ for the function

$$f(x,y)=4x^2+2xy-y^2+3x-2y+5.$$

Hint

Use Equations 14.3.1 and 14.3.2 from the definition of partial derivatives.

Answer

$$\frac{\partial f}{\partial x}=8x+2y+3$$
$$\frac{\partial f}{\partial y}=2x-2y-2$$

The idea to keep in mind when calculating partial derivatives is to treat all independent variables, other than the variable with respect to which we are differentiating, as constants. Then proceed to differentiate as with a function of a single variable. To see why this is true, first fix y and define $g(x)=f(x,y)$ as a function of x. Then

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} \\ &= \frac{\partial f}{\partial x}. \end{aligned}$$

The same is true for calculating the partial derivative of f with respect to y. This time, fix x and define $h(y) = f(x, y)$ as a function of y. Then

$$\begin{aligned} h'(x) &= \lim_{k\to 0} \frac{h(x+k)-h(x)}{k} \\ &= \lim_{k\to 0} \frac{f(x,y+k)-f(x,y)}{k} \\ &= \frac{\partial f}{\partial y}. \end{aligned}$$

All differentiation rules apply.

Example 14.3.2: Calculating Partial Derivatives

Calculate $\partial f/\partial x$ and $\partial f/\partial y$ for the following functions by holding the opposite variable constant then differentiating:

a. $f(x, y) = x^2 - 3xy + 2y^2 - 4x + 5y - 12$
b. $g(x, y) = \sin(x^2y - 2x + 4)$

Solution:

a. To calculate $\partial f/\partial x$, treat the variable y as a constant. Then differentiate $f(x, y)$ with respect to x using the sum, difference, and power rules:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left[x^2 - 3xy + 2y^2 - 4x + 5y - 12\right] \\ &= \frac{\partial}{\partial x}[x^2] - \frac{\partial}{\partial x}[3xy] + \frac{\partial}{\partial x}[2y^2] - \frac{\partial}{\partial x}[4x] + \frac{\partial}{\partial x}[5y] - \frac{\partial}{\partial x}[12] \\ &= 2x - 3y + 0 - 4 + 0 - 0 \\ &= 2x - 3y - 4. \end{aligned}$$

The derivatives of the third, fifth, and sixth terms are all zero because they do not contain the variable x, so they are treated as constant terms. The derivative of the second term is equal to the coefficient of x, which is $-3y$. Calculating $\partial f/\partial y$:

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left[x^2 - 3xy + 2y^2 - 4x + 5y - 12\right] \\ &= \frac{\partial}{\partial y}[x^2] - \frac{\partial}{\partial y}[3xy] + \frac{\partial}{\partial y}[2y^2] - \frac{\partial}{\partial y}[4x] + \frac{\partial}{\partial y}[5y] - \frac{\partial}{\partial y}[12] \\ &= 0 - 3x + 4y - 0 + 5 - 0 \\ &= -3x + 4y + 5. \end{aligned}$$

These are the same answers obtained in Example 14.3.1.

b. To calculate $\partial g/\partial x$, treat the variable y as a constant. Then differentiate $g(x, y)$ with respect to x using the chain rule and power rule:

$$\begin{aligned} \frac{\partial g}{\partial x} &= \frac{\partial}{\partial x}\left[\sin(x^2y - 2x + 4)\right] \\ &= \cos(x^2y - 2x + 4)\frac{\partial}{\partial x}[x^2y - 2x + 4] \\ &= (2xy - 2)\cos(x^2y - 2x + 4). \end{aligned}$$

To calculate $\partial g/\partial y$, treat the variable x as a constant. Then differentiate $g(x,y)$ with respect to y using the chain rule and power rule:

$$\begin{aligned}\frac{\partial g}{\partial y} &= \frac{\partial}{\partial y}\left[\sin(x^2y-2x+4)\right] \\ &= \cos(x^2y-2x+4)\frac{\partial}{\partial y}[x^2y-2x+4] \\ &= x^2\cos(x^2y-2x+4).\end{aligned}$$

? Exercise 14.3.2

Calculate $\partial f/\partial x$ and $\partial f/\partial y$ for the function

$$f(x,y)=\tan(x^3-3x^2y^2+2y^4)$$

by holding the opposite variable constant, then differentiating.

Hint

Use Equations 14.3.1 and 14.3.1 from the definition of partial derivatives.

Answer

$$\frac{\partial f}{\partial x}=(3x^2-6xy^2)\sec^2(x^3-3x^2y^2+2y^4)$$
$$\frac{\partial f}{\partial y}=(-6x^2y+8y^3)\sec^2(x^3-3x^2y^2+2y^4)$$

How can we interpret these partial derivatives? Recall that the graph of a function of two variables is a surface in R^3. If we remove the limit from the definition of the partial derivative with respect to x, the difference quotient remains:

$$\frac{f(x+h,y)-f(x,y)}{h}.$$

This resembles the difference quotient for the derivative of a function of one variable, except for the presence of the y variable. Figure 14.3.1 illustrates a surface described by an arbitrary function $z=f(x,y)$.

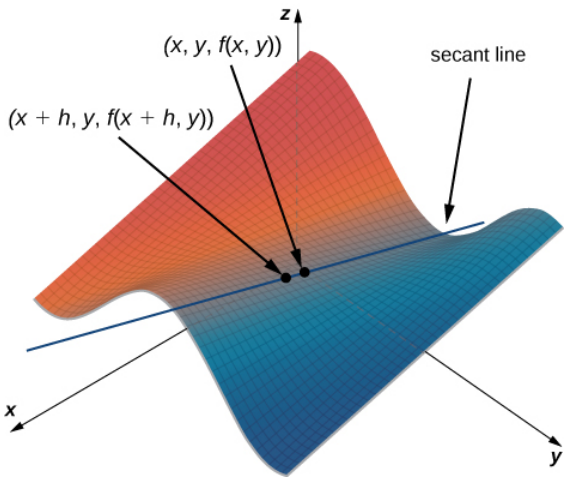

Figure 14.3.1: Secant line passing through the points $(x,y,f(x,y))$ and $(x+h,y,f(x+h,y))$.

In Figure 14.3.1, the value of h is positive. If we graph $f(x,y)$ and $f(x+h,y)$ for an arbitrary point (x,y), then the slope of the secant line passing through these two points is given by

$$\frac{f(x+h,y)-f(x,y)}{h}.$$

This line is parallel to the x-axis. Therefore, the slope of the secant line represents an average rate of change of the function f as we travel parallel to the x-axis. As h approaches zero, the slope of the secant line approaches the slope of the tangent line.

If we choose to change y instead of x by the same incremental value h, then the secant line is parallel to the y-axis and so is the tangent line. Therefore, $\partial f/\partial x$ represents the slope of the tangent line passing through the point $(x, y, f(x, y))$ parallel to the x-axis and $\partial f/\partial y$ represents the slope of the tangent line passing through the point $(x, y, f(x, y))$ parallel to the y-axis. If we wish to find the slope of a tangent line passing through the same point in any other direction, then we need what are called directional derivatives.

We now return to the idea of contour maps, which we introduced in Functions of Several Variables. We can use a **contour map** to estimate partial derivatives of a function $g(x, y)$.

Example 14.3.3: Partial Derivatives from a Contour Map

Use a contour map to estimate $\partial g/\partial x$ at the point $(\sqrt{5}, 0)$ for the function

$$g(x, y) = \sqrt{9 - x^2 - y^2}.$$

Solution

Figure 14.3.2 represents a contour map for the function $g(x, y)$.

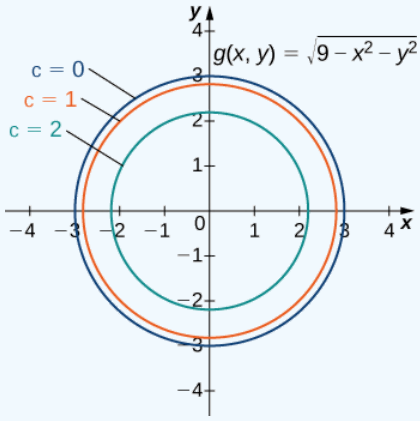

Figure 14.3.2: Contour map for the function $g(x, y) = \sqrt{9 - x^2 - y^2}$, using $c = 0, 1, 2$, and $3 (c = 3$ corresponds to the origin).

The inner circle on the contour map corresponds to $c = 2$ and the next circle out corresponds to $c = 1$. The first circle is given by the equation $2 = \sqrt{9 - x^2 - y^2}$; the second circle is given by the equation $1 = \sqrt{9 - x^2 - y^2}$. The first equation simplifies to $x^2 + y^2 = 5$ and the second equation simplifies to $x^2 + y^2 = 8$. The x-intercept of the first circle is $(\sqrt{5}, 0)$ and the x-intercept of the second circle is $(2\sqrt{2}, 0)$. We can estimate the value of $\partial g/\partial x$ evaluated at the point $(\sqrt{5}, 0)$ using the slope formula:

$$\begin{aligned} \frac{\partial g}{\partial x}\Big|_{(x,y)=(\sqrt{5},0)} &\approx \frac{g(\sqrt{5}, 0) - g(2\sqrt{2}, 0)}{\sqrt{5} - 2\sqrt{2}} \\ &= \frac{2 - 1}{\sqrt{5} - 2\sqrt{2}} \\ &= \frac{1}{\sqrt{5} - 2\sqrt{2}} \approx -1.688. \end{aligned}$$

To calculate the exact value of $\partial g/\partial x$ evaluated at the point $(\sqrt{5}, 0)$, we start by finding $\partial g/\partial x$ using the chain rule. First, we rewrite the function as

$$g(x, y) = \sqrt{9 - x^2 - y^2} = (9 - x^2 - y^2)^{1/2}$$

and then differentiate with respect to x while holding y constant:

$$\begin{aligned} \frac{\partial g}{\partial x} &= \frac{1}{2}(9 - x^2 - y^2)^{-1/2}(-2x) \\ &= -\frac{x}{\sqrt{9 - x^2 - y^2}}. \end{aligned}$$

Next, we evaluate this expression using $x = \sqrt{5}$ and $y = 0$:

$$\frac{\partial g}{\partial x}\Big|_{(x,y)=(\sqrt{5},0)} = -\frac{\sqrt{5}}{\sqrt{9-(\sqrt{5})^2-(0)^2}}$$
$$= -\frac{\sqrt{5}}{\sqrt{4}}$$
$$= -\frac{\sqrt{5}}{2} \approx -1.118.$$

The estimate for the partial derivative corresponds to the slope of the secant line passing through the points $(\sqrt{5}, 0, g(\sqrt{5}, 0))$ and $(2\sqrt{2}, 0, g(2\sqrt{2}, 0))$. It represents an approximation to the slope of the tangent line to the surface through the point $(\sqrt{5}, 0, g(\sqrt{5}, 0))$, which is parallel to the x-axis.

? Exercise 14.3.3

Use a contour map to estimate $\partial f/\partial y$ at point $(0, \sqrt{2})$ for the function

$$f(x, y) = x^2 - y^2.$$

Compare this with the exact answer.

Hint

Create a contour map for f using values of c from -3 to 3. Which of these curves passes through point $(0, \sqrt{2})$?

Answer

Using the curves corresponding to $c = -2$ and $c = -3$, we obtain

$$\frac{\partial f}{\partial y}\Big|_{(x,y)=(0,\sqrt{2})} \approx \frac{f(0,\sqrt{3}) - f(0,\sqrt{2})}{\sqrt{3}-\sqrt{2}}$$
$$= \frac{-3-(-2)}{\sqrt{3}-\sqrt{2}} \cdot \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}$$
$$= -\sqrt{3}-\sqrt{2} \approx -3.146.$$

The exact answer is

$$\frac{\partial f}{\partial y}\Big|_{(x,y)=(0,\sqrt{2})} = (-2y|_{(x,y)=(0,\sqrt{2})} = -2\sqrt{2} \approx -2.828.$$

Functions of More Than Two Variables

Suppose we have a function of three variables, such as $w = f(x, y, z)$. We can calculate partial derivatives of w with respect to any of the independent variables, simply as extensions of the definitions for partial derivatives of functions of two variables.

Definition: Partial Derivatives

Let $f(x, y, z)$ be a function of three variables. Then, the partial derivative of f with respect to x, written as $\partial f/\partial x$, or f_x, is defined to be

$$\frac{\partial f}{\partial x} = f_x(x, y, z) = \lim_{h\to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}. \tag{14.3.3}$$

The *partial derivative* of f with *respect* to y, written as $\partial f/\partial y$, or f_y, is defined to be

$$\frac{\partial f}{\partial y} = f_y(x, y, z) = \lim_{k\to 0} \frac{f(x, y+k, z) - f(x, y, z)}{k.} \tag{14.3.4}$$

The *partial derivative* of f with respect to z, written as $\partial f/\partial z$, or f_z, is defined to be

$$\frac{\partial f}{\partial z} = f_z(x, y, z) = \lim_{m\to 0} \frac{f(x, y, z+m) - f(x, y, z)}{m}. \tag{14.3.5}$$

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We can calculate a partial derivative of a function of three variables using the same idea we used for a function of two variables. For example, if we have a function f of x, y, and z, and we wish to calculate $\partial f/\partial x$, then we treat the other two independent variables as if they are constants, then differentiate with respect to x.

Example 14.3.4: Calculating Partial Derivatives for a Function of Three Variables

Use the limit definition of partial derivatives to calculate $\partial f/\partial x$ for the function

$$f(x, y, z) = x^2 - 3xy + 2y^2 - 4xz + 5yz^2 - 12x + 4y - 3z.$$

Then, find $\partial f/\partial y$ and $\partial f/\partial z$ by setting the other two variables constant and differentiating accordingly.

Solution:

We first calculate $\partial f/\partial x$ using Equation 14.3.3, then we calculate the other two partial derivatives by holding the remaining variables constant. To use the equation to find $\partial f/\partial x$, we first need to calculate $f(x + h, y, z)$:

$$\begin{aligned} f(x+h, y, z) &= (x+h)^2 - 3(x+h)y + 2y^2 - 4(x+h)z + 5yz^2 - 12(x+h) + 4y - 3z \\ &= x^2 + 2xh + h^2 - 3xy - 3xh + 2y^2 - 4xz - 4hz + 5yz^2 - 12x - 12h + 4y - 3z \end{aligned}$$

and recall that $f(x, y, z) = x^2 - 3xy + 2y^2 - 4zx + 5yz^2 - 12x + 4y - 3z$. Next, we substitute these two expressions into the equation:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \lim_{h\to 0} \\ &\quad \left[\frac{\begin{array}{r} x^2 + 2xh + h^2 - 3xy - 3hy + 2y^2 - 4xz - 4hz + 5yz^2 - 12x - 12h + 4y - 3zh - x^2 - 3xy \\ + 2y^2 - 4xz + 5yz^2 - 12x + 4y - 3z \end{array}}{h}\right] \\ &= \lim_{h\to 0}\left[\frac{2xh + h^2 - 3hy - 4hz - 12h}{h}\right] \\ &= \lim_{h\to 0}\left[\frac{h(2x + h - 3y - 4z - 12)}{h}\right] \\ &= \lim_{h\to 0}(2x + h - 3y - 4z - 12) \\ &= 2x - 3y - 4z - 12. \end{aligned}$$

Then we find $\partial f/\partial y$ by holding x and z constant. Therefore, any term that does not include the variable y is constant, and its derivative is zero. We can apply the sum, difference, and power rules for functions of one variable:

$$\begin{aligned} &\frac{\partial}{\partial y}\left[x^2 - 3xy + 2y^2 - 4xz + 5yz^2 - 12x + 4y - 3z\right] \\ &= \frac{\partial}{\partial y}[x^2] - \frac{\partial}{\partial y}[3xy] + \frac{\partial}{\partial y}[2y^2] - \frac{\partial}{\partial y}[4xz] + \frac{\partial}{\partial y}[5yz^2] - \frac{\partial}{\partial y}[12x] + \frac{\partial}{\partial y}[4y] - \frac{\partial}{\partial z}[3z] \\ &= 0 - 3x + 4y - 0 + 5z^2 - 0 + 4 - 0 \\ &= -3x + 4y + 5z^2 + 4. \end{aligned}$$

To calculate $\partial f/\partial z$, we hold x and y constant and apply the sum, difference, and power rules for functions of one variable:

$$\begin{aligned} &\frac{\partial}{\partial z}[x^2 - 3xy + 2y^2 - 4xz + 5yz^2 - 12x + 4y - 3z] \\ &= \frac{\partial}{\partial z}[x^2] - \frac{\partial}{\partial z}[3xy] + \frac{\partial}{\partial z}[2y^2] - \frac{\partial}{\partial z}[4xz] + \frac{\partial}{\partial z}[5yz^2] - \frac{\partial}{\partial z}[12x] + \frac{\partial}{\partial z}[4y] - \frac{\partial}{\partial z}[3z] \\ &= 0 - 0 + 0 - 4x + 10yz - 0 + 0 - 3 \\ &= -4x + 10yz - 3 \end{aligned}$$

Exercise 14.3.4

Use the limit definition of partial derivatives to calculate $\partial f/\partial x$ for the function

$$f(x,y,z) = 2x^2 - 4x^2y + 2y^2 + 5xz^2 - 6x + 3z - 8.$$

Then find $\partial f/\partial y$ and $\partial f/\partial z$ by setting the other two variables constant and differentiating accordingly.

Hint

Use the strategy in the preceding example.

Answer

$\dfrac{\partial f}{\partial x} = 4x - 8xy + 5z^2 - 6, \dfrac{\partial f}{\partial y} = -4x^2 + 4y, \dfrac{\partial f}{\partial z} = 10xz + 3$

Example 14.3.5: Calculating Partial Derivatives for a Function of Three Variables

Calculate the three partial derivatives of the following functions.

a. $f(x,y,z) = \dfrac{x^2y - 4xz + y^2}{x - 3yz}$

b. $g(x,y,z) = \sin(x^2y - z) + \cos(x^2 - yz)$

Solution

In each case, treat all variables as constants except the one whose partial derivative you are calculating.

a.

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left[\frac{x^2y - 4xz + y^2}{x - 3yz}\right] \\ &= \frac{\frac{\partial}{\partial x}(x^2y - 4xz + y^2)(x - 3yz) - (x^2y - 4xz + y^2)\frac{\partial}{\partial x}(x - 3yz)}{(x - 3yz)^2} \\ &= \frac{(2xy - 4z)(x - 3yz) - (x^2y - 4xz + y^2)(1)}{(x - 3yz)^2} \\ &= \frac{2x^2y - 6xy^2z - 4xz + 12yz^2 - x^2y + 4xz - y^2}{(x - 3yz)^2} \\ &= \frac{x^2y - 6xy^2z - 4xz + 12yz^2 + 4xz - y^2}{(x - 3yz)^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left[\frac{x^2y - 4xz + y^2}{x - 3yz}\right] \\ &= \frac{\frac{\partial}{\partial y}(x^2y - 4xz + y^2)(x - 3yz) - (x^2y - 4xz + y^2)\frac{\partial}{\partial y}(x - 3yz)}{(x - 3yz)^2} \\ &= \frac{(x^2 + 2y)(x - 3yz) - (x^2y - 4xz + y^2)(-3z)}{(x - 3yz)^2} \\ &= \frac{x^3 - 3x^2yz + 2xy - 6y^2z + 3x^2yz - 12xz^2 + 3y^2z}{(x - 3yz)^2} \\ &= \frac{x^3 + 2xy - 3y^2z - 12xz^2}{(x - 3yz)^2}\end{aligned}$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[\frac{x^2y - 4xz + y^2}{x - 3yz}\right]$$

$$= \frac{\frac{\partial}{\partial z}(x^2y - 4xz + y^2)(x - 3yz) - (x^2y - 4xz + y^2)\frac{\partial}{\partial z}(x - 3yz)}{(x - 3yz)^2}$$

$$= \frac{(-4x)(x - 3yz) - (x^2y - 4xz + y^2)(-3y)}{(x - 3yz)^2}$$

$$= \frac{-4x^2 + 12xyz + 3x^2y^2 - 12xyz + 3y^3}{(x - 3yz)^2}$$

$$= \frac{-4x^2 + 3x^2y^2 + 3y^3}{(x - 3yz)^2}$$

b.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[\sin(x^2y - z) + \cos(x^2 - yz)\right]$$

$$= (\cos(x^2y - z))\frac{\partial}{\partial x}(x^2y - z) - (\sin(x^2 - yz))\frac{\partial}{\partial x}(x^2 - yz)$$

$$= 2xy\cos(x^2y - z) - 2x\sin(x^2 - yz)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[\sin(x^2y - z) + \cos(x^2 - yz)]$$

$$= (\cos(x^2y - z))\frac{\partial}{\partial y}(x^2y - z) - (\sin(x^2 - yz))\frac{\partial}{\partial y}(x^2 - yz)$$

$$= x^2\cos(x^2y - z) + z\sin(x^2 - yz)$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}[\sin(x^2y - z) + \cos(x^2 - yz)]$$

$$= (\cos(x^2y - z))\frac{\partial}{\partial z}(x^2y - z) - (\sin(x^2 - yz))\frac{\partial}{\partial z}(x^2 - yz)$$

$$= -\cos(x^2y - z) + y\sin(x^2 - yz)$$

? Exercise 14.3.5

Calculate $\partial f/\partial x, \partial f/\partial y$, and $\partial f/\partial z$ for the function

$$f(x, y, z) = \sec(x^2y) - \tan(x^3yz^2).$$

Hint

Use the strategy in the preceding example.

Answer

$$\frac{\partial f}{\partial x} = 2xy\sec(x^2y)\tan(x^2y) - 3x^2yz^2\sec^2(x^3yz^2)$$

$$\frac{\partial f}{\partial y} = x^2\sec(x^2y)\tan(x^2y) - x^3z^2\sec^2(x^3yz^2)$$

$$\frac{\partial f}{\partial z} = -2x^3yz\sec^2(x^3yz^2)$$

Higher-Order Partial Derivatives

Consider the function

$$f(x, y) = 2x^3 - 4xy^2 + 5y^3 - 6xy + 5x - 4y + 12.$$

Its partial derivatives are

$$\frac{\partial f}{\partial x} = 6x^2 - 4y^2 - 6y + 5$$

and

$$\frac{\partial f}{\partial y} = -8xy + 15y^2 - 6x - 4.$$

Each of these partial derivatives is a function of two variables, so we can calculate partial derivatives of these functions. Just as with derivatives of single-variable functions, we can call these *second-order derivatives*, third-order derivatives, and so on. In general, they are referred to as **higher-order partial derivatives.** There are four second-order partial derivatives for any function (provided they all exist):

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] \\
\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] \\
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] \\
\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right].
\end{aligned}$$

An alternative notation for each is f_{xx}, f_{xy}, f_{yx}, and f_{yy}, respectively. Higher-order partial derivatives calculated with respect to different variables, such as f_{xy} and f_{yx}, are commonly called **mixed partial derivatives.**

Example 14.3.6: Calculating Second Partial Derivatives

Calculate all four second partial derivatives for the function

$$f(x, y) = xe^{-3y} + \sin(2x - 5y). \tag{14.3.6}$$

Solution:

To calculate $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y \partial x}$, we first calculate $\partial f/\partial x$:

$$\frac{\partial f}{\partial x} = e^{-3y} + 2\cos(2x - 5y). \tag{14.3.7}$$

To calculate $\frac{\partial^2 f}{\partial x^2}$, differentiate $\partial f/\partial x$ (Equation 14.3.7) with respect to x:

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] \\
&= \frac{\partial}{\partial x}[e^{-3y} + 2\cos(2x - 5y)] \\
&= -4\sin(2x - 5y).
\end{aligned}$$

To calculate $\frac{\partial^2 f}{\partial y \partial x}$, differentiate $\partial f/\partial x$ (Equation 14.3.7) with respect to y:

$$\begin{aligned}
\frac{\partial^2 f}{\partial y\, \partial x} &= \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] \\
&= \frac{\partial}{\partial y}[e^{-3y} + 2\cos(2x - 5y)] \\
&= -3e^{-3y} + 10\sin(2x - 5y).
\end{aligned}$$

To calculate $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y^2}$, first calculate $\partial f/\partial y$:

$$\frac{\partial f}{\partial y} = -3xe^{-3y} - 5\cos(2x - 5y). \tag{14.3.8}$$

To calculate $\frac{\partial^2 f}{\partial x \partial y}$, differentiate $\partial f/\partial y$ (Equation 14.3.8) with respect to x:

$$\begin{aligned}\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] \\ &= \frac{\partial}{\partial x}[-3xe^{-3y} - 5\cos(2x-5y)] \\ &= -3e^{-3y} + 10\sin(2x-5y).\end{aligned}$$

To calculate $\frac{\partial^2 f}{\partial y^2}$, differentiate $\partial f/\partial y$ (Equation 14.3.8) with respect to y:

$$\begin{aligned}\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] \\ &= \frac{\partial}{\partial y}[-3xe^{-3y} - 5\cos(2x-5y)] \\ &= 9xe^{-3y} - 25\sin(2x-5y).\end{aligned}$$

? Exercise 14.3.6

Calculate all four second partial derivatives for the function

$$f(x,y) = \sin(3x-2y) + \cos(x+4y).$$

Hint

Follow the same steps as in the previous example.

Answer

$$\frac{\partial^2 f}{\partial x^2} = -9\sin(3x-2y) - \cos(x+4y)$$

$$\frac{\partial^2 f}{\partial y \partial x} = 6\sin(3x-2y) - 4\cos(x+4y)$$

$$\frac{\partial^2 f}{\partial x \partial y} = 6\sin(3x-2y) - 4\cos(x+4y)$$

$$\frac{\partial^2 f}{\partial y^2} = -4\sin(3x-2y) - 16\cos(x+4y)$$

At this point we should notice that, in both Example 14.3.6 and the checkpoint, it was true that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$. Under certain conditions, this is always true. In fact, it is a direct consequence of the following theorem.

Theorem 14.3.1: Equality of Mixed Partial Derivatives (Clairaut's Theorem)

Suppose that $f(x,y)$ is defined on an open disk D that contains the point (a,b). If the functions f_{xy} and f_{yx} are continuous on D, then $f_{xy}(a,b) = f_{yx}(a,b)$.

Clairaut's theorem guarantees that as long as mixed second-order derivatives are continuous, the order in which we choose to differentiate the functions (i.e., which variable goes first, then second, and so on) does not matter. It can be extended to higher-order derivatives as well. The proof of Clairaut's theorem can be found in most advanced calculus books.

Two other second-order partial derivatives can be calculated for any function $f(x,y)$. The partial derivative f_{xx} is equal to the partial derivative of f_x with respect to x, and f_{yy} is equal to the partial derivative of f_y with respect to y.

Partial Differential Equations

Previously, we studied differential equations in which the unknown function had one independent variable. A partial differential equation is an equation that involves an unknown function of more than one independent variable and one or more of its partial derivatives. Examples of partial differential equations are

$$u_t = c^2(u_{xx} + u_{yy})$$
heat equation in two dimensions

$$u_{tt} = c^2(u_{xx} + u_{yy})$$
wave equation in two dimensions

$$u_{xx} + u_{yy} = 0$$
Laplace's equation in two dimensions

In the heat and wave equations, the unknown function u has three independent variables: t, x, and y with c is an arbitrary constant. The independent variables x and y are considered to be spatial variables, and the variable t represents time. In Laplace's equation, the unknown function u has two independent variables x and y.

Example 14.3.7: A Solution to the Wave Equation

Verify that

$$u(x, y, t) = 5\sin(3\pi x)\sin(4\pi y)\cos(10\pi t)$$

is a solution to the wave equation

$$u_{tt} = 4(u_{xx} + u_{yy}). \tag{14.3.9}$$

Solution

First, we calculate u_{tt}, u_{xx}, and u_{yy} :

$$\begin{aligned} u_{tt}(x, y, t) &= \frac{\partial}{\partial t}\left[\frac{\partial u}{\partial t}\right] \\ &= \frac{\partial}{\partial t}[5\sin(3\pi x)\sin(4\pi y)(-10\pi\sin(10\pi t))] \\ &= \frac{\partial}{\partial t}[-50\pi\sin(3\pi x)\sin(4\pi y)\sin(10\pi t)] \\ &= -500\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t) \end{aligned}$$

$$\begin{aligned} u_{xx}(x, y, t) &= \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\right] \\ &= \frac{\partial}{\partial x}[15\pi\cos(3\pi x)\sin(4\pi y)\cos(10\pi t)] \\ &= -45\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t) \end{aligned}$$

$$\begin{aligned} u_{yy}(x, y, t) &= \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial y}\right] \\ &= \frac{\partial}{\partial y}[5\sin(3\pi x)(4\pi\cos(4\pi y))\cos(10\pi t)] \\ &= \frac{\partial}{\partial y}[20\pi\sin(3\pi x)\cos(4\pi y)\cos(10\pi t)] \\ &= -80\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t). \end{aligned}$$

Next, we substitute each of these into the right-hand side of Equation 14.3.9 and simplify:

$$\begin{aligned} 4(u_{xx} + u_{yy}) &= 4(-45\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t) + -80\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t)) \\ &= 4(-125\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t)) \\ &= -500\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t) \\ &= u_{tt}. \end{aligned}$$

This verifies the solution.

? Exercise 14.3.7: A Solution to the Heat Equation

Verify that

$$u(x,y,t) = 2\sin\left(\frac{x}{3}\right)\sin\left(\frac{y}{4}\right)e^{-25t/16}$$

is a solution to the heat equation

$$u_t = 9(u_{xx} + u_{yy}).$$

Hint

Calculate the partial derivatives and substitute into the right-hand side.

Answer

TBA

Since the solution to the two-dimensional heat equation is a function of three variables, it is not easy to create a visual representation of the solution. We can graph the solution for fixed values of t, which amounts to snapshots of the heat distributions at fixed times. These snapshots show how the heat is distributed over a two-dimensional surface as time progresses. The graph of the preceding solution at time $t = 0$ appears in Figure 14.3.3. As time progresses, the extremes level out, approaching zero as t approaches infinity.

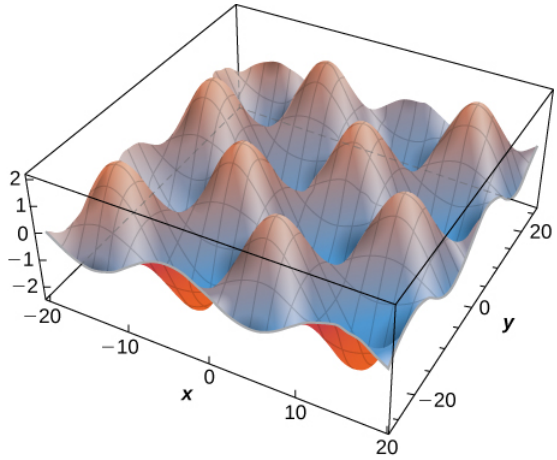


Figure 14.3.3

If we consider the heat equation in one dimension, then it is possible to graph the solution over time. The heat equation in one dimension becomes

$$u_t = c^2 u_{xx},$$

where c^2 represents the thermal diffusivity of the material in question. A solution of this differential equation can be written in the form

$$u_m(x,t) = e^{-\pi^2 m^2 c^2 t}\sin(m\pi x)$$

where m is any positive integer. A graph of this solution using $m = 1$ appears in Figure 14.3.4, where the initial temperature distribution over a wire of length 1 is given by $u(x,0) = \sin \pi x$. Notice that as time progresses, the wire cools off. This is seen because, from left to right, the highest temperature (which occurs in the middle of the wire) decreases and changes color from red to blue.

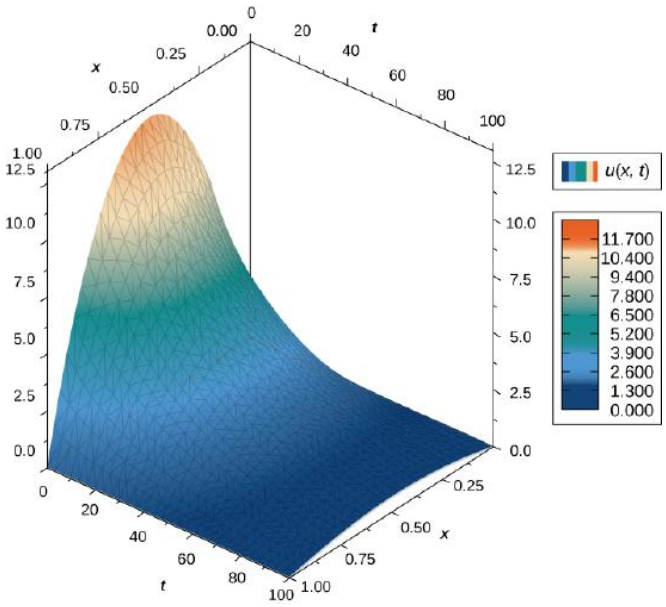
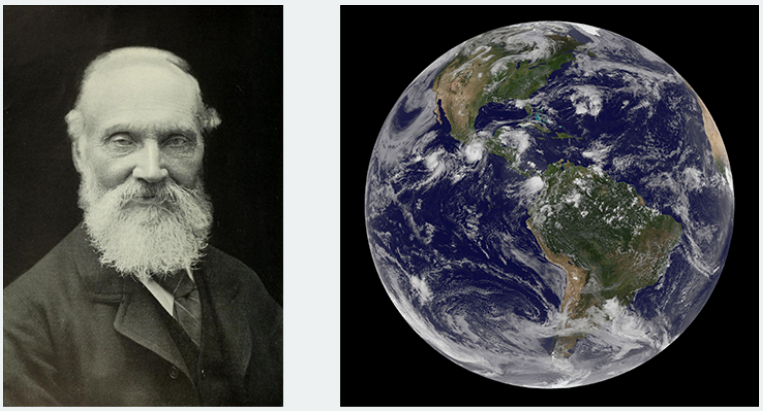


Figure 14.3.4: Graph of a solution of the heat equation in one dimension over time.

Lord Kelvin and the Age of Earth

During the late 1800s, the scientists of the new field of geology were coming to the conclusion that Earth must be “millions and millions” of years old. At about the same time, Charles Darwin had published his treatise on evolution. Darwin’s view was that evolution needed many millions of years to take place, and he made a bold claim that the Weald chalk fields, where important fossils were found, were the result of 300 million years of erosion.



(a) (b)

Figure 14.3.5: (a) William Thomson (Lord Kelvin), 1824-1907, was a British physicist and electrical engineer; (b) Kelvin used the heat diffusion equation to estimate the age of Earth (credit: modification of work by NASA).

At that time, eminent physicist William Thomson (Lord Kelvin) used an important partial differential equation, known as the heat diffusion equation, to estimate the age of Earth by determining how long it would take Earth to cool from molten rock to what we had at that time. His conclusion was a range of **20** to **400** million years, but most likely about **50** million years. For many decades, the proclamations of this irrefutable icon of science did not sit well with geologists or with Darwin.

- Read Kelvin’s paper on estimating the age of the Earth.

Kelvin made reasonable assumptions based on what was known in his time, but he also made several assumptions that turned out to be wrong. One incorrect assumption was that Earth is solid and that the cooling was therefore via conduction only, hence justifying the use of the diffusion equation. But the most serious error was a forgivable one—omission of the fact that Earth contains radioactive elements that continually supply heat beneath Earth’s mantle. The discovery of radioactivity came near the end of Kelvin’s life and he acknowledged that his calculation would have to be modified.

Kelvin used the simple one-dimensional model applied only to Earth's outer shell, and derived the age from graphs and the roughly known temperature gradient near Earth's surface. Let's take a look at a more appropriate version of the diffusion equation in radial coordinates, which has the form

$$\frac{\partial T}{\partial t} = K\left[\frac{\partial^2 T}{\partial^2 r} + \frac{2}{r}\frac{\partial T}{\partial r}\right] \tag{14.3.10}$$

.

Here, $T(r,t)$ is temperature as a function of r (measured from the center of Earth) and time $t.K$ is the heat conductivity—for molten rock, in this case. The standard method of solving such a partial differential equation is by separation of variables, where we express the solution as the product of functions containing each variable separately. In this case, we would write the temperature as

$$T(r,t) = R(r)f(t).$$

1. Substitute this form into Equation 14.3.10 and, noting that $f(t)$ is constant with respect to distance (r) and $R(r)$ is constant with respect to time (t), show that

$$\frac{1}{f}\frac{\partial f}{\partial t} = \frac{K}{R}\left[\frac{\partial^2 R}{\partial r^2} + \frac{2}{r}\frac{\partial R}{\partial r}\right].$$

2. This equation represents the separation of variables we want. The left-hand side is only a function of t and the right-hand side is only a function of r, and they must be equal for all values of r and t. Therefore, they both must be equal to a constant. Let's call that constant $-\lambda^2$. (The convenience of this choice is seen on substitution.) So, we have

$$\frac{1}{f}\frac{\partial f}{\partial t} = -\lambda^2 \text{and} \frac{K}{R}\left[\frac{\partial^2 R}{\partial r^2} + \frac{2}{r}\frac{\partial R}{\partial r}\right] = -\lambda^2.$$

3. Now, we can verify through direct substitution for each equation that the solutions are $f(t) = Ae^{-\lambda^2 t}$ and $R(r) = B\left(\frac{\sin \alpha r}{r}\right) + C\left(\frac{\cos \alpha r}{r}\right)$, where $\alpha = \lambda/\sqrt{K}$. Note that $f(t) = Ae^{+\lambda n^2 t}$ is also a valid solution, so we could have chosen $+\lambda^2$for our constant. Can you see why it would not be valid for this case as time increases?
4. Let's now apply boundary conditions.
   a. The temperature must be finite at the center of Earth, $r = 0$. Which of the two constants, B or C, must therefore be zero to keep R finite at $r = 0$? (Recall that $\sin(\alpha r)/r \to \alpha =$ as $r \to 0$, but $\cos(\alpha r)/r$ behaves very differently.)
   b. Kelvin argued that when magma reaches Earth's surface, it cools very rapidly. A person can often touch the surface within weeks of the flow. Therefore, the surface reached a moderate temperature very early and remained nearly constant at a surface temperature T_s. For simplicity, let's set $T = 0$ at $r = R_E$ and find α such that this is the temperature there for all time t. (Kelvin took the value to be $300K \approx 80°F$. We can add this $300K$ constant to our solution later.) For this to be true, the sine argument must be zero at $r = R_E$. Note that α has an infinite series of values that satisfies this condition. Each value of α represents a valid solution (each with its own value for A). The total or general solution is the sum of all these solutions.
   c. At $t = 0$, we assume that all of Earth was at an initial hot temperature T_0 (Kelvin took this to be about $7000K$.) The application of this boundary condition involves the more advanced application of Fourier coefficients. As noted in part b. each value of α_n represents a valid solution, and the general solution is a sum of all these solutions. This results in a series solution:

$$T(r,t) = \left(\frac{T_0 R_E}{\pi}\right)\sum_n \frac{(-1)^{n-1}}{n} e^{-\lambda n^2 t}\frac{\sin(\alpha_n r)}{r}$$

   where $\alpha_n = n\pi/R_E$.

Note how the values of α_n come from the boundary condition applied in part b. The term $\frac{-1^{n-1}}{n}$ is the constant A_n for each term in the series, determined from applying the Fourier method. Letting $\beta = \frac{\pi}{R_E}$, examine the first few terms of this solution shown here and note how λ^2 in the exponential causes the higher terms to decrease quickly as time progresses:

$$T(r,t) = \frac{T_0 R_E}{\pi r}\left(e^{-K\beta^2 t}(\sin \beta r) - \frac{1}{2}e^{-4K\beta^2 t}(\sin 2\beta r) + \frac{1}{3}e^{-9K\beta^2 t}(\sin 3\beta r) - \frac{1}{4}e^{-16K\beta^2 t}(\sin 4\beta r)\right.$$
$$\left. + \frac{1}{5}e^{-25K\beta^2 t}(\sin 5\beta r) \right)$$

$$+\frac{1}{5}e^{-25\beta^2 t}(\sin 5\beta r)\ldots\Big).$$

Near time $t = 0$, many terms of the solution are needed for accuracy. Inserting values for the conductivity K and $\beta = \pi/R_E$ for time approaching merely thousands of years, only the first few terms make a significant contribution. Kelvin only needed to look at the solution near Earth's surface (Figure 14.3.6) and, after a long time, determine what time best yielded the estimated temperature gradient known during his era ($1°F$ increase per $50\,ft$). He simply chose a range of times with a gradient close to this value. In Figure 14.3.6, the solutions are plotted and scaled, with the $300 - K$ surface temperature added. Note that the center of Earth would be relatively cool. At the time, it was thought Earth must be solid.

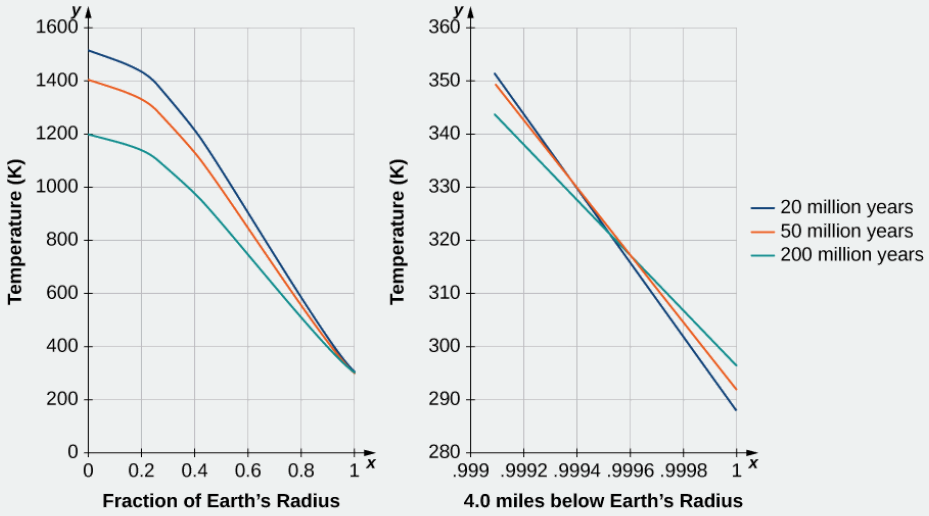


Figure 14.3.6: Temperature versus radial distance from the center of Earth. (a) Kelvin's results, plotted to scale. (b) A close-up of the results at a depth of 4.0 miles below Earth's surface.

Epilog

On May **20, 1904,** physicist **Ernest Rutherford** spoke at the Royal Institution to announce a revised calculation that included the contribution of radioactivity as a source of Earth's heat. In Rutherford's own words:

> "I came into the room, which was half-dark, and presently spotted Lord Kelvin in the audience, and realized that I was in for trouble at the last part of my speech dealing with the age of the Earth, where my views conflicted with his. To my relief, Kelvin fell fast asleep, but as I came to the important point, I saw the old bird sit up, open an eye and cock a baleful glance at me.
>
> Then a sudden inspiration came, and I said Lord Kelvin had limited the age of the Earth, provided no new source [of heat] was discovered. That prophetic utterance referred to what we are now considering tonight, radium! Behold! The old boy beamed upon me."

Rutherford calculated an age for Earth of about **500** million years. Today's accepted value of Earth's age is about **4.6** billion years.

Key Concepts

- A partial derivative is a derivative involving a function of more than one independent variable.
- To calculate a partial derivative with respect to a given variable, treat all the other variables as constants and use the usual differentiation rules.
- Higher-order partial derivatives can be calculated in the same way as higher-order derivatives.

Key Equations

Partial derivative of f with respect to x

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

Partial derivative of f with respect to y

$$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y + k) - f(x, y)}{k}$$

Glossary

higher-order partial derivatives

second-order or higher partial derivatives, regardless of whether they are mixed partial derivatives

mixed partial derivatives

second-order or higher partial derivatives, in which at least two of the differentiations are with respect to different variables

partial derivative

a derivative of a function of more than one independent variable in which all the variables but one are held constant

partial differential equation

an equation that involves an unknown function of more than one independent variable and one or more of its partial derivatives

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14.3E: Exercises for Section 14.3

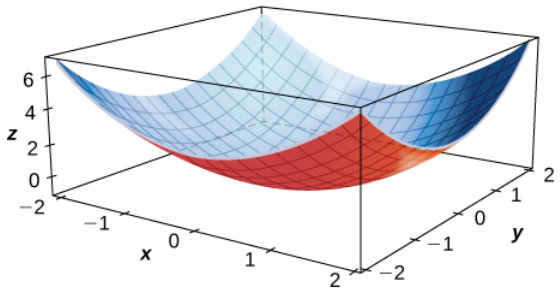
In the following exercise, calculate the partial derivative using the limit definitions only.

1) $\frac{\partial z}{\partial y}$ for $z = x^2 - 3xy + y^2$

Answer

$\frac{\partial z}{\partial y} = -3x + 2y$

For exercises 2 - 5, calculate the sign of the partial derivative using the graph of the surface.



2) $f_x(1,1)$

3) $f_x(-1,1)$

Answer

The sign is negative.

4) $f_y(1,1)$

5) $f_x(0,0)$

Answer

The partial derivative is zero at the origin.

In exercises 6 - 16, calculate the requested partial derivatives.

6) $\frac{\partial z}{\partial x}$ for $z = \sin(3x)\cos(3y)$

7) $\frac{\partial z}{\partial y}$ for $z = \sin(3x)\cos(3y)$

Answer

$\frac{\partial z}{\partial y} = -3\sin(3x)\sin(3y)$

8) $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $z = x^8 e^3 y$

9) $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $z = \ln(x^6 + y^4)$

Answer

$\frac{\partial z}{\partial x} = \frac{6x^5}{x^6+y^4}$; $\frac{\partial z}{\partial y} = \frac{4y^3}{x^6+y^4}$

10) Find $f_y(x,y)$ for $f(x,y) = e^{xy}\cos(x)\sin(y)$.

11) Let $z = e^{xy}$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

-

Answer

$\frac{\partial z}{\partial x} = ye^{xy}; \quad \frac{\partial z}{\partial y} = xe^{xy}$

12) Let $z = \ln(\frac{x}{y})$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

13) Let $z = \tan(2x - y)$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Answer

$\frac{\partial z}{\partial x} = 2\sec^2(2x - y), \quad \frac{\partial z}{\partial y} = -\sec^2(2x - y)$

14) Let $z = \sinh(2x + 3y)$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

15) Let $f(x, y) = \arctan(\frac{y}{x})$. Evaluate $f_x(2, -2)$ and $f_y(2, -2)$.

Answer

$f_x(2, -2) = \frac{1}{4} = f_y(2, -2)$

16) Let $f(x, y) = \frac{xy}{x - y}$. Find $f_x(2, -2)$ and $f_y(2, -2)$.

17) Find $\frac{\partial z}{\partial x}$ at $(0, 1)$ for $z = e^{-x}cos(y)$.

Answer

$\frac{\partial z}{\partial x} = -\cos(1)$

18) Given $f(x, y, z) = x^3yz^2$, find $\frac{\partial^2 f}{\partial x \partial y}$ and $f_z(1, 1, 1)$.

19) Given $f(x, y, z) = 2\sin(x + y)$, find $f_x(0, \frac{\pi}{2}, -4)$, $f_y(0, \frac{\pi}{2}, -4)$, and $f_z(0, \frac{\pi}{2}, -4)$.

Answer

$f_x(x, y, z) = 0, \quad f_y(x, y, z) = 0, \quad f_z(x, y, z) = 0$

20) The area of a parallelogram with adjacent side lengths that are a and b, and in which the angle between these two sides is θ, is given by the function $A(a, b, \theta) = ba\sin(\theta)$.Find the rate of change of the area of the parallelogram with respect to the following:

a. Side a

b. Side b

c. Angle θ

21) Express the volume of a right circular cylinder as a function of two variables:

a. its radius r and its height h.

b. Show that the rate of change of the volume of the cylinder with respect to its radius is the product of its circumference multiplied by its height.

c. Show that the rate of change of the volume of the cylinder with respect to its height is equal to the area of the circular base.

Answer

$a. V(r, h) = \pi r^2 h$
$b. \frac{\partial V}{\partial r} = 2\pi rh$
$c. \frac{\partial V}{\partial h} = \pi r^2$

22) Calculate $\frac{\partial w}{\partial z}$ for $w = z\sin(xy^2 + 2z)$.

In exercises 23 - 39, find the indicated higher-order partial derivatives.

23) $f_{xy}(x,y)$ for $f(x,y)=\ln(x-y)$

Answer

$f_{xy}(x,y)=\frac{1}{(x-y)^2}$

24) $f_{yx}(x,y)$ for $f(x,y)=\ln(x-y)$

25) Let $z=x^2+3xy+2y^2$. Find $\dfrac{\partial^2 z}{\partial x^2}$ and $\dfrac{\partial^2 z}{\partial y^2}$.

Answer

$\dfrac{\partial^2 z}{\partial x^2}=2, \quad \dfrac{\partial^2 z}{\partial y^2}=4$

26) Given $z=e^x\tan y$, find $\dfrac{\partial^2 z}{\partial x\partial y}$ and $\dfrac{\partial^2 z}{\partial y\partial x}$.

27) Given $f(x,y,z)=xyz$, find $f_{xyy}(x,y,z)$, $f_{yxy}(x,y,z)$, and $f_{yyx}(x,y,z)$.

Answer

$f_{xyy}(x,y,z)=f_{yxy}(x,y,z)=f_{yyx}(x,y,z)=0$

28) Given $f(x,y,z)=e^{-2x}\sin(z^2y)$, show that $f_{xyy}(x,y,z)=f_{yxy}(x,y,z)$.

29) Show that $z=\frac{1}{2}(e^y-e^{-y})\sin x$ is a solution of the differential equation $\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2}=0$.

Answer

$\dfrac{d^2z}{dx^2}=-\frac{1}{2}(e^y-e^{-y})\sin x$
$\dfrac{d^2z}{dy^2}=\frac{1}{2}(e^y-e^{-y})\sin x$
$\dfrac{d^2z}{dx^2}+\dfrac{d^2z}{dy^2}=0$

30) Find $f_{xx}(x,y)$ for $f(x,y)=\frac{4x^2}{y}+\frac{y^2}{2x}$.

31) Let $f(x,y,z)=x^2y^3z-3xy^2z^3+5x^2z-y^3z$. Find f_{xyz}.

Answer

$f_{xyz}(x,y,z)=6y^2x-18yz^2$

32) Let $F(x,y,z)=x^3yz^2-2x^2yz+3xz-2y^3z$. Find $F_{xyz}(x,y,z)$.

33) Given $f(x,y)=x^2+x-3xy+y^3-5$, find all points at which $f_x(x,y)=f_y(x,y)=0$ simultaneously.

Answer

$(\frac{1}{4},\frac{1}{2}), \quad (1,1)$

34) Given $f(x,y)=2x^2+2xy+y^2+2x-3$, find all points at which $\dfrac{\partial f}{\partial x}=0$ and $\dfrac{\partial f}{\partial y}=0$ simultaneously.

35) Given $f(x,y)=y^3-3yx^2-3y^2-3x^2+1$, find all points on f at which $f_x(x,y)=f_y(x,y)=0$ simultaneously.

Answer

$(0,0), \quad (0,2), \quad (\sqrt{3},-1), \quad (-\sqrt{3},-1)$

36) Given $f(x,y)=15x^3-3xy+15y^3$, find all points at which $f_x(x,y)=f_y(x,y)=0$ simultaneously.

37) Show that $z=e^x\sin y$ satisfies the equation $\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2}=0$.

Answer

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = e^x \sin y - e^x \sin y = 0$$

38) Show that $f(x, y) = \ln(x^2 + y^2)$ solves Laplace's equation $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$.

39) Show that $z = e^{-t}\cos(\frac{x}{c})$ satisfies the heat equation $\dfrac{\partial z}{\partial t} = -e^{-t}\cos(\frac{x}{c})$.

Answer

$$c^2 \frac{\partial^2 z}{\partial x^2} = e^{-t}\cos(\tfrac{x}{c})$$

40) Find $\lim\limits_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x, y)}{\Delta x}$ for $f(x, y) = -7x - 2xy + 7y$.

41) Find $\lim\limits_{\Delta y \to 0} \dfrac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$ for $f(x, y) = -7x - 2xy + 7y$.

Answer

$$\frac{\partial f}{\partial y} = -2x + 7$$

42) Find $\lim\limits_{\Delta x \to 0} \dfrac{\Delta f}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$ for $f(x, y) = x^2 y^2 + xy + y$.

43) Find $\lim\limits_{\Delta x \to 0} \dfrac{\Delta f}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$ for $f(x, y) = \sin(xy)$.

Answer

$$\frac{\partial f}{\partial x} = y \cos xy$$

44) The function $P(T, V) = \dfrac{nRT}{V}$ gives the pressure at a point in a gas as a function of temperature T and volume V. The letters n and R are constants. Find $\dfrac{\partial P}{\partial V}$ and $\dfrac{\partial P}{\partial T}$, and explain what these quantities represent.

45) The equation for heat flow in the xy-plane is $\dfrac{\partial f}{\partial t} = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2}$. Show that $f(x, y, t) = e^{-2t} \sin x \sin y$ is a solution.

46) The basic wave equation is $f_{tt} = f_{xx}$. Verify that $f(x, t) = \sin(x + t)$ and $f(x, t) = \sin(x - t)$ are solutions.

47) The law of cosines can be thought of as a function of three variables. Let x, y, and θ be two sides of any triangle where the angle θ is the included angle between the two sides. Then, $F(x, y, \theta) = x^2 + y^2 - 2xy\cos\theta$ gives the square of the third side of the triangle. Find $\dfrac{\partial F}{\partial \theta}$ and $\dfrac{\partial F}{\partial x}$ when $x = 2$, $y = 3$, and $\theta = \frac{\pi}{6}$.

Answer

$$\frac{\partial F}{\partial \theta} = 6, \quad \frac{\partial F}{\partial x} = 4 - 3\sqrt{3}$$

48) Suppose the sides of a rectangle are changing with respect to time. The first side is changing at a rate of 2in./sec whereas the second side is changing at the rate of 4 in/sec. How fast is the diagonal of the rectangle changing when the first side measures 16 in. and the second side measures 20 in.? (Round answer to three decimal places.)

49) A **Cobb-Douglas production function** is $f(x, y) = 200x^{0.7}y^{0.3}$, where x and y represent the amount of labor and capital available. Let $x = 500$ and $y = 1000$. Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at these values, which represent the marginal productivity of labor and capital, respectively.

Answer

$$\frac{\partial f}{\partial x} \text{ at } (500, 1000) = 172.36, \quad \frac{\partial f}{\partial y} \text{ at } (500, 1000) = 36.93$$

50) The apparent temperature index is a measure of how the temperature feels, and it is based on two variables: h, which is relative humidity, and t, which is the air temperature.

$A = 0.885t - 22.4h + 1.20th - 0.544$. Find $\frac{\partial A}{\partial t}$ and $\frac{\partial A}{\partial h}$ when $t = 20°F$ and $h = 0.90$.

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14.4: Tangent Planes and Linear Approximations

Learning Objectives

- Determine the equation of a plane tangent to a given surface at a point.
- Use the tangent plane to approximate a function of two variables at a point.
- Explain when a function of two variables is differentiable.
- Use the total differential to approximate the change in a function of two variables.

In this section, we consider the problem of finding the tangent plane to a surface, which is analogous to finding the equation of a tangent line to a curve when the curve is defined by the graph of a function of one variable, $y = f(x)$. The slope of the tangent line at the point $x = a$ is given by $m = f'(a)$; what is the slope of a tangent plane? We learned about the equation of a plane in Equations of Lines and Planes in Space; in this section, we see how it can be applied to the problem at hand.

Tangent Planes

Intuitively, it seems clear that, in a plane, only one line can be tangent to a curve at a point. However, in three-dimensional space, many lines can be tangent to a given point. If these lines lie in the same plane, they determine the tangent plane at that point. A more intuitive way to think of a tangent plane is to assume the surface is smooth at that point (no corners). Then, a tangent line to the surface at that point in any direction does not have any abrupt changes in slope because the direction changes smoothly. Therefore, in a small-enough neighborhood around the point, a tangent plane touches the surface at that point only.

Definition: Tangent Lines

Let $P_0 = (x_0, y_0, z_0)$ be a point on a surface S, and let C be any curve passing through P_0 and lying entirely in S. If the tangent lines to all such curves C at P_0 lie in the same plane, then this plane is called the *tangent plane* to S at P_0 (Figure 14.4.1).

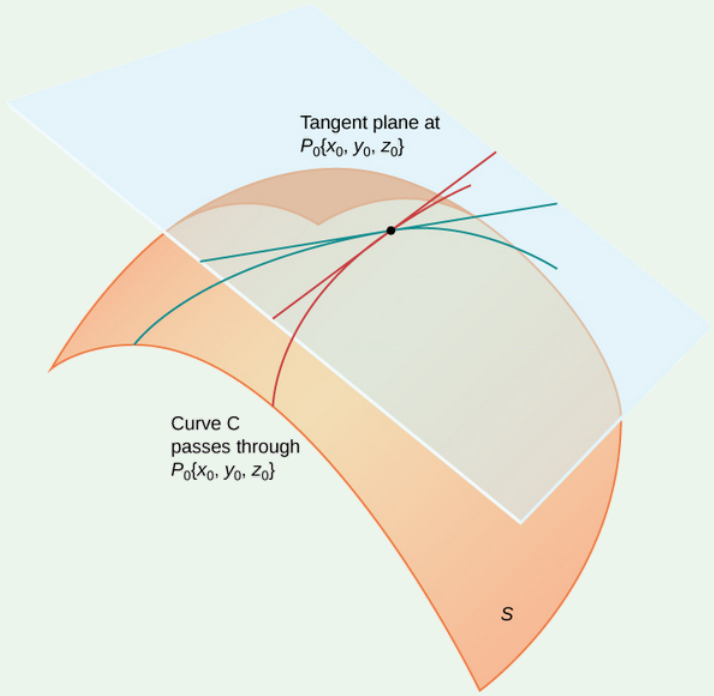

Figure 14.4.1: The tangent plane to a surface S at a point P_0 contains all the tangent lines to curves in S *that pass through* P_0.

For a tangent plane to a surface to exist at a point on that surface, it is sufficient for the function that defines the surface to be differentiable at that point. We define the term tangent plane here and then explore the idea intuitively.

Definition: Tangent Planes

Let S be a surface defined by a differentiable function $z = f(x, y)$, and let $P_0 = (x_0, y_0)$ be a point in the domain of f. Then, the equation of the tangent plane to S at P_0 is given by

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \tag{14.4.1}$$

To see why this formula is correct, let's first find two tangent lines to the surface S. The equation of the tangent line to the curve that is represented by the intersection of S with the vertical trace given by $x = x_0$ is $z = f(x_0, y_0) + f_y(x_0, y_0)(y - y_0)$. Similarly, the equation of the tangent line to the curve that is represented by the intersection of S with the vertical trace given by $y = y_0$ is $z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0)$. A parallel vector to the first tangent line is $\vec{\mathbf{a}} = \hat{\mathbf{j}} + f_y(x_0, y_0)\,\hat{\mathbf{k}}$; a parallel vector to the second tangent line is $\vec{\mathbf{b}} = \hat{\mathbf{i}} + f_x(x_0, y_0)\,\hat{\mathbf{k}}$. We can take the cross product of these two vectors:

$$\begin{aligned}
\vec{\mathbf{a}} \times \vec{\mathbf{b}} &= (\hat{\mathbf{j}} + f_y(x_0, y_0)\,\hat{\mathbf{k}}) \times (\hat{\mathbf{i}} + f_x(x_0, y_0)\,\hat{\mathbf{k}}) \\
&= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 1 & f_y(x_0, y_0) \\ 1 & 0 & f_x(x_0, y_0) \end{vmatrix} \\
&= f_x(x_0, y_0)\,\hat{\mathbf{i}} + f_y(x_0, y_0)\,\hat{\mathbf{j}} - \hat{\mathbf{k}}.
\end{aligned}$$

This vector is perpendicular to both lines and is therefore perpendicular to the tangent plane. We can use this vector as a normal vector to the tangent plane, along with the point $P_0 = (x_0, y_0, f(x_0, y_0))$ in the equation for a plane:

$$\begin{aligned}
\vec{\mathbf{n}} \cdot ((x - x_0)\,\hat{\mathbf{i}} + (y - y_0)\,\hat{\mathbf{j}} + (z - f(x_0, y_0))\,\hat{\mathbf{k}}) &= 0 \\
(f_x(x_0, y_0)\,\hat{\mathbf{i}} + f_y(x_0, y_0)\,\hat{\mathbf{j}} - \hat{\mathbf{k}}) \cdot ((x - x_0)\,\hat{\mathbf{i}} + (y - y_0)\,\hat{\mathbf{j}} + (z - f(x_0, y_0))\,\hat{\mathbf{k}}) &= 0 \\
f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - f(x_0, y_0)) &= 0.
\end{aligned}$$

Solving this equation for z gives Equation 14.4.1.

Example 14.4.1: Finding a Tangent Plane

Find the equation of the tangent plane to the surface defined by the function $f(x, y) = 2x^2 - 3xy + 8y^2 + 2x - 4y + 4$ at point $(2, -1)$.

Solution

First, we must calculate $f_x(x, y)$ and $f_y(x, y)$, then use Equation with $x_0 = 2$ and $y_0 = -1$:

$$\begin{aligned}
f_x(x, y) &= 4x - 3y + 2 \\
f_y(x, y) &= -3x + 16y - 4 \\
f(2, -1) &= 2(2)^2 - 3(2)(-1) + 8(-1)^2 + 2(2) - 4(-1) + 4 = 34 \\
f_x(2, -1) &= 4(2) - 3(-1) + 2 = 13 \\
f_y(2, -1) &= -3(2) + 16(-1) - 4 = -26.
\end{aligned}$$

Then Equation 14.4.1 becomes

$$\begin{aligned}
z &= f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\
z &= 34 + 13(x - 2) - 26(y - (-1)) \\
z &= 34 + 13x - 26 - 26y - 26 \\
z &= 13x - 26y - 18.
\end{aligned}$$

(See the following figure).

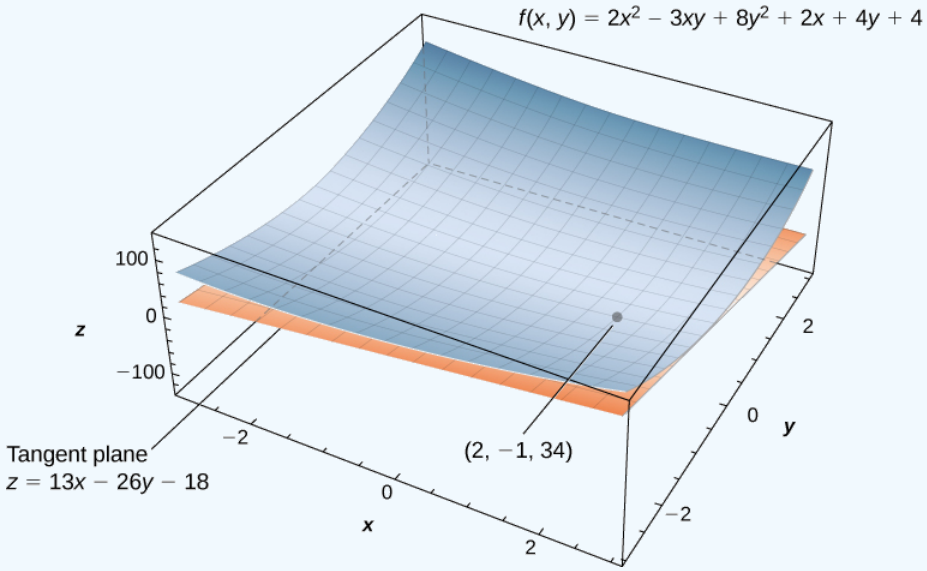

Figure 14.4.2: Calculating the equation of a tangent plane to a given surface at a given point.

Exercise 14.4.1

Find the equation of the tangent plane to the surface defined by the function $f(x,y) = x^3 - x^2y + y^2 - 2x + 3y - 2$ at point $(-1,3)$.

Hint

First, calculate $f_x(x,y)$ and $f_y(x,y)$, then use Equation 14.4.1.

Answer

$z = 7x + 8y - 3$

Example 14.4.2: Finding Another Tangent Plane

Find the equation of the tangent plane to the surface defined by the function $f(x,y) = \sin(2x)\cos(3y)$ at the point $(\pi/3, \pi/4)$.

Solution

First, calculate $f_x(x,y)$ and $f_y(x,y)$, then use Equation 14.4.1 with $x_0 = \pi/3$ and $y_0 = \pi/4$:

$$f_x(x,y) = 2\cos(2x)\cos(3y)$$
$$f_y(x,y) = -3\sin(2x)\sin(3y)$$
$$f\left(\frac{\pi}{3},\frac{\pi}{4}\right) = \sin\left(2\left(\frac{\pi}{3}\right)\right)\cos\left(3\left(\frac{\pi}{4}\right)\right) = \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{6}}{4}$$
$$f_x\left(\frac{\pi}{3},\frac{\pi}{4}\right) = 2\cos\left(2\left(\frac{\pi}{3}\right)\right)\cos\left(3\left(\frac{\pi}{4}\right)\right) = 2\left(-\frac{1}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}$$
$$f_y\left(\frac{\pi}{3},\frac{\pi}{4}\right) = -3\sin\left(2\left(\frac{\pi}{3}\right)\right)\sin\left(3\left(\frac{\pi}{4}\right)\right) = -3\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = -\frac{3\sqrt{6}}{4}.$$

Then Equation 14.4.1 becomes

$$\begin{aligned} z &= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ &= -\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{3}\right) - \frac{3\sqrt{6}}{4}\left(y - \frac{\pi}{4}\right) \\ &= \frac{\sqrt{2}}{2}x - \frac{3\sqrt{6}}{4}y - \frac{\sqrt{6}}{4} - \frac{\pi\sqrt{2}}{6} + \frac{3\pi\sqrt{6}}{16} \end{aligned}$$

A tangent plane to a surface does not always exist at every point on the surface. Consider the piecewise function

$$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}. \tag{14.4.2}$$

The graph of this function follows.

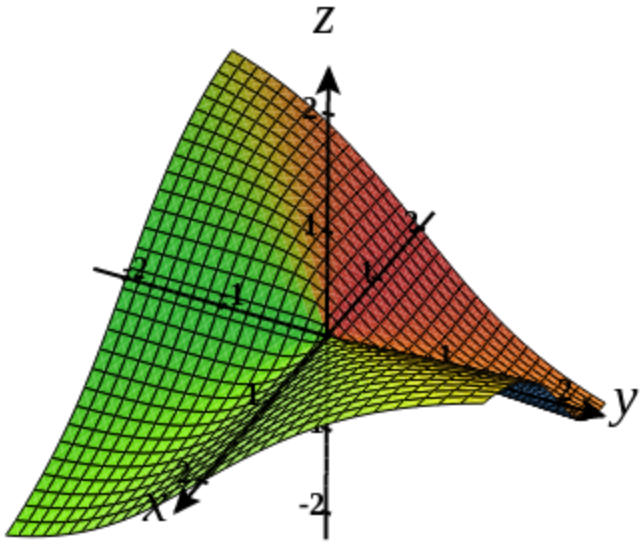

Figure 14.4.3***:*** *Graph of a function that does not have a tangent plane at the origin. Dynamic figure powered by CalcPlot3D.*

If either $x=0$ or $y=0$, then $f(x,y)=0$, so the value of the function does not change on either the x- or y-axis. Therefore, $f_x(x,0)=f_y(0,y)=0$, so as either x or y approach zero, these partial derivatives stay equal to zero. Substituting them into Equation gives $z=0$ as the equation of the tangent line. However, if we approach the origin from a different direction, we get a different story. For example, suppose we approach the origin along the line $y=x$. If we put $y=x$ into the original function, it becomes

$$f(x,x)=\frac{x(x)}{\sqrt{x^2+(x)^2}}=\frac{x^2}{\sqrt{2x^2}}=\frac{|x|}{\sqrt{2}}.$$

When $x>0$, the slope of this curve is equal to $\sqrt{2}/2$; when $x<0$, the slope of this curve is equal to $-(\sqrt{2}/2)$. This presents a problem. In the definition of tangent plane, we presumed that all tangent lines through point P (in this case, the origin) lay in the same plane. This is clearly not the case here. When we study differentiable functions, we will see that this function is not differentiable at the origin.

Linear Approximations

Recall from Linear Approximations and Differentials that the formula for the linear approximation of a function $f(x)$ at the point $x=a$ is given by

$$y\approx f(a)+f'(a)(x-a)$$

$y \approx f(a) + f'(a)(x - a)$.

The diagram for the linear approximation of a function of one variable appears in the following graph.

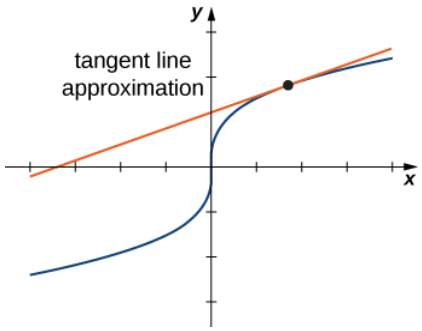

Figure 14.4.4: Linear approximation of a function in one variable.

The tangent line can be used as an approximation to the function $f(x)$ for values of x reasonably close to $x = a$. When working with a function of two variables, the tangent line is replaced by a tangent plane, but the approximation idea is much the same.

Definition: Linear Approximation

Given a function $z = f(x, y)$ with continuous partial derivatives that exist at the point (x_0, y_0), the linear approximation of f at the point (x_0, y_0) is given by the equation

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \tag{14.4.3}$$

Notice that this equation also represents the tangent plane to the surface defined by $z = f(x, y)$ at the point (x_0, y_0). The idea behind using a linear approximation is that, if there is a point (x_0, y_0) at which the precise value of $f(x, y)$ is known, then for values of (x, y) reasonably close to (x_0, y_0), the linear approximation (i.e., tangent plane) yields a value that is also reasonably close to the exact value of $f(x, y)$ (Figure 14.4.5). Furthermore the plane that is used to find the linear approximation is also the tangent plane to the surface at the point (x_0, y_0).

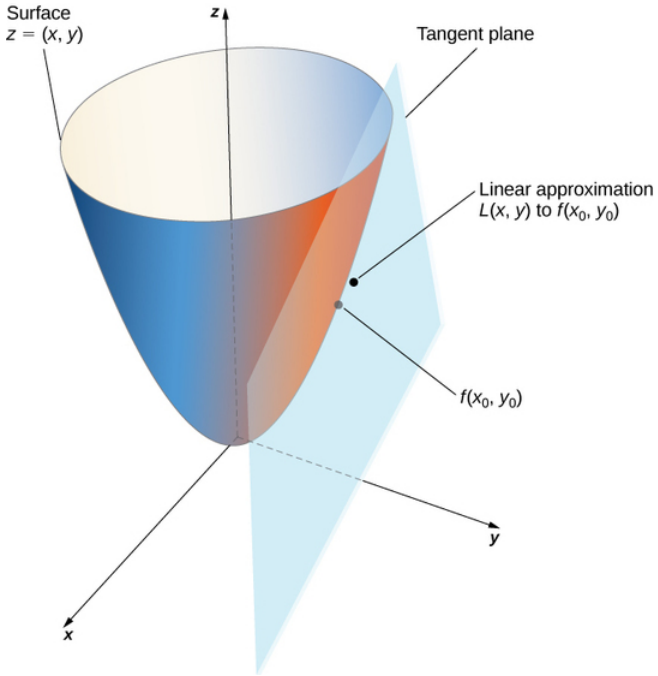

Figure 14.4.5: Using a tangent plane for linear approximation at a point.

Example 14.4.3: Using a Tangent Plane Approximation

Given the function $f(x,y) = \sqrt{41 - 4x^2 - y^2}$, approximate $f(2.1, 2.9)$ using point $(2,3)$ for (x_0, y_0). What is the approximate value of $f(2.1, 2.9)$ to four decimal places?

Solution

To apply Equation 14.4.3, we first must calculate $f(x_0,y_0)$, $f_x(x_0,y_0)$, and $f_y(x_0,y_0)$ using $x_0 = 2$ and $y_0 = 3$:

$$f(x_0,y_0) = f(2,3) = \sqrt{41 - 4(2)^2 - (3)^2} = \sqrt{41 - 16 - 9} = \sqrt{16} = 4$$

$$f_x(x,y) = -\frac{4x}{\sqrt{41 - 4x^2 - y^2}} \text{ so } f_x(x_0,y_0) = -\frac{4(2)}{\sqrt{41 - 4(2)^2 - (3)^2}} = -2$$

$$f_y(x,y) = -\frac{y}{\sqrt{41 - 4x^2 - y^2}} \text{ so } f_y(x_0,y_0) = -\frac{3}{\sqrt{41 - 4(2)^2 - (3)^2}} = -\frac{3}{4}.$$

Now we substitute these values into Equation 14.4.3:

$$\begin{aligned} L(x,y) &= f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0) \\ &= 4 - 2(x-2) - \frac{3}{4}(y-3) \\ &= \frac{41}{4} - 2x - \frac{3}{4}y. \end{aligned}$$

Last, we substitute $x = 2.1$ and $y = 2.9$ into $L(x,y)$:

$$L(2.1, 2.9) = \frac{41}{4} - 2(2.1) - \frac{3}{4}(2.9) = 10.25 - 4.2 - 2.175 = 3.875.$$

The approximate value of $f(2.1, 2.9)$ to four decimal places is

$$f(2.1, 2.9) = \sqrt{41 - 4(2.1)^2 - (2.9)^2} = \sqrt{14.95} \approx 3.8665,$$

which corresponds to a 0.2 error in approximation.

Exercise 14.4.2

Given the function $f(x,y) = e^{5-2x+3y}$, approximate $f(4.1, 0.9)$ using point $(4,1)$ for (x_0, y_0). Also state the approximate value of $f(4.1, 0.9)$ to four decimal places to compare the two values.

Hint

First calculate $f(x_0,y_0)$, $f_x(x_0,y_0)$, and $f_y(x_0,y_0)$ using $x_0 = 4$ and $y_0 = 1$, then use Equation 14.4.3.

Answer

$L(x,y) = 6 - 2x + 3y$, so $L(4.1, 0.9) = 6 - 2(4.1) + 3(0.9) = 0.5$
Note that $f(4.1, 0.9) = e^{5-2(4.1)+3(0.9)} = e^{-0.5} \approx 0.6065$.

Differentiability

When working with a function $y = f(x)$ of one variable, the function is said to be differentiable at a point $x = a$ if $f'(a)$ exists. Furthermore, if a function of one variable is differentiable at a point, the graph is "smooth" at that point (i.e., no corners exist) and a tangent line is well-defined at that point.

The idea behind differentiability of a function of two variables is connected to the idea of smoothness at that point. In this case, a surface is considered to be smooth at point P if a tangent plane to the surface exists at that point. If a function is differentiable at a point, then a tangent plane to the surface exists at that point. Recall the formula (Equation 14.4.1) for a tangent plane at a point (x_0, y_0) is given by

$$z = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$$

For a tangent plane to exist at the point (x_0, y_0), the partial derivatives must therefore exist at that point. However, this is not a sufficient condition for smoothness, as was illustrated in Figure 14.4.3. In that case, the partial derivatives existed at the origin, but the function also had a corner on the graph at the origin.

Definition: Differentiable Functions

A function $f(x, y)$ is *differentiable* at a point $P(x_0, y_0)$ if, for all points (x, y) in a δ disk around P, we can write

$$f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + E(x, y), \tag{14.4.4}$$

where the error term E satisfies

$$\lim_{(x,y)\to(x_0,y_0)} \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0. \tag{14.4.5}$$

The last term in Equation 14.4.4 is referred to as the **error term.** It represents how closely the tangent plane comes to the surface in a small neighborhood (δ disk) of point P. For the function f to be differentiable at P, the function must be smooth—that is, the graph of f must be close to the tangent plane for points near P.

Example 14.4.4: Demonstrating Differentiability

Show that the function $f(x, y) = 2x^2 - 4y$ is differentiable at point $(2, -3)$.

Solution

First, we calculate $f(x_0, y_0)$, $f_x(x_0, y_0)$, and $f_y(x_0, y_0)$ using $x_0 = 2$ and $y_0 = -3$, then we use Equation 14.4.4:

$$\begin{aligned} f(2, -3) &= 2(2)^2 - 4(-3) = 8 + 12 = 20 \\ f_x(2, -3) &= 4(2) = 8 \\ f_y(2, -3) &= -4. \end{aligned}$$

Therefore $m_1 = 8$ and $m_2 = -4$, and Equation 14.4.4 becomes

$$\begin{aligned} f(x, y) &= f(2, -3) + f_x(2, -3)(x - 2) + f_y(2, -3)(y + 3) + E(x, y) \\ 2x^2 - 4y &= 20 + 8(x - 2) - 4(y + 3) + E(x, y) \\ 2x^2 - 4y &= 20 + 8x - 16 - 4y - 12 + E(x, y) \\ 2x^2 - 4y &= 8x - 4y - 8 + E(x, y) \\ E(x, y) &= 2x^2 - 8x + 8. \end{aligned}$$

Next, we calculate the limit in Equation 14.4.5:

$$\begin{aligned} \lim_{(x,y)\to(x_0,y_0)} \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} &= \lim_{(x,y)\to(2,-3)} \frac{2x^2 - 8x + 8}{\sqrt{(x - 2)^2 + (y + 3)^2}} \\ &= \lim_{(x,y)\to(2,-3)} \frac{2(x^2 - 4x + 4)}{\sqrt{(x - 2)^2 + (y + 3)^2}} \\ &= \lim_{(x,y)\to(2,-3)} \frac{2(x - 2)^2}{\sqrt{(x - 2)^2 + (y + 3)^2}} \\ &\le \lim_{(x,y)\to(2,-3)} \frac{2((x - 2)^2 + (y + 3)^2)}{\sqrt{(x - 2)^2 + (y + 3)^2}} \\ &= \lim_{(x,y)\to(2,-3)} 2\sqrt{(x - 2)^2 + (y + 3)^2} \\ &= 0. \end{aligned}$$

Since $E(x, y) \ge 0$ for any value of x or y, the original limit must be equal to zero. Therefore, $f(x, y) = 2x^2 - 4y$ is differentiable at point $(2, -3)$.

Exercise 14.4.3

Show that the function $f(x,y) = 3x - 4y^2$ is differentiable at point $(-1,2)$.

Hint

First, calculate $f(x_0,y_0)$, $f_x(x_0,y_0)$, and $f_y(x_0,y_0)$ using $x_0 = -1$ and $y_0 = 2$, then use Equation 14.4.5 to find $E(x,y)$.

Since you will find that $E(x,y) \le 0$, show

$$\lim_{(x,y)\to(x_0,y_0)} \frac{|E(x,y)|}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$$

Answer

$$f(-1,2) = -19, \quad f_x(-1,2) = 3, \quad f_y(-1,2) = -16, \quad E(x,y) = -4(y-2)^2.$$

$$\begin{aligned}
\lim_{(x,y)\to(x_0,y_0)} \frac{|E(x,y)|}{\sqrt{(x-x_0)^2+(y-y_0)^2}} &= \lim_{(x,y)\to(-1,2)} \frac{|-4(y-2)^2|}{\sqrt{(x+1)^2+(y-2)^2}} \\
&= \lim_{(x,y)\to(-1,2)} \frac{4(y-2)^2}{\sqrt{(x+1)^2+(y-2)^2}} \\
&\le \lim_{(x,y)\to(-1,2)} \frac{4((x+1)^2+(y-2)^2)}{\sqrt{(x+1)^2+(y-2)^2}} \\
&= \lim_{(x,y)\to(-1,2)} 4\sqrt{(x+1)^2+(y-2)^2} \\
&= 0.
\end{aligned}$$

Since $\lim_{(x,y)\to(x_0,y_0)} \frac{|E(x,y)|}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$, we know it's also true that

$$\lim_{(x,y)\to(x_0,y_0)} \frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$$

Therefore, $f(x,y) = 3x - 4y^2$ is differentiable at point $(-1,2)$.

This function from (Equation 14.4.2)

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is not differentiable at the origin (Figure 14.4.3). We can see this by calculating the partial derivatives. This function appeared earlier in the section, where we showed that $f_x(0,0) = f_y(0,0) = 0$. Substituting this information into Equations 14.4.4 and 14.4.5 using $x_0 = 0$ and $y_0 = 0$, we get

$$\begin{aligned}
f(x,y) &= f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) + E(x,y) \\
E(x,y) &= \frac{xy}{\sqrt{x^2+y^2}}.
\end{aligned}$$

Calculating

$$\lim_{(x,y)\to(x_0,y_0)} \frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}$$

gives

$$\begin{aligned}
\lim_{(x,y)\to(x_0,y_0)} \frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} &= \lim_{(x,y)\to(0,0)} \frac{\dfrac{xy}{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} \\
&= \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}.
\end{aligned}$$

Depending on the path taken toward the origin, this limit takes different values. Therefore, the limit does not exist and the function f is not differentiable at the origin as shown in the following figure.

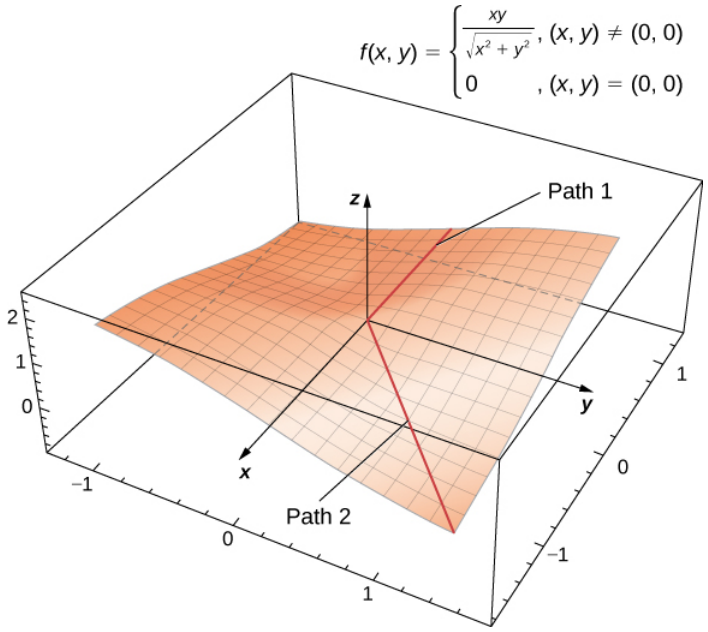

Figure 14.4.6: This function $f(x,y)$ (Equation 14.4.2) is not differentiable at the origin.

Differentiability and continuity for functions of two or more variables are connected, the same as for functions of one variable. In fact, with some adjustments of notation, the basic theorem is the same.

Theorem 14.4.1: Differentiability Implies Continuity

Let $z = f(x,y)$ be a function of two variables with (x_0, y_0) in the domain of f. If $f(x,y)$ is differentiable at (x_0, y_0), then $f(x,y)$ is continuous at (x_0, y_0).

Theorem 14.4.1 shows that if a function is differentiable at a point, then it is continuous there. However, if a function is continuous at a point, then it is not necessarily differentiable at that point. For example, the function discussed above (Equation 14.4.2)

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is *continuous* at the origin, but it is *not differentiable* at the origin. This observation is also similar to the situation in single-variable calculus.

We can further explore the connection between continuity and differentiability at a point. This next theorem says that if the function and its partial derivatives are continuous at a point, the function is differentiable.

Theorem 14.4.2: Continuity of First Partials Implies Differentiability

Let $z = f(x,y)$ be a function of two variables with (x_0, y_0) in the domain of f. If $f(x,y)$, $f_x(x,y)$, and $f_y(x,y)$ all exist in a neighborhood of (x_0, y_0) and are continuous at (x_0, y_0), then $f(x,y)$ is differentiable there.

Recall that earlier we showed that the function in Equation 14.4.2 was not differentiable at the origin. Let's calculate the partial derivatives f_x and f_y:

$$\frac{\partial f}{\partial x} = \frac{y^3}{(x^2+y^2)^{3/2}}$$

and

$$\frac{\partial f}{\partial} = \frac{x^3}{}.$$

$$\partial y \qquad (x^2+y^2)^{3/2}.$$

The contrapositive of the preceding theorem states that if a function is not differentiable, then at least one of the hypotheses must be false. Let's explore the condition that $f_x(0,0)$ must be continuous. For this to be true, it must be true that

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = f_x(0,0)$$

therefor

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = \lim_{(x,y)\to(0,0)} \frac{y^3}{(x^2+y^2)^{3/2}}.$$

Let $x = ky$. Then

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{y^3}{(x^2+y^2)^{3/2}} &= \lim_{y\to 0} \frac{y^3}{((ky)^2+y^2)^{3/2}} \\ &= \lim_{y\to 0} \frac{y^3}{(k^2y^2+y^2)^{3/2}} \\ &= \lim_{y\to 0} \frac{y^3}{|y|^3(k^2+1)^{3/2}} \\ &= \frac{1}{(k^2+1)^{3/2}} \lim_{y\to 0} \frac{|y|}{y}.\end{aligned}$$

If $y > 0$, then this expression equals $1/(k^2+1)^{3/2}$; if $y < 0$, then it equals $-(1/(k^2+1)^{3/2})$. In either case, the value depends on k, so the limit fails to exist.

Differentials

In Linear Approximations and Differentials we first studied the concept of differentials. The differential of y, written dy, is defined as $f'(x)dx$. The differential is used to approximate $\Delta y = f(x+\Delta x) - f(x)$, where $\Delta x = dx$. Extending this idea to the linear approximation of a function of two variables at the point (x_0, y_0) yields the formula for the total differential for a function of two variables.

Definition: Total Differential

Let $z = f(x,y)$ be a function of two variables with (x_0, y_0) in the domain of f, and let Δx and Δy be chosen so that $(x_0 + \Delta x, y_0 + \Delta y)$ is also in the domain of f. If f is differentiable at the point (x_0, y_0), then the differentials dx and dy are defined as

$$dx = \Delta x$$

and

$$dy = \Delta y.$$

The differential dz, also called the ***total differential*** of $z = f(x,y)$ at (x_0, y_0), is defined as

$$dz = f_x(x_0,y_0)dx + f_y(x_0,y_0)dy. \tag{14.4.6}$$

Notice that the symbol ∂ is not used to denote the total differential; rather, d appears in front of z. Now, let's define $\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$. We use dz to approximate Δz, so

$$\Delta z \approx dz = f_x(x_0,y_0)dx + f_y(x_0,y_0)dy.$$

Therefore, the differential is used to approximate the change in the function $z = f(x_0,y_0)$ at the point (x_0,y_0) for given values of Δx and Δy. Since $\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$, this can be used further to approximate $f(x+\Delta x, y+\Delta y)$:

$$f(x+\Delta x, y+\Delta y) = f(x,y) + \Delta z \approx f(x,y) + f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y.$$

See the following figure.

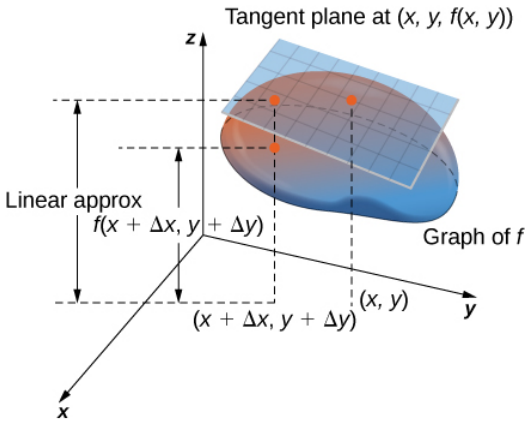

Figure 14.4.7: The linear approximation is calculated via the formula $f(x+\Delta x, y+\Delta y) \approx f(x,y) + f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y$.

One such application of this idea is to determine error propagation. For example, if we are manufacturing a gadget and are off by a certain amount in measuring a given quantity, the differential can be used to estimate the error in the total volume of the gadget.

Example 14.4.5: Approximation by Differentials

Find the differential dz of the function $f(x,y) = 3x^2 - 2xy + y^2$ and use it to approximate Δz at point $(2,-3)$. Use $\Delta x = 0.1$ and $\Delta y = -0.05$. What is the exact value of Δz?

Solution

First, we must calculate $f(x_0,y_0), f_x(x_0,y_0)$, and $f_y(x_0,y_0)$ using $x_0 = 2$ and $y_0 = -3$:

$$\begin{aligned} f(x_0,y_0) &= f(2,-3) = 3(2)^2 - 2(2)(-3) + (-3)^2 = 12 + 12 + 9 = 33 \\ f_x(x,y) &= 6x - 2y \end{aligned}$$

$$\begin{aligned} f_y(x,y) &= -2x + 2y \\ f_x(x_0,y_0) &= fx(2,-3) \\ &= 6(2) - 2(-3) = 12 + 6 = 18 \end{aligned}$$

$$\begin{aligned} f_y(x_0,y_0) &= f_y(2,-3) \\ &= -2(2) + 2(-3) \\ &= -4 - 6 = -10. \end{aligned}$$

Then, we substitute these quantities into Equation 14.4.6:

$$\begin{aligned} dz &= f_x(x_0,y_0)dx + f_y(x_0,y_0)dy \\ dz &= 18(0.1) - 10(-0.05) = 1.8 + 0.5 = 2.3. \end{aligned}$$

This is the approximation to $\Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)$. The exact value of Δz is given by

$$\begin{aligned} \Delta z &= f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0) \\ &= f(2+0.1, -3-0.05) - f(2,-3) \\ &= f(2.1,-3.05) - f(2,-3) \\ &= 2.3425. \end{aligned}$$

Exercise 14.4.4

Find the differential dz of the function $f(x,y) = 4y^2 + x^2y - 2xy$ and use it to approximate Δz at point $(1,-1)$. Use $\Delta x = 0.03$ and $\Delta y = -0.02$. What is the exact value of Δz?

Hint

First, calculate $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ using $x_0 = 1$ and $y_0 = -1$, then use Equation 14.4.6.

Answer

$dz = 0.18$

$\Delta z = f(1.03, -1.02) - f(1, -1) = 0.180682$

Differentiability of a Function of Three Variables

All of the preceding results for differentiability of functions of two variables can be generalized to functions of three variables. First, the definition:

Definition: Differentiability at a Point

A function $f(x, y, z)$ is differentiable at a point $P(x_0, y_0, z_0)$ if for all points (x, y, z) in a δ disk around P we can write

$$f(x, y) = f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) + E(x, y, z),$$

where the error term E satisfies

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} \frac{E(x, y, z)}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}} = 0.$$

If a function of three variables is differentiable at a point (x_0, y_0, z_0), then it is continuous there. Furthermore, continuity of first partial derivatives at that point guarantees differentiability.

Key Concepts

- The analog of a tangent line to a curve is a tangent plane to a surface for functions of two variables.
- Tangent planes can be used to approximate values of functions near known values.
- A function is differentiable at a point if it is ”smooth” at that point (i.e., no corners or discontinuities exist at that point).
- The total differential can be used to approximate the change in a function $z = f(x_0, y_0)$ at the point (x_0, y_0) for given values of Δx and Δy.

Key Equations

- **Tangent plane**

  $z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

- **Linear approximation**

  $L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

- **Total differential**

  $dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy.$

- **Differentiability (two variables)**

  $f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + E(x, y),$

  where the error term E satisfies

  $$\lim_{(x,y)\to(x_0,y_0)} \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0.$$

- **Differentiability (three variables)**

  $f(x, y) = f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) + E(x, y, z),$

  where the error term E satisfies

  $$\lim_{(x,y,z)\to(x_0,y_0,z_0)} \frac{E(x, y, z)}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}} = 0.$$

Glossary

differentiable

a function $f(x,y)$ is differentiable at (x_0,y_0) if $f(x,y)$ can be expressed in the form $f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + E(x,y)$,

where the error term $E(x,y)$ satisfies $\lim_{(x,y)\to(x_0,y_0)} \dfrac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$

linear approximation

given a function $f(x,y)$ and a tangent plane to the function at a point (x_0,y_0), we can approximate $f(x,y)$ for points near (x_0,y_0) using the tangent plane formula

tangent plane

given a function $f(x,y)$ that is differentiable at a point (x_0,y_0), the equation of the tangent plane to the surface $z=f(x,y)$ is given by $z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

total differential

the total differential of the function $f(x,y)$ at (x_0,y_0) is given by the formula $dz = f_x(x_0,y_0)dx + fy(x_0,y_0)dy$

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14.4E: Exercises for Section 14.4

In exercises 1 - 2, find a unit normal vector to the surface at the indicated point.

1) $f(x,y) = x^3, \quad (2,-1,8)$

Answer

$(\frac{\sqrt{145}}{145})(12\hat{\mathbf{i}} - \hat{\mathbf{k}})$

2) $\ln\left(\frac{x}{y-z}\right) = 0$ when $x = y = 1$

In exercises 3 - 7, find a normal vector and a tangent vector at point P.

3) $x^2 + xy + y^2 = 3, \quad P(-1,-1)$

Answer

Normal vector: $\hat{\mathbf{i}} + \hat{\mathbf{j}}$, tangent vector: $\hat{\mathbf{i}} - \hat{\mathbf{j}}$

4) $(x^2+y^2)^2 = 9(x^2-y^2), \quad P(\sqrt{2},1)$

5) $xy^2 - 2x^2 + y + 5x = 6, \quad P(4,2)$

Answer

Normal vector: $7\hat{\mathbf{i}} - 17\hat{\mathbf{j}}$, tangent vector: $17\hat{\mathbf{i}} + 7\hat{\mathbf{j}}$

6) $2x^3 - x^2y^2 = 3x - y - 7, \quad P(1,-2)$

7) $ze^{x^2-y^2} - 3 = 0, \quad P(2,2,3)$

Answer

Normal vector: $-12\hat{\mathbf{i}} + 12\hat{\mathbf{j}} - \hat{\mathbf{k}}$ or $12\hat{\mathbf{i}} - 12\hat{\mathbf{j}} + \hat{\mathbf{k}}$,
Tangent vector: $0\hat{\mathbf{i}} + 1\hat{\mathbf{j}} + 12\hat{\mathbf{k}}$ or $1\hat{\mathbf{i}} + 0\hat{\mathbf{j}} - 12\hat{\mathbf{k}}$

In exercises 8 - 19, find the equation for the tangent plane to the surface at the indicated point. (Hint: If the given function is not already solved for z, start by solving it for z in terms of x and y.)

8) $-8x - 3y - 7z = -19, \quad P(1,-1,2)$

9) $z = -9x^2 - 3y^2, \quad P(2,1,-39)$

Answer

$-36x - 6y - z = -39$

10) $x^2 + 10xyz + y^2 + 8z^2 = 0, \quad P(-1,-1,-1)$

11) $z = \ln(10x^2 + 2y^2 + 1), \quad P(0,0,0)$

Answer

$z = 0$

12) $z = e^{7x^2+4y^2}, \quad P(0,0,1)$

13) $xy + yz + zx = 11, \quad P(1,2,3)$

Answer

$5x + 4y + 3z - 22 = 0$

14) $x^2 + 4y^2 = z^2, \quad P(3,2,5)$

15) $x^3 + y^3 = 3xyz, \quad P(1,2,\frac{3}{2})$

Answer

$4x - 5y + 4z = 0$

16) $z = axy, \quad P(1, \frac{1}{a}, 1)$

17) $z = \sin x + \sin y + \sin(x + y), \quad P(0, 0, 0)$

Answer

$2x + 2y - z = 0$

18) $h(x, y) = \ln \sqrt{x^2 + y^2}, \quad P(3, 4)$

19) $z = x^2 - 2xy + y^2, \quad P(1, 2, 1)$

Answer

$-2(x - 1) + 2(y - 2) - (z - 1) = 0$

In exercises 20 - 25, find parametric equations for the normal line to the surface at the indicated point. (Recall that to find the equation of a line in space, you need a point on the line, $P_0(x_0, y_0, z_0)$, and a vector $\vec{\mathbf{v}} = \langle a, b, c \rangle$ that is parallel to the line. Then the equations of the line are: $x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$)

20) $-3x + 9y + 4z = -4, \quad P(1, -1, 2)$

21) $z = 5x^2 - 2y^2, \quad P(2, 1, 18)$

Answer

$x = 20t + 2, y = -4t + 1, z = -t + 18$

22) $x^2 - 8xyz + y^2 + 6z^2 = 0, \quad P(1, 1, 1)$

23) $z = \ln(3x^2 + 7y^2 + 1), \quad P(0, 0, 0)$

Answer

$x = 0, y = 0, z = t$

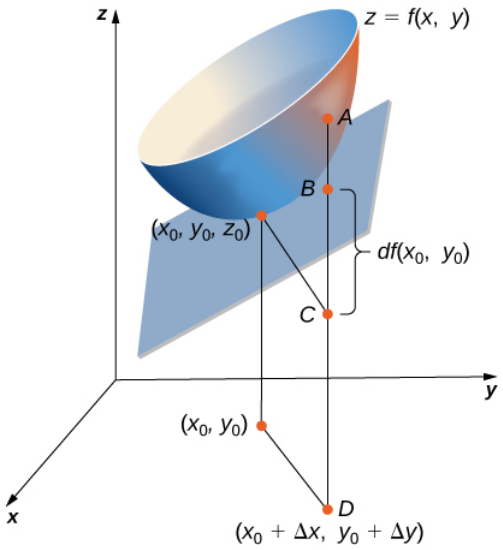
24) $z = e^{4x^2 + 6y^2}, \quad P(0, 0, 1)$

25) $z = x^2 - 2xy + y^2$ at point $P(1, 2, 1)$

Answer

$x - 1 = 2t; y - 2 = -2t; z - 1 = t$

In exercises 26 - 28, use the figure shown here.



26) The length of line segment AC is equal to what mathematical expression?

27) The length of line segment BC is equal to what mathematical expression?

Answer

The differential of the function $z(x,y) = dz = f_x dx + f_y dy$

28) Using the figure, explain what the length of line segment AB represents.

29) Show that $f(x,y) = e^{xy}x$ is differentiable at point $(1,0)$.

Answer

Using the definition of differentiability, we have $e^{xy}x \approx x + y$.

30) Show that $f(x,y) = x^2 + 3y$ is differentiable at every point. In other words, show that $\Delta z = f(x + \Delta x, y + \Delta y) - f(x,y) = f_x \Delta x + f_y \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$, where both ε_1 and ε_2 approach zero as $(\Delta x, \Delta y)$ approaches $(0,0)$.

Answer

$\Delta z = 2x\Delta x + 3\Delta y + (\Delta x)^2.(\Delta x)^2 \to 0$ for small Δx and z satisfies the definition of differentiability.

31) Find the total differential of each function:

a. $z = x^3 + y^3 - 5$
b. $z = e^{xy}$
c. $z = y\cos x + \sin y$
d. $P = t^2 + 3t + tu^3$
e. $w = e^y \cos(x) + z^2$

Answers:

a. $dz = 3x^2\,dx + 3y^2\,dy$
b. $dz = ye^{xy}\,dx + xe^{xy}\,dy$
c. $dz = -y\sin x\,dx + (\cos x + \cos y)\,dy$
d. $dP = (2t + 3 + u^3)\,dt + 3tu^2\,du$
e. $dw = -e^y \sin(x)\,dx + e^y \cos(x)\,dy + 2z\,dz$

32) a. Find the total differential dz of the function $z = \dfrac{xy}{y + x}$ and then

b. State its value where x changes from 10 to 10.5 and y changes from 15 to 13.

Answer

a. $dz = \dfrac{y^2}{(x+y)^2}\,dx + \dfrac{x^2}{(x+y)^2}\,dy$
b. $dx = 0.5$ and $dy = -2$ so

$$\begin{aligned} dz &= f_x(10,15)\,dx + f_y(10,15)\,dy \\ &= \frac{15^2}{25^2}\,dx + \frac{10^2}{25^2}\,dy \\ &= \frac{225}{625}\,(0.5) + \frac{100}{625}(-2) \\ &= \frac{9}{25}\left(\frac{1}{2}\right) + \frac{4}{25}(-2) \\ &= \frac{18}{100} - \frac{32}{100} \\ &= .18 - .32 = -0.14 \end{aligned}$$

33) Let $z = f(x,y) = xe^y$. State its total differential. Then compute Δz from $P(1,2)$ to $Q(1.05, 2.1)$ and then find the approximate change in z, dz, from point P to point Q. Recall $\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$, and dz and Δz should be approximately equal, if dx and dy are both reasonably small.

Answer

Total Differential: $dz = e^y\,dx + xe^y\,dy$
$\Delta z \approx 1.185422$ and $dz \approx 1.108$. Note that they are relatively close.

34) The volume of a right circular cylinder is given by $V(r,h) = \pi r^2 h$. Find the differential dV. Interpret the formula geometrically.

Answer

$dV = 2\pi rh\,dr + \pi r^2\,dh$

35) See the preceding problem. Use differentials to estimate the amount of aluminum in an enclosed aluminum can with diameter $8.0cm$ and height $12cm$ if the aluminum is 0.04 cm thick.

Answer

$16\,\text{cm}^3$

36) Use the differential dz to approximate the change in $z = \sqrt{4 - x^2 - y^2}$ as (x,y) moves from point $(1,1)$ to point $(1.01, 0.97)$. Compare this approximation with the actual change in the function.

37) Let $z = f(x,y) = x^2 + 3xy - y^2$. Find the exact change in the function and the approximate change in the function as x changes from 2.00 to 2.05 and y changes from 3.00 to 2.96.

Answer

$\Delta z =$ exact change $= 0.6449$, approximate change is $dz = 0.65$. The two values are close.

38) The centripetal acceleration of a particle moving in a circle is given by $a(r,v) = \frac{v^2}{r}$, where v is the velocity and r is the radius of the circle. Approximate the maximum percent error in measuring the acceleration resulting from errors of 3% in v and 2% in r. (Recall that the percentage error is the ratio of the amount of error over the original amount. So, in this case, the percentage error in a is given by $\frac{da}{a}$.)

39) The radius r and height h of a right circular cylinder are measured with possible errors of 4% and 5%, respectively. Approximate the maximum possible percentage error in measuring the volume (Recall that the percentage error is the ratio of the amount of error over the original amount. So, in this case, the percentage error in V is given by $\frac{dV}{V}$.)

Answer

13% or 0.13

40) The base radius and height of a right circular cone are measured as 10 in. and 25 in., respectively, with a possible error in measurement of as much as 0.1 in. each. Use differentials to estimate the maximum error in the calculated volume of the cone.

41) The **electrical resistance** R produced by wiring resistors R_1 and R_2 in parallel can be calculated from the formula $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. If R_1 and R_2 are measured to be 7Ω and 6Ω, respectively, and if these measurements are accurate to within 0.05Ω, estimate the maximum possible error in computing R. (The symbol Ω represents an ohm, the unit of electrical resistance.)

Answer

0.025

42) The area of an ellipse with axes of length $2a$ and $2b$ is given by the formula $A = \pi ab$. Approximate the percent change in the area when a increases by 2% and b increases by 1.5%.

43) The period T of a **simple pendulum** with small oscillations is calculated from the formula $T = 2\pi\sqrt{\frac{L}{g}}$, where L is the length of the pendulum and g is the acceleration resulting from gravity. Suppose that L and g have errors of, at most, 0.5% and 0.1%, respectively. Use differentials to approximate the maximum percentage error in the calculated value of T.

Answer

0.3%

44) **Electrical power** P is given by $P = \frac{V^2}{R}$, where V is the voltage and R is the resistance. Approximate the maximum percentage error in calculating power if $120V$ is applied to a $2000 - \Omega$ resistor and the possible percent errors in measuring V and R are 3% and 4%, respectively.

For exercises 45 - 49, find the linear approximation of each function at the indicated point.

45) $f(x, y) = x\sqrt{y}, \quad P(1, 4)$

Answer

$L(x, y) = 2x + \frac{1}{4}y - 1$

46) $f(x, y) = e^x \cos y; \quad P(0, 0)$

47) $f(x, y) = \arctan(x + 2y), \quad P(1, 0)$

Answer

$L(x, y) = \frac{1}{2}x + y + \frac{1}{4}\pi - \frac{1}{2}$

48) $f(x, y) = \sqrt{20 - x^2 - 7y^2}, \quad P(2, 1)$

49) $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}, \quad P(3, 2, 6)$

Answer

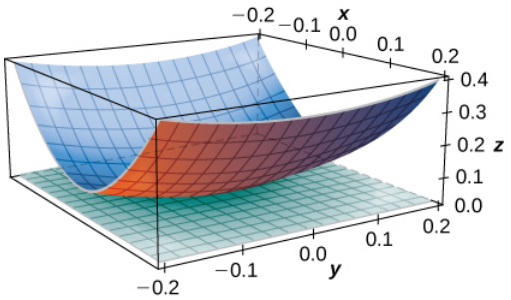
$L(x, y, z) = \frac{3}{7}x + \frac{2}{7}y + \frac{6}{7}z$

50) [T] Find the equation of the tangent plane to the surface $f(x, y) = x^2 + y^2$ at point $(1, 2, 5)$, and graph the surface and the tangent plane at the point.

51) [T] Find the equation for the tangent plane to the surface at the indicated point, and graph the surface and the tangent plane: $z = \ln(10x^2 + 2y^2 + 1), \quad P(0, 0, 0)$.

Answer

$z = 0$



52) [T] Find the equation of the tangent plane to the surface $z = f(x, y) = \sin(x + y^2)$ at point $\left(\frac{\pi}{4}, 0, \frac{\sqrt{2}}{2}\right)$, and graph the surface and the tangent plane.

Contributors

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.
- Paul Seeburger (Monroe Community College) created all but part e of exercise 31.

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14.5: The Chain Rule for Multivariable Functions

Learning Objectives

- State the chain rules for one or two independent variables.
- Use tree diagrams as an aid to understanding the chain rule for several independent and intermediate variables.
- Perform implicit differentiation of a function of two or more variables.

In single-variable calculus, we found that one of the most useful differentiation rules is the chain rule, which allows us to find the derivative of the composition of two functions. The same thing is true for multivariable calculus, but this time we have to deal with more than one form of the chain rule. In this section, we study extensions of the chain rule and learn how to take derivatives of compositions of functions of more than one variable.

Chain Rules for One or Two Independent Variables

Recall that the chain rule for the derivative of a composite of two functions can be written in the form

$$\frac{d}{dx}\Big(f(g(x))\Big) = f'\Big(g(x)\Big)g'(x).$$

In this equation, both $f(x)$ and $g(x)$ are functions of one variable. Now suppose that f is a function of two variables and g is a function of one variable. Or perhaps they are both functions of two variables, or even more. How would we calculate the derivative in these cases? The following theorem gives us the answer for the case of one independent variable.

Theorem 14.5.1: Chain Rule for One Independent Variable

Suppose that $x = g(t)$ and $y = h(t)$ are differentiable functions of t and $z = f(x, y)$ is a differentiable function of x and y. Then $z = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}, \tag{14.5.1}$$

where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y).

Proof

The proof of this theorem uses the definition of differentiability of a function of two variables. Suppose that f is differentiable at the point $P(x_0, y_0)$, where $x_0 = g(t_0)$ and $y_0 = h(t_0)$ for a fixed value of t_0. We wish to prove that $z = f\big(x(t), y(t)\big)$ is differentiable at $t = t_0$ and that Equation 14.5.1 holds at that point as well.

Since f is differentiable at P, we know that

$$z(t) = f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + E(x, y),$$

where

$$\lim_{(x,y)\to(x_0,y_0)} \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0.$$

We then subtract $z_0 = f(x_0, y_0)$ from both sides of this equation:

$$\begin{aligned} z(t) - z(t_0) &= f(x(t), y(t)) - f(x(t_0), y(t_0)) \\ &= f_x(x_0, y_0)(x(t) - x(t_0)) + f_y(x_0, y_0)(y(t) - y(t_0)) + E(x(t), y(t)). \end{aligned}$$

Next, we divide both sides by $t - t_0$:

$$\frac{z(t) - z(t_0)}{t - t_0} = f_x(x_0, y_0)\frac{x(t) - x(t_0)}{t - t_0} + f_y(x_0, y_0)\frac{y(t) - y(t_0)}{t - t_0} + \frac{E(x(t), y(t))}{t - t_0}.$$

Then we take the limit as t approaches t_0:

$$\lim_{t\to t_0}\frac{z(t)-z(t_0)}{t-t_0} = f_x(x_0,y_0)\lim_{t\to t_0}\left(\frac{x(t)-x(t_0)}{t-t_0}\right) + f_y(x_0,y_0)\lim_{t\to t_0}\left(\frac{y(t)-y(t_0)}{t-t_0}\right) + \lim_{t\to t_0}\frac{E(x(t),y(t))}{t-t_0}.$$

The left-hand side of this equation is equal to dz/dt, which leads to

$$\frac{dz}{dt} = f_x(x_0,y_0)\frac{dx}{dt} + f_y(x_0,y_0)\frac{dy}{dt} + \lim_{t\to t_0}\frac{E(x(t),y(t))}{t-t_0}.$$

The last term can be rewritten as

$$\begin{aligned}\lim_{t\to t_0}\frac{E(x(t),y(t))}{t-t_0} &= \lim_{t\to t_0}\frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}\frac{\sqrt{(x-x_0)^2+(y-y_0)^2}}{t-t_0})\\ &= \lim_{t\to t_0}\left(\frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}\right)\lim_{t\to t_0}\left(\frac{\sqrt{(x-x_0)^2+(y-y_0)^2}}{t-t_0}\right).\end{aligned}$$

As t approaches t_0, $\big(x(t),y(t)\big)$ approaches $\big(x(t_0),y(t_0)\big)$, so we can rewrite the last product as

$$\lim_{(x,y)\to(x_0,y_0)}\frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}\lim_{(x,y)\to(x_0,y_0)}\left(\frac{\sqrt{(x-x_0)^2+(y-y_0)^2}}{t-t_0}\right).$$

Since the first limit is equal to zero, we need only show that the second limit is finite:

$$\begin{aligned}\lim_{(x,y)\to(x_0,y_0)}\frac{\sqrt{(x-x_0)^2+(y-y_0)^2}}{t-t_0} &= \lim_{(x,y)\to(x_0,y_0)}\sqrt{\frac{(x-x_0)^2+(y-y_0)^2}{(t-t_0)^2}}\\ &= \lim_{(x,y)\to(x_0,y_0)}\sqrt{\left(\frac{x-x_0}{t-t_0}\right)^2+\left(\frac{y-y_0}{t-t_0}\right)^2}\\ &= \sqrt{\left[\lim_{(x,y)\to(x_0,y_0)}\left(\frac{x-x_0}{t-t_0}\right)\right]^2+\left[\lim_{(x,y)\to(x_0,y_0)}\left(\frac{y-y_0}{t-t_0}\right)\right]^2}.\end{aligned}$$

Since $x(t)$ and $y(t)$ are both differentiable functions of t, both limits inside the last radical exist. Therefore, this value is finite. This proves the chain rule at $t=t_0$; the rest of the theorem follows from the assumption that all functions are differentiable over their entire domains.

□

Closer examination of Equation 14.5.1 reveals an interesting pattern. The first term in the equation is $\frac{\partial f}{\partial x}\cdot\frac{dx}{dt}$ and the second term is $\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$. Recall that when multiplying fractions, cancelation can be used. If we treat these derivatives as fractions, then each product "simplifies" to something resembling $\partial f/dt$. The variables x and y that disappear in this simplification are often called **intermediate variables**: they are independent variables for the function f, but are dependent variables for the variable t. Two terms appear on the right-hand side of the formula, and f is a function of two variables. This pattern works with functions of more than two variables as well, as we see later in this section.

✔ Example 14.5.1: Using the Chain Rule

Calculate dz/dt for each of the following functions:

a. $z=f(x,y)=4x^2+3y^2, \quad x=x(t)=\sin t, \quad y=y(t)=\cos t$
b. $z=f(x,y)=\sqrt{x^2-y^2}, \quad x=x(t)=e^{2t}, \quad y=y(t)=e^{-t}$

Solution

a. To use the chain rule, we need four quantities—$\partial z/\partial x,\ \partial z/\partial y,\ dx/dt$, and dy/dt:

- $\frac{\partial z}{\partial x} = 8x$
- $\frac{dx}{dt} = \cos t$
- $\frac{\partial z}{\partial y} = 6y$
- $\frac{dy}{dt} = -\sin t$

Now, we substitute each of these into Equation 14.5.1:

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} \\ &= (8x)(\cos t) + (6y)(-\sin t) \\ &= 8x\cos t - 6y\sin t.\end{aligned}$$

This answer has three variables in it. To reduce it to one variable, use the fact that $x(t) = \sin t$ and $y(t) = \cos t$. We obtain

$$\begin{aligned}\frac{dz}{dt} &= 8x\cos t - 6y\sin t \\ &= 8(\sin t)\cos t - 6(\cos t)\sin t \\ &= 2\sin t\cos t.\end{aligned}$$

This derivative can also be calculated by first substituting $x(t)$ and $y(t)$ into $f(x, y)$, then differentiating with respect to t:

$$\begin{aligned}z = f(x, y) &= f\big(x(t), y(t)\big) \\ &= 4(x(t))^2 + 3(y(t))^2 \\ &= 4\sin^2 t + 3\cos^2 t.\end{aligned}$$

Then

$$\begin{aligned}\frac{dz}{dt} &= 2(4\sin t)(\cos t) + 2(3\cos t)(-\sin t) \\ &= 8\sin t\cos t - 6\sin t\cos t \\ &= 2\sin t\cos t,\end{aligned}$$

which is the same solution. However, it may not always be this easy to differentiate in this form.

b. To use the chain rule, we again need four quantities—$\partial z/\partial x, \partial z/dy, dx/dt$, and dy/dt :

- $\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2 - y^2}}$
- $\frac{dx}{dt} = 2e^{2t}$
- $\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{x^2 - y^2}}$
- $\frac{dy}{dt} = -e^{-t}.$

We substitute each of these into Equation 14.5.1:

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} \\ &= \left(\frac{x}{\sqrt{x^2 - y^2}}\right)(2e^{2t}) + \left(\frac{-y}{\sqrt{x^2 - y^2}}\right)(-e^{-t}) \\ &= \frac{2xe^{2t} - ye^{-t}}{\sqrt{x^2 - y^2}}.\end{aligned}$$

To reduce this to one variable, we use the fact that $x(t) = e^{2t}$ and $y(t) = e^{-t}$. Therefore,

$$\begin{aligned}\frac{dz}{dt} &= \frac{2xe^2t + ye^{-t}}{\sqrt{x^2 - y^2}} \\ &= \frac{2(e^{2t})e^{2t} + (e^{-t})e^{-t}}{\sqrt{e^{4t} - e^{-2t}}} \\ &= \frac{2e^{4t} + e^{-2t}}{\sqrt{e^{4t} - e^{-2t}}}.\end{aligned}$$

To eliminate negative exponents, we multiply the top by e^{2t} and the bottom by $\sqrt{e^{4t}}$:

$$\begin{aligned}\frac{dz}{dt} &= \frac{2e^{4t} + e^{-2t}}{\sqrt{e^{4t} - e^{-2t}}} \cdot \frac{e^{2t}}{\sqrt{e^{4t}}} \\ &= \frac{2e^{6t} + 1}{\sqrt{e^{8t} - e^{2t}}} \\ &= \frac{2e^{6t} + 1}{\sqrt{e^{2t}(e^{6t} - 1)}} \\ &= \frac{2e^{6t} + 1}{e^t\sqrt{e^{6t} - 1}}.\end{aligned}$$

Again, this derivative can also be calculated by first substituting $x(t)$ and $y(t)$ into $f(x, y)$, then differentiating with respect to t:

$$\begin{aligned}z &= f(x, y) \\ &= f(x(t), y(t)) \\ &= \sqrt{(x(t))^2 - (y(t))^2} \\ &= \sqrt{e^{4t} - e^{-2t}} \\ &= (e^{4t} - e^{-2t})^{1/2}.\end{aligned}$$

Then

$$\begin{aligned}\frac{dz}{dt} &= \frac{1}{2}(e^{4t} - e^{-2t})^{-1/2}\left(4e^{4t} + 2e^{-2t}\right) \\ &= \frac{2e^{4t} + e^{-2t}}{\sqrt{e^{4t} - e^{-2t}}}.\end{aligned}$$

This is the same solution.

? Exercise 14.5.1

Calculate dz/dt given the following functions. Express the final answer in terms of t.

$$\begin{aligned}z = f(x, y) &= x^2 - 3xy + 2y^2, \\ x = x(t) &= 3\sin 2t, \\ y = y(t) &= 4\cos 2t\end{aligned}$$

Hint

Calculate $\partial z/\partial x, \partial z/dy, dx/dt$, and dy/dt, then use Equation 14.5.1.

Answer

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= (2x - 3y)(6\cos 2t) + (-3x + 4y)(-8\sin 2t) \\ &= -92\sin 2t\cos 2t - 72(\cos^2 2t - \sin^2 2t) \\ &= -46\sin 4t - 72\cos 4t.\end{aligned}$$

It is often useful to create a visual representation of Equation 14.5.1 for the chain rule. This is called a **tree diagram** for the chain rule for functions of one variable and it provides a way to remember the formula (Figure 14.5.1). This diagram can be expanded for functions of more than one variable, as we shall see very shortly.

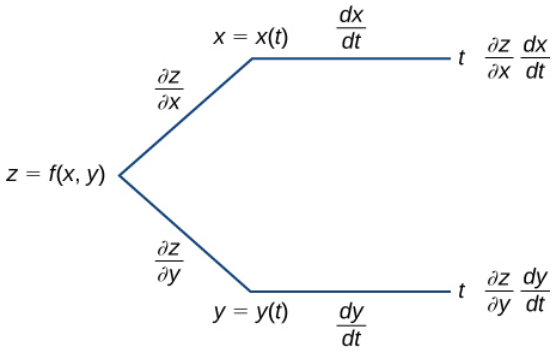


Figure 14.5.1: Tree diagram for the case $\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$.

In this diagram, the leftmost corner corresponds to $z = f(x, y)$. Since f has two **independent variables**, there are two lines coming from this corner. The upper branch corresponds to the variable x and the lower branch corresponds to the variable y. Since each of these variables is then dependent on one variable t, one branch then comes from x and one branch comes from y. Last, each of the branches on the far right has a label that represents the path traveled to reach that branch. The top branch is reached by following the x branch, then the t branch; therefore, it is labeled $(\partial z/\partial x) \times (dx/dt)$. The bottom branch is similar: first the y branch, then the t branch. This branch is labeled $(\partial z/\partial y) \times (dy/dt)$. To get the formula for dz/dt, add all the terms that appear on the rightmost side of the diagram. This gives us Equation.

In Chain Rule for Two Independent Variables, $z = f(x, y)$ is a function of x and y, and both $x = g(u, v)$ and $y = h(u, v)$ are functions of the independent variables u and v.

Theorem 14.5.2: Chain Rule for Two Independent Variables

Suppose $x = g(u, v)$ and $y = h(u, v)$ are differentiable functions of u and v, and $z = f(x, y)$ is a differentiable function of x and y. Then, $z = f(g(u, v), h(u, v))$ is a differentiable function of u and v, and

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \tag{14.5.2}$$

and

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}. \tag{14.5.3}$$

We can draw a tree diagram for each of these formulas as well as follows.

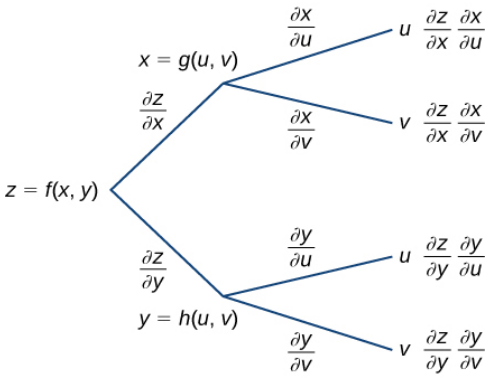


Figure 14.5.2: Tree diagram for $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$ and $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$.

To derive the formula for $\partial z/\partial u$, start from the left side of the diagram, then follow only the branches that end with u and add the terms that appear at the end of those branches. For the formula for $\partial z/\partial v$, follow only the branches that end with v and add the terms that appear at the end of those branches.

There is an important difference between these two chain rule theorems. In Chain Rule for One Independent Variable, the left-hand side of the formula for the derivative is not a partial derivative, but in Chain Rule for Two Independent Variables it is. The reason is that, in Chain Rule for One Independent Variable, z is ultimately a function of t alone, whereas in Chain Rule for Two Independent Variables, z is a function of both u and v.

Example 14.5.2: Using the Chain Rule for Two Variables

Calculate $\partial z/\partial u$ and $\partial z/\partial v$ using the following functions:

$$z = f(x,y) = 3x^2 - 2xy + y^2,\ x = x(u,v) = 3u + 2v,\ y = y(u,v) = 4u - v.$$

Solution

To implement the chain rule for two variables, we need six partial derivatives—$\partial z/\partial x$, $\partial z/\partial y$, $\partial x/\partial u$, $\partial x/\partial v$, $\partial y/\partial u$, and $\partial y/\partial v$:

$$\frac{\partial z}{\partial x} = 6x - 2y \qquad \frac{\partial z}{\partial y} = -2x + 2y$$
$$\frac{\partial x}{\partial u} = 3 \qquad \frac{\partial x}{\partial v} = 2$$
$$\frac{\partial y}{\partial u} = 4 \qquad \frac{\partial y}{\partial v} = -1.$$

To find $\partial z/\partial u$, we use Equation 14.5.2:

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}\\ &= 3(6x - 2y) + 4(-2x + 2y)\\ &= 10x + 2y.\end{aligned}$$

Next, we substitute $x(u,v) = 3u + 2v$ and $y(u,v) = 4u - v$:

$$\begin{aligned}\frac{\partial z}{\partial u} &= 10x + 2y\\ &= 10(3u + 2v) + 2(4u - v)\\ &= 38u + 18v.\end{aligned}$$

To find $\partial z/\partial v$, we use Equation 14.5.3:

$$\begin{aligned}\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}\\ &= 2(6x - 2y) + (-1)(-2x + 2y)\\ &= 14x - 6y.\end{aligned}$$

Then we substitute $x(u,v) = 3u + 2v$ and $y(u,v) = 4u - v$:

$$\begin{aligned}\frac{\partial z}{\partial v} &= 14x - 6y\\ &= 14(3u + 2v) - 6(4u - v)\\ &= 18u + 34v\end{aligned}$$

Exercise 14.5.2

Calculate $\partial z/\partial u$ and $\partial z/\partial v$ given the following functions:

$$z = f(x,y) = \frac{2x - y}{x + 3y}, \quad x(u,v) = e^{2u}\cos 3v, \quad y(u,v) = e^{2u}\sin 3v.$$

Hint

Calculate $\partial z/\partial x,\ \partial z/\partial y,\ \partial x/\partial u,\ \partial x/\partial v,\ \partial y/\partial u$, and $\partial y/\partial v$, then use Equation 14.5.2 and Equation 14.5.3.

Answer

$$\frac{\partial z}{\partial u} = 0, \quad \frac{\partial z}{\partial v} = \frac{-21}{(3\sin 3v + \cos 3v)^2}$$

The Generalized Chain Rule

Now that we've see how to extend the original chain rule to functions of two variables, it is natural to ask: Can we extend the rule to more than two variables? The answer is yes, as the *generalized chain rule* states.

Theorem 14.5.3: Generalized Chain Rule

Let $w = f(x_1, x_2, \ldots, x_m)$ be a differentiable function of m independent variables, and for each $i \in 1, \ldots, m$, let $x_i = x_i(t_1, t_2, \ldots, t_n)$ be a differentiable function of n independent variables. Then

$$\frac{\partial w}{\partial t_j} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial w}{\partial x_m}\frac{\partial x_m}{\partial t_j}$$

for any $j \in 1, 2, \ldots, n$.

In the next example we calculate the derivative of a function of three independent variables in which each of the three variables is dependent on two other variables.

Example 14.5.3: Using the Generalized Chain Rule

Calculate $\partial w/\partial u$ and $\partial w/\partial v$ using the following functions:

$$\begin{aligned} w &= f(x, y, z) = 3x^2 - 2xy + 4z^2 \\ x &= x(u, v) = e^u \sin v \\ y &= y(u, v) = e^u \cos v \\ z &= z(u, v) = e^u. \end{aligned}$$

Solution

The formulas for $\partial w/\partial u$ and $\partial w/\partial v$ are

$$\begin{aligned} \frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u} \\ \frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}. \end{aligned}$$

Therefore, there are nine different partial derivatives that need to be calculated and substituted. We need to calculate each of them:

$$\begin{aligned} \frac{\partial w}{\partial x} &= 6x - 2y \quad \frac{\partial w}{\partial y} = -2x \quad \frac{\partial w}{\partial z} = 8z \\ \frac{\partial x}{\partial u} &= e^u \sin v \quad \frac{\partial y}{\partial u} = e^u \cos v \quad \frac{\partial z}{\partial u} = e^u \\ \frac{\partial x}{\partial v} &= e^u \cos v \quad \frac{\partial y}{\partial v} = -e^u \sin v \quad \frac{\partial z}{\partial v} = 0. \end{aligned}$$

Now, we substitute each of them into the first formula to calculate $\partial w/\partial u$:

$$\begin{aligned} \frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u} \\ &= (6x - 2y)e^u \sin v - 2xe^u \cos v + 8ze^u, \end{aligned}$$

then substitute $x(u, v) = e^u \sin v$, $y(u, v) = e^u \cos v$, and $z(u, v) = e^u$ into this equation:

$$\begin{aligned}\frac{\partial w}{\partial u} &= (6x-2y)e^u \sin v - 2xe^u \cos v + 8ze^u \\ &= (6e^u \sin v - 2eu\cos v)e^u \sin v - 2(e^u \sin v)e^u \cos v + 8e^{2u} \\ &= 6e^{2u}\sin^2 v - 4e^{2u}\sin v \cos v + 8e^{2u} \\ &= 2e^{2u}(3\sin^2 v - 2\sin v \cos v + 4).\end{aligned}$$

Next, we calculate $\partial w/\partial v$:

$$\begin{aligned}\frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v} \\ &= (6x-2y)e^u \cos v - 2x(-e^u \sin v) + 8z(0),\end{aligned}$$

then we substitute $x(u,v) = e^u \sin v$, $y(u,v) = e^u \cos v$, and $z(u,v) = e^u$ into this equation:

$$\begin{aligned}\frac{\partial w}{\partial v} &= (6x-2y)e^u \cos v - 2x(-e^u \sin v) \\ &= (6e^u \sin v - 2e^u \cos v)e^u \cos v + 2(e^u \sin v)(e^u \sin v) \\ &= 2e^{2u}\sin^2 v + 6e^{2u}\sin v \cos v - 2e^{2u}\cos^2 v \\ &= 2e^{2u}(\sin^2 v + \sin v \cos v - \cos^2 v).\end{aligned}$$

? Exercise 14.5.3

Calculate $\partial w/\partial u$ and $\partial w/\partial v$ given the following functions:

$$\begin{aligned}w = f(x,y,z) &= \frac{x+2y-4z}{2x-y+3z} \\ x = x(u,v) &= e^{2u}\cos 3v \\ y = y(u,v) &= e^{2u}\sin 3v \\ z = z(u,v) &= e^{2u}.\end{aligned}$$

Hint

Calculate nine partial derivatives, then use the same formulas from Example 14.5.3.

Answer

$$\frac{\partial w}{\partial u} = 0$$

$$\frac{\partial w}{\partial v} = \frac{15 - 33\sin 3v + 6\cos 3v}{(3 + 2\cos 3v - \sin 3v)^2}$$

✓ Example 14.5.4: Drawing a Tree Diagram

Create a tree diagram for the case when

$$w = f(x,y,z), \quad x = x(t,u,v), \quad y = y(t,u,v), \quad z = z(t,u,v)$$

and write out the formulas for the three partial derivatives of w.

Solution

Starting from the left, the function f has three independent variables: x, y, and z. Therefore, three branches must be emanating from the first node. Each of these three branches also has three branches, for each of the variables t, u, and v.

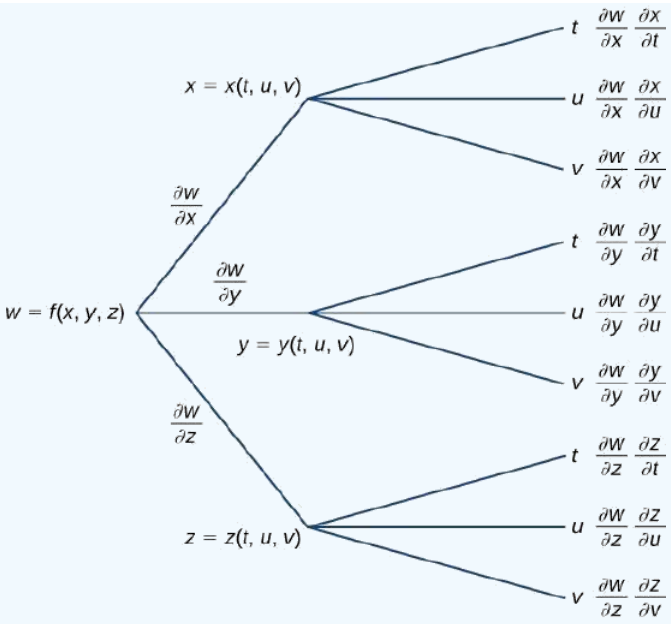


Figure 14.5.3: Tree diagram for a function of three variables, each of which is a function of three independent variables.

The three formulas are

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$
$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$
$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}.$$

Exercise 14.5.4

Create a tree diagram for the case when

$$w = f(x, y), \quad x = x(t, u, v), \quad y = y(t, u, v)$$

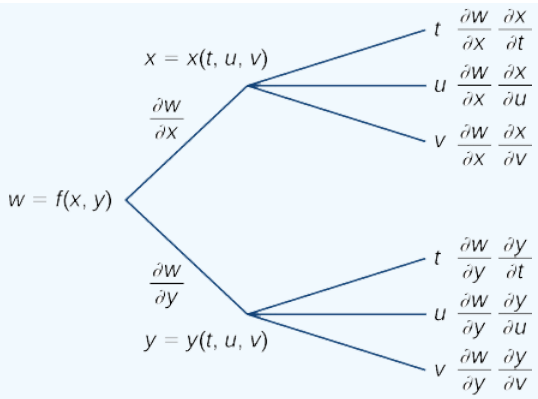
and write out the formulas for the three partial derivatives of w.

Hint

Determine the number of branches that emanate from each node in the tree.

Answer

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$$
$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u}$$
$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v}$$



Implicit Differentiation

Recall from implicit differentiation provides a method for finding dy/dx when y is defined implicitly as a function of x. The method involves differentiating both sides of the equation defining the function with respect to x, then solving for dy/dx. Partial derivatives provide an alternative to this method.

Consider the ellipse defined by the equation $x^2 + 3y^2 + 4y - 4 = 0$ as follows.

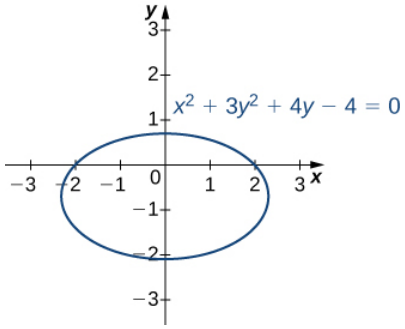


Figure 14.5.4: Graph of the ellipse defined by $x^2 + 3y^2 + 4y - 4 = 0$.

This equation implicitly defines y as a function of x. As such, we can find the derivative dy/dx using the method of implicit differentiation:

$$\begin{aligned} \frac{d}{dx}(x^2 + 3y^2 + 4y - 4) &= \frac{d}{dx}(0) \\ 2x + 6y\frac{dy}{dx} + 4\frac{dy}{dx} &= 0 \\ (6y + 4)\frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= -\frac{x}{3y + 2} \end{aligned}$$

We can also define a function $z = f(x, y)$ by using the left-hand side of the equation defining the ellipse. Then $f(x, y) = x^2 + 3y^2 + 4y - 4$. The ellipse $x^2 + 3y^2 + 4y - 4 = 0$ can then be described by the equation $f(x, y) = 0$. Using this function and the following theorem gives us an alternative approach to calculating dy/dx.

Theorem 14.5.4: Implicit Differentiation of a Function of Two or More Variables

Suppose the function $z = f(x, y)$ defines y implicitly as a function $y = g(x)$ of x via the equation $f(x, y) = 0$. Then

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} \tag{14.5.4}$$

provided $f_y(x, y) \neq 0$.

If the equation $f(x, y, z) = 0$ defines z implicitly as a differentiable function of x and y, then

$$\frac{dz}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial z} \quad \text{and} \quad \frac{dz}{dy} = -\frac{\partial f/\partial y}{\partial f/\partial z} \tag{14.5.5}$$

as long as $f_z(x, y, z) \neq 0$.

Equation 14.5.4 is a direct consequence of Equation 14.5.2. In particular, if we assume that y is defined implicitly as a function of x via the equation $f(x, y) = 0$, we can apply the chain rule to find dy/dx :

$$\begin{aligned} \frac{d}{dx} f(x, y) &= \frac{d}{dx}(0) \\ \frac{\partial f}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} &= 0 \\ \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} &= 0. \end{aligned}$$

Solving this equation for dy/dx gives Equation 14.5.4. Equation 14.5.4 can be derived in a similar fashion.

Let's now return to the problem that we started before the previous theorem. Using Theorem 14.5.4 and the function $f(x, y) = x^2 + 3y^2 + 4y - 4$, we obtain

$$\begin{aligned} \frac{\partial f}{\partial x} &= 2x \\ \frac{\partial f}{\partial y} &= 6y + 4. \end{aligned}$$

Then Equation 14.5.4 gives

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{2x}{6y + 4} = -\frac{x}{3y + 2},$$

which is the same result obtained by the earlier use of implicit differentiation.

Example 14.5.5: Implicit Differentiation by Partial Derivatives

a. Calculate dy/dx if y is defined implicitly as a function of x via the equation $3x^2 - 2xy + y^2 + 4x - 6y - 11 = 0$. What is the equation of the tangent line to the graph of this curve at point $(2, 1)$?

b. Calculate $\partial z/\partial x$ and $\partial z/\partial y$, given $x^2 e^y - yze^x = 0$.

Solution

a. Set $f(x, y) = 3x^2 - 2xy + y^2 + 4x - 6y - 11 = 0$, then calculate f_x and f_y : $f_x(x, y) = 6x - 2y + 4$ and $f_y(x, y) = -2x + 2y - 6$.

The derivative is given by

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = \frac{6x - 2y + 4}{-2x + 2y - 6} = \frac{3x - y + 2}{x - y + 3}.$$

The slope of the tangent line at point $(2, 1)$ is given by

$$\left.\frac{dy}{dx}\right|_{(x,y)=(2,1)} = \frac{3(2) - 1 + 2}{2 - 1 + 3} = \frac{7}{4}$$

To find the equation of the tangent line, we use the point-slope form (Figure 14.5.5):

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - 1 &= \frac{7}{4}(x - 2) \\ y &= \frac{7}{4}x - \frac{7}{2} + 1 \\ y &= \frac{7}{4}x - \frac{5}{2}. \end{aligned}$$

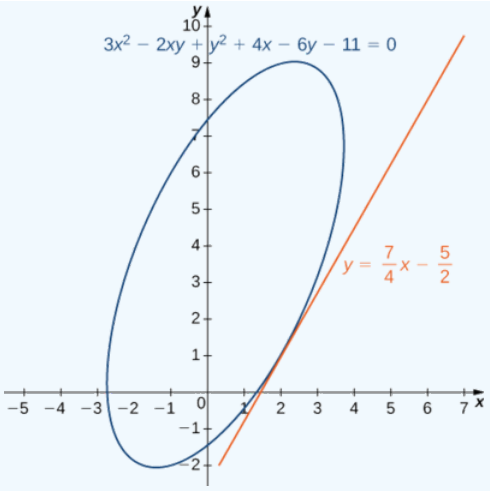


Figure 14.5.5: Graph of the rotated ellipse defined by $3x^2 - 2xy + y^2 + 4x - 6y - 11 = 0$.

b. We have $f(x, y, z) = x^2e^y - yze^x$. Therefore,

$$\frac{\partial f}{\partial x} = 2xe^y - yze^x$$
$$\frac{\partial f}{\partial y} = x^2e^y - ze^x$$
$$\frac{\partial f}{\partial z} = -ye^x$$

Using Equation 14.5.5,

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial y} \qquad \text{and} \qquad \frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z}$$
$$= -\frac{2xe^y - yze^x}{-ye^x} \qquad = -\frac{x^2e^y - ze^x}{-ye^x}$$
$$= \frac{2xe^y - yze^x}{ye^x} \qquad = \frac{x^2e^y - ze^x}{ye^x}$$

? Exercise 14.5.5

Find dy/dx if y is defined implicitly as a function of x by the equation $x^2 + xy - y^2 + 7x - 3y - 26 = 0$. What is the equation of the tangent line to the graph of this curve at point $(3, -2)$?

Hint

Calculate $\partial f/dx$ and $\partial f/dy$, then use Equation 14.5.4.

Solution

$$\frac{dy}{dx} = \frac{2x + y + 7}{2y - x + 3}\Big|_{(3,-2)} = \frac{2(3) + (-2) + 7}{2(-2) - (3) + 3} = -\frac{11}{4}$$

Equation of the tangent line: $y = -\frac{11}{4}x + \frac{25}{4}$

Key Concepts

- The chain rule for functions of more than one variable involves the partial derivatives with respect to all the independent variables.
- Tree diagrams are useful for deriving formulas for the chain rule for functions of more than one variable, where each independent variable also depends on other variables.

Key Equations

- **Chain rule, one independent variable**

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

- **Chain rule, two independent variables**

$$\frac{dz}{du} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} \frac{dz}{dv} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

- **Generalized chain rule**

$$\frac{\partial w}{\partial t_j} = \frac{\partial w}{\partial x_1} \frac{\partial x_1}{\partial t_j} + \frac{\partial w}{\partial x_2} \frac{\partial x_1}{\partial t_j} + \cdots + \frac{\partial w}{\partial x_m} \frac{\partial x_m}{\partial t_j}$$

Glossary

generalized chain rule

the chain rule extended to functions of more than one independent variable, in which each independent variable may depend on one or more other variables

intermediate variable

given a composition of functions (e.g., $f(x(t), y(t))$), the intermediate variables are the variables that are independent in the outer function but dependent on other variables as well; in the function $f(x(t), y(t))$, the variables x and y are examples of intermediate variables

tree diagram

illustrates and derives formulas for the generalized chain rule, in which each independent variable is accounted for

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14.5E: Exercises for Section 14.5

In exercises 1 - 6, use the information provided to solve the problem.

1) Let $w(x,y,z) = xy\cos z$, where $x = t, y = t^2$, and $z = \arcsin t$. Find $\frac{dw}{dt}$.

Answer

$$\frac{dw}{dt} = y\cos z + x\cos z(2t) - \frac{xy\sin z}{\sqrt{1-t^2}}$$

2) Let $w(t,v) = e^{tv}$ where $t = r + s$ and $v = rs$. Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$.

3) If $w = 5x^2 + 2y^2, \quad x = -3u + v$, and $y = u - 4v$, find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$.

Answer

$$\frac{\partial w}{\partial u} = -30x + 4y \quad = \quad -30(-3u+v) + 4(u-4v) \quad = \quad 90u - 30v + 4u - 16v \quad = \quad 94u - 46v,$$
$$\frac{\partial w}{\partial v} = 10x - 16y \quad = \quad 10(-3u+v) - 16(u-4v) \quad = \quad -30u + 10v - 16u + 64v \quad = \quad -46u + 74v$$

4) If $w = xy^2, x = 5\cos(2t)$, and $y = 5\sin(2t)$, find $\frac{\partial w}{\partial t}$.

5) If $f(x,y) = xy, x = r\cos\theta$, and $y = r\sin\theta$, find $\frac{\partial f}{\partial r}$ and express the answer in terms of r and θ.

Answer

$$\frac{\partial f}{\partial r} = r\sin(2\theta)$$

6) Suppose $f(x,y) = x + y, u = e^x\sin y, \quad x = t^2$ and $y = \pi t$, where $x = r\cos\theta$ and $y = r\sin\theta$. Find $\frac{\partial f}{\partial \theta}$.

In exercises 7 - 12, find $\frac{dz}{dt}$ in two ways, first using the chain rule and then by direct substitution.

7) $z = x^2 + y^2, \quad x = t, y = t^2$

Answer

$$\frac{dz}{dt} = 2t + 4t^3$$

8) $z = \sqrt{x^2 + y^2}, \quad y = t^2, x = t$

9) $z = xy, \quad x = 1 - \sqrt{t}, y = 1 + \sqrt{t}$

Answer

$$\frac{dz}{dt} = -1$$

10) $z = \frac{x}{y}, \quad x = e^t, y = 2e^t$

11) $z = \ln(x + y), \quad x = e^t, y = e^t$

Answer

$$\frac{dz}{dt} = 1$$

12) $z = x^4, \quad x = t, y = t$

13) Let $w(x,y,z) = x^2 + y^2 + z^2, \quad x = cost, y = sint$, and $z = e^t$. Express w as a function of t and find $\frac{dw}{dt}$ directly. Then, find $\frac{dw}{dt}$ using the chain rule.

Answer

$\frac{dw}{dt} = 2e^{2t}$ in both cases

14) Let $z = x^2y$, where $x = t^2$ and $y = t^3$. Find $\frac{dz}{dt}$.

15) Let $u = e^x \sin y$, where $x = -\ln 2t$ and $y = \pi t$. Find $\frac{du}{dt}$ when $x = \ln 2$ and $y = \frac{\pi}{4}$.

Answer

$\frac{du}{dt} = \sqrt{2}(\pi - 4)$

In exercises 16 - 33, find $\frac{dy}{dx}$ using partial derivatives.

16) $\sin(6x) + \tan(8y) + 5 = 0$

17) $x^3 + y^2x - 3 = 0$

Answer

$\frac{dy}{dx} = -\frac{3x^2 + y^2}{2xy}$

18) $\sin(x + y) + \cos(x - y) = 4$

19) $x^2 - 2xy + y^4 = 4$

Answer

$\frac{dy}{dx} = \frac{y - x}{-x + 2y^3}$

20) $xe^y + ye^x - 2x^2y = 0$

21) $x^{2/3} + y^{2/3} = a^{2/3}$

Answer

$\frac{dy}{dx} = -\sqrt[3]{\frac{y}{x}}$

22) $x\cos(xy) + y\cos x = 2$

23) $e^{xy} + ye^y = 1$

Answer

$\frac{dy}{dx} = -\frac{ye^{xy}}{xe^{xy} + e^y(1 + y)}$

24) $x^2y^3 + \cos y = 0$

25) Find $\frac{dz}{dt}$ using the chain rule where $z = 3x^2y^3$, $x = t^4$, and $y = t^2$.

Answer

$\frac{dz}{dt} = 42t^{13}$

26) Let $z = 3\cos x - \sin(xy), x = \frac{1}{t}$, and $y = 3t$. Find $\frac{dz}{dt}$.

27) Let $z = e^{1-xy}$, $x = t^{1/3}$, and $y = t^3$. Find $\frac{dz}{dt}$.

Answer

$\frac{dz}{dt} = -\frac{10}{3}t^{7/3} \times e^{1-t^{10/3}}$

28) Find $\frac{dz}{dt}$ by the chain rule where $z = \cosh^2(xy)$, $x = \frac{1}{2}t$, and $y = e^t$.

29) Let $z = \dfrac{x}{y},\ x = 2\cos u$, and $y = 3\sin v$. Find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

Answer

$\dfrac{\partial z}{\partial u} = \dfrac{-2\sin u}{3\sin v}$ and $\dfrac{\partial z}{\partial v} = \dfrac{-2\cos u\cos v}{3\sin^2 v}$

30) Let $z = e^{x^2y}$, where $x = \sqrt{uv}$ and $y = \frac{1}{v}$. Find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

31) If $z = xye^{x/y},\ x = r\cos\theta$, and $y = r\sin\theta$, find $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial z}{\partial \theta}$ when $r = 2$ and $\theta = \frac{\pi}{6}$.

Answer

$\dfrac{\partial z}{\partial r} = \sqrt{3}e^{\sqrt{3}}, \dfrac{\partial z}{\partial \theta} = (2 - 4\sqrt{3})e^{\sqrt{3}}$

32) Find $\dfrac{\partial w}{\partial s}$ if $w = 4x + y^2 + z^3,\ x = e^{rs^2},\ y = \ln(\frac{r+s}{t})$, and $z = rst^2$.

33) If $w = \sin(xyz),\ x = 1 - 3t,\ y = e^{1-t}$, and $z = 4t$, find $\dfrac{\partial w}{\partial t}$.

Answer

$\dfrac{\partial w}{\partial t} = -3yz\cos(xyz) - xze^{1-t}\cos(xyz) + 4xy\cos(xyz)$

In exercises 34 - 36, use this information: A function $f(x,y)$ is said to be homogeneous of degree n if $f(tx,ty) = t^n f(x,y)$. For all homogeneous functions of degree n, the following equation is true: $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = nf(x,y)$. Show that the given function is homogeneous and verify that $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = nf(x,y)$.

34) $f(x,y) = 3x^2 + y^2$

35) $f(x,y) = \sqrt{x^2 + y^2}$

Answer

$f(tx,ty) = \sqrt{t^2x^2 + t^2y^2} = t^1 f(x,y), \quad \dfrac{\partial f}{\partial y} = x\frac{1}{2}(x^2+y^2)^{-1/2}\times 2x + y\frac{1}{2}(x^2+y^2)^{-1/2}\times 2y = 1f(x,y)$

36) $f(x,y) = x^2y - 2y^3$

37) The volume of a right circular cylinder is given by $V(x,y) = \pi x^2 y$, where x is the radius of the cylinder and y is the cylinder height. Suppose x and y are functions of t given by $x = \frac{1}{2}t$ and $y = \frac{1}{3}t$ so that x and y are both increasing with time. How fast is the volume increasing when $x = 2$ and $y = 5$? Assume time is measured in seconds.

Answer

\ddfrac$dVdt = \frac{34\pi}{3}$ units^3/s

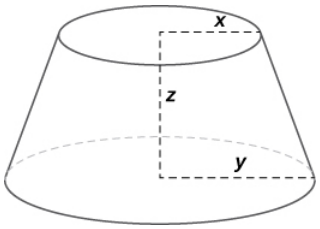
38) The pressure P of a gas is related to the volume and temperature by the formula $PV = kT$, where temperature is expressed in kelvins. Express the pressure of the gas as a function of both V and T. Find $\dfrac{dP}{dt}$ when $k = 1, \dfrac{dV}{dt} = 2$ cm^3/min, $\dfrac{dT}{dt} = 12$ K/min, $V = 20cm^3$, and $T = 20°F$.

39) The radius of a right circular cone is increasing at 3 cm/min whereas the height of the cone is decreasing at 2 cm/min. Find the rate of change of the volume of the cone when the radius is 13 cm and the height is 18 cm.

Answer

$\dfrac{dV}{dt} = \frac{1066\pi}{3}$ cm^3/min

40) The volume of a frustum of a cone is given by the formula $V = \frac{1}{3}\pi z(x^2 + y^2 + xy)$, where x is the radius of the smaller circle, y is the radius of the larger circle, and z is the height of the frustum (see figure). Find the rate of change of the volume of this frustum when $x = 10$ in., $y = 12$ in., and $z = 18$ in.



41) A closed box is in the shape of a rectangular solid with dimensions $x, y,$ and z. (Dimensions are in inches.) Suppose each dimension is changing at the rate of 0.5 in./min. Find the rate of change of the total surface area of the box when $x = 2$ in., $y = 3$ in., and $z = 1$ in.

Answer

$\frac{dA}{dt} = 12\,\text{in.}^2/\text{min}$

42) The total resistance in a circuit that has three individual resistances represented by $x, y,$ and z is given by the formula $R(x, y, z) = \frac{xyz}{yz + xz + xy}$. Suppose at a given time the x resistance is $100\,\Omega$, the y resistance is $200\,\Omega$, and the z resistance is $300\,\Omega$. Also, suppose the x resistance is changing at a rate of $2\,\Omega/\text{min}$, the y resistance is changing at the rate of $1\,\Omega/\text{min}$, and the z resistance has no change. Find the rate of change of the total resistance in this circuit at this time.

43) The temperature T at a point (x, y) is $T(x, y)$ and is measured using the Celsius scale. A fly crawls so that its position after t seconds is given by $x = \sqrt{1 + t}$ and $y = 2 + \frac{1}{3}t$, where x and y are measured in centimeters. The temperature function satisfies $T_x(2, 3) = 4$ and $T_y(2, 3) = 3$. How fast is the temperature increasing on the fly's path after 3 sec?

Answer

2°C/sec

44) The x and y components of a fluid moving in two dimensions are given by the following functions: $u(x, y) = 2y$ and $v(x, y) = -2x$ with $x \geq 0$ and $y \geq 0$. The speed of the fluid at the point (x, y) is $s(x, y) = \sqrt{u(x, y)^2 + v(x, y)^2}$. Find $\frac{\partial s}{\partial x}$ and $\frac{\partial s}{\partial y}$ using the chain rule.

45) Let $u = u(x, y, z)$, where $x = x(w, t)$, $y = y(w, t)$, $z = z(w, t)$, $w = w(r, s)$, and $t = t(r, s)$. Use a tree diagram and the chain rule to find an expression for $\frac{\partial u}{\partial r}$.

Answer

$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\left(\frac{\partial x}{\partial w}\frac{\partial w}{\partial r} + \frac{\partial x}{\partial t}\frac{\partial t}{\partial r}\right) + \frac{\partial u}{\partial y}\left(\frac{\partial y}{\partial w}\frac{\partial w}{\partial r} + \frac{\partial y}{\partial t}\frac{\partial t}{\partial r}\right) + \frac{\partial u}{\partial z}\left(\frac{\partial z}{\partial w}\frac{\partial w}{\partial r} + \frac{\partial z}{\partial t}\frac{\partial t}{\partial r}\right)$

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14.6: Directional Derivatives and the Gradient

Learning Objectives

- Determine the directional derivative in a given direction for a function of two variables.
- Determine the gradient vector of a given real-valued function.
- Explain the significance of the gradient vector with regard to direction of change along a surface.
- Use the gradient to find the tangent to a level curve of a given function.
- Calculate directional derivatives and gradients in three dimensions.

A function $z = f(x, y)$ has two partial derivatives: $\partial z/\partial x$ and $\partial z/\partial y$. These derivatives correspond to each of the independent variables and can be interpreted as instantaneous rates of change (that is, as slopes of a tangent line). For example, $\partial z/\partial x$ represents the slope of a tangent line passing through a given point on the surface defined by $z = f(x, y)$, assuming the tangent line is parallel to the x-**axis**. Similarly, $\partial z/\partial y$ represents the slope of the tangent line parallel to the y-**axis.** Now we consider the possibility of a tangent line parallel to neither axis.

Directional Derivatives

We start with the graph of a surface defined by the equation $z = f(x, y)$. Given a point (a, b) in the domain of f, we choose a direction to travel from that point. We measure the direction using an angle θ, which is measured counterclockwise in the xy-plane, starting at zero from the positive x-axis (Figure 14.6.1). The distance we travel is h and the direction we travel is given by the unit vector $\vec{\mathbf{u}} = (\cos\theta)\,\hat{\mathbf{i}} + (\sin\theta)\,\hat{\mathbf{j}}$. Therefore, the z-coordinate of the second point on the graph is given by $z = f(a + h\cos\theta, b + h\sin\theta)$.

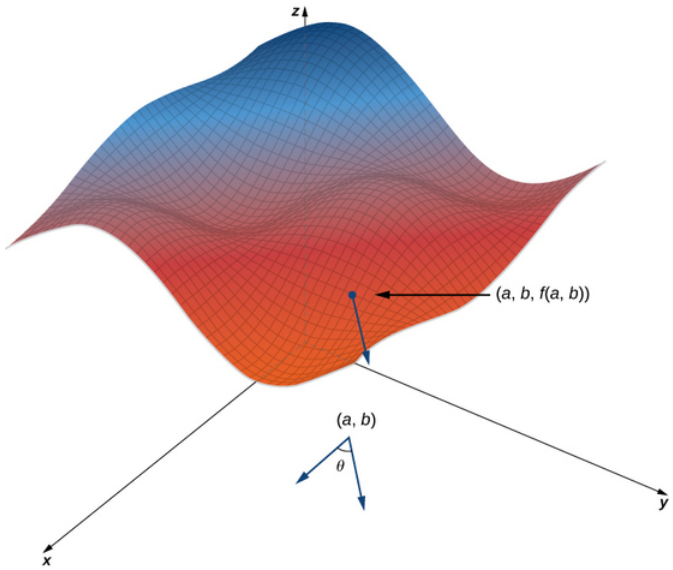


Figure 14.6.1: Finding the directional derivative at a point on the graph of $z = f(x, y)$. The slope of the blue arrow on the graph indicates the value of the directional derivative at that point.

We can calculate the slope of the secant line by dividing the difference in z-values by the length of the line segment connecting the two points in the domain. The length of the line segment is h. Therefore, the slope of the secant line is

$$m_{sec} = \frac{f(a + h\cos\theta, b + h\sin\theta) - f(a, b)}{h}$$

To find the slope of the tangent line in the same direction, we take the limit as h approaches zero.

Definition: Directional Derivatives

Suppose $z = f(x, y)$ is a function of two variables with a domain of D. Let $(a, b) \in D$ and define $\vec{\mathbf{u}} = (\cos\theta)\,\hat{\mathbf{i}} + (\sin\theta)\,\hat{\mathbf{j}}$. Then the directional derivative of f in the direction of $\vec{\mathbf{u}}$ is given by

$$D_{\vec{\mathbf{u}}} f(a,b) = \lim_{h \to 0} \frac{f(a + h\cos\theta, b + h\sin\theta) - f(a,b)}{h} \tag{14.6.1}$$

provided the limit exists.

Equation 14.6.1 provides a formal definition of the directional derivative that can be used in many cases to calculate a directional derivative.

Note that since the point (a,b) is chosen randomly from the domain D of the function f, we can use this definition to find the directional derivative as a function of x and y.

That is,

$$D_{\vec{\mathbf{u}}} f(x,y) = \lim_{h \to 0} \frac{f(x + h\cos\theta, y + h\sin\theta) - f(x,y)}{h} \tag{14.6.2}$$

Example 14.6.1: Finding a Directional Derivative from the Definition

Let $\theta = \arccos(3/5)$. Find the directional derivative $D_{\vec{\mathbf{u}}} f(x,y)$ of $f(x,y) = x^2 - xy + 3y^2$ in the direction of $\vec{\mathbf{u}} = (\cos\theta)\,\hat{\mathbf{i}} + (\sin\theta)\,\hat{\mathbf{j}}$.

Then determine $D_{\vec{\mathbf{u}}} f(-1,2)$.

Solution

First of all, since $\cos\theta = 3/5$ and θ is acute, this implies

$$\sin\theta = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25}} = \frac{4}{5}.$$

Using $f(x,y) = x^2 - xy + 3y^2$, we first calculate $f(x + h\cos\theta, y + h\sin\theta)$:

$$\begin{aligned} f(x + h\cos\theta, y + h\sin\theta) &= (x + h\cos\theta)^2 - (x + h\cos\theta)(y + h\sin\theta) + 3(y + h\sin\theta)^2 \\ &= x^2 + 2xh\cos\theta + h^2\cos^2\theta - xy - xh\sin\theta - yh\cos\theta - h^2\sin\theta\cos\theta \\ &\quad + 3y^2 + 6yh\sin\theta + 3h^2\sin^2\theta \\ &= x^2 + 2xh(\frac{3}{5}) + \frac{9h^2}{25} - xy - \frac{4xh}{5} - \frac{3yh}{5} - \frac{12h^2}{25} \\ &\quad + 3y^2 + 6yh(\frac{4}{5}) + 3h^2(\frac{16}{25}) \\ &= x^2 - xy + 3y^2 + \frac{2xh}{5} + \frac{9h^2}{5} + \frac{21yh}{5}. \end{aligned}$$

We substitute this expression into Equation 14.6.1 with $a = x$ and $b = y$:

$$\begin{aligned} D_{\vec{\mathbf{u}}} f(x,y) &= \lim_{h \to 0} \frac{f(x + h\cos\theta, y + h\sin\theta) - f(x,y)}{h} \\ &= \lim_{h \to 0} \frac{(x^2 - xy + 3y^2 + \frac{2xh}{5} + \frac{9h^2}{5} + \frac{21yh}{5}) - (x^2 - xy + 3y^2)}{h} \\ &= \lim_{h \to 0} \frac{\frac{2xh}{5} + \frac{9h^2}{5} + \frac{21yh}{5}}{h} \\ &= \lim_{h \to 0} \frac{2x}{5} + \frac{9h}{5} + \frac{21y}{5} \\ &= \frac{2x + 21y}{5}. \end{aligned}$$

To calculate $D_{\vec{\mathbf{u}}} f(-1,2)$, we substitute $x = -1$ and $y = 2$ into this answer (Figure 14.6.2):

$$D_{\vec{\mathbf{u}}} f(-1,2) = \frac{2(-1) + 21(2)}{5} = \frac{-2 + 42}{5} = 8.$$

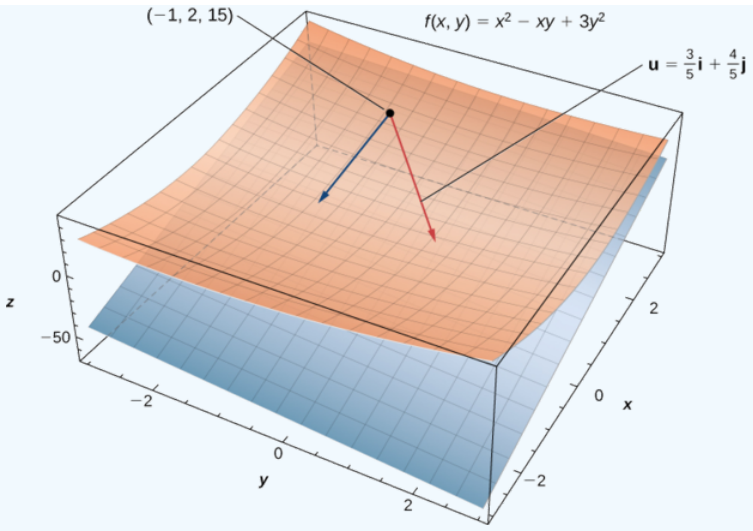


Figure 14.6.2: Finding the directional derivative in a given direction $\vec{\mathbf{u}}$ at a given point on a surface. The plane is tangent to the surface at the given point $(-1, 2, 15)$.

An easier approach to calculating directional derivatives that involves partial derivatives is outlined in the following theorem.

Theorem 14.6.1: Directional Derivative of a Function of Two Variables

Let $z = f(x, y)$ be a function of two variables x and y, and assume that f_x and f_y exist. Then the directional derivative of f in the direction of $\vec{\mathbf{u}} = (\cos\theta)\,\hat{\mathbf{i}} + (\sin\theta)\,\hat{\mathbf{j}}$ is given by

$$D_{\vec{\mathbf{u}}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta. \tag{14.6.3}$$

Proof

Applying the definition of a directional derivative stated above in Equation 14.6.1, the directional derivative of f in the direction of $\vec{\mathbf{u}} = (\cos\theta)\,\hat{\mathbf{i}} + (\sin\theta)\,\hat{\mathbf{j}}$ at a point (x_0, y_0) in the domain of f can be written

$$D_{\vec{\mathbf{u}}} f((x_0, y_0)) = \lim_{t\to 0} \frac{f(x_0 + t\cos\theta, y_0 + t\sin\theta) - f(x_0, y_0)}{t}.$$

Let $x = x_0 + t\cos\theta$ and $y = y_0 + t\sin\theta$, and define $g(t) = f(x, y)$. Since f_x and f_y both exist, we can use the chain rule for functions of two variables to calculate $g'(t)$:

$$g'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta.$$

If $t = 0$, then $x = x_0$ and $y = y_0$, so

$$g'(0) = f_x(x_0, y_0)\cos\theta + f_y(x_0, y_0)\sin\theta$$

By the definition of $g'(t)$, it is also true that

$$g'(0) = \lim_{t\to 0}\frac{g(t) - g(0)}{t} = \lim_{t\to 0}\frac{f(x_0 + t\cos\theta, y_0 + t\sin\theta) - f(x_0, y_0)}{t}.$$

Therefore, $D_{\vec{\mathbf{u}}} f(x_0, y_0) = f_x(x_0, y_0)\cos\theta + f_y(x_0, y_0)\sin\theta$.

Since the point (x_0, y_0) is an arbitrary point from the domain of f, this result holds for all points in the domain of f for which the partials f_x and f_y exist.

Therefore,

$$D_{\vec{\mathbf{u}}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta.$$

□

Example 14.6.2: Finding a Directional Derivative: Alternative Method

Let $\theta = \arccos(3/5)$. Find the directional derivative $D_{\vec{\mathbf{u}}} f(x,y)$ of $f(x,y) = x^2 - xy + 3y^2$ in the direction of $\vec{\mathbf{u}} = (\cos\theta)\,\hat{\mathbf{i}} + (\sin\theta)\,\hat{\mathbf{j}}$.

Then determine $D_{\vec{\mathbf{u}}} f(-1,2)$.

Solution

First, we must calculate the partial derivatives of f:

$$\begin{aligned} f_x(x,y) &= 2x - y \\ f_y(x,y) &= -x + 6y, \end{aligned}$$

Then we use Equation 14.6.3 with $\theta = \arccos(3/5)$:

$$\begin{aligned} D_{\vec{\mathbf{u}}} f(x,y) &= f_x(x,y)\cos\theta + f_y(x,y)\sin\theta \\ &= (2x-y)\frac{3}{5} + (-x+6y)\frac{4}{5} \\ &= \frac{6x}{5} - \frac{3y}{5} - \frac{4x}{5} + \frac{24y}{5} \\ &= \frac{2x+21y}{5}. \end{aligned}$$

To calculate $D_{\vec{\mathbf{u}}} f(-1,2)$, let $x = -1$ and $y = 2$:

$$D_{\vec{\mathbf{u}}} f(-1,2) = \frac{2(-1)+21(2)}{5} = \frac{-2+42}{5} = 8.$$

This is the same answer obtained in Example 14.6.1.

Exercise 14.6.1:

Find the directional derivative $D_{\vec{\mathbf{u}}} f(x,y)$ of $f(x,y) = 3x^2y - 4xy^3 + 3y^2 - 4x$ in the direction of $\vec{\mathbf{u}} = (\cos\frac{\pi}{3})\,\hat{\mathbf{i}} + (\sin\frac{\pi}{3})\,\hat{\mathbf{j}}$ using Equation 14.6.3.

What is $D_{\vec{\mathbf{u}}} f(3,4)$?

Hint

Calculate the partial derivatives and determine the value of θ.

Answer

$$D_{\vec{\mathbf{u}}} f(x,y) = \frac{(6xy - 4y^3 - 4)(1)}{2} + \frac{(3x^2 - 12xy^2 + 6y)\sqrt{3}}{2}$$

$$D_{\vec{\mathbf{u}}} f(3,4) = \frac{72 - 256 - 4}{2} + \frac{(27 - 576 + 24)\sqrt{3}}{2} = -94 - \frac{525\sqrt{3}}{2}$$

If the vector that is given for the direction of the derivative is not a unit vector, then it is only necessary to divide by the norm of the vector. For example, if we wished to find the directional derivative of the function in Example 14.6.2 in the direction of the vector $\langle -5, 12\rangle$, we would first divide by its magnitude to get $\vec{\mathbf{u}}$. This gives us $\vec{\mathbf{u}} = \langle -\frac{5}{13}, \frac{12}{13}\rangle$.

Then

$$\begin{aligned} D_{\vec{\mathbf{u}}} f(x,y) &= f_x(x,y)\cos\theta + f_y(x,y)\sin\theta \\ &= -\frac{5}{13}(2x-y) + \frac{12}{13}(-x+6y) \\ &= -\frac{22}{13}x + \frac{17}{13}y \end{aligned}$$

Gradient

The right-hand side of Equation 14.6.3 is equal to $f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$, which can be written as the dot product of two vectors. Define the first vector as $\vec{\nabla} f(x,y) = f_x(x,y)\,\hat{\mathbf{i}} + f_y(x,y)\,\hat{\mathbf{j}}$ and the second vector as $\vec{\mathbf{u}} = (\cos\theta)\,\hat{\mathbf{i}} + (\sin\theta)\,\hat{\mathbf{j}}$. Then the right-hand side of the equation can be written as the dot product of these two vectors:

$$D_{\vec{\mathbf{u}}} f(x,y) = \vec{\nabla} f(x,y) \cdot \vec{\mathbf{u}}. \tag{14.6.4}$$

The first vector in Equation 14.6.4 has a special name: the gradient of the function f. The symbol ∇ is called **nabla** and the vector $\vec{\nabla} f$ is read "**del** f."

Definition: The Gradient

Let $z = f(x,y)$ be a function of x and y such that f_x and f_y exist. The vector $\vec{\nabla} f(x,y)$ is called the **gradient** of f and is defined as

$$\vec{\nabla} f(x,y) = f_x(x,y)\,\hat{\mathbf{i}} + f_y(x,y)\,\hat{\mathbf{j}}. \tag{14.6.5}$$

The vector $\vec{\nabla} f(x,y)$ is also written as "**grad** f."

Example 14.6.3: Finding Gradients

Find the gradient $\vec{\nabla} f(x,y)$ of each of the following functions:

a. $f(x,y) = x^2 - xy + 3y^2$
b. $f(x,y) = \sin 3x \cos 3y$

Solution

For both parts a. and b., we first calculate the partial derivatives f_x and f_y, then use Equation 14.6.5.

a. $f_x(x,y) = 2x - y$ and $f_y(x,y) = -x + 6y$, so

$$\begin{aligned}\vec{\nabla} f(x,y) &= f_x(x,y)\,\hat{\mathbf{i}} + f_y(x,y)\,\hat{\mathbf{j}} \\ &= (2x - y)\,\hat{\mathbf{i}} + (-x + 6y)\,\hat{\mathbf{j}}.\end{aligned}$$

b. $f_x(x,y) = 3\cos 3x \cos 3y$ and $f_y(x,y) = -3\sin 3x \sin 3y$, so

$$\begin{aligned}\vec{\nabla} f(x,y) &= f_x(x,y)\,\hat{\mathbf{i}} + f_y(x,y)\,\hat{\mathbf{j}} \\ &= (3\cos 3x \cos 3y)\,\hat{\mathbf{i}} - (3\sin 3x \sin 3y)\,\hat{\mathbf{j}}.\end{aligned}$$

Exercise 14.6.2

Find the gradient $\vec{\nabla} f(x,y)$ of $f(x,y) = \dfrac{x^2 - 3y^2}{2x + y}$.

Hint

Calculate the partial derivatives, then use Equation 14.6.5.

Answer

$$\vec{\nabla} f(x,y) = \frac{2x^2 + 2xy + 6y^2}{(2x+y)^2}\,\hat{\mathbf{i}} - \frac{x^2 + 12xy + 3y^2}{(2x+y)^2}\,\hat{\mathbf{j}}$$

The gradient has some important properties. We have already seen one formula that uses the gradient: the formula for the directional derivative. Recall from The Dot Product that if the angle between two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is φ, then $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \|\vec{\mathbf{a}}\|\|\vec{\mathbf{b}}\|\cos\varphi$. Therefore, if the angle between $\vec{\nabla} f(x_0,y_0)$ and $\vec{\mathbf{u}} = (cos\theta)\,\hat{\mathbf{i}} + (sin\theta)\,\hat{\mathbf{j}}$ is φ, we have

$$D_{\vec{\mathbf{u}}} f(x_0,y_0) = \vec{\nabla} f(x_0,y_0) \cdot \vec{\mathbf{u}} = \|\vec{\nabla} f(x_0,y_0)\|\|\vec{\mathbf{u}}\|\cos\varphi = \|\vec{\nabla} f(x_0,y_0)\|\cos\varphi.$$

The $\|\vec{\mathbf{u}}\|$ disappears because $\vec{\mathbf{u}}$ is a unit vector. Therefore, the directional derivative is equal to the magnitude of the gradient evaluated at (x_0, y_0) multiplied by $\cos\varphi$. Recall that $\cos\varphi$ ranges from -1 to 1.

If $\varphi = 0$, then $\cos\varphi = 1$ and $\vec{\nabla} f(x_0, y_0)$ and $\vec{\mathbf{u}}$ both point in the same direction.

If $\varphi = \pi$, then $\cos\varphi = -1$ and $\vec{\nabla} f(x_0, y_0)$ and $\vec{\mathbf{u}}$ point in opposite directions.

In the first case, the value of $D_{\vec{\mathbf{u}}} f(x_0, y_0)$ is maximized; in the second case, the value of $D_{\vec{\mathbf{u}}} f(x_0, y_0)$ is minimized.

We can also see that if $\vec{\nabla} f(x_0, y_0) = \vec{0}$, then

$$D_{\vec{\mathbf{u}}} f(x_0, y_0) = \vec{\nabla} f(x_0, y_0) \cdot \vec{\mathbf{u}} = 0$$

for any vector $\vec{\mathbf{u}}$. These three cases are outlined in the following theorem.

Theorem 14.6.2: Properties of the Gradient

Suppose the function $z = f(x, y)$ is differentiable at (x_0, y_0) (Figure 14.6.3).

i. If $\vec{\nabla} f(x_0, y_0) = \vec{0}$, then $D_{\vec{\mathbf{u}}} f(x_0, y_0) = 0$ for any unit vector $\vec{\mathbf{u}}$.
ii. If $\vec{\nabla} f(x_0, y_0) \neq \vec{0}$, then $D_{\vec{\mathbf{u}}} f(x_0, y_0)$ is **maximized** when $\vec{\mathbf{u}}$ points in the same direction as $\vec{\nabla} f(x_0, y_0)$. The maximum value of $D_{\vec{\mathbf{u}}} f(x_0, y_0)$ is $\|\vec{\nabla} f(x_0, y_0)\|$.
iii. If $\vec{\nabla} f(x_0, y_0) \neq \vec{0}$, then $D_{\vec{\mathbf{u}}} f(x_0, y_0)$ is **minimized** when $\vec{\mathbf{u}}$ points in the opposite direction from $\vec{\nabla} f(x_0, y_0)$. The minimum value of $D_{\vec{\mathbf{u}}} f(x_0, y_0)$ is $-\|\vec{\nabla} f(x_0, y_0)\|$.

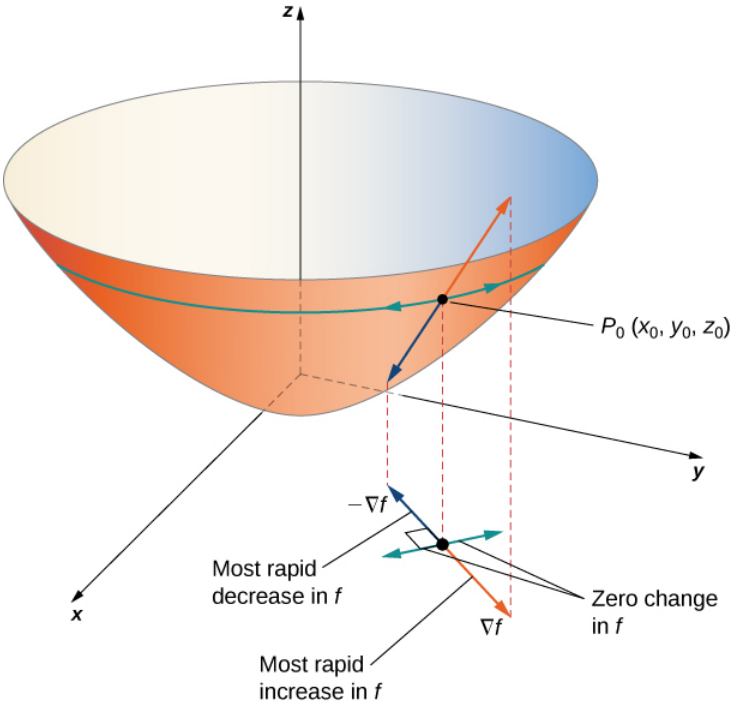


Figure 14.6.3: The gradient indicates the maximum and minimum values of the directional derivative at a point.

Example 14.6.4: Finding a Maximum Directional Derivative

Find the direction for which the directional derivative of $f(x, y) = 3x^2 - 4xy + 2y^2$ at $(-2, 3)$ is a maximum. What is the maximum value?

Solution

The maximum value of the directional derivative occurs when $\vec{\nabla} f$ and the unit vector point in the same direction. Therefore, we start by calculating $\vec{\nabla} f(x, y)$:

$$f_x(x,y) = 6x - 4y \text{ and } f_y(x,y) = -4x + 4y$$

so

$$\vec{\nabla} f(x,y) = f_x(x,y)\,\hat{\mathbf{i}} + f_y(x,y)\,\hat{\mathbf{j}} = (6x - 4y)\,\hat{\mathbf{i}} + (-4x + 4y)\,\hat{\mathbf{j}}.$$

Next, we evaluate the gradient at $(-2,3)$:

$$\vec{\nabla} f(-2,3) = (6(-2) - 4(3))\,\hat{\mathbf{i}} + (-4(-2) + 4(3))\,\hat{\mathbf{j}} = -24\,\hat{\mathbf{i}} + 20\,\hat{\mathbf{j}}.$$

We need to find a unit vector that points in the same direction as $\vec{\nabla} f(-2,3)$, so the next step is to divide $\vec{\nabla} f(-2,3)$ by its magnitude, which is $\sqrt{(-24)^2 + (20)^2} = \sqrt{976} = 4\sqrt{61}$. Therefore,

$$\frac{\vec{\nabla} f(-2,3)}{\|\vec{\nabla} f(-2,3)\|} = \frac{-24}{4\sqrt{61}}i + \frac{20}{4\sqrt{61}}j = -\frac{6\sqrt{61}}{61}\,\hat{\mathbf{i}} + \frac{5\sqrt{61}}{61}\,\hat{\mathbf{j}}.$$

This is the unit vector that points in the same direction as $\vec{\nabla} f(-2,3)$. To find the angle corresponding to this unit vector, we solve the equations

$$\cos\theta = \frac{-6\sqrt{61}}{61} \text{ and } \sin\theta = \frac{5\sqrt{61}}{61}$$

for θ. Since cosine is negative and sine is positive, the angle must be in the second quadrant. Therefore, $\theta = \pi - \arcsin((5\sqrt{61})/61) \approx 2.45$ rad.

The maximum value of the directional derivative at $(-2,3)$ is $\|\vec{\nabla} f(-2,3)\| = 4\sqrt{61}$ (Figure 14.6.4).

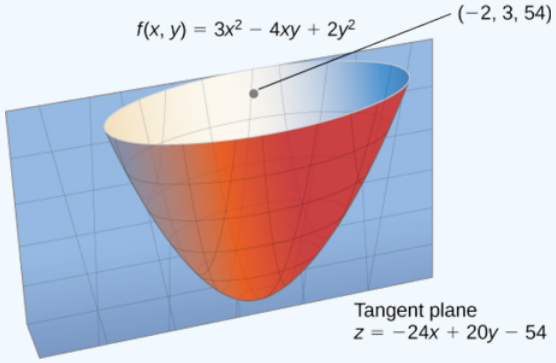


Figure 14.6.4: The maximum value of the directional derivative at $(-2,3)$ is in the direction of the gradient.

Exercise 14.6.3

Find the direction for which the directional derivative of $g(x,y) = 4x - xy + 2y^2$ at $(-2,3)$ is a maximum. What is the maximum value?

Hint

Evaluate the gradient of g at point $(-2,3)$.

Answer

The gradient of g at $(-2,3)$ is $\vec{\nabla} g(-2,3) = \hat{\mathbf{i}} + 14\,\hat{\mathbf{j}}$. The unit vector that points in the same direction as $\vec{\nabla} g(-2,3)$ is

$$\frac{\vec{\nabla} g(-2,3)}{\|\vec{\nabla} g(-2,3)\|} = \frac{1}{\sqrt{197}}\,\hat{\mathbf{i}} + \frac{14}{\sqrt{197}}\,\hat{\mathbf{j}} = \frac{\sqrt{197}}{197}\,\hat{\mathbf{i}} + \frac{14\sqrt{197}}{197}\,\hat{\mathbf{j}},$$

which gives an angle of $\theta = \arcsin((14\sqrt{197})/197) \approx 1.499$ rad.

The maximum value of the directional derivative is $\|\vec{\nabla} g(-2,3)\| = \sqrt{197}$.

Figure 14.6.5 shows a portion of the graph of the function $f(x, y) = 3 + \sin x \sin y$. Given a point (a, b) in the domain of f, the maximum value of the directional derivative at that point is given by $\|\vec{\nabla} f(a, b)\|$. This would equal the rate of greatest ascent if the surface represented a topographical map. If we went in the opposite direction, it would be the rate of greatest descent.

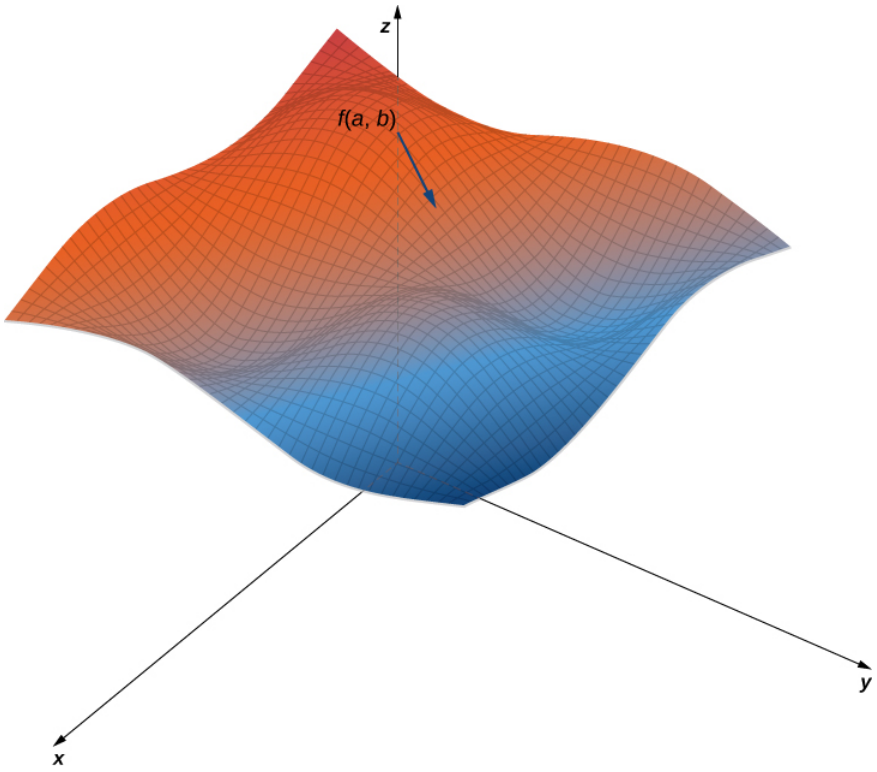


Figure 14.6.5: A typical surface in $\mathbb{R}^3$. Given a point on the surface, the directional derivative can be calculated using the gradient.

When using a topographical map, the steepest slope is always in the direction where the contour lines are closest together (Figure 14.6.6). This is analogous to the contour map of a function, assuming the level curves are obtained for equally spaced values throughout the range of that function.

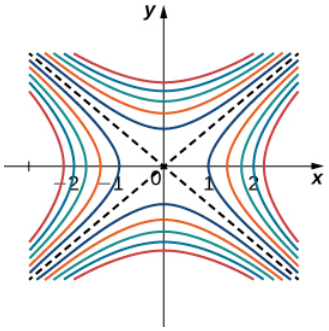


Figure 14.6.6: Contour map for the function $f(x, y) = x^2 - y^2$ using level values between -5 and 5.

Gradients and Level Curves

Recall that if a curve is defined parametrically by the function pair $(x(t), y(t))$, then the vector $x'(t)\,\hat{\mathbf{i}} + y'(t)\,\hat{\mathbf{j}}$ is tangent to the curve for every value of t in the domain. Now let's assume $z = f(x, y)$ is a differentiable function of x and y, and (x_0, y_0) is in its domain. Let's suppose further that $x_0 = x(t_0)$ and $y_0 = y(t_0)$ for some value of t, and consider the level curve $f(x, y) = k$. Define $g(t) = f(x(t), y(t))$ and calculate $g'(t)$ on the level curve. By the chain Rule,

$$g'(t) = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t).$$

But $g'(t) = 0$ because $g(t) = k$ for all t. Therefore, on the one hand,

$$f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t) = 0;$$

on the other hand,

$$f_x(x(t),y(t))x'(t)+f_y(x(t),y(t))y'(t)=\vec{\nabla}f(x,y)\cdot\langle x'(t),y'(t)\rangle.$$

Therefore,

$$\vec{\nabla}f(x,y)\cdot\langle x'(t),y'(t)\rangle=0.$$

Thus, the dot product of these vectors is equal to zero, which implies they are orthogonal. However, the second vector is tangent to the level curve, which implies the gradient must be normal to the level curve, which gives rise to the following theorem.

Theorem 14.6.3: Gradient Is Normal to the Level Curve

Suppose the function $z=f(x,y)$ has continuous first-order partial derivatives in an open disk centered at a point (x_0,y_0). If $\vec{\nabla}f(x_0,y_0)\neq 0$, then $\vec{\nabla}f(x_0,y_0)$ is normal to the level curve of f at (x_0,y_0).

We can use this theorem to find tangent and normal vectors to level curves of a function.

Example 14.6.5: Finding Tangents to Level Curves

For the function $f(x,y)=2x^2-3xy+8y^2+2x-4y+4$, find a tangent vector to the level curve at point $(-2,1)$. Graph the level curve corresponding to $f(x,y)=18$ and draw in $\vec{\nabla}f(-2,1)$ and a tangent vector.

Solution

First, we must calculate $\vec{\nabla}f(x,y)$:

$$f_x(x,y)=4x-3y+2 \text{ and } f_y=-3x+16y-4 \text{ so}$$
$$\vec{\nabla}f(x,y)=(4x-3y+2)\,\hat{\mathbf{i}}+(-3x+16y-4)\,\hat{\mathbf{j}}.$$

Next, we evaluate $\vec{\nabla}f(x,y)$ at $(-2,1)$:

$$\vec{\nabla}f(-2,1)=(4(-2)-3(1)+2)\,\hat{\mathbf{i}}+(-3(-2)+16(1)-4)\,\hat{\mathbf{j}}=-9\,\hat{\mathbf{i}}+18\,\hat{\mathbf{j}}.$$

This vector is orthogonal to the curve at point $(-2,1)$. We can obtain a tangent vector by reversing the components and multiplying either one by -1. Thus, for example, $-18\,\hat{\mathbf{i}}-9\,\hat{\mathbf{j}}$ is a tangent vector (Figure 14.6.7).

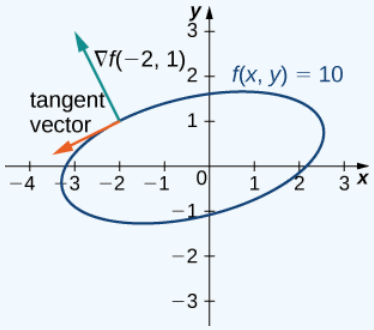


Figure 14.6.7: Tangent and normal vectors to $2x^2-3xy+8y^2+2x-4y+4=18$ at point $(-2,1)$.

Exercise 14.6.4

For the function $f(x,y)=x^2-2xy+5y^2+3x-2y+3$, find the tangent to the level curve at point $(1,1)$. Draw the graph of the level curve corresponding to $f(x,y)=8$ and draw $\vec{\nabla}f(1,1)$ and a tangent vector.

Hint

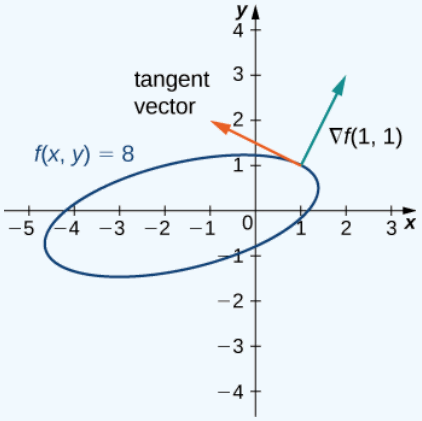
Calculate the gradient at point $(1,1)$.

Answer

$\vec{\nabla} f(x, y) = (2x - 2y + 3)\,\hat{\mathbf{i}} + (-2x + 10y - 2)\,\hat{\mathbf{j}}$

$\vec{\nabla} f(1, 1) = 3\,\hat{\mathbf{i}} + 6\,\hat{\mathbf{j}}$

Tangent vector: $6\,\hat{\mathbf{i}} - 3\,\hat{\mathbf{j}}$ or $-6\,\hat{\mathbf{i}} + 3\,\hat{\mathbf{j}}$



Three-Dimensional Gradients and Directional Derivatives

The definition of a gradient can be extended to functions of more than two variables.

Definition: Gradients in 3D

Let $w = f(x, y, z)$ be a function of three variables such that f_x, f_y, and f_z exist. The vector $\vec{\nabla} f(x, y, z)$ is called the gradient of f and is defined as

$$\vec{\nabla} f(x, y, z) = f_x(x, y, z)\,\hat{\mathbf{i}} + f_y(x, y, z)\,\hat{\mathbf{j}} + f_z(x, y, z)\,\hat{\mathbf{k}}. \tag{14.6.6}$$

$\vec{\nabla} f(x, y, z)$ can also be written as **grad** $f(x, y, z)$.

Calculating the gradient of a function in three variables is very similar to calculating the gradient of a function in two variables. First, we calculate the partial derivatives f_x, f_y, and f_z, and then we use Equation 14.6.6.

Example 14.6.6: Finding Gradients in Three Dimensions

Find the gradient $\vec{\nabla} f(x, y, z)$ of each of the following functions:

a. $f(x, y, z) = 5x^2 - 2xy + y^2 - 4yz + z^2 + 3xz$
b. $f(x, y, z) = e^{-2z} \sin 2x \cos 2y$

Solution

For both parts a. and b., we first calculate the partial derivatives f_x, f_y, and f_z, then use Equation 14.6.6.

a. $f_x(x, y, z) = 10x - 2y + 3z$, $f_y(x, y, z) = -2x + 2y - 4z$, and $f_z(x, y, z) = 3x - 4y + 2z$, so

$$\begin{aligned}\vec{\nabla} f(x, y, z) &= f_x(x, y, z)\,\hat{\mathbf{i}} + f_y(x, y, z)\,\hat{\mathbf{j}} + f_z(x, y, z)\,\hat{\mathbf{k}} \\ &= (10x - 2y + 3z)\,\hat{\mathbf{i}} + (-2x + 2y - 4z)\,\hat{\mathbf{j}} + (3x - 4y + 2z)\,\hat{\mathbf{k}}.\end{aligned}$$

b. $f_x(x, y, z) = 2e^{-2z} \cos 2x \cos 2y$, $f_y(x, y, z) = -2e^{-2z} \sin 2x \sin 2y$, and $f_z(x, y, z) = -2e^{-2z} \sin 2x \cos 2y$, so

$$\begin{aligned}\vec{\nabla} f(x, y, z) &= f_x(x, y, z)\,\hat{\mathbf{i}} + f_y(x, y, z)\,\hat{\mathbf{j}} + f_z(x, y, z)\,\hat{\mathbf{k}} \\ &= (2e^{-2z} \cos 2x \cos 2y)\,\hat{\mathbf{i}} + (-2e^{-2z} \sin 2x \sin 2y)\,\hat{\mathbf{j}} + (-2e^{-2z} \sin 2x \cos 2y)\,\hat{\mathbf{k}} \\ &= 2e^{-2z}(\cos 2x \cos 2y\,\hat{\mathbf{i}} - \sin 2x \sin 2y\,\hat{\mathbf{j}} - \sin 2x \cos 2y\,\hat{\mathbf{k}}).\end{aligned}$$

Exercise 14.6.5:

Find the gradient $\vec{\nabla} f(x,y,z)$ of $f(x,y,z) = \dfrac{x^2 - 3y^2 + z^2}{2x + y - 4z.}$

Answer

$$\vec{\nabla} f(x,y,z) = \frac{2x^2 + 2xy + 6y^2 - 8xz - 2z^2}{(2x + y - 4z)^2}\,\hat{\mathbf{i}} - \frac{x^2 + 12xy + 3y^2 - 24yz + z^2}{(2x + y - 4z)^2}\,\hat{\mathbf{j}} + \frac{4x^2 - 12y^2 - 4z^2 + 4xz + 2yz}{(2x + y - 4z)^2}\,\hat{\mathbf{k}}$$

The directional derivative can also be generalized to functions of three variables. To determine a direction in three dimensions, a vector with three components is needed. This vector is a unit vector, and the components of the unit vector are called *directional cosines*. Given a three-dimensional unit vector $\vec{\mathbf{u}}$ in standard form (i.e., the initial point is at the origin), this vector forms three different angles with the positive x-, y-, and z-axes. Let's call these angles $\alpha, \beta,$ and γ. Then the directional cosines are given by $\cos\alpha, \cos\beta,$ and $\cos\gamma$. These are the components of the unit vector $\vec{\mathbf{u}}$; since $\vec{\mathbf{u}}$ is a unit vector, it is true that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Definition: Directional Derivative of a Function of Three variables

Suppose $w = f(x,y,z)$ is a function of three variables with a domain of D. Let $(x_0, y_0, z_0) \in D$ and let $\vec{\mathbf{u}} = \cos\alpha\,\hat{\mathbf{i}} + \cos\beta\,\hat{\mathbf{j}} + \cos\gamma\,\hat{\mathbf{k}}$ be a unit vector. Then, the directional derivative of f in the direction of u is given by

$$D_{\vec{\mathbf{u}}} f(x_0, y_0, z_0) = \lim_{t \to 0} \frac{f(x_0 + t\cos\alpha, y_0 + t\cos\beta, z_0 + t\cos\gamma) - f(x_0, y_0, z_0)}{t}$$

provided the limit exists.

We can calculate the directional derivative of a function of three variables by using the gradient, leading to a formula that is analogous to Equation 14.6.3.

Theorem 14.6.4: Directional Derivative of a Function of Three Variables

Let $f(x,y,z)$ be a differentiable function of three variables and let $\vec{\mathbf{u}} = \cos\alpha\,\hat{\mathbf{i}} + \cos\beta\,\hat{\mathbf{j}} + \cos\gamma\,\hat{\mathbf{k}}$ be a unit vector. Then, the directional derivative of f in the direction of $\vec{\mathbf{u}}$ is given by

$$\begin{aligned} D_{\vec{\mathbf{u}}} f(x,y,z) &= \vec{\nabla} f(x,y,z) \cdot \vec{\mathbf{u}} \\ &= f_x(x,y,z)\cos\alpha + f_y(x,y,z)\cos\beta + f_z(x,y,z)\cos\gamma. \end{aligned} \tag{14.6.7}$$

The three angles $\alpha, \beta,$ and γ determine the unit vector $\vec{\mathbf{u}}$. In practice, we can use an arbitrary (nonunit) vector, then divide by its magnitude to obtain a unit vector in the desired direction.

Example 14.6.7: Finding a Directional Derivative in Three Dimensions

Calculate $D_{\vec{\mathbf{v}}} f(1,-2,3)$ in the direction of $\vec{\mathbf{v}} = -\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ for the function

$$f(x,y,z) = 5x^2 - 2xy + y^2 - 4yz + z^2 + 3xz.$$

Solution:

First, we find the magnitude of v:

$$\|\vec{\mathbf{v}}\| = \sqrt{(-1)^2 + (2)^2 + (2)^2} = \sqrt{9} = 3.$$

Therefore, $\dfrac{\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|} = \dfrac{-\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}}{3} = -\dfrac{1}{3}\,\hat{\mathbf{i}} + \dfrac{2}{3}\,\hat{\mathbf{j}} + \dfrac{2}{3}\,\hat{\mathbf{k}}$ is a unit vector in the direction of $\vec{\mathbf{v}}$, so $\cos\alpha = -\dfrac{1}{3}, \cos\beta = \dfrac{2}{3},$ and $\cos\gamma = \dfrac{2}{3}$. Next, we calculate the partial derivatives of f:

$$f_x(x,y,z) = 10x - 2y + 3z$$
$$f_y(x,y,z) = -2x + 2y - 4z$$
$$f_z(x,y,z) = -4y + 2z + 3x,$$

then substitute them into Equation 14.6.7:

$$\begin{aligned} D_{\vec{\mathbf{v}}} f(x,y,z) &= f_x(x,y,z)\cos\alpha + f_y(x,y,z)\cos\beta + f_z(x,y,z)\cos\gamma \\ &= (10x - 2y + 3z)(-\frac{1}{3}) + (-2x + 2y - 4z)(\frac{2}{3}) + (-4y + 2z + 3x)(\frac{2}{3}) \\ &= -\frac{10x}{3} + \frac{2y}{3} - \frac{3z}{3} - \frac{4x}{3} + \frac{4y}{3} - \frac{8z}{3} - \frac{8y}{3} + \frac{4z}{3} + \frac{6x}{3} \\ &= -\frac{8x}{3} - \frac{2y}{3} - \frac{7z}{3}. \end{aligned}$$

Last, to find $D_{\vec{\mathbf{v}}} f(1,-2,3)$, we substitute $x = 1$, $y = -2$, and $z = 3$:

$$\begin{aligned} D_{\vec{\mathbf{v}}} f(1,-2,3) &= -\frac{8(1)}{3} - \frac{2(-2)}{3} - \frac{7(3)}{3} \\ &= -\frac{8}{3} + \frac{4}{3} - \frac{21}{3} \\ &= -\frac{25}{3}. \end{aligned}$$

? Exercise 14.6.6:

Calculate $D_{\vec{\mathbf{v}}} f(x,y,z)$ and $D_{\vec{\mathbf{v}}} f(0,-2,5)$ in the direction of $\vec{\mathbf{v}} = -3\,\hat{\mathbf{i}} + 12\,\hat{\mathbf{j}} - 4\,\hat{\mathbf{k}}$ for the function

$$f(x,y,z) = 3x^2 + xy - 2y^2 + 4yz - z^2 + 2xz.$$

Hint

First, divide $\vec{\mathbf{v}}$ by its magnitude, calculate the partial derivatives of f, then use Equation 14.6.7.

Answer

$$D_{\vec{\mathbf{v}}} f(x,y,z) = -\frac{3}{13}(6x + y + 2z) + \frac{12}{13}(x - 4y + 4z) - \frac{4}{13}(2x + 4y - 2z)$$

$$D_{\vec{\mathbf{v}}} f(0,-2,5) = \frac{384}{13}$$

Summary

- A directional derivative represents a rate of change of a function in any given direction.
- The gradient can be used in a formula to calculate the directional derivative.
- The gradient indicates the direction of greatest change of a function of more than one variable.

Key Equations

- **directional derivative (two dimensions)**

$$D_{\vec{\mathbf{u}}} f(a,b) = \lim_{h \to 0} \frac{f(a + h\cos\theta, b + h\sin\theta) - f(a,b)}{h}$$

or

$$D_{\vec{\mathbf{u}}} f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$$

- **gradient (two dimensions)**

$$\vec{\nabla} f(x,y) = f_x(x,y)\,\hat{\mathbf{i}} + f_y(x,y)\,\hat{\mathbf{j}}$$

- **gradient (three dimensions)**

$$\vec{\nabla} f(x,y,z) = f_x(x,y,z)\,\hat{\mathbf{i}} + f_y(x,y,z)\,\hat{\mathbf{j}} + f_z(x,y,z)\,\hat{\mathbf{k}}$$

- **directional derivative (three dimensions)**

$$D_{\vec{\mathbf{u}}} f(x,y,z) = \vec{\nabla} f(x,y,z) \cdot \vec{\mathbf{u}} = f_x(x,y,z)\cos\alpha + f_y(x,y,z)\cos\beta + f_x(x,y,z)\cos\gamma$$

Glossary

directional derivative

the derivative of a function in the direction of a given unit vector

gradient

the gradient of the function $f(x,y)$ is defined to be $\vec{\nabla} f(x,y) = (\partial f/\partial x)\,\hat{\mathbf{i}} + (\partial f/\partial y)\,\hat{\mathbf{j}}$, which can be generalized to a function of any number of independent variables

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14.6E: Exercises for Section 14.6

In exercise 1, find the directional derivative using the limit definition only.

1) a. $f(x,y) = 5 - 2x^2 - \frac{1}{2}y^2$ at point $P(3,4)$ in the direction of $\vec{\mathbf{u}} = (\cos\frac{\pi}{4})\,\hat{\mathbf{i}} + (\sin\frac{\pi}{4})\,\hat{\mathbf{j}}$

Answer

a. $D_{\vec{\mathbf{u}}} f(3,4) = -8\sqrt{2}$

1) b. $f(x,y) = y^2\cos(2x)$ at point $P(\frac{\pi}{3},2)$ in the direction of $\vec{\mathbf{u}} = (\cos\frac{\pi}{4})\,\hat{\mathbf{i}} + (\sin\frac{\pi}{4})\,\hat{\mathbf{j}}$

Answer

b. $D_{\vec{\mathbf{u}}} f(\frac{\pi}{3},2) = -2\sqrt{6} - \sqrt{2}$

2) Find the directional derivative of $f(x,y) = y^2\sin(2x)$ at point $P(\frac{\pi}{4},2)$ in the direction of $\vec{\mathbf{u}} = 5\,\hat{\mathbf{i}} + 12\,\hat{\mathbf{j}}$.

In exercises 3 - 13, find the directional derivative of the function in the direction of $\vec{\mathbf{v}}$ as a function of x and y. Remember that you first need to find a unit vector in the direction of the direction vector. Then find the value of the directional derivative at point P.

3) $f(x,y) = xy, \quad P(-2,0), \quad \vec{\mathbf{v}} = \frac{1}{2}\,\hat{\mathbf{i}} + \frac{\sqrt{3}}{2}\,\hat{\mathbf{j}}$

Answer

$D_{\vec{\mathbf{v}}} f(x,y) = \frac{1}{2}y + \frac{\sqrt{3}}{2}x$
$D_{\vec{\mathbf{v}}} f(-2,0) = -\sqrt{3}$

4) $h(x,y) = e^x\sin y, \quad P(1,\frac{\pi}{2}), \quad \vec{\mathbf{v}} = -\hat{\mathbf{i}}$

5) $f(x,y) = x^2y, \quad P(-5,5), \quad \vec{\mathbf{v}} = 3\,\hat{\mathbf{i}} - 4\,\hat{\mathbf{j}}$

Answer

$D_{\vec{\mathbf{v}}} f(x,y) = \frac{6}{5}xy - \frac{4}{5}x^2$
$D_{\vec{\mathbf{v}}} f(-5,5) = -50$

6) $f(x,y) = xy, \quad P(1,1), \quad \vec{\mathbf{u}} = \langle\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\rangle$

7) $f(x,y) = x^2 - y^2, \quad P(1,0), \quad \vec{\mathbf{u}} = \langle\frac{\sqrt{3}}{2},\frac{1}{2}\rangle$

Answer

$D_{\vec{\mathbf{u}}} f(x,y) = x\sqrt{3} - y$
$D_{\vec{\mathbf{u}}} f(1,0) = \sqrt{3}$

8) $f(x,y) = 3x + 4y + 7, \quad P(0,\frac{\pi}{2}), \quad \vec{\mathbf{u}} = \langle\frac{3}{5},\frac{4}{5}\rangle$

9) $f(x,y) = e^x\cos y, \quad P = (0,\frac{\pi}{2}), \quad \vec{\mathbf{u}} = \langle 0,5\rangle$

Answer

$D_{\vec{\mathbf{u}}} f(x,y) = -e^x\sin y$
$D_{\vec{\mathbf{u}}} f(0,\frac{\pi}{2}) = -1$

10) $f(x,y) = y^{10}, \quad \vec{\mathbf{u}} = \langle 0,-3\rangle, \quad P = (1,-1)$

11) $f(x,y) = \ln(x^2+y^2), \quad \vec{\mathbf{u}} = \langle 2,-5\rangle, \quad P(1,2)$

Answer

$D_{\vec{\mathbf{u}}} f(x,y) = \frac{\sqrt{29}}{29}\left(\frac{4x-10y}{x^2+y^2}\right)$
$D_{\vec{\mathbf{u}}} f(1,2) = -\frac{16\sqrt{29}}{145}$

12) $h(x,y,z) = xyz, \quad P(2,1,1), \quad \vec{\mathbf{v}} = 2\,\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$

Answer

$D_{\vec{\mathbf{v}}}h(x,y,z) = \frac{\sqrt{6}}{6}(2yz + xz - xy)$
$D_{\vec{\mathbf{v}}}h(2,1,1) = \frac{2}{\sqrt{6}} = \frac{\sqrt{6}}{3}$

13) $f(x,y,z) = y^2 + xz, \quad P(1,2,2), \quad \vec{\mathbf{v}} = \langle 2,-1,2\rangle$

Answer

$D_{\vec{\mathbf{v}}}f(x,y,z) = \frac{2}{3}(z - y + x)$
$D_{\vec{\mathbf{v}}}f(1,2,2) = \frac{2}{3}$

In exercises 14 - 19, find the directional derivative of the function in the direction of the unit vector $\vec{\mathbf{u}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}$.

14) $f(x,y) = x^2 + 2y^2, \quad \theta = \frac{\pi}{6}$

15) $f(x,y) = \dfrac{y}{x+2y}, \quad \theta = -\frac{\pi}{4}$

Answer

$$D_{\vec{\mathbf{u}}}f(x,y) = \frac{-\sqrt{2}(x+y)}{2(x+2y)^2}$$

16) $f(x,y) = \cos(3x+y), \quad \theta = \frac{\pi}{4}$

17) $w(x,y) = ye^x, \quad \theta = \frac{\pi}{3}$

Answer

$$D_{\vec{\mathbf{u}}}f(x,y) = \frac{e^x(y+\sqrt{3})}{2}$$

18) $f(x,y) = x\arctan(y), \quad \theta = \frac{\pi}{2}$

19) $f(x,y) = \ln(x+2y), \quad \theta = \frac{\pi}{3}$

Answer

$$D_{\vec{\mathbf{u}}}f(x,y) = \frac{1+2\sqrt{3}}{2(x+2y)}$$

In exercises 20 - 23, find the gradient $\vec{\nabla}f$.

20) Find the gradient of $f(x,y) = 3x^2 + y^3 - 3x + y$. Then find it's value at the point $P(2,3)$.

21) Find the gradient of $f(x,y) = \dfrac{14 - x^2 - y^2}{3}$. Then, find the gradient at point $P(1,2)$.

Answer

$\vec{\nabla}f(x,y) = -\frac{2}{3}x\,\hat{\mathbf{i}} - \frac{2}{3}y\hat{\mathbf{j}}$
$\vec{\nabla}f(1,2) = -\frac{2}{3}\,\hat{\mathbf{i}} - \frac{4}{3}\,\hat{\mathbf{j}}$

22) Find the gradient of $f(x,y) = \ln(4x^3 - 3y)$. Then, find the gradient at point $P(1,1)$.

23) Find the gradient of $f(x,y,z) = xy + yz + xz$. Then find the gradient at point $P(1,2,3)$.

Answer

$\vec{\nabla}f(x,y,z) = \langle y+z, x+z, y+x\rangle$
$\vec{\nabla}f(1,2,3) = \langle 5,4,3\rangle$

In exercises 24 - 25, find the directional derivative of the function at point P in the direction of Q.

24) $f(x,y) = x^2 + 3y^2, \quad P(1,1), \quad Q(4,5)$

25) $f(x,y,z) = \dfrac{y}{x+z}, \quad P(2,1,-1), \quad Q(-1,2,0)$

Answer

$D_{\vec{PQ}} f(x,y) = \frac{3}{\sqrt{11}}$

26) Find the directional derivative of $f(x,y,z))$ at P and in the direction of $\vec{\mathbf{u}}$: $f(x,y,z) = \ln(x^2+2y^2+3z^2)$, $P(2,1,4)$, $\vec{\mathbf{u}} = \frac{-3}{13}\hat{\mathbf{i}} - \frac{4}{13}\hat{\mathbf{j}} - \frac{12}{13}\hat{\mathbf{k}}$.

In exercises 27 - 29, find the directional derivative of the function at P in the direction of $\vec{\mathbf{u}}$.

27) $f(x,y) = \ln(5x+4y)$, $P(3,9)$, $\vec{\mathbf{u}} = 6\hat{\mathbf{i}} + 8\hat{\mathbf{j}}$

Answer

$D_{\vec{\mathbf{u}}} f(3,9) = \frac{31}{255}$

28) $f(x,y) = -7x+2y$, $P(2,-4)$, $\vec{\mathbf{u}} = 4\hat{\mathbf{i}} - 3\hat{\mathbf{j}}$

29) $f(x,y,z) = 4x^5y^2z^3$, $P(2,-1,1)$, $\vec{\mathbf{u}} = \frac{1}{3}\hat{\mathbf{i}} + \frac{2}{3}\hat{\mathbf{j}} - \frac{2}{3}\hat{\mathbf{k}}$

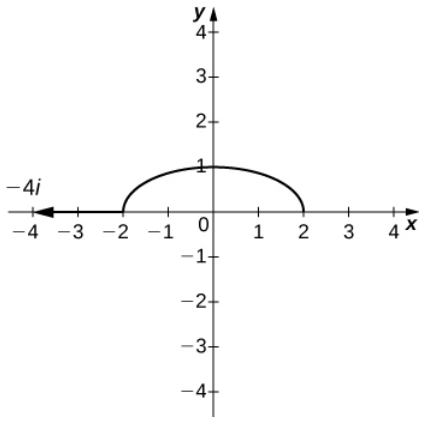
Answer

$D_{\vec{\mathbf{u}}} f(2,-1,1) = -320$

30) [T] Use technology to sketch the level curve of $f(x,y) = 4x - 2y + 3$ that passes through $P(1,2)$ and draw the gradient vector at P.

31) [T] Use technology to sketch the level curve of $f(x,y) = x^2 + 4y^2$ that passes through $P(-2,0)$ and draw the gradient vector at P.

Answer



In exercises 32 - 35, find the gradient vector at the indicated point.

32) $f(x,y) = xy^2 - yx^2$, $P(-1,1)$

33) $f(x,y) = xe^y - \ln(x)$, $P(-3,0)$

Answer

$\vec{\nabla} f(-3,0) = \frac{4}{3}\hat{\mathbf{i}} - 3\hat{\mathbf{j}}$

34) $f(x,y,z) = xy - \ln(z)$, $P(2,-2,2)$

35) $f(x,y,z) = x\sqrt{y^2+z^2}$, $P(-2,-1,-1)$

Answer

$\vec{\nabla} f(-2,-1,-1) = \sqrt{2}\hat{\mathbf{i}} + \sqrt{2}\hat{\mathbf{j}} + \sqrt{2}\hat{\mathbf{k}}$

In exercises 36 - 40, find the indicated directional derivative of the function.

36) $f(x,y) = x^2 + xy + y^2$ at point $(-5,-4)$ in the direction the function increases most rapidly.

37) $f(x,y) = e^{xy}$ at point $(6,7)$ in the direction the function increases most rapidly.

Answer

$1.6(10^{19})$

38) $f(x,y) = \arctan\left(\frac{y}{x}\right)$ at point $(-9,9)$ in the direction the function increases most rapidly.

39) $f(x,y,z) = \ln(xy+yz+zx)$ at point $(-9,-18,-27)$ in the direction the function increases most rapidly.

Answer

$\frac{5\sqrt{2}}{99}$

40) $f(x,y,z) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$ at point $(5,-5,5)$ in the direction the function increases most rapidly.

In exercises 41 - 43, find the maximum rate of change of f at the given point and the direction in which it occurs.

41) $f(x,y) = xe^{-y}, \quad (-2,0)$

Answer

$\max\{D_{\vec{u}} f(-2,0)\} = \sqrt{5}, \quad \langle 1,2\rangle$

Solution:

$\vec{\nabla} f(x,y) = e^{-y}\hat{\mathbf{i}} - xe^{-y}\hat{\mathbf{j}}$. So $\max\{D_{\vec{u}} f(-2,0)\} = \|\vec{\nabla} f(-2,0)\| = \sqrt{5}$.
The direction in which it occurs will just be the direction of the gradient vector: $\vec{\nabla} f(-2,0) = \langle 1,2\rangle$.

42) $f(x,y) = \sqrt{x^2+2y}, \quad (4,10)$

43) $f(x,y) = \cos(3x+2y), \quad (\frac{\pi}{6}, -\frac{\pi}{8})$

Answer

$\max\{D_{\vec{u}} f\left(\frac{\pi}{6}, -\frac{\pi}{8}\right)\} = \sqrt{\frac{13}{2}} = \frac{\sqrt{26}}{2}, \quad \langle -3,-2\rangle$

In exercises 44 - 47, find equations of

a. the tangent plane and

b. the normal line to the given surface at the given point.

44) The level curve $f(x,y,z) = 12$ for $f(x,y,z) = 4x^2 - 2y^2 + z^2$ at point $(2,2,2)$.

45) $f(x,y,z) = xy + yz + xz = 3$ at point $(1,1,1)$

Answer

a. tangent plane equation: $x+y+z=3$,
b. normal line equations: $x-1 = y-1 = z-1$

46) $f(x,y,z) = xyz = 6$ at point $(1,2,3)$

47) $f(x,y,z) = xe^y \cos z - z = 1$ at point $(1,0,0)$

Answer

a. tangent plane equation: $x+y-z=1$,
b. normal line equations: $x-1 = y = -z$

In exercises 48 - 51, solve the stated problem.

48) The temperature T in a metal sphere is inversely proportional to the distance from the center of the sphere (the origin: $(0,0,0)$). The temperature at point $(1,2,2)$ is $120°$C.

a. Find the rate of change of the temperature at point $(1,2,2)$ in the direction toward point $(2,1,3)$.

b. Show that, at any point in the sphere, the direction of greatest increase in temperature is given by a vector that points toward the origin.

49) The **electrical potential** (voltage) in a certain region of space is given by the function $V(x, y, z) = 5x^2 - 3xy + xyz$.

a. Find the rate of change of the voltage at point $(3, 4, 5)$ in the direction of the vector $\langle 1, 1, -1 \rangle$.

b. In which direction does the voltage change most rapidly at point $(3, 4, 5)$?

c. What is the maximum rate of change of the voltage at point $(3, 4, 5)$?

Answer

a. $\frac{32}{\sqrt{3}}$,
b. $\langle 38, 6, 12 \rangle$,
c. $2\sqrt{406}$

50) If the electric potential at a point (x, y) in the xy-plane is $V(x, y) = e^{-2x}\cos(2y)$, then the electric intensity vector at (x, y) is $E = -\vec{\nabla}V(x, y)$.

a. Find the electric intensity vector at $(\frac{\pi}{4}, 0)$.

b. Show that, at each point in the plane, the electric potential decreases most rapidly in the direction of the vector E.

51) In two dimensions, the motion of an ideal fluid is governed by a velocity potential φ. The velocity components of the fluid u in the x-direction and v in the y-direction, are given by $\langle u, v \rangle = \vec{\nabla}\varphi$. Find the velocity components associated with the velocity potential $\varphi(x, y) = \sin \pi x \sin 2\pi y$.

Answer

$\langle u, v \rangle = \langle \pi \cos(\pi x)\sin(2\pi y), 2\pi \sin(\pi x)\cos(2\pi y) \rangle$

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- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.
- Paul Seeburger (Monroe Community College) created problems 20 and 22 and added a solution to problem 41.

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14.7: Maxima/Minima Problems

Learning Objectives

- Use partial derivatives to locate critical points for a function of two variables.
- Apply a second derivative test to identify a critical point as a local maximum, local minimum, or saddle point for a function of two variables.
- Examine critical points and boundary points to find absolute maximum and minimum values for a function of two variables.

One of the most useful applications for derivatives of a function of one variable is the determination of maximum and/or minimum values. This application is also important for functions of two or more variables, but as we have seen in earlier sections of this chapter, the introduction of more independent variables leads to more possible outcomes for the calculations. The main ideas of finding critical points and using derivative tests are still valid, but new wrinkles appear when assessing the results.

Critical Points

For functions of a single variable, we defined critical points as the values of the variable at which the function's derivative equals zero or does not exist. For functions of two or more variables, the concept is essentially the same, except for the fact that we are now working with partial derivatives.

Definition: Critical Points

Let $z = f(x, y)$ be a function of two variables that is differentiable on an open set containing the point (x_0, y_0). The point (x_0, y_0) is called a *critical point* of a function of two variables f if one of the two following conditions holds:

1. $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$
2. Either $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

Example 14.7.1: Finding Critical Points

Find the critical points of each of the following functions:

a. $f(x, y) = \sqrt{4y^2 - 9x^2 + 24y + 36x + 36}$
b. $g(x, y) = x^2 + 2xy - 4y^2 + 4x - 6y + 4$

Solution

a. First, we calculate $f_x(x, y)$ and $f_y(x, y)$:

$$f_x(x, y) = \frac{1}{2}(-18x + 36)(4y^2 - 9x^2 + 24y + 36x + 36)^{-1/2}$$
$$= \frac{-9x + 18}{\sqrt{4y^2 - 9x^2 + 24y + 36x + 36}}$$

$$f_y(x, y) = \frac{1}{2}(8y + 24)(4y^2 - 9x^2 + 24y + 36x + 36)^{-1/2}$$
$$= \frac{4y + 12}{\sqrt{4y^2 - 9x^2 + 24y + 36x + 36}}.$$

Next, we set each of these expressions equal to zero:

$$\frac{-9x + 18}{\sqrt{4y^2 - 9x^2 + 24y + 36x + 36}} = 0$$
$$\frac{4y + 12}{\sqrt{4y^2 - 9x^2 + 24y + 36x + 36}} = 0.$$

Then, multiply each equation by its common denominator:

$$-9x + 18 = 0$$
$$4y + 12 = 0.$$

Therefore, $x = 2$ and $y = -3$, so $(2, -3)$ is a critical point of f.

We must also check for the possibility that the denominator of each partial derivative can equal zero, thus causing the partial derivative not to exist. Since the denominator is the same in each partial derivative, we need only do this once:

$$4y^2 - 9x^2 + 24y + 36x + 36 = 0. \tag{14.7.1}$$

Equation 14.7.1 represents a hyperbola. We should also note that the domain of f consists of points satisfying the inequality

$$4y^2 - 9x^2 + 24y + 36x + 36 \geq 0.$$

Therefore, any points on the hyperbola are not only critical points, they are also on the boundary of the domain. To put the hyperbola in standard form, we use the method of completing the square:

$$\begin{aligned} 4y^2 - 9x^2 + 24y + 36x + 36 &= 0 \\ 4y^2 - 9x^2 + 24y + 36x &= -36 \\ 4y^2 + 24y - 9x^2 + 36x &= -36 \\ 4(y^2 + 6y) - 9(x^2 - 4x) &= -36 \\ 4(y^2 + 6y + 9) - 9(x^2 - 4x + 4) &= -36 - 36 + 36 \\ 4(y + 3)^2 - 9(x - 2)^2 &= -36. \end{aligned}$$

Dividing both sides by -36 puts the equation in standard form:

$$\begin{aligned} \frac{4(y+3)^2}{-36} - \frac{9(x-2)^2}{-36} &= 1 \\ \frac{(x-2)^2}{4} - \frac{(y+3)^2}{9} &= 1. \end{aligned}$$

Notice that point $(2, -3)$ is the center of the hyperbola.

Thus, the critical points of the function f are $(2, -3)$ and all points on the hyperbola, $\frac{(x-2)^2}{4} - \frac{(y+3)^2}{9} = 1$.

b. First, we calculate $g_x(x, y)$ and $g_y(x, y)$:

$$\begin{aligned} g_x(x, y) &= 2x + 2y + 4 \\ g_y(x, y) &= 2x - 8y - 6. \end{aligned}$$

Next, we set each of these expressions equal to zero, which gives a system of equations in x and y:

$$\begin{aligned} 2x + 2y + 4 &= 0 \\ 2x - 8y - 6 &= 0. \end{aligned}$$

Subtracting the second equation from the first gives $10y + 10 = 0$, so $y = -1$. Substituting this into the first equation gives $2x + 2(-1) + 4 = 0$, so $x = -1$.

Therefore $(-1, -1)$ is a critical point of g. There are no points in $\mathbb{R}^2$ that make either partial derivative not exist.

Figure 14.7.1 shows the behavior of the surface at the critical point.

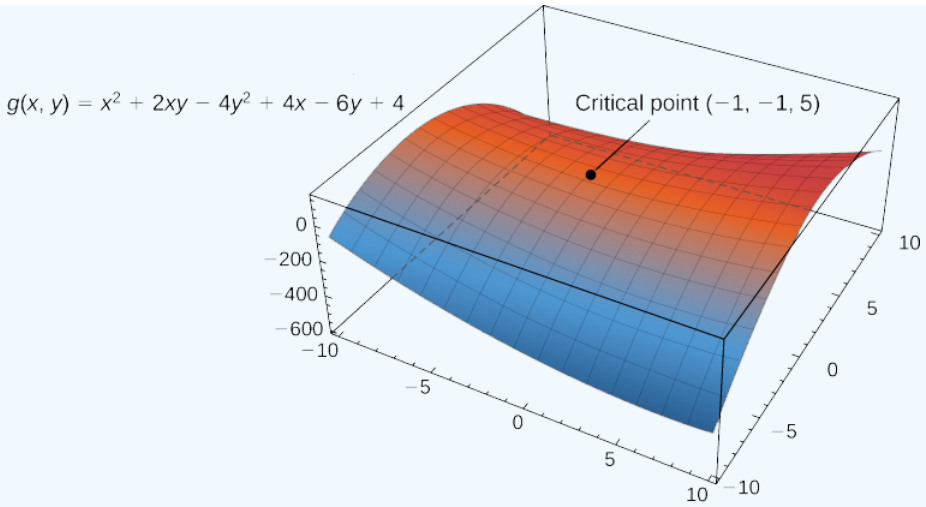

Figure 14.7.1: The function $g(x,y)$ has a critical point at $(-1,-1,5)$.

Exercise 14.7.1

Find the critical point of the function $f(x,y)=x^3+2xy-2x-4y$.

Hint

Calculate $f_x(x,y)$ and $f_y(x,y)$, then set them equal to zero.

Answer

The only critical point of f is $(2,-5)$.

The main purpose for determining critical points is to locate relative maxima and minima, as in single-variable calculus. When working with a function of one variable, the definition of a local extremum involves finding an interval around the critical point such that the function value is either greater than or less than all the other function values in that interval. When working with a function of two or more variables, we work with an open disk around the point.

Definition: Global and Local Extrema

Let $z=f(x,y)$ be a function of two variables that is defined and continuous on an open set containing the point (x_0,y_0). Then f has a **local maximum** at (x_0,y_0) if

$$f(x_0,y_0)\geq f(x,y)$$

for all points (x,y) within some disk centered at (x_0,y_0). The number $f(x_0,y_0)$ is called a local maximum value. If the preceding inequality holds for every point (x,y) in the domain of f, then f has a **global maximum** (also called an absolute maximum) at (x_0,y_0).

The function f has a local minimum at (x_0,y_0) if

$$f(x_0,y_0)\leq f(x,y)$$

for all points (x,y) within some disk centered at (x_0,y_0). The number $f(x_0,y_0)$ is called a local minimum value. If the preceding inequality holds for every point (x,y) in the domain of f, then f has a **global minimum** (also called an absolute minimum) at (x_0,y_0).

If $f(x_0,y_0)$ is either a local maximum or local minimum value, then it is called a **local extremum** (see the following figure).

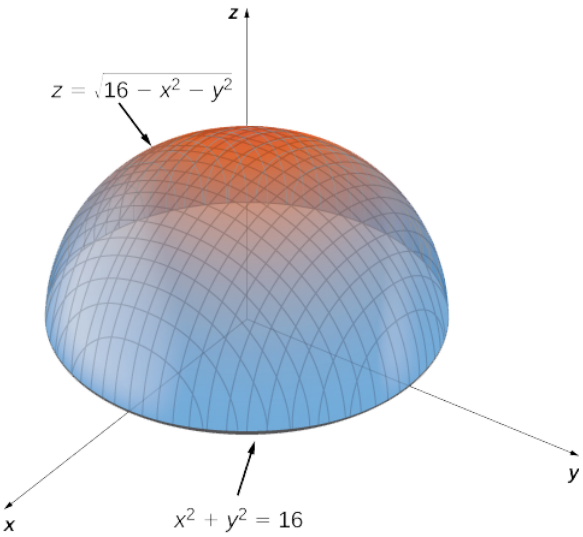

Figure 14.7.2: The graph of $z = \sqrt{16 - x^2 - y^2}$ has a maximum value when $(x, y) = (0, 0)$. It attains its minimum value at the boundary of its domain, which is the circle $x^2 + y^2 = 16$.

In Calculus 1, we showed that extrema of functions of one variable occur at critical points. The same is true for functions of more than one variable, as stated in the following theorem.

Fermat's Theorem for Functions of Two Variables

Let $z = f(x, y)$ be a function of two variables that is defined and continuous on an open set containing the point (x_0, y_0). Suppose f_x and f_y each exist at (x_0, y_0). If f has a local extremum at (x_0, y_0), then (x_0, y_0) is a critical point of f.

Second Derivative Test

Consider the function $f(x) = x^3$. This function has a critical point at $x = 0$, since $f'(0) = 3(0)^2 = 0$. However, f does not have an extreme value at $x = 0$. Therefore, the existence of a critical value at $x = x_0$ does not guarantee a local extremum at $x = x_0$. The same is true for a function of two or more variables. One way this can happen is at a **saddle point**. An example of a saddle point appears in the following figure.

$$z = x^2 - y^2$$

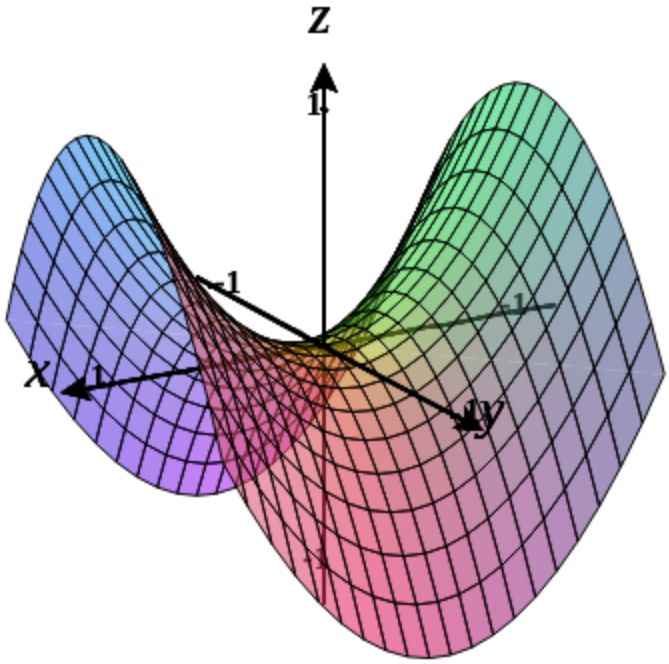

Figure 14.7.3**:** Graph of the function $z = x^2 - y^2$. This graph has a saddle point at the origin.

In this graph, the origin is a saddle point. This is because the first partial derivatives of $f(x, y) = x^2 - y^2$ are both equal to zero at this point, but it is neither a maximum nor a minimum for the function. Furthermore the vertical trace corresponding to $y = 0$ is $z = x^2$ (a parabola opening upward), but the vertical trace corresponding to $x = 0$ is $z = -y^2$ (a parabola opening downward). Therefore, it is both a global maximum for one trace and a global minimum for another.

Definition: Saddle Point

Given the function $z = f(x, y)$, the point $\big(x_0, y_0, f(x_0, y_0)\big)$ is a saddle point if both $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$, but f does not have a local extremum at (x_0, y_0).

The second derivative test for a function of one variable provides a method for determining whether an extremum occurs at a critical point of a function. When extending this result to a function of two variables, an issue arises related to the fact that there are, in fact, four different second-order partial derivatives, although equality of mixed partials reduces this to three. The second derivative test for a function of two variables, stated in the following theorem, uses a **discriminant** D that replaces $f''(x_0)$ in the second derivative test for a function of one variable.

Second Derivative Test

Let $z = f(x, y)$ be a function of two variables for which the first- and second-order partial derivatives are continuous on some disk containing the point (x_0, y_0). Suppose $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$. Define the quantity

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - \big(f_{xy}(x_0, y_0)\big)^2.$$

Then:

i. If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a local minimum at (x_0, y_0).
ii. If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a local maximum at (x_0, y_0).
iii. If $D < 0$, then f has a saddle point at (x_0, y_0).
iv. If $D = 0$, then the test is inconclusive.

See Figure 14.7.4.

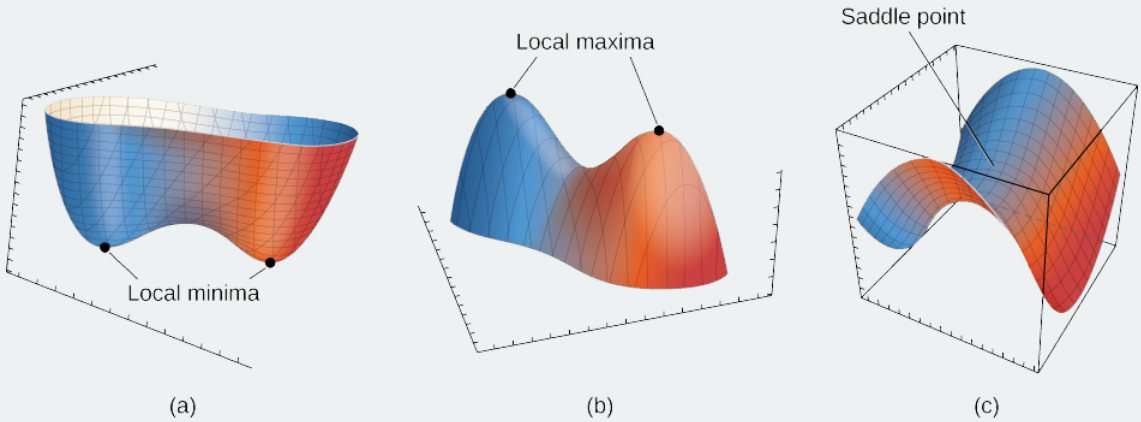

Figure 14.7.4: The second derivative test can often determine whether a function of two variables has local minima (a), local maxima (b), or a saddle point (c).

To apply the second derivative test, it is necessary that we first find the critical points of the function. There are several steps involved in the entire procedure, which are outlined in a problem-solving strategy.

Problem-Solving Strategy: Using the Second Derivative Test for Functions of Two Variables

Let $z = f(x, y)$ be a function of two variables for which the first- and second-order partial derivatives are continuous on some disk containing the point (x_0, y_0). To apply the second derivative test to find local extrema, use the following steps:

1. Determine the critical points (x_0, y_0) of the function f where $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. Discard any points where at least one of the partial derivatives does not exist.
2. Calculate the discriminant $D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - \big(f_{xy}(x_0, y_0)\big)^2$ for each critical point of f.
3. Apply the four cases of the test to determine whether each critical point is a local maximum, local minimum, or saddle point, or whether the theorem is inconclusive.

Example 14.7.2: Using the Second Derivative Test

Find the critical points for each of the following functions, and use the second derivative test to find the local extrema:

a. $f(x, y) = 4x^2 + 9y^2 + 8x - 36y + 24$
b. $g(x, y) = \frac{1}{3}x^3 + y^2 + 2xy - 6x - 3y + 4$

Solution

a. **Step 1** of the problem-solving strategy involves finding the critical points of f. To do this, we first calculate $f_x(x, y)$ and $f_y(x, y)$, then set each of them equal to zero:

$$f_x(x, y) = 8x + 8$$
$$f_y(x, y) = 18y - 36.$$

Setting them equal to zero yields the system of equations

$$8x + 8 = 0$$
$$18y - 36 = 0.$$

The solution to this system is $x = -1$ and $y = 2$. Therefore $(-1, 2)$ is a critical point of f.

Step 2 of the problem-solving strategy involves calculating D. To do this, we first calculate the second partial derivatives of f:

$$f_{xx}(x, y) = 8$$
$$f_{xy}(x, y) = 0$$
$$f_{yy}(x, y) = 18.$$

Therefore, $D = f_{xx}(-1,2)f_{yy}(-1,2) - \big(f_{xy}(-1,2)\big)^2 = (8)(18) - (0)^2 = 144$.

Step 3 states to apply the four cases of the test to classify the function's behavior at this critical point.

Since $D > 0$ and $f_{xx}(-1,2) > 0$, this corresponds to case 1. Therefore, f has a local minimum at $(-1,2)$ as shown in the following figure.

$$z = 4x^2 + 9y^2 + 8x - 36y + 24$$

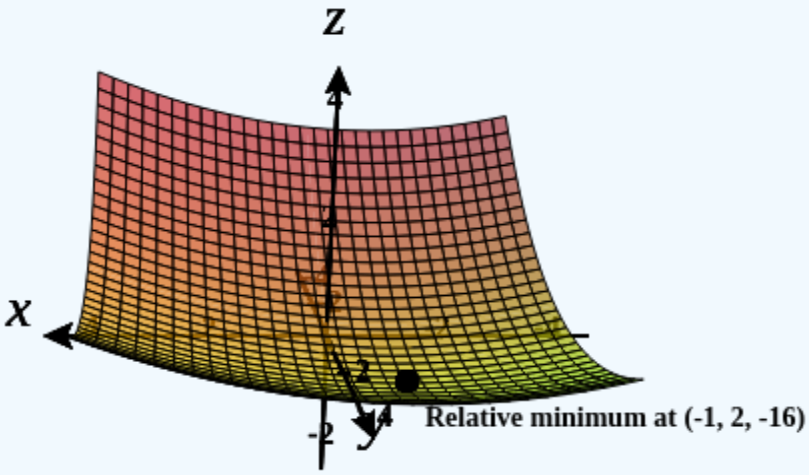

Figure 14.7.5**:** The function $f(x,y)$ has a local minimum at $(-1,2,-16)$. Note the scale on the y-axis in this plot is in thousands.

b. For **step 1**, we first calculate $g_x(x,y)$ and $g_y(x,y)$, then set each of them equal to zero:

$$g_x(x,y) = x^2 + 2y - 6$$
$$g_y(x,y) = 2y + 2x - 3.$$

Setting them equal to zero yields the system of equations

$$x^2 + 2y - 6 = 0$$
$$2y + 2x - 3 = 0.$$

To solve this system, first solve the second equation for y. This gives $y = \dfrac{3-2x}{2}$. Substituting this into the first equation gives

$$x^2 + 3 - 2x - 6 = 0$$
$$x^2 - 2x - 3 = 0$$
$$(x-3)(x+1) = 0.$$

Therefore, $x = -1$ or $x = 3$. Substituting these values into the equation $y = \dfrac{3-2x}{2}$ yields the critical points $(-1, \frac{5}{2})$ and $(3, -\frac{3}{2})$.

Step 2 involves calculating the second partial derivatives of g:

$$g_{xx}(x,y) = 2x$$
$$g_{xy}(x,y) = 2$$
$$g_{yy}(x,y) = 2.$$

Then, we find a general formula for D:

$$\begin{aligned} D(x_0,y_0) &= g_{xx}(x_0,y_0)g_{yy}(x_0,y_0) - \big(g_{xy}(x_0,y_0)\big)^2 \\ &= (2x_0)(2) - 2^2 \\ &= 4x_0 - 4. \end{aligned}$$

Next, we substitute each critical point into this formula:

$$D\left(-1,\tfrac{5}{2}\right) = (2(-1))(2) - (2)^2 = -4 - 4 = -8$$
$$D\left(3,-\tfrac{3}{2}\right) = (2(3))(2) - (2)^2 = 12 - 4 = 8.$$

In step 3, we note that, applying Note to point $\left(-1,\frac{5}{2}\right)$ leads to case 3, which means that $\left(-1,\frac{5}{2}\right)$ is a saddle point. Applying the theorem to point $\left(3,-\frac{3}{2}\right)$ leads to case 1, which means that $\left(3,-\frac{3}{2}\right)$ corresponds to a local minimum as shown in the following figure.

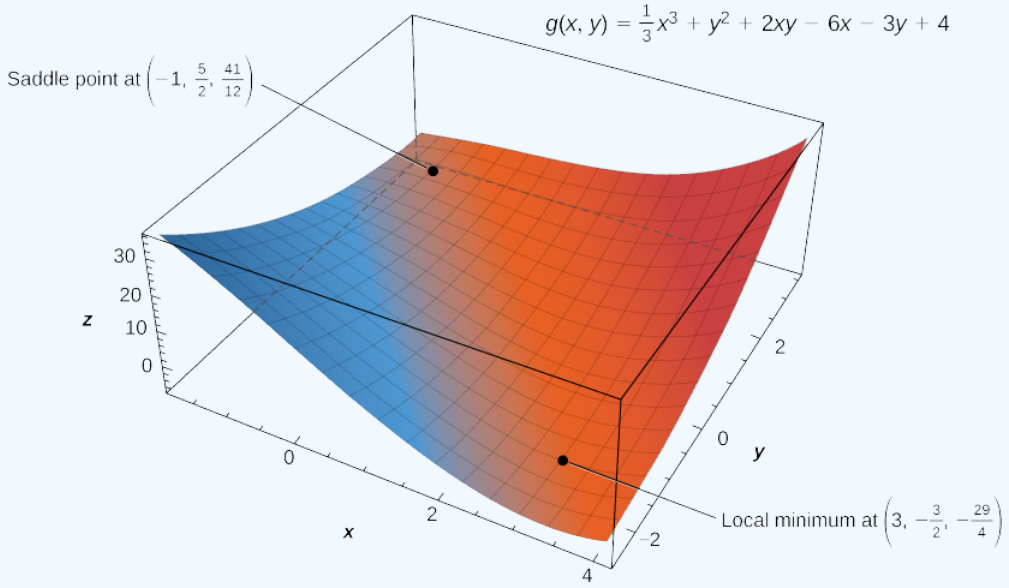

Figure 14.7.6: The function $g(x,y)$ has a local minimum and a saddle point.

Exercise 14.7.2

Use the second derivative test to find the local extrema of the function

$$f(x,y) = x^3 + 2xy - 6x - 4y^2.$$

Hint

Follow the problem-solving strategy for applying the second derivative test.

Answer

$\left(\frac{4}{3},\frac{1}{3}\right)$ is a saddle point, $\left(-\frac{3}{2},-\frac{3}{8}\right)$ is a local maximum.

Absolute Maxima and Minima

When finding global extrema of functions of one variable on a closed interval, we start by checking the critical values over that interval and then evaluate the function at the endpoints of the interval. When working with a function of two variables, the closed interval is replaced by a closed, bounded set. A set is bounded if all the points in that set can be contained within a ball (or disk) of

finite radius. First, we need to find the critical points inside the set and calculate the corresponding critical values. Then, it is necessary to find the maximum and minimum value of the function on the boundary of the set. When we have all these values, the largest function value corresponds to the global maximum and the smallest function value corresponds to the absolute minimum. First, however, we need to be assured that such values exist. The following theorem does this.

Extreme Value Theorem

A continuous function $f(x,y)$ on a closed and bounded set D in the plane attains an absolute maximum value at some point of D and an absolute minimum value at some point of D.

Now that we know any continuous function f defined on a closed, bounded set attains its extreme values, we need to know how to find them.

Finding Extreme Values of a Function of Two Variables

Assume $z = f(x,y)$ is a differentiable function of two variables defined on a closed, bounded set D. Then f will attain the absolute maximum value and the absolute minimum value, which are, respectively, the largest and smallest values found among the following:

1. The values of f at the critical points of f in D.
2. The values of f on the boundary of D.

The proof of this theorem is a direct consequence of the extreme value theorem and Fermat's theorem. In particular, if either extremum is not located on the boundary of D, then it is located at an interior point of D. But an interior point (x_0, y_0) of D that's an absolute extremum is also a local extremum; hence, (x_0, y_0) is a critical point of f by Fermat's theorem. Therefore the only possible values for the global extrema of f on D are the extreme values of f on the interior or boundary of D.

Problem-Solving Strategy: Finding Absolute Maximum and Minimum Values

Let $z = f(x,y)$ be a continuous function of two variables defined on a closed, bounded set D, and assume f is differentiable on D. To find the absolute maximum and minimum values of f on D, do the following:

1. Determine the critical points of f in D.
2. Calculate f at each of these critical points.
3. Determine the maximum and minimum values of f on the boundary of its domain.
4. The maximum and minimum values of f will occur at one of the values obtained in steps 2 and 3.

Finding the maximum and minimum values of f on the boundary of D can be challenging. If the boundary is a rectangle or set of straight lines, then it is possible to parameterize the line segments and determine the maxima on each of these segments, as seen in Example 14.7.3. The same approach can be used for other shapes such as circles and ellipses.

If the boundary of the set D is a more complicated curve defined by a function $g(x,y) = c$ for some constant c, and the first-order partial derivatives of g exist, then the method of Lagrange multipliers can prove useful for determining the extrema of f on the boundary which is introduced in Lagrange Multipliers.

Example 14.7.3: Finding Absolute Extrema

Use the problem-solving strategy for finding absolute extrema of a function to determine the absolute extrema of each of the following functions:

a. $f(x,y) = x^2 - 2xy + 4y^2 - 4x - 2y + 24$ on the domain defined by $0 \le x \le 4$ and $0 \le y \le 2$
b. $g(x,y) = x^2 + y^2 + 4x - 6y$ on the domain defined by $x^2 + y^2 \le 16$

Solution

a. Using the problem-solving strategy, step 1 involves finding the critical points of f on its domain. Therefore, we first calculate $f_x(x,y)$ and $f_y(x,y)$, then set them each equal to zero:

$$f_x(x,y) = 2x - 2y - 4$$
$$f_y(x,y) = -2x + 8y - 2.$$

Setting them equal to zero yields the system of equations

$$2x - 2y - 4 = 0$$
$$-2x + 8y - 2 = 0.$$

The solution to this system is $x = 3$ and $y = 1$. Therefore $(3,1)$ is a critical point of f. Calculating $f(3,1)$ gives $f(3,1) = 17$.

The next step involves finding the extrema of f on the boundary of its domain. The boundary of its domain consists of four line segments as shown in the following graph:

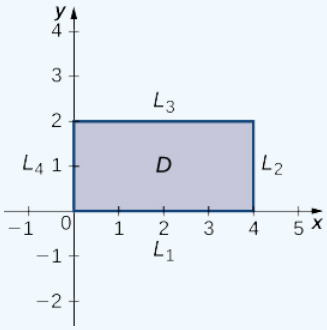

Figure 14.7.7: Graph of the domain of the function $f(x,y) = x^2 - 2xy + 4y^2 - 4x - 2y + 24$.

L_1 is the line segment connecting $(0,0)$ and $(4,0)$, and it can be parameterized by the equations $x(t) = t, y(t) = 0$ for $0 \le t \le 4$. Define $g(t) = f\big(x(t), y(t)\big)$. This gives $g(t) = t^2 - 4t + 24$. Differentiating g leads to $g'(t) = 2t - 4$. Therefore, g has a critical value at $t = 2$, which corresponds to the point $(2,0)$. Calculating $f(2,0)$ gives the z-value 20.

L_2 is the line segment connecting $(4,0)$ and $(4,2)$, and it can be parameterized by the equations $x(t) = 4, y(t) = t$ for $0 \le t \le 2$. Again, define $g(t) = f\big(x(t), y(t)\big)$. This gives $g(t) = 4t^2 - 10t + 24$. Then, $g'(t) = 8t - 10$. g has a critical value at $t = \frac{5}{4}$, which corresponds to the point $\left(4, \frac{5}{4}\right)$. Calculating $f\left(4, \frac{5}{4}\right)$ gives the z-value 17.75.

L_3 is the line segment connecting $(0,2)$ and $(4,2)$, and it can be parameterized by the equations $x(t) = t, y(t) = 2$ for $0 \le t \le 4$. Again, define $g(t) = f\big(x(t), y(t)\big)$. This gives $g(t) = t^2 - 8t + 36$. The critical value corresponds to the point $(4,2)$. So, calculating $f(4,2)$ gives the z-value 20.

L_4 is the line segment connecting $(0,0)$ and $(0,2)$, and it can be parameterized by the equations $x(t) = 0, y(t) = t$ for $0 \le t \le 2$. This time, $g(t) = 4t^2 - 2t + 24$ and the critical value $t = \frac{1}{4}$ correspond to the point $\left(0, \frac{1}{4}\right)$. Calculating $f\left(0, \frac{1}{4}\right)$ gives the z-value 23.75.

We also need to find the values of $f(x,y)$ at the corners of its domain. These corners are located at $(0,0), (4,0), (4,2)$ and $(0,2)$:

$$f(0,0) = (0)^2 - 2(0)(0) + 4(0)^2 - 4(0) - 2(0) + 24 = 24$$
$$f(4,0) = (4)^2 - 2(4)(0) + 4(0)^2 - 4(4) - 2(0) + 24 = 24$$
$$f(4,2) = (4)^2 - 2(4)(2) + 4(2)^2 - 4(4) - 2(2) + 24 = 20$$
$$f(0,2) = (0)^2 - 2(0)(2) + 4(2)^2 - 4(0) - 2(2) + 24 = 36.$$

The absolute maximum value is 36, which occurs at $(0,2)$, and the global minimum value is 20, which occurs at both $(4,2)$ and $(2,0)$ as shown in the following figure.

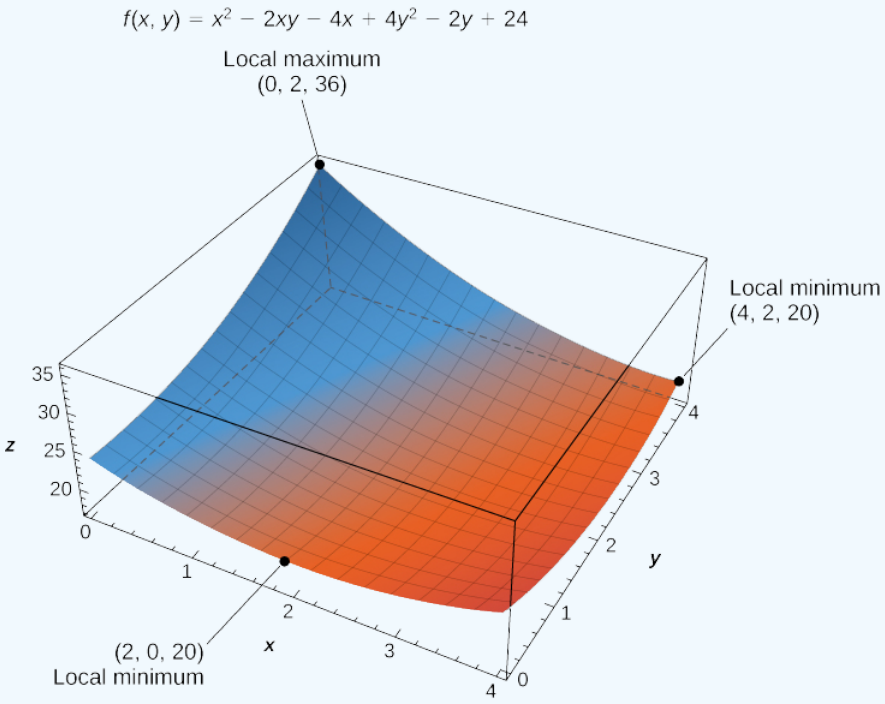

Figure 14.7.8: The function $f(x,y)$ has two global minima and one global maximum over its domain.

b. Using the problem-solving strategy, step 1 involves finding the critical points of g on its domain. Therefore, we first calculate $g_x(x,y)$ and $g_y(x,y)$, then set them each equal to zero:

$$g_x(x,y) = 2x + 4$$
$$g_y(x,y) = 2y - 6.$$

Setting them equal to zero yields the system of equations

$$2x + 4 = 0$$
$$2y - 6 = 0.$$

The solution to this system is $x = -2$ and $y = 3$. Therefore, $(-2,3)$ is a critical point of g. Calculating $g(-2,3)$, we get

$$g(-2,3) = (-2)^2 + 3^2 + 4(-2) - 6(3) = 4 + 9 - 8 - 18 = -13.$$

The next step involves finding the extrema of g on the boundary of its domain. The boundary of its domain consists of a circle of radius 4 centered at the origin as shown in the following graph.

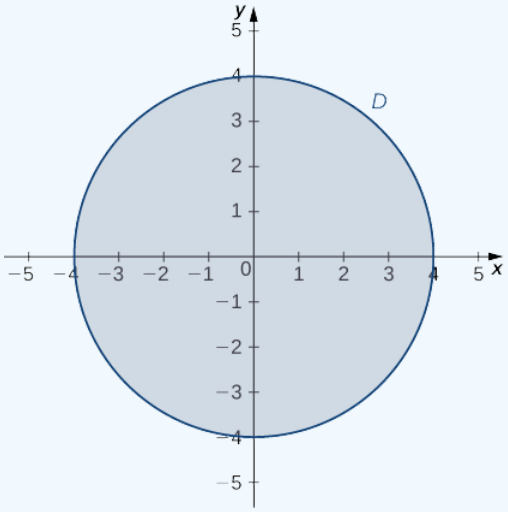

Figure 14.7.9: Graph of the restricted domain of the function $g(x,y) = x^2 + y^2 + 4x - 6y$.

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The boundary of the domain of g can be parameterized using the functions $x(t) = 4\cos t,\ y(t) = 4\sin t$ for $0 \le t \le 2\pi$. Define $h(t) = g\big(x(t), y(t)\big)$:

$$\begin{aligned} h(t) &= g\big(x(t), y(t)\big) \\ &= (4\cos t)^2 + (4\sin t)^2 + 4(4\cos t) - 6(4\sin t) \\ &= 16\cos^2 t + 16\sin^2 t + 16\cos t - 24\sin t \\ &= 16 + 16\cos t - 24\sin t. \end{aligned}$$

Setting $h'(t) = 0$ leads to

$$\begin{aligned} -16\sin t - 24\cos t &= 0 \\ -16\sin t &= 24\cos t \\ \frac{-16\sin t}{-16\cos t} &= \frac{24\cos t}{-16\cos t} \\ \tan t &= -\frac{3}{2}. \end{aligned}$$

This equation has two solutions over the interval $0 \le t \le 2\pi$. One is $t = \pi - \arctan(\frac{3}{2})$ and the other is $t = 2\pi - \arctan(\frac{3}{2})$. For the first angle,

$$\sin t = \sin(\pi - \arctan(\tfrac{3}{2})) = \sin(\arctan(\tfrac{3}{2})) = \frac{3\sqrt{13}}{13}$$

$$\cos t = \cos(\pi - \arctan(\tfrac{3}{2})) = -\cos(\arctan(\tfrac{3}{2})) = -\frac{2\sqrt{13}}{13}.$$

Therefore, $x(t) = 4\cos t = -\frac{8\sqrt{13}}{13}$ and $y(t) = 4\sin t = \frac{12\sqrt{13}}{13}$, so $\left(-\frac{8\sqrt{13}}{13}, \frac{12\sqrt{13}}{13}\right)$ is a critical point on the boundary and

$$\begin{aligned} g\left(-\tfrac{8\sqrt{13}}{13}, \tfrac{12\sqrt{13}}{13}\right) &= \left(-\tfrac{8\sqrt{13}}{13}\right)^2 + \left(\tfrac{12\sqrt{13}}{13}\right)^2 + 4\left(-\tfrac{8\sqrt{13}}{13}\right) - 6\left(\tfrac{12\sqrt{13}}{13}\right) \\ &= \frac{144}{13} + \frac{64}{13} - \frac{32\sqrt{13}}{13} - \frac{72\sqrt{13}}{13} \\ &= \frac{208 - 104\sqrt{13}}{13} \approx -12.844. \end{aligned}$$

For the second angle,

$$\sin t = \sin(2\pi - \arctan(\tfrac{3}{2})) = -\sin(\arctan(\tfrac{3}{2})) = -\frac{3\sqrt{13}}{13}$$

$$\cos t = \cos(2\pi - \arctan(\tfrac{3}{2})) = \cos(\arctan(\tfrac{3}{2})) = \frac{2\sqrt{13}}{13}.$$

Therefore, $x(t) = 4\cos t = \frac{8\sqrt{13}}{13}$ and $y(t) = 4\sin t = -\frac{12\sqrt{13}}{13}$, so $\left(\frac{8\sqrt{13}}{13}, -\frac{12\sqrt{13}}{13}\right)$ is a critical point on the boundary and

$$\begin{aligned} g\left(\tfrac{8\sqrt{13}}{13}, -\tfrac{12\sqrt{13}}{13}\right) &= \left(\tfrac{8\sqrt{13}}{13}\right)^2 + \left(-\tfrac{12\sqrt{13}}{13}\right)^2 + 4\left(\tfrac{8\sqrt{13}}{13}\right) - 6\left(-\tfrac{12\sqrt{13}}{13}\right) \\ &= \frac{144}{13} + \frac{64}{13} + \frac{32\sqrt{13}}{13} + \frac{72\sqrt{13}}{13} \\ &= \frac{208 + 104\sqrt{13}}{13} \approx 44.844. \end{aligned}$$

The absolute minimum of g is -13, which is attained at the point $(-2, 3)$, which is an interior point of D. The absolute maximum of g is approximately equal to 44.844, which is attained at the boundary point $\left(\frac{8\sqrt{13}}{13}, -\frac{12\sqrt{13}}{13}\right)$. These are the absolute extrema of g on D as shown in the following figure.

$$f(x,y) = x^2 + 4x + y^2 - 6y$$

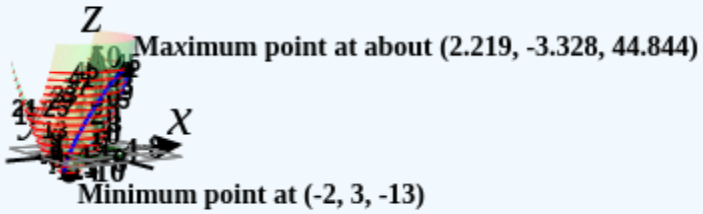

Figure 14.7.10: The function $f(x,y)$ has a local minimum and a local maximum.

? Exercise 14.7.3:

Use the problem-solving strategy for finding absolute extrema of a function to find the absolute extrema of the function

$$f(x,y) = 4x^2 - 2xy + 6y^2 - 8x + 2y + 3$$

on the domain defined by $0 \le x \le 2$ and $-1 \le y \le 3$.

Hint

Calculate $f_x(x,y)$ and $f_y(x,y)$, and set them equal to zero. Then, calculate f for each critical point and find the extrema of f on the boundary of D.

Answer

The absolute minimum occurs at $(1,0)$: $f(1,0) = -1$.

The absolute maximum occurs at $(0,3)$: $f(0,3) = 63$.

✓ Example 14.7.4: Profitable Golf Balls

Pro-T company has developed a profit model that depends on the number x of golf balls sold per month (measured in thousands), and the number of hours per month of advertising y, according to the function

$$z = f(x,y) = 48x + 96y - x^2 - 2xy - 9y^2,$$

where z is measured in thousands of dollars. The maximum number of golf balls that can be produced and sold is $50,000$, and the maximum number of hours of advertising that can be purchased is 25. Find the values of x and y that maximize profit, and find the maximum profit.

Figure 14.7.11: (credit: modification of work by oatsy40, Flickr)

Solution

Using the problem-solving strategy, step 1 involves finding the critical points of f on its domain. Therefore, we first calculate $f_x(x,y)$ and $f_y(x,y)$, then set them each equal to zero:

$$f_x(x,y) = 48 - 2x - 2y$$
$$f_y(x,y) = 96 - 2x - 18y.$$

Setting them equal to zero yields the system of equations

$$48 - 2x - 2y = 0$$
$$96 - 2x - 18y = 0.$$

The solution to this system is $x = 21$ and $y = 3$. Therefore $(21,3)$ is a critical point of f. Calculating $f(21,3)$ gives $f(21,3) = 48(21) + 96(3) - 21^2 - 2(21)(3) - 9(3)^2 = 648$.

The domain of this function is $0 \le x \le 50$ and $0 \le y \le 25$ as shown in the following graph.

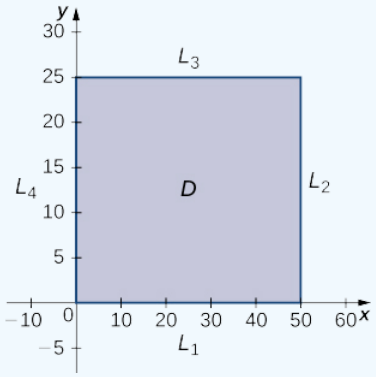

Figure 14.7.12: Graph of the domain of the function $f(x,y) = 48x + 96y - x^2 - 2xy - 9y^2$.

L_1 is the line segment connecting $(0,0)$ and $(50,0)$, and it can be parameterized by the equations $x(t) = t, y(t) = 0$ for $0 \le t \le 50$. We then define $g(t) = f(x(t), y(t))$:

$$\begin{aligned} g(t) &= f(x(t), y(t)) \\ &= f(t,0) \\ &= 48t + 96(0) - y^2 - 2(t)(0) - 9(0)^2 \\ &= 48t - t^2. \end{aligned}$$

Setting $g'(t) = 0$ yields the critical point $t = 24$, which corresponds to the point $(24,0)$ in the domain of f. Calculating $f(24,0)$ gives 576.

L_2 is the line segment connecting $(50,0)$ and $(50,25)$, and it can be parameterized by the equations $x(t) = 50, y(t) = t$ for $0 \le t \le 25$. Once again, we define $g(t) = f\big(x(t), y(t)\big)$:

$$\begin{aligned} g(t) &= f\big(x(t), y(t)\big) \\ &= f(50, t) \\ &= 48(50) + 96t - 50^2 - 2(50)t - 9t^2 \\ &= -9t^2 - 4t - 100. \end{aligned}$$

This function has a critical point at $t = -\frac{2}{9}$, which corresponds to the point $(50, -29)$. This point is not in the domain of f.

L_3 is the line segment connecting $(0, 25)$ and $(50, 25)$, and it can be parameterized by the equations $x(t) = t, y(t) = 25$ for $0 \leq t \leq 50$. We define $g(t) = f\big(x(t), y(t)\big)$:

$$\begin{aligned} g(t) &= f\big(x(t), y(t)\big) \\ &= f(t, 25) \\ &= 48t + 96(25) - t^2 - 2t(25) - 9(25^2) \\ &= -t^2 - 2t - 3225. \end{aligned}$$

This function has a critical point at $t = -1$, which corresponds to the point $(-1, 25)$, which is not in the domain.

L_4 is the line segment connecting $(0, 0)$ to $(0, 25)$, and it can be parameterized by the equations $x(t) = 0, y(t) = t$ for $0 \leq t \leq 25$. We define $g(t) = f\big(x(t), y(t)\big)$:

$$\begin{aligned} g(t) &= f\big(x(t), y(t)\big) \\ &= f(0, t) \\ &= 48(0) + 96t - (0)^2 - 2(0)t - 9t^2 \\ &= 96t - 9t^2. \end{aligned}$$

This function has a critical point at $t = \frac{16}{3}$, which corresponds to the point $\left(0, \frac{16}{3}\right)$, which is on the boundary of the domain. Calculating $f\left(0, \frac{16}{3}\right)$ gives 256.

We also need to find the values of $f(x, y)$ at the corners of its domain. These corners are located at $(0, 0), (50, 0), (50, 25)$ and $(0, 25)$:

$$\begin{aligned} f(0, 0) &= 48(0) + 96(0) - (0)^2 - 2(0)(0) - 9(0)^2 = 0 \\ f(50, 0) &= 48(50) + 96(0) - (50)^2 - 2(50)(0) - 9(0)^2 = -100 \\ f(50, 25) &= 48(50) + 96(25) - (50)^2 - 2(50)(25) - 9(25)^2 = -5825 \\ f(0, 25) &= 48(0) + 96(25) - (0)^2 - 2(0)(25) - 9(25)^2 = -3225. \end{aligned}$$

The maximum value is 648, which occurs at $(21, 3)$. Therefore, a maximum profit of $\$648,000$ is realized when $21,000$ golf balls are sold and 3 hours of advertising are purchased per month as shown in the following figure.

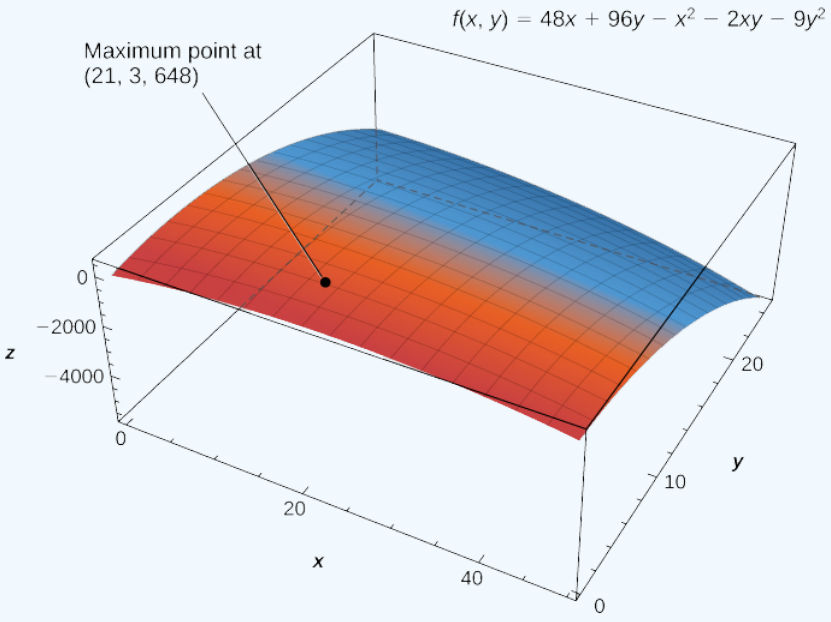

Figure 14.7.13: The profit function $f(x, y)$ has a maximum at $(21, 3, 648)$.

Key Concepts

- A critical point of the function $f(x, y)$ is any point (x_0, y_0) where either $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$, or at least one of $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ do not exist.
- A saddle point is a point (x_0, y_0) where $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$, but $f(x_0, y_0)$ is neither a maximum nor a minimum at that point.
- To find extrema of functions of two variables, first find the critical points, then calculate the discriminant and apply the second derivative test.

Key Equations

- **Discriminant**

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$$

Glossary

critical point of a function of two variables

the point (x_0, y_0) is called a critical point of $f(x, y)$ if one of the two following conditions holds:

1. $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$
2. At least one of $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ do not exist

discriminant

the discriminant of the function $f(x, y)$ is given by the formula $D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$

saddle point

given the function $z = f(x, y)$, the point $(x_0, y_0, f(x_0, y_0))$ is a saddle point if both $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$, but f does not have a local extremum at (x_0, y_0)

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14.7E: Exercises for Section 14.7

Finding Critical Points

In exercises 1 - 5, find all critical points.

1) $f(x,y) = 1 + x^2 + y^2$

Answer

$(0,0)$

2) $f(x,y) = 1 - (x-2)^2 + (y+3)^2$

3) $f(x,y) = (3x-2)^2 + (y-4)^2$

Answer

$\left(\frac{2}{3}, 4\right)$

4) $f(x,y) = x^4 + y^4 - 16xy$

Answer

$(0,0), \quad (-2,-2), \quad (2,2)$

5) $f(x,y) = 15x^3 - 3xy + 15y^3$

Answer

$(0,0), \quad \left(\frac{1}{15}, \frac{1}{15}\right)$

Finding Extrema & the Second Partials Test

In exercises 6 - 9, find the critical points of the function and test for extrema or saddle points by using algebraic techniques (completing the square) or by examining the form of the equation. Where possible, verify your results using the Second Partials Test.

6) $f(x,y) = -\sqrt{x^2 + y^2}$

Answer

Crit. pts.: $(0,0)$
Extrema: f has a relative maximum of 0 at $(0,0)$.
To justify this, consider the fact that the square root function cannot give a negative value, so this function cannot possibly return a positive value. Since it's value is 0 at the critical point $(0,0)$, we know it must be the function's **absolute maximum** value.

7) $f(x,y) = -x^2 - 5y^2 + 8x - 10y - 13$

Answer

Crit. pts.: $(4,-1)$
Extrema: f has a relative maximum of 8 at $(4,-1)$.
To justify this, we complete the square on this function, being careful to factor out the coefficient of the squared terms before completing the square.

$$\begin{aligned} f(x,y) &= -x^2 - 5y^2 + 8x - 10y - 13 \\ &= -(x^2 - 8x \quad\quad) - 5(y^2 + 2y \quad\quad) - 13 \\ &= -(x^2 - 8x + 16) - 5(y^2 + 2y + 1) - 13 + 16 + 5 \\ &= -(x-4)^2 - 5(y+1)^2 + 8 \end{aligned}$$

Note that this quadratic polynomial function takes the form $z = -(x^2 + y^2)$, so we can see that it will have a **relative** (and, in fact, absolute) **maximum** at its vertex (the critical point $(4,-1)$). We can also argue that since we are subtracting

squared terms from 8, we cannot possibly obtain a function value larger than 8, and since we do get a value of 8 at the critical point $(4,-1)$, we know it will be the absolute maximum value of this function.

8) $f(x,y)=x^2+y^2+2x-6y+6$

9) $f(x,y)=\sqrt{x^2+y^2}+1$

Answer

Crit. Pts.: $(0,0)$
Extrema: f has a relative minimum of 1 at $(0,0)$.
To justify this, consider the fact that the square root function cannot give a negative value, so this function cannot possibly return a value less than 1. Since it's value is 1 at the critical point $(0,0)$, we know 1 must be the function's **absolute minimum** value.

In exercises 10 - 34, identify any critical points and use the Second Partials Test to determine the behavior of the function at each critical point, whether there is a maximum, minimum, saddle point, or none of these. If the Second Partials Test fails, determine the function's behavior at the point using another method and justify your answer clearly.

10) $f(x,y)=-x^3+4xy-2y^2+1$

11) $f(x,y)=x^2y^2$

Answer

Crit. pts.: All points on the lines $x=0$ and $y=0$ are critical points of this function.
Exrema: The Second Partials Test fails.
Since $x^2y^2>0$ for all x and y different from zero, and $x^2y^2=0$ when either x or y equals zero (or both), then the absolute minimum of 0 occurs at all points on the x- or y-axes, that is, for all points on the lines $x=0$ and $y=0$.

12) $f(x,y)=x^2-6x+y^2+4y-8$

13) $f(x,y)=2xy+3x+4y$

Answer

Crit. pts.: $\left(-2,-\frac{3}{2}\right)$
Exrema: f has a saddle point at $\left(-2,-\frac{3}{2},-6\right)$.

14) $f(x,y)=8xy(x+y)+7$

15) $f(x,y)=x^2+4xy+y^2$

Answer

Crit. pts.: $(0,0)$
Exrema: f has a saddle point at $(0,0,0)$.

16) $f(x,y)=x^3+y^3-300x-75y-3$

17) $f(x,y)=9-x^4y^4$

Answer

Crit. pts.: All points on the lines $x=0$ and $y=0$ are critical points of this function.
Extrema: The Second Partials Test fails.
Since the term $-x^4y^4<0$ for all x and y different from zero, and $-x^4y^4=0$ when either x or y equals zero (or both), then this function cannot attain a value greater than 9 anywhere, but is 9 at the critical points. Thus f has an absolute maximum of 9 at all points on the x- or y-axes, that is, for all points on the lines $x=0$ and $y=0$.

18) $f(x,y)=x^2+10xy+y^2$

Answer

Crit. pts.: $(0,0)$
Extrema: f has a saddle point at $(0,0,0)$.

19) $f(x,y) = x^4 + y^2 + 2xy + 3$

Answer

Crit. pts.: $(0,0), \quad \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \quad \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
Extrema: f has a saddle point at $(0,0,3)$,
f has a local minimum of 2.75 at the point $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.
f has a local minimum of 2.75 at the point $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

20) $f(x,y) = 7x^2y + 9xy^2$

21) $f(x,y) = 3x^2 - 2xy + y^2 - 8y$

Answer

Crit. pts.: $(2,6)$
Extrema: f has a relative minimum of -24 located at $(2,6)$.

22) $f(x,y) = 3x^2 + 2xy + y^2$

23) $f(x,y) = y^2 + xy + 3y + 2x + 3$

Answer

Crit. pts.: $(1,-2)$
Extrema: f has a saddle point at $(1,-2,1)$.

24) $f(x,y) = x^2 + xy + y^2 - 3x$

25) $f(x,y) = x^2 + 2y^2 - x^2y$

Answer

Crit. pts.: $(0,0), \quad (-2,1), \quad (2,1)$
Extrema: f has a relative minimum of 0 at $(0,0)$ and saddle points at $(2,1,2)$ and $(-2,1,2)$.

26) $f(x,y) = x^2 + y - e^y$

27) $f(x,y) = e^{-(x^2+y^2+2x)}$

Answer

Crit. pts.: $(-1,0)$
Extrema: f has a relative maximum of e located at $(-1,0)$.
See this problem illustrated in CalcPlot3D.

28) $f(x,y) = x^2 + xy + y^2 - x - y + 1$

29) $f(x,y) = x^2y(9 - x + y)$

Answer

Crit. pts.: $\left(\frac{9}{2}, -\frac{9}{4}\right), \quad (9,0)$, and all points on the line $x = 0$
Extrema: f has a saddle point at $(9,0,0)$ and a relative minimum of -102.515625 at $\left(\frac{9}{2}, -\frac{9}{4}\right)$.
At the critical points on the line $x = 0$, f has neither relative extrema nor saddle points, but they do represent a sort of trough on the surface.

30) $f(x,y) = -x^2 - 5y^2 + 10x - 30y - 62$

31) $f(x,y) = 120x + 120y - xy - x^2 - y^2$

Answer

Crit. pts.: $(40, 40)$
Extrema: f has a relative maximum of 4800 located at $(40, 40)$.

32) $f(x, y) = 2x^2 + 2xy + y^2 + 2x - 3$

33) $f(x, y) = x^2 + x - 3xy + y^3 - 5$

Answer

Crit. pts.: $\left(\frac{1}{4}, \frac{1}{2}\right)$ and $(1, 1)$
Extrema: f has a saddle point at $\left(\frac{1}{4}, \frac{1}{2}, -\frac{79}{16}\right)$ and a relative minimum of -5 at $(1, 1)$.

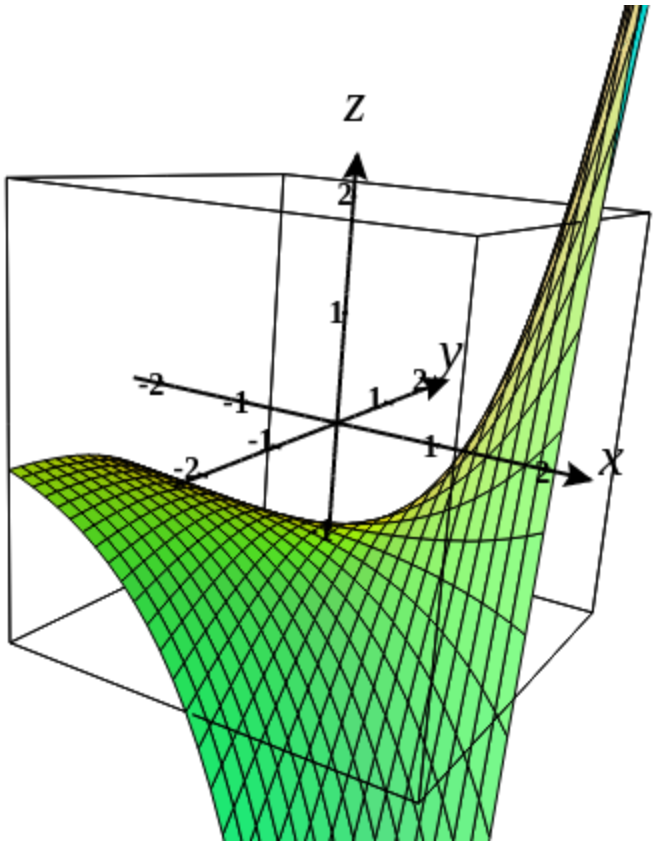
34) $f(x, y) = 2xye^{-x^2-y^2}$

In exercises 35 - 37, determine the extreme values and the saddle points. Use a CAS to graph the function.

35) [T] $f(x, y) = ye^x - e^y$

Answer

A saddle point is located at $(0, 0, -1)$.

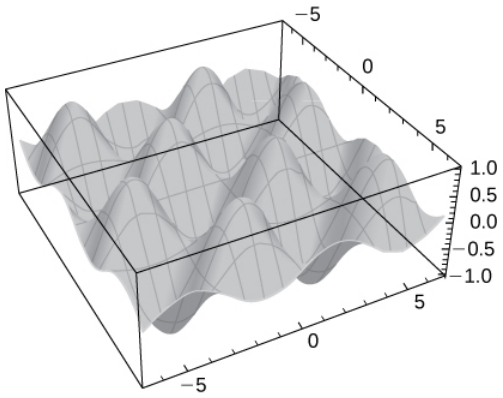


36) [T] $f(x, y) = x \sin(y)$

37) [T] $f(x, y) = \sin(x) \sin(y), \quad x \in (0, 2\pi), \quad y \in (0, 2\pi)$

Answer

There is a saddle point at (π, π), local maxima at $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$, and local minima at $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, \frac{\pi}{2}\right)$.



In exercises 38 - 41, find the absolute extrema of the given function on the indicated closed and bounded set R.

38) $f(x,y) = xy - x - 3y$ where R is the triangular region with vertices $(0,0)$, $(0,4)$, and $(5,0)$.

39) Find the absolute maximum and minimum values of $f(x,y) = x^2 + y^2 - 2y + 1$ on the region $R = \{(x,y) \,|\, x^2 + y^2 \leq 4\}$.

Answer

$(0,1,0)$ is the absolute minimum and $(0,-2,9)$ is the absolute maximum.

40) $f(x,y) = x^3 - 3xy - y^3$ on $R = \{(x,y) \,|\, -2 \leq x \leq 2,\ -2 \leq y \leq 2\}$

41) $f(x,y) = \dfrac{-2y}{x^2 + y^2 + 1}$ on $R = \{(x,y) \,|\, x^2 + y^2 \leq 4\}$

Answer

There is an absolute minimum at $(0,1,-1)$ and an absolute maximum at $(0,-1,1)$.

42) Find three positive numbers the sum of which is 27, such that the sum of their squares is as small as possible.

43) Find the points on the surface $x^2 - yz = 5$ that are closest to the origin.

Answer

$\left(\sqrt{5},0,0\right),\ \left(-\sqrt{5},0,0\right)$

44) Find the maximum volume of a rectangular box with three faces in the coordinate planes and a vertex in the first octant on the line $x + y + z = 1$.

45) The sum of the length and the girth (perimeter of a cross-section) of a package carried by a delivery service cannot exceed 108 in. Find the dimensions of the rectangular package of largest volume that can be sent.

Answer

18 by 36 by 18 in.

46) A cardboard box without a lid is to be made with a volume of 4 ft^3. Find the dimensions of the box that requires the least amount of cardboard.

47) Find the point on the surface $f(x,y) = x^2 + y^2 + 10$ nearest the plane $x + 2y - z = 0$. Identify the point on the plane.

Answer

$\left(\frac{47}{24}, \frac{47}{12}, \frac{235}{24}\right)$

48) Find the point in the plane $2x - y + 2z = 16$ that is closest to the origin.

49) A company manufactures two types of athletic shoes: jogging shoes and cross-trainers. The total revenue from x units of jogging shoes and y units of cross-trainers is given by $R(x,y) = -5x^2 - 8y^2 - 2xy + 42x + 102y$, where x and y are in thousands of units. Find the values of x and y to maximize the total revenue.

Answer

$x = 3$ and $y = 6$

50) A shipping company handles rectangular boxes provided the sum of the length, width, and height of the box does not exceed 96in. Find the dimensions of the box that meets this condition and has the largest volume.

51) Find the maximum volume of a cylindrical soda can such that the sum of its height and circumference is 120 cm.

Answer

$V = \dfrac{64,000}{\pi} \approx 20,372 \text{ cm}^3$

Contributors

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.
- Paul Seeburger (Monroe Community College) created problems 19 and 29, and added dynamic figures for problems 27 and 35.

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14.8: Lagrange Multipliers

Learning Objectives

- Use the method of Lagrange multipliers to solve optimization problems with one constraint.
- Use the method of Lagrange multipliers to solve optimization problems with two constraints.

Solving optimization problems for functions of two or more variables can be similar to solving such problems in single-variable calculus. However, techniques for dealing with multiple variables allow us to solve more varied optimization problems for which we need to deal with additional conditions or constraints. In this section, we examine one of the more common and useful methods for solving optimization problems with constraints.

Lagrange Multipliers

In the previous section, an applied situation was explored involving maximizing a profit function, subject to certain **constraints.** In that example, the constraints involved a maximum number of golf balls that could be produced and sold in 1 month (x), and a maximum number of advertising hours that could be purchased per month (y). Suppose these were combined into a single budgetary constraint, such as $20x + 4y \leq 216$, that took into account both the cost of producing the golf balls and the number of advertising hours purchased per month. The goal is still to maximize profit, but now there is a different type of constraint on the values of x and y. This constraint and the corresponding profit function

$$f(x, y) = 48x + 96y - x^2 - 2xy - 9y^2$$

is an example of an **optimization problem**, and the function $f(x, y)$ is called the **objective function**. A graph of various level curves of the function $f(x, y)$ follows.

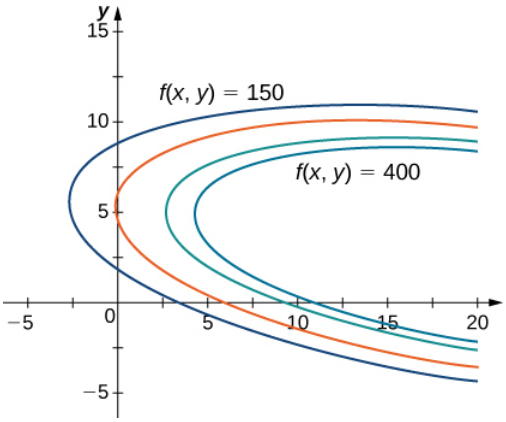

Figure 14.8.1: Graph showing level curves of the function $f(x, y) = 48x + 96y - x^2 - 2xy - 9y^2$ corresponding to $c = 150, 250, 350,$ and 400.

In Figure $14.8.1$, the value c represents different profit levels (i.e., values of the function f). As the value of c increases, the curve shifts to the right. Since our goal is to maximize profit, we want to choose a curve as far to the right as possible. If there were no restrictions on the number of golf balls the company could produce or the number of units of advertising available, then we could produce as many golf balls as we want, and advertise as much as we want, and there would be not be a maximum profit for the company. Unfortunately, we have a budgetary constraint that is modeled by the inequality $20x + 4y \leq 216$. To see how this constraint interacts with the profit function, Figure $14.8.2$ shows the graph of the line $20x + 4y = 216$ superimposed on the previous graph.

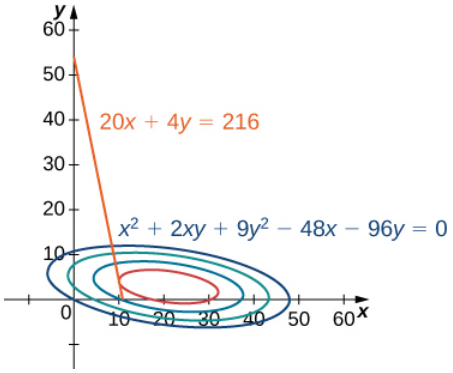

Figure 14.8.2: Graph of level curves of the function $f(x,y) = 48x + 96y - x^2 - 2xy - 9y^2$ corresponding to $c = 150, 250, 350,$ and 395. The red graph is the constraint function.

As mentioned previously, the maximum profit occurs when the level curve is as far to the right as possible. However, the level of production corresponding to this maximum profit must also satisfy the budgetary constraint, so the point at which this profit occurs must also lie on (or to the left of) the red line in Figure 14.8.2. Inspection of this graph reveals that this point exists where the line is tangent to the level curve of f. Trial and error reveals that this profit level seems to be around 395, when x and y are both just less than 5. We return to the solution of this problem later in this section. From a theoretical standpoint, at the point where the profit curve is tangent to the constraint line, the gradient of both of the functions evaluated at that point must point in the same (or opposite) direction. Recall that the gradient of a function of more than one variable is a vector. If two vectors point in the same (or opposite) directions, then one must be a constant multiple of the other. This idea is the basis of the **method of Lagrange multipliers.**

Theorem 14.8.1: Method of Lagrange Multipliers with One Constraint

Let f and g be functions of two variables with continuous partial derivatives at every point of some open set containing the smooth curve $g(x,y) = 0$. Suppose that f, when restricted to points on the curve $g(x,y) = 0$, has a local extremum at the point (x_0, y_0) and that $\vec{\nabla} g(x_0, y_0) \neq 0$. Then there is a number λ called a **Lagrange multiplier,** for which

$$\vec{\nabla} f(x_0, y_0) = \lambda \vec{\nabla} g(x_0, y_0).$$

Proof

Assume that a constrained extremum occurs at the point (x_0, y_0). Furthermore, we assume that the equation $g(x,y) = 0$ can be smoothly parameterized as

$$x = x(s) \text{ and } y = y(s)$$

where s is an arc length parameter with reference point (x_0, y_0) at $s = 0$. Therefore, the quantity $z = f(x(s), y(s))$ has a relative maximum or relative minimum at $s = 0$, and this implies that $\dfrac{dz}{ds} = 0$ at that point. From the chain rule,

$$\begin{aligned} \frac{dz}{ds} &= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} \\ &= \left(\frac{\partial f}{\partial x} \hat{\mathbf{i}} + \frac{\partial f}{\partial y} \hat{\mathbf{j}} \right) \cdot \left(\frac{\partial x}{\partial s} \hat{\mathbf{i}} + \frac{\partial y}{\partial s} \hat{\mathbf{j}} \right) \\ &= 0, \end{aligned}$$

where the derivatives are all evaluated at $s = 0$. However, the first factor in the dot product is the gradient of f, and the second factor is the unit tangent vector $\vec{\mathbf{T}}(0)$ to the constraint curve. Since the point (x_0, y_0) corresponds to $s = 0$, it follows from this equation that

$$\vec{\nabla} f(x_0, y_0) \cdot \vec{\mathbf{T}}(0) = 0,$$

which implies that the gradient is either the zero vector $\vec{\mathbf{0}}$ or it is normal to the constraint curve at a constrained relative extremum. However, the constraint curve $g(x,y) = 0$ is a level curve for the function $g(x,y)$ so that if $\vec{\nabla} g(x_0, y_0) \neq 0$ then $\vec{\nabla} g(x_0, y_0)$ is normal to this curve at (x_0, y_0) It follows, then, that there is some scalar λ such that

$$\vec{\nabla} f(x_0, y_0) = \lambda \vec{\nabla} g(x_0, y_0)$$

□

To apply Theorem 14.8.1 to an optimization problem similar to that for the golf ball manufacturer, we need a problem-solving strategy.

Problem-Solving Strategy: Steps for Using Lagrange Multipliers

1. Determine the objective function $f(x, y)$ and the constraint function $g(x, y)$. Does the optimization problem involve maximizing or minimizing the objective function?
2. Set up a system of equations using the following template:

$$\vec{\nabla} f(x_0, y_0) = \lambda \vec{\nabla} g(x_0, y_0) \tag{14.8.1}$$
$$g(x_0, y_0) = 0 \tag{14.8.2}$$

3. Solve for x_0 and y_0.
4. The largest of the values of f at the solutions found in step 3 maximizes f; the smallest of those values minimizes f.

Example 14.8.1: Using Lagrange Multipliers

Use the method of Lagrange multipliers to find the minimum value of $f(x, y) = x^2 + 4y^2 - 2x + 8y$ subject to the constraint $x + 2y = 7$.

Solution

Let's follow the problem-solving strategy:

1. The objective function is $f(x, y) = x^2 + 4y^2 - 2x + 8y$. To determine the constraint function, we must first subtract 7 from both sides of the constraint. This gives $x + 2y - 7 = 0$. The constraint function is equal to the left-hand side, so $g(x, y) = x + 2y - 7$. The problem asks us to solve for the minimum value of f, subject to the constraint (Figure 14.8.3).

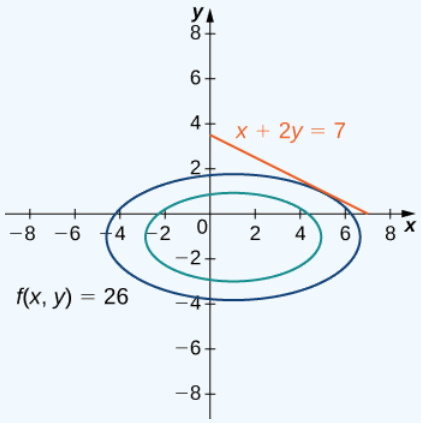

Figure 14.8.3: Graph of level curves of the function $f(x, y) = x^2 + 4y^2 - 2x + 8y$ corresponding to $c = 10$ and 26. The red graph is the constraint function.

2. We then must calculate the gradients of both f and g:

$$\vec{\nabla} f(x, y) = (2x - 2)\,\hat{\mathbf{i}} + (8y + 8)\,\hat{\mathbf{j}}$$
$$\vec{\nabla} g(x, y) = \hat{\mathbf{i}} + 2\hat{\mathbf{j}}.$$

The equation $\vec{\nabla} f(x_0, y_0) = \lambda \vec{\nabla} g(x_0, y_0)$ becomes

$$(2x_0 - 2)\,\hat{\mathbf{i}} + (8y_0 + 8)\,\hat{\mathbf{j}} = \lambda\left(\hat{\mathbf{i}} + 2\hat{\mathbf{j}}\right),$$

which can be rewritten as

$$(2x_0 - 2)\,\hat{\mathbf{i}} + (8y_0 + 8)\,\hat{\mathbf{j}} = \lambda\hat{\mathbf{i}} + 2\lambda\hat{\mathbf{j}}.$$

Next, we set the coefficients of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ equal to each other:

$$\begin{aligned} 2x_0 - 2 &= \lambda \\ 8y_0 + 8 &= 2\lambda. \end{aligned}$$

The equation $g(x_0, y_0) = 0$ becomes $x_0 + 2y_0 - 7 = 0$. Therefore, the system of equations that needs to be solved is

$$\begin{aligned} 2x_0 - 2 &= \lambda \\ 8y_0 + 8 &= 2\lambda \\ x_0 + 2y_0 - 7 &= 0. \end{aligned}$$

3. This is a linear system of three equations in three variables. We start by solving the second equation for λ and substituting it into the first equation. This gives $\lambda = 4y_0 + 4$, so substituting this into the first equation gives

$$2x_0 - 2 = 4y_0 + 4.$$

Solving this equation for x_0 gives $x_0 = 2y_0 + 3$. We then substitute this into the third equation:

$$\begin{aligned} (2y_0 + 3) + 2y_0 - 7 &= 0 \\ 4y_0 - 4 &= 0 \\ y_0 &= 1. \end{aligned}$$

Since $x_0 = 2y_0 + 3$, this gives $x_0 = 5$.

4. Next, we evaluate $f(x, y) = x^2 + 4y^2 - 2x + 8y$ at the point $(5, 1)$,

$$f(5, 1) = 5^2 + 4(1)^2 - 2(5) + 8(1) = 27.$$

To ensure this corresponds to a minimum value on the constraint function, let's try some other points on the constraint from either side of the point $(5, 1)$, such as the intercepts of $g(x, y) = 0$, Which are $(7, 0)$ and $(0, 3.5)$.

We get $f(7, 0) = 35 > 27$ and $f(0, 3.5) = 77 > 27$.

So it appears that f has a relative minimum of 27 at $(5, 1)$, subject to the given constraint.

? Exercise 14.8.1

Use the method of Lagrange multipliers to find the maximum value of

$$f(x, y) = 9x^2 + 36xy - 4y^2 - 18x - 8y$$

subject to the constraint $3x + 4y = 32$.

Hint

Use the problem-solving strategy for the method of Lagrange multipliers.

Answer

Subject to the given constraint, f has a maximum value of 976 at the point $(8, 2)$.

Let's now return to the problem posed at the beginning of the section.

✓ Example 14.8.2: Golf Balls and Lagrange Multipliers

The golf ball manufacturer, Pro-T, has developed a profit model that depends on the number x of golf balls sold per month (measured in thousands), and the number of hours per month of advertising y, according to the function

$$z = f(x, y) = 48x + 96y - x^2 - 2xy - 9y^2,$$

where z is measured in thousands of dollars. The budgetary constraint function relating the cost of the production of thousands golf balls and advertising units is given by $20x + 4y = 216$. Find the values of x and y that maximize profit, and find the maximum profit.

Solution:

Again, we follow the problem-solving strategy:

1. The objective function is $f(x,y) = 48x + 96y - x^2 - 2xy - 9y^2$. To determine the constraint function, we first subtract 216 from both sides of the constraint, then divide both sides by 4, which gives $5x + y - 54 = 0$. The constraint function is equal to the left-hand side, so $g(x,y) = 5x + y - 54$. The problem asks us to solve for the maximum value of f, subject to this constraint.
2. So, we calculate the gradients of both f and g:

$$\vec{\nabla} f(x,y) = (48 - 2x - 2y)\hat{\mathbf{i}} + (96 - 2x - 18y)\hat{\mathbf{j}}$$
$$\vec{\nabla} g(x,y) = 5\hat{\mathbf{i}} + \hat{\mathbf{j}}.$$

The equation $\vec{\nabla} f(x_0,y_0) = \lambda \vec{\nabla} g(x_0,y_0)$ becomes

$$(48 - 2x_0 - 2y_0)\hat{\mathbf{i}} + (96 - 2x_0 - 18y_0)\hat{\mathbf{j}} = \lambda(5\hat{\mathbf{i}} + \hat{\mathbf{j}}),$$

which can be rewritten as

$$(48 - 2x_0 - 2y_0)\hat{\mathbf{i}} + (96 - 2x_0 - 18y_0)\hat{\mathbf{j}} = \lambda 5\hat{\mathbf{i}} + \lambda\hat{\mathbf{j}}.$$

We then set the coefficients of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ equal to each other:

$$48 - 2x_0 - 2y_0 = 5\lambda$$
$$96 - 2x_0 - 18y_0 = \lambda.$$

The equation $g(x_0,y_0) = 0$ becomes $5x_0 + y_0 - 54 = 0$. Therefore, the system of equations that needs to be solved is

$$48 - 2x_0 - 2y_0 = 5\lambda$$
$$96 - 2x_0 - 18y_0 = \lambda$$
$$5x_0 + y_0 - 54 = 0.$$

3. We use the left-hand side of the second equation to replace λ in the first equation:

$$48 - 2x_0 - 2y_0 = 5(96 - 2x_0 - 18y_0)$$
$$48 - 2x_0 - 2y_0 = 480 - 10x_0 - 90y_0$$
$$8x_0 = 432 - 88y_0$$
$$x_0 = 54 - 11y_0.$$

Then we substitute this into the third equation:

$$5(54 - 11y_0) + y_0 - 54 = 0$$
$$270 - 55y_0 + y_0 - 54 = 0$$
$$216 - 54y_0 = 0$$
$$y_0 = 4.$$

Since $x_0 = 54 - 11y_0$, this gives $x_0 = 10$.

4. We then substitute $(10,4)$ into $f(x,y) = 48x + 96y - x^2 - 2xy - 9y^2$, which gives

$$\begin{aligned} f(10,4) &= 48(10) + 96(4) - (10)^2 - 2(10)(4) - 9(4)^2 \\ &= 480 + 384 - 100 - 80 - 144 \\ &= 540. \end{aligned}$$

Therefore the maximum profit that can be attained, subject to budgetary constraints, is $\$540,000$ with a production level of $10,000$ golf balls and 4 hours of advertising bought per month. Let's check to make sure this truly is a maximum. The endpoints of the line that defines the constraint are $(10.8, 0)$ and $(0, 54)$ Let's evaluate f at both of these points:

$$\begin{aligned} f(10.8,0) &= 48(10.8) + 96(0) - 10.8^2 - 2(10.8)(0) - 9(0^2) \\ &= 401.76 \\ f(0,54) &= 48(0) + 96(54) - 0^2 - 2(0)(54) - 9(54^2) \\ &= -21,060. \end{aligned}$$

The second value represents a loss, since no golf balls are produced. Neither of these values exceed 540, so it seems that our extremum is a maximum value of f, subject to the given constraint.

? Exercise 14.8.2: Optimizing the Cobb-Douglas function

A company has determined that its production level is given by the Cobb-Douglas function $f(x,y) = 2.5x^{0.45}y^{0.55}$ where x represents the total number of labor hours in 1 year and y represents the total capital input for the company. Suppose 1 unit of labor costs $\$40$ and 1 unit of capital costs $\$50$. Use the method of Lagrange multipliers to find the maximum value of $f(x,y) = 2.5x^{0.45}y^{0.55}$ subject to a budgetary constraint of $\$500,000$ per year.

Hint

Use the problem-solving strategy for the method of Lagrange multipliers.

Answer

Subject to the given constraint, a maximum production level of 13890 occurs with 5625 labor hours and $\$5500$ of total capital input.

In the case of an objective function with three variables and a single constraint function, it is possible to use the method of Lagrange multipliers to solve an optimization problem as well. An example of an objective function with three variables could be the **Cobb-Douglas function** in Exercise 14.8.2: $f(x,y,z) = x^{0.2}y^{0.4}z^{0.4}$, where x represents the cost of labor, y represents capital input, and z represents the cost of advertising. The method is the same as for the method with a function of two variables; the equations to be solved are

$$\vec{\nabla} f(x,y,z) = \lambda \vec{\nabla} g(x,y,z)$$
$$g(x,y,z) = 0.$$

✓ Example 14.8.3: Lagrange Multipliers with a Three-Variable objective function

Maximize the function $f(x,y,z) = x^2 + y^2 + z^2$ subject to the constraint $x + y + z = 1$.

Solution

1. The objective function is $f(x,y,z) = x^2 + y^2 + z^2$. To determine the constraint function, we subtract 1 from each side of the constraint: $x + y + z - 1 = 0$ which gives the constraint function as $g(x,y,z) = x + y + z - 1$.

2. Next, we calculate $\vec{\nabla} f(x,y,z)$ and $\vec{\nabla} g(x,y,z)$:

$$\vec{\nabla} f(x,y,z) = \langle 2x, 2y, 2z \rangle$$
$$\vec{\nabla} g(x,y,z) = \langle 1,1,1 \rangle.$$

This leads to the equations

$$\langle 2x_0, 2y_0, 2z_0 \rangle = \lambda \langle 1,1,1 \rangle$$
$$x_0 + y_0 + z_0 - 1 = 0$$

which can be rewritten in the following form:

$$2x_0 = \lambda$$
$$2y_0 = \lambda$$
$$2z_0 = \lambda$$
$$x_0 + y_0 + z_0 - 1 = 0.$$

3. Since each of the first three equations has λ on the right-hand side, we know that $2x_0 = 2y_0 = 2z_0$ and all three variables are equal to each other. Substituting $y_0 = x_0$ and $z_0 = x_0$ into the last equation yields $3x_0 - 1 = 0$, so $x_0 = \frac{1}{3}$ and $y_0 = \frac{1}{3}$ and $z_0 = \frac{1}{3}$ which corresponds to a critical point on the constraint curve.

4. Then, we evaluate f at the point $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$:

$$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^2 + \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^2 + \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^2 = \frac{3}{9} = \frac{1}{3}$$

$$f\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)=\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2=\frac{1}{9}=\frac{1}{3}$$

Therefore, a possible extremum of the function is $\frac{1}{3}$. To verify it is a minimum, choose other points that satisfy the constraint from either side of the point we obtained above and calculate f at those points. For example,

$$f(1,0,0)=1^2+0^2+0^2=1$$
$$f(0,-2,3)=0^2+(-2)^2+3^2=13.$$

Both of these values are greater than $\frac{1}{3}$, leading us to believe the extremum is a minimum, subject to the given constraint.

? Exercise 14.8.3

Use the method of Lagrange multipliers to find the minimum value of the function

$$f(x,y,z)=x+y+z$$

subject to the constraint $x^2+y^2+z^2=1$.

Hint

Use the problem-solving strategy for the method of Lagrange multipliers with an objective function of three variables.

Answer

Evaluating f at both points we obtained, gives us,

$$f\left(\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right)=\frac{\sqrt{3}}{3}+\frac{\sqrt{3}}{3}+\frac{\sqrt{3}}{3}=\sqrt{3}$$
$$f\left(-\frac{\sqrt{3}}{3},-\frac{\sqrt{3}}{3},-\frac{\sqrt{3}}{3}\right)=-\frac{\sqrt{3}}{3}-\frac{\sqrt{3}}{3}-\frac{\sqrt{3}}{3}=-\sqrt{3}$$

Since the constraint is continuous, we compare these values and conclude that f has a relative minimum of $-\sqrt{3}$ at the point $\left(-\frac{\sqrt{3}}{3},-\frac{\sqrt{3}}{3},-\frac{\sqrt{3}}{3}\right)$, subject to the given constraint.

Problems with Two Constraints

The method of Lagrange multipliers can be applied to problems with more than one constraint. In this case the objective function, w is a function of three variables:

$$w=f(x,y,z)$$

and it is subject to two constraints:

$$g(x,y,z)=0 \text{ and } h(x,y,z)=0.$$

There are two Lagrange multipliers, λ_1 and λ_2, and the system of equations becomes

$$\vec{\nabla}f(x_0,y_0,z_0)=\lambda_1\vec{\nabla}g(x_0,y_0,z_0)+\lambda_2\vec{\nabla}h(x_0,y_0,z_0)$$
$$g(x_0,y_0,z_0)=0$$
$$h(x_0,y_0,z_0)=0$$

✓ Example 14.8.4: Lagrange Multipliers with Two Constraints

Find the maximum and minimum values of the function

$$f(x,y,z)=x^2+y^2+z^2$$

subject to the constraints $z^2=x^2+y^2$ and $x+y-z+1=0$.

Solution

Let's follow the problem-solving strategy:

1. The objective function is $f(x,y,z) = x^2 + y^2 + z^2$. To determine the constraint functions, we first subtract z^2 from both sides of the first constraint, which gives $x^2 + y^2 - z^2 = 0$, so $g(x,y,z) = x^2 + y^2 - z^2$. The second constraint function is $h(x,y,z) = x + y - z + 1$.
2. We then calculate the gradients of f, g, and h:

$$\vec{\nabla} f(x,y,z) = 2x\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} + 2z\hat{\mathbf{k}}$$
$$\vec{\nabla} g(x,y,z) = 2x\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} - 2z\hat{\mathbf{k}}$$
$$\vec{\nabla} h(x,y,z) = \hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}.$$

The equation $\vec{\nabla} f(x_0,y_0,z_0) = \lambda_1 \vec{\nabla} g(x_0,y_0,z_0) + \lambda_2 \vec{\nabla} h(x_0,y_0,z_0)$ becomes

$$2x_0\hat{\mathbf{i}} + 2y_0\hat{\mathbf{j}} + 2z_0\hat{\mathbf{k}} = \lambda_1(2x_0\hat{\mathbf{i}} + 2y_0\hat{\mathbf{j}} - 2z_0\hat{\mathbf{k}}) + \lambda_2(\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}),$$

which can be rewritten as

$$2x_0\hat{\mathbf{i}} + 2y_0\hat{\mathbf{j}} + 2z_0\hat{\mathbf{k}} = (2\lambda_1 x_0 + \lambda_2)\hat{\mathbf{i}} + (2\lambda_1 y_0 + \lambda_2)\hat{\mathbf{j}} - (2\lambda_1 z_0 + \lambda_2)\hat{\mathbf{k}}.$$

Next, we set the coefficients of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ equal to each other:

$$2x_0 = 2\lambda_1 x_0 + \lambda_2$$
$$2y_0 = 2\lambda_1 y_0 + \lambda_2$$
$$2z_0 = -2\lambda_1 z_0 - \lambda_2.$$

The two equations that arise from the constraints are $z_0^2 = x_0^2 + y_0^2$ and $x_0 + y_0 - z_0 + 1 = 0$. Combining these equations with the previous three equations gives

$$\begin{aligned} 2x_0 &= 2\lambda_1 x_0 + \lambda_2 \\ 2y_0 &= 2\lambda_1 y_0 + \lambda_2 \\ 2z_0 &= -2\lambda_1 z_0 - \lambda_2 \\ z_0^2 &= x_0^2 + y_0^2 \\ x_0 + y_0 - z_0 + 1 &= 0. \end{aligned}$$

3. The first three equations contain the variable λ_2. Solving the third equation for λ_2 and replacing into the first and second equations reduces the number of equations to four:

$$\begin{aligned} 2x_0 &= 2\lambda_1 x_0 - 2\lambda_1 z_0 - 2z_0 \\ 2y_0 &= 2\lambda_1 y_0 - 2\lambda_1 z_0 - 2z_0 \\ z_0^2 &= x_0^2 + y_0^2 \\ x_0 + y_0 - z_0 + 1 &= 0. \end{aligned}$$

Next, we solve the first and second equation for λ_1. The first equation gives $\lambda_1 = \dfrac{x_0 + z_0}{x_0 - z_0}$, the second equation gives $\lambda_1 = \dfrac{y_0 + z_0}{y_0 - z_0}$. We set the right-hand side of each equation equal to each other and cross-multiply:

$$\begin{aligned} \frac{x_0 + z_0}{x_0 - z_0} &= \frac{y_0 + z_0}{y_0 - z_0} \\ (x_0 + z_0)(y_0 - z_0) &= (x_0 - z_0)(y_0 + z_0) \\ x_0y_0 - x_0z_0 + y_0z_0 - z_0^2 &= x_0y_0 + x_0z_0 - y_0z_0 - z_0^2 \\ 2y_0z_0 - 2x_0z_0 &= 0 \\ 2z_0(y_0 - x_0) &= 0. \end{aligned}$$

Therefore, either $z_0 = 0$ or $y_0 = x_0$. If $z_0 = 0$, then the first constraint becomes $0 = x_0^2 + y_0^2$. The only real solution to this equation is $x_0 = 0$ and $y_0 = 0$, which gives the ordered triple $(0,0,0)$. This point does not satisfy the second constraint, so it is not a solution. Next, we consider $y_0 = x_0$, which reduces the number of equations to three:

$$y_0 = x_0$$
$$z_0^2 = x_0^2 + y_0^2$$
$$x_0 + y_0 - z_0 + 1 = 0.$$

We substitute the first equation into the second and third equations:

$$\begin{aligned} z_0^2 &= x_0^2 + x_0^2 \\ &= x_0 + x_0 - z_0 + 1 \qquad = 0. \end{aligned}$$

Then, we solve the second equation for z_0, which gives $z_0 = 2x_0 + 1$. We then substitute this into the first equation,

$$\begin{aligned} z_0^2 &= 2x_0^2 \\ (2x_0 + 1)^2 &= 2x_0^2 \\ 4x_0^2 + 4x_0 + 1 &= 2x_0^2 \\ 2x_0^2 + 4x_0 + 1 &= 0, \end{aligned}$$

and use the quadratic formula to solve for x_0:

$$x_0 = \frac{-4 \pm \sqrt{4^2 - 4(2)(1)}}{2(2)} = \frac{-4 \pm \sqrt{8}}{4} = \frac{-4 \pm 2\sqrt{2}}{4} = -1 \pm \frac{\sqrt{2}}{2}.$$

Recall $y_0 = x_0$, so this solves for y_0 as well. Then, $z_0 = 2x_0 + 1$, so

$$z_0 = 2x_0 + 1 = 2\left(-1 \pm \frac{\sqrt{2}}{2}\right) + 1 = -2 + 1 \pm \sqrt{2} = -1 \pm \sqrt{2}.$$

Therefore, there are two ordered triplet solutions:

$$\left(-1 + \frac{\sqrt{2}}{2}, -1 + \frac{\sqrt{2}}{2}, -1 + \sqrt{2}\right) \text{ and } \left(-1 - \frac{\sqrt{2}}{2}, -1 - \frac{\sqrt{2}}{2}, -1 - \sqrt{2}\right).$$

4. We substitute $\left(-1 + \frac{\sqrt{2}}{2}, -1 + \frac{\sqrt{2}}{2}, -1 + \sqrt{2}\right)$ into $f(x, y, z) = x^2 + y^2 + z^2$, which gives

$$\begin{aligned} f\left(-1 + \frac{\sqrt{2}}{2}, -1 + \frac{\sqrt{2}}{2}, -1 + \sqrt{2}\right) &= \left(-1 + \frac{\sqrt{2}}{2}\right)^2 + \left(-1 + \frac{\sqrt{2}}{2}\right)^2 + (-1 + \sqrt{2})^2 \\ &= \left(1 - \sqrt{2} + \frac{1}{2}\right) + \left(1 - \sqrt{2} + \frac{1}{2}\right) + (1 - 2\sqrt{2} + 2) \\ &= 6 - 4\sqrt{2}. \end{aligned}$$

Then, we substitute $\left(-1 - \frac{\sqrt{2}}{2}, -1 + \frac{\sqrt{2}}{2}, -1 + \sqrt{2}\right)$ into $f(x, y, z) = x^2 + y^2 + z^2$, which gives

$$\begin{aligned} f\left(-1 - \frac{\sqrt{2}}{2}, -1 + \frac{\sqrt{2}}{2}, -1 + \sqrt{2}\right) &= \left(-1 - \frac{\sqrt{2}}{2}\right)^2 + \left(-1 - \frac{\sqrt{2}}{2}\right)^2 + (-1 - \sqrt{2})^2 \\ &= \left(1 + \sqrt{2} + \frac{1}{2}\right) + \left(1 + \sqrt{2} + \frac{1}{2}\right) + (1 + 2\sqrt{2} + 2) \\ &= 6 + 4\sqrt{2}. \end{aligned}$$

$6 + 4\sqrt{2}$ is the maximum value and $6 - 4\sqrt{2}$ is the minimum value of $f(x, y, z)$, subject to the given constraints.

Exercise 14.8.4

Use the method of Lagrange multipliers to find the minimum value of the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

subject to the constraints $2x + y + 2z = 9$ and $5x + 5y + 7z = 29$.

Hint

Use the problem-solving strategy for the method of Lagrange multipliers with two constraints.

Answer

$f(2,1,2)=9$ is a minimum value of f, subject to the given constraints.

Key Concepts

- An objective function combined with one or more constraints is an example of an optimization problem.
- To solve optimization problems, we apply the method of Lagrange multipliers using a four-step problem-solving strategy.

Key Equations

- **Method of Lagrange multipliers, one constraint**

  $$\vec{\nabla} f(x_0,y_0)=\lambda\vec{\nabla} g(x_0,y_0)$$

  $$g(x_0,y_0)=0$$

- **Method of Lagrange multipliers, two constraints**

  $$\vec{\nabla} f(x_0,y_0,z_0)=\lambda_1\vec{\nabla} g(x_0,y_0,z_0)+\lambda_2\vec{\nabla} h(x_0,y_0,z_0)$$

  $$g(x_0,y_0,z_0)=0$$

  $$h(x_0,y_0,z_0)=0$$

Glossary

constraint
an inequality or equation involving one or more variables that is used in an optimization problem; the constraint enforces a limit on the possible solutions for the problem

Lagrange multiplier
the constant (or constants) used in the method of Lagrange multipliers; in the case of one constant, it is represented by the variable λ

method of Lagrange multipliers
a method of solving an optimization problem subject to one or more constraints

objective function
the function that is to be maximized or minimized in an optimization problem

optimization problem
calculation of a maximum or minimum value of a function of several variables, often using Lagrange multipliers

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14.8E: Exercises for Section 14.8

In exercises 1-15, use the method of Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint.

1) Objective function: $f(x,y) = 4xy$ Constraint: $\frac{x^2}{9} + \frac{y^2}{16} = 1$

Answer

Subject to the given constraint, the function f has a relative minimum of -24 at both $\left(-\frac{3\sqrt{2}}{2}, 2\sqrt{2}\right)$ and $\left(\frac{3\sqrt{2}}{2}, -2\sqrt{2}\right)$ and a relative maximum of 24 at both $\left(\frac{3\sqrt{2}}{2}, 2\sqrt{2}\right)$ and $\left(-\frac{3\sqrt{2}}{2}, -2\sqrt{2}\right)$

2) Objective function: $f(x,y) = x^2y$ Constraint: $x^2 + 2y^2 = 6$

3) Objective function: $f(x,y) = x^2 + y^2 + 2x - 2y + 1$ Constraint: $g(x,y) = x^2 + y^2 = 2$

Answer

Subject to the given constraint, f has a relative minimum of -1 at $(-1,1)$ and a relative maximum of 7 at $(1,-1)$.

4) Objective function: $f(x,y) = xy$ Constraint: $4x^2 + 8y^2 = 16$

5) Objective function: $f(x,y) = x^2 + y^2$ Constraint: $xy = 1$

Answer

f has a relative minimum of 2 at both $(-1,-1)$ and $(1,1)$, subject to the given constraint.

6) Objective function: $f(x,y) = x^2 - y^2$ Constraint: $x - 2y + 6 = 0$

7) Objective function: $f(x,y) = x^2 + y^2$ Constraint: $x + 2y - 5 = 0$

Answer

Subject to the given constraint, f has a relative minimum of $f(1,2) = 5$ at the point $(1,2)$.

8) Objective function: $f(x,y) = x^2 + y^2$ Constraint: $(x-1)^2 + 4y^2 = 4$

9) Objective function: $f(x,y) = 4x^3 + y^2$ Constraint: $2x^2 + y^2 = 1$

Answer

Subject to the given constraint, the function f has a relative minimum of $-\sqrt{2}$ at $\left(-\frac{\sqrt{2}}{2}, 0\right)$,
a relative minimum of $\frac{25}{27}$ at both points $\left(\frac{1}{3}, -\frac{\sqrt{7}}{3}\right)$ and $\left(\frac{1}{3}, \frac{\sqrt{7}}{3}\right)$,
a relative maximum of $\sqrt{2}$ at $\left(\frac{\sqrt{2}}{2}, 0\right)$, and a relative maximum of 1 at both points $(0,1)$ and $(0,-1)$.

Solution:

Let $g(x,y) = 2x^2 + y^2$ be the constraint function. Then:

$\vec{\nabla} f(x,y) = 12x^2\,\hat{\mathbf{i}} + 2y\,\hat{\mathbf{j}}$ and $\vec{\nabla} g(x,y) = 4x\,\hat{\mathbf{i}} + 2y\,\hat{\mathbf{j}}$

Using the Lagrange Multiplier equation,

$$\vec{\nabla} f(x,y) = \lambda \vec{\nabla} g(x,y),$$

we have:

$$12x^2\,\hat{\mathbf{i}} + 2y\,\hat{\mathbf{j}} = 4x\lambda\,\hat{\mathbf{i}} + 2y\lambda\,\hat{\mathbf{j}}$$

giving us the system of equations:

$12x^2 = 4x\lambda$, $2y = 2y\lambda$, and the constraint $2x^2 + y^2 = 1$

$12x = 4x\lambda$, $2y = 2y\lambda$, and the constraint $2x + y = 1$

Rewriting the first two equations as zero-products (by moving to one-side and factoring), we get:

$$\begin{array}{lll} 4x(3x-\lambda)=0 & \text{and} & 2y(1-\lambda)=0 \\ x=0 \quad \text{or} \quad \lambda=3x & & y=0 \quad \text{or} \quad \lambda=1 \end{array}$$

Now we consider the combinations of these solutions to the above two equations and plug each of these into the constraint equation to solve for the corresponding Lagrange points.

The combination $x=0$ and $y=0$ produces a contradiction when placed in the constraint equation, since this point is not on the ellipse.

Taking the combination $x=0$ and $\lambda=1$, we put 0 in for x in the constraint and solve for y, obtaining: $y=\pm 1$. This gives us two Lagrange points: $(0,1)$ and $(0,-1)$.

Taking the combination $\lambda=3x$ and $y=0$, we put 0 in for y in the constraint and solve for x, obtaining: $x=\pm\frac{\sqrt{2}}{2}$. This gives us two Lagrange points: $\left(-\frac{\sqrt{2}}{2},0\right)$ and $\left(\frac{\sqrt{2}}{2},0\right)$.

Taking the combination $\lambda=3x$ and $\lambda=1$, we substitute 1 into the first equation for λ, giving us $1=3x$ so $x=\frac{1}{3}$. Plugging this value in for x in the constraint equation and solving for y, we obtain $y=\pm\frac{\sqrt{7}}{3}$ which gives us the two Lagrange points: $\left(\frac{1}{3},-\frac{\sqrt{7}}{3}\right)$ and $\left(\frac{1}{3},\frac{\sqrt{7}}{3}\right)$.

Evaluating the function f at these Lagrange points, we find:

$$\begin{array}{ll} f(0,-1)=1 & f(0,1)=1 \\ f\left(-\frac{\sqrt{2}}{2},0\right)=\frac{-4(\sqrt{2})^3}{8}=-\sqrt{2} & f\left(\frac{\sqrt{2}}{2},0\right)=\frac{4(\sqrt{2})^3}{8}=\sqrt{2} \\ f\left(\frac{1}{3},-\frac{\sqrt{7}}{3}\right)=\frac{25}{27} & f\left(\frac{1}{3},\frac{\sqrt{7}}{3}\right)=\frac{25}{27} \end{array}$$

Comparing these values with where the corresponding Lagrange points lie on the constraint curve, we conclude the results stated in the answer above.

10) Objective function: $f(x,y)=2x^2+y^2$ Constraint: $g(x,y)=x^2+y^2=1$

11) Objective function: $f(x,y,z)=x+3y-z$ Constraint: $x^2+y^2+z^2=4$

Answer

Subject to the given constraint, f has a relative minimum of $-2\sqrt{11}$ at the point $\left(-\frac{2\sqrt{11}}{11},-\frac{6\sqrt{11}}{11},\frac{2\sqrt{11}}{11}\right)$ and a relative maximum of $2\sqrt{11}$ at the point $\left(\frac{2\sqrt{11}}{11},\frac{6\sqrt{11}}{11},-\frac{2\sqrt{11}}{11}\right)$.

12) Objective function: $f(x,y,z)=x+y+z$ Constraint: $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$

13) Objective function: $f(x,y,z)=xyz$ Constraint: $x^2+2y^2+3z^2=6$

Answer

Subject to the given constraint, f has a relative minimum of $-\frac{2\sqrt{3}}{3}$ at $\left(\sqrt{2},1,-\frac{\sqrt{6}}{3}\right)$, $\left(\sqrt{2},-1,\frac{\sqrt{6}}{3}\right)$, $\left(-\sqrt{2},1,\frac{\sqrt{6}}{3}\right)$, and $\left(-\sqrt{2},-1,-\frac{\sqrt{6}}{3}\right)$ and a relative maximum of $\frac{2\sqrt{3}}{3}$ at $\left(\sqrt{2},1,\frac{\sqrt{6}}{3}\right)$, $\left(\sqrt{2},-1,-\frac{\sqrt{6}}{3}\right)$, $\left(-\sqrt{2},-1,\frac{\sqrt{6}}{3}\right)$, and $\left(-\sqrt{2},1,-\frac{\sqrt{6}}{3}\right)$.

14) Objective function: $f(x,y,z)=x^2+y^2+z^2$ Constraint: $x^4+y^4+z^4=1$

15) Objective function: $f(x,y,z)=x^2+y^2+z^2$ Constraint: $xyz=4$

Answer

Subject to the given constraint, f has a relative minimum of $6\sqrt[3]{2}$ at the points $\left(\sqrt[3]{4}, \sqrt[3]{4}, \sqrt[3]{4}\right)$, $\left(\sqrt[3]{4}, -\sqrt[3]{4}, -\sqrt[3]{4}\right)$, $\left(-\sqrt[3]{4}, \sqrt[3]{4}, -\sqrt[3]{4}\right)$, and $\left(-\sqrt[3]{4}, -\sqrt[3]{4}, \sqrt[3]{4}\right)$.
To see a 3D visualization of this problem, see: CalcPlot3D for Problem 15.

In exercises 16-21, use the method of Lagrange multipliers to find the requested extremum of the given function subject to the given constraint.

16) Maximize $f(x,y) = \sqrt{6 - x^2 - y^2}$ subject to the constraint, $x + y - 2 = 0$.

17) Maximize $f(x,y) = x^2 - y^2$ subject to the constraints, $g(x,y) = y - x^2 = 0, \quad x > 0, \quad y > 0$.

Answer

Subject to the given constraints, f has a relative maximum of $f\left(\frac{\sqrt{2}}{2}, \frac{1}{2}\right) = \frac{1}{4}$ at the point $\left(\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$. If $x > 0$ were not a constraint, there would have been two other Lagrange points with relative extrema of f subject to the other two constraints. These would have been $(0,0)$ and $\left(-\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$.

To verify that f really has a relative **maximum** at the point $\left(\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$, we would need to check the value of f on either side of this point on the constraint curve, $y - x^2 = 0$.

If $x = 0.5$ which is less than $\frac{\sqrt{2}}{2}$, y would be $y = (0.5)^2 = 0.25$.
If $x = 1$ which is greater than $\frac{\sqrt{2}}{2}$, y would be $y = (1)^2 = 1$.

Then we compare the value of f at the Lagrange point, $\left(\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$, with the values of f at these other points on the constraint.
We have $f(0.5, 0.25) = (0.5)^2 - (0.25)^2 = 0.25 - 0.0625 = 0.1875 < \frac{1}{4}$ and $f(1,1) = (1)^2 - (1)^2 = 0 < \frac{1}{4}$.

Therefore, we can conclude that f indeed has a relative maximum of $\frac{1}{4}$ at the point $\left(\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$.

18) Maximize $U(x,y) = 8x^{4/5}y^{1/5}$ subject to the constraint, $4x + 2y = 12$.

19) Minimize $f(x,y,z) = x^2 + y^2 + z^2$ subject to the constraint, $x + y + z = 1$.

Answer

Subject to the given constraint, f has a relative minimum of $f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3}$ at the point $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

20) Minimize $f(x,y) = xy$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

21) Maximize $f(x,y,z) = 2x + 3y + 5z$ on the sphere $x^2 + y^2 + z^2 = 19$.

Answer

Subject to the given constraint, f has a relative maximum of $19\sqrt{2}$ at the point $\left(\sqrt{2}, \frac{3\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right)$.
Note that, subject to this constraint, f also has a relative minimum of $-19\sqrt{2}$ at the point $\left(-\sqrt{2}, -\frac{3\sqrt{2}}{2}, -\frac{5\sqrt{2}}{2}\right)$.
To see a 3D visualization of this problem, see: CalcPlot3D for Problem 21.

In exercises 22-23, use the method of Lagrange multipliers with two constraints.

22) Optimize $f(x,y,z) = yz + xy$ subject to the constraints: $xy = 1, \quad y^2 + z^2 = 1$.

Answer

maximum: $\frac{3}{2}$, minimum: $\frac{1}{2}$

23) Minimize $f(x,y,z) = x^2 + y^2 + z^2$ when $x + y + z = 9$ and $x + 2y + 3z = 20$.

Answer

minimum: $f(2,3,4) = 29$

Use the method of Lagrange multipliers to solve the following applied problems.

24) A large container in the shape of a rectangular solid must have a volume of 480 m^3. The bottom of the container costs \$5/m^2 to construct whereas the top and sides cost \$3/m^2 to construct. Use Lagrange multipliers to find the dimensions of the container of this size that has the minimum cost.

25) A rectangular box without a top (a topless box) is to be made from 12 ft^2 of cardboard. Find the maximum volume of such a box.

Answer

The maximum volume is 4 ft^3. The dimensions are $1 \times 2 \times 2$ ft.

26) Find the minimum distance from the parabola $y = x^2$ to point $(0, 3)$.

27) Find the point on the line $y = 2x + 3$ that is closest to point $(4, 2)$.

Answer

$(25, 195)$

29) Find the minimum distance from point $(0, 1)$ to the parabola $x^2 = 4y$.

Answer

1.0 unit

30) Find the minimum and maximum distances between the ellipse $x^2 + xy + 2y^2 = 1$ and the origin.

31) Find the minimum distance from the plane $x + y + z = 1$ to point $(2, 1, 1)$.

Answer

$\sqrt{3}$ units

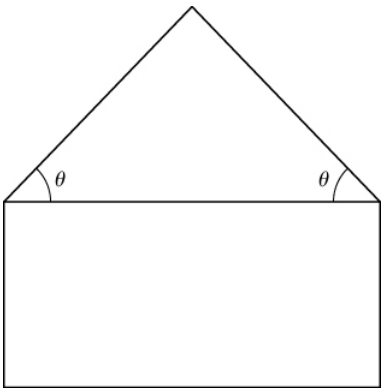
32) Find the point on the plane $4x + 3y + z = 2$ that is closest to the point $(1, -1, 1)$.

33) Find the point on the surface $x^2 - 2xy + y^2 - x + y = 0$ closest to the point $(1, 2, -3)$.

Answer

$\left(1, \frac{1}{2}, -3\right)$

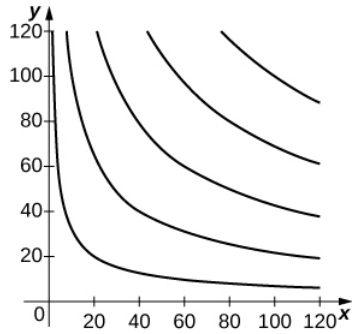
34) A pentagon is formed by placing an isosceles triangle on a rectangle, as shown in the diagram. If the perimeter of the pentagon is 10 in., find the lengths of the sides of the pentagon that will maximize the area of the pentagon.



35) **[T]** By investing x units of labor and y units of capital, a watch manufacturer can produce $P(x, y) = 50x^{0.4}y^{0.6}$ watches. Find the maximum number of watches that can be produced on a budget of \$20,000 if labor costs \$100/unit and capital costs \$200/unit. Use a grapher like CalcPlot3D to sketch a contour plot of the function.

Answer

Roughly 3365 watches at the critical point $(80, 60)$.

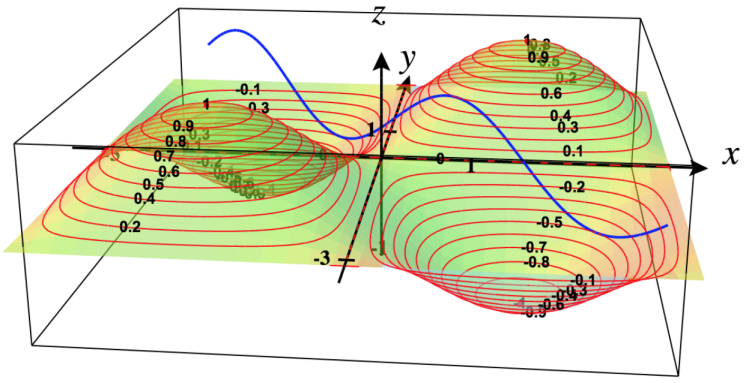


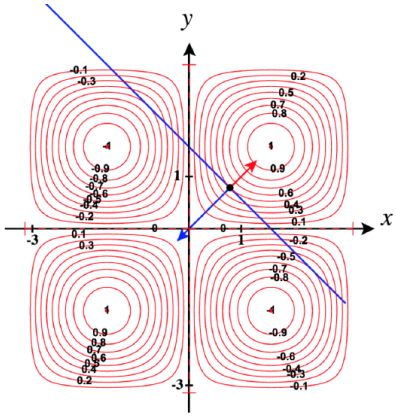
36) A rectangular solid is contained within a tetrahedron with vertices at $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and the origin. The base of the box has dimensions x and y, and the height of the box is z. If the sum of x, y, and z is 1, find the dimensions that maximizes the volume of the rectangular solid.

37) Find the maximum value of $f(x, y) = \sin x \sin y$, where x and y denote the acute angles of a right triangle. Draw the surface plot and contour plot of the function using a CAS.

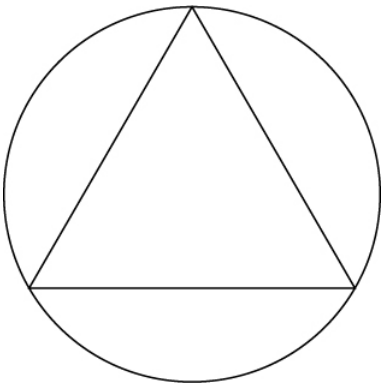
Answer

Subject to this constraint, f has a relative maximum of $\frac{1}{2}$ when $x = \frac{\pi}{4}$ and $y = \frac{\pi}{4}$.
Surface plot and contour plot for f:





38) Show that, of all the triangles inscribed in a circle of radius R (see diagram), the equilateral triangle has the largest perimeter.



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Contributors

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.
- Paul Seeburger (Monroe Community College) reordered these problems, adding problems 3 and 10 and answers for problems 15 and 17. He also added a full worked-out solution for problem 9 and a link to CalcPlot3D in problems 15, 21 and 35. He also created new images for Problem 37 and expanded the answers for many problems.

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14.R: Chapter 14 Review Exercises

For the following exercises, determine whether the statement is *true or false*. Justify your answer with a proof or a counterexample.

1. The domain of $f(x,y) = x^3 \arcsin(y)$ is $\{(x,y) \mid x \in \mathbb{R} \text{ and } -\pi \leq y \leq \pi\}$.

2. If the function $f(x,y)$ is continuous everywhere, then $f_{xy}(x,y) = f_{yx}(x,y)$.

Answer

True, by Clairaut's theorem

3. The linear approximation to the function of $f(x,y) = 5x^2 + x\tan y$ at the point $(2,\pi)$ is given by $L(x,y) = 22 + 21(x-2) + (y-\pi)$.

4. $(34, 916)$ is a critical point of $g(x,y) = 4x^3 - 2x^2y + y^2 - 2$.

Answer

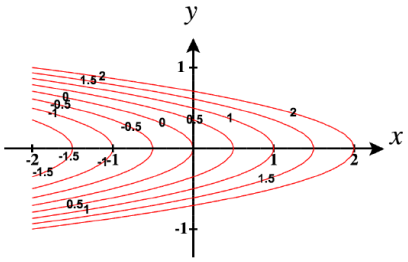
False

For the following exercises, sketch the function in one graph and, in a second, sketch several level curves.

5. $f(x,y) = e^{-(x^2+2y^2)}$

6. $f(x,y) = x + 4y^2$

Answer



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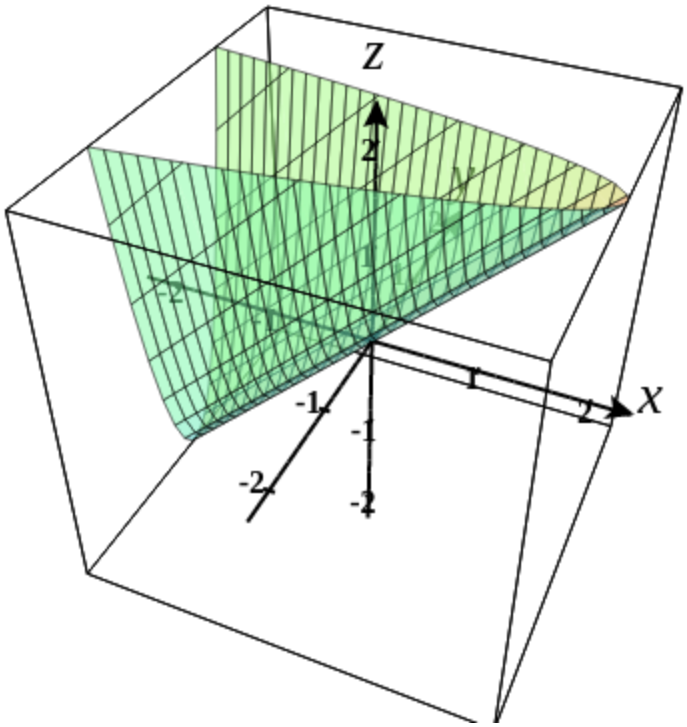

For the following exercises, evaluate the following limits, if they exist. If they do not exist, prove it.

7. $\lim\limits_{(x,y)\to(1,1)} \dfrac{4xy}{x-2y^2}$

8. $\lim\limits_{(x,y)\to(0,0)} \dfrac{4xy}{x-2y^2}$

Answer

Does not exist.

For the following exercises, find the largest interval of continuity for the function.

9. $f(x,y)=x^3 \arcsin y$

10. $g(x,y)=\ln(4-x^2-y^2)$

Answer

Continuous at all points on the xy-plane, except where $x^2+y^2>4$.

For the following exercises, find all first partial derivatives.

11. $f(x,y)=x^2-y^2$

12. $u(x,y)=x^4-3xy+1$, with $x=2t$ and $y=t^3$

Answer

$\dfrac{\partial u}{\partial x}=4x^3-3y,$

$\dfrac{\partial u}{\partial y}=-3x,$

$\dfrac{dx}{dt}=2$ and $\dfrac{dy}{dt}=3t^2$

$$\begin{aligned}\frac{du}{dt} &= \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}\\ &= 8x^3 - 6y - 9xt^2\\ &= 8\big(2t\big)^3 - 6(t^3) - 9(2t)t^2\\ &= 64t^3 - 6t^3 - 18t^3\\ &= 40t^3\end{aligned}$$

For the following exercises, find all second partial derivatives.

13. $g(t,x) = 3t^2 - \sin(x+t)$

14. $h(x,y,z) = \dfrac{x^3e^{2y}}{z}$

Answer

$$\begin{aligned}h_{xx}(x,y,z) &= \frac{6xe^{2y}}{z},\\ h_{xy}(x,y,z) &= \frac{6x^2e^{2y}}{z},\\ h_{xz}(x,y,z) &= -\frac{3x^2e^{2y}}{z^2},\\ h_{yx}(x,y,z) &= \frac{6x^2e^{2y}}{z},\\ h_{yy}(x,y,z) &= \frac{4x^3e^{2y}}{z},\\ h_{yz}(x,y,z) &= -\frac{2x^3e^{2y}}{z^2},\\ h_{zx}(x,y,z) &= -\frac{3x^2e^{2y}}{z^2},\\ h_{zy}(x,y,z) &= -\frac{2x^3e^{2y}}{z^2},\\ h_{zz}(x,y,z) &= \frac{2x^3e^{2y}}{z^3}\end{aligned}$$

For the following exercises, find the equation of the tangent plane to the specified surface at the given point.

15. $z = x^3 - 2y^2 + y - 1$ at point $(1,1,-1)$

16. $z = e^x + \dfrac{2}{y}$ at point $(0,1,3)$

Answer

$z = x - 2y + 5$

17. Approximate $f(x,y) = e^{x^2} + \sqrt{y}$ at $(0.1, 9.1)$. Write down your linear approximation function $L(x,y)$. How accurate is the approximation to the exact answer, rounded to four digits?

18. Find the differential dz of $h(x,y) = 4x^2 + 2xy - 3y$ and approximate Δz at the point $(1,-2)$. Let $\Delta x = 0.1$ and $\Delta y = 0.01$.

Answer

$dz = 4\,dx - dy,\ dz(0.1, 0.01) = 0.39,\ \Delta z = 0.432$

19. Find the directional derivative of $f(x,y) = x^2 + 6xy - y^2$ in the direction $\vec{\mathbf{v}} = \hat{\mathbf{i}} + 4\,\hat{\mathbf{j}}$.

20. Find the maximal directional derivative magnitude and direction for the function $f(x,y) = x^3 + 2xy - \cos(\pi y)$ at point $(3,0)$.

Answer

$3\sqrt{85}\langle 27, 6\rangle$

For the following exercises, find the gradient.

21. $c(x,t) = e(t-x)^2 + 3\cos t$

22. $f(x,y) = \dfrac{\sqrt{x}+y^2}{xy}$

Answer

$$\vec{\nabla} f(x,y) = -\frac{\sqrt{x}+2y^2}{2x^2y}\,\hat{\mathrm{i}} + \left(\frac{1}{x} + \frac{1}{\sqrt{x}y^2}\right)\hat{\mathrm{j}}$$

For the following exercise, find and classify the critical points.

23. $z = x^3 - xy + y^2 - 1$

For the following exercises, use Lagrange multipliers to find the maximum and minimum values for the functions with the given constraints.

24. $f(x,y) = x^2y$, subject to the constraint: $x^2 + y^2 = 4$

Answer

maximum: $\dfrac{16}{3\sqrt{3}}$, minimum: $-\dfrac{16}{3\sqrt{3}}$,

25. $f(x,y) = x^2 - y^2$, subject to the constraint: $x + 6y = 4$

26. A machinist is constructing a right circular cone out of a block of aluminum. The machine gives an error of 5% in height and 2% in radius. Find the maximum error in the volume of the cone if the machinist creates a cone of height 6 cm and radius 2 cm.

Answer

2.3228 cm^3

27. A trash compactor is in the shape of a cuboid. Assume the trash compactor is filled with incompressible liquid. The length and width are decreasing at rates of 2 ft/sec and 3 ft/sec, respectively. Find the rate at which the liquid level is rising when the length is 14 ft, the width is 10 ft, and the height is 4 ft.

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CHAPTER OVERVIEW

15: Multiple Integration

In this chapter we extend the concept of a definite integral of a single variable to double and triple integrals of functions of two and three variables, respectively. We examine applications involving integration to compute volumes, masses, and centroids of more general regions. We will also see how the use of other coordinate systems (such as polar, cylindrical, and spherical coordinates) makes it simpler to compute multiple integrals over some types of regions and functions. In the preceding chapter, we discussed differential calculus with multiple independent variables. Now we examine integral calculus in multiple dimensions. Just as a partial derivative allows us to differentiate a function with respect to one variable while holding the other variables constant, we will see that an iterated integral allows us to integrate a function with respect to one variable while holding the other variables constant.

15.0: Prelude to Multiple Integration
15.1: Double Integrals over Rectangular Regions
15.1E: Exercises for Section 15.1
15.2: Double Integrals over General Regions
15.2E: Exercises for Section 15.2
15.3: Double Integrals in Polar Coordinates
15.3E: Exercises for Section 15.3
15.4: Triple Integrals
15.4E: Exercises for Section 15.4
15.5: Triple Integrals in Cylindrical and Spherical Coordinates
15.5E: Exercises for Section 15.5
15.6: Calculating Centers of Mass and Moments of Inertia
15.6E: Exercises for Section 15.6
15.7: Change of Variables in Multiple Integrals
15.7E: Exercises for Section 15.7
15.R: Chapter 15 Review Exercises

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15.0: Prelude to Multiple Integration

In this chapter we extend the concept of a definite integral of a single variable to double and triple integrals of functions of two and three variables, respectively. We examine applications involving integration to compute volumes, masses, and centroids of more general regions. We will also see how the use of other coordinate systems (such as polar, cylindrical, and spherical coordinates) makes it simpler to compute multiple integrals over some types of regions and functions. As an example, we will use polar coordinates to find the volume of structures such as l'Hemisfèric (Figure 15.0.1).

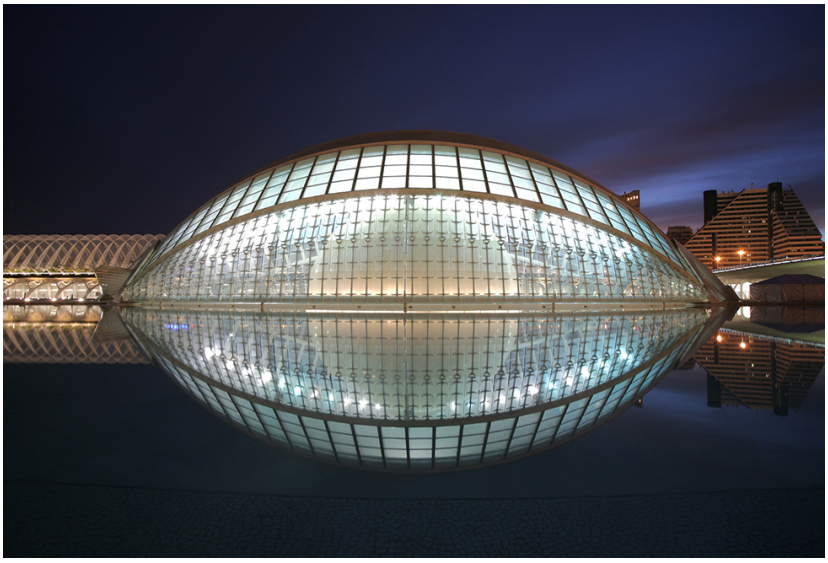

Figure 15.0.1: The City of Arts and Sciences in Valencia, Spain, has a unique structure along an axis of just two kilometers that was formerly the bed of the River Turia. The l'Hemisfèric has an IMAX cinema with three systems of modern digital projections onto a concave screen of 900 square meters. An oval roof over 100 meters long has been made to look like a huge human eye that comes alive and opens up to the world as the "Eye of Wisdom." (credit: modification of work by Javier Yaya Tur, Wikimedia Commons)

In the preceding chapter, we discussed differential calculus with multiple independent variables. Now we examine integral calculus in multiple dimensions. Just as a partial derivative allows us to differentiate a function with respect to one variable while holding the other variables constant, we will see that an iterated integral allows us to integrate a function with respect to one variable while holding the other variables constant.

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15.1: Double Integrals over Rectangular Regions

Learning Objectives

- Recognize when a function of two variables is integrable over a rectangular region.
- Recognize and use some of the properties of double integrals.
- Evaluate a double integral over a rectangular region by writing it as an iterated integral.
- Use a double integral to calculate the area of a region, volume under a surface, or average value of a function over a plane region.

In this section we investigate double integrals and show how we can use them to find the volume of a solid over a rectangular region in the xy-plane. Many of the properties of double integrals are similar to those we have already discussed for single integrals.

Volumes and Double Integrals

We begin by considering the space above a rectangular region R. Consider a continuous function $f(x,y) \geq 0$ of two variables defined on the closed rectangle R:

$$R = [a,b] \times [c,d] = \{(x,y) \in \mathbb{R}^2 | \, a \leq x \leq b, \, c \leq y \leq d\}$$

Here $[a,b] \times [c,d]$ denotes the Cartesian product of the two closed intervals $[a,b]$ and $[c,d]$. It consists of rectangular pairs (x,y) such that $a \leq x \leq b$ and $c \leq y \leq d$. The graph of f represents a surface above the xy-plane with equation $z = f(x,y)$ where z is the height of the surface at the point (x,y). Let S be the solid that lies above R and under the graph of f (Figure 15.1.1). The base of the solid is the rectangle R in the xy-plane. We want to find the volume V of the solid S.

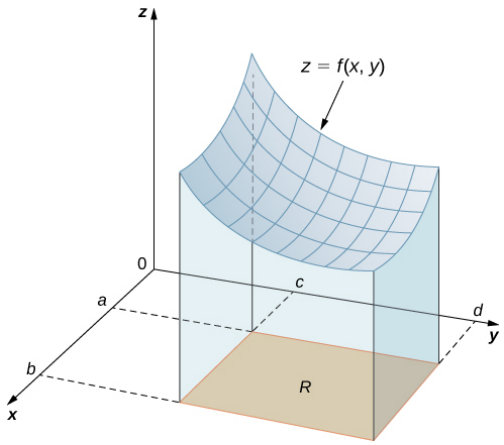


Figure 15.1.1: The graph of $f(x,y)$ *over the rectangle R in the* xy*-plane is a curved surface.*

We divide the region R into small rectangles R_{ij}, each with area ΔA and with sides Δx and Δy (Figure 15.1.2). We do this by dividing the interval $[a,b]$ into m subintervals and dividing the interval $[c,d]$ into n subintervals. Hence $\Delta x = \frac{b-a}{m}$, $\Delta y = \frac{d-c}{n}$, and $\Delta A = \Delta x \Delta y$.

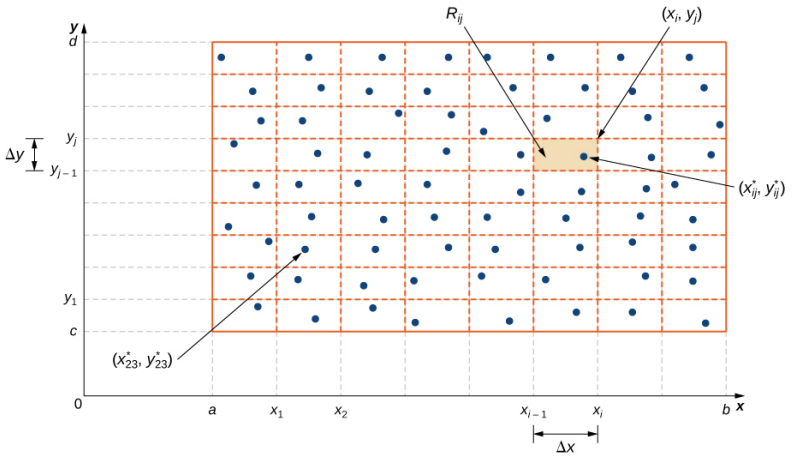


Figure 15.1.2: Rectangle R is divided into small rectangles R_{ij} each with area ΔA.

The volume of a thin rectangular box above R_{ij} is $f(x_{ij}^*, y_{ij}^*)\,\Delta A$, where (x_{ij}^*, y_{ij}^*) is an arbitrary sample point in each R_{ij} as shown in the following figure, $f(x_{ij}^*, y_{ij}^*)$ is the height of the corresponding thin rectangular box, and ΔA is the area of each rectangle R_{ij}.

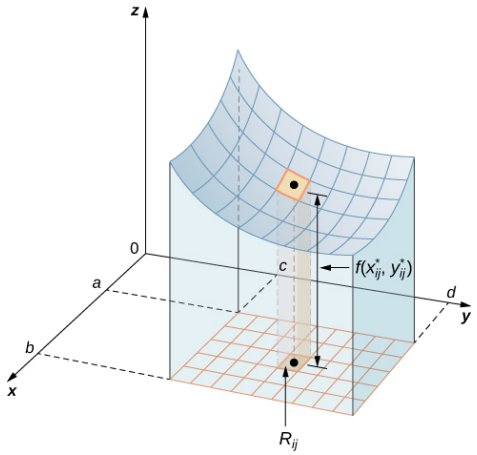


Figure 15.1.3: A thin rectangular box above R_{ij} with height $f(x_{ij}^*, y_{ij}^*)$.

Using the same idea for all the subrectangles, we obtain an approximate volume of the solid S as

$$V \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A.$$

This sum is known as a **double Riemann sum** and can be used to approximate the value of the volume of the solid. Here the double sum means that for each subrectangle we evaluate the function at the chosen point, multiply by the area of each rectangle, and then add all the results.

As we have seen in the single-variable case, we obtain a better approximation to the actual volume if m and n become larger.

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

or

$$V = \lim_{\Delta x,\, \Delta y\to 0} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A.$$

Note that the sum approaches a limit in either case and the limit is the volume of the solid with the base R. Now we are ready to define the double integral.

Definition: Double Integral over a Rectangular Region R

The double integral of the function $f(x, y)$ over the rectangular region R in the xy-plane is defined as

$$\iint_R f(x, y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A.$$

If $f(x, y) \geq 0$, then the volume V of the solid S, which lies above R in the xy-plane and under the graph of f, is the double integral of the function $f(x, y)$ over the rectangle R. If the function is ever negative, then the double integral can be considered a "signed" volume in a manner similar to the way we defined net signed area in The Definite Integral.

Example 15.1.1: Setting up a Double Integral and Approximating It by Double Sums

Consider the function $z = f(x, y) = 3x^2 - y$ over the rectangular region $R = [0, 2] \times [0, 2]$ (Figure 15.1.4).

a. Set up a double integral for finding the value of the signed volume of the solid S that lies above R and "under" the graph of f.

b. Divide R into four squares with $m = n = 2$, and choose the sample point as the upper right corner point of each square (1,1), (2,1),(1,2), and (2,2) (Figure 15.1.4) to approximate the signed volume of the solid S that lies above R and "under" the graph of f.

c. Divide R into four squares with $m = n = 2$, and choose the sample point as the midpoint of each square: (1/2, 1/2), (3/2, 1/2), (1/2,3/2), and (3/2, 3/2) to approximate the signed volume.

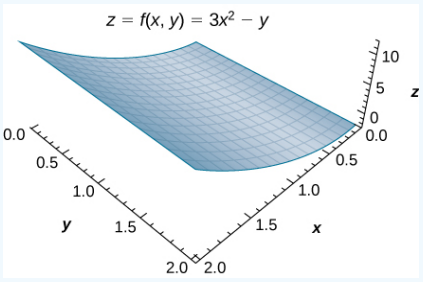


Figure 15.1.4: The function $z = f(x, y)$ graphed over the rectangular region $R = [0, 2] \times [0, 2]$.

Solution

a. As we can see, the function $z = f(x, y) = 3x^2 - y$ is above the plane. To find the signed volume of S, we need to divide the region R into small rectangles R_{ij}, each with area ΔA and with sides Δx and Δy, and choose (x_{ij}^*, y_{ij}^*) as sample points in each R_{ij}. Hence, a double integral is set up as

$$V = \iint_R (3x^2 - y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} [3(x_{ij}^*)^2 - y_{ij}^*]\Delta A.$$

b. Approximating the signed volume using a Riemann sum with $m = n = 2$ we have $\Delta A = \Delta x \Delta y = 1 \times 1 = 1$. Also, the sample points are (1, 1), (2, 1), (1, 2), and (2, 2) as shown in the following figure.

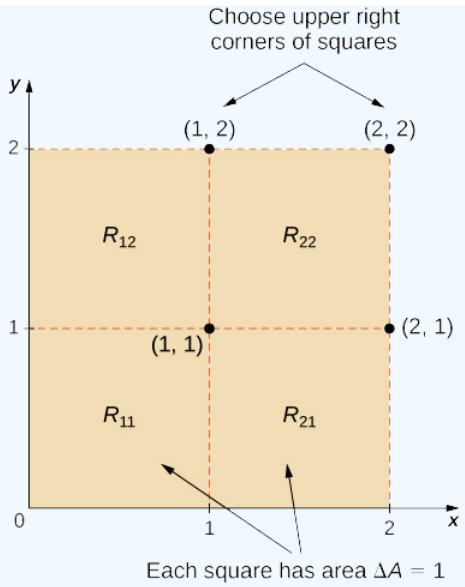


Figure 15.1.5: Subrectangles for the rectangular region $R = [0, 2] \times [0, 2]$.

Hence,

$$\begin{aligned} V &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\Delta A \\ &= \sum_{i=1}^{2}(f(x_{i1}^*, y_{i1}^*) + f(x_{i2}^*, y_{i2}^*))\Delta A \\ &= f(x_{11}^*, y_{11}^*)\Delta A + f(x_{21}^*, y_{21}^*)\Delta A + f(x_{12}^*, y_{12}^*)\Delta A + f(x_{22}^*, y_{22}^*)\Delta A \\ &= f(1,1)(1) + f(2,1)(1) + f(1,2)(1) + f(2,2)(1) \\ &= (3-1)(1) + (12-1)(1) + (3-2)(1) + (12-2)(1) \\ &= 2 + 11 + 1 + 10 = 24. \end{aligned}$$

c. Approximating the signed volume using a Riemann sum with $m = n = 2$ we have$\Delta A = \Delta x \Delta y = 1 \times 1 = 1$. In this case the sample points are (1/2, 1/2), (3/2, 1/2), (1/2, 3/2), and (3/2, 3/2).
Hence,

$$\begin{aligned} V &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\Delta A \\ &= f(x_{11}^*, y_{11}^*)\Delta A + f(x_{21}^*, y_{21}^*)\Delta A + f(x_{12}^*, y_{12}^*)\Delta A + f(x_{22}^*, y_{22}^*)\Delta A \\ &= f(1/2, 1/2)(1) + f(3/2, 1/2)(1) + f(1/2, 3/2)(1) + f(3/2, 3/2)(1) \\ &= \left(\frac{3}{4} - \frac{1}{4}\right)(1) + \left(\frac{27}{4} - \frac{1}{2}\right)(1) + \left(\frac{3}{4} - \frac{3}{2}\right)(1) + \left(\frac{27}{4} - \frac{3}{2}\right)(1) \\ &= \frac{2}{4} + \frac{25}{4} + \left(-\frac{3}{4}\right) + \frac{21}{4} = \frac{45}{4} = 11. \end{aligned}$$

Analysis

Notice that the approximate answers differ due to the choices of the sample points. In either case, we are introducing some error because we are using only a few sample points. Thus, we need to investigate how we can achieve an accurate answer.

? Exercise 15.1.1

Use the same function$z = f(x, y) = 3x^2 - y$ over the rectangular region $R = [0, 2] \times [0, 2]$.

Divide R into the same four squares with $m = n = 2$, and choose the sample points as the upper left corner point of each square (0,1), (1,1), (0,2), and (1,2) (Figure 15.1.5) to approximate the signed volume of the solid S that lies above R and "under" the graph of f.

Hint

Follow the steps of the previous example.

Answer

$$V \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\,\Delta A = 0$$

Note that we developed the concept of double integral using a rectangular region R. This concept can be extended to any general region. However, when a region is not rectangular, the subrectangles may not all fit perfectly into R, particularly if the base area is curved. We examine this situation in more detail in the next section, where we study regions that are not always rectangular and subrectangles may not fit perfectly in the region R. Also, the heights may not be exact if the surface $z = f(x, y)$ is curved. However, the errors on the sides and the height where the pieces may not fit perfectly within the solid S approach 0 as m and n approach infinity. Also, the double integral of the function $z = f(x, y)$ exists provided that the function f is not too discontinuous. If the function is bounded and continuous over R except on a finite number of smooth curves, then the double integral exists and we say that f is integrable over R.

Since $\Delta A = \Delta x \Delta y = \Delta y \Delta x$, we can express dA as $dx\,dy$ or $dy\,dx$. This means that, when we are using rectangular coordinates, the double integral over a region R denoted by

$$\iint_R f(x, y)\,dA$$

can be written as

$$\iint_R f(x, y)\,dx\,dy$$

or

$$\iint_R f(x, y)\,dy\,dx.$$

Now let's list some of the properties that can be helpful to compute double integrals.

Properties of Double Integrals

The properties of double integrals are very helpful when computing them or otherwise working with them. We list here six properties of double integrals. Properties 1 and 2 are referred to as the linearity of the integral, property 3 is the additivity of the integral, property 4 is the monotonicity of the integral, and property 5 is used to find the bounds of the integral. Property 6 is used if $f(x, y)$ is a product of two functions $g(x)$ and $h(y)$.

Theorem: Properties of Double Integrals

Assume that the functions $f(x, y)$ and $g(x, y)$ are integrable over the rectangular region R; S and T are subregions of R; and assume that m and M are real numbers.

i. The sum $f(x, y) + g(x, y)$ is integrable and

$$\iint_R [f(x, y) + g(x, y)]\,dA = \iint_R f(x, y)\,dA + \iint_R g(x, y)\,dA.$$

ii. If c is a constant, then $cf(x, y)$ is integrable and

$$\iint_R cf(x, y)\,dA = c\iint_R f(x, y)\,dA.$$

iii. If $R = S \cup T$ and $S \cap T = \emptyset$ except an overlap on the boundaries, then

$$\iint_R f(x, y)\,dA = \iint_S f(x, y)\,dA + \iint_T f(x, y)\,dA.$$

iv. If $f(x,y) \geq g(x,y)$ for (x,y) in R, then

$$\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA.$$

v. If $m \leq f(x,y) \leq M$ and $A(R) = \text{ the area of } R$, then

$$m \cdot A(R) \leq \iint_R f(x,y)\,dA \leq M \cdot A(R).$$

vi. In the case where $f(x,y)$ can be factored as a product of a function $g(x)$ of x only and a function $h(y)$ of y only, then over the region $R = \{(x,y) \mid a \leq x \leq b,\ c \leq y \leq d\}$, the double integral can be written as

$$\iint_R f(x,y)\,dA = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right).$$

These properties are used in the evaluation of double integrals, as we will see later. We will become skilled in using these properties once we become familiar with the computational tools of double integrals. So let's get to that now.

Iterated Integrals

So far, we have seen how to set up a double integral and how to obtain an approximate value for it. We can also imagine that evaluating double integrals by using the definition can be a very lengthy process if we choose larger values for m and n.Therefore, we need a practical and convenient technique for computing double integrals. In other words, we need to learn how to compute double integrals without employing the definition that uses limits and double sums.

The basic idea is that the evaluation becomes easier if we can break a double integral into single integrals by integrating first with respect to one variable and then with respect to the other. The key tool we need is called an iterated integral.

Definitions: Iterated Integrals

Assume a, b, c, and d are real numbers. We define an *iterated integral* for a function $f(x,y)$ over the rectangular region $R = [a,b] \times [c,d]$ as

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$$

or

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy.$$

The notation $\int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$ means that we integrate $f(x,y)$ with respect to y while holding x constant. Similarly, the notation $\int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$ means that we integrate $f(x,y)$ with respect to x while holding y constant. The fact that double integrals can be split into iterated integrals is expressed in Fubini's theorem. Think of this theorem as an essential tool for evaluating double integrals.

Theorem: Fubini's Theorem

Suppose that $f(x,y)$ is a function of two variables that is continuous over a rectangular region $R = \{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b,\ c \leq y \leq d\}$. Then we see from Figure 15.1.6 that the double integral of f over the region equals an iterated integral,

$$\iint_R f(x,y)\,dA = \iint_R f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

More generally, Fubini's theorem is true if f is bounded on R and f is discontinuous only on a finite number of continuous curves. In other words, f has to be integrable over R.

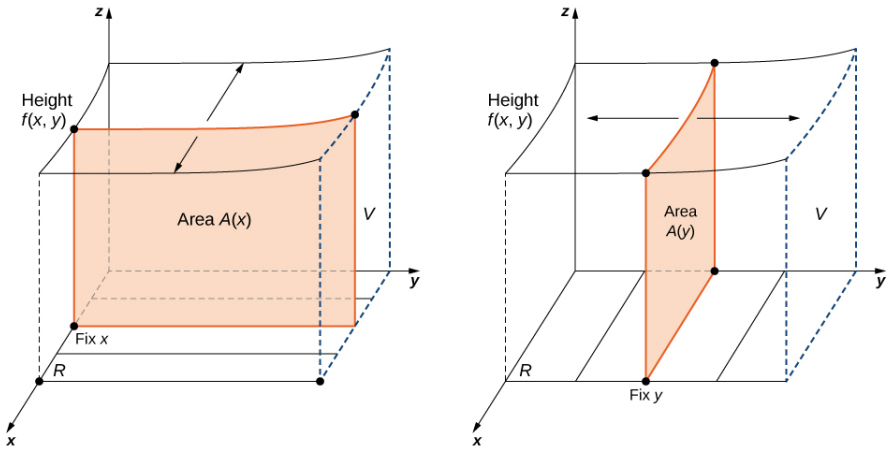


Figure 15.1.6: (a) Integrating first with respect to y and then with respect to x to find the area $A(x)$ and then the volume V; (b) integrating first with respect to x and then with respect to y to find the area $A(y)$ and then the volume V.

Example 15.1.2: Using Fubini's Theorem

Use Fubini's theorem to compute the double integral $\iint_R f(x,y)\,dA$ where $f(x,y)=x$ and $R=[0,2]\times[0,1]$.

Solution

Fubini's theorem offers an easier way to evaluate the double integral by the use of an iterated integral. Note how the boundary values of the region R become the upper and lower limits of integration.

$$\begin{aligned}\iint_R f(x,y)\,dA &= \iint_R f(x,y)\,dx\,dy \\ &= \int_{y=0}^{y=1}\int_{x=0}^{x=2} x\,dx\,dy \\ &= \int_{y=0}^{y=1}\left[\frac{x^2}{2}\Big|_{x=0}^{x=2}\right]dy \\ &= \int_{y=0}^{y=1} 2\,dy = 2y\Big|_{y=0}^{y=1} = 2\end{aligned}$$

The double integration in this example is simple enough to use Fubini's theorem directly, allowing us to convert a double integral into an iterated integral. Consequently, we are now ready to convert all double integrals to iterated integrals and demonstrate how the properties listed earlier can help us evaluate double integrals when the function $f(x,y)$ is more complex. Note that the order of integration can be changed (see Example 7).

Example 15.1.3: Illustrating Properties i and ii

Evaluate the double integral

$$\iint_R (xy-3xy^2)\,dA, \text{ where } R=\{(x,y)\,|\,0\le x\le 2,\ 1\le y\le 2\}.$$

Solution

This function has two pieces: one piece is xy and the other is $3xy^2$. Also, the second piece has a constant 3. Notice how we use properties i and ii to help evaluate the double integral.

$$\iint_R (xy - 3xy^2)\,dA = \iint_R xy\,dA + \iint_R (-3xy^2)\,dA$$ Property i: Integral of a sum is the sum of the integrals.

$$= \int_{y=1}^{y=2}\int_{x=0}^{x=2} xy\,dx\,dy - \int_{y=1}^{y=2}\int_{x=0}^{x=2} 3xy^2\,dx\,dy$$ Convert double integrals to iterated integrals.

$$= \int_{y=1}^{y=2}\left(\frac{x^2}{2}y\right)\Big|_{x=0}^{x=2} dy - 3\int_{y=1}^{y=2}\left(\frac{x^2}{2}y^2\right)\Big|_{x=0}^{x=2} dy$$ Integrate with respect to x, holding y constant.

$$= \int_{y=1}^{y=2} 2y\,dy - \int_{y=1}^{y=2} 6y^2 dy$$ Property ii: Placing the constant before the integral.

$$= 2\int_1^2 y\,dy - 6\int_1^2 y^2\,dy$$ Integrate with respect to y.

$$= 2\frac{y^2}{2}\Big|_1^2 - 6\frac{y^3}{3}\Big|_1^2$$

$$= y^2\Big|_1^2 - 2y^3\Big|_1^2$$

$$= (4-1) - 2(8-1) = 3 - 2(7)$$
$$= 3 - 14 = -11.$$

Example 15.1.4: Illustrating Property v.

Over the region $R = \{(x,y) \mid 1 \le x \le 3,\ 1 \le y \le 2\}$, we have $2 \le x^2 + y^2 \le 13$. Find a lower and an upper bound for the integral $\iint_R (x^2 + y^2)\,dA$.

Solution

For a lower bound, integrate the constant function 2 over the region R. For an upper bound, integrate the constant function 13 over the region R.

$$\int_1^2\int_1^3 2\,dx\,dy = \int_1^2 [2x\Big|_1^3]\,dy = \int_1^2 2(2)dy = 4y\Big|_1^2 = 4(2-1) = 4$$
$$\int_1^2\int_1^3 13dx\,dy = \int_1^2 [13x\Big|_1^3]\,dy = \int_1^2 13(2)\,dy = 26y\Big|_1^2 = 26(2-1) = 26.$$

Hence, we obtain $4 \le \iint_R (x^2 + y^2)\,dA \le 26$.

Example 15.1.5: Illustrating Property vi

Evaluate the integral $\iint_R e^y \cos x\,dA$ over the region $R = \{(x,y) \mid 0 \le x \le \frac{\pi}{2},\ 0 \le y \le 1\}$.

Solution

This is a great example for property vi because the function $f(x,y)$ is clearly the product of two single-variable functions e^y and $\cos x$. Thus we can split the integral into two parts and then integrate each one as a single-variable integration problem.

$$\iint_R e^y \cos x \, dA = \int_0^1 \int_0^{\pi/2} e^y \cos x \, dx \, dy$$
$$= \left(\int_0^1 e^y dy \right) \left(\int_0^{\pi/2} \cos x \, dx \right)$$
$$= (e^y \Big|_0^1)(\sin x \Big|_0^{\pi/2})$$
$$= e - 1.$$

? Exercise 15.1.2

a. Use the properties of the double integral and Fubini's theorem to evaluate the integral

$$\int_0^1 \int_{-1}^3 (3 - x + 4y) \, dy \, dx.$$

b. Show that $0 \leq \iint_R \sin \pi x \, \cos \pi y \, dA \leq \frac{1}{32}$ where $R = (0, \frac{1}{4}) \, (\frac{1}{4}, \frac{1}{2})$.

Hint

Use properties i. and ii. and evaluate the iterated integral, and then use property v.

Answer

a. 26

b. Answers may vary.

As we mentioned before, when we are using rectangular coordinates, the double integral over a region R denoted by $\iint_R f(x, y) \, dA$ can be written as $\iint_R \, f(x, y) \, dx \, dy$ or $\iint_R \, f(x, y) \, dy \, dx$. The next example shows that the results are the same regardless of which order of integration we choose.

✓ Example 15.1.6: Evaluating an Iterated Integral in Two Ways

Let's return to the function $f(x, y) = 3x^2 - y$ from Example 1, this time over the rectangular region $R = [0, 2] \times [0, 3]$. Use Fubini's theorem to evaluate $\iint_R f(x, y) \, dA$ in two different ways:

a. First integrate with respect to y and then with respect to x;
b. First integrate with respect to x and then with respect to y.

Solution

Figure 15.1.6 shows how the calculation works in two different ways.

a. First integrate with respect to y and then integrate with respect to x:

$$\iint_R f(x, y) \, dA = \int_{x=0}^{x=2} \int_{y=0}^{y=3} (3x^2 - y) \, dy \, dx$$
$$= \int_{x=0}^{x=2} \left(\int_{y=0}^{y=3} (3x^2 - y) \, dy \right) dx = \int_{x=0}^{x=2} \left[3x^2 y - \frac{y^2}{2} \Big|_{y=0}^{y=3} \right] dx$$
$$= \int_{x=0}^{x=2} \left(9x^2 - \frac{9}{2} \right) dx = 3x^3 - \frac{9}{2} x \Big|_{x=0}^{x=2} = 15.$$

b. First integrate with respect to x and then integrate with respect to y:

$$\begin{aligned}\iint_R f(x,y)\,dA &= \int_{y=0}^{y=3}\int_{x=0}^{x=2}(3x^2-y)\,dx\,dy \\ &= \int_{y=0}^{y=3}\left(\int_{x=0}^{x=2}(3x^2-y)\,dx\right)\,dy \\ &= \int_{y=0}^{y=3}\left[x^3-xy\Big|_{x=0}^{x=2}\right]dy \\ &= \int_{y=0}^{y=3}(8-2y)\,dy = 8y-y^2\Big|_{y=0}^{y=3} = 15.\end{aligned}$$

Analysis

With either order of integration, the double integral gives us an answer of 15. We might wish to interpret this answer as a volume in cubic units of the solid S below the function $f(x,y)=3x^2-y$ over the region $R=[0,2]\times[0,3]$. However, remember that the interpretation of a double integral as a (non-signed) volume works only when the integrand f is a nonnegative function over the base region R.

? Exercise 15.1.3

Evaluate

$$\int_{y=-3}^{y=2}\int_{x=3}^{x=5}(2-3x^2+y^2)\,dx\,dy.$$

Hint

Use Fubini's theorem.

Answer

$-\frac{1340}{3}$

In the next example we see that it can actually be beneficial to switch the order of integration to make the computation easier. We will come back to this idea several times in this chapter.

✓ Example 15.1.7: Switching the Order of Integration

Consider the double integral $\iint_R x\,\sin(xy)\,dA$ over the region $R=\{(x,y)\,|\,0\le x\le\pi,\ 1\le y\le 2\}$ (Figure 15.1.7).

a. Express the double integral in two different ways.
b. Analyze whether evaluating the double integral in one way is easier than the other and why.
c. Evaluate the integral.

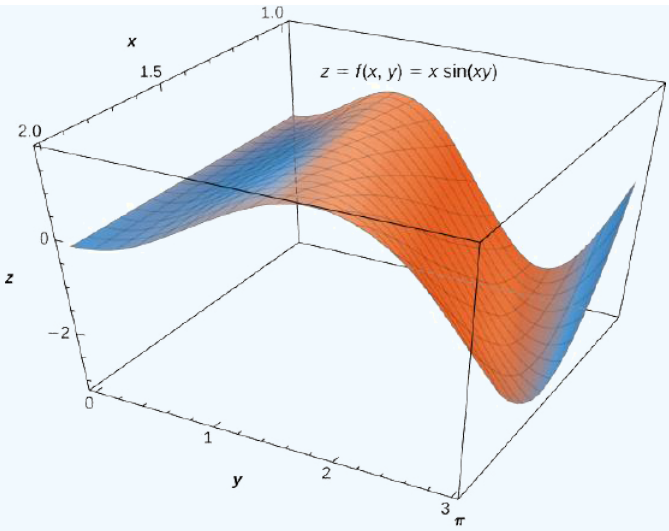


Figure 15.1.7: The function $z = f(x, y) = x\ \sin(xy)$ over the rectangular region $R = [0, \pi] \times [1, 2]$.

a. We can express $\iint_R x\ \sin(xy)\, dA$ in the following two ways: first by integrating with respect to y and then with respect to x; second by integrating with respect to x and then with respect to y.

$$\iint_R x\ \sin(xy)\, dA = \int_{x=0}^{x=\pi} \int_{y=1}^{y=2} x\ \sin(xy)\, dy\, dx$$

Integrate first with respect to y.

$$= \int_{y=1}^{y=2} \int_{x=0}^{x=\pi} x\ \sin(xy)\, dx\, dy$$

Integrate first with respect to x.

b. If we want to integrate with respect to y first and then integrate with respect to x, we see that we can use the substitution $u = xy$, which gives $du = x\, dy$. Hence the inner integral is simply $\int \sin u\, du$ and we can change the limits to be functions of x,

$$\iint_R x\ \sin(xy)\, dA = \int_{x=0}^{x=\pi} \int_{y=1}^{y=2} x\ \sin(xy)\, dy\, dx = \int_{x=0}^{x=\pi} \left[\int_{u=x}^{u=2x} \sin(u)\, du \right] dx.$$

However, integrating with respect to x first and then integrating with respect to y requires integration by parts for the inner integral, with $u = x$ and $dv = \sin(xy)dx$

Then $du = dx$ and $v = -\frac{\cos(xy)}{y}$, so

$$\iint_R x \sin(xy)\, dA = \int_{y=1}^{y=2} \int_{x=0}^{x=\pi} x \sin(xy)\, dx\, dy$$
$$= \int_{y=1}^{y=2} \left[-\frac{x\ \cos(xy)}{y} \Big|_{x=0}^{x=\pi} + \frac{1}{y} \int_{x=0}^{x=\pi} \cos(xy)\, dx \right] dy.$$

Since the evaluation is getting complicated, we will only do the computation that is easier to do, which is clearly the first method.

c. Evaluate the double integral using the easier way.

$$\iint_R x\,\sin(xy)\,dA = \int_{x=0}^{x=\pi}\int_{y=1}^{y=2} x\,\sin(xy)\,dy\,dx$$
$$= \int_{x=0}^{x=\pi}\left[\int_{u=x}^{u=2x}\sin(u)\,du\right]dx = \int_{x=0}^{x=\pi}\left[-\cos u\Big|_{u=x}^{u=2x}\right]dx$$
$$= \int_{x=0}^{x=\pi}(-\cos 2x+\cos x)\,dx$$
$$= \left(-\frac{1}{2}\sin 2x+\sin x\right)\Big|_{x=0}^{x=\pi} = 0.$$

? Exercise 15.1.4

Evaluate the integral $\iint_R xe^{xy}\,dA$ where $R=[0,1]\times[0,\ln 5]$.

Hint

Integrate with respect to y first.

Answer

$\frac{4-\ln 5}{\ln 5}$

Applications of Double Integrals

Double integrals are very useful for finding the area of a region bounded by curves of functions. We describe this situation in more detail in the next section. However, if the region is a rectangular shape, we can find its area by integrating the constant function $f(x,y)=1$ over the region R.

Definition: Area of a Region R

The area of the region R is given by

$$A(R)=\iint_R 1\,dA.$$

This definition makes sense because using $f(x,y)=1$ and evaluating the integral make it a product of length and width. Let's check this formula with an example and see how this works.

Example 15.1.8: Finding Area Using a Double Integral

Find the area of the region $R=\{(x,y)\,|\,0\le x\le 3,\ 0\le y\le 2\}$ by using a double integral, that is, by integrating 1 over the region R.

Solution

The region is rectangular with length 3 and width 2, so we know that the area is 6. We get the same answer when we use a double integral:

$$A(R)=\int_0^2\int_0^3 1\,dx\,dy=\int_0^2\left[x\Big|_0^3\right]dy=\int_0^2 3dy=3\int_0^2 dy=3y\Big|_0^2=3(2)=6\text{ units}^2.$$

We have already seen how double integrals can be used to find the volume of a solid bounded above by a function $f(x,y)\ge 0$ over a region R provided $f(x,y)\ge 0$ for all (x,y) in R. Here is another example to illustrate this concept.

Example 15.1.9: Volume of an Elliptic Paraboloid

Find the volume V of the solid S that is bounded by the elliptic paraboloid $2x^2 + y^2 + z = 27$, the planes $x = 3$ and $y = 3$, and the three coordinate planes.

Solution

First notice the graph of the surface $z = 27 - 2x^2 - y^2$ in Figure 15.1.8(a) and above the square region $R_1 = [-3, 3] \times [-3, 3]$. However, we need the volume of the solid bounded by the elliptic paraboloid $2x^2 + y^2 + z = 27$, the planes $x = 3$ and $y = 3$, and the three coordinate planes.

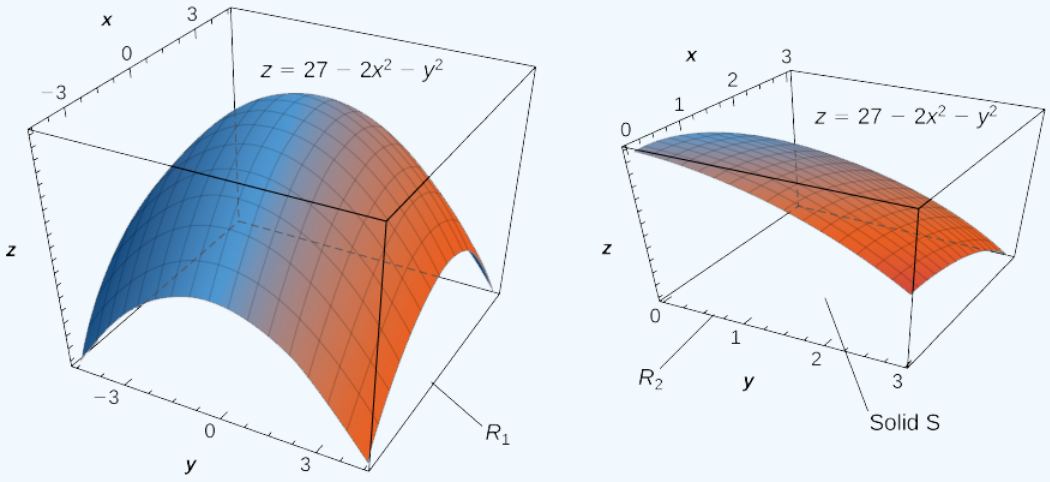


Figure 15.1.8: (a) The surface $z = 27 - 2x^2 - y^2$ above the square region $R1 = [-3, 3] \times [-3, 3]$. (b) The solid S lies under the surface $z = 27 - 2x^2 - y^2$ above the square region $R1 = [0, 3] \times [0, 3]$.

Now let's look at the graph of the surface in Figure 15.1.8(b). We determine the volume V by evaluating the double integral over R_2:

$$
\begin{aligned}
V &= \iint_R z\,dA = \iint_R (27 - 2x^2 - y^2)\,dA \\
&= \int_{y=0}^{y=3} \int_{x=0}^{x=3} (27 - 2x^2 - y^2)\,dx\,dy && \text{Convert to literal integral.} \\
&= \int_{y=0}^{y=3} \left[27x - \frac{2}{3}x^3 - y^2 x\right]\Big|_{x=0}^{x=3} dy && \text{Integrate with respect to } x. \\
&= \int_{y=0}^{y=3} (63 - 3y^2)dy = 63y - y^3\Big|_{y=0}^{y=3} = 162.
\end{aligned}
$$

Exercise 15.1.5

Find the volume of the solid bounded above by the graph of $f(x, y) = xy\sin(x^2 y)$ and below by the xy-plane on the rectangular region $R = [0, 1] \times [0, \pi]$.

Hint

Graph the function, set up the integral, and use an iterated integral.

Answer

$\frac{\pi}{2}$

Recall that we defined the average value of a function of one variable on an interval $[a, b]$ as

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Similarly, we can define the average value of a function of two variables over a region R. The main difference is that we divide by an area instead of the width of an interval.

Definition: Average Value of a Function

The **average value of a function** of two variables over a region R is

$$F_{ave} = \frac{1}{\text{Area of } R}\iint_R f(x,y)\,dx\,dy.$$

In the next example we find the average value of a function over a rectangular region. This is a good example of obtaining useful information for an integration by making individual measurements over a grid, instead of trying to find an algebraic expression for a function.

Example 15.1.10: Calculating Average Storm Rainfall

The weather map in Figure 15.1.9 shows an unusually moist storm system associated with the remnants of Hurricane Karl, which dumped 4–8 inches (100–200 mm) of rain in some parts of the Midwest on September 22–23, 2010. The area of rainfall measured 300 miles east to west and 250 miles north to south. Estimate the average rainfall over the entire area in those two days.

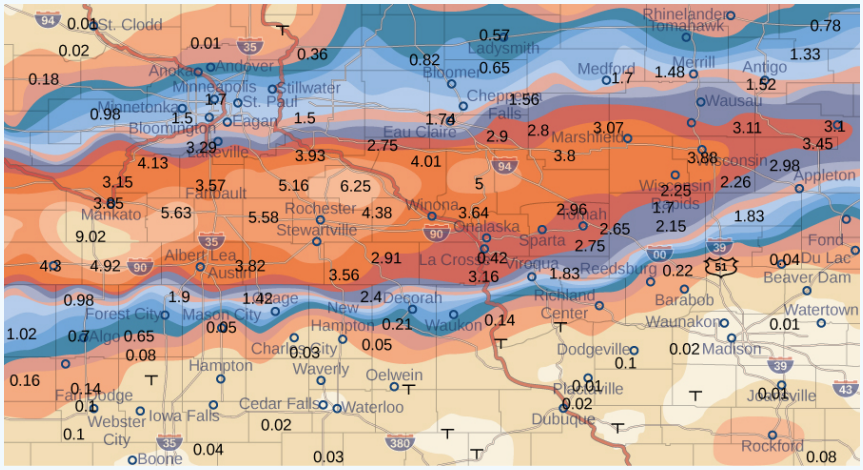


Figure 15.1.9: Effects of Hurricane Karl, which dumped 4–8 inches (100–200 mm) of rain in some parts of southwest Wisconsin, southern Minnesota, and southeast South Dakota over a span of 300 miles east to west and 250 miles north to south.

Solution

Place the origin at the southwest corner of the map so that all the values can be considered as being in the first quadrant and hence all are positive. Now divide the entire map into six rectangles ($m = 2$ and $n = 3$), as shown in Figure 15.1.9. Assume $f(x,y)$ denotes the storm rainfall in inches at a point approximately x miles to the east of the origin and y miles to the north of the origin. Let R represent the entire area of $250 \times 300 = 75000$ square miles. Then the area of each subrectangle is

$$\Delta A = \frac{1}{6}(75000) = 12500.$$

Assume $(x_{ij}*, y_{ij}*)$ are approximately the midpoints of each subrectangle R_{ij}. Note the color-coded region at each of these points, and estimate the rainfall. The rainfall at each of these points can be estimated as:

- At (x_{11}, y_{11}), the rainfall is 0.08.
- At (x_{12}, y_{12}), the rainfall is 0.08.
- At (x_{13}, y_{13}), the rainfall is 0.01.

- At (x_{21}, y_{21}), the rainfall is 1.70.
- At (x_{22}, y_{22}), the rainfall is 1.74.
- At (x_{23}, y_{23}), the rainfall is 3.00.

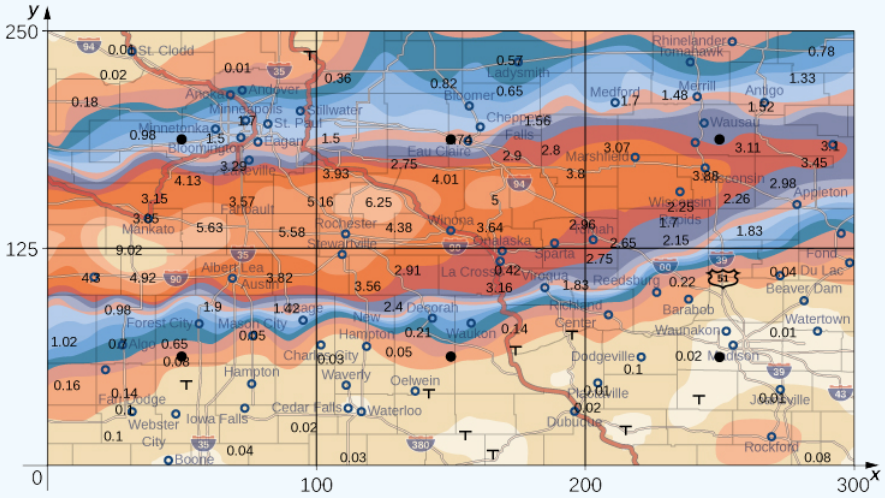


Figure 15.1.10: Storm rainfall with rectangular axes and showing the midpoints of each subrectangle.

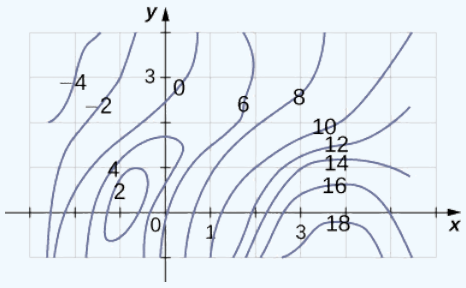
According to our definition, the average storm rainfall in the entire area during those two days was

$$
\begin{aligned}
f_{ave} &= \frac{1}{Area\, R} \\
\iint_R f(x,y)\,dx\,dy &= \frac{1}{75000}\iint_R f(x,y)\,dx\,dy \\
&\approx \frac{1}{75000}\sum_{i=1}^{3}\sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\Delta A \\
&= \frac{1}{75000}\Big[f(x_{11}^*, y_{11}^*)\Delta A + f(x_{12}^*, y_{12}^*)\Delta A + f(x_{13}^*, y_{13}^*)\Delta A \\
&\qquad + f(x_{21}^*, y_{21}^*)\Delta A + f(x_{22}^*, y_{22}^*)\Delta A + f(x_{23}^*, y_{23}^*)\Delta A\Big] \\
&\approx \frac{1}{75000}\Big[0.08 + 0.08 + 0.01 + 1.70 + 1.74 + 3.00\Big]\Delta A \\
&= \frac{1}{75000}\Big[0.08 + 0.08 + 0.01 + 1.70 + 1.74 + 3.00\Big]12500 \\
&= \frac{1}{6}\Big[0.08 + 0.08 + 0.01 + 1.70 + 1.74 + 3.00\Big] \\
&\approx 1.10 \text{ in.}
\end{aligned}
$$

During September 22–23, 2010 this area had an average storm rainfall of approximately 1.10 inches.

? Exercise 15.1.6

A contour map is shown for a function $f(x, y)$ on the rectangle $R = [-3, 6] \times [-1, 4]$.



a. Use the midpoint rule with $m = 3$ and $n = 2$ to estimate the value of $\iint_R f(x, y)\, dA$.

b. Estimate the average value of the function $f(x, y)$.

Hint

Divide the region into six rectangles, and use the contour lines to estimate the values for $f(x, y)$.

Answer

Answers to both parts a. and b. may vary.

Key Concepts

- We can use a double Riemann sum to approximate the volume of a solid bounded above by a function of two variables over a rectangular region. By taking the limit, this becomes a double integral representing the volume of the solid.
- Properties of double integral are useful to simplify computation and find bounds on their values.
- We can use Fubini's theorem to write and evaluate a double integral as an iterated integral.
- Double integrals are used to calculate the area of a region, the volume under a surface, and the average value of a function of two variables over a rectangular region.

Key Equations

- $$\iint_R f(x, y)\, dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i j*, y_i j*)\, \Delta A$$
- $$\int_a^b \int_c^d f(x, y)\, dx\, dy = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$

  or

  $$\int_c^d \int_a^b f(x, y)\, dx\, dy = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy$$
- $$f_{ave} = \frac{1}{\text{Area of } R} \iint_R f(x, y)\, dx\, dy$$

Glossary

double integral

of the function $f(x, y)$ over the region R in the xy-plane is defined as the limit of a double Riemann sum,

$$\iint_R f(x, y)\, dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A.$$

double Riemann sum

of the function $f(x, y)$ over a rectangular region R is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A,$$

where R is divided into smaller subrectangles R_{ij} and (x_{ij}^*, y_{ij}^*) is an arbitrary point in R_{ij}

Fubini's theorem

if $f(x, y)$ is a function of two variables that is continuous over a rectangular region $R = \{(x, y) \in \mathbb{R}^2 \mid a \le x \le b,\ c \le y \le d\}$, then the double integral of f over the region equals an iterated integral,

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dx\, dy = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

iterated integral

for a function $f(x, y)$ over the region R is

a. $\int_a^b \int_c^d f(x, y)\, dx\, dy = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx,$

b. $\int_c^d \int_a^b f(x, y)\, dx\, dy = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy,$

where a, b, c, and d are any real numbers and $R = [a, b] \times [c, d]$

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15.1E: Exercises for Section 15.1

In exercises 1 and 2, use the midpoint rule with $m = 4$ and $n = 2$ to estimate the volume of the solid bounded by the surface $z = f(x, y)$, the vertical planes $x = 1$, $x = 2$, $y = 1$, and $y = 2$, and the horizontal plane $z = 0$.

1) $f(x, y) = 4x + 2y + 8xy$

Answer

27

2) $f(x, y) = 16x^2 + \frac{y}{2}$

In exercises 3 and 4, estimate the volume of the solid under the surface $z = f(x, y)$ and above the rectangular region R by using a Riemann sum with $m = n = 2$ and the sample points to be the lower left corners of the sub-rectangles of the partition.

3) $f(x, y) = \sin x - \cos y$, $R = [0, \pi] \times [0, \pi]$

Answer

0

4) $f(x, y) = \cos x + \cos y$, $R = [0, \pi] \times [0, \frac{\pi}{2}]$

5) Use the midpoint rule with $m = n = 2$ to estimate $\iint_R f(x, y)\, dA$, where the values of the function f on $R = [8, 10] \times [9, 11]$ are given in the following table.

	y				
x	9	9.5	10	10.5	11
8	9.8	5	6.7	5	5.6
8.5	9.4	4.5	8	5.4	3.4
9	8.7	4.6	6	5.5	3.4
9.5	6.7	6	4.5	5.4	6.7
10	6.8	6.4	5.5	5.7	6.8

Answer

21.3

6) The values of the function f on the rectangle $R = [0, 2] \times [7, 9]$ are given in the following table. Estimate the double integral $\iint_R f(x, y)\, dA$ by using a Riemann sum with $m = n = 2$. Select the sample points to be the upper right corners of the sub-squares of R.

	$y_0 = 7$	$y_1 = 8$	$y_2 = 9$
$x_0 = 0$	10.22	10.21	9.85
$x_1 = 1$	6.73	9.75	9.63
$x_2 = 2$	5.62	7.83	8.21

7) The depth of a children's 4-ft by 4-ft swimming pool, measured at 1-ft intervals, is given in the following table.

a. Estimate the volume of water in the swimming pool by using a Riemann sum with $m = n = 2$. Select the sample points using the midpoint rule on $R = [0, 4] \times [0, 4]$.
b. Find the average depth of the swimming pool.

	y

	y				
x	0	1	2	3	4
0	1	1.5	2	2.5	3
1	1	1.5	2	2.5	3
2	1	1.5	1.5	2.5	3
3	1	1	1.5	2	2.5
4	1	1	1	1.5	2

Answer

a. 28 ft^3

b. 1.75 ft.

8) The depth of a 3-ft by 3-ft hole in the ground, measured at 1-ft intervals, is given in the following table.

a. Estimate the volume of the hole by using a Riemann sum with $m = n = 3$ and the sample points to be the upper left corners of the sub-squares of R.

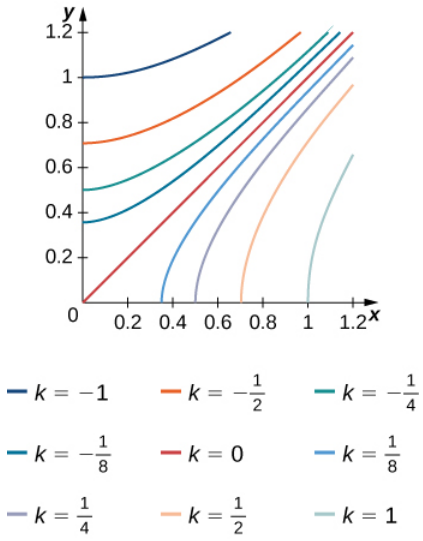
b. Find the average depth of the hole.

	y			
x	0	1	2	3
0	6	6.5	6.4	6
1	6.5	7	7.5	6.5
2	6.5	6.7	6.5	6
3	6	6.5	5	5.6

9) The level curves $f(x, y) = k$ of the function f are given in the following graph, where k is a constant.

a. Apply the midpoint rule with $m = n = 2$ to estimate the double integral $\iint_R f(x, y)\, dA$, where $R = [0.2, 1] \times [0, 0.8]$.

b. Estimate the average value of the function f on R.



Answer

a. Answers will vary somewhat, but should be close to these: $f(0.4, 0.2) \approx 0.125$, $f(0.4, 0.6) \approx -0.2$, $f(0.8, 0.2) \approx 0.6$, and $f(0.8, 0.6) \approx 0.3$.

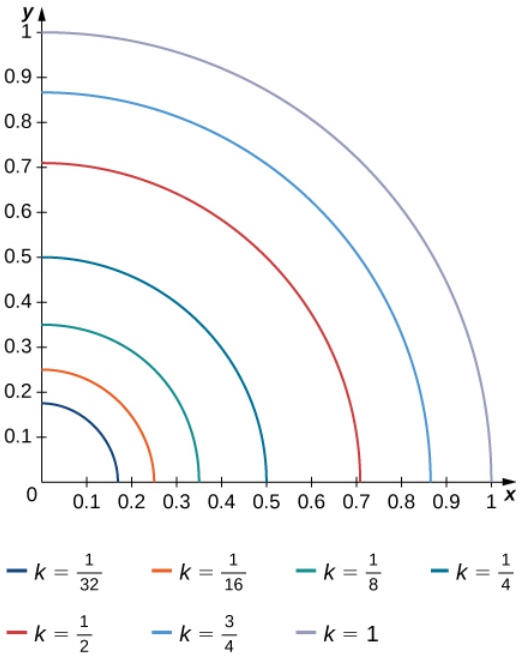
Since the area of each rectangle is $\Delta A_i = (0.4)(0.4) = 0.16$ units^2, $\iint_R f(x, y)\, dA = \int_{0.2}^{1} \int_{0}^{0.8} f(x, y)\, dy\, dx \approx f(0.4, 0.2)(0.16) + f(0.4, 0.6)(0.16) + f(0.8, 0.2)(0.16) + f(0.8, 0.6)(0.16) = 0.132$

b. $f_{ave} = \dfrac{\int_{0.2}^{1} \int_{0}^{0.8} f(x, y)\, dy\, dx}{\text{area of } R} \approx \dfrac{\int_{0.2}^{1} \int_{0}^{0.8} f(x, y)\, dy\, dx}{0.64} \approx 0.206$

10) The level curves $f(x, y) = k$ of the function f are given in the following graph, where k is a constant.

a. Apply the midpoint rule with $m = n = 2$ to estimate the double integral $\iint_R f(x, y)\, dA$, where $R = [0.1, 0.5] \times [0.1, 0.5]$.

b. Estimate the average value of the function f on R.



11) The solid lying under the surface $z = \sqrt{4 - y^2}$ and above the rectangular region $R = [0, 2] \times [0, 2]$ is illustrated in the following graph. Evaluate the double integral $\iint_R f(x, y)$, where $f(x, y) = \sqrt{4 - y^2}$ by finding the volume of the corresponding solid.

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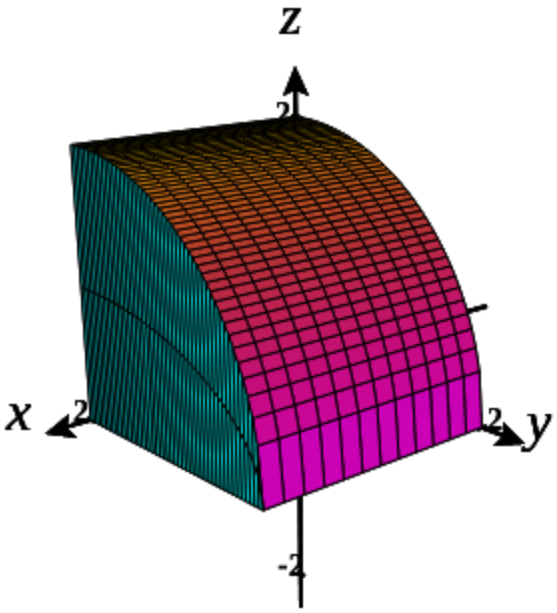
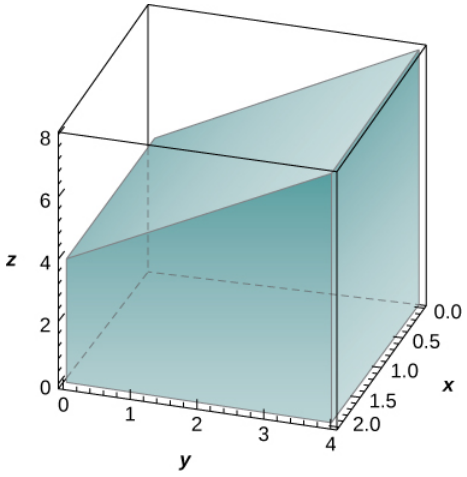


Answer

2π

12) The solid lying under the plane $z = y + 4$ and above the rectangular region $R = [0, 2] \times [0, 4]$ is illustrated in the following graph. Evaluate the double integral $\iint_R f(x, y)\, dA$, where $f(x, y) = y + 4$, by finding the volume of the corresponding solid.



In the exercises 13 - 20, calculate the integrals by reversing the order of integration.

13) $\int_{-1}^{1} \left(\int_{-2}^{2} (2x + 3y + 5)\, dx \right)\, dy$

Answer

40

14) $\int_0^2 \left(\int_0^1 (x + 2e^y + 3)\, dx \right)\, dy$

15) $\int_1^{27} \left(\int_1^2 (\sqrt[3]{x} + \sqrt[3]{y})\, dy \right)\, dx$

Answer

$\frac{81}{2} + 39\sqrt[3]{2}$

16) $\int_1^{16} \left(\int_1^8 (\sqrt[4]{x} + 2\sqrt[3]{y})\, dy \right)\, dx$

17) $\int_{\ln 2}^{\ln 3} \left(\int_0^1 e^{x+y}\, dy \right)\, dx$

Answer

$e - 1$

18) $\int_0^2 \left(\int_0^1 3^{x+y}\, dy \right)\, dx$

19) $\int_1^6 \left(\int_2^9 \frac{\sqrt{y}}{x^2}\, dy \right)\, dx$

Answer

$15 - \frac{10\sqrt{2}}{9}$

20) $\int_1^9 \left(\int_4^2 \frac{\sqrt{x}}{y^2}\, dy \right)\, dx$

In exercises 21 - 34, evaluate the iterated integrals by choosing the order of integration.

21) $\int_0^{\pi} \int_0^{\pi/2} \sin(2x)\cos(3y)\, dx\, dy$

Answer

0

22) $\int_{\pi/12}^{\pi/8} \int_{\pi/4}^{\pi/3} [\cot x + \tan(2y)]\, dx\, dy$

23) $\int_1^e \int_1^e \left[\frac{1}{x}\sin(\ln x) + \frac{1}{y}\cos(\ln y) \right]\, dx\, dy$

Answer

$(e-1)(1 + \sin 1 - \cos 1)$

24) $\int_1^e \int_1^e \frac{\sin(\ln x)\cos(\ln y)}{xy}\, dx\, dy$

25) $\int_1^2 \int_1^2 \left(\frac{\ln y}{x} + \frac{x}{2y+1} \right)\, dy\, dx$

Answer

$\frac{3}{4}\ln\left(\frac{5}{3}\right) + 2(\ln 2)^2 - \ln 2$

26) $\int_1^e \int_1^2 x^2 \ln(x)\, dy\, dx$

27) $\int_1^{\sqrt{3}} \int_1^2 y \arctan\left(\frac{1}{x}\right)\, dy\, dx$

Answer

$\frac{1}{8}[(2\sqrt{3}-3)\pi+6\ \ln 2]$

28) $\int_0^1 \int_0^{1/2} (\arcsin x + \arcsin y)\, dy\, dx$

29) $\int_0^1 \int_1^2 xe^{x+4y}\, dy\, dx$

Answer

$\frac{1}{4}e^4(e^4-1)$

30) $\int_1^2 \int_0^1 xe^{x-y}\, dy\, dx$

31) $\int_1^e \int_1^e \left(\frac{\ln y}{\sqrt{y}} + \frac{\ln x}{\sqrt{x}}\right) dy\, dx$

Answer

$4(e-1)(2-\sqrt{e})$

32) $\int_1^e \int_1^e \left(\frac{x\ \ln y}{\sqrt{y}} + \frac{y\ \ln x}{\sqrt{x}}\right) dy\, dx$

33) $\int_0^1 \int_1^2 \left(\frac{x}{x^2+y^2}\right) dy\, dx$

Answer

$-\frac{\pi}{4} + \ln\left(\frac{5}{4}\right) - \frac{1}{2}\ln 2 + \arctan 2$

34) $\int_0^1 \int_1^2 \frac{y}{x+y^2}\, dy\, dx$

In exercises 35 - 38, find the average value of the function over the given rectangles.

35) $f(x,y) = -x+2y$, $R = [0,1]\times[0,1]$

Answer

$\frac{1}{2}$

36) $f(x,y) = x^4+2y^3$, $R = [1,2]\times[2,3]$

37) $f(x,y) = \sinh x + \sinh y$, $R = [0,1]\times[0,2]$

Answer

$\frac{1}{2}(2\ \cosh 1 + \cosh 2 - 3)$.

38) $f(x,y) = \arctan(xy)$, $R = [0,1]\times[0,1]$

39) Let f and g be two continuous functions such that $0 \le m_1 \le f(x) \le M_1$ for any $x \in [a,b]$ and $0 \le m_2 \le g(y) \le M_2$ for any $y \in [c,d]$. Show that the following inequality is true:

$$m_1 m_2 (b-a)(c-d) \le \int_a^b \int_c^d f(x)g(y)\, dydx \le M_1 M_2 (b-a)(c-d).$$

In exercises 40 - 43, use property v. of double integrals and the answer from the preceding exercise to show that the following inequalities are true.

40) $\frac{1}{e^2} \le \iint_R e^{-x^2-y^2}\, dA \le 1$, where $R = [0,1]\times[0,1]$

41) $\frac{\pi^2}{144} \le \iint_R \sin x \cos y\, dA \le \frac{\pi^2}{48}$, where $R = [\frac{\pi}{6}, \frac{\pi}{3}]\times[\frac{\pi}{6}, \frac{\pi}{3}]$

42) $0 \le \iint_R e^{-y}\ \cos x\, dA \le \frac{\pi}{2}$, where $R = [0, \frac{\pi}{2}]\times[0, \frac{\pi}{2}]$

43) $0 \le \iint_R (\ln x)(\ln y)\, dA \le (e-1)^2$, where $R = [1,e]\times[1,e]$

44) Let f and g be two continuous functions such that $0 \le m_1 \le f(x) \le M_1$ for any $x \in [a,b]$ and $0 \le m_2 \le g(y) \le M_2$ for any $y \in [c,d]$. Show that the following inequality is true:

$$(m_1+m_2)(b-a)(c-d) \le \int_a^b \int_c^d |f(x)+g(y)|\, dy\, dx \le (M_1+M_2)(b-a)(c-d)$$

In exercises 45 - 48, use property v. of double integrals and the answer from the preceding exercise to show that the following inequalities are true.

45) $\frac{2}{e} \le \iint_R (e^{-x^2}+e^{-y^2})\, dA \le 2$, where $R=[0,1]\times[0,1]$

46) $\frac{\pi^2}{36} \iint_R (\sin x+\cos y)\, dA \le \frac{\pi^2\sqrt{3}}{36}$, where $R=[\frac{\pi}{6},\frac{\pi}{3}]\times[\frac{\pi}{6},\frac{\pi}{3}]$

47) $\frac{\pi}{2}e^{-\pi/2} \le \iint_R (\cos x+e^{-y})\, dA \le \pi$, where $R=[0,\frac{\pi}{2}]\times[0,\frac{\pi}{2}]$

48) $\frac{1}{e} \le \iint_R (e^{-y}-\ln x)\, dA \le 2$, where $R=[0,1]\times[0,1]$

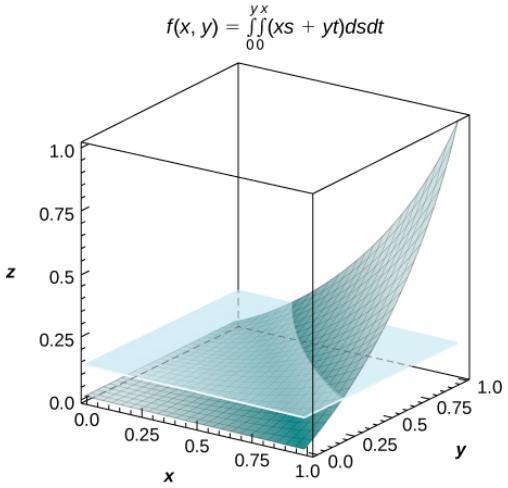
In exercises 49 - 50, the function f is given in terms of double integrals.

a. Determine the explicit form of the function f.
b. Find the volume of the solid under the surface $z=f(x,y)$ and above the region R.
c. Find the average value of the function f on R.
d. Use a computer algebra system (CAS) to plot $z=f(x,y)$ and $z=f_{ave}$ in the same system of coordinates.

49) [T] $f(x,y)=\int_0^y \int_0^x (xs+yt)ds\, dt$, where $(x,y) \in R=[0,1]\times[0,1]$

Answer

a. $f(x,y)=\frac{1}{2}xy(x^2+y^2)$;
b. $V=\int_0^1 \int_0^1 f(x,y)\, dx\, dy=\frac{1}{8}$;
c. $f_{ave}=\frac{1}{8}$;
d.



50) [T] $f(x,y)=\int_0^x \int_0^y [\cos(s)+\cos(t)]\, dt\, ds$, where $(x,y) \in R=[0,3]\times[0,3]$

51) Show that if f and g are continuous on $[a,b]$ and $[c,d]$, respectively, then

$$\int_a^b \int_c^d |f(x)+g(y)|dy\, dx = (d-c)\int_a^b f(x)\, dx$$

$$+\int_a^b \int_c^d g(y)\, dy\, dx = (b-a)\int_c^d g(y)\, dy + \int_c^d \int_a^b f(x)\, dx\, dy.$$

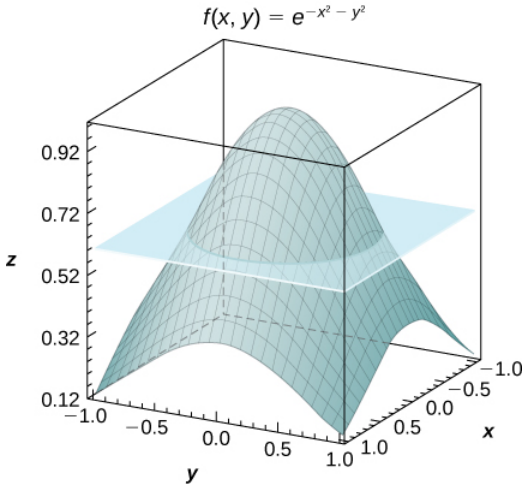
52) Show that $\int_a^b \int_c^d yf(x)+xg(y)\, dy\, dx = \frac{1}{2}(d^2-c^2)\left(\int_a^b f(x)\, dx\right)+\frac{1}{2}(b^2-a^2)\left(\int_c^d g(y)\, dy\right)$.

53) [T] Consider the function $f(x,y)=e^{-x^2-y^2}$, where $(x,y) \in R=[-1,1]\times[-1,1]$.

a. Use the midpoint rule with $m = n = 2, 4, ..., 10$ to estimate the double integral $I = \iint_R e^{-x^2-y^2} dA$. Round your answers to the nearest hundredths.
b. For $m = n = 2$, find the average value of f over the region R. Round your answer to the nearest hundredths.
c. Use a CAS to graph in the same coordinate system the solid whose volume is given by $\iint_R e^{-x^2-y^2} dA$ and the plane $z = f_{ave}$.

Answer

a. For $m = n = 2$, $I = 4e^{-0.5} \approx 2.43$
b. $f_{ave} = e^{-0.5} \simeq 0.61$;
c.



54) [T] Consider the function $f(x, y) = \sin(x^2)\ \cos(y^2)$, where $(x, y \in R = [-1, 1] \times [-1, 1]$.

a. Use the midpoint rule with $m = n = 2, 4, ..., 10$ to estimate the double integral $I = \iint_R \sin(x^2) \cos(y^2)\ dA$. Round your answers to the nearest hundredths.
b. For $m = n = 2$, find the average value of $f$over the region R. Round your answer to the nearest hundredths.
c. Use a CAS to graph in the same coordinate system the solid whose volume is given by $\iint_R \sin(x^2) \cos(y^2)\ dA$ and the plane $z = f_{ave}$.

In exercises 55 - 56, the functions f_n are given, where $n \geq 1$ is a natural number.

a. Find the volume of the solids S_n under the surfaces $z = f_n(x, y)$ and above the region R.
b. Determine the limit of the volumes of the solids S_n as n increases without bound.

55) $f(x, y) = x^n + y^n + xy,\ (x, y) \in R = [0, 1] \times [0, 1]$

Answer

a. $\frac{2}{n+1} + \frac{1}{4}$
b. $\frac{1}{4}$

56) $f(x, y) = \frac{1}{x^n} + \frac{1}{y^n},\ (x, y) \in R = [1, 2] \times [1, 2]$

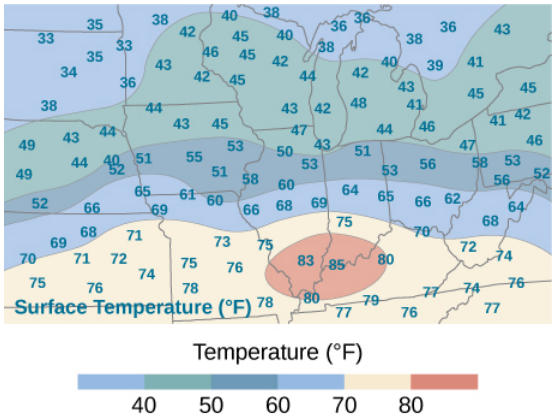
57) Show that the average value of a function f on a rectangular region $R = [a, b] \times [c, d]$ is $f_{ave} \approx \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)$,where (x_{ij}^*, y_{ij}^*) are the sample points of the partition of R, where $1 \leq i \leq m$ and $1 \leq j \leq n$.

58) Use the midpoint rule with $m = n$ to show that the average value of a function f on a rectangular region $R = [a, b] \times [c, d]$ is approximated by

$$f_{ave} \approx \frac{1}{n^2} \sum_{i,j=1}^{n} f\left(\frac{1}{2}(x_{i=1} + x_i),\ \frac{1}{2}(y_{j=1} + y_j)\right).$$

59) An isotherm map is a chart connecting points having the same temperature at a given time for a given period of time. Use the preceding exercise and apply the midpoint rule with $m = n = 2$ to find the average temperature over the region given in the

following figure.



Answer

56.5° F; here $f(x_1^*, y_1^*) = 71,\ f(x_2^*, y_1^*) = 72,\ f(x_2^*, y_1^*) = 40,\ f(x_2^*, y_2^*) = 43$, where x_i^* and y_j^* are the midpoints of the subintervals of the partitions of $[a, b]$ and $[c, d]$, respectively.

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15.2: Double Integrals over General Regions

Learning Objectives

- Recognize when a function of two variables is integrable over a general region.
- Evaluate a double integral by computing an iterated integral over a region bounded by two vertical lines and two functions of x, or two horizontal lines and two functions of y.
- Simplify the calculation of an iterated integral by changing the order of integration.
- Use double integrals to calculate the volume of a region between two surfaces or the area of a plane region.
- Solve problems involving double improper integrals.

Previously, we studied the concept of double integrals and examined the tools needed to compute them. We learned techniques and properties to integrate functions of two variables over rectangular regions. We also discussed several applications, such as finding the volume bounded above by a function over a rectangular region, finding area by integration, and calculating the average value of a function of two variables.

In this section we consider double integrals of functions defined over a general bounded region D on the plane. Most of the previous results hold in this situation as well, but some techniques need to be extended to cover this more general case.

General Regions of Integration

An example of a general bounded region D on a plane is shown in Figure 15.2.1. Since D is bounded on the plane, there must exist a rectangular region R on the same plane that encloses the region D that is, a rectangular region R exists such that D is a subset of $R (D \subseteq R)$.

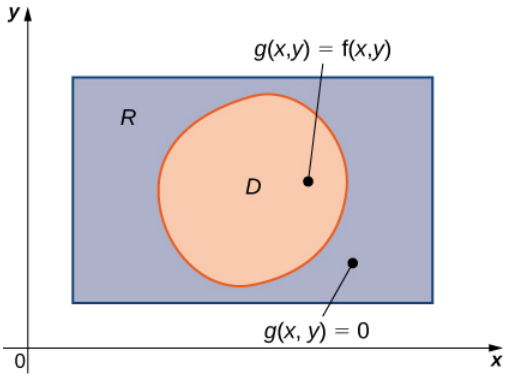

Figure 15.2.1: For a region D that is a subset of R, we can define a function $g(x, y)$ to equal $f(x, y)$ at every point in D and 0 at every point of R not in D.

Suppose $z = f(x, y)$ is defined on a general planar bounded region D as in Figure 15.2.1. In order to develop double integrals of f over D we extend the definition of the function to include all points on the rectangular region R and then use the concepts and tools from the preceding section. But how do we extend the definition of f to include all the points on R? We do this by defining a new function $g(x, y)$ on R as follows:

$$g(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \text{ is in } D \\ 0, & \text{if } (x,y) \text{ is in } R \text{ but not in } D \end{cases}$$

Note that we might have some technical difficulties if the boundary of D is complicated. So we assume the boundary to be a piecewise smooth and continuous simple closed curve. Also, since all the results developed in the section on Double Integrals over Rectangular Regions used an integrable function $f(x, y)$ we must be careful about $g(x, y)$ and verify that $g(x, y)$ is an integrable function over the rectangular region R. This happens as long as the region D is bounded by simple closed curves. For now we will concentrate on the descriptions of the regions rather than the function and extend our theory appropriately for integration.

We consider two types of planar bounded regions.

Definition: Type I and Type II regions

A region D in the (x, y)-plane is of Type I if it lies between two vertical lines and the graphs of two continuous functions $g_1(x)$ and $g_2(x)$. That is (Figure 15.2.2),

$$D = \{(x, y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}.$$

A region D in the xy-plane is of Type II if it lies between two horizontal lines and the graphs of two continuous functions $h_1(y)$ and $h_2(y)$. That is (Figure 15.2.3),

$$D = \{(x, y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}.$$

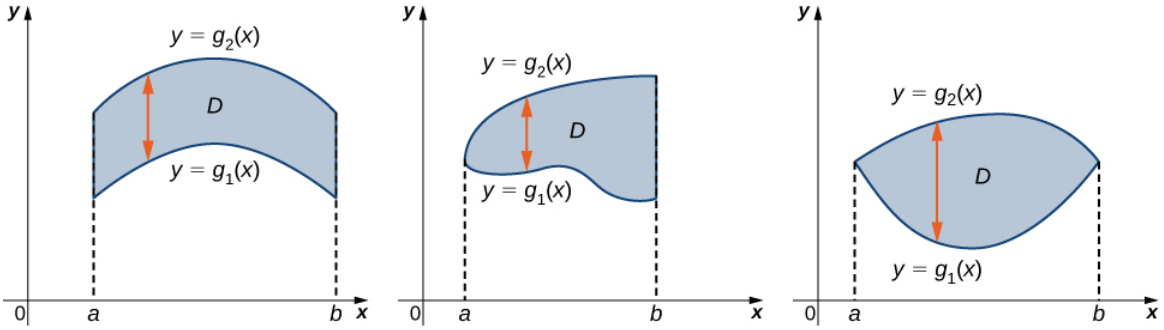

Figure 15.2.2. A Type I region lies between two vertical lines and the graphs of two functions of x.

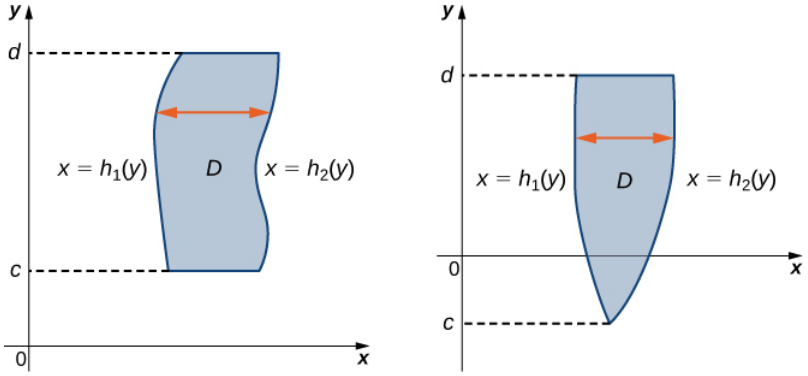

Figure 15.2.3: A Type II region lies between two horizontal lines and the graphs of two functions of y.

Example 15.2.1: Describing a Region as Type I and Also as Type II

Consider the region in the first quadrant between the functions $y = \sqrt{x}$ and $y = x^3$ (Figure 15.2.4). Describe the region first as Type I and then as Type II.

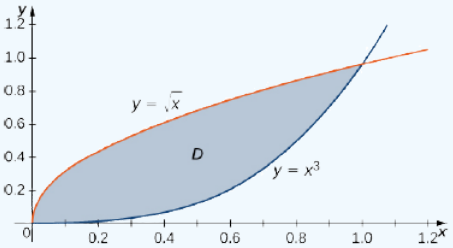

Figure 15.2.4: Region D can be described as Type I or as Type II.

When describing a region as Type I, we need to identify the function that lies above the region and the function that lies below the region. Here, region D is bounded above by $y = \sqrt{x}$ and below by $y = x^3$ in the interval for x in $[0, 1]$. Hence, as Type I, D is described as the set $\{(x, y) \mid 0 \le x \le 1,\ x^3 \le y \le \sqrt[3]{x}\}$.

However, when describing a region as Type II, we need to identify the function that lies on the left of the region and the function that lies on the right of the region. Here, the region D is bounded on the left by $x = y^2$ and on the right by $x = \sqrt[3]{y}$ in

the interval for y in $[0,1]$. Hence, as Type II, D is described as the set $\{(x,y)\,|\,0\le y\le 1,\ y^2\le x\le \sqrt[3]{y}\}$.

Exercise 15.2.1

Consider the region in the first quadrant between the functions $y=2x$ and $y=x^2$. Describe the region first as Type I and then as Type II.

Hint

Graph the functions, and draw vertical and horizontal lines.

Answer

Type I and Type II are expressed as $\{(x,y)\,|\,0\le x\le 2,\ x^2\le y\le 2x\}$ and $\{(x,y)|\,0\le y\le 4,\ \frac{1}{2}y\le x\le \sqrt{y}\}$, respectively.

Double Integrals over Non-rectangular Regions

To develop the concept and tools for evaluation of a double integral over a general, nonrectangular region, we need to first understand the region and be able to express it as Type I or Type II or a combination of both. Without understanding the regions, we will not be able to decide the limits of integrations in double integrals. As a first step, let us look at the following theorem.

Theorem: Double Integrals over Nonrectangular Regions

Suppose $g(x,y)$ is the extension to the rectangle R of the function $f(x,y)$ defined on the regions D and R as shown in Figure 15.2.1 inside R. Then $g(x,y)$ is integrable and we define the double integral of $f(x,y)$ over D by

$$\iint_D f(x,y)\,dA=\iint_R g(x,y)\,dA.$$

The right-hand side of this equation is what we have seen before, so this theorem is reasonable because R is a rectangle and $\iint_R g(x,y)dA$ has been discussed in the preceding section. Also, the equality works because the values of $g(x,y)$ are 0 for any point (x,y) that lies outside D and hence these points do not add anything to the integral. However, it is important that the rectangle R contains the region D.

As a matter of fact, if the region D is bounded by smooth curves on a plane and we are able to describe it as Type I or Type II or a mix of both, then we can use the following theorem and not have to find a rectangle R containing the region.

Theorem: Fubini's Theorem (Strong Form)

For a function $f(x,y)$ that is continuous on a region D of Type I, we have

$$\iint_D f(x,y)\,dA=\iint_D f(x,y)\,dy\,dx=\int_a^b\left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right]dx.$$

Similarly, for a function $f(x,y)$ that is continuous on a region D of Type II, we have

$$\iint_D f(x,y)\,dA=\iint_D f(x,y)\,dx\,dy=\int_c^d\left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right]dy.$$

The integral in each of these expressions is an iterated integral, similar to those we have seen before. Notice that, in the inner integral in the first expression, we integrate $f(x,y)$ with x being held constant and the limits of integration being $g_1(x)$ and $g_2(x)$. In the inner integral in the second expression, we integrate $f(x,y)$ with y being held constant and the limits of integration are $h_1(x)$ and $h_2(x)$.

Example 15.2.2: Evaluating an Iterated Integral over a Type I Region

Evaluate the integral $\iint_D x^2 e^{xy}\, dA$ where D is shown in Figure 15.2.5.

Solution

First construct the region as a Type I region (Figure 15.2.5). Here $D = \{(x, y) \mid 0 \le x \le 2,\ \frac{1}{2}x \le y \le 1\}$. Then we have

$$\iint_D x^2 e^{xy}\, dA = \int_{x=0}^{x=2} \int_{y=1/2x}^{y=1} x^2 e^{xy}\, dy\, dx.$$

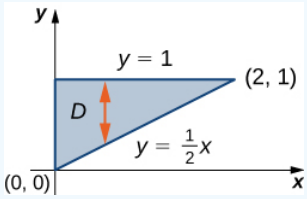

Figure 15.2.5: We can express region D as a Type I region and integrate from $y = \frac{1}{2}x$ to $y = 1$ between the lines $x = 0$ and $x = 2$.

Therefore, we have

$$\begin{aligned}
\int_{x=0}^{x=2} \int_{y=\frac{1}{2}x}^{y=1} x^2 e^{xy}\, dy\, dx &= \int_{x=0}^{x=2} \left[\int_{y=\frac{1}{2}x}^{y=1} x^2 e^{xy}\, dy \right] dx && \text{Iterated integral for a Type I region.} \\
&= \int_{x=0}^{x=2} \left[x^2 \frac{e^{xy}}{x} \right] \Big|_{y=1/2x}^{y=1} dx && \text{Integrate with respect to } y \\
&= \int_{x=0}^{x=2} \left[xe^x - xe^{x^2/2} \right] dx && \text{Integrate with respect to } x \\
&= \left[xe^x - e^x - e^{\frac{1}{2}x^2} \right] \Big|_{x=0}^{x=2} = 2.
\end{aligned}$$

In Example 15.2.2, we could have looked at the region in another way, such as $D = \{(x, y) \mid 0 \le y \le 1,\ 0 \le x \le 2y\}$ (Figure 15.2.6).

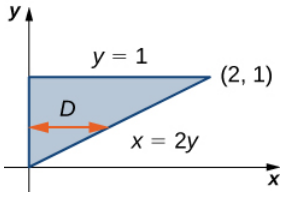

Figure 15.2.6.

This is a Type II region and the integral would then look like

$$\iint_D x^2 e^{xy}\, dA = \int_{y=0}^{y=1} \int_{x=0}^{x=2y} x^2 e^{xy}\, dx\ dy.$$

However, if we integrate first with respect to x this integral is lengthy to compute because we have to use integration by parts twice.

Example 15.2.3: Evaluating an Iterated Integral over a Type II Region

Evaluate the integral

$$\iint_D (3x^2 + y^2)\, dA$$

where $D = \{(x, y) \mid -2 \le y \le 3,\ y^2 - 3 \le x \le y + 3\}$.

Solution

Notice that D can be seen as either a Type I or a Type II region, as shown in Figure 15.2.7. However, in this case describing D as Type I is more complicated than describing it as Type II. Therefore, we use D as a Type II region for the integration.

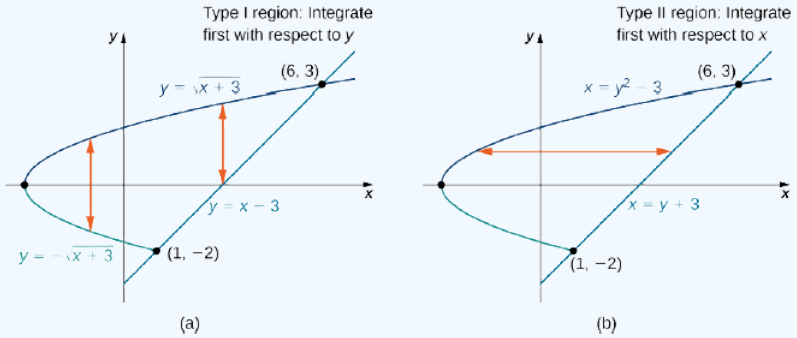

Figure 15.2.7: The region D in this example can be either (a) Type I or (b) Type II.

Choosing this order of integration, we have

$$\begin{aligned}
\iint_D (3x^2 + y^2)\, dA &= \int_{y=-2}^{y=3} \int_{x=y^2-3}^{x=y+3} (3x^2 + y^2)\, dx\ dy \\
&= \int_{y=-2}^{y=3} (x^3 + xy^2)\Big|_{y^2-3}^{y+3}\, dy && \text{Iterated integral, Type II region} \\
&= \int_{y=-2}^{y=3} \big((y+3)^3 + (y+3)y^2 - (y^2-3)^3 \\
&\quad - (y^2-3)y^2\big)\, dy \\
&= \int_{-2}^{3} (54 + 27y - 12y^2 + 2y^3 + 8y^4 && \text{Integrate with respect to } x. \\
&\quad - y^6)\, dy \\
&= \left[54y + \frac{27y^2}{2} - 4y^3 + \frac{y^4}{2} + \frac{8y^5}{5}\right. \\
&\quad \left. - \frac{y^7}{7}\right]_{-2}^{3} \\
&= \frac{2375}{7}.
\end{aligned}$$

? Exercise 15.2.2

Sketch the region D and evaluate the iterated integral

$$\iint_D xy\ dy\ dx$$

where D is the region bounded by the curves $y = \cos\ x$ and $y = \sin\ x$ in the interval $[-3\pi/4,\ \pi/4]$.

Hint

Express D as a Type I region, and integrate with respect to y first.

Answer

$\frac{\pi}{4}$

Recall from Double Integrals over Rectangular Regions the properties of double integrals. As we have seen from the examples here, all these properties are also valid for a function defined on a non-rectangular bounded region on a plane. In particular,

property 3 states:

If $R = S \cup T$ and $S \cap T = 0$ except at their boundaries, then

$$\iint_R f(x,y)\,dA = \iint_S f(x,y)\,dA + \iint_T f(x,y)\,dA.$$

Similarly, we have the following property of double integrals over a non-rectangular bounded region on a plane.

Theorem: Decomposing Regions into Smaller Regions

Suppose the region D can be expressed as $D = D_1 \cup D_2$ where D_1 and D_2 do not overlap except at their boundaries. Then

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA.$$

This theorem is particularly useful for non-rectangular regions because it allows us to split a region into a union of regions of Type I and Type II. Then we can compute the double integral on each piece in a convenient way, as in the next example.

Example 15.2.4: Decomposing Regions

Express the region D shown in Figure 15.2.8 as a union of regions of Type I or Type II, and evaluate the integral

$$\iint_D (2x + 5y)\,dA.$$

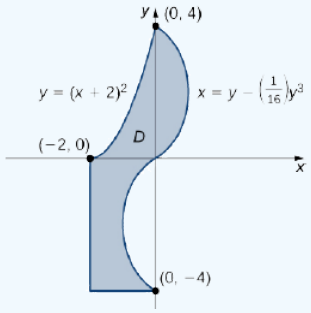

Figure 15.2.8: This region can be decomposed into a union of three regions of Type I or Type II.

Solution

The region D is not easy to decompose into any one type; it is actually a combination of different types. So we can write it as a union of three regions D_1, D_2, and D_3 where, $D_1 = \{(x,y) \mid -2 \le x \le 0,\ 0 \le y \le (x+2)^2\}$, $D_2 = \{(x,y) \mid 0 \le y \le 4,\ 0 \le x \le \left(y - \frac{1}{16}y^3\right)\}$, and $D_3 = \{(x,y) \mid -4 \le y \le 0,\ -2 \le x \le \left(y - \frac{1}{16}y^3\right)\}$. These regions are illustrated more clearly in Figure 15.2.9.

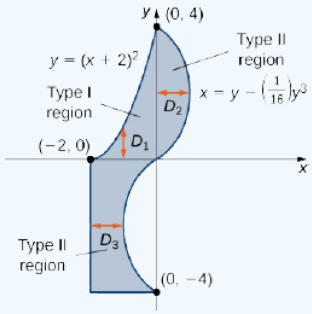

Figure 15.2.9: Breaking the region into three subregions makes it easier to set up the integration.

Here D_1 is Type I and D_2 and D_3 are both of Type II. Hence,

$$\begin{aligned}\iint_D (2x+5y)\,dA &= \iint_{D_1} (2x+5y)\,dA + \iint_{D_2} (2x+5y)\,dA + \iint_{D_3} (2x+5y)\,dA \\ &= \int_{x=-2}^{x=0}\int_{y=0}^{y=(x+2)^2} (2x+5y)\,dy\,dx + \int_{y=0}^{y=4}\int_{x=0}^{x=y-(1/16)y^3} (2x+5y)\,dx\,dy \\ &\quad + \int_{y=-4}^{y=0}\int_{x=-2}^{x=y-(1/16)y^3} (2x+5y)\,dx\,dy \\ &= \int_{x=-2}^{x=0}\left[\frac{1}{2}(2+x)^2(20+24x+5x^2)\right]dx + \int_{y=0}^{y=4}\left[\frac{1}{256}y^6 - \frac{7}{16}y^4 + 6y^2\right]dy \\ &\quad + \int_{y=-4}^{y=0}\left[\frac{1}{256}y^6 - \frac{7}{16}y^4 + 6y^2 + 10y - 4\right]dy \\ &= \frac{40}{3} + \frac{1664}{35} - \frac{1696}{35} = \frac{1304}{105}.\end{aligned}$$

Now we could redo this example using a union of two Type II regions (see Exercise 15.2.4 below).

? Exercise 15.2.3

Consider the region bounded by the curves $y = \ln x$ and $y = e^x$ in the interval $[1, 2]$. Decompose the region into smaller regions of Type II.

Hint

Sketch the region, and split it into three regions to set it up.

Answer

$$\{(x,y)\,|\,0 \le y \le 1,\ 1 \le x \le e^y\} \cup \{(x,y)\,|\,1 \le y \le e,\ 1 \le x \le 2\} \cup \{(x,y)\,|\,e \le y \le e^2,\ \ln y \le x \le 2\}$$

? Exercise 15.2.4

Redo Example 15.2.4 using a union of two Type II regions.

Hint

$$\{(x,y)\,|\,0 \le y \le 4,\ \sqrt{y} - 2 \le x \le \left(y - \frac{1}{16}y^3\right)\} \cup \{(x,y)\,|\,-4 \le y \le 0,\ -2 \le x \le \left(y - \frac{1}{16}y^3\right)\}$$

Answer

Same as in the example shown.

Changing the Order of Integration

As we have already seen when we evaluate an iterated integral, sometimes one order of integration leads to a computation that is significantly simpler than the other order of integration. Sometimes the order of integration does not matter, but it is important to learn to recognize when a change in order will simplify our work.

✓ Example 15.2.5: Changing the Order of Integration

Reverse the order of integration in the iterated integral

$$\int_{x=0}^{x=\sqrt{2}}\int_{y=0}^{y=2-x^2} xe^{x^2}\,dy\,dx.$$

Then evaluate the new iterated integral.

Solution

The region as presented is of Type I. To reverse the order of integration, we must first express the region as Type II. Refer to Figure 15.2.10.

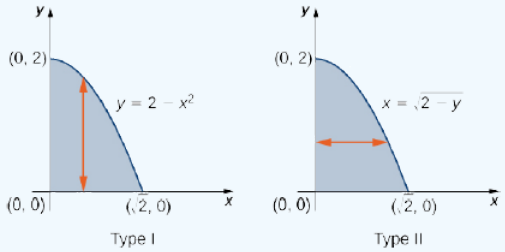

Figure 15.2.10: Converting a region from Type I to Type II.

We can see from the limits of integration that the region is bounded above by $y = 2 - x^2$ and below by $y = 0$ where x is in the interval $[0, \sqrt{2}]$. By reversing the order, we have the region bounded on the left by $x = 0$ and on the right by $x = \sqrt{2 - y}$ where y is in the interval $[0, 2]$. We solved $y = 2 - x^2$ in terms of x to obtain $x = \sqrt{2 - y}$.

Hence

$$\int_0^{\sqrt{2}} \int_0^{2-x^2} xe^{x^2}\, dy\, dx = \int_0^2 \int_0^{\sqrt{2-y}} xe^{x^2}\, dx\, dy \qquad \text{Reverse the order of integration then use substitution.}$$

$$= \int_0^2 \left[\frac{1}{2} e^{x^2} \Big|_0^{\sqrt{2-y}} \right] dy$$

$$= \int_0^2 \frac{1}{2}(e^{2-y} - 1)\, dy$$

$$= -\frac{1}{2}(e^{2-y} + y)\Big|_0^2$$

$$= \frac{1}{2}(e^2 - 3).$$

Example 15.2.6: Evaluating an Iterated Integral by Reversing the Order of Integration

Consider the iterated integral

$$\iint_R f(x, y)\, dx\, dy$$

where $z = f(x, y) = x - 2y$ over a triangular region R that has sides on $x = 0,\ y = 0$, and the line $x + y = 1$. Sketch the region, and then evaluate the iterated integral by

a. integrating first with respect to y and then
b. integrating first with respect to x.

Solution

A sketch of the region appears in Figure 15.2.11.

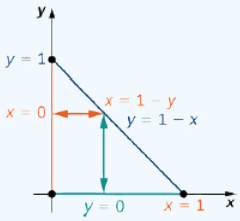

Figure 15.2.11: A triangular region R for integrating in two ways.

We can complete this integration in two different ways.

a. One way to look at it is by first integrating y from $y = 0$ to $y = 1 - x$ vertically and then integrating x from $x = 0$ to $x = 1$:

$$\iint_R f(x,y)\,dx\,dy = \int_{x=0}^{x=1}\int_{y=0}^{y=1-x}(x-2y)\,dy\,dx = \int_{x=0}^{x=1}\left(xy-y^2\right)\Big|_{y=0}^{y=1-x}dx$$

$$= \int_{x=0}^{x=1}\left[x(1-x)-(1-x)^2\right]\,dx = \int_{x=0}^{x=1}[-1+3x-2x^2]dx$$

$$= \left[-x+\frac{3}{2}x^2-\frac{2}{3}x^3\right]\Big|_{x=0}^{x=1} = -\frac{1}{6}.$$

b. The other way to do this problem is by first integrating x from $x=0$ to $x=1-y$ horizontally and then integrating y from $y=0$ to $y=1$:

$$\iint_R f(x,y)\,dx\,dy = \int_{y=0}^{y=1}\int_{x=0}^{x=1-y}(x-2y)\,dx\,dy = \int_{y=0}^{y=1}\left[\frac{1}{2}x^2-2xy\right]_{x=0}^{x=1-y}dy$$

$$= \int_{y=0}^{y=1}\left[\frac{1}{2}(1-y)^2-2y(1-y)\right]dy = \int_{y=0}^{y=1}\left[\frac{1}{2}-3y+\frac{5}{2}y^2\right]\,dy$$

$$= \left[\frac{1}{2}y-\frac{3}{2}y^2+\frac{5}{6}y^3\right]\Big|_{y=0}^{y=1} = -\frac{1}{6}.$$

? Exercise 15.2.5

Evaluate the iterated integral $\iint_D (x^2+y^2)\,dA$ over the region D in the first quadrant between the functions $y=2x$ and $y=x^2$.

Evaluate the iterated integral by integrating first with respect to y and then integrating first with resect to x.

Hint

Sketch the region and follow Example 15.2.6.

Answer

$\frac{216}{35}$

Calculating Volumes, Areas, and Average Values

We can use double integrals over general regions to compute volumes, areas, and average values. The methods are the same as those in Double Integrals over Rectangular Regions, but without the restriction to a rectangular region, we can now solve a wider variety of problems.

✓ Example 15.2.7: Finding the Volume of a Tetrahedron

Find the volume of the solid bounded by the planes $x=0,\ y=0,\ z=0$, and $2x+3y+z=6$.

Solution

The solid is a tetrahedron with the base on the xy-plane and a height $z=6-2x-3y$. The base is the region D bounded by the lines, $x=0$, $y=0$ and $2x+3y=6$ where $z=0$ (Figure 15.2.12). Note that we can consider the region D as Type I or as Type II, and we can integrate in both ways.

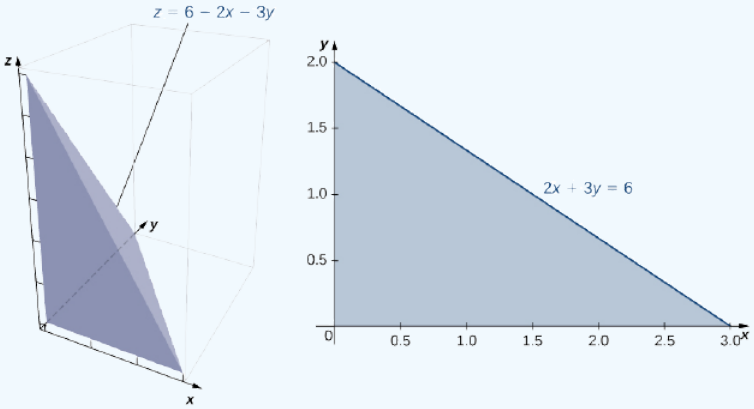

Figure 15.2.12: A tetrahedron consisting of the three coordinate planes and the plane $z = 6 - 2x - 3y$, with the base bound by $x = 0,\ y = 0$, and $2x + 3y = 6$.

First, consider D as a Type I region, and hence $D = \{(x, y) \,|\, 0 \le x \le 3,\ 0 \le y \le 2 - \frac{2}{3}x\}$.

Therefore, the volume is

$$V = \int_{x=0}^{x=3} \int_{y=0}^{y=2-(2x/3)} (6 - 2x - 3y)\, dy\, dx = \int_{x=0}^{x=3} \left[\left(6y - 2xy - \frac{3}{2}y^2 \right) \Big|_{y=0}^{y=2-(2x/3)} \right] dx$$

$$= \int_{x=0}^{x=3} \left[\frac{2}{3}(x-3)^2 \right] dx = 6.$$

Now consider D as a Type II region, so $D = \{(x, y) \,|\, 0 \le y \le 2,\ 0 \le x \le 3 - \frac{3}{2}y\}$. In this calculation, the volume is

$$V = \int_{y=0}^{y=2} \int_{x=0}^{x=3-(3y/2)} (6 - 2x - 3y)\, dx\, dy = \int_{y=0}^{y=2} \left[(6x - x^2 - 3xy) \Big|_{x=0}^{x=3-(3y/2)} \right] dy$$

$$= \int_{y=0}^{y=2} \left[\frac{9}{4}(y-2)^2 \right] dy = 6.$$

Therefore, the volume is 6 cubic units.

Exercise 15.2.6

Find the volume of the solid bounded above by $f(x, y) = 10 - 2x + y$ over the region enclosed by the curves $y = 0$ and $y = e^x$ where x is in the interval $[0, 1]$.

Hint

Sketch the region, and describe it as Type I.

Answer

$\frac{e^2}{4} + 10e - \frac{49}{4}$ cubic units

Finding the area of a rectangular region is easy, but finding the area of a non-rectangular region is not so easy. As we have seen, we can use double integrals to find a rectangular area. As a matter of fact, this comes in very handy for finding the area of a general non-rectangular region, as stated in the next definition.

Definition: Double Integrals

The area of a plane-bounded region D is defined as the double integral

$$\iint_D 1\, dA.$$

We have already seen how to find areas in terms of single integration. Here we are seeing another way of finding areas by using double integrals, which can be very useful, as we will see in the later sections of this chapter.

Example 15.2.8: Finding the Area of a Region

Find the area of the region bounded below by the curve $y = x^2$ and above by the line $y = 2x$ in the first quadrant (Figure 15.2.13).

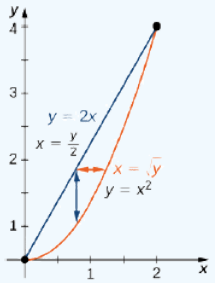

Figure 15.2.13: The region bounded by $y = x^2$ and $y = 2x$.

Solution

We just have to integrate the constant function $f(x, y) = 1$ over the region. Thus, the area A of the bounded region is $\int_{x=0}^{x=2} \int_{y=x^2}^{y=2x} dy\ dx$ or $\int_{y=0}^{y=4} \int_{x=y/2}^{x=\sqrt{y}} dx\ dy$:

$$\begin{aligned} A &= \iint_D 1\, dx\ dy \\ &= \int_{x=0}^{x=2} \int_{y=x^2}^{y=2x} 1\, dy\ dx \\ &= \int_{x=0}^{x=2} \left(y \Big|_{y=x^2}^{y=2x} \right) dx \\ &= \int_{x=0}^{x=2} (2x - x^2)\, dx \\ &= \left(x^2 - \frac{x^3}{3} \right) \Big|_0^2 = \frac{4}{3}. \end{aligned}$$

Exercise 15.2.7

Find the area of a region bounded above by the curve $y = x^3$ and below by $y = 0$ over the interval $[0, 3]$.

Hint

Sketch the region.

Answer

$\frac{81}{4}$ square units

We can also use a double integral to find the average value of a function over a general region. The definition is a direct extension of the earlier formula.

Definition: The Average Value of a Function

If $f(x, y)$ is integrable over a plane-bounded region D with positive area $A(D)$, then the average value of the function is

$$f_{ave} = \frac{1}{A(D)} \iint_D f(x, y)\, dA.$$

Note that the area is $A(D) = \iint_D 1\,dA$.

Example 15.2.9: Finding an Average Value

Find the average value of the function $f(x,y) = 7xy^2$ on the region bounded by the line $x = y$ and the curve $x = \sqrt{y}$ (Figure 15.2.14).

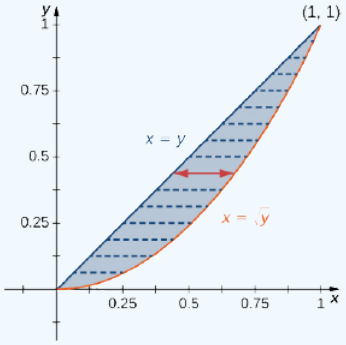

Figure 15.2.14: The region bounded by $x = y$ and $x = \sqrt{y}$.

Solution

First find the area $A(D)$ where the region D is given by the figure. We have

$$A(D) = \iint_D 1\,dA = \int_{y=0}^{y=1} \int_{x=y}^{x=\sqrt{y}} 1\,dx\,dy = \int_{y=0}^{y=1} \left[x\Big|_{x=y}^{x=\sqrt{y}}\right] dy$$
$$= \int_{y=0}^{y=1} (\sqrt{y} - y)\,dy = \frac{2}{3}y^{2/3} - \frac{y^2}{2}\Big|_0^1 = \frac{1}{6}$$

Then the average value of the given function over this region is

$$f_{ave} = \frac{1}{A(D)} \iint_D f(x,y)\,dA = \frac{1}{A(D)} \int_{y=0}^{y=1} \int_{x=y}^{x=\sqrt{y}} 7xy^2\,dx\,dy = \frac{1}{1/6} \int_{y=0}^{y=1} \left[\frac{7}{2}x^2y^2\Big|_{x=y}^{x=\sqrt{y}}\right] dy$$
$$= 6\int_{y=0}^{y=1} \left[\frac{7}{2}y^2(y - y^2)\right] dy = 6\int_{y=0}^{y=1} \left[\frac{7}{2}(y^3 - y^4)\right] dy = \frac{42}{2}\left(\frac{y^4}{4} - \frac{y^5}{5}\right)\Big|_0^1 = \frac{42}{40} = \frac{21}{20}.$$

Exercise 15.2.8

Find the average value of the function $f(x,y) = xy$ over the triangle with vertices $(0,0)$, $(1,0)$ and $(1,3)$.

Hint

Express the line joining $(0,0)$ and $(1,3)$ as a function $y = g(x)$.

Answer

$\frac{3}{4}$

Improper Double Integrals

An **improper double integral** is an integral $\iint_D f\,dA$ where either D is an unbounded region or f is an unbounded function. For example, $D = \{(x,y)\,|\,|x - y| \geq 2\}$ is an unbounded region, and the function $f(x,y) = 1/(1 - x^2 - 2y^2)$ over the ellipse $x^2 + 3y^2 \geq 1$ is an unbounded function. Hence, both of the following integrals are improper integrals:

i. $\iint_D xy\ dA$ where $D = \{(x,y) || x-y| \geq 2\}$;

ii. $\iint_D \frac{1}{1-x^2-2y^2}\ dA$ where $D = \{(x,y) | x^2+3y^2 \leq 1\}$.

In this section we would like to deal with improper integrals of functions over rectangles or simple regions such that f has only finitely many discontinuities. Not all such improper integrals can be evaluated; however, a form of Fubini's theorem does apply for some types of improper integrals.

Theorem: Fubini's Theorem for Improper Integrals

If D is a bounded rectangle or simple region in the plane defined by

$\{(x,y)\ : a \leq x \leq b,\ g(x) \leq y \leq h(x)\}$ and also by

$\{(x,y)\ : c \leq y \leq d,\ j(y) \leq x \leq k(y)\}$ and f is a nonnegative function on D with finitely many discontinuities in the interior of D then

$$\iint_D f\ dA = \int_{x=a}^{x=b}\int_{y=g(x)}^{y=h(x)} f(x,y)\,dy\,dx = \int_{y=c}^{y=d}\int_{x=j(y)}^{x=k(y)} f(x,y)\,dx\,dy$$

It is very important to note that we required that the function be nonnegative on D for the theorem to work. We consider only the case where the function has finitely many discontinuities inside D.

Example 15.2.10: Evaluating a Double Improper Integral

Consider the function $f(x,y) = \frac{e^y}{y}$ over the region $D = \{(x,y)\ : 0 \leq x \leq 1,\ x \leq y \leq \sqrt{x}\}$.

Notice that the function is nonnegative and continuous at all points on D except $(0,0)$. Use Fubini's theorem to evaluate the improper integral.

Solution

First we plot the region D (Figure 15.2.15); then we express it in another way.

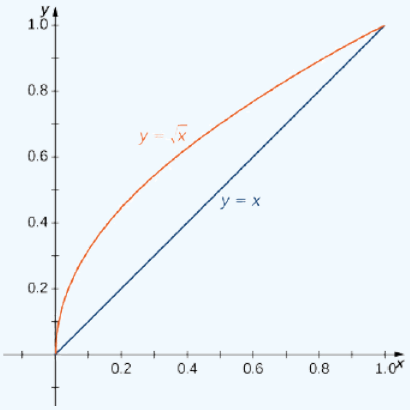

Figure 15.2.15: The function f is continuous at all points of the region D except $(0,0)$.

The other way to express the same region D is

$$D = \{(x,y)\ :\ 0 \leq y \leq 1,\ y^2 \leq x \leq y\}.$$

Thus we can use Fubini's theorem for improper integrals and evaluate the integral as

$$\int_{y=0}^{y=1}\int_{x=y^2}^{x=y} \frac{e^y}{y}\,dx\,dy.$$

Therefore, we have

$$\int_{y=0}^{y=1}\int_{x=y^2}^{x=y}\frac{e^y}{y}\,dx\,dy=\int_{y=0}^{y=1}\frac{e^y}{y}x\Big|_{x=y^2}^{x=y}dy=\int_{y=0}^{y=1}\frac{e^y}{y}(y-y^2)\,dy=\int_0^1(e^y-ye^y)\,dy=e-2.$$

As mentioned before, we also have an improper integral if the region of integration is unbounded. Suppose now that the function f is continuous in an unbounded rectangle R.

Theorem: Improper Integrals on an Unbounded Region

If R is an unbounded rectangle such as $R=\{(x,y)\,:\,a\le x\le\infty,\ c\le y\le\infty\}$, then when the limit exists, we have

$$\iint_R f(x,y)\,dA=\lim_{(b,d)\to(\infty,\infty)}\int_a^b\left(\int_c^d f(x,y)\,dy\right)dx=\lim_{(b,d)\to(\infty,\infty)}\int_c^d\left(\int_a^b f(x,y)\,dx\right)dy.$$

The following example shows how this theorem can be used in certain cases of improper integrals.

Example 15.2.11

Evaluate the integral $\iint_R xye^{-x^2-y^2}\,dA$ where R is the first quadrant of the plane.

Solution

The region R is the first quadrant of the plane, which is unbounded. So

$$\begin{aligned}
\iint_R xye^{-x^2-y^2}\,dA&=\lim_{(b,d)\to(\infty,\infty)}\int_{x=0}^{x=b}\left(\int_{y=0}^{y=d}xye^{-x^2-y^2}dy\right)dx\\
&=\lim_{(b,d)\to(\infty,\infty)}\int_{x=0}^{x=b}\left(-\frac{1}{2}x\int_{y=0}^{y=d}-2ye^{-x^2-y^2}dy\right)dx\\
&=\lim_{(b,d)\to(\infty,\infty)}\int_{x=0}^{x=b}-\frac{1}{2}x\left(e^{-x^2-y^2}\right)\Big|_{y=0}^{y=d}dx\\
&=\lim_{(b,d)\to(\infty,\infty)}\int_{x=0}^{x=b}-\frac{1}{2}x\left(e^{-x^2-d^2}-e^{-x^2}\right)dx\\
&=\lim_{(b,d)\to(\infty,\infty)}\int_{x=0}^{x=b}\frac{1}{2}xe^{-x^2}\left(1-e^{-d^2}\right)dx\\
&=\lim_{(b,d)\to(\infty,\infty)}\left(-\frac{1}{4}\right)\left(1-e^{-d^2}\right)\int_{x=0}^{x=b}-2xe^{-x^2}\,dx\\
&=\lim_{(b,d)\to(\infty,\infty)}\left(-\frac{1}{4}\right)\left(1-e^{-d^2}\right)e^{-x^2}\Big|_{x=0}^{x=b}dx\\
&=\lim_{(b,d)\to(\infty,\infty)}\frac{1}{4}\left(1-e^{-b^2}\right)\left(1-e^{-d^2}\right)\\
&=\lim_{(b,d)\to(\infty,\infty)}\frac{1}{4}\left(1-\cancelto{0}{e^{-b^2}}\right)\left(1-\cancelto{0}{e^{-d^2}}\right)\\
&=\frac{1}{4}
\end{aligned}$$

Thus,

$$\iint_R xye^{-x^2-y^2}\,dA$$

is convergent and the value is $\frac{1}{4}$.

Exercise 15.2.9

$$\iint_D \frac{y}{\sqrt{1-x^2-y^2}}\,dA$$

where $D = \{(x,y) : x \geq 0,\ y \geq 0,\ x^2+y^2 \leq 1\}$.

Hint

Notice that the integral is nonnegative and discontinuous on $x^2+y^2=1$. Express the region D as $D = \{(x,y) : 0 \leq x \leq 1,\ 0 \leq y \leq \sqrt{1-x^2}\}$ and integrate using the method of substitution.

Answer

$\frac{\pi}{4}$

In some situations in probability theory, we can gain insight into a problem when we are able to use double integrals over general regions. Before we go over an example with a double integral, we need to set a few definitions and become familiar with some important properties.

Definition: Joint Density Function

Consider a pair of continuous random variables X and Y such as the birthdays of two people or the number of sunny and rainy days in a month. The **joint density function** f of X and Y satisfies the probability that (X,Y) lies in a certain region D:

$$P((X,Y) \in D) = \iint_D f(x,y)\,dA.$$

Since the probabilities can never be negative and must lie between 0 and 1 the joint density function satisfies the following inequality and equation:

$$f(x,y) \geq 0 \text{ and } \iint_R f(x,y)\,dA = 1.$$

Definition: Independent Random Variables

The variables X and Y are said to be **independent random variables** if their joint density function is the product of their individual density functions:

$$f(x,y) = f_1(x)f_2(y).$$

Example 15.2.12: Application to Probability

At Sydney's Restaurant, customers must wait an average of 15 minutes for a table. From the time they are seated until they have finished their meal requires an additional 40 minutes, on average. What is the probability that a customer spends less than an hour and a half at the diner, assuming that waiting for a table and completing the meal are independent events?

Solution

Waiting times are mathematically modeled by exponential density functions, with m being the average waiting time, as

$$f(t) = \begin{cases} 0, & \text{if } t < 0 \\ \frac{1}{m}e^{-t/m}, & \text{if } t \geq 0. \end{cases}$$

if X and Y are random variables for 'waiting for a table' and 'completing the meal,' then the probability density functions are, respectively,

$$f_1(x) = \begin{cases} 0, & \text{if } x < 0. \\ \frac{1}{15}e^{-x/15}, & \text{if } x \geq 0. \end{cases} \quad \text{and} \quad f_2(y) = \begin{cases} 0, & \text{if } y < 0 \\ \frac{1}{40}e^{-y/40}, & \text{if } y \geq 0. \end{cases}$$

Clearly, the events are independent and hence the joint density function is the product of the individual functions

$$f(x,y) = f_1(x)f_2(y) = \begin{cases} 0, & \text{if } x < 0 \text{ or } y < 0, \\ \frac{1}{600}e^{-x/15}e^{-y/40}, & \text{if } x, y \geq 0 \end{cases}$$

We want to find the probability that the combined time $X + Y$ is less than 90 minutes. In terms of geometry, it means that the region D is in the first quadrant bounded by the line $x + y = 90$ (Figure 15.2.16).

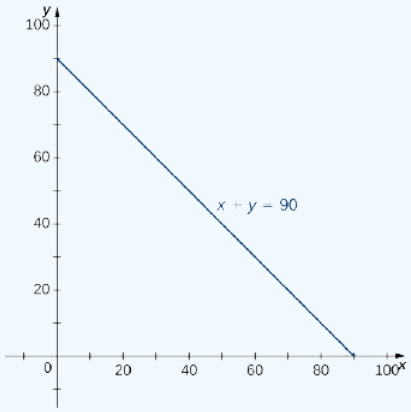

Figure 15.2.16: The region of integration for a joint probability density function.

Hence, the probability that (X, Y) is in the region D is

$$P(X + Y \leq 90) = P((X,Y) \in D) = \iint_D f(x,y)\,dA = \iint_D \frac{1}{600}e^{-x/15}e^{-y/40}\,dA.$$

Since $x + y = 90$ is the same as $y = 90 - x$, we have a region of Type I, so

$$\begin{aligned} D &= \{(x,y) \mid 0 \leq x \leq 90,\ 0 \leq y \leq 90 - x\}, \\ P(X + Y \leq 90) &= \frac{1}{600}\int_{x=0}^{x=90}\int_{y=0}^{y=90-x} e^{-x/15}e^{-y/40}dy\,dx \\ &= -\frac{1}{15}\int_{x=0}^{x=90} e^{-x/15}\int_{y=0}^{y=90-x} -\frac{1}{40}e^{-y/40}dy\,dx \\ &= -\frac{1}{15}\int_{x=0}^{x=90} e^{-x/15}e^{-y/40}\Big|_{y=0}^{y=90-x}\,dx \\ &= \quad \ldots \\ &\approx 0.8328 \end{aligned}$$

Thus, there is an 83.3% chance that a customer spends less than an hour and a half at the restaurant.

Another important application in probability that can involve improper double integrals is the calculation of expected values. First we define this concept and then show an example of a calculation.

Definition: Expected Values

In probability theory, we denote the **expected values** $E(X)$ and $E(Y)$ respectively, as the most likely outcomes of the events. The expected values $E(X)$ and $E(Y)$ are given by

$$E(X) = \iint_S x\,f(x,y)\,dA \text{ and } E(Y) = \iint_S y\,f(x,y)\,dA,$$

where S is the sample space of the random variables X and Y.

Example 15.2.13: Finding Expected Value

Find the expected time for the events ‘waiting for a table’ and ‘completing the meal’ in Example 15.2.12.

Solution

Using the first quadrant of the rectangular coordinate plane as the sample space, we have improper integrals for $E(X)$ and $E(Y)$. The expected time for a table is

$$\begin{aligned} E(X) &= \iint_S x\frac{1}{600}e^{-x/15}e^{-y/40}\,dA \\ &= \frac{1}{600}\int_{x=0}^{x=\infty}\int_{y=0}^{y=\infty} xe^{-x/15}e^{-y/40}dA \\ &= \frac{1}{600}\lim_{(a,b)\to(\infty,\infty)}\int_{x=0}^{x=a}\int_{y=0}^{y=b} xe^{-x/15}e^{-y/40}dx\;dy \\ &= \frac{1}{600}\left(\lim_{a\to\infty}\int_{x=0}^{x=a} xe^{-x/15}dx\right)\left(\lim_{b\to\infty}\int_{y=0}^{y=b} e^{-y/40}dy\right) \\ &= \frac{1}{600}\left((\lim_{a\to\infty}(-15e^{-x/15}(x+15)))\Big|_{x=0}^{x=a}\right)\left((\lim_{b\to\infty}(-40e^{-y/40}))\Big|_{y=0}^{y=b}\right) \\ &= \frac{1}{600}\left(\lim_{a\to\infty}(-15e^{-a/15}(a+15)+225)\right)\left(\lim_{b\to\infty}(-40e^{-b/40}+40)\right) \\ &= \frac{1}{600}(225)(40) = 15. \end{aligned}$$

A similar calculation shows that $E(Y) = 40$. This means that the expected values of the two random events are the average waiting time and the average dining time, respectively.

Exercise 15.2.10

The joint density function for two random variables X and Y is given by

$$f(x,y) = \begin{cases} \frac{1}{16250}(x^2+y^2), & \text{if } 0\le x\le 15,\ 0\le y\le 10 \\ 0, & \text{otherwise} \end{cases}$$

Find the probability that X is at most 10 and Y is at least 5.

Hint

Compute the probability

$$P(X\le 10,\ Y\ge 5) = \int_{x=0}^{x=10}\int_{y=5}^{y=10}\frac{1}{16250}(x^2+y^2)dy\;dx.$$

Answer

$P(X\le 10,\ Y\ge 5) = \frac{11}{39}\approx 0.28205$

Key Concepts

- A general bounded region D on the plane is a region that can be enclosed inside a rectangular region. We can use this idea to define a double integral over a general bounded region.
- To evaluate an iterated integral of a function over a general nonrectangular region, we sketch the region and express it as a Type I or as a Type II region or as a union of several Type I or Type II regions that overlap only on their boundaries.
- We can use double integrals to find volumes, areas, and average values of a function over general regions, similarly to calculations over rectangular regions.
- We can use Fubini’s theorem for improper integrals to evaluate some types of improper integrals.

Key Equations

- **Iterated integral over a Type I region**

$$\iint_D f(x,y)\,dA = \iint_D f(x,y)\,dy\,dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right] dx$$

- **Iterated integral over a Type II region**

$$\iint_D f(x,y)\,dA = \iint_D (x,y)\,dx\,dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right] dy$$

Glossary

improper double integral

a double integral over an unbounded region or of an unbounded function

Type I

a region D in the xy- plane is Type I if it lies between two vertical lines and the graphs of two continuous functions $g_1(x)$ and $g_2(x)$

Type II

a region D in the xy-plane is Type II if it lies between two horizontal lines and the graphs of two continuous functions $h_1(y)$ and $h_2(h)$

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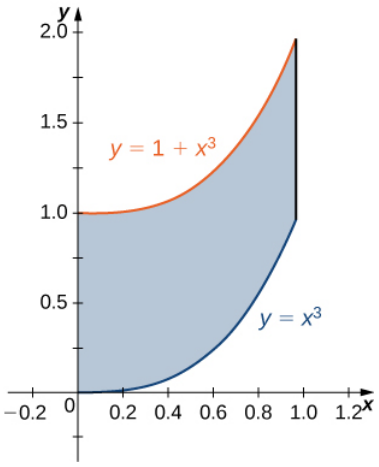
15.2E: Exercises for Section 15.2

1) The region D bounded by $y = x^3,\ y = x^3 + 1,\ x = 0$, and $x = 1$ as given in the following figure.



a. Classify this region as vertically simple (Type I) or horizontally simple (Type II).

Type:

Type I but not Type II

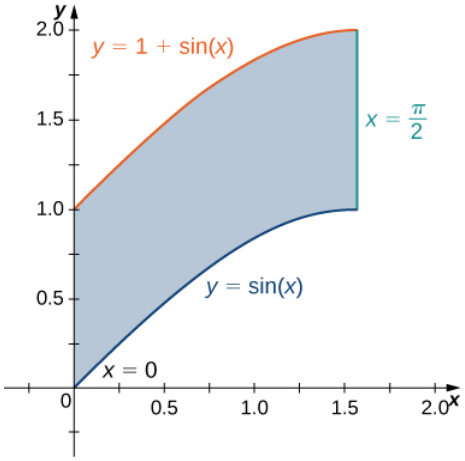
b. Find the area of the region D.

c. Find the average value of the function $f(x, y) = 3xy$ on the region graphed in the previous exercise.

Answer

$\frac{27}{20}$

2) The region D bounded by $y = \sin x,\ y = 1 + \sin x,\ x = 0$, and $x = \frac{\pi}{2}$ as given in the following figure.



a. Classify this region as vertically simple (Type I) or horizontally simple (Type II).

Type:

Type I but not Type II

b. Find the area of the region D.

Answer

$\frac{\pi}{2}$ units^2

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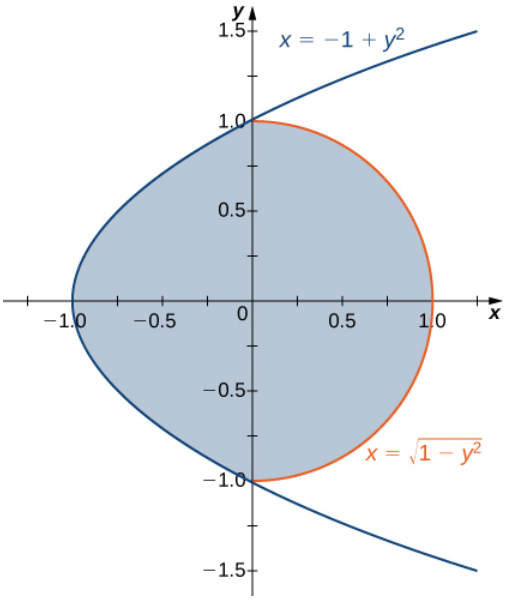
c. Find the average value of the function $f(x, y) = \cos x$ on the region D.

3) The region D bounded by $x = y^2 - 1$ and $x = \sqrt{1 - y^2}$ as given in the following figure.



a. Classify this region as vertically simple (Type I) or horizontally simple (Type II).

Type:

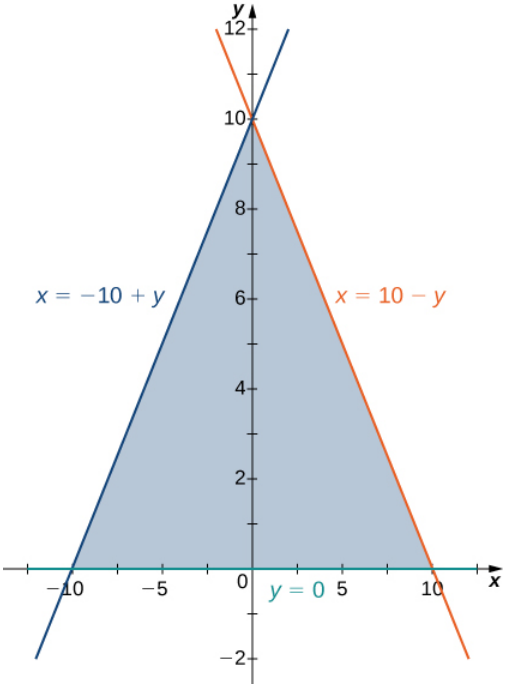
Type II but not Type I

b. Find the volume of the solid under the graph of the function $f(x, y) = xy + 1$ and above the region D.

Answer

$\frac{1}{6}(8 + 3\pi)$ units^3

4) The region D bounded by $y = 0,\ x = -10 + y$, and $x = 10 - y$ as given in the following figure.



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a. Classify this region as vertically simple (Type I) or horizontally simple (Type II).

Type:

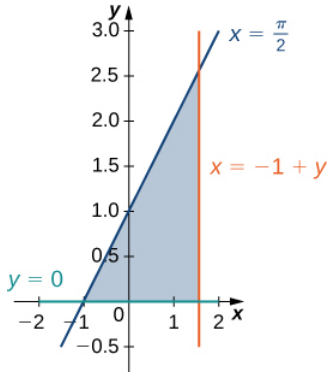
Type II but not Type I

b. Find the volume of the solid under the graph of the function $f(x,y) = x + y$ and above the region in the figure from the previous exercise.

Answer

$\frac{1000}{3}$ units^3

5) The region D bounded by $y = 0,\ x = y - 1,\ x = \frac{\pi}{2}$ as given in the following figure.

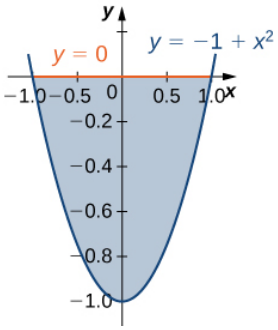


Classify this region as vertically simple (Type I) or horizontally simple (Type II).

Type:

Type I and Type II

6) The region D bounded by $y = 0$ and $y = x^2 - 1$ as given in the following figure.



Classify this region as vertically simple (Type I) or horizontally simple (Type II).

Type:

Type I and Type II

7) Let D be the region bounded by the curves of equations $y = \cos x$ and $y = 4 - x^2$ and the x-axis. Explain why D is neither of Type I nor II.

Answer

The region D is not of Type I: it does not lie between two vertical lines and the graphs of two continuous functions $g_1(x)$ and $g_2(x)$. The region is not of Type II: it does not lie between two horizontal lines and the graphs of two continuous functions $h_1(y)$ and $h_2(y)$.

8) Let D be the region bounded by the curves of equations $y = x,\ y = -x$ and $y = 2 - x^2$. Explain why D is neither of Type I nor II.

In exercises 9 - 14, evaluate the double integral $\iint_D f(x,y)\,dA$ over the region D.

9) $f(x,y)=1$ and

$D=\{(x,y)|\,0\le x\le \frac{\pi}{2},\ \sin x\le y\le 1+\sin x\}$

Answer

$\frac{\pi}{2}$

10) $f(x,y)=2$ and

$D=\{(x,y)|\,0\le y\le 1,\ y-1\le x\le \arccos y\}$

11) $f(x,y)=xy$ and

$D=\{(x,y)|\,-1\le y\le 1,\ y^2-1\le x\le \sqrt{1-y^2}\}$

Answer

0

12) $f(x,y)=\sin y$ and D is the triangular region with vertices $(0,0),\ (0,3)$, and $(3,0)$

13) $f(x,y)=-x+1$ and D is the triangular region with vertices $(0,0),\ (0,2)$, and $(2,2)$

Answer

$\frac{2}{3}$

14) $f(x,y)=2x+4y$ and

$D=\{(x,y)|\,0\le x\le 1,\ x^3\le y\le x^3+1\}$

In exercises 15 - 20, evaluate the iterated integrals.

15) $\int_0^1\int_{2\sqrt{x}}^{2\sqrt{x}+1}(xy+1)\,dy\,dx$

Answer

$\frac{41}{20}$

16) $\int_0^3\int_{2x}^{3x}(x+y^2)\,dy\,dx$

17) $\int_1^2\int_{-u^2-1}^{-u}(8uv)\,dv\,du$

Answer

-63

18) $\int_e^{e^2}\int_{\ln u}^{2}(v+\ln u)\,dv\,du$

19) $\int_0^{1/2}\int_{-\sqrt{1-4y^2}}^{\sqrt{1-4y^2}}4\,dx\,dy$

Answer

π

20) $\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}(2x+4y^3)\,dx\,dy$

21) Let D be the region bounded by $y=1-x^2,\ y=4-x^2$, and the x- and y-axes.

a. Show that $\iint_D x\,dA=\int_0^1\int_{1-x^2}^{4-x^2}x\,dy\,dx+\int_1^2\int_0^{4-x^2}x\,dy\,dx$ by dividing the region D into two regions of Type I.

b. Evaluate the integral $\iint_D x\,dA$.

22) Let D be the region bounded by $y = 1,\ y = x,\ y = \ln x$, and the x-axis.

a. Show that $\iint_D y^2\,dA = \int_{-1}^{0}\int_{-x}^{2-x^2} y^2 dy\,dx + \int_{0}^{1}\int_{x}^{2-x^2} y^2 dy\,dx$ by dividing the region D into two regions of Type I, where $D = \{(x,y)\,|\,y \geq x, y \geq -x,\ y \leq 2 - x^2\}$.

b. Evaluate the integral $\iint_D y^2\,dA$.

23) Let D be the region bounded by $y = x^2$, $y = x + 2$, and $y = -x$.

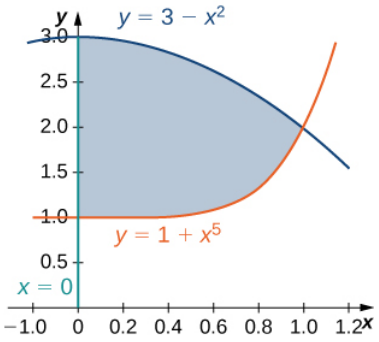
a. Show that $\iint_D x\,dA = \int_{0}^{1}\int_{-y}^{\sqrt{y}} x\,dx\,dy + \int_{1}^{4}\int_{y-2}^{\sqrt{y}} x\,dx\,dy$ by dividing the region D into two regions of Type II, where $D = \{(x,y)\,|\,y \geq x^2,\ y \geq -x,\ y \leq x + 2\}$.

b. Evaluate the integral $\iint_D x\,dA$.

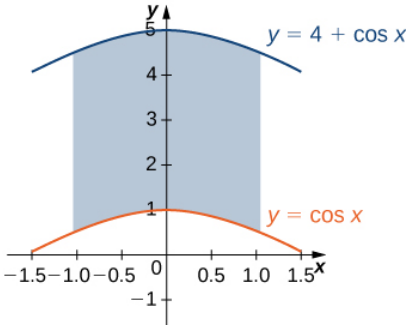
Answer

a. Answers may vary;
b. $\frac{7}{3}$

24) The region D bounded by $x = 0, y = x^5 + 1$, and $y = 3 - x^2$ is shown in the following figure. Find the area $A(D)$ of the region D.



25) The region D bounded by $y = \cos x,\ y = 4 + \cos x$, and $x = \pm\frac{\pi}{3}$ is shown in the following figure. Find the area $A(D)$ of the region D.



Answer

$\frac{8\pi}{3}$

26) Find the area $A(D)$ of the region $D = \{(x,y)|\,y \geq 1 - x^2, y \leq 4 - x^2,\ y \geq 0,\ x \geq 0\}$.

27) Let D be the region bounded by $y = 1,\ y = x,\ y = \ln x$, and the x-axis. Find the area $A(D)$ of the region D.

Answer

$\left(e - \frac{3}{2}\right)$ units^2

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28) Find the average value of the function $f(x,y) = \sin y$ on the triangular region with vertices $(0,0)$, $(0,3)$, and $(3,0)$.

29) Find the average value of the function $f(x,y) = -x+1$ on the triangular region with vertices $(0,0)$, $(0,2)$, and $(2,2)$.

Answer

The average value of f on this triangular region is $\frac{1}{3}$.

In exercises 30 - 33, change the order of integration and evaluate the integral.

30) $\int_{-1}^{\pi/2} \int_{0}^{x+1} \sin x \, dy \, dx$

31) $\int_{0}^{1} \int_{x-1}^{1-x} x \, dy \, dx$

Answer

$$\int_0^1 \int_{x-1}^{1-x} x \; dy \; dx = \int_{-1}^0 \int_0^{y+1} x \; dx \; dy + \int_0^1 \int_0^{1-y} x \; dx \; dy = \frac{1}{3}$$

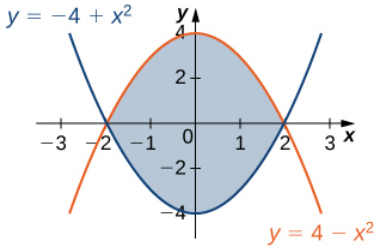
32) $\int_{-1}^{0} \int_{-\sqrt{y+1}}^{\sqrt{y+1}} y^2 dx \; dy$

33) $\int_{-1/2}^{1/2} \int_{-\sqrt{y^2+1}}^{\sqrt{y^2+1}} y \; dx \; dy$

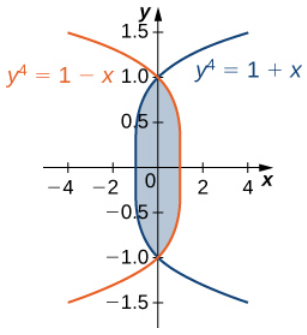
Answer

$$\int_{-1/2}^{1/2} \int_{-\sqrt{y^2+1}}^{\sqrt{y^2+1}} y \; dx \; dy = \int_1^2 \int_{-\sqrt{x^2-1}}^{\sqrt{x^2-1}} y \; dy \; dx = 0$$

34) The region D is shown in the following figure. Evaluate the double integral $\iint_D (x^2 + y) \, dA$ by using the easier order of integration.



35) The region D is shown in the following figure. Evaluate the double integral $\iint_D (x^2 - y^2) \, dA$ by using the easier order of integration.



Answer

$$\iint_D (x^2 - y^2) dA = \int_{-1}^{1} \int_{y^4-1}^{1-y^4} (x^2 - y^2) dx \; dy = \frac{464}{4095}$$

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36) Find the volume of the solid under the surface $z = 2x + y^2$ and above the region bounded by $y = x^5$ and $y = x$.

37) Find the volume of the solid under the plane $z = 3x + y$ and above the region determined by $y = x^7$ and $y = x$.

Answer

$\frac{4}{5}$ units^3

38) Find the volume of the solid under the plane $z = 3x + y$ and above the region bounded by $x = \tan y,\ x = -\tan y$, and $x = 1$.

39) Find the volume of the solid under the surface $z = x^3$ and above the plane region bounded by $x = \sin y,\ x = -\sin y$, and $x = 1$.

Answer

$\frac{5\pi}{32}$ units^3

40) Let g be a positive, increasing, and differentiable function on the interval $[a, b]$. Show that the volume of the solid under the surface $z = g'(x)$ and above the region bounded by $y = 0,\ y = g(x),\ x = a$, and $x = b$ is given by $\frac{1}{2}(g^2(b) - g^2(a))$.

41) Let g be a positive, increasing, and differentiable function on the interval $[a, b]$ and let k be a positive real number. Show that the volume of the solid under the surface $z = g'(x)$ and above the region bounded by $y = g(x),\ y = g(x) + k,\ x = a$, and $x = b$ is given by $k(g(b) - g(a))$.

42) Find the volume of the solid situated in the first octant and determined by the planes $z = 2$, $z = 0,\ x + y = 1,\ x = 0$, and $y = 0$.

43) Find the volume of the solid situated in the first octant and bounded by the planes $x + 2y = 1$, $x = 0,\ z = 4$, and $z = 0$.

Answer

1 units^3

44) Find the volume of the solid bounded by the planes $x + y = 1,\ x - y = 1,\ x = 0,\ z = 0$, and $z = 10$.

45) Find the volume of the solid bounded by the planes $x + y = 1,\ x - y = 1,\ x + y = -1\ x - y = -1,\ z = 1$, and $z = 0$

Answer

2 units^3

46) Let S_1 and S_2 be the solids situated in the first octant under the planes $x + y + z = 1$ and $x + y + 2z = 1$ respectively, and let S be the solid situated between $S_1,\ S_2,\ x = 0$, and $y = 0$.

a. Find the volume of the solid S_1.
b. Find the volume of the solid S_2.
c. Find the volume of the solid S by subtracting the volumes of the solids S_1 and S_2.

47) Let S_1 and S_2 be the solids situated in the first octant under the planes $2x + 2y + z = 2$ and $x + y + z = 1$ respectively, and let S be the solid situated between $S_1,\ S_2,\ x = 0$, and $y = 0$.

a. Find the volume of the solid S_1.
b. Find the volume of the solid S_2.
c. Find the volume of the solid S by subtracting the volumes of the solids S_1 and S_2.

Answer

a. $\frac{1}{3}$ units^3
b. $\frac{1}{6}$ units^3
c. $\frac{1}{6}$ units^3

48) Let S_1 and S_2 be the solids situated in the first octant under the plane $x + y + z = 2$ and under the sphere $x^2 + y^2 + z^2 = 4$, respectively. If the volume of the solid S_2 is $\frac{4\pi}{3}$ determine the volume of the solid S situated between S_1 and S_2 by subtracting the volumes of these solids.

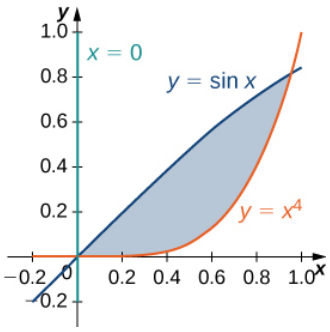
49) Let S_1 and S_2 be the solids situated in the first octant under the plane $x + y + z = 2$ and bounded by the cylinder $x^2 + y^2 = 4$, respectively.

a. Find the volume of the solid S_1.
b. Find the volume of the solid S_2.
c. Find the volume of the solid S situated between S_1 and S_2 by subtracting the volumes of the solids S_1 and S_2.

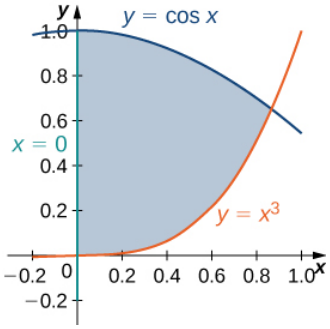
Answer

a. $\frac{4}{3}$ units^3
b. 2π units^3
c. $\frac{6\pi-4}{3}$ units^3

50) [T] The following figure shows the region D bounded by the curves $y=\sin x,\ x=0$, and $y=x^4$. Use a graphing calculator or CAS to find the x-coordinates of the intersection points of the curves and to determine the area of the region D. Round your answers to six decimal places.



51) [T] The region D bounded by the curves $y=\cos x,\ x=0$, and $y=x^3$ is shown in the following figure. Use a graphing calculator or CAS to find the x-coordinates of the intersection points of the curves and to determine the area of the region D. Round your answers to six decimal places.



Answer

0 and 0.865474; $A(D)=0.621135$ units^3

52) Suppose that (X,Y) is the outcome of an experiment that must occur in a particular region S in the xy-plane. In this context, the region S is called the sample space of the experiment and X and Y are random variables. If D is a region included in S, then the probability of (X,Y) being in D is defined as $P[(X,Y)\in D]=\iint_D p(x,y)dx\ dy$, where $p(x,y)$ is the joint probability density of the experiment. Here, $p(x,y)$ is a nonnegative function for which $\iint_S p(x,y)dx\ dy=1$. Assume that a point (X,Y) is chosen arbitrarily in the square $[0,3]\times[0,3]$ with the probability density

$$p(x,y)=\frac{1}{9}(x,y)\in[0,3]\times[0,3],$$

$$p(x,y)=0 \text{ otherwise}$$

Find the probability that the point (X,Y) is inside the unit square and interpret the result.

53) Consider X and Y two random variables of probability densities $p_1(x)$ and $p_2(x)$, respectively. The random variables X and Y are said to be independent if their joint density function is given by $p_(x,y)=p_1(x)p_2(y)$. At a drive-thru restaurant, customers spend, on average, 3 minutes placing their orders and an additional 5 minutes paying for and picking up their meals. Assume that placing

the order and paying for/picking up the meal are two independent events X and Y. If the waiting times are modeled by the exponential probability densities

$$p_1(x) = \frac{1}{3}e^{-x/3} \; x \geq 0,$$

$$p_1(x) = 0 \text{ otherwise}$$

$$p_2(y) = \frac{1}{5}e^{-y/5} \; y \geq 0$$

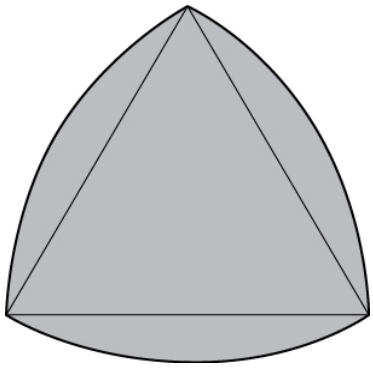
$$p_2(y) = 0 \text{ otherwise}$$

respectively, the probability that a customer will spend less than 6 minutes in the drive-thru line is given by $P[X + Y \leq 6] = \iint_D p(x, y)dx \; dy$, where $D = \{(x, y)|x \geq 0, \; y \geq 0, \; x + y \leq 6\}$. Find $P[X + Y \leq 6]$ and interpret the result.

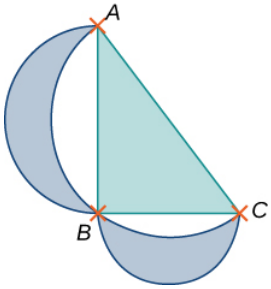
Answer

$P[X + Y \leq 6] = 1 + \frac{3}{2e^2} - \frac{5}{e^{6/5}} \approx 0.45$; there is a 45% chance that a customer will spend 6 minutes in the drive-thru line.

54) [T] The Reuleaux triangle consists of an equilateral triangle and three regions, each of them bounded by a side of the triangle and an arc of a circle of radius s centered at the opposite vertex of the triangle. Show that the area of the Reuleaux triangle in the following figure of side length s is $\frac{s^2}{2}(\pi - \sqrt{3})$.



55) [T] Show that the area of the lunes of Alhazen, the two blue lunes in the following figure, is the same as the area of the right triangle ABC. The outer boundaries of the lunes are semicircles of diameters AB and AC respectively, and the inner boundaries are formed by the circumcircle of the triangle ABC.



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15.3: Double Integrals in Polar Coordinates

Learning Objectives

- Recognize the format of a double integral over a polar rectangular region.
- Evaluate a double integral in polar coordinates by using an iterated integral.
- Recognize the format of a double integral over a general polar region.
- Use double integrals in polar coordinates to calculate areas and volumes.

Double integrals are sometimes much easier to evaluate if we change rectangular coordinates to polar coordinates. However, before we describe how to make this change, we need to establish the concept of a double integral in a polar rectangular region.

Polar Rectangular Regions of Integration

When we defined the double integral for a continuous function in rectangular coordinates—say, g over a region R in the xy-plane—we divided R into subrectangles with sides parallel to the coordinate axes. These sides have either constant x-values and/or constant y-values. In polar coordinates, the shape we work with is a polar rectangle, whose sides have constant r-values and/or constant θ-values. This means we can describe a polar rectangle as in Figure 15.3.1a, with $R = \{(r, \theta) \mid a \le r \le b,\ \alpha \le \theta \le \beta\}$.

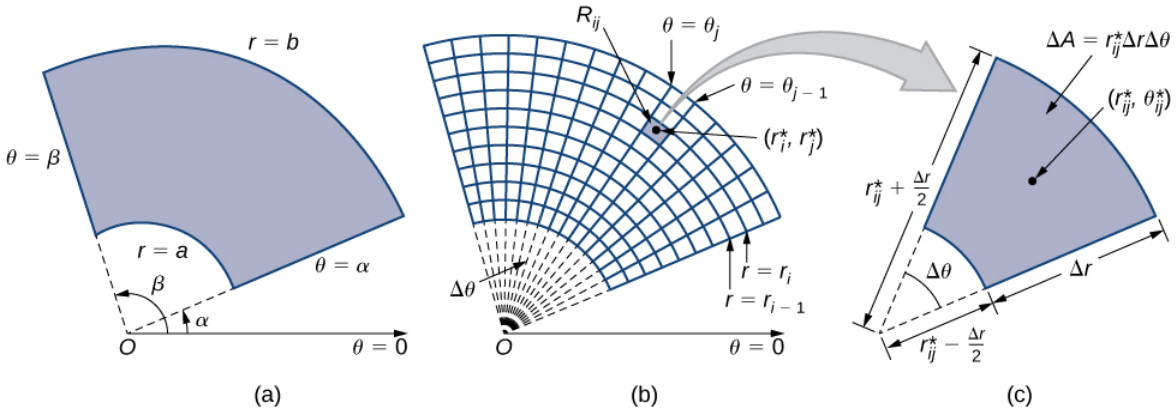

Figure 15.3.1: (a) A polar rectangle R (b) divided into subrectangles R_{ij} (c) Close-up of a subrectangle.

In this section, we are looking to integrate over polar rectangles. Consider a function $f(r, \theta)$ over a polar rectangle R. We divide the interval $[a, b]$ into m subintervals $[r_{i-1}, r_i]$ of length $\Delta r = (b - a)/m$ and divide the interval $[\alpha, \beta]$ into n subintervals $[\theta_{i-1}, \theta_i]$ of width $\Delta\theta = (\beta - \alpha)/n$. This means that the circles $r = r_i$ and rays $\theta = \theta_i$ for $1 \le i \le m$ and $1 \le j \le n$ divide the polar rectangle R into smaller polar subrectangles R_{ij} (Figure 15.3.1b).

As before, we need to find the area ΔA of the polar subrectangle R_{ij} and the "polar" volume of the thin box above R_{ij}. Recall that, in a circle of radius r the length s of an arc subtended by a central angle of θ radians is $s = r\theta$. Notice that the polar rectangle R_{ij} looks a lot like a trapezoid with parallel sides $r_{i-1}\Delta\theta$ and $r_i\Delta\theta$ and with a width Δr. Hence the area of the polar subrectangle R_{ij} is

$$\Delta A = \frac{1}{2}\Delta r(r_{i-1}\Delta\theta + r_i\Delta\theta).$$

Simplifying and letting

$$r^*_{ij} = \frac{1}{2}(r_{i-1} + r_i)$$

we have $\Delta A = r^*_{ij}\Delta r\Delta\theta$.

Therefore, the polar volume of the thin box above R_{ij} (Figure 15.3.2) is

$$f(r^*_{ij}, \theta^*_{ij})\Delta A = f(r^*_{ij}, \theta^*_{ij})r^*_{ij}\Delta r\Delta\theta.$$

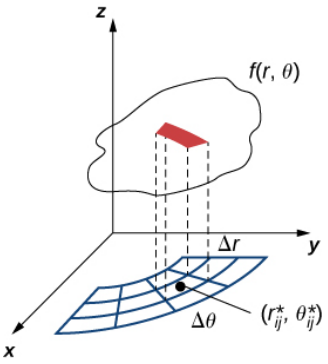

Figure 15.3.2: Finding the volume of the thin box above polar rectangle R_{ij}.

Using the same idea for all the subrectangles and summing the volumes of the rectangular boxes, we obtain a double Riemann sum as

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*, \theta_{ij}^*) r_{ij}^* \Delta r \Delta\theta.$$

As we have seen before, we obtain a better approximation to the polar volume of the solid above the region R when we let m and n become larger. Hence, we define the polar volume as the limit of the double Riemann sum,

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*, \theta_{ij}^*) r_{ij}^* \Delta r \Delta\theta.$$

This becomes the expression for the double integral.

Definition: The double integral in polar coordinates

The double integral of the function $f(r,\theta)$ over the polar rectangular region R in the $r\theta$-plane is defined as

$$\iint_R f(r,\theta) dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*, \theta_{ij}^*) \Delta A \tag{15.3.1}$$

$$= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*, \theta_{ij}^*) r_{ij}^* \Delta r \Delta\theta. \tag{15.3.2}$$

Again, just as in section on Double Integrals over Rectangular Regions, the double integral over a polar rectangular region can be expressed as an iterated integral in polar coordinates. Hence,

$$\iint_R f(r,\theta)\, dA = \iint_R f(r,\theta)\, r\, dr\, d\theta = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=a}^{r=b} f(r,\theta)\, r\, dr\, d\theta.$$

Notice that the expression for dA is replaced by $r\, dr\, d\theta$ when working in polar coordinates. Another way to look at the polar double integral is to change the double integral in rectangular coordinates by substitution. When the function f is given in terms of x and y using $x = r\cos\theta$, $y = r\sin\theta$, and $dA = r\, dr\, d\theta$ changes it to

$$\iint_R f(x,y)\, dA = \iint_R f(r\cos\theta,\, r\sin\theta)\, r\, dr\, d\theta.$$

Note that all the properties listed in section on Double Integrals over Rectangular Regions for the double integral in rectangular coordinates hold true for the double integral in polar coordinates as well, so we can use them without hesitation.

Example 15.3.1A: Sketching a Polar Rectangular Region

Sketch the polar rectangular region

$$R = \{(r,\theta) \mid 1 \le r \le 3, 0 \le \theta \le \pi\}.$$

Solution

As we can see from Figure 15.3.3, $r = 1$ and $r = 3$ are circles of radius 1 and 3 and $0 \le \theta \le \pi$ covers the entire top half of the plane. Hence the region R looks like a semicircular band.

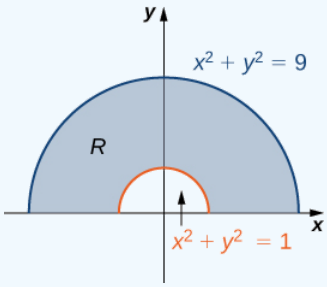

Figure 15.3.3: The polar region R lies between two semicircles.

Now that we have sketched a polar rectangular region, let us demonstrate how to evaluate a double integral over this region by using polar coordinates.

Example 15.3.1B: Evaluating a Double Integral over a Polar Rectangular Region

Evaluate the integral $\iint_R 3x\,dA$ over the region $R = \{(r,\theta)\,|\,1 \le r \le 2,\ 0 \le \theta \le \pi\}$.

Solution

First we sketch a figure similar to Figure 15.3.3, but with outer radius $r = 2$. From the figure we can see that we have

$$\begin{aligned}\iint_R 3x\,dA &= \int_{\theta=0}^{\theta=\pi}\int_{r=1}^{r=2} 3r\cos\theta\, r\,dr\,d\theta \quad \text{Use an integral with correct limits of integration.}\\ &= \int_{\theta=0}^{\theta=\pi}\cos\theta\left[r^3\Big|_{r=1}^{r=2}\right]d\theta \quad \text{Integrate first with respect to } r.\\ &= \int_{\theta=0}^{\theta=\pi} 7\cos\theta\,d\theta\\ &= 7\sin\theta\Big|_{\theta=0}^{\theta=\pi} = 0.\end{aligned}$$

Exercise 15.3.1

Sketch the region $D = \{(r,\theta)|1 \le r \le 2,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\}$, and evaluate $\iint_R x\,dA$.

Hint

Follow the steps in Example 15.3.1A.

Answer

$\frac{14}{3}$

Example 15.3.2A: Evaluating a Double Integral by Converting from Rectangular Coordinates

Evaluate the integral

$$\iint_R (1 - x^2 - y^2)\,dA$$

where R is the unit circle on the xy-plane.

Solution

The region R is a unit circle, so we can describe it as $R = \{(r,\theta)\,|\,0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$.

Using the conversion $x = r\cos\theta$, $y = r\sin\theta$, and $dA = r\,dr\,d\theta$, we have

$$\iint_R (1 - x^2 - y^2)\, dA = \int_0^{2\pi} \int_0^1 (1 - r^2)\, r\, dr\, d\theta$$
$$= \int_0^{2\pi} \int_0^1 (r - r^3)\, dr\, d\theta$$
$$= \int_0^{2\pi} \left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 d\theta$$
$$= \int_0^{2\pi} \frac{1}{4}\, d\theta = \frac{\pi}{2}.$$

Example 15.3.2B: Evaluating a Double Integral by Converting from Rectangular Coordinates

Evaluate the integral

$$\iint_R (x + y)\, dA$$

where $R = \{(x, y) \mid 1 \le x^2 + y^2 \le 4,\ x \le 0\}$.

Solution

We can see that R is an annular region that can be converted to polar coordinates and described as $R = \{(r, \theta) \mid 1 \le r \le 2,\ \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\}$ (see the following graph).

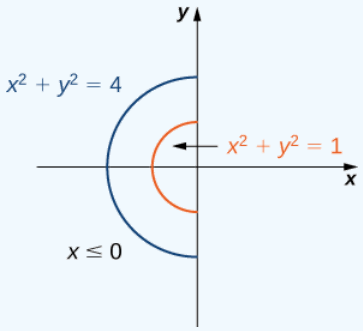

Figure 15.3.4: The annular region of integration R.

Hence, using the conversion $x = r\cos\theta$, $y = r\sin\theta$, and $dA = r\, dr\, d\theta$, we have

$$\iint_R (x + y)\, dA = \int_{\theta=\pi/2}^{\theta=3\pi/2} \int_{r=1}^{r=2} (r\cos\theta + r\sin\theta) r\, dr\, d\theta$$
$$= \left(\int_{r=1}^{r=2} r^2\, dr\right)\left(\int_{\pi/2}^{3\pi/2} (\cos\theta + \sin\theta)\, d\theta\right)$$
$$= \left[\frac{r^3}{3}\right]_1^2 [\sin\theta - \cos\theta]\Big|_{\pi/2}^{3\pi/2}$$
$$= -\frac{14}{3}.$$

Exercise 15.3.2

Evaluate the integral

$$\iint_R (4 - x^2 - y^2)\, dA$$

where R is the circle of radius 2 on the xy-plane.

Hint

Follow the steps in the previous example.

Answer

8π

General Polar Regions of Integration

To evaluate the double integral of a continuous function by iterated integrals over general polar regions, we consider two types of regions, analogous to Type I and Type II as discussed for rectangular coordinates in section on Double Integrals over General Regions. It is more common to write polar equations as $r = f(\theta)$ than $\theta = f(r)$, so we describe a general polar region as $R = \{(r, \theta) \mid \alpha \leq \theta \leq \beta,\ h_1(\theta) \leq r \leq h_2(\theta)\}$ (Figure 15.3.5).

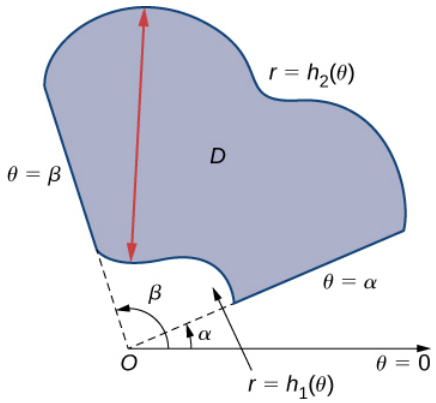

Figure 15.3.5: A general polar region between $\alpha \leq \theta \leq \beta$ and $h_1(\theta) \leq r \leq h_2(\theta)$.

Theorem: Double Integrals over General Polar Regions

If $f(r, \theta)$ is continuous on a general polar region D as described above, then

$$\iint_D f(r, \theta)\, r\, dr\, d\theta = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} f(r, \theta)\, r\, dr\, d\theta.$$

Example 15.3.3: Evaluating a Double Integral over a General Polar Region

Evaluate the integral

$$\iint_D r^2 \sin\theta\, r\, dr\, d\theta$$

where D is the region bounded by the polar axis and the upper half of the cardioid $r = 1 + \cos\theta$.

Solution

We can describe the region D as $\{(r, \theta) \mid 0 \leq \theta \leq \pi,\ 0 \leq r \leq 1 + \cos\theta\}$ as shown in Figure 15.3.6.

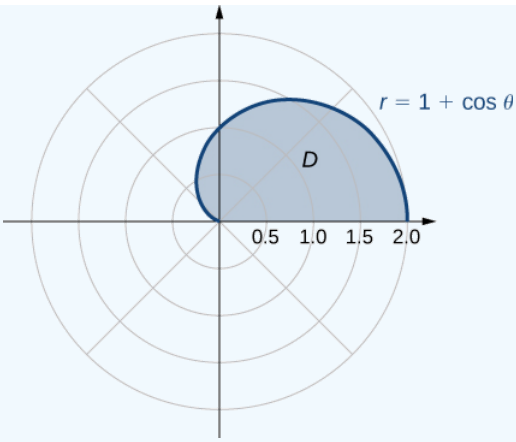

Figure 15.3.6: The region D is the top half of a cardioid.

Hence, we have

$$\begin{aligned}\iint_D r^2 \sin\,\theta\, r\,dr\,d\theta &= \int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=1+\cos\theta}(r^2\sin\,\theta)\,r\,dr\,d\theta \\ &= \frac{1}{4}\int_{\theta=0}^{\theta=\pi}[r^4]\Big|_{r=0}^{r=1+\cos\,\theta}\sin\,\theta\,d\theta \\ &= \frac{1}{4}\int_{\theta=0}^{\theta=\pi}(1+\cos\,\theta)^4\sin\,\theta\,d\theta \\ &= -\frac{1}{4}\left[\frac{(1+\cos\,\theta)^5}{5}\right]_0^{\pi} = \frac{8}{5}.\end{aligned}$$

? Exercise 15.3.3

Evaluate the integral

$$\iint_D r^2\sin^2 2\theta\, r\,dr\,d\theta$$

where $D = \left\{(r,\theta)\,|\,0\le\theta\le\pi,\ 0\le r\le 2\sqrt{\cos\,2\theta}\right\}$.

Hint

Graph the region and follow the steps in the previous example.

Answer

$\frac{\pi}{8}$

Polar Areas and Volumes

As in rectangular coordinates, if a solid S is bounded by the surface $z = f(r,\theta)$, as well as by the surfaces $r = a$, $r = b$, $\theta = \alpha$, and $\theta = \beta$, we can find the volume V of S by double integration, as

$$V = \iint_R f(r,\theta)\,r\,dr\,d\theta = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=a}^{r=b} f(r,\theta)\,r\,dr\,d\theta.$$

If the base of the solid can be described as $D = \{(r,\theta)|\alpha\le\theta\le\beta,\ h_1(\theta)\le r\le h_2(\theta)\}$, then the double integral for the volume becomes

$$V = \iint_D f(r,\theta)\,r\,dr\,d\theta = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=h_1(\theta)}^{r=h_2(\theta)} f(r,\theta)\,r\,dr\,d\theta.$$

We illustrate this idea with some examples.

Example 15.3.4A: Finding a Volume Using a Double Integral

Find the volume of the solid that lies under the paraboloid $z = 1 - x^2 - y^2$ and above the unit circle on the xy-plane (Figure 15.3.7).

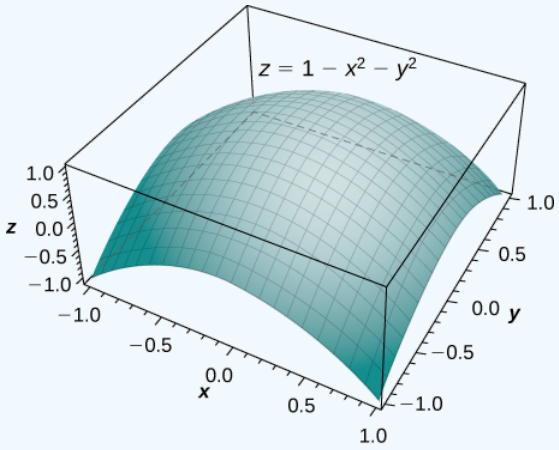

Figure 15.3.7: Finding the volume of a solid under a paraboloid and above the unit circle.

Solution

By the method of double integration, we can see that the volume is the iterated integral of the form

$$\iint_R (1 - x^2 - y^2)\, dA$$

where $R = \{(r, \theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$.

This integration was shown before in Example 15.3.2A, so the volume is $\frac{\pi}{2}$ cubic units.

Example 15.3.4B: Finding a Volume Using Double Integration

Find the volume of the solid that lies under the paraboloid $z = 4 - x^2 - y^2$ and above the disk $(x - 1)^2 + y^2 = 1$ on the xy-plane. See the paraboloid in Figure 15.3.8 intersecting the cylinder $(x - 1)^2 + y^2 = 1$ above the xy-plane.

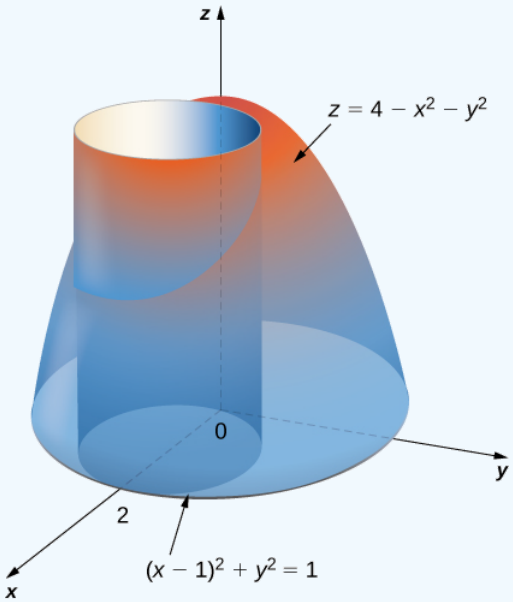

Figure 15.3.8: Finding the volume of a solid with a paraboloid cap and a circular base.

Solution

First change the disk $(x-1)^2+y^2=1$ to polar coordinates. Expanding the square term, we have $x^2-2x+1+y^2=1$. Then simplify to get $x^2+y^2=2x$, which in polar coordinates becomes $r^2=2r\cos\theta$ and then either $r=0$ or $r=2\cos\theta$. Similarly, the equation of the paraboloid changes to $z=4-r^2$. Therefore we can describe the disk $(x-1)^2+y^2=1$ on the xy -plane as the region

$$D=\{(r,\theta)\,|\,0\le\theta\le\pi,\ 0\le r\le 2\cos\theta\}.$$

Hence the volume of the solid bounded above by the paraboloid $z=4-x^2-y^2$ and below by $r=2\cos\theta$ is

$$\begin{aligned}V&=\iint_D f(r,\theta)\,r\,dr\,d\theta\\&=\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=2\cos\theta}(4-r^2)\,r\,dr\,d\theta\\&=\int_{\theta=0}^{\theta=\pi}\left[4\frac{r^2}{2}-\frac{r^4}{4}\Big|_0^{2\cos\theta}\right]d\theta\\&=\int_0^{\pi}[8\cos^2\theta-4\cos^4\theta]\,d\theta\\&=\left[\frac{5}{2}\theta+\frac{5}{2}\sin\theta\cos\theta-\sin\theta\cos^3\theta\right]_0^{\pi}=\frac{5}{2}\pi\text{ units}^3.\end{aligned}$$

Notice in the next example that integration is not always easy with polar coordinates. Complexity of integration depends on the function and also on the region over which we need to perform the integration. If the region has a more natural expression in polar coordinates or if f has a simpler antiderivative in polar coordinates, then the change in polar coordinates is appropriate; otherwise, use rectangular coordinates.

Example 15.3.5A: Finding a Volume Using a Double Integral

Find the volume of the region that lies under the paraboloid $z=x^2+y^2$ and above the triangle enclosed by the lines $y=x$, $x=0$, and $x+y=2$ in the xy-plane.

Solution

First examine the region over which we need to set up the double integral and the accompanying paraboloid.

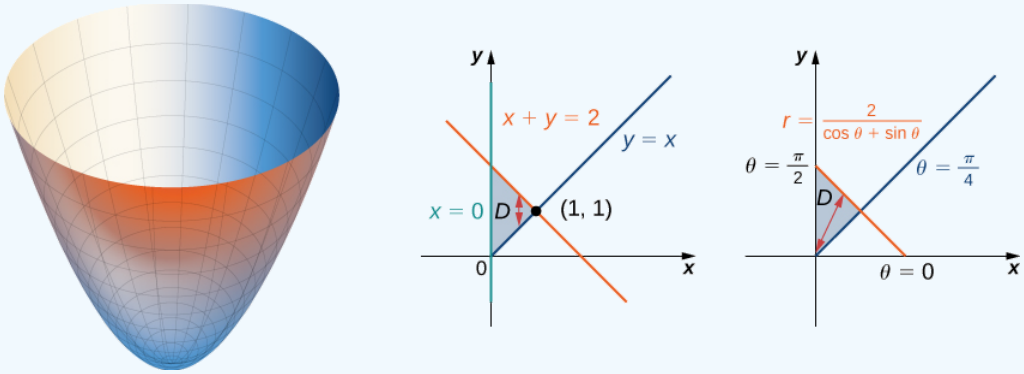

Figure 15.3.9: Finding the volume of a solid under a paraboloid and above a given triangle.

The region D is $\{(x,y)\,|\,0\le x\le 1,\ x\le y\le 2-x\}$. Converting the lines $y=x$, $x=0$, and $x+y=2$ in the xy-plane to functions of r and θ we have $\theta=\pi/4$, $\theta=\pi/2$, and $r=2/(\cos\theta+\sin\theta)$, respectively. Graphing the region on the xy- plane, we see that it looks like $D=\{(r,\theta)\,|\,\pi/4\le\theta\le\pi/2,\ 0\le r\le 2/(\cos\theta+\sin\theta)\}$.

Now converting the equation of the surface gives $z=x^2+y^2=r^2$. Therefore, the volume of the solid is given by the double integral

$$
\begin{aligned}
V &= \iint_D f(r,\theta)\, r\, dr\, d\theta \\
&= \int_{\theta=\pi/4}^{\theta=\pi/2} \int_{r=0}^{r=2/(\cos\,\theta+\sin\,\theta)} r^2 r\, dr d\theta \\
&= \int_{\pi/4}^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2/(\cos\,\theta+\sin\,\theta)} d\theta \\
&= \frac{1}{4}\int_{\pi/4}^{\pi/2} \left(\frac{2}{\cos\,\theta+\sin\,\theta}\right)^4 d\theta \\
&= \frac{16}{4}\int_{\pi/4}^{\pi/2} \left(\frac{1}{\cos\,\theta+\sin\,\theta}\right)^4 d\theta \\
&= 4\int_{\pi/4}^{\pi/2} \left(\frac{1}{\cos\,\theta+\sin\,\theta}\right)^4 d\theta.
\end{aligned}
$$

As you can see, this integral is very complicated. So, we can instead evaluate this double integral in rectangular coordinates as

$$V = \int_0^1 \int_x^{2-x} (x^2+y^2)\, dy\, dx.$$

Evaluating gives

$$
\begin{aligned}
V &= \int_0^1 \int_x^{2-x} (x^2+y^2)\, dy\, dx \\
&= \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]\Big|_x^{2-x} dx \\
&= \int_0^1 \frac{8}{3} - 4x + 4x^2 - \frac{8x^3}{3}\, dx \\
&= \left[\frac{8x}{3} - 2x^2 + \frac{4x^3}{3} - \frac{2x^4}{3}\right]\Big|_0^1 \\
&= \frac{4}{3}\ \text{units}^3.
\end{aligned}
$$

To answer the question of how the formulas for the volumes of different standard solids such as a sphere, a cone, or a cylinder are found, we want to demonstrate an example and find the volume of an arbitrary cone.

Example 15.3.5B: Finding a Volume Using a Double Integral

Use polar coordinates to find the volume inside the cone $z = 2 - \sqrt{x^2+y^2}$ and above the xy-plane.

Solution

The region D for the integration is the base of the cone, which appears to be a circle on the xy-plane (Figure 15.3.10).

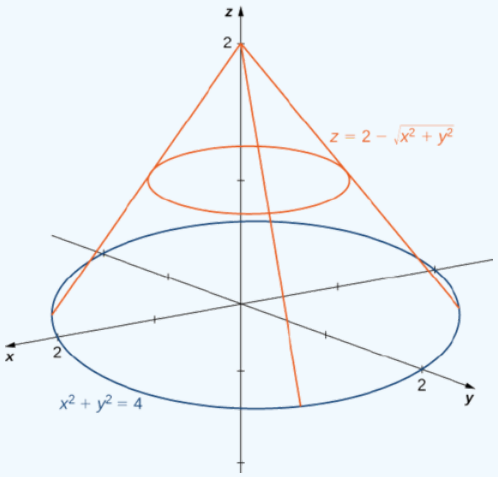

Figure 15.3.10: Finding the volume of a solid inside the cone and above the xy-plane.

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We find the equation of the circle by setting $z = 0$:

$$\begin{aligned} 0 &= 2 - \sqrt{x^2 + y^2} \\ 2 &= \sqrt{x^2 + y^2} \\ x^2 + y^2 &= 4. \end{aligned}$$

This means the radius of the circle is 2 so for the integration we have $0 \le \theta \le 2\pi$ and $0 \le r \le 2$. Substituting $x = r\cos\theta$ and $y = r\sin\theta$ in the equation $z = 2 - \sqrt{x^2 + y^2}$ we have $z = 2 - r$. Therefore, the volume of the cone is

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} (2 - r)\, r\, dr\, d\theta = 2\pi \frac{4}{3} = \frac{8\pi}{3} \text{ cubic units.}$$

Analysis

Note that if we were to find the volume of an arbitrary cone with radius α units and height h units, then the equation of the cone would be $z = h - \frac{h}{a}\sqrt{x^2 + y^2}$.

We can still use Figure $15.3.10$ and set up the integral as

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a} \left(h - \frac{h}{a}r\right) r\, dr\, d\theta.$$

Evaluating the integral, we get $\frac{1}{3}\pi a^2 h$.

Exercise $15.3.5$

Use polar coordinates to find an iterated integral for finding the volume of the solid enclosed by the paraboloids $z = x^2 + y^2$ and $z = 16 - x^2 - y^2$.

Hint

Sketching the graphs can help.

Answer

$$V = \int_0^{2\pi} \int_0^{2\sqrt{2}} (16 - 2r^2)\, r\, dr\, d\theta = 64\pi \text{ cubic units.}$$

As with rectangular coordinates, we can also use polar coordinates to find areas of certain regions using a double integral. As before, we need to understand the region whose area we want to compute. Sketching a graph and identifying the region can be helpful to realize the limits of integration. Generally, the area formula in double integration will look like

$$\text{Area of } A = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} 1\, r\, dr\, d\theta.$$

Example $15.3.6A$: Finding an Area Using a Double Integral in Polar Coordinates

Evaluate the area bounded by the curve $r = \cos 4\theta$.

Solution

Sketching the graph of the function $r = \cos 4\theta$ reveals that it is a polar rose with eight petals (see the following figure).

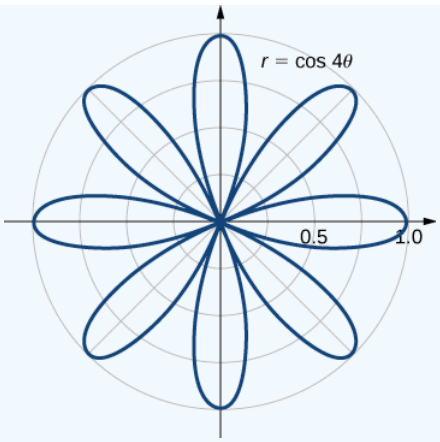

Figure 15.3.11: Finding the area of a polar rose with eight petals.

Using symmetry, we can see that we need to find the area of one petal and then multiply it by 8. Notice that the values of θ for which the graph passes through the origin are the zeros of the function $\cos\ 4\theta$, and these are odd multiples of $\pi/8$. Thus, one of the petals corresponds to the values of θ in the interval $[-\pi/8, \pi/8]$. Therefore, the area bounded by the curve $r = \cos\ 4\theta$ is

$$\begin{aligned} A &= 8\int_{\theta=-\pi/8}^{\theta=\pi/8}\int_{r=0}^{r=\cos\ 4\theta} 1\, r\, dr\, d\theta \\ &= 8\int_{\theta=-\pi/8}^{\theta=\pi/8}\left[\frac{1}{2}r^2\Big|_0^{\cos\ 4\theta}\right] d\theta \\ &= 8\int_{-\pi/8}^{\pi/8}\frac{1}{2}\cos^2 4\theta\, d\theta \\ &= 8\left[\frac{1}{4}\theta + \frac{1}{16}\sin\ 4\theta\ \cos\ 4\theta\Big|_{-\pi/8}^{\pi/8}\right] \\ &= 8\left[\frac{\pi}{16}\right] = \frac{\pi}{2}\ \text{units}^2. \end{aligned}$$

Example 15.3.6B: Finding Area Between Two Polar Curves

Find the area enclosed by the circle $r = 3\ \cos\ \theta$ and the cardioid $r = 1 + \cos\ \theta$.

Solution

First and foremost, sketch the graphs of the region (Figure 15.3.12).

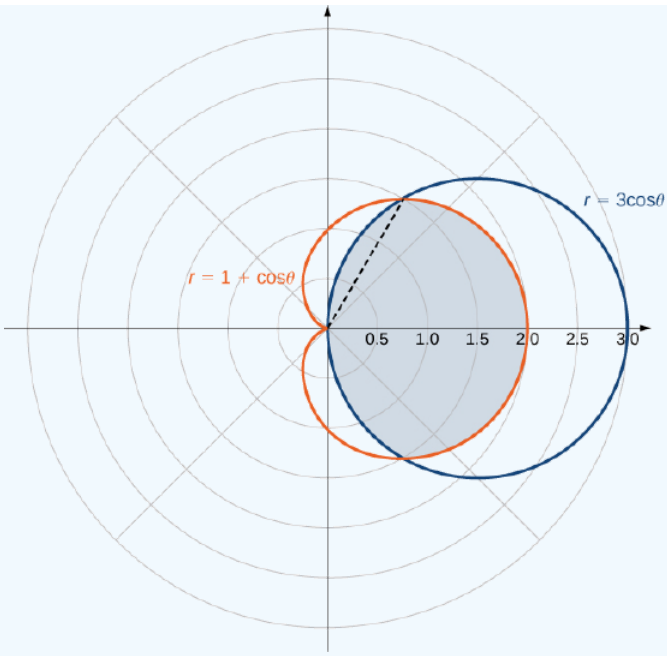

Figure 15.3.12: Finding the area enclosed by both a circle and a cardioid.

We can from see the symmetry of the graph that we need to find the points of intersection. Setting the two equations equal to each other gives

$$3\cos\theta = 1 + \cos\theta.$$

One of the points of intersection is $\theta = \pi/3$. The area above the polar axis consists of two parts, with one part defined by the cardioid from $\theta = 0$ to $\theta = \pi/3$ and the other part defined by the circle from $\theta = \pi/3$ to $\theta = \pi/2$. By symmetry, the total area is twice the area above the polar axis. Thus, we have

$$A = 2\left[\int_{\theta=0}^{\theta=\pi/3}\int_{r=0}^{r=1+\cos\theta} 1\, r\, dr\, d\theta + \int_{\theta=\pi/3}^{\theta=\pi/2}\int_{r=0}^{r=3\cos\theta} 1\, r\, dr\, d\theta\right].$$

Evaluating each piece separately, we find that the area is

$$A = 2\left(\frac{1}{4}\pi + \frac{9}{16}\sqrt{3} + \frac{3}{8}\pi - \frac{9}{16}\sqrt{3}\right) = 2\left(\frac{5}{8}\pi\right) = \frac{5}{4}\pi \text{ square units.}$$

Exercise 15.3.6

Find the area enclosed inside the cardioid $r = 3 - 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.

Hint

Sketch the graph, and solve for the points of intersection.

Answer

$$A = 2\int_{-\pi/2}^{\pi/6}\int_{1+\sin\theta}^{3-3\sin\theta} r\, dr\, d\theta = \left(8\pi + 9\sqrt{3}\right) \text{ units}^2$$

Example 15.3.7: Evaluating an Improper Double Integral in Polar Coordinates

Evaluate the integral

$$\iint_{R^2} e^{-10(x^2+y^2)}\, dx\, dy.$$

Solution

This is an improper integral because we are integrating over an unbounded region R^2. In polar coordinates, the entire plane R^2 can be seen as $0 \le \theta \le 2\pi$, $0 \le r \le \infty$.

Using the changes of variables from rectangular coordinates to polar coordinates, we have

$$\begin{aligned}\iint_{R^2} e^{-10(x^2+y^2)}\,dx\,dy &= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=\infty} e^{-10r^2}\,r\,dr\,d\theta = \int_{\theta=0}^{\theta=2\pi}\left(\lim_{a\to\infty}\int_{r=0}^{r=a} e^{-10r^2}r\,dr\right)d\theta \\ &= \left(\int_{\theta=0}^{\theta=2\pi}\right)d\theta\left(\lim_{a\to\infty}\int_{r=0}^{r=a} e^{-10r^2}r\,dr\right) \\ &= 2\pi\left(\lim_{a\to\infty}\int_{r=0}^{r=a} e^{-10r^2}r\,dr\right) \\ &= 2\pi\lim_{a\to\infty}\left(-\frac{1}{20}\right)\left(e^{-10r^2}\Big|_0^a\right) \\ &= 2\pi\left(-\frac{1}{20}\right)\lim_{a\to\infty}\left(e^{-10a^2}-1\right) \\ &= \frac{\pi}{10}.\end{aligned}$$

? Exercise 15.3.7

Evaluate the integral

$$\iint_{R^2} e^{-4(x^2+y^2)}dx\,dy.$$

Hint

Convert to the polar coordinate system.

Answer

$\frac{\pi}{4}$

Key Concepts

- To apply a double integral to a situation with circular symmetry, it is often convenient to use a double integral in polar coordinates. We can apply these double integrals over a polar rectangular region or a general polar region, using an iterated integral similar to those used with rectangular double integrals.
- The area dA in polar coordinates becomes $r\,dr\,d\theta$.
- Use $x = r\cos\theta$, $y = r\sin\theta$, and $dA = r\,dr\,d\theta$ to convert an integral in rectangular coordinates to an integral in polar coordinates.
- Use $r^2 = x^2 + y^2$ and $\theta = tan^{-1}\left(\frac{y}{x}\right)$ to convert an integral in polar coordinates to an integral in rectangular coordinates, if needed.
- To find the volume in polar coordinates bounded above by a surface $z = f(r,\theta)$ over a region on the xy-plane, use a double integral in polar coordinates.

Key Equations

- **Double integral over a polar rectangular region** R

$$\iint_R f(r,\theta)dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*,\theta_{ij}^*)\Delta A = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*,\theta_{ij}^*)r_{ij}^*\Delta r\Delta\theta$$

- **Double integral over a general polar region**

$$\iint_D f(r,\theta)\,r\,dr\,d\theta = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=h_1(\theta)}^{r_2(\theta)} f(r,\theta)\,r\,dr\,d\theta$$

Glossary

polar rectangle

the region enclosed between the circles $r = a$ and $r = b$ and the angles $\theta = \alpha$ and $\theta = \beta$; it is described as $R = \{(r, \theta) \mid a \leq r \leq b, \alpha \leq \theta \leq \beta\}$

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15.3E: EXERCISES FOR SECTION 15.3

TERMS AND CONCEPTS

1. When evaluating $\int\int_R f(x,y)\,dA$ using polar coordinates, $f(x,y)$ is replaced with _____ and dA is replaced with _____.

Answer

$f(x,y)$ is replaced with $f(r\cos\theta, r\sin\theta)$ and dA is replaced with $r\,dr\,d\theta$.

2. Why would one be interested in evaluating a double integral with polar coordinates?

DEFINING POLAR REGIONS

In exercises 3 - 6, express the region R in polar coordinates.

3) R is the region of the disk of radius 2 centered at the origin that lies in the first quadrant.

Answer

$R = \{(r,\theta)\,|\,0 \le r \le 2,\ 0 \le \theta \le \frac{\pi}{2}\}$

4) R is the region of the disk of radius 3 centered at the origin.

5) R is the region between the circles of radius 4 and radius 5 centered at the origin that lies in the second quadrant.

Answer

$R = \{(r,\theta)\,|\,4 \le r \le 5,\ \frac{\pi}{2} \le \theta \le \pi\}$

6) R is the region bounded by the y-axis and $x = \sqrt{1-y^2}$.

7) R is the region bounded by the x-axis and $y = \sqrt{2-x^2}$.

Answer

$R = \{(r,\theta)\,|\,0 \le r \le \sqrt{2},\ 0 \le \theta \le \pi\}$

8) $R = \{(x,y)\,|\,x^2+y^2 \le 4x\}$

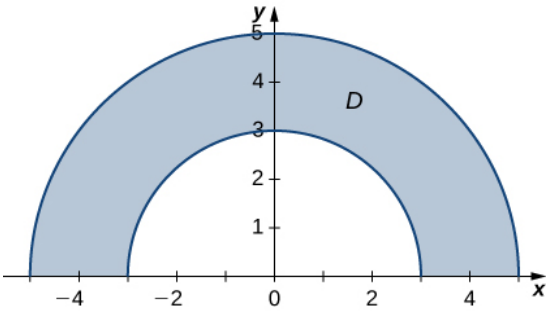
9) $R = \{(x,y)\,|\,x^2+y^2 \le 4y\}$

Answer

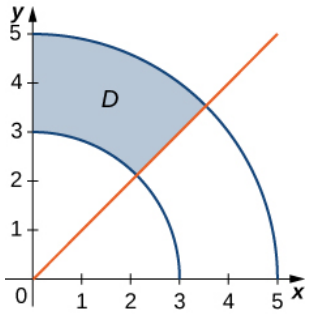
$R = \{(r,\theta)\,|\,0 \le r \le 4\sin\theta,\ 0 \le \theta \le \pi\}$

In exercises 10 - 15, the graph of the polar rectangular region D is given. Express D in polar coordinates. In exercises 10 - 13 the regions are bounded by circles centered at the origin, the coordinate axes, and/or lines of the form $y = \pm x$ (each of which should be able to be identified by inspection).

10)



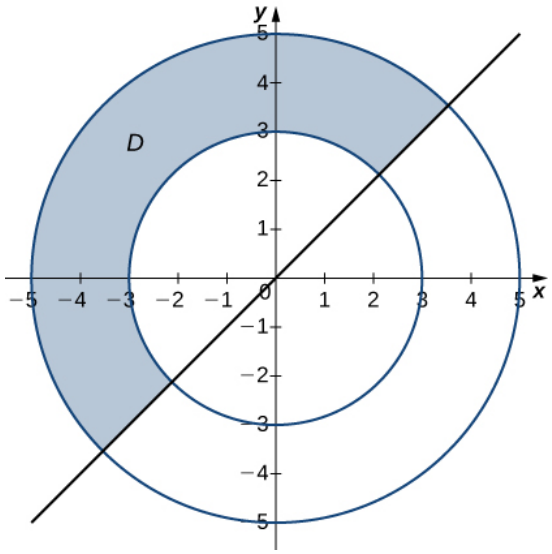
11)



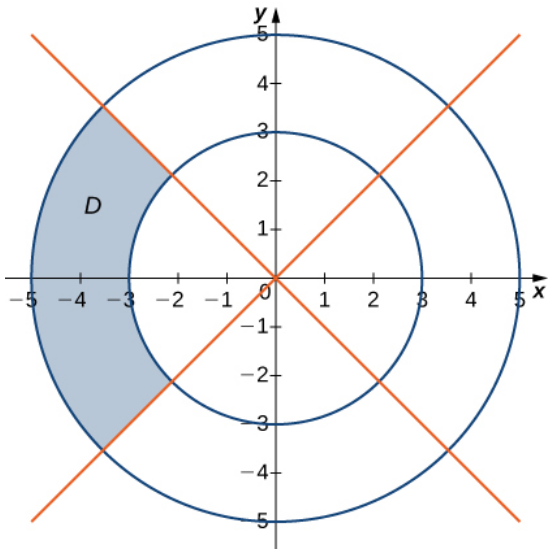
Answer

$D = \{(r, \theta) \mid 3 \leq r \leq 5,\ \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}\}$

12)



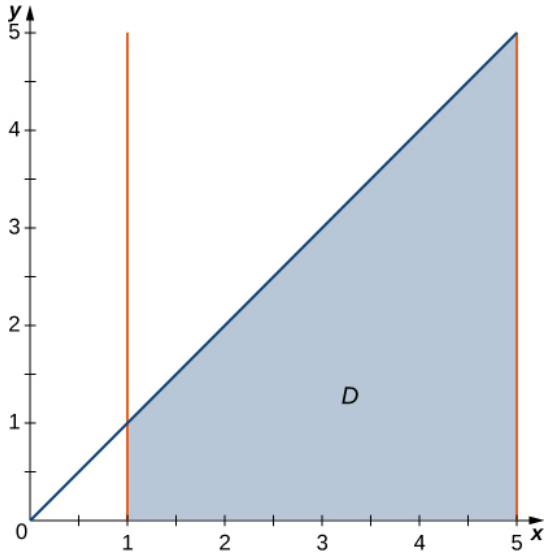
13)



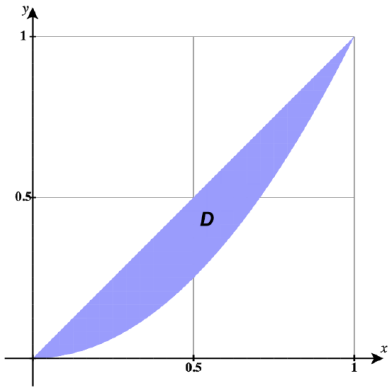
Answer

$D = \{(r, \theta) \mid 3 \leq r \leq 5,\ \frac{3\pi}{4} \leq \theta \leq \frac{5\pi}{4}\}$

14) In the following graph, the region D is situated below $y = x$ and is bounded by $x = 1,\ x = 5$, and $y = 0$.



15) In the following graph, the region D is bounded by $y = x$ and $y = x^2$.



Answer

$D = \{(r,\theta) \mid 0 \le r \le \tan\theta \sec\theta,\ 0 \le \theta \le \frac{\pi}{4}\}$

EVALUATING POLAR DOUBLE INTEGRALS

In exercises 16 - 25, evaluate the double integral $\iint_R f(x,y)\,dA$ over the polar rectangular region R.

16) $f(x,y) = x^2 + y^2$, $R = \{(r,\theta) \mid 3 \le r \le 5,\ 0 \le \theta \le 2\pi\}$

17) $f(x,y) = x + y$, $R = \{(r,\theta) \mid 3 \le r \le 5,\ 0 \le \theta \le 2\pi\}$

Answer

0

18) $f(x,y) = x^2 + xy$, $R = \{(r,\theta) \mid 1 \le r \le 2,\ \pi \le \theta \le 2\pi\}$

19) $f(x,y) = x^4 + y^4$, $R = \{(r,\theta) \mid 1 \le r \le 2,\ \frac{3\pi}{2} \le \theta \le 2\pi\}$

Answer

$\frac{63\pi}{16}$

20) $f(x,y) = \sqrt[3]{x^2 + y^2}$, where $R = \{(r,\theta) \mid 0 \le r \le 1,\ \frac{\pi}{2} \le \theta \le \pi\}$.

21) $f(x,y) = x^4 + 2x^2y^2 + y^4$, where $R = \{(r,\theta) \mid 3 \le r \le 4,\ \frac{\pi}{3} \le \theta \le \frac{2\pi}{3}\}$.

Answer

$\frac{3367\pi}{18}$

22) $f(x,y) = \sin(\arctan \frac{y}{x})$, where $R = \{(r,\theta) \mid 1 \le r \le 2,\ \frac{\pi}{6} \le \theta \le \frac{\pi}{3}\}$

23) $f(x,y) = \arctan\left(\frac{y}{x}\right)$, where $R = \{(r,\theta) \mid 2 \le r \le 3,\ \frac{\pi}{4} \le \theta \le \frac{\pi}{3}\}$

Answer

$\frac{35\pi^2}{576}$

24) $\iint_R e^{x^2+y^2}\left[1 + 2\ \arctan\left(\frac{y}{x}\right)\right] dA,\ R = \{(r,\theta) \mid 1 \le r \le 2,\ \frac{\pi}{6} \le \theta \frac{\pi}{3}\}$

25) $\iint_R \left(e^{x^2+y^2} + x^4 + 2x^2y^2 + y^4\right) \arctan\left(\frac{y}{x}\right) dA,\ R = \{(r,\theta) \mid 1 \le r \le 2,\ \frac{\pi}{4} \le \theta \le \frac{\pi}{3}\}$

Answer

$\frac{7}{576}\pi^2(21 - e + e^4)$

CONVERTING DOUBLE INTEGRALS TO POLAR FORM

In exercises 26 - 29, the integrals have been converted to polar coordinates. Verify that the identities are true and choose the easiest way to evaluate the integrals, in rectangular or polar coordinates.

26) $\int_1^2 \int_0^x (x^2 + y^2)\, dy\, dx = \int_0^{\frac{\pi}{4}} \int_{\sec\theta}^{2\ \sec\theta} r^3\, dr\, d\theta$

27) $\int_2^3 \int_0^x \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx = \int_0^{\pi/4} \int_{2\sec\theta}^{3\sec\theta} r\ \cos\theta\, dr\, d\theta$

Answer

$\frac{5}{2}\ln(1+\sqrt{2})$

28) $\int_0^1 \int_{x^2}^x \frac{1}{\sqrt{x^2+y^2}}\, dy\, dx = \int_0^{\pi/4} \int_0^{\tan\theta\ \sec\theta} dr\, d\theta$

29) $\int_0^1 \int_{x^2}^x \frac{y}{\sqrt{x^2+y^2}}\, dy\, dx = \int_0^{\pi/4} \int_0^{\tan\theta\ \sec\theta} r\ \sin\theta\, dr\, d\theta$

Answer

$\frac{1}{6}(2 - \sqrt{2})$

In exercises 30 - 37, draw the region of integration, R, labeling all limits of integration, convert the integrals to polar coordinates and evaluate them.

30) $\int_0^3 \int_0^{\sqrt{9-y^2}} (x^2 + y^2)\, dx\, dy$

31) $\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (x^2 + y^2)^2\, dx\, dy$

Answer

$\int_0^{\pi} \int_0^2 r^5\, dr\, d\theta \quad = \quad \frac{32\pi}{3}$

32) $\int_0^1 \int_0^{\sqrt{1-x^2}} (x + y)\, dy\, dx$

33) $\int_0^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \sin(x^2 + y^2)\, dy\, dx$

Answer

$\int_{-\pi/2}^{\pi/2} \int_0^4 r\ \sin(r^2)\, dr\, d\theta \quad = \quad \pi\ \sin^2 8$

34) $\int_0^5 \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} \sqrt{x^2+y^2}\, dy\, dx$

35) $\int_{-4}^4 \int_{-\sqrt{16-y^2}}^0 (2y - x)\, dx\, dy$

Answer

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^4 \left(2r\sin\theta - r\cos\theta\right) r\,dr\,d\theta \quad = \quad \frac{128}{3}$$

36) $\int_0^2 \int_y^{\sqrt{8-y^2}} (x+y)\,dx\,dy$

37) $\int_{-2}^{-1} \int_0^{\sqrt{4-x^2}} (x+5)\,dy\,dx + \int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} (x+5)\,dy\,dx + \int_1^2 \int_0^{\sqrt{4-x^2}} (x+5)\,dy\,dx$

Answer

$$\int_0^{\pi} \int_1^2 \left(r\cos\theta + 5\right) r\,dr\,d\theta \quad = \quad \frac{15\pi}{2}$$

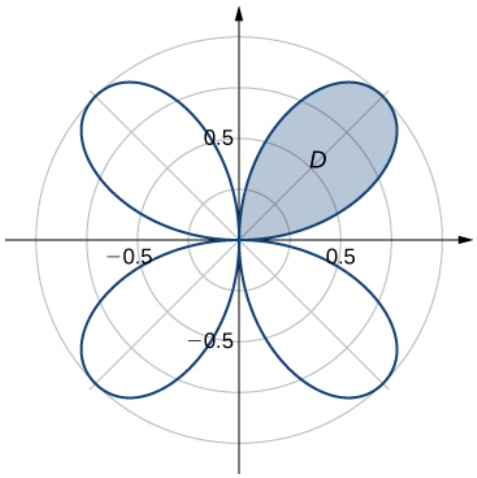
38) Evaluate the integral $\iint_D r\,dA$ where D is the region bounded by the polar axis and the upper half of the cardioid $r = 1 + \cos\theta$.

39) Find the area of the region D bounded by the polar axis and the upper half of the cardioid $r = 1 + \cos\theta$.

Answer

$\frac{3\pi}{4}$

40) Evaluate the integral $\iint_D r\,dA$, where D is the region bounded by the part of the four-leaved rose $r = \sin 2\theta$ situated in the first quadrant (see the following figure).



41) Find the total area of the region enclosed by the four-leaved rose $r = \sin 2\theta$ (see the figure in the previous exercise).

Answer

$\frac{\pi}{2}$

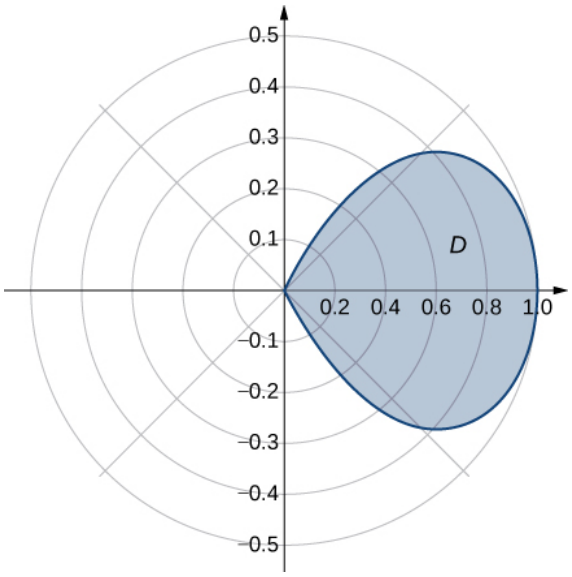
42) Find the area of the region D which is the region bounded by $y = \sqrt{4 - x^2}$, $x = \sqrt{3}$, $x = 2$, and $y = 0$.

43) Find the area of the region D, which is the region inside the disk $x^2 + y^2 \leq 4$ and to the right of the line $x = 1$.

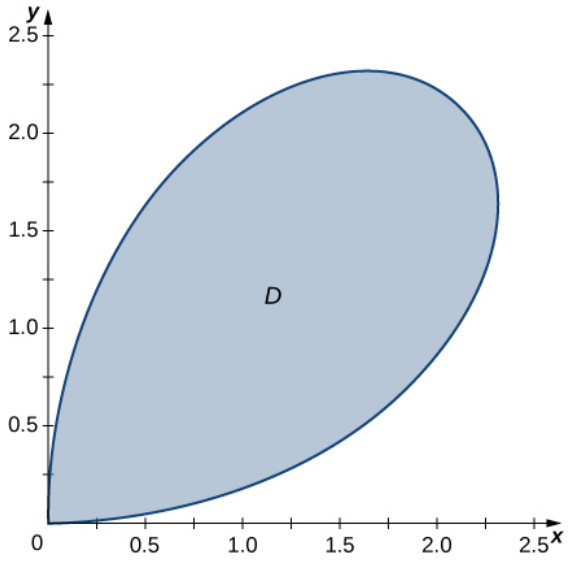
Answer

$\frac{1}{3}(4\pi - 3\sqrt{3})$

44) Determine the average value of the function $f(x, y) = x^2 + y^2$ over the region D bounded by the polar curve $r = \cos 2\theta$, where $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$ (see the following graph).



45) Determine the average value of the function $f(x,y)=\sqrt{x^2+y^2}$ over the region D bounded by the polar curve $r=3\sin 2\theta$, where $0\le\theta\le\frac{\pi}{2}$ (see the following graph).



Answer

$\frac{16}{3\pi}$

46) Find the volume of the solid situated in the first octant and bounded by the paraboloid $z=1-4x^2-4y^2$ and the planes $x=0,\ y=0$, and $z=0$.

47) Find the volume of the solid bounded by the paraboloid $z=2-9x^2-9y^2$ and the plane $z=1$.

Answer

$\frac{\pi}{18}$

48)

1. Find the volume of the solid S_1 bounded by the cylinder $x^2+y^2=1$ and the planes $z=0$ and $z=1$.
2. Find the volume of the solid S_2 outside the double cone $z^2=x^2+y^2$ inside the cylinder $x^2+y^2=1$, and above the plane $z=0$.
3. Find the volume of the solid inside the cone $z^2=x^2+y^2$ and below the plane $z=1$ by subtracting the volumes of the solids S_1 and S_2.

49)

1. Find the volume of the solid S_1 inside the unit sphere $x^2+y^2+z^2=1$ and above the plane $z=0$.

2. Find the volume of the solid S_2 inside the double cone $(z-1)^2 = x^2 + y^2$ and above the plane $z = 0$.
3. Find the volume of the solid outside the double cone $(z-1)^2 = x^2 + y^2$ and inside the sphere $x^2 + y^2 + z^2 = 1$.

Answer

a. $\frac{2\pi}{3}$; b. $\frac{\pi}{2}$; c. $\frac{\pi}{6}$

In Exercises 50-51, special double integrals are presented that are especially well suited for evaluation in polar coordinates.

50) The surface of a right circular cone with height h and base radius a can be described by the equation $f(x,y) = h - h\sqrt{\frac{x^2}{a^2} + \frac{y^2}{a^2}}$, where the tip of the cone lies at $(0,0,h)$ and the circular base lies in the xy-plane, centered at the origin.
Confirm that the volume of a right circular cone with height h and base radius a is $V = \frac{1}{3}\pi a^2 h$ by evaluating $\int\int_R f(x,y)\, dA$ in polar coordinates.

51) Consider $\int\int_R e^{-(x^2+y^2)}\, dA$.
(a) Why is this integral difficult to evaluate in rectangular coordinates, regardless of the region R?
(b) Let R be the region bounded by the circle of radius a centered at the origin. Evaluate the double integral using polar coordinates.
(c) Take the limit of your answer from (b), as $a \to \infty$. What does this imply about the volume under the surface of $e^{-(x^2+y^2)}$ over the entire xy-plane?

For the following two exercises, consider a spherical ring, which is a sphere with a cylindrical hole cut so that the axis of the cylinder passes through the center of the sphere (see the following figure).

52) If the sphere has radius 4 and the cylinder has radius 2 find the volume of the spherical ring.

53) A cylindrical hole of diameter 6 cm is bored through a sphere of radius 5 cm such that the axis of the cylinder passes through the center of the sphere. Find the volume of the resulting spherical ring.

Answer

$\frac{256\pi}{3}$ cm^3

54) Find the volume of the solid that lies under the double cone $z^2 = 4x^2 + 4y^2$, inside the cylinder $x^2 + y^2 = x$, and above the plane $z = 0$.

55) Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, inside the cylinder $x^2 + y^2 = 1$ and above the plane $z = 0$.

Answer

$\frac{3\pi}{32}$

56) Find the volume of the solid that lies under the plane $x + y + z = 10$ and above the disk $x^2 + y^2 = 4x$.

57) Find the volume of the solid that lies under the plane $2x + y + 2z = 8$ and above the unit disk $x^2 + y^2 = 1$.

Answer

4π

58) A radial function f is a function whose value at each point depends only on the distance between that point and the origin of the system of coordinates; that is, $f(x,y) = g(r)$, where $r = \sqrt{x^2 + y^2}$. Show that if f is a continuous radial function, then

$$\iint_D f(x,y)dA = (\theta_2 - \theta_1)[G(R_2) - G(R_1)],\ \textit{where}\ G'(r) = rg(r)$$

and $(x,y) \in D = \{(r,\theta) \mid R_1 \le r \le R_2,\ 0 \le \theta \le 2\pi\}$, with $0 \le R_1 < R_2$ and $0 \le \theta_1 < \theta_2 \le 2\pi$.

59) Use the information from the preceding exercise to calculate the integral $\iint_D (x^2+y^2)^3 dA$, where D is the unit disk.

Answer

$\frac{\pi}{4}$

60) Let $f(x,y) = \dfrac{F'(r)}{r}$ be a continuous radial function defined on the annular region $D = \{(r,\theta) \mid R_1 \le r \le R_2,\ 0 \le \theta \le 2\pi\}$, where $r = \sqrt{x^2+y^2}$, $0 < R_1 < R_2$, and F is a differentiable function.

Show that $\iint_D f(x,y)\,dA = 2\pi[F(R_2) - F(R_1)]$.

61) Apply the preceding exercise to calculate the integral $\iint_D \dfrac{e^{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}\,dx\,dy$ where D is the annular region between the circles of radii 1 and 2 situated in the third quadrant.

Answer

$\frac{1}{2}\pi e(e-1)$

62) Let f be a continuous function that can be expressed in polar coordinates as a function of θ only; that is, $f(x,y) = h(\theta)$, where $(x,y) \in D = \{(r,\theta) \mid R_1 \le r \le R_2,\ \theta_1 \le \theta \le \theta_2\}$, with $0 \le R_1 < R_2$ and $0 \le \theta_1 < \theta_2 \le 2\pi$.

Show that $\iint_D f(x,y)\,dA = \dfrac{1}{2}(R_2^2 - R_1^2)[H(\theta_2) - H(\theta_1)]$, where H is an antiderivative of h.

63) Apply the preceding exercise to calculate the integral $\iint_D \dfrac{y^2}{x^2}\,dA$, where $D = \{(r,\theta) \mid 1 \le r \le 2,\ \frac{\pi}{6} \le \theta \le \frac{\pi}{3}\}$.

Answer

$\sqrt{3} - \frac{\pi}{4}$

64) Let f be a continuous function that can be expressed in polar coordinates as a function of θ only; that is $f(x,y) = g(r)h(\theta)$, where $(x,y) \in \{(r,\theta) \mid R_1 \le r \le R_2,\ \theta_1 \le \theta \le \theta_2\}$ with $0 \le R_1 < R_2$ and $0 \le \theta_1 < \theta_2 \le 2\pi$. Show that

$$\iint_D f(x,y)\,dA = [G(R_2) - G(R_1)]\ [H(\theta_2) - H(\theta_1)],$$

where G and H are antiderivatives of g and h, respectively.

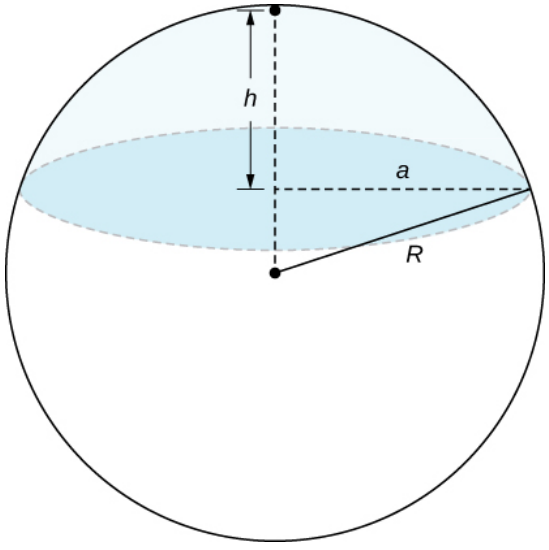
65) Evaluate $\iint_D \arctan\left(\dfrac{y}{x}\right)\sqrt{x^2+y^2}\,dA$, where $D = \{(r,\theta) \mid 2 \le r \le 3,\ \frac{\pi}{4} \le \theta \le \frac{\pi}{3}\}$.

Answer

$\frac{133}{864}\pi^2$

66) A spherical cap is the region of a sphere that lies above or below a given plane.

a. Show that the volume of the spherical cap in the figure below is $\frac{1}{6}\pi h(3a^2 + h^2)$.



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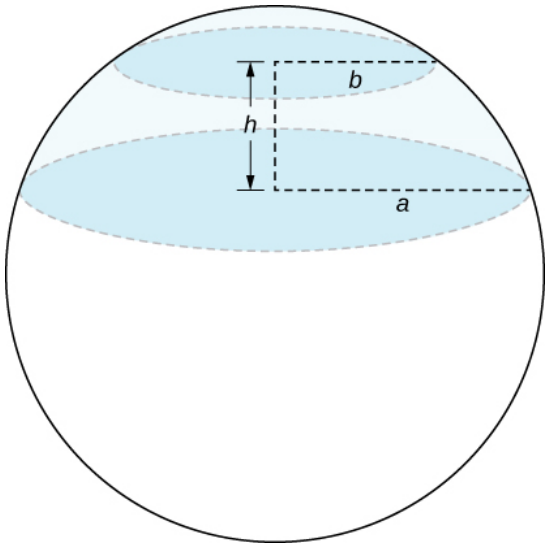
b. A spherical segment is the solid defined by intersecting a sphere with two parallel planes. If the distance between the planes is h show that the volume of the spherical segment in the figure below is $\frac{1}{6}\pi h(3a^2+3b^2+h^2)$.



67) In statistics, the joint density for two independent, normally distributed events with a mean $\mu=0$ and a standard distribution σ is defined by $p(x,y)=\frac{1}{2\pi\sigma^2}e^{-\frac{x^2+y^2}{2\sigma^2}}$. Consider (X,Y), the Cartesian coordinates of a ball in the resting position after it was released from a position on the z-axis toward the xy-plane. Assume that the coordinates of the ball are independently normally distributed with a mean $\mu=0$ and a standard deviation of σ (in feet). The probability that the ball will stop no more than a feet from the origin is given by

$$P[X^2+Y^2\le a^2]=\iint_D p(x,y)dy\,dx,$$

where D is the disk of radius a centered at the origin. Show that

$$P[X^2+Y^2\le a^2]=1-e^{-a^2/2\sigma^2}.$$

68) The double improper integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-x^2+y^2/2}\,dy\,dx$$

may be defined as the limit value of the double integrals $\iint_D e^{-x^2+y^2/2}\,dA$ over disks D_a of radii a centered at the origin, as a increases without bound; that is,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-x^2+y^2/2}dy\,dx=\lim_{a\to\infty}\iint_{D_a}e^{-x^2+y^2/2}\,dA.$$

Use polar coordinates to show that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-x^2+y^2/2}\,dy\,dx=2\pi$.

69) Show that $\int_{-\infty}^{\infty}e^{-x^2/2}\,dx=\sqrt{2\pi}$ by using the relation

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-x^2+y^2/2}\,dy\,dx=\left(\int_{-\infty}^{\infty}e^{-x^2/2}dx\right)\left(\int_{-\infty}^{\infty}e^{-y^2/2}dy\right).$$

CONTRIBUTORS

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.
- Problems 1, 2, 34 - 37 and 50 - 51 are from Apex Calculus, Chapter 13.3
- Edited by Paul Seeburger (Monroe Community College)

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15.4: Triple Integrals

Learning Objectives

- Recognize when a function of three variables is integrable over a rectangular box.
- Evaluate a triple integral by expressing it as an iterated integral.
- Recognize when a function of three variables is integrable over a closed and bounded region.
- Simplify a calculation by changing the order of integration of a triple integral.
- Calculate the average value of a function of three variables.

Previously, we discussed the double integral of a function $f(x, y)$ of two variables over a rectangular region in the plane. In this section we define the triple integral of a function $f(x, y, z)$ of three variables over a rectangular solid box in space, $\mathbb{R}^3$. Later in this section we extend the definition to more general regions in $\mathbb{R}^3$.

Integrable Functions of Three Variables

We can define a rectangular box B in $\mathbb{R}^3$ as

$$B = \{(x, y, z) \,|\, a \leq x \leq b,\, c \leq y \leq d,\, e \leq z \leq f\}.$$

We follow a similar procedure to what we did in previously. We divide the interval $[a, b]$ into l subintervals $[x_{i-1}, x_i]$ of equal length Δx with

$$\Delta x = \frac{x_i - x_{i-1}}{l},$$

divide the interval $[c, d]$ into m subintervals $[y_{i-1}, y_i]$ of equal length Δy with

$$\Delta y = \frac{y_j - y_{j-1}}{m},$$

and divide the interval $[e, f]$ into n subintervals $[z_{i-1}, z_i]$ of equal length Δz with

$$\Delta z = \frac{z_k - z_{k-1}}{n}$$

Then the rectangular box B is subdivided into lmn subboxes:

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{i-1}, y_i] \times [z_{i-1}, z_i],$$

as shown in Figure 15.4.1.

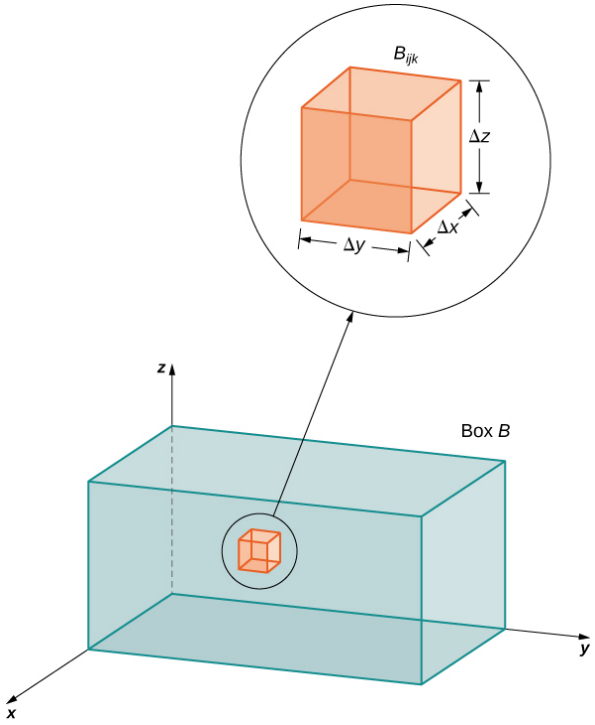

Figure 15.4.1: A rectangular box in $\mathbb{R}^3$ divided into subboxes by planes parallel to the coordinate planes.

For each i, j, and k, consider a sample point $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ in each sub-box B_{ijk}. We see that its volume is $\Delta V = \Delta x \Delta y \Delta z$. Form the triple Riemann sum

$$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\,\Delta x \Delta y \Delta z.$$

We define the triple integral in terms of the limit of a triple Riemann sum, as we did for the double integral in terms of a double Riemann sum.

Definition: The triple integral

The triple integral of a function $f(x, y, z)$ over a rectangular box B is defined as

$$\lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\,\Delta x \Delta y \Delta z = \iiint_B f(x, y, z)\,dV$$

if this limit exists.

When the triple integral exists on B the function $f(x, y, z)$ is said to be integrable on B. Also, the triple integral exists if $f(x, y, z)$ is continuous on B. Therefore, we will use continuous functions for our examples. However, continuity is sufficient but not necessary; in other words, f is bounded on B and continuous except possibly on the boundary of B. The sample point $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ can be any point in the rectangular sub-box B_{ijk} and all the properties of a double integral apply to a triple integral. Just as the double integral has many practical applications, the triple integral also has many applications, which we discuss in later sections.

Now that we have developed the concept of the triple integral, we need to know how to compute it. Just as in the case of the double integral, we can have an iterated triple integral, and consequently, a version of **Fubini's theorem** for triple integrals exists.

Fubini's Theorem for Triple Integrals

If $f(x, y, z)$ is continuous on a rectangular box $B = [a, b] \times [c, d] \times [e, f]$, then

$$\iiint_B f(x, y, z)\,dV = \int_e^f \int_c^d \int_a^b f(x, y, z)\,dx\,dy\,dz.$$

This integral is also equal to any of the other five possible orderings for the iterated triple integral.

For a, b, c, d, e and f real numbers, the iterated triple integral can be expressed in six different orderings:

$$\int_e^f \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz = \int_e^f \left(\int_c^d \left(\int_a^b f(x,y,z)\,dx \right) dy \right) dz \tag{15.4.1}$$

$$= \int_c^d \left(\int_e^f \left(\int_a^b f(x,y,z)\,dx \right) dz \right) dy \tag{15.4.2}$$

$$= \int_a^b \left(\int_e^f \left(\int_c^d f(x,y,z)\,dy \right) dz \right) dx \tag{15.4.3}$$

$$= \int_e^f \left(\int_a^b \left(\int_c^d f(x,y,z)\,dy \right) dx \right) dz \tag{15.4.4}$$

$$= \int_c^d \left(\int_a^b \left(\int_e^f f(x,y,z)\,dz \right) dx \right) dy \tag{15.4.5}$$

$$= \int_a^b \left(\int_c^d \left(\int_e^f f(x,y,z)\,dz \right) dy \right) dx \tag{15.4.6}$$

For a rectangular box, the order of integration does not make any significant difference in the level of difficulty in computation. We compute triple integrals using Fubini's Theorem rather than using the Riemann sum definition. We follow the order of integration in the same way as we did for double integrals (that is, from inside to outside).

Example 15.4.1: Evaluating a Triple Integral

Evaluate the triple integral

$$\int_{z=0}^{z=1} \int_{y=2}^{y=4} \int_{x=-1}^{x=5} (x + yz^2)\,dx\,dy\,dz.$$

Solution

The order of integration is specified in the problem, so integrate with respect to x first, then y, and then z.

$$\begin{aligned}
&\int_{z=0}^{z=1} \int_{y=2}^{y=4} \int_{x=-1}^{x=5} (x + yz^2)\,dx\,dy\,dz \\
&= \int_{z=0}^{z=1} \int_{y=2}^{y=4} \left[\frac{x^2}{2} + xyz^2 \Big|_{x=-1}^{x=5} \right] dy\,dz && \text{Integrate with respect to } x. \\
&= \int_{z=0}^{z=1} \int_{y=2}^{y=4} \left[12 + 6yz^2 \right]\, dy\,dz && \text{Evaluate.} \\
&= \int_{z=0}^{z=1} \left[12y + 6\frac{y^2}{2}z^2 \Big|_{y=2}^{y=4} \right] dz && \text{Integrate with respect to } y. \\
&= \int_{z=0}^{z=1} \left[24 + 36z^2 \right] dz && \text{Evaluate.} \\
&= \left[24z + 36\frac{z^3}{3} \right]_{z=0}^{z=1} && \text{Integrate with respect to } z. \\
&= 36. && \text{Evaluate.}
\end{aligned}$$

Example 15.4.2: Evaluating a Triple Integral

Evaluate the triple integral

$$\iiint_B x^2yz\,dV$$

where $B = \{(x,y,z) \mid -2 \le x \le 1,\ 0 \le y \le 3,\ 1 \le z \le 5\}$ as shown in Figure 15.4.2.

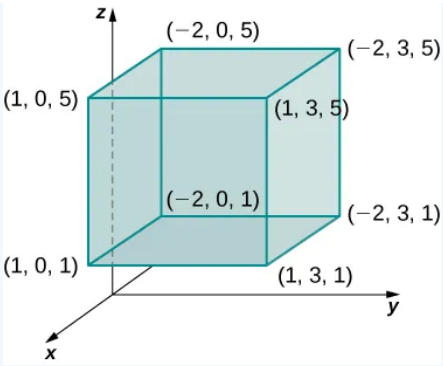

Figure 15.4.2: Evaluating a triple integral over a given rectangular box.

Solution

The order is not specified, but we can use the iterated integral in any order without changing the level of difficulty. Choose, say, to integrate y first, then x, and then z.

$$\begin{aligned}\iiint_B x^2yz\,dV &= \int_1^5\int_{-2}^1\int_0^3 [x^2yz]\,dy\,dx\,dz\\ &= \int_1^5\int_{-2}^1 \left[x^2\frac{y^2}{2}z\Big|_0^3\right] dx\,dz\\ &= \int_1^5\int_{-2}^1 \frac{9}{2}x^2z\,dx\,dz\\ &= \int_1^5 \left[\frac{9}{2}\frac{x^3}{3}z\Big|_{-2}^1\right] dz\\ &= \int_1^5 \frac{27}{2}z\,dz\\ &= \frac{27}{2}\frac{z^2}{2}\Big|_1^5 = \frac{27}{4}(25-1) = 162.\end{aligned}$$

Now try to integrate in a different order just to see that we get the same answer. Choose to integrate with respect to x first, then z, then y

$$\begin{aligned}\iiint_B x^2yz\,dV &= \int_0^3\int_1^5\int_{-2}^1 [x^2yz]\,dx\,dz\,dy\\ &= \int_0^3\int_1^5 \left[\frac{x^3}{3}yz\Big|_{-2}^1\right] dz\,dy\\ &= \int_0^3\int_1^5 3yz\,dz\,dy\\ &= \int_0^3 \left[3y\frac{z^2}{2}\Big|_1^5\right] dy\\ &= \int_0^3 36y\,dy\\ &= 36\frac{y^2}{2}\Big|_0^3 = 18(9-0) = 162.\end{aligned}$$

? Exercise 15.4.1

Evaluate the triple integral

$$\iiint_B z\sin x\cos y\,dV$$

where $B = \{(x,y,z)\,|\,0\le x\le\pi,\ \frac{3\pi}{2}\le y\le 2\pi,\ 1\le z\le 3\}$.

Hint

Follow the steps in the previous example.

Answer

$$\iiint_B z \sin x \cos y \, dV = 8$$

Triple Integrals over a General Bounded Region

We now expand the definition of the triple integral to compute a triple integral over a more **general bounded region** E in $\mathbb{R}^3$. The general bounded regions we will consider are of three types. First, let D be the bounded region that is a projection of E onto the xy-plane. Suppose the region E in $\mathbb{R}^3$ has the form

$$E = \{(x, y, z) \,|\, (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}.$$

For two functions $z = u_1(x, y)$ and $u_2(x, y)$, such that $u_1(x, y) \leq u_2(x, y)$ for all (x, y) in D as shown in the following figure.

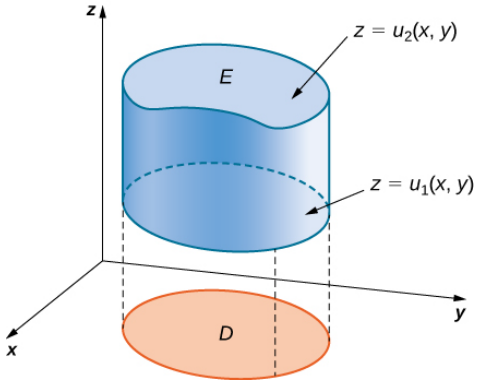

Figure 15.4.3: We can describe region E as the space between $u_1(x, y)$ and $u_2(x, y)$ above the projection D of E onto the xy-plane.

Triple Integral over a General Region

The triple integral of a continuous function $f(x, y, z)$ over a general three-dimensional region

$$E = \{(x, y, z) \,|\, (x, y) \in D, \, u_1(x, y) \leq z \leq u_2(x, y)\}$$

in $\mathbb{R}^3$, where D is the projection of E onto the xy-plane, is

$$\iiint_E f(x, y, z) \, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z) \, dz \right] dA.$$

Similarly, we can consider a general bounded region D in the xy-plane and two functions $y = u_1(x, z)$ and $y = u_2(x, z)$ such that $u_1(x, z) \leq u_2(x, z)$ for all (x, z) in D. Then we can describe the solid region E in $\mathbb{R}^3$ as

$$E = \{(x, y, z) \,|\, (x, z) \in D, \, u_1(x, z) \leq z \leq u_2(x, z)\}$$

where D is the projection of E onto the xy-plane and the triple integral is

$$\iiint_E f(x, y, z) \, dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z) \, dy \right] dA.$$

Finally, if D is a general bounded region in the xy-plane and we have two functions $x = u_1(y, z)$ and $x = u_2(y, z)$ such that $u_1(y, z) \leq u_2(y, z)$ for all (y, z) in D, then the solid region E in $\mathbb{R}^3$ can be described as

$$E = \{(x, y, z) \,|\, (y, z) \in D, \, u_1(y, z) \leq z \leq u_2(y, z)\}$$

where D is the projection of E onto the xy-plane and the triple integral is

$$\iiint_E f(x, y, z) \, dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z) \, dx \right] dA.$$

Note that the region D in any of the planes may be of Type I or Type II as described in previously. If D in the xy-plane is of Type I (Figure 15.4.4), then

$$E = \{(x, y, z) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x),\ u_1(x, y) \le z \le u_2(x, y)\}.$$

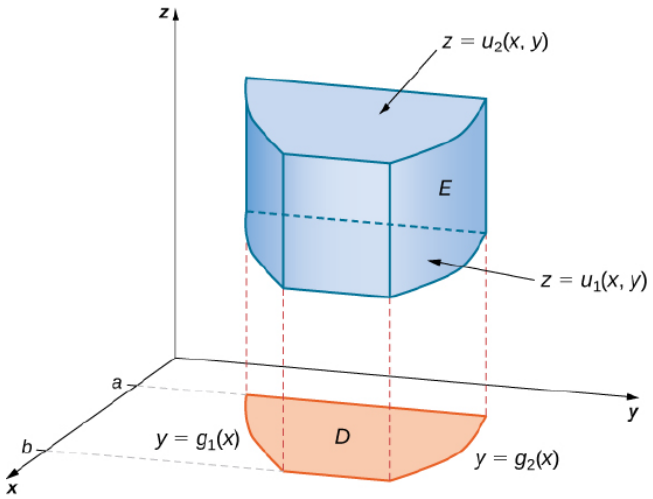

Figure 15.4.4: A box E where the projection D in the xy-plane is of Type I.

Then the triple integral becomes

$$\iiint_E f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\, dy\, dx.$$

If D in the xy-plane is of Type II (Figure 15.4.5), then

$$E = \{(x, y, z) \mid c \le x \le d, h_1(x) \le y \le h_2(x),\ u_1(x, y) \le z \le u_2(x, y)\}.$$

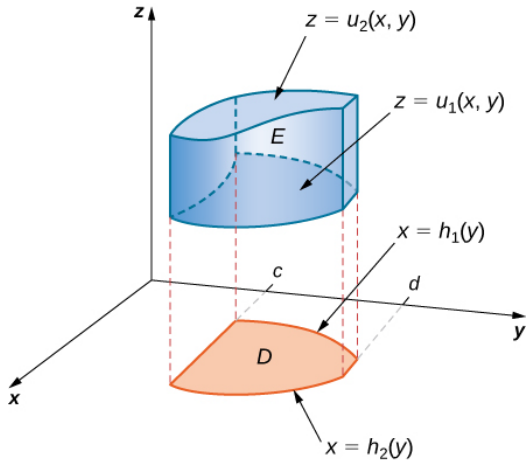

Figure 15.4.5: A box E where the projection D in the xy-plane is of Type II.

Then the triple integral becomes

$$\iiint_E f(x, y, z)\, dV = \int_{y=c}^{y=d} \int_{x=h_1(y)}^{x=h_2(y)} \int_{z=u_1(x,y)}^{z=u_2(x,y)} f(x, y, z)\, dz\, dx\, dy.$$

Example 15.4.3A: Evaluating a Triple Integral over a General Bounded Region

Evaluate the triple integral of the function $f(x, y, z) = 5x - 3y$ over the solid tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

Solution

Figure 15.4.6 shows the solid tetrahedron E and its projection D on the xy-plane.

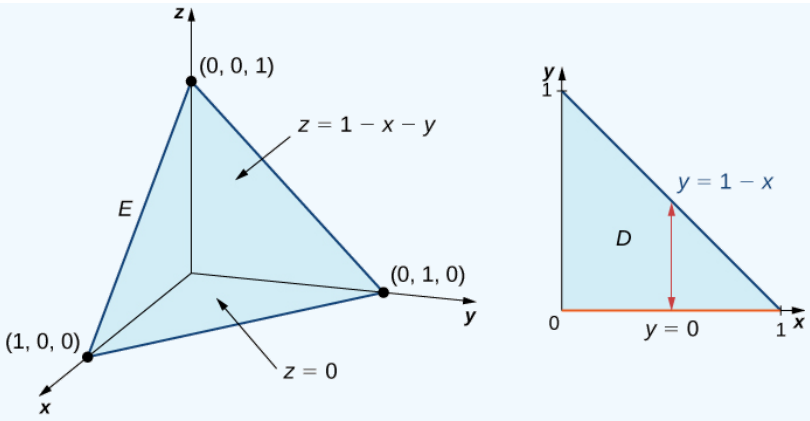

Figure 15.4.6: The solid E has a projection D on the xy-plane of Type I.

We can describe the solid region tetrahedron as

$$E = \{(x, y, z) \mid 0 \le x \le 1,\ 0 \le y \le 1 - x,\ 0 \le z \le 1 - x - y\}.$$

Hence, the triple integral is

$$\iiint_E f(x, y, z)\, dV = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} (5x - 3y)\, dz\, dy\, dx.$$

To simplify the calculation, first evaluate the integral $\int_{z=0}^{z=1-x-y} (5x - 3y)\, dz$. We have

$$\int_{z=0}^{z=1-x-y} (5x - 3y)\, dz = (5x - 3y)z\Big|_{z=0}^{z=1-x-y} = (5x - 3y)(1 - x - y).$$

Now evaluate the integral

$$\int_{y=0}^{y=1-x} (5x - 3y)(1 - x - y)\, dy,$$

obtaining

$$\int_{y=0}^{y=1-x} (5x - 3y)(1 - x - y)\, dy = \frac{1}{2}(x - 1)^2(6x - 1).$$

Finally evaluate

$$\int_{x=0}^{x=1} \frac{1}{2}(x - 1)^2(6x - 1)\, dx = \frac{1}{12}.$$

Putting it all together, we have

$$\iiint_E f(x, y, z)\, dV = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} (5x - 3y)\, dz\, dy\, dx = \frac{1}{12}.$$

Just as we used the double integral

$$\iint_D 1\, dA$$

to find the area of a general bounded region D we can use

$$\iiint_E 1\, dV$$

to find the volume of a general solid bounded region E. The next example illustrates the method.

Example 15.4.3B: Finding a Volume by Evaluating a Triple Integral

Find the volume of a right pyramid that has the square base in the xy-plane $[-1,1]\times[-1,1]$ and vertex at the point $(0,0,1)$ as shown in the following figure.

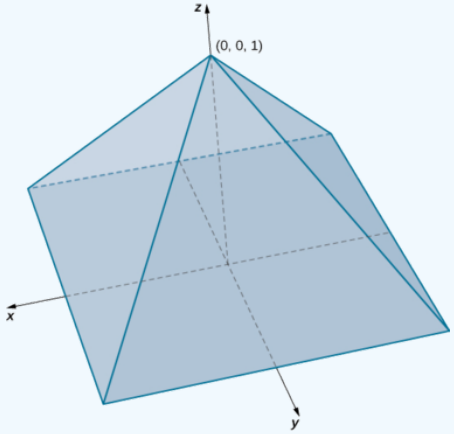

Figure 15.4.7: Finding the volume of a pyramid with a square base.

Solution

In this pyramid the value of z changes from 0 to 1 and at each height z the cross section of the pyramid for any value of z is the square

$$[-1+z,\,1-z]\times[-1+z,\,1-z].$$

Hence, the volume of the pyramid is

$$\iiint_E 1\,dV$$

where

$$E=\{(x,y,z)\,|\,0\le z\le 1,\,-1+z\le y\le 1-z,\,-1+z\le x\le 1-z\}.$$

Thus, we have

$$\begin{aligned}\iiint_E 1\,dV &= \int_{z=0}^{z=1}\int_{y=-1+z}^{y=1-z}\int_{x=-1+z}^{x=1-z} 1\,dx\,dy\,dz\\ &= \int_{z=0}^{z=1}\int_{y=-1+z}^{y=1-z}(2-2z)\,dy\,dz\\ &= \int_{z=0}^{z=1}(2-2z)^2\,dz=\frac{4}{3}.\end{aligned}$$

Hence, the volume of the pyramid is $\frac{4}{3}$ cubic units.

Exercise 15.4.3

Consider the solid sphere $E=\{(x,y,z)\,|\,x^2+y^2+z^2=9\}$. Write the triple integral

$$\iiint_E f(x,y,z)\,dV$$

for an arbitrary function f as an iterated integral. Then evaluate this triple integral with $f(x,y,z)=1$. Notice that this gives the volume of a sphere using a triple integral.

Hint

Follow the steps in the previous example. Use symmetry.

Answer

$$\iiint_E 1\,dV = 8\int_{x=-3}^{x=3}\int_{y=-\sqrt{9-z^2}}^{y=\sqrt{9-z^2}}\int_{z=-\sqrt{9-x^2-y^2}}^{z=\sqrt{9-x^2-y^2}} 1\,dz\,dy\,dx$$
$$= 36\pi \text{ cubic units.}$$

Changing the Order of Integration

As we have already seen in double integrals over general bounded regions, changing the order of the integration is done quite often to simplify the computation. With a triple integral over a rectangular box, the order of integration does not change the level of difficulty of the calculation. However, with a triple integral over a general bounded region, choosing an appropriate order of integration can simplify the computation quite a bit. Sometimes making the change to polar coordinates can also be very helpful. We demonstrate two examples here.

Example 15.4.4: Changing the Order of Integration

Consider the iterated integral

$$\int_{x=0}^{x=1}\int_{y=0}^{y=x^2}\int_{z=0}^{z=y} f(x,y,z)\,dz\,dy\,dx.$$

The order of integration here is first with respect to z, then y, and then x. Express this integral by changing the order of integration to be first with respect to x, then z, and then y. Verify that the value of the integral is the same if we let $f(x,y,z) = xyz$.

Solution

The best way to do this is to sketch the region E and its projections onto each of the three coordinate planes. Thus, let

$$E = \{(x,y,z)\,|\,0 \le x \le 1,\ 0 \le y \le x^2,\ 0 \le z \le y\}.$$

and

$$\int_{x=0}^{x=1}\int_{y=0}^{y=x^2}\int_{z=0}^{z=x^2} f(x,y,z)\,dz\,dy\,dx = \iiint_E f(x,y,z)\,dV.$$

We need to express this triple integral as

$$\int_{y=c}^{y=d}\int_{z=v_1(y)}^{z=v_2(y)}\int_{x=u_1(y,z)}^{x=u_2(y,z)} f(x,y,z)\,dx\,dz\,dy.$$

Knowing the region E we can draw the following projections (Figure 15.4.8):

on the xy-plane is $D_1 = \{(x,y)\,|\,0 \le x \le 1,\ 0 \le y \le x^2\} = \{(x,y)\,|\,0 \le y \le 1,\ \sqrt{y} \le x \le 1\}$,

on the yz-plane is $D_2 = \{(y,z)\,|\,0 \le y \le 1,\ 0 \le z \le y^2\}$, and

on the xz-plane is $D_3 = \{(x,z)\,|\,0 \le x \le 1,\ 0 \le z \le x^2\}$.

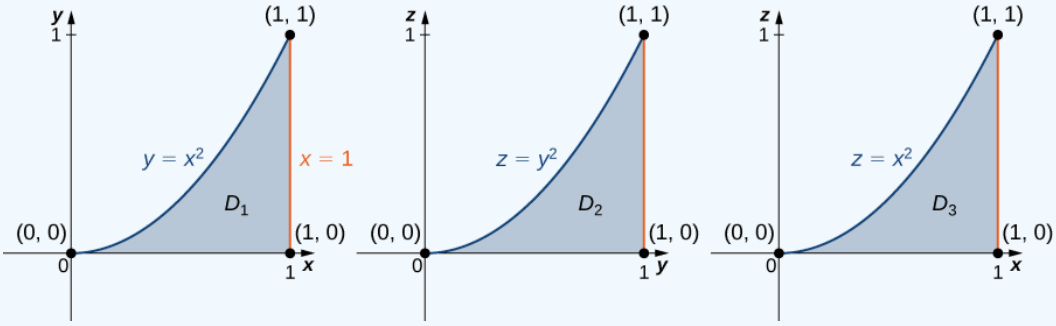

Figure 15.4.8. The three cross sections of E on the three coordinate planes.

Now we can describe the same region E as $\{(x,y,z)\,|\,0 \le y \le 1,\ 0 \le z \le y^2,\ \sqrt{y} \le x \le 1\}$, and consequently, the triple integral becomes

$$\int_{y=c}^{y=d}\int_{z=v_1(y)}^{z=v_2(y)}\int_{x=u_1(y,z)}^{x=u_2(y,z)} f(x,y,z)\,dx\,dz\,dy = \int_{y=0}^{y=1}\int_{z=0}^{z=x^2}\int_{x=\sqrt{y}}^{x=1} f(x,y,z)\,dx\,dz\,dy$$

Now assume that $f(x,y,z)=xyz$ in each of the integrals. Then we have

$$\begin{aligned}\int_{x=0}^{x=1}\int_{y=0}^{y=x^2}\int_{z=0}^{z=y^2} xyz\,dz\,dy\,dx &= \int_{x=0}^{x=1}\int_{y=0}^{y=x^2}\left[xy\frac{z^2}{2}\Big|_{z=0}^{z=y^2}\right]dy\,dx\\ &= \int_{x=0}^{x=1}\int_{y=0}^{y=x^2}\left(x\frac{y^5}{2}\right)dy\,dx\\ &= \int_{x=0}^{x=1}\left[x\frac{y^6}{12}\Big|_{y=0}^{y=x^2}\right]dx\\ &= \int_{x=0}^{x=1}\frac{x^{13}}{12}dx = \frac{x^{14}}{168}\Big|_{x=0}^{x=1}\\ &= \frac{1}{168},\end{aligned}$$

$$\begin{aligned}\int_{y=0}^{y=1}\int_{z=0}^{z=y^2}\int_{x=\sqrt{y}}^{x=1} xyz\,dx\,dz\,dy &= \int_{y=0}^{y=1}\int_{z=0}^{z=y^2}\left[yz\frac{x^2}{2}\Big|_{\sqrt{y}}^{1}\right]dz\,dy\\ &= \int_{y=0}^{y=1}\int_{z=0}^{z=y^2}\left(\frac{yz}{2}-\frac{y^2z}{2}\right)dz\,dy\\ &= \int_{y=0}^{y=1}\left[\frac{yz^2}{4}-\frac{y^2z^2}{4}\Big|_{z=0}^{z=y^2}\right]dy\\ &= \int_{y=0}^{y=1}\left(\frac{y^5}{4}-\frac{y^6}{4}\right)dy\\ &= \left(\frac{y^6}{24}-\frac{y^7}{28}\right)\Big|_{y=0}^{y=1}\\ &= \frac{1}{168}.\end{aligned}$$

The answers match.

? Exercise 15.4.4

Write five different iterated integrals equal to the given integral

$$\int_{z=0}^{z=4}\int_{y=0}^{y=4-z}\int_{x=0}^{x=\sqrt{y}} f(x,y,z)\,dx\,dy\,dz.$$

Hint

Follow the steps in the previous example, using the region E as $\{(x,y,z)\,|\,0\le z\le 4,\ 0\le y\le 4-z,\ 0\le x\le\sqrt{y}\}$, and describe and sketch the projections onto each of the three planes, five different times.

Answer

$$(i)\ \int_{z=0}^{z=4}\int_{x=0}^{x=\sqrt{4-z}}\int_{y=x^2}^{y=4-z} f(x,y,z)\,dy\,dx\,dz,\ (ii)\ \int_{y=0}^{y=4}\int_{z=0}^{z=4-y}\int_{x=0}^{x=\sqrt{y}} f(x,y,z)\,dx\,dz\,dy,\ (iii)\ \int_{y=0}^{y=4}\int_{x=0}^{x=\sqrt{y}}\int_{z=0}^{Z=4-y} f(x,y,z)\,dz\,dx\,dy,$$

$$(iv)\ \int_{x=0}^{x=2}\int_{y=x^2}^{y=4}\int_{z=0}^{z=4-y} f(x,y,z)\,dz\,dy\,dx,\ (v)\ \int_{x=0}^{x=2}\int_{z=0}^{z=4-x^2}\int_{y=x^2}^{y=4-z} f(x,y,z)\,dy\,dz\,dx$$

✓ Example 15.4.5: Changing Integration Order and Coordinate Systems

Evaluate the triple integral

$$\iiint_E \sqrt{x^2+z^2}\,dV,$$

where E is the region bounded by the paraboloid $y=x^2+z^2$ (Figure 15.4.9) and the plane $y=4$.

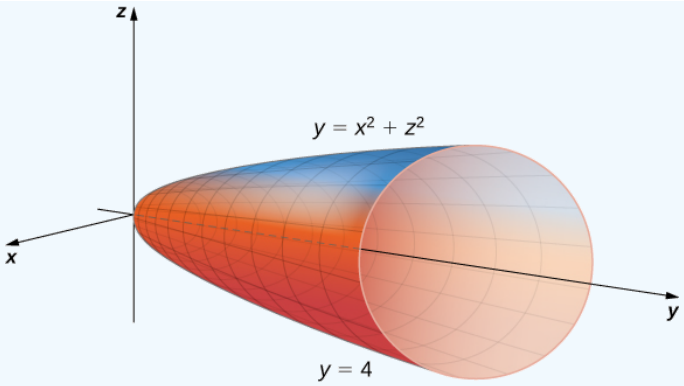

Figure 15.4.9. Integrating a triple integral over a paraboloid.

Solution

The projection of the solid region E onto the xy-plane is the region bounded above by $y = 4$ and below by the parabola $y = x^2$ as shown.

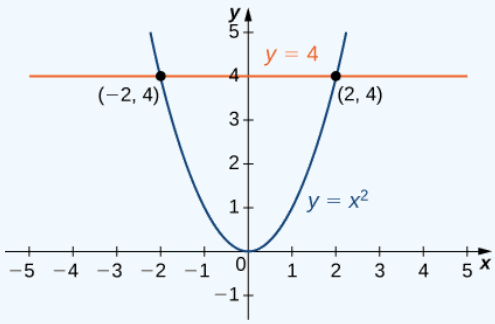

Figure 15.4.10. Cross section in the xy-plane of the paraboloid in Figure 15.4.9.

Thus, we have

$$E = \{(x, y, z) \mid -2 \le x \le 2,\ x^2 \le y \le 4,\ -\sqrt{y - x^2} \le z \le \sqrt{y - x^2}\}.$$

The triple integral becomes

$$\iiint_E \sqrt{x^2 + z^2}\, dV = \int_{x=-2}^{x=2} \int_{y=x^2}^{y=4} \int_{z=-\sqrt{y-x^2}}^{z=\sqrt{y-x^2}} \sqrt{x^2 + z^2}\, dz\, dy\, dx.$$

This expression is difficult to compute, so consider the projection of E onto the xz-plane. This is a circular disc $x^2 + z^2 \le 4$. So we obtain

$$\begin{aligned}\iiint_E \sqrt{x^2 + z^2}\, dV &= \int_{x=-2}^{x=2} \int_{y=x^2}^{y=4} \int_{z=-\sqrt{y-x^2}}^{z=\sqrt{y-x^2}} \sqrt{x^2 + z^2}\, dz\, dy\, dx \\ &= \int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} \int_{y=x^2+z^2}^{y=4} \sqrt{x^2 + z^2}\, dy\, dz\, dx.\end{aligned}$$

Here the order of integration changes from being first with respect to z then y and then x to being first with respect to y then to z and then to x. It will soon be clear how this change can be beneficial for computation. We have

$$\int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} \int_{y=x^2+z^2}^{y=4} \sqrt{x^2 + z^2}\, dy\, dz\, dx = \int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} (4 - x^2 - z^2)\sqrt{x^2 + z^2}\, dz\, dx.$$

Now use the polar substitution $x = r\cos\theta$, $z = r\sin\theta$, and $dz\, dx = r\, dr\, d\theta$ in the xz-plane. This is essentially the same thing as when we used polar coordinates in the xy-plane, except we are replacing y by z. Consequently the limits of integration change and we have, by using $r^2 = x^2 + z^2$,

$$\begin{aligned}\int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} (4 - x^2 - z^2)\sqrt{x^2 + z^2}\, dz\, dx &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} (4 - r^2) r r\, dr\, d\theta \\ &= \int_0^{2\pi} \left[\frac{4r^3}{3} - \frac{r^5}{5}\Big|^2\right] d\theta = \int_0^{2\pi} \frac{64}{15}\, d\theta = \frac{128\pi}{15}\end{aligned}$$

Average Value of a Function of Three Variables

Recall that we found the average value of a function of two variables by evaluating the double integral over a region on the plane and then dividing by the area of the region. Similarly, we can find the average value of a function in three variables by evaluating the triple integral over a solid region and then dividing by the volume of the solid.

Average Value of a Function of Three Variables

If $f(x, y, z)$ is integrable over a solid bounded region E with positive volume $V(E)$, then the average value of the function is

$$f_{ave} = \frac{1}{V(E)} \iiint_E f(x, y, z)\, dV.$$

Note that the volume is

$$V(E) = \iiint_E 1\, dV.$$

Example 15.4.6: Finding an Average Temperature

The temperature at a point (x, y, z) of a solid E bounded by the coordinate planes and the plane $x + y + z = 1$ is $T(x, y, z) = (xy + 8z + 20)$ °C. Find the average temperature over the solid.

Solution

Use the theorem given above and the triple integral to find the numerator and the denominator. Then do the division. Notice that the plane $x + y + z = 1$ has intercepts $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. The region E looks like

$$E = \{(x, y, z) \mid 0 \le x \le 1,\ 0 \le y \le 1 - x,\ 0 \le z \le 1 - x - y\}.$$

Hence the triple integral of the temperature is

$$\iiint_E f(x, y, z)\, dV = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} (xy + 8z + 20)\, dz\, dy\, dx = \frac{147}{40}.$$

The volume evaluation is

$$V(E) = \iiint_E 1\, dV = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} 1\, dz\, dy\, dx = \frac{1}{6}.$$

Hence the average value is

$$T_{ave} = \frac{147/40}{1/6} = \frac{6(147)}{40} = \frac{441}{20}\ ^\circ\text{C}$$

.

Exercise 15.4.6

Find the average value of the function $f(x, y, z) = xyz$ over the cube with sides of length 4 units in the first octant with one vertex at the origin and edges parallel to the coordinate axes.

Hint

Follow the steps in the previous example.

Answer

$f_{ave} = 8$

Key Concepts

- To compute a triple integral we use Fubini's theorem, which states that if $f(x, y, z)$ is continuous on a rectangular box $B = [a, b] \times [c, d] \times [e, f]$, then

$$\iiint_B f(x, y, z)\, dV = \int_e^f \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

  and is also equal to any of the other five possible orderings for the iterated triple integral.
- To compute the volume of a general solid bounded region E we use the triple integral

$$V(E) = \iiint_E 1\, dV.$$

- Interchanging the order of the iterated integrals does not change the answer. As a matter of fact, interchanging the order of integration can help simplify the computation.
- To compute the average value of a function over a general three-dimensional region, we use

$$f_{ave} = \frac{1}{V(E)} \iiint_E f(x, y, z)\, dV.$$

Key Equations

- **Triple integral**

$$\lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta x \Delta y \Delta z = \iiint_B f(x, y, z)\, dV$$

Glossary

triple integral

the triple integral of a continuous function $f(x, y, z)$ over a rectangular solid box B is the limit of a Riemann sum for a function of three variables, if this limit exists

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15.4E: Exercises for Section 15.4

In exercises 1 - 4, evaluate the triple integrals over the rectangular solid box B.

1. $\iiint_B (2x + 3y^2 + 4z^3)\, dV$, where $B = \{(x, y, z) \mid 0 \le x \le 1,\ 0 \le y \le 2,\ 0 \le z \le 3\}$

   Answer

   192

2. $\iiint_B (xy + yz + xz)\, dV$, where $B = \{(x, y, z) \mid 1 \le x \le 2,\ 0 \le y \le 2,\ 1 \le z \le 3\}$
3. $\iiint_B (x \cos y + z)\, dV$, where $B = \{(x, y, z) \mid 0 \le x \le 1,\ 0 \le y \le \pi,\ -1 \le z \le 1\}$

   Answer

   0

4. $\iiint_B (z \sin x + y^2)\, dV$, where $B = \{(x, y, z) \mid 0 \le x \le \pi,\ 0 \le y \le 1,\ -1 \le z \le 2\}$

In exercises 5 - 8, change the order of integration by integrating first with respect to z, then x, then y.

5. $\int_0^1 \int_1^2 \int_2^3 (x^2 + \ln y + z)\, dx\, dy\, dz$

   Answer

   $$\int_0^1 \int_1^2 \int_2^3 (x^2 + \ln y + z)\, dx\, dy\, dz = \frac{35}{6} + 2 \ln 2$$

6. $\int_0^1 \int_{-1}^1 \int_0^3 (ze^x + 2y)\, dx\, dy\, dz$
7. $\int_{-1}^2 \int_1^3 \int_0^4 \left(x^2 z + \frac{1}{y}\right) dx\, dy\, dz$

   Answer

   $$\int_{-1}^2 \int_1^3 \int_0^4 \left(x^2 z + \frac{1}{y}\right) dx\, dy\, dz = 64 + 12 \ln 3$$

8. $\int_1^2 \int_{-2}^{-1} \int_0^1 \frac{x + y}{z}\, dx\, dy\, dz$

9. Let F, G, and H be continuous functions on $[a, b]$, $[c, d]$, and $[e, f]$, respectively, where a, b, c, d, e, and f are real numbers such that $a < b$, $c < d$, and $e < f$. Show that

$$\int_a^b \int_c^d \int_e^f F(x)\, G(y)\, H(z)\, dz\, dy\, dx = \left(\int_a^b F(x)\, dx\right) \left(\int_c^d G(y)\, dy\right) \left(\int_e^f H(z)\, dz\right).$$

10. Let F, G, and H be differential functions on $[a, b]$, $[c, d]$, and $[e, f]$, respectively, where a, b, c, d, e, and f are real numbers such that $a < b$, $c < d$, and $e < f$. Show that

$$\int_a^b \int_c^d \int_e^f F'(x)\, G'(y)\, H'(z)\, dz\, dy\, dx = [F(b) - F(a)]\, [G(d) - G(c)]\, H(f) - H(e)].$$

In exercises 11 - 14, evaluate the triple integrals over the bounded region

$E = \{(x, y, z) \mid a \le x \le b,\ h_1(x) \le y \le h_2(x),\ e \le z \le f\}$.

11. $\iiint_E (2x + 5y + 7z)\, dV$, where $E = \{(x, y, z) \mid 0 \le x \le 1,\ 0 \le y \le -x + 1,\ 1 \le z \le 2\}$

    Answer

    $\frac{77}{12}$

12. $\iiint_E (y \ln x + z)\, dV$, where $E = \{(x, y, z) \mid 1 \le x \le e,\ 0 \le y \ln x,\ 0 \le z \le 1\}$

13. $\iiint_E (\sin x + \sin y)\, dV$, where $E = \{(x, y, z) \mid 0 \le x \le \frac{\pi}{2},\ -\cos x \le y \cos x,\ -1 \le z \le 1\}$

Answer

2

14. $\iiint_E (xy + yz + xz) dV$ where $E = \{(x, y, z) \mid 0 \le x \le 1,\ -x^2 \le y \le x^2,\ 0 \le z \le 1\}$

In exercises 15 - 18, evaluate the triple integrals over the indicated bounded region E.

15. $\iiint_E (x + 2yz)\, dV$, where $E = \{(x, y, z) \mid 0 \le x \le 1,\ 0 \le y \le x,\ 0 \le z \le 5 - x - y\}$

Answer

$\frac{430}{120}$

16. $\iiint_E (x^3 + y^3 + z^3)\, dV$, where $E = \{(x, y, z) \mid 0 \le x \le 2,\ 0 \le y \le 2x,\ 0 \le z \le 4 - x - y\}$

17. $\iiint_E y\, dV$, where $E = \{(x, y, z) \mid -1 \le x \le 1,\ -\sqrt{1 - x^2} \le y \le \sqrt{1 - x^2},\ 0 \le z \le 1 - x^2 - y^2\}$

Answer

0

18. $\iiint_E x\, dV$, where $E = \{(x, y, z) \mid -2 \le x \le 2,\ -4\sqrt{1 - x^2} \le y \le \sqrt{4 - x^2},\ 0 \le z \le 4 - x^2 - y^2\}$

In exercises 19 - 22, evaluate the triple integrals over the bounded region E of the form

$E = \{(x, y, z) \mid g_1(y) \le x \le g_2(y),\ c \le y \le d,\ e \le z \le f\}$.

19. $\iiint_E x^2\, dV$, where $E = \{(x, y, z) \mid 1 - y^2 \le x \le y^2 - 1,\ -1 \le y \le 1,\ 1 \le z \le 2\}$

Answer

$-\frac{64}{105}$

20. $\iiint_E (y + \sin x)\, dV$, where $E = \{(x, y, z) \mid -y^4 \le x \le y^4,\ 0 \le y \le 2,\ 0 \le z \le 4\}$

21. $\iiint_E (x - yz)\, dV$, where $E = \{(x, y, z) \mid -y^6 \le x \le \sqrt{y},\ 0 \le y \le 1x,\ -1 \le z \le 1\}$

Answer

$\frac{11}{26}$

22. $\iiint_E z\, dV$, where $E = \{(x, y, z) \mid 2 - 2y \le x \le 2 + \sqrt{y},\ 0 \le y \le 1x,\ 2 \le z \le 3\}$

In exercises 23 - 26, evaluate the triple integrals over the bounded region

$E = \{(x, y, z) \mid g_1(y) \le x \le g_2(y),\ c \le y \le d,\ u_1(x, y) \le z \le u_2(x, y)\}$

23. $\iiint_E z\, dV$, where $E = \{(x, y, z) \mid -y \le x \le y,\ 0 \le y \le 1,\ 0 \le z \le 1 - x^4 - y^4\}$

Answer

$\frac{113}{450}$

24. $\iiint_E (xz + 1)\, dV$, where $E = \{(x, y, z) \mid 0 \le x \le \sqrt{y},\ 0 \le y \le 2,\ 0 \le z \le 1 - x^2 - y^2\}$

25. $\iiint_E (x - z)\, dV$, where $E = \{(x, y, z) \mid -\sqrt{1 - y^2} \le x \le y,\ 0 \le y \le \frac{1}{2}x,\ 0 \le z \le 1 - x^2 - y^2\}$

Answer

$\frac{1}{160}(6\sqrt{3}-41)$

26. $\iiint_E (x+y)\,dV$, where $E=\{(x,y,z)\,|\,0\le x\le\sqrt{1-y^2},\ 0\le y\le 1x,\ 0\le z\le 1-x\}$

In exercises 27 - 30, evaluate the triple integrals over the bounded region

$E=\{(x,y,z)\,|\,(x,y)\in D,\ u_1(x,y)x\le z\le u_2(x,y)\}$**, where** D **is the projection of** E **onto the** xy**-plane.**

27. $\iint_D \left(\int_1^2 (x+y)\,dz\right)\,dA$, where $D=\{(x,y)\,|\,x^2+y^2\le 1\}$

Answer

$\frac{3\pi}{2}$

28. $\iint_D \left(\int_1^3 x(z+1)\,dz\right)\,dA$, where $D=\{(x,y)\,|\,x^2-y^2\ge 1,\ x\le\sqrt{5}\}$

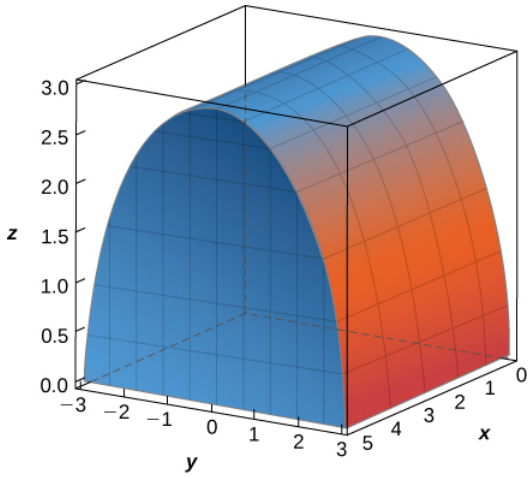
29. $\iint_D \left(\int_0^{10-x-y} (x+2z)\,dz\right)\,dA$, where $D=\{(x,y)\,|\,y\ge 0,\ x\ge 0,\ x+y\le 10\}$

Answer

1250

30. $\iint_D \left(\int_0^{4x^2+4y^2} y\,dz\right)\,dA$, where $D=\{(x,y)\,|\,x^2+y^2\le 4,\ y\ge 1,\ x\ge 0\}$

31. The solid E bounded by $y^2+z^2=9$, $z=0$, and $x=5$ is shown in the following figure. Evaluate the integral $\iiint_E z\,dV$ by integrating first with respect to z, then y, and then x.



Answer

$\int_0^5\int_{-3}^3\int_0^{\sqrt{9-y^2}} z\,dz\,dy\,dx=90$

32. The solid E bounded by $y=\sqrt{x}$, $x=4$, $y=0$, and $z=1$ is given in the following figure. Evaluate the integral $\iiint_E xyz\,dV$ by integrating first with respect to x, then y, and then z.

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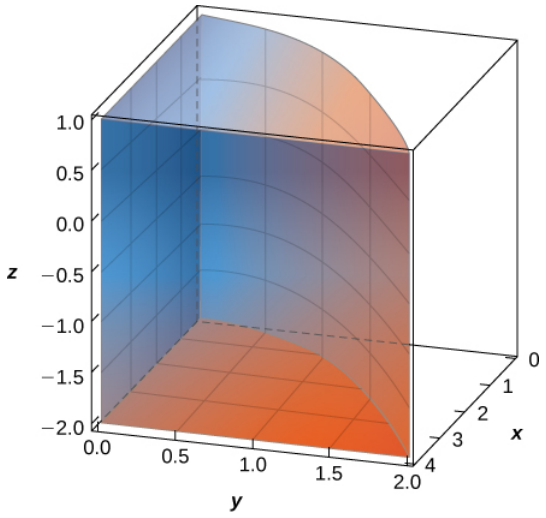
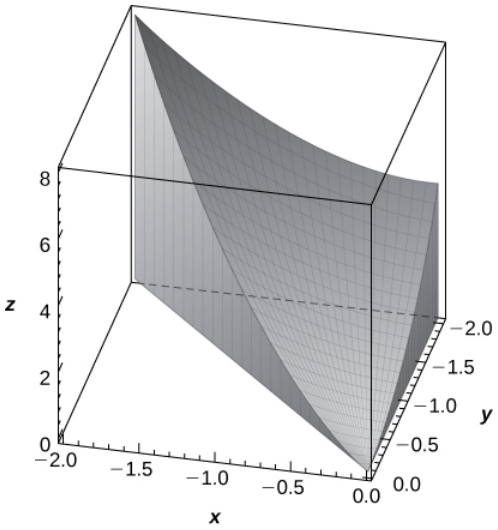


33. [T] The volume of a solid E is given by the integral $\int_{-2}^{0}\int_{x}^{0}\int_{0}^{x^2+y^2} dz\,dy\,dx$. Use a computer algebra system (CAS) to graph E and find its volume. Round your answer to two decimal places.

Answer

$V \approx 5.33 \text{ units}^3$



34. [T] The volume of a solid E is given by the integral $\int_{-1}^{0}\int_{-x^3}^{0}\int_{0}^{1+\sqrt{x^2+y^2}} dz\,dy\,dx$. Use a CAS to graph E and find its volume V. Round your answer to two decimal places.

In exercises 35 - 38, use two circular permutations of the variables x, y, and z to write new integrals whose values equal the value of the original integral. A circular permutation of x, y, and z is the arrangement of the numbers in one of the following orders: y, z, and x or z, x, and y.

35. $\int_{0}^{1}\int_{1}^{3}\int_{2}^{4}(x^2z^2+1)\,dx\,dy\,dz$

Answer

$\int_{0}^{1}\int_{1}^{3}\int_{2}^{4}(y^2z^2+1)\,dz\,dx\,dy;$
$\int_{0}^{1}\int_{1}^{3}\int_{2}^{4}(x^2y^2+1)\,dy\,dz\,dx$

36. $\int_0^3 \int_0^1 \int_0^{-x+1} (2x+5y+7z)dy\,dx\,dz$

37. $\int_0^1 \int_{-y}^{y} \int_0^{1-x^4-y^4} \ln x \; dz\,dx\,dy$

38. $\int_{-1}^1 \int_0^1 \int_{-y^6}^{\sqrt{y}} (x+yz)\,dx\,dy\,dz$

39. Set up the integral that gives the volume of the solid E bounded by $y^2 = x^2 + z^2$ and $y = a^2$, where $a > 0$.

Answer

$$V = \int_{-a}^{a} \int_{-\sqrt{a^2-z^2}}^{\sqrt{a^2-z^2}} \int_{\sqrt{x^2+z^2}}^{a^2} dy\,dx\,dz$$

40. Set up the integral that gives the volume of the solid E bounded by $x = y^2 + z^2$ and $x = a^2$, where $a > 0$.

Average Value of a Function

41. Find the average value of the function $f(x, y, z) = x + y + z$ over the parallelepiped determined by $x + 0,\ x = 1,\ y = 0,\ y = 3,\ z = 0$, and $z = 5$.

Answer

$\frac{9}{2}$

42. Find the average value of the function $f(x, y, z) = xyz$ over the solid $E = [0, 1] \times [0, 1] \times [0, 1]$ situated in the first octant.

Finding Volumes using Triple Integrals

43. Find the volume of the solid E that lies under the plane $x + y + z = 9$ and whose projection onto the xy-plane is bounded by $x = sqrty - 1,\ x = 0$, and $x + y = 7$.

Answer

$\frac{156}{5}$ units^3

44. Find the volume of the solid E that lies under the plane $2x + y + z = 8$ and whose projection onto the xy-plane is bounded by $x = sin^{-1}y,\ y = 0$, and $x = \frac{\pi}{2}$.

45. Consider the pyramid with the base in the xy-plane of $[-2, 2] \times [-2, 2]$ and the vertex at the point $(0, 0, 8)$.

a. Show that the equations of the planes of the lateral faces of the pyramid are $4y + z = 8,\ 4y - z = -8,\ 4x + z = 8$, and $-4x + z = 8$.

b. Find the volume of the pyramid.

Answer

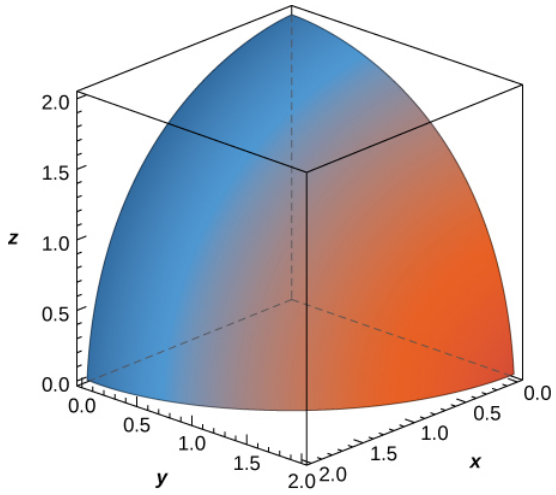
a. Answers may vary;
b. $\frac{128}{3}$ units^3

46. Consider the pyramid with the base in the xy-plane of $[-3, 3] \times [-3, 3]$ and the vertex at the point $(0, 0, 9)$.

a. Show that the equations of the planes of the side faces of the pyramid are $3y + z = 9,\ 3y + z = 9,\ y = 0$ and $x = 0$.

b. Find the volume of the pyramid.

47. The solid E bounded by the sphere of equation $x^2 + y^2 + z^2 = r^2$ with $r > 0$ and located in the first octant is represented in the following figure.



a. Write the triple integral that gives the volume of E by integrating first with respect to z, then with y, and then with x.

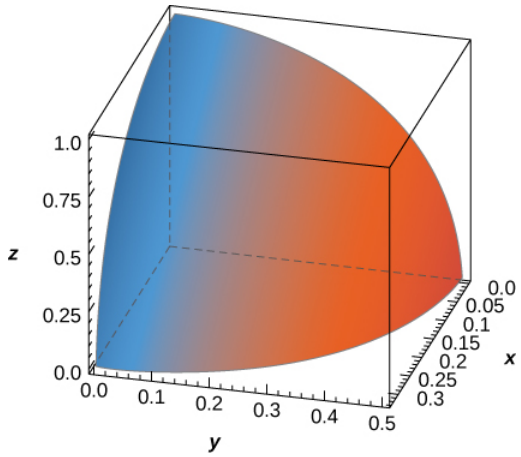
b. Rewrite the integral in part a. as an equivalent integral in five other orders.

Answer

a. $\int_0^4 \int_0^{\sqrt{r^2-x^2}} \int_0^{\sqrt{r^2-x^2-y^2}} dz\, dy\, dx$

b. $\int_0^2 \int_0^{\sqrt{r^2-x^2}} \int_0^{\sqrt{r^2-x^2-y^2}} dz\, dx\, dy$,
$\int_0^r \int_0^{\sqrt{r^2-x^2}} \int_0^{\sqrt{r^2-x^2-y^2}} dy\, dx\, dz$,
$\int_0^r \int_0^{\sqrt{r^2-x^2}} \int_0^{\sqrt{r^2-x^2-y^2}} dy\, dz\, dx$,
$\int_0^r \int_0^{\sqrt{r^2-x^2}} \int_0^{\sqrt{r^2-x^2-y^2}} dx\, dy\, dz$,
$\int_0^r \int_0^{\sqrt{r^2-x^2}} \int_0^{\sqrt{r^2-x^2-y^2}} dx\, dz\, dy$

48. The solid E bounded by the sphere of equation $9x^2 + 4y^2 + z^2 = 1$ and located in the first octant is represented in the following figure.



a. Write the triple integral that gives the volume of E by integrating first with respect to z then with y and then with x.

b. Rewrite the integral in part a. as an equivalent integral in five other orders.

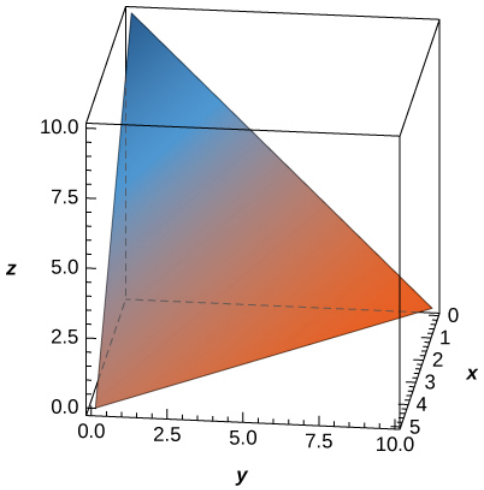
49. Find the volume of the prism with vertices $(0,0,0)$, $(2,0,0)$, $(2,3,0)$, $(0,3,0)$, $(0,0,1)$, and $(2,0,1)$.

Answer

3 units^3

50. Find the volume of the prism with vertices $(0,0,0)$, $(4,0,0)$, $(4,6,0)$, $(0,6,0)$, $(0,0,1)$, and $(4,0,1)$.

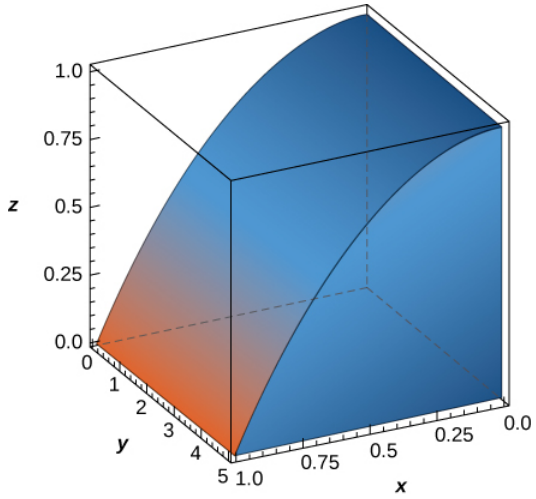
51. The solid E bounded by $z = 10 - 2x - y$ and situated in the first octant is given in the following figure. Find the volume of the solid.



Answer

$\frac{250}{3}$ units^3

52. The solid E bounded by $z = 1 - x^2$ and situated in the first octant is given in the following figure. Find the volume of the solid.



Approximating Triple Integrals

53. The midpoint rule for the triple integral $\iiint_B f(x,y,z)\,dV$ over the rectangular solid box B is a generalization of the midpoint rule for double integrals. The region B is divided into subboxes of equal sizes and the integral is approximated by the triple Riemann sum

$$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(\bar{x}_i,\bar{y}_j,\bar{z}_k)\Delta V,$$

where $(\bar{x}_i,\bar{y}_j,\bar{z}_k)$ is the center of the box B_{ijk} and ΔV is the volume of each subbox. Apply the midpoint rule to approximate

$$\iiint_B x^2\,dV$$

over the solid $B = \{(x, y, z) \,|\, 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$ by using a partition of eight cubes of equal size. Round your answer to three decimal places.

Answer

$\iiint_B f(x, y, z)\, dV \quad \approx \quad \frac{5}{16} \approx 0.313 \text{ units}^3$

54. [T]

a. Apply the midpoint rule to approximate $\iiint_B e^{-x^2}\, dV$ over the solid $B = \{(x, y, z) \,|\, 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$ by using a partition of eight cubes of equal size. Round your answer to three decimal places.

b. Use a CAS to improve the above integral approximation in the case of a partition of n^3 cubes of equal size, where $n = 3,\ 4,\ \ldots,\ 10$.

Applications

55. Suppose that the temperature in degrees Celsius at a point (x, y, z) of a solid E bounded by the coordinate planes and the plane $x + y + z = 5$ is given by:

$$T(x, y, z) = xz + 5z + 10$$

Find the average temperature over the solid.

Answer

17.5° C

56. Suppose that the temperature in degrees Fahrenheit at a point (x, y, z) of a solid E bounded by the coordinate planes and the plane $x + y + z = 5$ is given by:

$$T(x, y, z) = x + y + xy$$

Find the average temperature over the solid.

57. Show that the volume of a right square pyramid of height h and side length a is $v = \dfrac{ha^2}{3}$ by using triple integrals.

58. Show that the volume of a regular right hexagonal prism of edge length a is $\dfrac{3a^3\sqrt{3}}{2}$ by using triple integrals.

59. Show that the volume of a regular right hexagonal pyramid of edge length a is $\dfrac{a^3\sqrt{3}}{2}$ by using triple integrals.

60. If the charge density at an arbitrary point (x, y, z) of a solid E is given by the function $\rho(x, y, z)$, then the total charge inside the solid is defined as the triple integral $\iiint_E \rho(x, y, z)\, dV$. Assume that the charge density of the solid E enclosed by the paraboloids $x = 5 - y^2 - z^2$ and $x = y^2 + z^2 - 5$ is equal to the distance from an arbitrary point of E to the origin. Set up the integral that gives the total charge inside the solid E.

Answer

Total Charge inside the Solid $E \quad = \quad \int_{-\sqrt{5}}^{\sqrt{5}} \int_{-\sqrt{5-y^2}}^{\sqrt{5-y^2}} \int_{y^2+z^2-5}^{5-y^2-z^2} \sqrt{x^2 + y^2 + z^2}\, dx\, dz\, dy$

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15.5: Triple Integrals in Cylindrical and Spherical Coordinates

Learning Objectives

- Evaluate a triple integral by changing to cylindrical coordinates.
- Evaluate a triple integral by changing to spherical coordinates.

Earlier in this chapter we showed how to convert a double integral in rectangular coordinates into a double integral in polar coordinates in order to deal more conveniently with problems involving circular symmetry. A similar situation occurs with triple integrals, but here we need to distinguish between cylindrical symmetry and spherical symmetry. In this section we convert triple integrals in rectangular coordinates into a triple integral in either cylindrical or spherical coordinates.

Also recall the chapter prelude, which showed the opera house l'Hemisphèric in Valencia, Spain. It has four sections with one of the sections being a theater in a five-story-high sphere (ball) under an oval roof as long as a football field. Inside is an IMAX screen that changes the sphere into a planetarium with a sky full of 9000 twinkling stars. Using triple integrals in spherical coordinates, we can find the volumes of different geometric shapes like these.

Review of Cylindrical Coordinates

As we have seen earlier, in two-dimensional space $\mathbb{R}^2$ a point with rectangular coordinates (x, y) can be identified with (r, θ) in polar coordinates and vice versa, where $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$ are the relationships between the variables.

In three-dimensional space $\mathbb{R}^3$ a point with rectangular coordinates (x, y, z) can be identified with cylindrical coordinates (r, θ, z) and vice versa. We can use these same conversion relationships, adding z as the vertical distance to the point from the $(xy$-plane as shown in $15.5.1$.

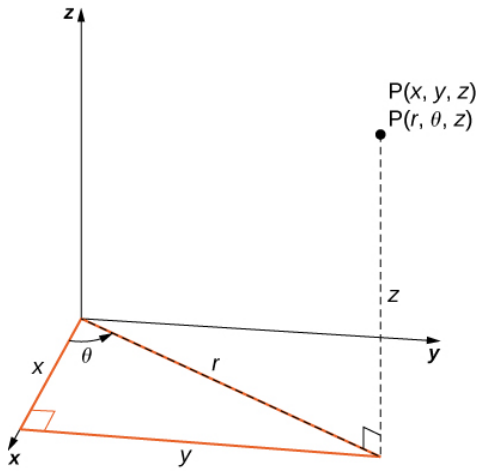

Figure $15.5.1$: Cylindrical coordinates are similar to polar coordinates with a vertical z coordinate added.

To convert from rectangular to cylindrical coordinates, we use the conversion

- $x = r\cos\theta$
- $y = r\sin\theta$
- $z = z$

To convert from cylindrical to rectangular coordinates, we use

- $r^2 = x^2 + y^2$ and
- $\theta = \tan^{-1}\left(\frac{y}{x}\right)$
- $z = z$

Note that that z-coordinate remains the same in both cases.

In the two-dimensional plane with a rectangular coordinate system, when we say $x = k$ (constant) we mean an unbounded vertical line parallel to the y-axis and when $y = l$ (constant) we mean an unbounded horizontal line parallel to the x-axis. With the polar

coordinate system, when we say $r = c$ (constant), we mean a circle of radius c units and when $\theta = \alpha$ (constant) we mean an infinite ray making an angle α with the positive x-axis.

Similarly, in three-dimensional space with rectangular coordinates (x, y, z) the equations $x = k$, $y = l$ and $z = m$ where k, l and m are constants, represent unbounded planes parallel to the yz-plane, xz-plane and xy-plane, respectively. With cylindrical coordinates (r, θ, z), by $r = c$, $\theta = \alpha$, and $z = m$, where c, α, and m are constants, we mean an unbounded vertical cylinder with the z-axis as its radial axis; a plane making a constant angle α with the xy-plane; and an unbounded horizontal plane parallel to the xy-plane, respectively. This means that the circular cylinder $x^2 + y^2 = c^2$ in rectangular coordinates can be represented simply as $r = c$ in cylindrical coordinates. (Refer to Cylindrical and Spherical Coordinates for more review.)

Integration in Cylindrical Coordinates

Triple integrals can often be more readily evaluated by using cylindrical coordinates instead of rectangular coordinates. Some common equations of surfaces in rectangular coordinates along with corresponding equations in cylindrical coordinates are listed in Table 15.5.1. These equations will become handy as we proceed with solving problems using triple integrals.

Table 15.5.1: Equations of Some Common Shapes

	Circular cylinder	Circular cone	Sphere	Paraboloid
Rectangular	$x^2 + y^2 = c^2$	$z^2 = c^2(x^2 + y^2)$	$x^2 + y^2 + z^2 = c^2$	$z = c(x^2 + y^2)$
Cylindrical	$r = c$	$z = cr$	$r^2 + z^2 = c^2$	$z = cr^2$

As before, we start with the simplest bounded region B in $\mathbb{R}^3$ to describe in cylindrical coordinates, in the form of a cylindrical box, $B = \{(r, \theta, z) | a \le r \le b, \alpha \le \theta \le \beta, c \le z \le d\}$ (Figure 15.5.2). Suppose we divide each interval into l, m, and n subdivisions such that $\Delta r = \frac{b-a}{l}$, $\Delta\theta = \frac{\beta-\alpha}{m}$, and $\Delta z = \frac{d-c}{n}$. Then we can state the following definition for a triple integral in cylindrical coordinates.

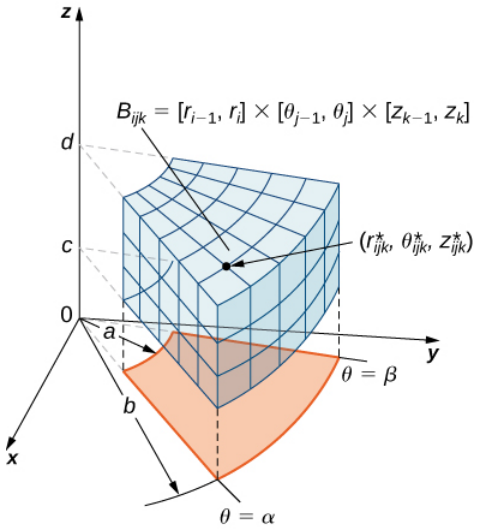

Figure 15.5.2: A cylindrical box B described by cylindrical coordinates.

DEFINITION: triple integral in cylindrical coordinates

Consider the cylindrical box (expressed in cylindrical coordinates)

$$B = \{(r, \theta, z) | a \le r \le b, \alpha \le \theta \le \beta, c \le z \le d\}.$$

If the function $f(r, \theta, z)$ is continuous on B and if $(r^*_{ijk}, \theta^*_{ijk}, z^*_{ijk})$ is any sample point in the cylindrical subbox $B_{ijk} = |r_{i-1}, r_i| \times |\theta_{j-1}, \theta_j| \times |z_{k-1}, k_i|$ (Figure 15.5.2), then we can define the *triple integral in cylindrical coordinates* as the limit of a triple Riemann sum, provided the following limit exists:

$$\lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(r^*_{ijk}, \theta^*_{ijk}, z^*_{ijk})\, \Delta r \Delta\theta \Delta z.$$

Note that if $g(x, y, z)$ is the function in rectangular coordinates and the box B is expressed in rectangular coordinates, then the triple integral

$$\iiint_B g(x, y, z)\, dV$$

is equal to the triple integral

$$\iiint_B g(r\, \cos\theta,\, r\, \sin\, \theta,\, z) r\, dr\, d\theta\, dz$$

and we have

$$\iiint_B g(x, y, z)\, dV = \iiint_B g(r\, \cos\theta,\, r\, \sin\, \theta,\, z)\, r\, dr\, d\theta\, dz = \iiint_B f(r, \theta\, z)\, r\, dr\, d\theta\, dz.$$

As mentioned in the preceding section, all the properties of a double integral work well in triple integrals, whether in rectangular coordinates or cylindrical coordinates. They also hold for iterated integrals. To reiterate, in cylindrical coordinates, Fubini's theorem takes the following form:

Theorem: Fubini's Theorem in Cylindrical Coordinates

Suppose that $g(x, y, z)$ is continuous on a rectangular box B which when described in cylindrical coordinates looks like $B = \{(r, \theta, z) | a \le r \le b,\, \alpha \le \theta \le \beta,\, c \le z \le d\}$.

Then $g(x, y, z) = g(r\, \cos\theta, r\, \sin\, \theta, z) = f(r, \theta, z)$ and

$$\iiint_B g(x, y, z)\, dV = \int_c^d \int_\beta^\alpha \int_a^b f(r, \theta, z)\, r\, dr\, d\theta\, dz.$$

The iterated integral may be replaced equivalently by any one of the other five iterated integrals obtained by integrating with respect to the three variables in other orders.

Cylindrical coordinate systems work well for solids that are symmetric around an axis, such as cylinders and cones. Let us look at some examples before we define the triple integral in cylindrical coordinates on general cylindrical regions.

Example 15.5.1: Evaluating a Triple Integral over a Cylindrical Box

Evaluate the triple integral

$$\iiint_B (zr\, \sin\, \theta) r\, dr\, d\theta\, dz$$

where the cylindrical box B is $B = \{(r, \theta, z) | 0 \le r \le 2,\, 0 \le \theta \le \pi/2,\, 0, \le z \le 4\}$.

Solution

As stated in Fubini's theorem, we can write the triple integral as the iterated integral

$$\iiint_B (zr\, \sin\, \theta) r\, dr\, d\theta\, dz = \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=2} \int_{z=0}^{z=4} (zr\, \sin\, \theta)\, r\, dz\, dr\, d\theta.$$

The evaluation of the iterated integral is straightforward. Each variable in the integral is independent of the others, so we can integrate each variable separately and multiply the results together. This makes the computation much easier:

$$\begin{aligned}\int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=2} \int_{z=0}^{z=4} (zr\, \sin\, \theta)\, r\, dz\, dr\, d\theta &= \left(\int_0^{\pi/2} \sin\, \theta\, d\theta\right) \left(\int_0^2 r^2 dr\right) \left(\int_0^4 z\, dz\right) \\ &= \left(-\cos\theta|_0^{\pi/2}\right) \left(\frac{r^3}{3}\Big|_0^2\right) \left(\frac{z^2}{2}\Big|_0^4\right) = \frac{64}{3}.\end{aligned}$$

Exercise 15.5.1:

Evaluate the triple integral

$$\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=1}\int_{z=0}^{z=4}(rz\,\sin\,\theta)\,r\,dz\,dr\,d\theta.$$

Hint

Follow the same steps as in the previous example.

Answer

8

If the cylindrical region over which we have to integrate is a general solid, we look at the projections onto the coordinate planes. Hence the triple integral of a continuous function $f(r,\theta,z)$ over a general solid region $E=\{(r,\theta,z)|(r,\theta)\in D, u_1(r,\theta)\le z\le u_2(r,\theta)\}$ in $\mathbb{R}^3$ where D is the projection of E onto the $r\theta$-plane, is

$$\iiint_E f(r,\theta,z)\,r\,dr\,d\theta\,dz=\iint_D\left[\int_{u_1(r,\theta)}^{u_2(r,\theta)}f(r,\theta,z)dz\right]r\,dr\,d\theta.$$

In particular, if $D=\{(r,\theta)|g_1(\theta)\le r\le g_2(\theta),\alpha\le\theta\le\beta\}$, then we have

$$\iiint_E f(r,\theta,z)\,r\,dr\,d\theta=\int_{\theta=\alpha}^{\theta=\beta}\int_{r=g_1(\theta)}^{r=g_2(\theta)}\int_{z=u_1(r,\theta)}^{z=u_2(r,\theta)}f(r,\theta,z)\,r\,dz\,dr\,d\theta.$$

Similar formulas exist for projections onto the other coordinate planes. We can use polar coordinates in those planes if necessary.

Example 15.5.2: Setting up a Triple Integral in Cylindrical Coordinates over a General Region

Consider the region E inside the right circular cylinder with equation $r=2\,\sin\,\theta$, bounded below by the $r\theta$-plane and bounded above by the sphere with radius 4 centered at the origin (Figure 15.5.3). Set up a triple integral over this region with a function $f(r,\theta,z)$ in cylindrical coordinates.

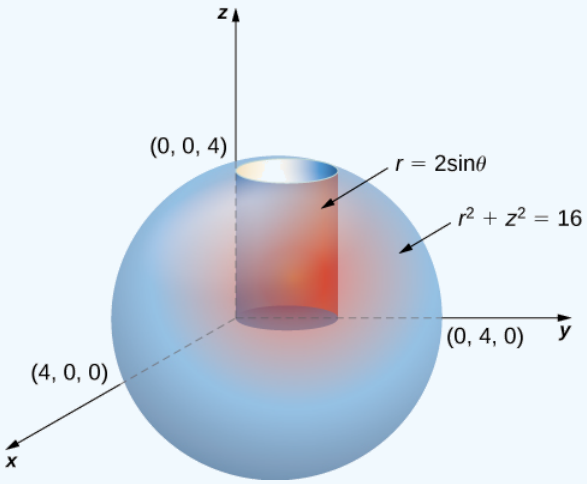

Figure 15.5.3: Setting up a triple integral in cylindrical coordinates over a cylindrical region.

Solution

First, identify that the equation for the sphere is $r^2+z^2=16$. We can see that the limits for z are from 0 to $z=\sqrt{16-r^2}$. Then the limits for r are from 0 to $r=2\,\sin\,\theta$. Finally, the limits for θ are from 0 to π. Hence the region is $E=\{(r,\theta,z)|0\le\theta\le\pi,\ 0\le r\le 2\,\sin\,\theta,\ 0\le z\le\sqrt{16-r^2}\}$. Therefore, the triple integral is

$$\iiint_E f(r,\theta,z)\,r\,dz\,dr\,d\theta=\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=2\,\sin\,\theta}\int_{z=0}^{z=\sqrt{16-r^2}}f(r,\theta,z)\,r\,dz\,dr\,d\theta.$$

Exercise 15.5.2:

Consider the region inside the right circular cylinder with equation $r = 2 \sin \theta$ bounded below by the $r\theta$-plane and bounded above by $z = 4 - y$. Set up a triple integral with a function $f(r, \theta, z)$ in cylindrical coordinates.

Hint

Analyze the region, and draw a sketch.

Answer

$$\iiint_E f(r,\theta,z) r\,dz\,dr\,d\theta = \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=2\sin\theta} \int_{z=0}^{z=4-r\sin\theta} f(r,\theta,z) r\,dz\,dr\,d\theta.$$

Example 15.5.3: Setting up a Triple Integral in Two Ways

Let E be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$. (Figure 15.5.4). Set up a triple integral in cylindrical coordinates to find the volume of the region, using the following orders of integration:

a. $dz\,dr\,d\theta$

b. $dr\,dz\,d\theta$

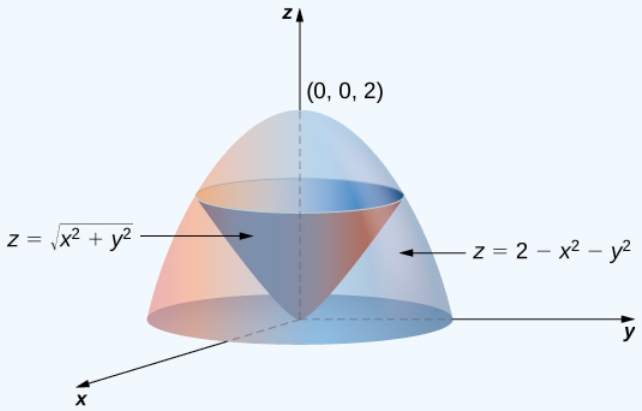

Figure 15.5.4: Setting up a triple integral in cylindrical coordinates over a conical region.

Solution

a. The cone is of radius 1 where it meets the paraboloid. Since $z = 2 - x^2 - y^2 = 2 - r^2$ and $z = \sqrt{x^2 + y^2} = r^2$ (assuming r is nonnegative), we have $2 - r^2 = r$. Solving, we have $r^2 + r - 2 = (r + 2)(r - 1) = 0$. Since $r \geq 0$, we have $r = 1$. Therefore $z = 1$. So the intersection of these two surfaces is a circle of radius 1 in the plane $z = 1$. The cone is the lower bound for z and the paraboloid is the upper bound. The projection of the region onto the xy-plane is the circle of radius 1 centered at the origin.

Thus, we can describe the region as $E = \{(r, \theta, z) | 0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 1,\ r \leq z \leq 2 - r^2\}$.

Hence the integral for the volume is

$$V = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=r}^{z=2-r^2} r\,dz\,dr\,d\theta.$$

b. We can also write the cone surface as $r = z$ and the paraboloid as $r^2 = 2 - z$. The lower bound for r is zero, but the upper bound is sometimes the cone and the other times it is the paraboloid. The plane $z = 1$ divides the region into two regions. Then the region can be described as

$$E = \{(r, \theta, z) | 0 \leq \theta \leq 2\pi,\ 0 \leq z \leq 1,\ 0 \leq r \leq z\}$$
$$\cup \{(r, \theta, z) | 0 \leq \theta \leq 2\pi,\ 1 \leq z \leq 2,\ 0 \leq r \leq \sqrt{2 - z}\}.$$

Now the integral for the volume becomes

$$V = \int_{\theta=0}^{\theta=2\pi} \int_{z=0}^{z=1} \int_{r=0}^{r=z} r\,dr\,dz\,d\theta + \int_{\theta=0}^{\theta=2\pi} \int_{z=1}^{z=2} \int_{r=0}^{r=\sqrt{2-z}} r\,dr\,dz\,d\theta.$$

Exercise 15.5.3:

Redo the previous example with the order of integration $d\theta\, dz\, dr$.

Hint

Note that θ is independent of r and z.

Answer

$E = \{(r, \theta, z) | 0 \leq \theta \leq 2\pi,\ 0 \leq z \leq 1,\ 0 \leq r \leq 2 - z^2\}$ and

$$V = \int_{r=0}^{r=1} \int_{z=0}^{z=2-r^2} \int_{\theta=0}^{\theta=2\pi} r\, d\theta\, dz\, dr.$$

Example 15.5.4: Finding a Volume with Triple Integrals in Two Ways

Let E be the region bounded below by the $r\theta$-plane, above by the sphere $x^2 + y^2 + z^2 = 4$, and on the sides by the cylinder $x^2 + y^2 = 1$ (Figure 15.5.5). Set up a triple integral in cylindrical coordinates to find the volume of the region using the following orders of integration, and in each case find the volume and check that the answers are the same:

a. $dz\, dr\, d\theta$
b. $dr\, dz\, d\theta$.

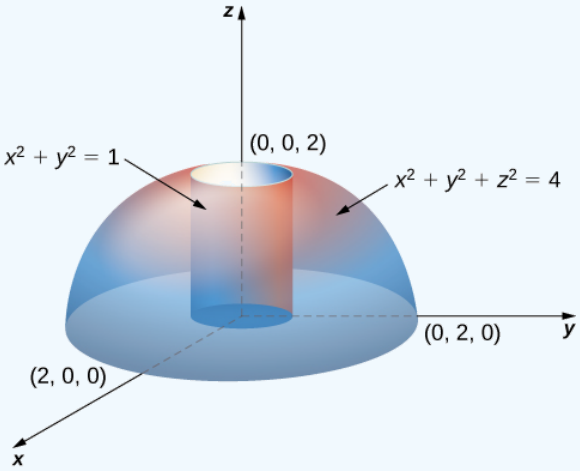

Figure 15.5.5: Finding a cylindrical volume with a triple integral in cylindrical coordinates.

Solution

a. Note that the equation for the sphere is

$$x^2 + y^2 + z^2 = 4 \text{ or } r^2 + z^2 = 4$$

and the equation for the cylinder is

$$x^2 + y^2 = 1 \text{ or } r^2 = 1.$$

Thus, we have for the region E

$$E = \{(r, \theta, z) | 0 \leq z \leq \sqrt{4 - r^2},\ 0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi\}$$

Hence the integral for the volume is

$$\begin{aligned} V(E) &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=0}^{z=\sqrt{4-r^2}} r\,dz\,dr\,d\theta \\ &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \left[rz \Big|_{z=0}^{z=\sqrt{4-r^2}} \right] dr\,d\theta \\ &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \left(r\sqrt{4-r^2} \right) dr\,d\theta \\ &= \int_0^{2\pi} \left(\frac{8}{3} - \sqrt{3} \right) d\theta = 2\pi \left(\frac{8}{3} - \sqrt{3} \right) \text{ cubic units.} \end{aligned}$$

b. Since the sphere is $x^2 + y^2 + z^2 = 4$, which is $r^2 + z^2 = 4$, and the cylinder is $x^2 + y^2 = 1$, which is $r^2 = 1$, we have $1 + z^2 = 4$, that is, $z^2 = 3$. Thus we have two regions, since the sphere and the cylinder intersect at $(1, \sqrt{3})$ in the rz-plane

$$E_1 = \{(r, \theta, z) | 0 \le r \le \sqrt{4-r^2},\ \sqrt{3} \le z \le 2,\ 0 \le \theta \le 2\pi\}$$

and

$$E_2 = \{(r, \theta, z) | 0 \le r \le 1,\ 0 \le z \le \sqrt{3},\ 0 \le \theta \le 2\pi\}.$$

Hence the integral for the volume is

$$\begin{aligned} V(E) &= \int_{\theta=0}^{\theta=2\pi} \int_{z=\sqrt{3}}^{z=2} \int_{r=0}^{r=\sqrt{4-r^2}} r\,dr\,dz\,d\theta + \int_{\theta=0}^{\theta=2\pi} \int_{z=0}^{z=\sqrt{3}} \int_{r=0}^{r=1} r\,dr\,dz\,d\theta \\ &= \sqrt{3}\pi + \left(\frac{16}{3} - 3\sqrt{3} \right) \pi \\ &= 2\pi \left(\frac{8}{3} - \sqrt{3} \right) \text{ cubic units.} \end{aligned}$$

? Exercise 15.5.4

Redo the previous example with the order of integration $d\theta\,dz\,dr$.

Hint

A figure can be helpful. Note that θ is independent of r and z.

Answer

$E_2 = \{(r, \theta, z) | 0 \le \theta \le 2\pi,\ 0 \le r \le 1,\ r \le z \le \sqrt{4-r^2}\}$ and

$$V = \int_{r=0}^{r=1} \int_{z=r}^{z=\sqrt{4-r^2}} \int_{\theta=0}^{\theta=2\pi} r\,d\theta\,dz\,dr.$$

Review of Spherical Coordinates

In three-dimensional space $\mathbb{R}^3$ in the spherical coordinate system, we specify a point P by its distance ρ from the origin, the polar angle θ from the positive x-axis (same as in the cylindrical coordinate system), and the angle φ from the positive z-axis and the line OP (Figure 15.5.6). Note that $\rho > 0$ and $0 \le \varphi \le \pi$. (Refer to Cylindrical and Spherical Coordinates for a review.) Spherical coordinates are useful for triple integrals over regions that are symmetric with respect to the origin.

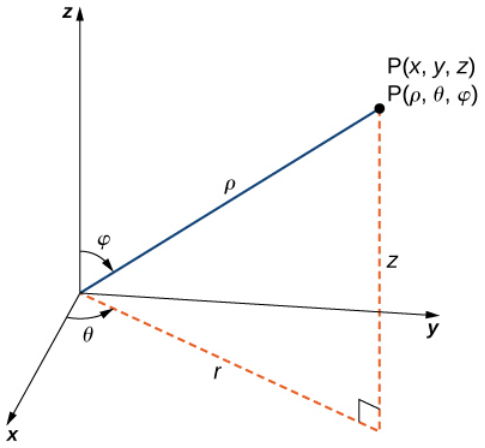

Figure 15.5.6: The spherical coordinate system locates points with two angles and a distance from the origin.

Recall the relationships that connect rectangular coordinates with spherical coordinates.

From spherical coordinates to rectangular coordinates:

$$x = \rho \sin \varphi \cos\theta,\ y = \rho \sin \varphi \sin \theta,\ and\, z = \rho \cos \varphi.$$

From rectangular coordinates to spherical coordinates:

$$\rho^2 = x^2 + y^2 + z^2,\ \tan \theta = \frac{y}{x},\ \varphi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right).$$

Other relationships that are important to know for conversions are

- $r = \rho \sin \varphi$
- $\theta = \theta$ These equations are used to convert from spherical coordinates to cylindrical coordinates.
- $z = \rho \cos \varphi$

and

- $\rho = \sqrt{r^2 + z^2}$
- $\theta = \theta$ These equations are used to convert from cylindrical coordinates to spherical coordinates.
- $\varphi = \arccos\left(\frac{z}{\sqrt{r^2+z^2}}\right)$

15.5.7 shows a few solid regions that are convenient to express in spherical coordinates.

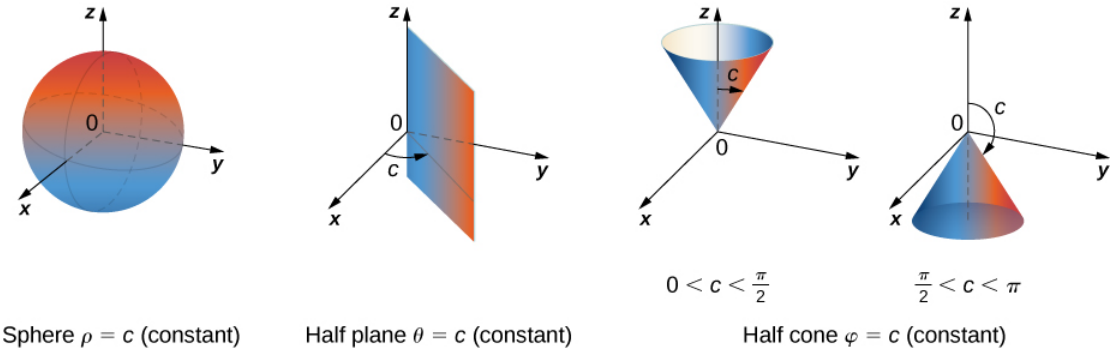

Figure 15.5.7: Spherical coordinates are especially convenient for working with solids bounded by these types of surfaces. (The letter c indicates a constant.)

Integration in Spherical Coordinates

We now establish a triple integral in the spherical coordinate system, as we did before in the cylindrical coordinate system. Let the function $f(\rho, \theta, \varphi)$ be continuous in a bounded spherical box, $B = \{(\rho, \theta, \varphi) | a \le \rho \le b,\ \alpha \le \theta \le \beta,\ \gamma \le \varphi \le \psi\}$. We then divide each interval into l, m, n and n subdivisions such that $\Delta\rho = \frac{b-a}{l}$, $\Delta\theta = \frac{\beta-\alpha}{m}$. $\Delta\varphi = \frac{\psi-\gamma}{n}$. Now we can illustrate the following theorem for

triple integrals in spherical coordinates with $(\rho^*_{ijk}, \theta^*_{ijk}, \varphi^*_{ijk})$ being any sample point in the spherical subbox B_{ijk}. For the volume element of the subbox ΔV in spherical coordinates, we have $\Delta V = (\Delta\rho)\,(\rho\Delta\varphi)\,(\rho\,\sin\,\varphi\,\Delta\theta)$, as shown in the following figure.

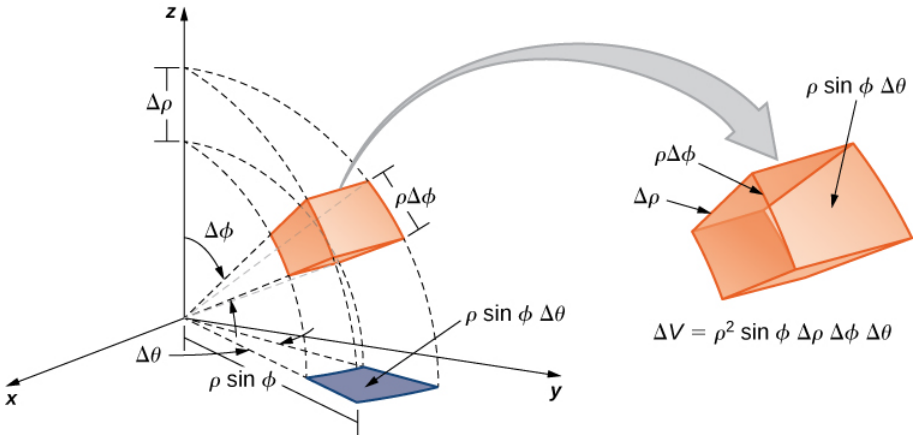

Figure 15.5.8: The volume element of a box in spherical coordinates.

Definition: triple integral in spherical coordinates

The **triple integral in spherical coordinates** is the limit of a triple Riemann sum,

$$\lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(\rho^*_{ijk}, \theta^*_{ijk}, \varphi^*_{ijk})(\rho^*_{ijk})^2 \sin\,\varphi\Delta\rho\,\Delta\theta\Delta\varphi$$

provided the limit exists.

As with the other multiple integrals we have examined, all the properties work similarly for a triple integral in the spherical coordinate system, and so do the iterated integrals. Fubini's theorem takes the following form.

Theorem: Fubini's Theorem for Spherical Coordinates

If $f(\rho, \theta, \varphi)$ is continuous on a spherical solid box $B = [a, b] \times [\alpha, \beta] \times [\gamma, \psi]$, then

$$\iiint_B f(\rho, \theta, \varphi)\,\rho^2 \sin\,\varphi\,d\rho\,d\varphi\,d\theta = \int_{\varphi=\gamma}^{\varphi=\psi}\int_{\theta=\alpha}^{\theta=\beta}\int_{\rho=a}^{\rho=b} f(\rho, \theta, \varphi)\,\rho^2 \sin\,\varphi\,d\rho\,d\varphi\,d\theta.$$

This iterated integral may be replaced by other iterated integrals by integrating with respect to the three variables in other orders.

As stated before, spherical coordinate systems work well for solids that are symmetric around a point, such as spheres and cones. Let us look at some examples before we consider triple integrals in spherical coordinates on general spherical regions.

Example 15.5.5: Evaluating a Triple Integral in Spherical Coordinates

Evaluate the iterated triple integral

$$\int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\pi/2}\int_{\rho=0}^{\rho=1} \rho^2 \sin\,\varphi\,d\rho\,d\varphi\,d\theta.$$

Solution

As before, in this case the variables in the iterated integral are actually independent of each other and hence we can integrate each piece and multiply:

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^2 \sin\,\varphi\,d\rho\,d\varphi\,d\theta = \int_0^{2\pi} d\theta \int_0^{\pi/2} \sin\,\varphi\,d\varphi \int_0^1 \rho^2 d\rho = (2\pi)\,(1)\,\left(\frac{1}{3}\right) = \frac{2\pi}{3}$$

The concept of triple integration in spherical coordinates can be extended to integration over a general solid, using the projections onto the coordinate planes. Note that dV and dA mean the increments in volume and area, respectively. The variables V and A are used as the variables for integration to express the integrals.

The triple integral of a continuous function $f(\rho, \theta, \varphi)$ over a general solid region

$$E = \{(\rho, \theta, \varphi) | (\rho, \theta) \in D, u_1(\rho, \theta) \leq \varphi \leq u_2(\rho, \theta)\}$$

in $\mathbb{R}^3$, where D is the projection of E onto the $\rho\theta$-plane, is

$$\iiint_E f(\rho, \theta, \varphi)\, dV = \iint_D \left[\int_{u_1(\rho,\theta)}^{u_2(\rho,\theta)} f(\rho, \theta, \varphi)\, d\varphi \right] dA.$$

In particular, if $D = \{(\rho, \theta) | g_1(\theta) \leq \rho \leq g_2(\theta),\ \alpha \leq \theta \leq \beta\}$, the we have

$$\iiint_E f(\rho, \theta, \varphi)\, dV = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \int_{u_1(\rho,\theta)}^{u_2(\rho,\theta)} f(\rho, \theta, \varphi) \rho^2 \sin\varphi\, d\varphi\, d\rho\, d\theta.$$

Similar formulas occur for projections onto the other coordinate planes.

Example 15.5.6: Setting up a Triple Integral in Spherical Coordinates

Set up an integral for the volume of the region bounded by the cone $z = \sqrt{3(x^2 + y^2)}$ and the hemisphere $z = \sqrt{4 - x^2 - y^2}$ (see the figure below).

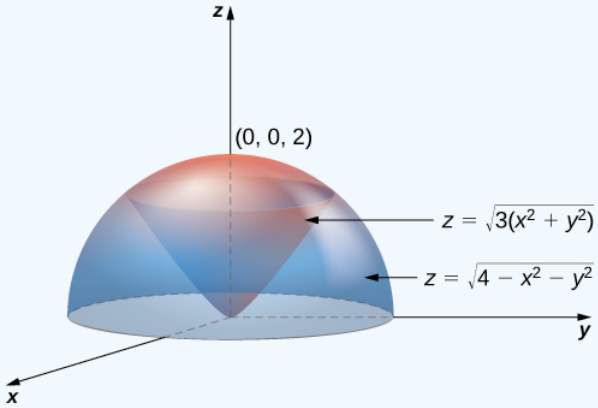

Figure 15.5.9: A region bounded below by a cone and above by a hemisphere.

Solution

Using the conversion formulas from rectangular coordinates to spherical coordinates, we have:

For the cone: $z = \sqrt{3(x^2 + y^2)}$ or $\rho \cos\varphi = \sqrt{3}\rho \sin\varphi$ or $\tan\varphi = \frac{1}{\sqrt{3}}$ or $\varphi = \frac{\pi}{6}$.

For the sphere: $z = \sqrt{4 - x^2 - y^2}$ or $z^2 + x^2 + y^2 = 4$ or $\rho^2 = 4$ or $\rho = 2$.

Thus, the triple integral for the volume is

$$V(E) = \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi/6} \int_{\rho=0}^{\rho=2} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta.$$

Exercise 15.5.5

Set up a triple integral for the volume of the solid region bounded above by the sphere $\rho = 2$ and bounded below by the cone $\varphi = \pi/3$.

Hint

Follow the steps of the previous example.

Answer

$$V(E) = \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi/3} \int_{\rho=0}^{\rho=2} \rho^2 \sin\,\varphi \, d\rho \, d\varphi \, d\theta$$

Example 15.5.7: Interchanging Order of Integration in Spherical Coordinates

Let E be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the sphere $z = x^2 + y^2 + z^2$ (Figure 15.5.10). Set up a triple integral in spherical coordinates and find the volume of the region using the following orders of integration:

a. $d\rho \, d\phi \, d\theta$
b. $d\varphi \, d\rho \, d\theta$

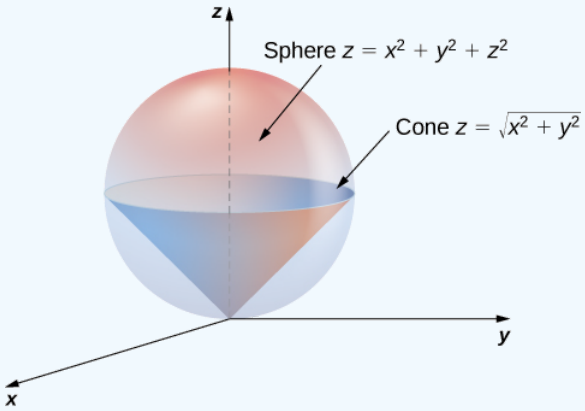

Figure 15.5.10:. A region bounded below by a cone and above by a sphere.

Solution

a. Use the conversion formulas to write the equations of the sphere and cone in spherical coordinates.

For the sphere:

$$\begin{aligned} x^2 + y^2 + z^2 &= z \\ \rho^2 &= \rho \cos\,\varphi \\ \rho &= \cos\,\varphi. \end{aligned}$$

For the cone:

$$\begin{aligned} z &= \sqrt{x^2 + y^2} \\ \rho \cos\,\varphi &= \sqrt{\rho^2 \sin^2\,\varphi \cos^2\theta + \rho^2 \sin^2\,\varphi \sin^2\theta} \\ \rho \cos\,\varphi &= \sqrt{\rho^2 \sin^2\varphi\,(\cos^2\theta + \sin^2\theta)} \\ \rho \cos\,\varphi &= \rho \sin\,\varphi \\ \cos\,\varphi &= \sin\,\varphi \\ \varphi &= \pi/4. \end{aligned}$$

Hence the integral for the volume of the solid region E becomes

$$V(E) = \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi/4} \int_{\rho=0}^{\rho=\cos\,\varphi} \rho^2 \sin\,\varphi \, d\rho \, d\varphi \, d\theta.$$

b. Consider the $\varphi\rho$-plane. Note that the ranges for φ and ρ (from part a.) are

$$\begin{aligned} 0 \le \varphi \le \pi/4 \\ 0 \le \rho \le \cos\varphi \end{aligned}$$

The curve $\rho = \cos\,\varphi$ meets the line $\varphi = \pi/4$ at the point $(\pi/4, \sqrt{2}/2)$. Thus, to change the order of integration, we need to use two pieces:

$$0 \le \rho \le \sqrt{2}/2,\ 0 \le \varphi \le \pi/4$$

and

$$\sqrt{2}/2 \le \rho \le 1,\ 0 \le \varphi \le \cos^{-1}\rho.$$

Hence the integral for the volume of the solid region E becomes

$$V(E) = \int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=\sqrt{2}/2}\int_{\varphi=0}^{\varphi=\pi/4}\rho^2\sin\,\varphi\,d\varphi\,d\rho\,d\theta + \int_{\theta=0}^{\theta=2\pi}\int_{\rho=\sqrt{2}/2}^{\rho=1}\int_{\varphi=0}^{\varphi=\cos^{-1}\rho}\rho^2\sin\,\varphi\,d\varphi\,d\rho\,d\theta$$

In each case, the integration results in $V(E) = \frac{\pi}{8}$.

Before we end this section, we present a couple of examples that can illustrate the conversion from rectangular coordinates to cylindrical coordinates and from rectangular coordinates to spherical coordinates.

Example 15.5.8: Converting from Rectangular Coordinates to Cylindrical Coordinates

Convert the following integral into cylindrical coordinates:

$$\int_{y=-1}^{y=1}\int_{x=0}^{x=\sqrt{1-y^2}}\int_{z=x^2+y^2}^{z=\sqrt{x^2+y^2}} xyz\,dz\,dx\,dy.$$

Solution

The ranges of the variables are

$$\begin{aligned} -1 &\le y \le y \\ 0 &\le x \le \sqrt{1-y^2} \\ x^2+y^2 &\le z \le \sqrt{x^2+y^2}. \end{aligned}$$

The first two inequalities describe the right half of a circle of radius 1. Therefore, the ranges for θ and r are

$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \text{ and } 0 \le r \le 1.$$

The limits of z are $r^2 \le z \le r$, hence

$$\int_{y=-1}^{y=1}\int_{x=0}^{x=\sqrt{1-y^2}}\int_{z=x^2+y^2}^{z=\sqrt{x^2+y^2}} xyz\,dz\,dx\,dy = \int_{\theta=-\pi/2}^{\theta=\pi/2}\int_{r=0}^{r=1}\int_{z=r^2}^{z=r} r(r\,\cos\theta)\,(r\,\sin\,\theta)\,z\,dz\,dr\,d\theta.$$

Example 15.5.9: Converting from Rectangular Coordinates to Spherical Coordinates

Convert the following integral into spherical coordinates:

$$\int_{y=0}^{y=3}\int_{x=0}^{x=\sqrt{9-y^2}}\int_{z=\sqrt{x^2+y^2}}^{z=\sqrt{18-x^2-y^2}} (x^2+y^2+z^2)\,dz\,dx\,dy.$$

Solution

The ranges of the variables are

$$\begin{aligned} 0 &\le y \le 3 \\ 0 &\le x \le \sqrt{9-y^2} \\ \sqrt{x^2+y^2} &\le z \le \sqrt{18-x^2-y^2}. \end{aligned}$$

The first two ranges of variables describe a quarter disk in the first quadrant of the xy-plane. Hence the range for θ is $0 \le \theta \le \frac{\pi}{2}$.

The lower bound $z = \sqrt{x^2+y^2}$ is the upper half of a cone and the upper bound $z = \sqrt{18-x^2-y^2}$ is the upper half of a sphere. Therefore, we have $0 \le \rho \le \sqrt{18}$, which is $0 \le \rho \le 3\sqrt{2}$.

For the ranges of φ we need to find where the cone and the sphere intersect, so solve the equation

$$\begin{aligned} r^2 + z^2 &= 18 \\ \left(\sqrt{x^2+y^2}\right)^2 + z^2 &= 18 \\ z^2 + z^2 &= 18 \\ 2z^2 &= 18 \\ z^2 &= 9 \\ z &= 3. \end{aligned}$$

This gives

$$\begin{aligned} 3\sqrt{2}\cos\varphi &= 3 \\ \cos\varphi &= \frac{1}{\sqrt{2}} \\ \varphi &= \frac{\pi}{4}. \end{aligned}$$

Putting this together, we obtain

$$\int_{y=0}^{y=3}\int_{x=0}^{x=\sqrt{9-y^2}}\int_{z=\sqrt{x^2+y^2}}^{z=\sqrt{18-x^2-y^2}}(x^2+y^2+z^2)\,dz\,dx\,dy = \int_{\varphi=0}^{\varphi=\pi/4}\int_{\theta=0}^{\theta=\pi/2}\int_{\rho=0}^{\rho=3\sqrt{2}}\rho^4\sin\varphi\,d\rho\,d\theta\,d\varphi.$$

? Exercise 15.5.6:

Use rectangular, cylindrical, and spherical coordinates to set up triple integrals for finding the volume of the region inside the sphere $x^2+y^2+z^2=4$ but outside the cylinder $x^2+y^2=1$.

Answer: Rectangular

$$\int_{x=-2}^{x=2}\int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}}\int_{z=-\sqrt{4-x^2-y^2}}^{z=\sqrt{4-x^2-y^2}}dz\,dy\,dx - \int_{x=-1}^{x=1}\int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}}\int_{z=-\sqrt{4-x^2-y^2}}^{z=\sqrt{4-x^2-y^2}}dz\,dy\,dx.$$

Answer: Cylindrical

$$\int_{\theta=0}^{\theta=2\pi}\int_{r=1}^{r=2}\int_{z=-\sqrt{4-r^2}}^{z=\sqrt{4-r^2}}r\,dz\,dr\,d\theta.$$

Answer: Spherical

$$\int_{\varphi=\pi/6}^{\varphi=5\pi/6}\int_{\theta=0}^{\theta=2\pi}\int_{\rho=\csc\varphi}^{\rho=2}\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi.$$

Now that we are familiar with the spherical coordinate system, let's find the volume of some known geometric figures, such as spheres and ellipsoids.

✓ Example 15.5.10: Chapter Opener: Finding the Volume of l'Hemisphèric

Find the volume of the spherical planetarium in l'Hemisphèric in Valencia, Spain, which is five stories tall and has a radius of approximately 50 ft, using the equation $x^2+y^2+z^2=r^2$.

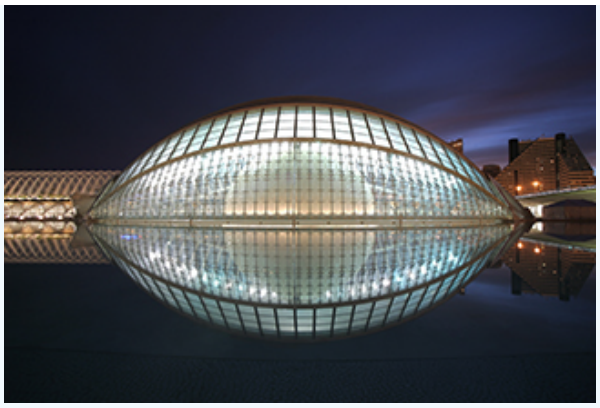

Figure 15.5.11: (credit: modification of work by Javier Yaya Tur, Wikimedia Commons)

Solution

We calculate the volume of the ball in the first octant, where $x \leq 0$, $y \leq 0$, and $z \leq 0$, using spherical coordinates, and then multiply the result by 8 for symmetry. Since we consider the region D as the first octant in the integral, the ranges of the variables are

$$0 \leq \varphi \leq \frac{\pi}{2}, \, 0 \leq \rho \leq r, \, 0 \leq \theta \leq \frac{\pi}{2}.$$

Therefore,

$$\begin{aligned} V = \iiint_D dx\, dy\, dz &= 8 \int_{\theta=0}^{\theta=\pi/2} \int_{\rho=0}^{\rho=\pi} \int_{\varphi=0}^{\varphi=\pi/2} \rho^2 \sin\,\theta \, d\varphi \, d\rho \, d\varphi \\ &= 8 \int_{\varphi=0}^{\varphi=\pi/2} d\varphi \int_{\rho=0}^{\rho=r} \rho^2 d\rho \int_{\theta=0}^{\theta=\pi/2} \sin\,\theta \, d\theta \\ &= 8 \left(\frac{\pi}{2}\right) \left(\frac{r^3}{3}\right) (1) \\ &= \frac{4}{3}\pi r^3. \end{aligned}$$

This exactly matches with what we knew. So for a sphere with a radius of approximately 50 ft, the volume is $\frac{4}{3}\pi(50)^3 \approx 523,600$ ft^3.

For the next example we find the volume of an ellipsoid.

Example 15.5.11: Finding the Volume of an Ellipsoid

Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Solution

We again use symmetry and evaluate the volume of the ellipsoid using spherical coordinates. As before, we use the first octant $x \leq 0$, $y \leq 0$, and $z \leq 0$ and then multiply the result by 8.

In this case the ranges of the variables are

$$0 \leq \varphi \leq \frac{\pi}{2} \, 0 \leq \rho \leq 1, \text{ and } 0 \leq \theta \leq \frac{\pi}{2}.$$

Also, we need to change the rectangular to spherical coordinates in this way:

$$x = a\rho \cos\,\varphi \sin\,\theta, \, y = b\rho \sin\,\varphi \sin\,\theta, \text{ and } z = cp \cos\theta.$$

Then the volume of the ellipsoid becomes

$$
\begin{aligned}
V &= \iiint_D dx\,dy\,dz \\
&= 8\int_{\theta=0}^{\theta=\pi/2}\int_{\rho=0}^{\rho=1}\int_{\varphi=0}^{\varphi=\pi/2} abc\,\rho^2\sin\,\theta\,d\varphi\,d\rho\,d\theta \\
&= 8abc\int_{\varphi=0}^{\varphi=\pi/2} d\varphi\int_{\rho=0}^{\rho=1}\rho^2 d\rho\int_{\theta=0}^{\theta=\pi/2}\sin\,\theta\,d\theta \\
&= 8abc\left(\frac{\pi}{2}\right)\left(\frac{1}{3}\right)(1) \\
&= \frac{4}{3}\pi abc.
\end{aligned}
$$

Example 15.5.12: Finding the Volume of the Space Inside an Ellipsoid and Outside a Sphere

Find the volume of the space inside the ellipsoid $\frac{x^2}{75^2}+\frac{y^2}{80^2}+\frac{z^2}{90^2}=1$ and outside the sphere $x^2+y^2+z^2=50^2$.

Solution

This problem is directly related to the l'Hemisphèric structure. The volume of space inside the ellipsoid and outside the sphere might be useful to find the expense of heating or cooling that space. We can use the preceding two examples for the volume of the sphere and ellipsoid and then substract.

First we find the volume of the ellipsoid using $a=75$ ft, $b=80$ ft, and $c=90$ ft in the result from Example. Hence the volume of the ellipsoid is

$$V_{ellipsoid}=\frac{4}{3}\pi(75)(80)(90)\approx 2,262,000\,ft^3.$$

From Example, the volume of the sphere is

$$V_{sphere}\approx 523,600\,ft^3.$$

Therefore, the volume of the space inside the ellipsoid $\frac{x^2}{75^2}+\frac{y^2}{80^2}+\frac{z^2}{90^2}=1$ and outside the sphere $x^2+y^2+z^2=50^2$ is approximately

$$V_{Hemispheric}=V_{ellipsoid}-V_{sphere}=1,738,400\,ft^3.$$

Student Project: Hot air balloons

Hot air ballooning is a relaxing, peaceful pastime that many people enjoy. Many balloonist gatherings take place around the world, such as the Albuquerque International Balloon Fiesta. The Albuquerque event is the largest hot air balloon festival in the world, with over 500 balloons participating each year.

Figure 15.5.12: Balloons lift off at the 2001 Albuquerque International Balloon Fiesta. (credit: David Herrera, Flickr)

As the name implies, hot air balloons use hot air to generate lift. (Hot air is less dense than cooler air, so the balloon floats as long as the hot air stays hot.) The heat is generated by a propane burner suspended below the opening of the basket. Once the balloon takes off, the pilot controls the altitude of the balloon, either by using the burner to heat the air and ascend or by using a vent near the top of the balloon to release heated air and descend. The pilot has very little control over where the balloon goes, however—balloons are at the mercy of the winds. The uncertainty over where we will end up is one of the reasons balloonists are attracted to the sport.

In this project we use triple integrals to learn more about hot air balloons. We model the balloon in two pieces. The top of the balloon is modeled by a half sphere of radius 28

feet. The bottom of the balloon is modeled by a frustum of a cone (think of an ice cream cone with the pointy end cut off). The radius of the large end of the frustum is 28 feet and the radius of the small end of the frustum is 28 feet. A graph of our balloon model and a cross-sectional diagram showing the dimensions are shown in the following figure.

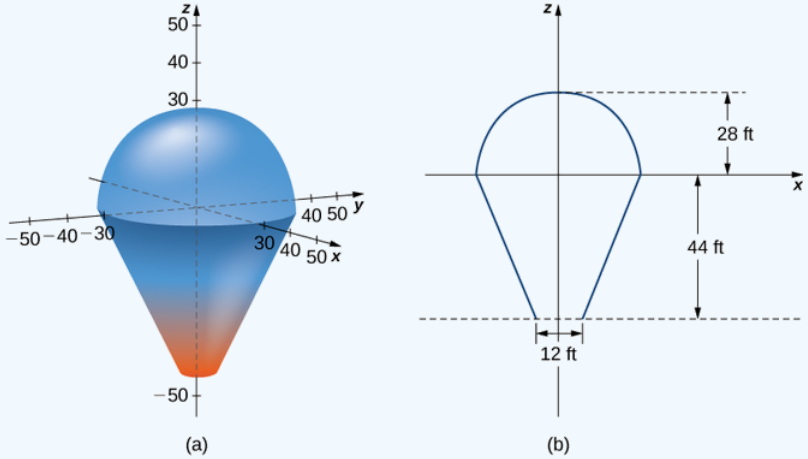

Figure 15.5.13: (a) Use a half sphere to model the top part of the balloon and a frustum of a cone to model the bottom part of the balloon. (b) A cross section of the balloon showing its dimensions.

We first want to find the volume of the balloon. If we look at the top part and the bottom part of the balloon separately, we see that they are geometric solids with known volume formulas. However, it is still worthwhile to set up and evaluate the integrals we would need to find the volume. If we calculate the volume using integration, we can use the known volume formulas to check our answers. This will help ensure that we have the integrals set up correctly for the later, more complicated stages of the project.

1. Find the volume of the balloon in two ways.

a. Use triple integrals to calculate the volume. Consider each part of the balloon separately. (Consider using spherical coordinates for the top part and cylindrical coordinates for the bottom part.)

b. Verify the answer using the formulas for the volume of a sphere, $V = \frac{4}{3}\pi r^3$, and for the volume of a cone, $V = \frac{1}{3}\pi r^2 h$.

In reality, calculating the temperature at a point inside the balloon is a tremendously complicated endeavor. In fact, an entire branch of physics (thermodynamics) is devoted to studying heat and temperature. For the purposes of this project, however, we are going to make some simplifying assumptions about how temperature varies from point to point within the balloon. Assume that just prior to liftoff, the temperature (in degrees Fahrenheit) of the air inside the balloon varies according to the function

$$T_0(r, \theta, z) = \frac{z - r}{10} + 210.$$

2. What is the average temperature of the air in the balloon just prior to liftoff? (Again, look at each part of the balloon separately, and do not forget to convert the function into spherical coordinates when looking at the top part of the balloon.)

Now the pilot activates the burner for 10 seconds. This action affects the temperature in a 12-foot-wide column 20 feet high, directly above the burner. A cross section of the balloon depicting this column in shown in the following figure

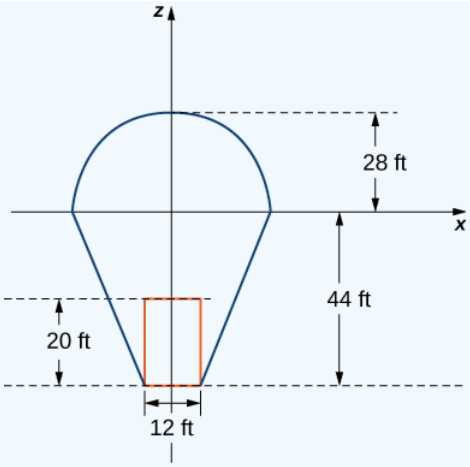

Figure 15.5.14: Activating the burner heats the air in a 20-foot-high, 12-foot-wide column directly above the burner.

Assume that after the pilot activates the burner for 10 seconds, the temperature of the air in the column described above *increases* according to the formula

$$H(r,\theta,z) = -2z - 48.$$

Then the temperature of the air in the column is given by

$$T_1(r,\theta,z) = \frac{z-r}{10} + 210 + (-2z - 48),$$

while the temperature in the remainder of the balloon is still given by

$$T_0(r,\theta,z) = \frac{z-r}{10} + 210.$$

3. Find the average temperature of the air in the balloon after the pilot has activated the burner for 10 seconds.

Key Concepts

- To evaluate a triple integral in cylindrical coordinates, use the iterated integral

$$\int_{\theta=\alpha}^{\theta=\beta}\int_{r=g_1(\theta)}^{r=g_2(\theta)}\int_{z=u_1(r,\theta)}^{u_2(r,\theta)} f(r,\theta,z)r\,dz\,dr\,d\theta.$$

- To evaluate a triple integral in spherical coordinates, use the iterated integral

$$\int_{\theta=\alpha}^{\theta=\beta}\int_{\rho=g_1(\theta)}^{\rho=g_2(\theta)}\int_{\varphi=u_1(r,\theta)}^{u_2(r,\theta)} f(\rho,\theta,\varphi)\,\rho^2\sin\varphi\,d\varphi\,d\rho\,d\theta.$$

Key Equations

- **Triple integral in cylindrical coordinates**

$$\iiint_B g(s,y,z)dV = \iiint_B g(r\cos\theta,\ r\sin\theta,\ z)r\,dr\,d\theta\,dz = \iiint_B f(r,\theta,z)r\,dr\,d\theta\,dz$$

- **Triple integral in spherical coordinates**

$$\iiint_B f(\rho,\theta,\varphi)\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = \int_{\varphi=\gamma}^{\varphi=\psi}\int_{\theta=\alpha}^{\theta=\beta}\int_{\rho=a}^{\rho=b} f(\rho,\theta,\varphi)\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

Glossary

triple integral in cylindrical coordinates

the limit of a triple Riemann sum, provided the following limit exists:

$$lim_{l,m,n\to\infty}\sum^{l}\sum^{m}\sum^{n} f(r^*_{ijk},\theta^*_{ijk},s^*_{ijk})r^*_{ijk}\Delta r\Delta\theta\Delta z$$

$$\lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \Delta x \Delta y \Delta z$$

triple integral in spherical coordinates

the limit of a triple Riemann sum, provided the following limit exists:

$$lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(\rho_{ijk}^*, \theta_{ijk}^*, \varphi_{ijk}^*)(\rho_{ijk}^*)^2 \sin\, \varphi \Delta\rho\Delta\theta\Delta\varphi$$

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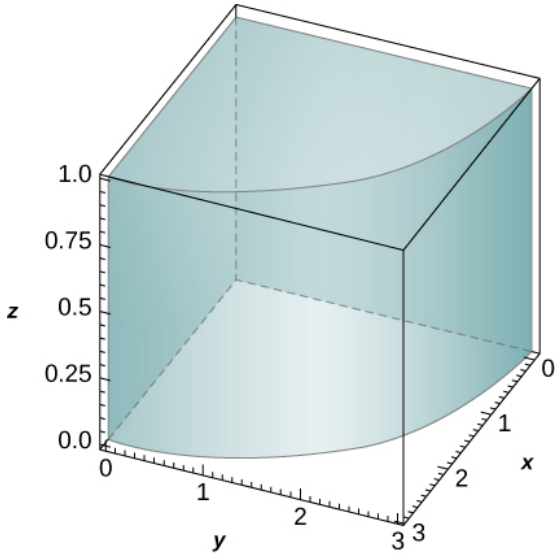
15.5E: EXERCISES FOR SECTION 15.5

In exercises 1 - 8, evaluate the triple integrals $\iiint_E f(x,y,z)\,dV$ over the solid E.

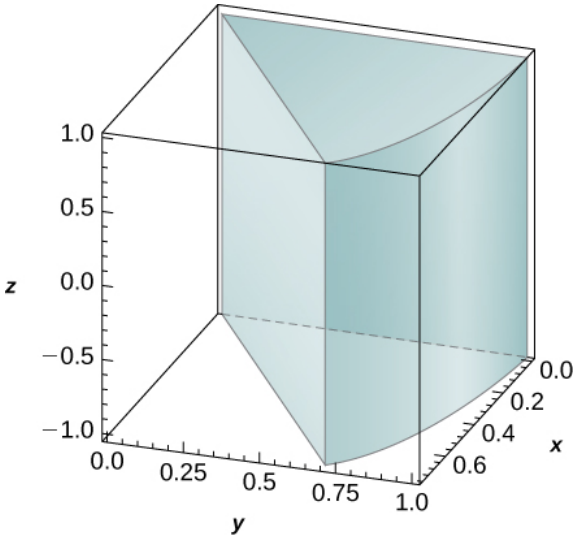
1. $f(x,y,z)=z,\quad B=\{(x,y,z)\,|\,x^2+y^2\le 9,\quad x\le 0,\quad y\le 0,\quad 0\le z\le 1\}$



Answer

$\frac{9\pi}{8}$

2. $f(x,y,z)=xz^2,\ B=\{(x,y,z)\,|\,x^2+y^2\le 16,\ x\ge 0,\ y\le 0,\ -1\le z\le 1\}$
3. $f(x,y,z)=xy,\ B=\{(x,y,z)\,|\,x^2+y^2\le 1,\ x\ge 0,\ x\ge y,\ -1\le z\le 1\}$



Answer

$\frac{1}{8}$

4. $f(x,y,z)=x^2+y^2,\ B=\{(x,y,z)\,|\,x^2+y^2\le 4,\ x\ge 0,\ x\le y,\ 0\le z\le 3\}$
5. $f(x,y,z)=e^{\sqrt{x^2+y^2}},\ B=\{(x,y,z)\,|\,1\le x^2+y^2\le 4,\ y\le 0,\ x\le y\sqrt{3},\ 2\le z\le 3\}$

Answer

$\frac{\pi e^2}{6}$

6. $f(x,y,z)=\sqrt{x^2+y^2},\ B=\{(x,y,z)\,|\,1\le x^2+y^2\le 9,\ y\le 0,\ 0\le z\le 1\}$

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7. a. Let B be a cylindrical shell with inner radius a outer radius b, and height c where $0 < a < b$ and $c > 0$. Assume that a function F defined on B can be expressed in cylindrical coordinates as $F(x, y, z) = f(r) + h(z)$, where f and h are differentiable functions. If $\int_a^b \bar{f}(r)\, dr = 0$ and $\bar{h}(0) = 0$, where $\bar{f}$ and $\bar{h}$ are antiderivatives of f and h, respectively, show that $\iiint_B F(x, y, z)\, dV = 2\pi c(b\bar{f}(b) - a\bar{f}(a)) + \pi(b^2 - a^2)\bar{h}(c)$.

b. Use the previous result to show that $\iiint_B \left(z + \sin\sqrt{x^2 + y^2}\right)\, dx\, dy\, dz = 6\pi^2(\pi - 2)$, where B is a cylindrical shell with inner radius π outer radius 2π, and height 2.

8. a. Let B be a cylindrical shell with inner radius a outer radius b and height c where $0 < a < b$ and $c > 0$. Assume that a function F defined on B can be expressed in cylindrical coordinates as $F(x, y, z) = f(r)g(\theta)f(z)$, where f, g, and h are differentiable functions. If $\int_a^b \tilde{f}(r)\, dr = 0$, where $\tilde{f}$ is an antiderivative of f, show that $\iiint_B F(x, y, z)\, dV = [b\tilde{f}(b) - a\tilde{f}(a)][\tilde{g}(2\pi) - \tilde{g}(0)][\tilde{h}(c) - \tilde{h}(0)]$, where $\tilde{g}$ and $\tilde{h}$ are antiderivatives of g and h, respectively.

b. Use the previous result to show that $\iiint_B z\sin\sqrt{x^2 + y^2}\, dx\, dy\, dz = -12\pi^2$, where B is a cylindrical shell with inner radius π outer radius 2π, and height 2.

In exercises 9 - 12, the boundaries of the solid E are given in cylindrical coordinates.

a. Express the region E in cylindrical coordinates.

b. Convert the integral $\iiint_E f(x, y, z)\, dV$ to cylindrical coordinates.

9. E is bounded by the right circular cylinder $r = 4\sin\theta$, the $r\theta$-plane, and the sphere $r^2 + z^2 = 16$.

Answer

a. $E = \{(r, \theta, z) \mid 0 \le \theta \le \pi,\ 0 \le r \le 4\sin\theta,\ 0 \le z \le \sqrt{16 - r^2}\}$

b. $\int_0^{\pi} \int_0^{4\sin\theta} \int_0^{\sqrt{16 - r^2}} f(r, \theta, z) r\, dz\, dr\, d\theta$

10. E is bounded by the right circular cylinder $r = \cos\theta$, the $r\theta$-plane, and the sphere $r^2 + z^2 = 9$.

11. E is located in the first octant and is bounded by the circular paraboloid $z = 9 - 3r^2$, the cylinder $r = \sqrt{3}$, and the plane $r(\cos\theta + \sin\theta) = 20 - z$.

Answer

a. $E = \{(r, \theta, z) \mid 0 \le \theta \le \frac{\pi}{2},\ 0 \le r \le \sqrt{3},\ 9 - r^2 \le z \le 10 - r(\cos\theta + \sin\theta)\}$

b. $\int_0^{\pi/2} \int_0^{\sqrt{3}} \int_{9 - r^2}^{10 - r(\cos\theta + \sin\theta)} f(r, \theta, z) r\, dz\, dr\, d\theta$

12. E is located in the first octant outside the circular paraboloid $z = 10 - 2r^2$ and inside the cylinder $r = \sqrt{5}$ and is bounded also by the planes $z = 20$ and $\theta = \frac{\pi}{4}$.

In exercises 13 - 16, the function f and region E are given.

a. Express the region E and the function f in cylindrical coordinates.

b. Convert the integral $\iiint_B f(x, y, z)\, dV$ into cylindrical coordinates and evaluate it.

13. $f(x, y, z) = x^2 + y^2$, $E = \{(x, y, z) \mid 0 \le x^2 + y^2 \le 9,\ x \ge 0,\ y \ge 0,\ 0 \le z \le x + 3\}$

Answer

a. $E = \{(r, \theta, z) \mid 0 \le r \le 3,\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le z \le r\ \cos\theta + 3\}$,
$f(r, \theta, z) = \frac{1}{r\ \cos\theta + 3}$

b. $\int_0^3 \int_0^{\pi/2} \int_0^{r\ \cos\theta + 3} \frac{r}{r\ \cos\theta + 3}\, dz\, d\theta\, dr = \frac{9\pi}{4}$

14. $f(x, y, z) = x^2 + y^2$, $E = \{(x, y, z) | 0 \le x^2 + y^2 \le 4,\ y \ge 0,\ 0 \le z \le 3 - x\}$

15. $f(x, y, z) = x$, $E = \{(x, y, z) \mid 1 \le y^2 + z^2 \le 9,\ 0 \le x \le 1 - y^2 - z^2\}$

Answer

a. $y = r\ \cos\theta,\ z = r\ \sin\theta,\ x = z,\ E = \{(r, \theta, z) \mid 1 \le r \le 3,\ 0 \le \theta \le 2\pi,\ 0 \le z \le 1 - r^2\},\ f(r, \theta, z) = z;$

b. $\int_1^3 \int_0^{2\pi} \int_0^{1 - r^2} zr\, dz\, d\theta\, dr = \frac{356\pi}{3}$

16. $f(x,y,z)=y,\ E=\{(x,y,z)\,|\,1\le x^2+z^2\le 9,\ 0\le y\le 1-x^2-z^2\}$

In exercises 17 - 24, find the volume of the solid E whose boundaries are given in rectangular coordinates.

17. E is above the xy-plane, inside the cylinder $x^2+y^2=1$, and below the plane $z=1$.

Answer

π

18. E is below the plane $z=1$ and inside the paraboloid $z=x^2+y^2$.

19. E is bounded by the circular cone $z=\sqrt{x^2+y^2}$ and $z=1$.

Answer

$\frac{\pi}{3}$

20. E is located above the xy-plane, below $z=1$, outside the one-sheeted hyperboloid $x^2+y^2-z^2=1$, and inside the cylinder $x^2+y^2=2$.

21. E is located inside the cylinder $x^2+y^2=1$ and between the circular paraboloids $z=1-x^2-y^2$ and $z=x^2+y^2$.

Answer

π

22. E is located inside the sphere $x^2+y^2+z^2=1$, above the xy-plane, and inside the circular cone $z=\sqrt{x^2+y^2}$.

23. E is located outside the circular cone $x^2+y^2=(z-1)^2$ and between the planes $z=0$ and $z=2$.

Answer

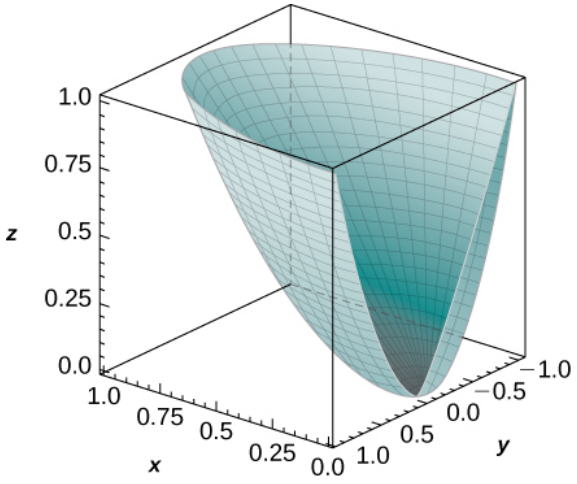
$\frac{4\pi}{3}$

24. E is located outside the circular cone $z=1-\sqrt{x^2+y^2}$, above the xy-plane, below the circular paraboloid, and between the planes $z=0$ and $z=2$.

25. [T] Use a computer algebra system (CAS) to graph the solid whose volume is given by the iterated integral in cylindrical coordinates $\int_{-\pi/2}^{\pi/2}\int_0^1\int_{r^2}^{r} r\,dz\,dr\,d\theta$. Find the volume V of the solid. Round your answer to four decimal places.

Answer

$V=\frac{pi}{12}\approx 0.2618$



26. [T] Use a CAS to graph the solid whose volume is given by the iterated integral in cylindrical coordinates $\int_0^{\pi/2}\int_0^1\int_{r^4}^{r} r\,dz\,dr\,d\theta$. Find the volume E of the solid. Round your answer to four decimal places.

27. Convert the integral $\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} xz\,dz\,dx\,dy$ into an integral in cylindrical coordinates.

Answer

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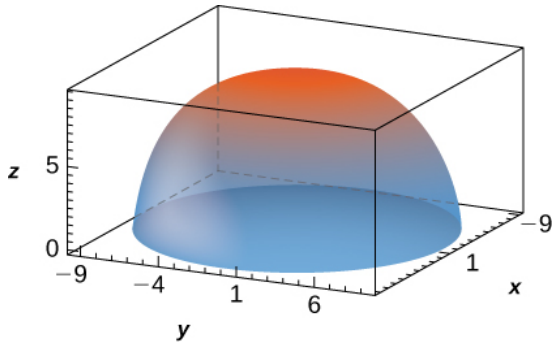
$$\int_0^1 \int_0^{\pi} \int_{r^2}^{r} zr^2 \cos\theta \, dz \, d\theta \, dr$$

28. Convert the integral $\int_0^2 \int_0^y \int_0^1 (xy+z)\, dz\, dx\, dy$ into an integral in cylindrical coordinates.

In exercises 29 - 32, evaluate the triple integral $\iiint_B f(x,y,z)\, dV$ over the solid B.

29. $f(x,y,z) = 1,\ B = \{(x,y,z) \mid x^2+y^2+z^2 \le 90,\ z \ge 0\}$

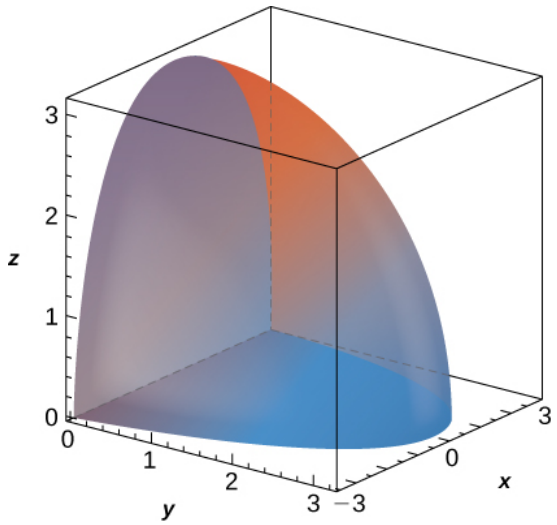


[Hide Solution]

Answer

$180\pi\sqrt{10}$

30. $f(x,y,z) = 1 - \sqrt{x^2+y^2+z^2},\ B = \{(x,y,z) \mid x^2+y^2+z^2 \le 9,\ y \ge 0,\ z \ge 0\}$



31. $f(x,y,z) = \sqrt{x^2+y^2},\ B$ is bounded above by the half-sphere $x^2+y^2+z^2 = 9$ with $z \ge 0$ and below by the cone $2z^2 = x^2+y^2$.

Answer

$\frac{81\pi(\pi-2)}{16}$

32. $f(x,y,z) = \sqrt{x^2+y^2},\ B$ is bounded above by the half-sphere $x^2+y^2+z^2 = 16$ with $z \ge 0$ and below by the cone $2z^2 = x^2+y^2$.

33. Show that if $F(\rho,\theta,\varphi) = f(\rho)g(\theta)h(\varphi)$ is a continuous function on the spherical box $B = \{(\rho,\theta,\varphi) \mid a \le \rho \le b,\ \alpha \le \theta \le \beta,\ \gamma \le \varphi \le \psi\}$, then $\iiint_B F\, dV = \left(\int_a^b \rho^2 f(\rho)\, dr\right)\left(\int_\alpha^\beta g(\theta)\, d\theta\right)\left(\int_\gamma^\psi h(\varphi)\, \sin\varphi\, d\varphi\right)$.

34. A function F is said to have spherical symmetry if it depends on the distance to the origin only, that is, it can be expressed in spherical coordinates as $F(x,y,z) = f(\rho)$, where $\rho = \sqrt{x^2+y^2+z^2}$. Show that $\iiint_B F(x,y,z)\, dV = 2\pi \int_a^b \rho^2 f(\rho)\, d\rho$, where B is the region between the upper concentric hemispheres of radii a and b centered at the origin, with $0 < a < b$ and F a spherical function defined on B.

Use the previous result to show that $\iiint_B (x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}\, dV = 21\pi$, where $B = \{(x,y,z) \mid 1 \le x^2+y^2+z^2 \le 2,\ z \ge 0\}$.

35. Let B be the region between the upper concentric hemispheres of radii a and b centered at the origin and situated in the first octant, where $0 < a < b$. Consider F a function defined on B whose form in spherical coordinates (ρ, θ, φ) is $F(x, y, z) = f(\rho)\cos\varphi$. Show that if $g(a) = g(b) = 0$ and $\int_a^b h(\rho)\, d\rho = 0$, then $\iiint_B F(x, y, z)\, dV = \frac{\pi^2}{4}[ah(a) - bh(b)]$, where g is an antiderivative of f and h is an antiderivative of g.

Use the previous result to show that $\iiint_B = \frac{z\cos\sqrt{x^2+y^2+z^2}}{\sqrt{x^2+y^2+z^2}}\, dV = \frac{3\pi^2}{2}$, where B is the region between the upper concentric hemispheres of radii π and 2π centered at the origin and situated in the first octant.

In exercises 36 - 39, the function f and region E are given.

a. Express the region E and function f in cylindrical coordinates.

b. Convert the integral $\iiint_B f(x, y, z)\, dV$ into cylindrical coordinates and evaluate it.

36. $f(x, y, z) = z;\ E = \{(x, y, z) \mid 0 \le x^2 + y^2 + z^2 \le 1,\ z \ge 0\}$

37. $f(x, y, z) = x + y;\ E = \{(x, y, z) \mid 1 \le x^2 + y^2 + z^2 \le 2,\ z \ge 0,\ y \ge 0\}$

Answer

a. $f(\rho, \theta, \varphi) = \rho\ \sin\varphi\ (\cos\theta + \sin\theta),\ E = \{(\rho, \theta, \varphi) \mid 1 \le \rho \le 2,\ 0 \le \theta \le \pi,\ 0 \le \varphi \le \frac{\pi}{2}\}$;

b. $\int_0^{\pi}\int_0^{\pi/2}\int_1^2 \rho^3\cos\varphi\ \sin\varphi\ d\rho\, d\varphi\, d\theta = \frac{15\pi}{8}$

38. $f(x, y, z) = 2xy;\ E = \{(x, y, z) \mid \sqrt{x^2+y^2} \le z \le \sqrt{1 - x^2 - y^2},\ x \ge 0,\ y \ge 0\}$

39. $f(x, y, z) = z;\ E = \{(x, y, z) \mid x^2 + y^2 + z^2 - 2x \le 0,\ \sqrt{x^2+y^2} \le z\}$

Answer

a. $f(\rho, \theta, \varphi) = \rho\ \cos\varphi;\ E = \{(\rho, \theta, \varphi) \mid 0 \le \rho \le 2\ \cos\varphi,\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le \varphi \le \frac{\pi}{4}\}$;

b. $\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{2\ \cos\varphi} \rho^3\sin\varphi\ \cos\varphi\ d\rho\, d\varphi\, d\theta = \frac{7\pi}{24}$

In exercises 40 - 41, find the volume of the solid E whose boundaries are given in rectangular coordinates.

40. $E = \{(x, y, z) \mid \sqrt{x^2+y^2} \le z \le \sqrt{16 - x^2 - y^2},\ x \ge 0,\ y \ge 0\}$

41. $E = \{(x, y, z) \mid x^2 + y^2 + z^2 - 2z \le 0,\ \sqrt{x^2+y^2} \le z\}$

Answer

$\frac{\pi}{4}$

42. Use spherical coordinates to find the volume of the solid situated outside the sphere $\rho = 1$ and inside the sphere $\rho = \cos\varphi$, with $\varphi \in [0, \frac{\pi}{2}]$.

43. Use spherical coordinates to find the volume of the ball $\rho \le 3$ that is situated between the cones $\varphi = \frac{\pi}{4}$ and $\varphi = \frac{\pi}{3}$.

Answer

$9\pi(\sqrt{2} - 1)$

44. Convert the integral $\int_{-4}^{4}\int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}}\int_{-\sqrt{16-x^2-y^2}}^{\sqrt{16-x^2-y^2}} (x^2 + y^2 + z^2)\, dz\, dx\, dy$ into an integral in spherical coordinates.

45. Convert the integral $\int_0^4\int_0^{\sqrt{16-x^2}}\int_{-\sqrt{16-x^2-y^2}}^{\sqrt{16-x^2-y^2}} (x^2 + y^2 + z^2)^2\, dz\, dy\, dx$ into an integral in spherical coordinates.

Answer

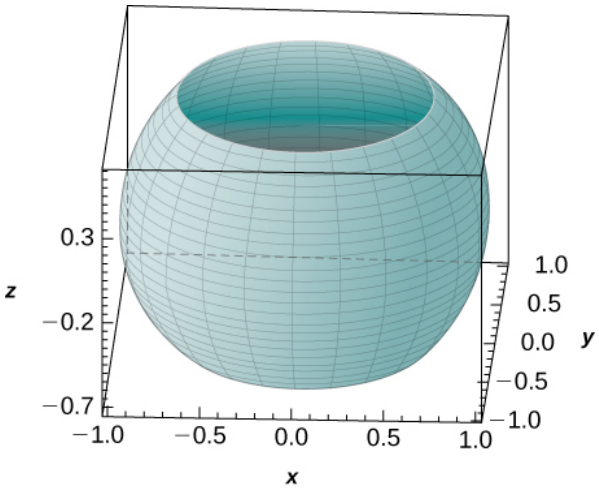
$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^4 \rho^6\sin\varphi\, d\rho\, d\phi\, d\theta$

46. Convert the integral $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}} dz\, dy\, dx$ into an integral in spherical coordinates and evaluate it.

47. [T] Use a CAS to graph the solid whose volume is given by the iterated integral in spherical coordinates $\int_{\pi/2}^{\pi}\int_{5\pi}^{\pi/6}\int_0^2 \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$. Find the volume V of the solid. Round your answer to three decimal places.

Answer

$V = \frac{4\pi\sqrt{3}}{3} \approx 7.255$



48. [T] Use a CAS to graph the solid whose volume is given by the iterated integral in spherical coordinates as $\int_0^{2\pi}\int_{3\pi/4}^{\pi/4}\int_0^1 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$. Find the volume V of the solid. Round your answer to three decimal places.

49. [T] Use a CAS to evaluate the integral $\iiint_E (x^2+y^2)\,dV$ where E lies above the paraboloid $z=x^2+y^2$ and below the plane $z=3y$.

Answer

$\frac{343\pi}{32}$

50. [T]

a. Evaluate the integral $\iiint_E e^{\sqrt{x^2+y^2+z^2}}\,dV$, where E is bounded by spheres $4x^2+4y^2+4z^2=1$ and $x^2+y^2+z^2=1$.

b. Use a CAS to find an approximation of the previous integral. Round your answer to two decimal places.

51. Express the volume of the solid inside the sphere $x^2+y^2+z^2=16$ and outside the cylinder $x^2+y^2=4$ as triple integrals in cylindrical coordinates and spherical coordinates, respectively.

Answer

$\int_0^{2\pi}\int_2^4\int_{-\sqrt{16-r^2}}^{\sqrt{16-r^2}} r\,dz\,dr\,d\theta$ and $\int_{\pi/6}^{5\pi/6}\int_0^{2\pi}\int_{2\csc\phi}^4 \rho^2\sin\rho\,d\rho\,d\theta\,d\phi$

52. Express the volume of the solid inside the sphere $x^2+y^2+z^2=16$ and outside the cylinder $x^2+y^2=4$ that is located in the first octant as triple integrals in cylindrical coordinates and spherical coordinates, respectively.

53. The power emitted by an antenna has a power density per unit volume given in spherical coordinates by $p(\rho,\theta,\varphi)=\frac{P_0}{\rho^2}\cos^2\theta\,\sin^4\varphi$, where P_0 is a constant with units in watts. The total power within a sphere B of radius r meters is defined as $P=\iiint_B p(\rho,\theta,\varphi)\,dV$. Find the total power P.

Answer

$P=\frac{32P_0\pi}{3}$ watts

54. Use the preceding exercise to find the total power within a sphere B of radius 5 meters when the power density per unit volume is given by $p(\rho,\theta,\varphi)=\frac{30}{\rho^2}\cos^2\theta\sin^4\varphi$.

55. A charge cloud contained in a sphere B of radius r centimeters centered at the origin has its charge density given by $q(x,y,z)=k\sqrt{x^2+y^2+z^2}\,\frac{\mu C}{cm^3}$, where $k>0$. The total charge contained in B is given by $Q=\iiint_B q(x,y,z)\,dV$. Find the total charge Q.

Answer

$Q=kr^4\pi\mu C$

56. Use the preceding exercise to find the total charge cloud contained in the unit sphere if the charge density is $q(x,y,z)=20\sqrt{x^2+y^2+z^2}\,\frac{\mu C}{cm^3}$.

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15.6: Calculating Centers of Mass and Moments of Inertia

Learning Objectives

- Use double integrals to locate the center of mass of a two-dimensional object.
- Use double integrals to find the moment of inertia of a two-dimensional object.
- Use triple integrals to locate the center of mass of a three-dimensional object.

We have already discussed a few applications of multiple integrals, such as finding areas, volumes, and the average value of a function over a bounded region. In this section we develop computational techniques for finding the center of mass and moments of inertia of several types of physical objects, using double integrals for a lamina (flat plate) and triple integrals for a three-dimensional object with variable density. The density is usually considered to be a constant number when the lamina or the object is homogeneous; that is, the object has uniform density.

Center of Mass in Two Dimensions

The center of mass is also known as the center of gravity if the object is in a uniform gravitational field. If the object has uniform density, the center of mass is the geometric center of the object, which is called the centroid. Figure $15.6.1$ shows a point P as the center of mass of a lamina. The lamina is perfectly balanced about its center of mass.

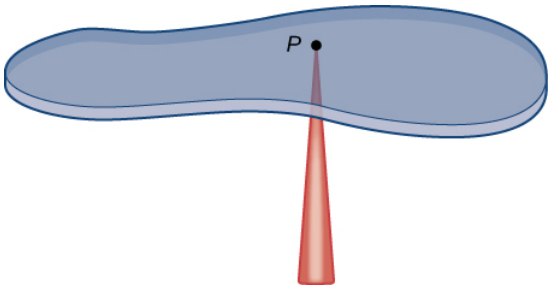

Figure $15.6.1$: A lamina is perfectly balanced on a spindle if the lamina's center of mass sits on the spindle.

To find the coordinates of the center of mass $P(\bar{x},\bar{y})$ of a lamina, we need to find the moment M_x of the lamina about the x-axis and the moment M_y about the y-axis. We also need to find the mass m of the lamina. Then

$$\bar{x}=\frac{M_y}{m}$$

and

$$\bar{y}=\frac{M_x}{m}.$$

Refer to Moments and Centers of Mass for the definitions and the methods of single integration to find the center of mass of a one-dimensional object (for example, a thin rod). We are going to use a similar idea here except that the object is a two-dimensional lamina and we use a double integral.

If we allow a constant density function, then $\bar{x}=\frac{M_y}{m}$ and $\bar{y}=\frac{M_x}{m}$ give the *centroid* of the lamina.

Suppose that the lamina occupies a region R in the xy-plane and let $\rho(x,y)$ be its density (in units of mass per unit area) at any point (x,y). Hence,

$$\rho(x,y)=\lim_{\Delta A\to 0}\frac{\Delta m}{\Delta A}$$

where Δm and ΔA are the mass and area of a small rectangle containing the point (x,y) and the limit is taken as the dimensions of the rectangle go to 0 (see the following figure).

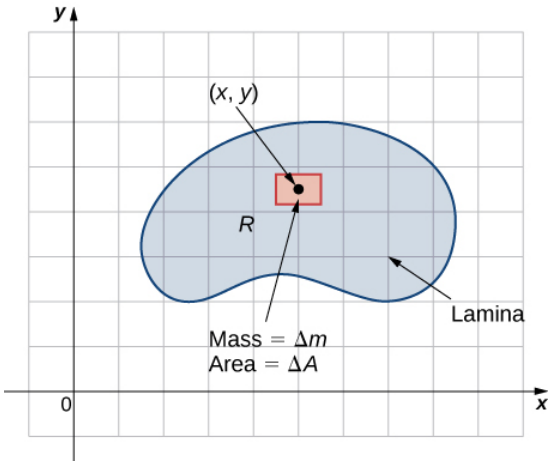

Figure 15.6.2: The density of a lamina at a point is the limit of its mass per area in a small rectangle about the point as the area goes to zero.

Just as before, we divide the region R into tiny rectangles R_{ij} with area ΔA and choose (x_{ij}^*, y_{ij}^*) as sample points. Then the mass m_{ij} of each R_{ij} is equal to $\rho(x_{ij}^*, y_{ij}^*)\Delta A$ (Figure 15.6.2). Let k and l be the number of subintervals in x and y respectively. Also, note that the shape might not always be rectangular but the limit works anyway, as seen in previous sections.

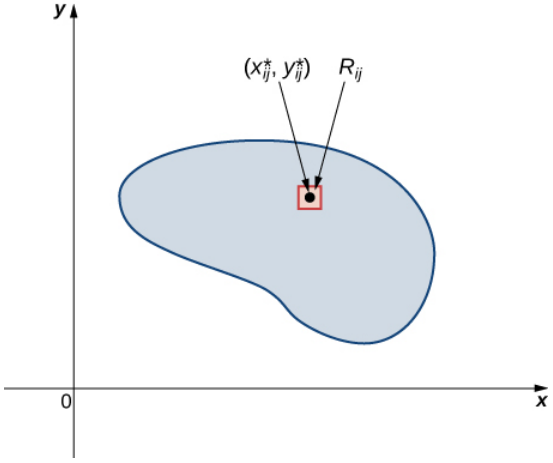

Figure 15.6.3: Subdividing the lamina into tiny rectangles R_{ij} each containing a sample point (x_{ij}^*, y_{ij}^*).

Hence, the mass of the lamina is

$$m = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} m_{ij} = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_R \rho(x,y)dA.$$

Let's see an example now of finding the total mass of a triangular lamina.

Example 15.6.1: Finding the Total Mass of a Lamina

Consider a triangular lamina R with vertices $(0,0)$, $(0,3)$, $(3,0)$ and with density $\rho(x,y) = xy\ kg/m^2$. Find the total mass.

Solution

A sketch of the region R is always helpful, as shown in the following figure.

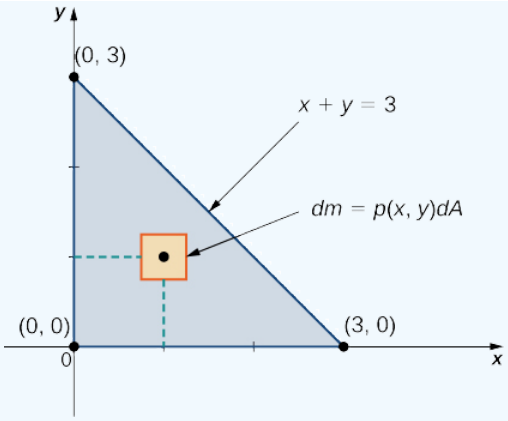

Figure 15.6.4: A lamina in the xy-plane with density $\rho(x,y) = xy$.

Using the expression developed for mass, we see that

$$m = \iint_R dm = \iint_R \rho(x,y)dA = \int_{x=0}^{x=3}\int_{y=0}^{y=3-x} xy\,dy\,dx = \int_{x=0}^{x=3}\left[x\frac{y^2}{2}\Big|_{y=0}^{y=3}\right]dx$$
$$= \int_{x=0}^{x=3}\frac{1}{2}x(3-x)^2dx = \left[\frac{9x^2}{4} - x^3 + \frac{x^4}{8}\right]\Big|_{x=0}^{x=3} = \frac{27}{8}.$$

The computation is straightforward, giving the answer $m = \dfrac{27}{8}\ kg$.

Exercise 15.6.1

Consider the same region R as in the previous example, and use the density function $\rho(x,y) = \sqrt{xy}$. Find the total mass.

Answer

$\dfrac{9\pi}{8}\ kg$

Now that we have established the expression for mass, we have the tools we need for calculating moments and centers of mass. The moment M_z about the x-axis for R is the limit of the sums of moments of the regions R_{ij} about the x-axis. Hence

$$M_x = \lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}(y_{ij}^*)m_{ij} = \lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}(y_{ij}^*)\rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_R y\rho(x,y)\,dA$$

Similarly, the moment M_y about the y-axis for R is the limit of the sums of moments of the regions R_{ij} about the y-axis. Hence

$$M_y = \lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}(x_{ij}^*)m_{ij} = \lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}(x_{ij}^*)\rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_R x\rho(x,y)\,dA$$

Example 15.6.2: Finding Moments

Consider the same triangular lamina R with vertices $(0,0)$, $(0,3)$, $(3,0)$ and with density $\rho(x,y) = xy$. Find the moments M_x and M_y.

Solution

Use double integrals for each moment and compute their values:

$$M_x = \iint_R y\rho(x,y)\,dA = \int_{x=0}^{x=3}\int_{y=0}^{y=3-x} xy^2\,dy\,dx = \frac{81}{20},$$

$$M_y = \iint_R x\rho(x,y)\,dA = \int_{x=0}^{x=3}\int_{y=0}^{y=3-x} x^2y\,dy\,dx = \frac{81}{20},$$

The computation is quite straightforward.

Exercise 15.6.2

Consider the same lamina R as above and use the density function $\rho(x,y)=\sqrt{xy}$. Find the moments M_x and M_y.

Answer

$M_x=\dfrac{81\pi}{64}$ and $M_y=\dfrac{81\pi}{64}$

Finally we are ready to restate the expressions for the center of mass in terms of integrals. We denote the *x*-coordinate of the center of mass by $\bar{x}$ and the *y*-coordinate by $\bar{y}$. Specifically,

$$\bar{x}=\frac{M_y}{m}=\frac{\iint_R x\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA}$$

and

$$\bar{y}=\frac{M_x}{m}=\frac{\iint_R y\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA}$$

Example 15.6.3: center of mass

Again consider the same triangular region R with vertices $(0,0),\ (0,3),\ (3,0)$ and with density function $\rho(x,y)=xy$. Find the center of mass.

Solution

Using the formulas we developed, we have

$$\bar{x}=\frac{M_y}{m}=\frac{\iint_R x\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA}=\frac{81/20}{27/8}=\frac{6}{5},$$

$$\bar{y}=\frac{M_x}{m}=\frac{\iint_R y\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA}=\frac{81/20}{27/8}=\frac{6}{5}.$$

Therefore, the center of mass is the point $\left(\dfrac{6}{5},\dfrac{6}{5}\right)$.

Analysis

If we choose the density $\rho(x,y)$ instead to be uniform throughout the region (i.e., constant), such as the value 1 (any constant will do), then we can compute the centroid,

$$x_c=\frac{M_y}{m}=\frac{\iint_R x\,dA}{\iint_R dA}=\frac{9/2}{9/2}=1,$$

$$y_c=\frac{M_x}{m}=\frac{\iint_R y\,dA}{\iint_R dA}=\frac{9/2}{9/2}=1.$$

Notice that the center of mass $\left(\dfrac{6}{5},\dfrac{6}{5}\right)$ is not exactly the same as the centroid $(1,1)$ of the triangular region. This is due to the variable density of R If the density is constant, then we just use $\rho(x,y)=c$ (constant). This value cancels out from the formulas, so for a constant density, the center of mass coincides with the centroid of the lamina.

Exercise 15.6.3

Again use the same region R as above and use the density function $\rho(x,y)=\sqrt{xy}$. Find the center of mass.

Answer

$\bar{x}=\dfrac{M_y}{m}=\dfrac{81\pi/64}{9\pi/8}=\dfrac{9}{8}$ and $\bar{y}=\dfrac{M_x}{m}=\dfrac{81\pi}{9\pi/8}=\dfrac{0}{8}$.

Once again, based on the comments at the end of Example 15.6.3, we have expressions for the centroid of a region on the plane:

$$x_c = \frac{M_y}{m} = \frac{\iint_R x\,dA}{\iint_R dA} \text{ and } y_c = \frac{M_x}{m} = \frac{\iint_R y\,dA}{\iint_R dA}.$$

We should use these formulas and verify the centroid of the triangular region referred to in the last three examples.

Example 15.6.4: Finding Mass, Moments, and Center of Mass

Find the mass, moments, and the center of mass of the lamina of density $\rho(x,y) = x + y$ occupying the region R under the curve $y = x^2$ in the interval $0 \le x \le 2$ (see the following figure).

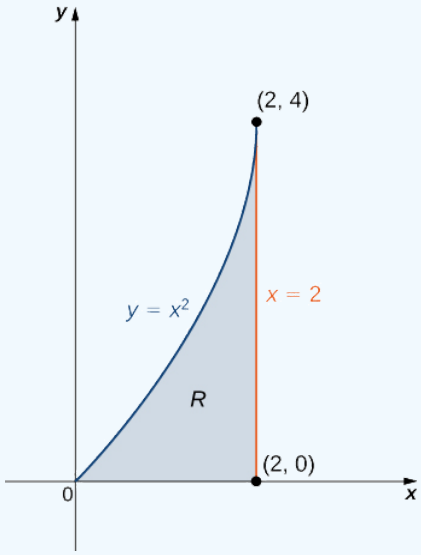

Figure 15.6.5: Locating the center of mass of a lamina R with density $\rho(x,y) = x + y$.

Solution

First we compute the mass m. We need to describe the region between the graph of $y = x^2$ and the vertical lines $x = 0$ and $x = 2$:

$$m = \iint_R dm = \iint_R \rho(x,y)\,dA = \int_{x=0} x = 2 \int_{y=0}^{y=x^2} (x+y)dy\,dx = \int_{x=0}^{x=2} \left[xy + \frac{y^2}{2} \Big|_{y=0}^{y=x^2} \right] dx$$

$$= \int_{x=0}^{x=2} \left[x^3 + \frac{x^4}{2} \right] dx = \left[\frac{x^4}{4} + \frac{x^5}{10} \right] \Big|_{x=0}^{x=2} = \frac{36}{5}.$$

Now compute the moments M_x and M_y:

$$M_x = \iint_R y\rho(x,y)\,dA = \int_{x=0}^{x=2} \int_{y=0}^{y=x^2} y(x+y)\,dy\,dx = \frac{80}{7},$$

$$M_y = \iint_R x\rho(x,y)\,dA = \int_{x=0}^{x=2} \int_{y=0}^{y=x^2} x(x+y)\,dy\,dx = \frac{176}{15}.$$

Finally, evaluate the center of mass,

$$\bar{x} = \frac{M_y}{m} = \frac{\iint_R x\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA} = \frac{176/15}{36/5} = \frac{44}{27},$$

$$\bar{y} = \frac{M_x}{m} = \frac{\iint_R y\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA} = \frac{80/7}{36/5} = \frac{100}{63}.$$

Hence the center of mass is $(\bar{x}, \bar{y}) = \left(\frac{44}{27}, \frac{100}{63} \right)$.

Exercise 15.6.4

Calculate the mass, moments, and the center of mass of the region between the curves $y = x$ and $y = x^2$ with the density function $\rho(x, y) = x$ in the interval $0 \le x \le 1$.

Answer

$$\bar{x} = \frac{M_y}{m} = \frac{1/20}{1/12} = \frac{3}{5} \text{ and } \bar{y} = \frac{M_x}{m} = \frac{1/24}{1/12} = \frac{1}{2}$$

Example 15.6.5: Finding a Centroid

Find the centroid of the region under the curve $y = e^x$ over the interval $1 \le x \le 3$ (Figure 15.6.6).

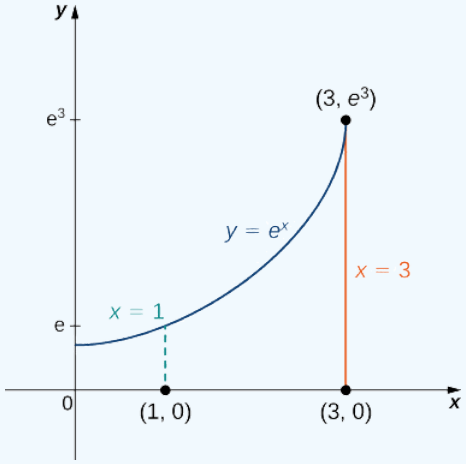

Figure 15.6.6: Finding a centroid of a region below the curve $y = e^x$.

Solution

To compute the centroid, we assume that the density function is constant and hence it cancels out:

$$x_c = \frac{M_y}{m} = \frac{\iint_R x\,dA}{\iint_R dA} \; and \; y_c = \frac{M_x}{m} = \frac{\iint_R y\,dA}{\iint_R dA},$$

$$x_c = \frac{M_y}{m} = \frac{\iint_R x\,dA}{\iint_R dA} = \frac{\int_{x=1}^{x=3}\int_{y=0}^{y=e^x} x\,dy\,dx}{\int_{x=1}^{x=3}\int_{y=0}^{y=e^x} dy\,dx} = \frac{\int_{x=1}^{x=3} xe^x dx}{\int_{x=1}^{x=3} e^x dx} = \frac{2e^3}{e^3 - e} = \frac{2e^2}{e^2 - 1},$$

$$y_c = \frac{M_x}{m} = \frac{\iint_R y\,dA}{\iint_R dA} = \frac{\int_{x=1}^{x=3}\int_{y=0}^{y=e^x} y\,dy\,dx}{\int_{x=1}^{x=3}\int_{y=0}^{y=e^x} dy\,dx} = \frac{\int_{x=1}^{x=3} \frac{e^{2x}}{2} dx}{\int_{x=1}^{x=3} e^x dx} = \frac{\frac{1}{4}e^2(e^4 - 1)}{e(e^2 - 1)} = \frac{1}{4}e(e^2 + 1).$$

Thus the centroid of the region is

$$(x_c, y_c) = \left(\frac{2e^2}{e^2 - 1}, \frac{1}{4}e(e^2 + 1)\right).$$

Exercise 15.6.5

Calculate the centroid of the region between the curves $y = x$ and $y = \sqrt{x}$ with uniform density in the interval $0 \le x \le 1$.

Answer

$$x_c = \frac{M_y}{m} = \frac{1/15}{1/6} = \frac{2}{5} \text{ and } y_c = \frac{M_x}{m} = \frac{1/12}{1/6} = \frac{1}{2}$$

Moments of Inertia

For a clear understanding of how to calculate moments of inertia using double integrals, we need to go back to the general definition in Section 6.6. The moment of inertia of a particle of mass m about an axis is mr^2 where r is the distance of the particle from the axis. We can see from Figure 15.6.3 that the moment of inertia of the subrectangle R_{ij} about the x-axis is $(y_{ij}^*)^2\rho(x_{ij}^*, y_{ij}^*)\Delta A$. Similarly, the moment of inertia of the subrectangle R_{ij} about the y-axis is $(x_{ij}^*)^2\rho(x_{ij}^*, y_{ij}^*)\Delta A$. The moment of inertia is related to the rotation of the mass; specifically, it measures the tendency of the mass to resist a change in rotational motion about an axis.

The moment of inertia I_x about the x-axis for the region R is the limit of the sum of moments of inertia of the regions R_{ij} about the x-axis. Hence

$$I_x = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} (y_{ij}^*)^2 m_{ij} = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} (y_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_R y^2\rho(x,y)\,dA.$$

Similarly, the moment of inertia I_y about the y-axis for R is the limit of the sum of moments of inertia of the regions R_{ij} about the y-axis. Hence

$$I_y = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} (x_{ij}^*)^2 m_{ij} = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} (x_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_R x^2\rho(x,y)\,dA.$$

Sometimes, we need to find the moment of inertia of an object about the origin, which is known as the polar moment of inertia. We denote this by I_0 and obtain it by adding the moments of inertia I_x and I_y. Hence

$$I_0 = I_x + I_y = \iint_R (x^2+y^2)\rho(x,y)\,dA.$$

All these expressions can be written in polar coordinates by substituting $x = r\cos\theta$, $y = r\sin\theta$, and $dA = r\,dr\,d\theta$. For example, $I_0 = \iint_R r^2\rho(r\cos\theta, r\sin\theta)\,dA$.

Example 15.6.6: Finding Moments of Inertia for a Triangular Lamina

Use the triangular region R with vertices $(0,0)$, $(2,2)$, and $(2,0)$ and with density $\rho(x,y) = xy$ as in previous examples. Find the moments of inertia.

Solution

Using the expressions established above for the moments of inertia, we have

$$I_x = \iint_R y^2\rho(x,y)\,dA = \int_{x=0}^{x=2}\int_{y=0}^{y=x} xy^3\,dy\,dx = \frac{8}{3},$$

$$I_y = \iint_R x^2\rho(x,y)\,dA = \int_{x=0}^{x=2}\int_{y=0}^{y=x} x^3y\,dy\,dx = \frac{16}{3},$$

$$I_0 = \iint_R (x^2+y^2)\rho(x,y)\,dA = \int_0^2\int_0^x (x^2+y^2)xy\,dy\,dx = I_x + I_y = 8$$

Exercise 15.6.6

Again use the same region R as above and the density function $\rho(x,y) = \sqrt{xy}$. Find the moments of inertia.

Answer

$$I_x = \int_{x=0}^{x=2}\int_{y=0}^{y=x} y^2\sqrt{xy}\,dy\,dx = \frac{64}{35}$$

and

$$I_y = \int_{x=0}^{x=2}\int_{y=0}^{y=x} x^2\sqrt{xy}\,dy\,dx = \frac{64}{35}.$$

Also,

$$I_0 = \int_{x=0}^{x=2} \int_{y=0}^{y=x} (x^2 + y^2)\sqrt{xy}\, dy\, dx = \frac{128}{21}$$

As mentioned earlier, the moment of inertia of a particle of mass m about an axis is mr^2 where r is the distance of the particle from the axis, also known as the **radius of gyration**.

Hence the radii of gyration with respect to the x-axis, the y-axis and the origin are

$$R_x = \sqrt{\frac{I_x}{m}},\ R_y = \sqrt{\frac{I_y}{m}},\ and\ R_0 = \sqrt{\frac{I_0}{m}},$$

respectively. In each case, the radius of gyration tells us how far (perpendicular distance) from the axis of rotation the entire mass of an object might be concentrated. The moments of an object are useful for finding information on the balance and torque of the object about an axis, but radii of gyration are used to describe the distribution of mass around its centroidal axis. There are many applications in engineering and physics. Sometimes it is necessary to find the radius of gyration, as in the next example.

Example 15.6.7: Finding the Radius of Gyration for a Triangular Lamina

Consider the same triangular lamina R with vertices $(0,0)$, $(2,2)$, and $(2,0)$ and with density $\rho(x,y) = xy$ as in previous examples. Find the radii of gyration with respect to the x-axis the y-axis and the origin.

Solution

If we compute the mass of this region we find that $m = 2$. We found the moments of inertia of this lamina in Example 15.6.4. From these data, the radii of gyration with respect to the x-axis, y-axis and the origin are, respectively,

$$R_x = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{8/3}{2}} = \sqrt{\frac{8}{6}} = \frac{2\sqrt{3}}{3}, \tag{15.6.1}$$

$$R_y = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{16/3}{2}} = \sqrt{\frac{8}{3}} = \frac{2\sqrt{6}}{3}, \tag{15.6.2}$$

$$R_0 = \sqrt{\frac{I_0}{m}} = \sqrt{\frac{8}{2}} = \sqrt{4} = 2. \tag{15.6.3}$$

Exercise 15.6.7

Use the same region R from Example 15.6.7 and the density function $\rho(x,y) = \sqrt{xy}$. Find the radii of gyration with respect to the x-axis, the y-axis, and the origin.

Hint

Follow the steps shown in the previous example.

Answer

$R_x = \frac{6\sqrt{35}}{35}$, $R_y = \frac{6\sqrt{15}}{15}$, and $R_0 = \frac{4\sqrt{42}}{7}$.

Center of Mass and Moments of Inertia in Three Dimensions

All the expressions of double integrals discussed so far can be modified to become triple integrals.

Definition

If we have a solid object Q with a density function $\rho(x,y,z)$ at any point (x,y,z) in space, then its mass is

$$m = \iiint_Q \rho(x,y,z)\, dV.$$

Its moments about the xy-plane the xz-plane and the yz-plane are

$$M_{xy} = \iiint_Q z\rho(x,y,z)\,dV,\ M_{xz} = \iiint_Q y\rho(x,y,z)\,dV,\ M_{yz} = \iiint_Q x\rho(x,y,z)\,dV.$$

If the center of mass of the object is the point $(\bar{x},\bar{y},\bar{z})$, then

$$\bar{x} = \frac{M_{yz}}{m},\ \bar{y} = \frac{M_{xz}}{m},\ \bar{z} = \frac{M_{xy}}{m}.$$

Also, if the solid object is homogeneous (with constant density), then the center of mass becomes the centroid of the solid. Finally, the moments of inertia about the yz-plane, xz-plane, and the xy-plane are

$$I_x = \iiint_Q (y^2+z^2)\,\rho(x,y,z)\,dV,$$

$$I_y = \iiint_Q (x^2+z^2)\,\rho(x,y,z)\,dV,$$

$$I_z = \iiint_Q (x^2+y^2)\,\rho(x,y,z)\,dV.$$

Example 15.6.8: Finding the Mass of a Solid

Suppose that Q is a solid region bounded by $x+2y+3z=6$ and the coordinate planes and has density $\rho(x,y,z) = x^2yz$. Find the total mass.

Solution

The region Q is a tetrahedron (Figure 15.6.7) meeting the axes at the points $(6,0,0)$, $(0,3,0)$, and $(0,0,2)$. To find the limits of integration, let $z=0$ in the slanted plane $z = \frac{1}{3}(6-x-2y)$. Then for x and y find the projection of Q onto the xy-plane, which is bounded by the axes and the line $x+2y=6$. Hence the mass is

$$m = \iiint_Q \rho(x,y,z)\,dV = \int_{x=0}^{x=6}\int_{y=0}^{y=1/2(6-x)}\int_{z=0}^{z=1/3(6-x-2y)} x^2yz\,dz\,dy\,dx = \frac{108}{35}$$

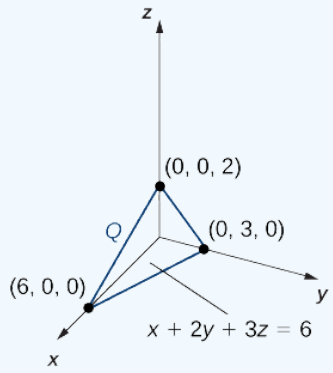

Figure 15.6.7: Finding the mass of a three-dimensional solid Q.

Exercise 15.6.8

Consider the same region Q (Figure 15.6.7), and use the density function $\rho(x,y,z) = xy^2z$. Find the mass.

Hint

Follow the steps in the previous example.

Answer

$\frac{54}{35} = 1.543$

Example 15.6.9: Finding the Center of Mass of a Solid

Suppose Q is a solid region bounded by the plane $x + 2y + 3z = 6$ and the coordinate planes with density $\rho(x, y, z) = x^2yz$ (see Figure 15.6.7). Find the center of mass using decimal approximation.

Solution

We have used this tetrahedron before and know the limits of integration, so we can proceed to the computations right away. First, we need to find the moments about the xy-plane, the xz-plane, and the yz-plane:

$$M_{xy} = \iiint_Q z\rho(x, y, z)\, dV = \int_{x=0}^{x=6} \int_{y=0}^{y=\frac{1}{2}(6-x)} \int_{z=0}^{z=\frac{1}{3}(6-x-2y)} x^2yz^2\, dz\, dy\, dx = \frac{54}{35} \approx 1.543,$$

$$M_{xz} = \iiint_Q y\rho(x, y, z)\, dV = \int_{x=0}^{x=6} \int_{y=0}^{y=\frac{1}{2}(6-x)} \int_{z=0}^{z=\frac{1}{3}(6-x-2y)} x^2y^2z\, dz\, dy\, dx = \frac{81}{35} \approx 2.314,$$

$$M_{yz} = \iiint_Q x\rho(x, y, z)\, dV = \int_{x=0}^{x=6} \int_{y=0}^{y=\frac{1}{2}(6-x)} \int_{z=0}^{z=\frac{1}{3}(6-x-2y)} x^3yz\, dz\, dy\, dx = \frac{243}{35} \approx 6.943.$$

Hence the center of mass is

$$\bar{x} = \frac{M_{yz}}{m},\ \bar{y} = \frac{M_{xz}}{m},\ \bar{z} = \frac{M_{xy}}{m},$$

$$\bar{x} = \frac{M_{yz}}{m} = \frac{243/35}{108/35} = \frac{243}{108} = 2.25,$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{81/35}{108/35} = \frac{81}{108} = 0.75,$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{54/35}{108/35} = \frac{54}{108} = 0.5$$

The center of mass for the tetrahedron Q is the point $(2.25, 0.75, 0.5)$.

Exercise 15.6.9

Consider the same region Q (Figure 15.6.7) and use the density function $\rho(x, y, z) = xy^2z$. Find the center of mass.

Hint

Check that $M_{xy} = \frac{27}{35}$, $M_{xz} = \frac{243}{140}$, and $M_{yz} = \frac{81}{35}$. Then use m from a previous checkpoint question.

Answer

$\left(\frac{3}{2}, \frac{9}{8}, \frac{1}{2}\right)$

We conclude this section with an example of finding moments of inertia I_x, I_y, and I_z.

Example 15.6.10: Finding the Moments of Inertia of a Solid

Suppose that Q is a solid region and is bounded by $x + 2y + 3z = 6$ and the coordinate planes with density $\rho(x, y, z) = x^2yz$ (see Figure 15.6.7). Find the moments of inertia of the tetrahedron Q about the yz-plane, the xz-plane, and the xy-plane.

Solution

Once again, we can almost immediately write the limits of integration and hence we can quickly proceed to evaluating the moments of inertia. Using the formula stated before, the moments of inertia of the tetrahedron Q about the yz-plane, the xz-plane, and the xy-plane are

$$I_x = \iiint_Q (y^2 + z^2)\rho(x, y, z)\, dV,$$

$$I_y = \iiint_Q (x^2 + z^2)\rho(x, y, z)\, dV,$$

and

$$I_z = \iiint_Q (x^2 + y^2)\rho(x, y, z)\, dV \text{ with } \rho(x, y, z) = x^2yz.$$

Proceeding with the computations, we have

$$I_x = \iiint_Q (y^2 + z^2)x^2\rho(x, y, z)\, dV$$

$$= \int_{x=0}^{x=6} \int_{y=0}^{y=\frac{1}{2}(6-x)} \int_{z=0}^{z=\frac{1}{3}(6-x-2y)} (y^2 + z^2)x^2yz\, dz\, dy\, dx = \frac{117}{35} \approx 3.343,$$

$$I_y = \iiint_Q (x^2 + z^2)x^2\rho(x, y, z)\, dV$$

$$= \int_{x=0}^{x=6} \int_{y=0}^{y=\frac{1}{2}(6-x)} \int_{z=0}^{z=\frac{1}{3}(6-x-2y)} (x^2 + z^2)x^2yz\, dz\, dy\, dx = \frac{684}{35} \approx 19.543,$$

$$I_z = \iiint_Q (x^2 + y^2)x^2\rho(x, y, z)\, dV$$

$$= \int_{x=0}^{x=6} \int_{y=0}^{y=\frac{1}{2}(6-x)} \int_{z=0}^{z=\frac{1}{3}(6-x-2y)} (x^2 + y^2)x^2yz\, dz\, dy\, dx = \frac{729}{35} \approx 20.829.$$

Thus, the moments of inertia of the tetrahedron Q about the yz-plane, the xz-plane, and the xy-plane are $117/35$, $684/35$, and $729/35$, respectively.

? Exercise 15.6.10

Consider the same region Q (Figure 15.6.7), and use the density function $\rho(x, y, z) = xy^2z$. Find the moments of inertia about the three coordinate planes.

Answer

The moments of inertia of the tetrahedron Q about the yz-plane, the xz-plane, and the xy-plane are $99/35$, $36/7$ and $243/35$, respectively.

Key Concepts

Finding the mass, center of mass, moments, and moments of inertia in double integrals:

- For a lamina R with a density function $\rho(x, y)$ at any point (x, y) in the plane, the mass is

$$m = \iint_R \rho(x, y)\, dA.$$

- The moments about the x-axis and y-axis are

$$M_x = \iint_R y\rho(x, y)\, dA \text{ and } M_y = \iint_R x\rho(x, y)\, dA.$$

- The center of mass is given by $\bar{x} = \dfrac{M_y}{m}$, $\bar{y} = \dfrac{M_x}{m}$.
- The center of mass becomes the centroid of the plane when the density is constant.
- The moments of inertia about the x-axis, y-axis, and the origin are

$$I_x = \iint_R y^2\rho(x, y)\, dA,\ I_y = \iint_R x^2\rho(x, y)\, dA, \text{ and } I_0 = I_x + I_y = \iint_R (x^2 + y^2)\rho(x, y)\, dA.$$

Finding the mass, center of mass, moments, and moments of inertia in triple integrals:

- For a solid object Q with a density function $\rho(x, y, z)$ at any point (x, y, z) in space, the mass is

$$m = \iiint_Q \rho(x, y, z)\, dV.$$

- The moments about the xy-plane, the xz-plane, and the yz-plane are

$$M_{xy} = \iiint_Q z\rho(x, y, z)\, dV,\ M_{xz} = \iiint_Q y\rho(x, y, z)\, dV,\ M_{yz} = \iiint_Q x\rho(x, y, z)\, dV$$

- The center of mass is given by $\bar{x} = \dfrac{M_{yz}}{m}$, $\bar{y} = \dfrac{M_{xz}}{m}$, $\bar{z} = \dfrac{M_{xy}}{m}$.
- The center of mass becomes the centroid of the solid when the density is constant.
- The moments of inertia about the yz-plane, the xz-plane, and the xy-plane are

$$I_x = \iiint_Q (y^2 + z^2)\, \rho(x, y, z)\, dV,\ I_y = \iiint_Q (x^2 + z^2)\, \rho(x, y, z)\, dV,$$
$$I_z = \iiint_Q (x^2 + y^2)\, \rho(x, y, z)\, dV.$$

Key Equations

- **Mass of a lamina**

$$m = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} m_{ij} = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_R \rho(x, y)\, dA$$

- **Moment about the *x*-axis**

$$M_x = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} (y_{ij}^*) m_{ij} = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} (y_{ij}^*) \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_R y\rho(x, y)\, dA$$

- **Moment about the *y*-axis**

$$M_y = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} (x_{ij}^*) m_{ij} = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} (x_{ij}^*) \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_R x\rho(x, y)\, dA$$

- **Center of mass of a lamina**

$$\bar{x} = \frac{M_y}{m} = \frac{\iint_R x\rho(x, y)\, dA}{\iint_R \rho(x, y)\, dA} \text{ and } \bar{y} = \frac{M_x}{m} = \frac{\iint_R y\rho(x, y)\, dA}{\iint_R \rho(x, y)\, dA}$$

Glossary

radius of gyration

the distance from an object's center of mass to its axis of rotation

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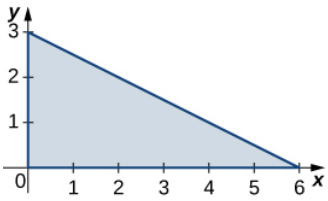
15.6E: Exercises for Section 15.6

In exercises 1 - 12, the region R occupied by a lamina is shown in a graph. Find the mass of R with the density function ρ.

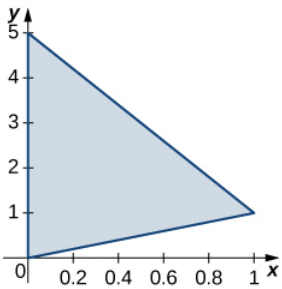
1. R is the triangular region with vertices $(0,0),\ (0,3)$, and $(6,0);\ \rho(x,y)=xy$.



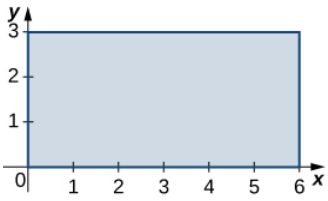
Answer

$\frac{27}{2}$

2. R is the triangular region with vertices $(0,0),\ (1,1)$, and $(0,5);\ \rho(x,y)=x+y$.



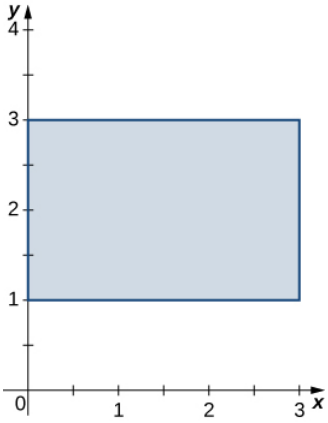
3. R is the rectangular region with vertices $(0,0),\ (0,3),\ (6,3)$ and $(6,0);\ \rho(x,y)=\sqrt{xy}$.



Answer

$24\sqrt{2}$

4. R is the rectangular region with vertices $(0,1),\ (0,3),\ (3,3)$ and $(3,1);\ \rho(x,y)=x^2y$.



5. R is the trapezoidal region determined by the lines $y=-\frac{1}{4}x+\frac{5}{2},\ y=0,\ y=2$, and $x=0;\ \rho(x,y)=3xy$.

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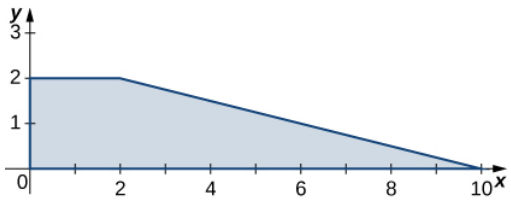
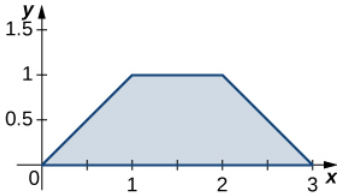


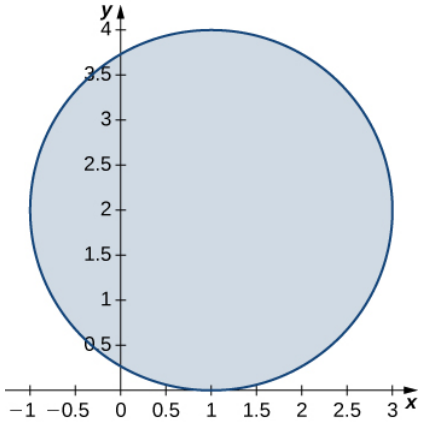
Answer

76

6. R is the trapezoidal region determined by the lines $y = 0,\ y = 1,\ y = x$ and $y = -x + 3;\ \rho(x, y) = 2x + y$.



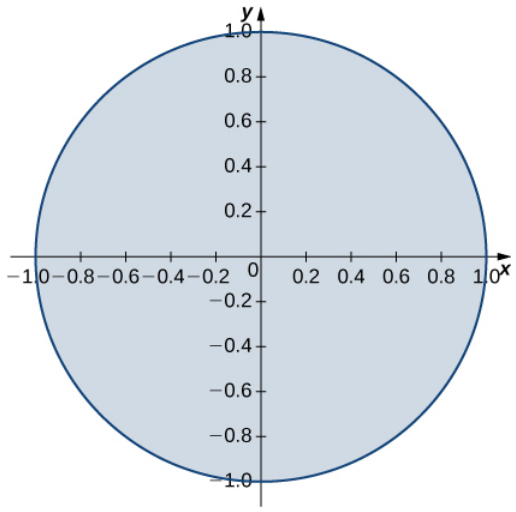
7. R is the disk of radius 2 centered at $(1, 2);\ \rho(x, y) = x^2 + y^2 - 2x - 4y + 5$.



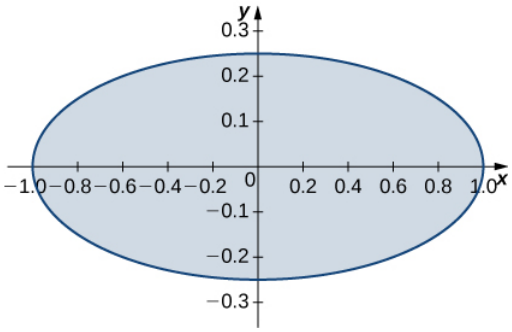
Answer

8π

8. R is the unit disk; $\rho(x, y) = 3x^4 + 6x^2y^2 + 3y^4$.



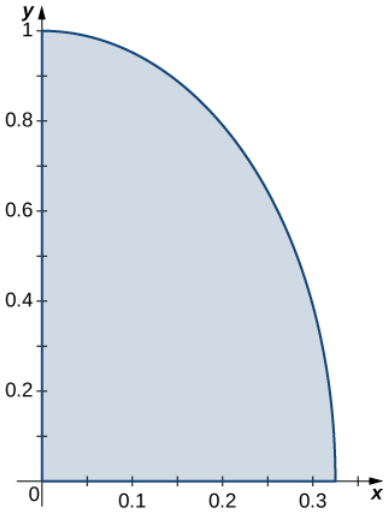
9. R is the region enclosed by the ellipse $x^2 + 4y^2 = 1;\ \rho(x, y) = 1$.



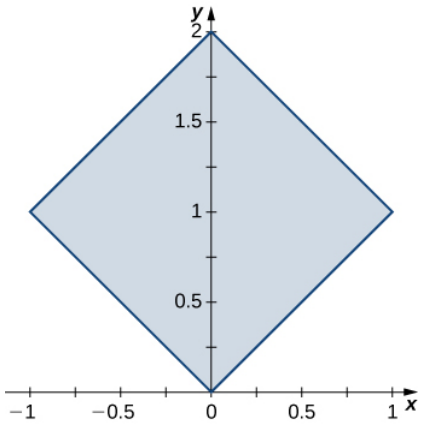
Answer

$\frac{\pi}{2}$

10. $R = \{(x, y) \mid 9x^2 + y^2 \le 1,\ x \ge 0,\ y \ge 0\};\ \rho(x, y) = \sqrt{9x^2 + y^2}$.



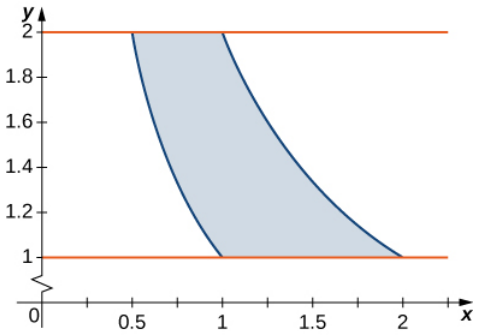
11. R is the region bounded by $y = x,\ y = -x,\ y = x + 2,\ y = -x + 2;\ \rho(x, y) = 1$.



Answer

2

12. R is the region bounded by $y = \frac{1}{x},\ y = \frac{2}{x},\ y = 1$, and $y = 2;\ \rho(x, y) = 4(x + y)$.



In exercises 13 - 24, consider a lamina occupying the region R and having the density function ρ given in the preceding group of exercises. Use a computer algebra system (CAS) to answer the following questions.

a. Find the moments M_x and M_y about the x-axis and y-axis, respectively.

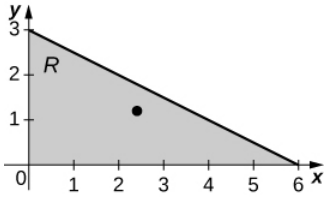
b. Calculate and plot the center of mass of the lamina.

c. [T] Use a CAS to locate the center of mass on the graph of R.

13. [T] R is the triangular region with vertices $(0,0),\ (0,3)$, and $(6,0);\ \rho(x,y)=xy$.

Answer

a. $M_x=\frac{81}{5},\ M_y=\frac{162}{5}$;
b. $\bar{x}=\frac{12}{5},\ \bar{y}=\frac{6}{5}$;
c.

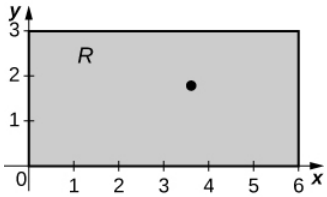


14. [T] R is the triangular region with vertices $(0,0),\ (1,1)$, and $(0,5);\ \rho(x,y)=x+y$.

15. [T] R is the rectangular region with vertices $(0,0),\ (0,3),\ (6,3)$, and $(6,0);\ \rho(x,y)=\sqrt{xy}$.

Answer

a. $M_x=\frac{216\sqrt{2}}{5},\ M_y=\frac{432\sqrt{2}}{5}$;
b. $\bar{x}=\frac{18}{5},\ \bar{y}=\frac{9}{5}$;
c.

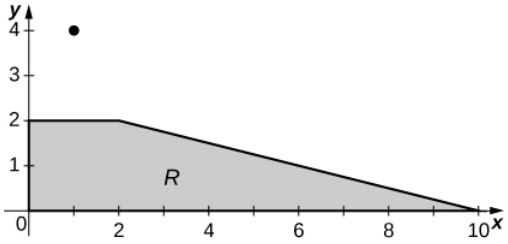


16. [T] R is the rectangular region with vertices $(0,1),\ (0,3),\ (3,3)$, and $(3,1);\ \rho(x,y)=x^2y$.

17. [T] R is the trapezoidal region determined by the lines $y=-\frac{1}{4}x+\frac{5}{2},\ y=0,\ y=2$, and $x=0;\ \rho(x,y)=3xy$.

Answer

a. $M_x=\frac{368}{5},\ M_y=\frac{1552}{5}$;
b. $\bar{x}=\frac{92}{95},\ \bar{y}=\frac{388}{95}$;
c.

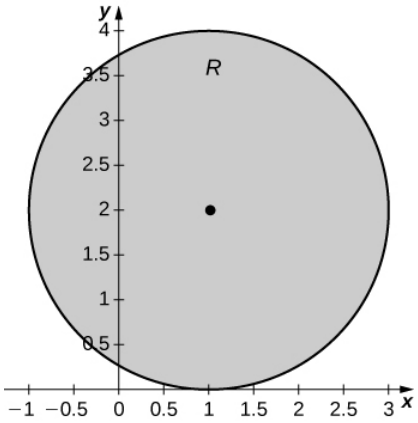


18. [T] R is the trapezoidal region determined by the lines $y = 0,\ y = 1,\ y = x$, and $y = -x + 3;\ \rho(x, y) = 2x + y$.

19. [T] R is the disk of radius 2 centered at $(1, 2);\ \rho(x, y) = x^2 + y^2 - 2x - 4y + 5$.

Answer

a. $M_x = 16\pi,\ M_y = 8\pi$;
b. $\bar{x} = 1,\ \bar{y} = 2$;
c.

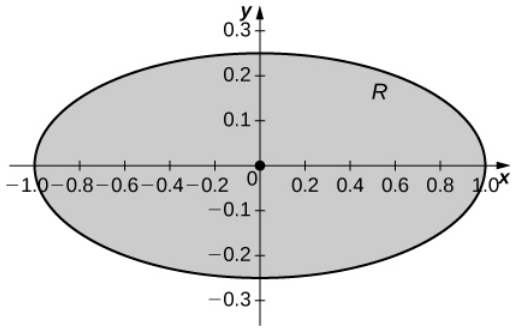


20. [T] R is the unit disk; $\rho(x, y) = 3x^4 + 6x^2y^2 + 3y^4$.

21. [T] R is the region enclosed by the ellipse $x^2 + 4y^2 = 1;\ \rho(x, y) = 1$.

Answer

a. $M_x = 0,\ M_y = 0)$;
b. $\bar{x} = 0,\ \bar{y} = 0$;
c.

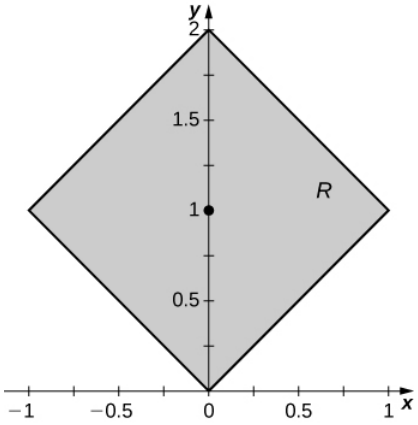


22. [T] $R = \{(x, y) \mid 9x^2 + y^2 \le 1,\ x \ge 0,\ y \ge 0\};\ \rho(x, y) = \sqrt{9x^2 + y^2}$.

23. [T] R is the region bounded by $y = x,\ y = -x,\ y = x + 2$, and $y = -x + 2;\ \rho(x, y) = 1$.

Answer

a. $M_x = 2,\ M_y = 0)$;
b. $\bar{x} = 0,\ \bar{y} = 1$;
c.



24. [T] R is the region bounded by $y = \frac{1}{x},\ y = \frac{2}{x},\ y = 1$, and $y = 2;\ \rho(x, y) = 4(x + y)$.

In exercises 25 - 36, consider a lamina occupying the region R and having the density function ρ given in the first two groups of Exercises.

a. Find the moments of inertia I_x, I_y and I_0 about the x-axis, y-axis, and origin, respectively.

b. Find the radii of gyration with respect to the x-axis, y-axis, and origin, respectively.

25. R is the triangular region with vertices $(0, 0),\ (0, 3)$, and $(6, 0);\ \rho(x, y) = xy$.

Answer

a. $I_x = \frac{243}{10},\ I_y = \frac{486}{5}$, and $I_0 = \frac{243}{2}$;
b. $R_x = \frac{3\sqrt{5}}{5},\ R_y = \frac{6\sqrt{5}}{5}$, and $R_0 = 3$

26. R is the triangular region with vertices $(0, 0),\ (1, 1)$, and $(0, 5);\ \rho(x, y) = x + y$.

27. R is the rectangular region with vertices $(0, 0),\ (0, 3),\ (6, 3)$, and $(6, 0);\ \rho(x, y) = \sqrt{xy}$.

Answer

a. $I_x = \frac{2592\sqrt{2}}{7},\ I_y = \frac{648\sqrt{2}}{7}$, and $I_0 = \frac{3240\sqrt{2}}{7}$;
b. $R_x = \frac{6\sqrt{21}}{7},\ R_y = \frac{3\sqrt{21}}{7}$, and $R_0 = \frac{3\sqrt{106}}{7}$

28. R is the rectangular region with vertices $(0, 1),\ (0, 3),\ (3, 3)$, and $(3, 1);\ \rho(x, y) = x^2y$.

29. R is the trapezoidal region determined by the lines $y = -\frac{1}{4}x + \frac{5}{2},\ y = 0,\ y = 2$, and x = 0; \space \rho (x,y) = 3xy\).

Answer

a. $I_x = 88,\ I_y = 1560$, and $I_0 = 1648$;
b. $R_x = \frac{\sqrt{418}}{19},\ R_y = \frac{\sqrt{7410}}{10}$, and $R_0 = \frac{2\sqrt{1957}}{19}$

30. R is the trapezoidal region determined by the lines $y = 0,\ y = 1,\ y = x$, and y = -x + 3; \space \rho (x,y) = 2x + y\).

31. R is the disk of radius 2 centered at $(1, 2);\ \rho(x, y) = x^2 + y^2 - 2x - 4y + 5$.

Answer

a. $I_x = \frac{128\pi}{3},\ I_y = \frac{56\pi}{3}$, and $I_0 = \frac{184\pi}{3}$;
b. $R_x = \frac{4\sqrt{3}}{3},\ R_y = \frac{\sqrt{21}}{2}$, and $R_0 = \frac{\sqrt{69}}{3}$

32. R is the unit disk; $\rho(x, y) = 3x^4 + 6x^2y^2 + 3y^4$.

33. R is the region enclosed by the ellipse $x^2 + 4y^2 = 1;\ \rho(x, y) = 1$.

Answer

a. $I_x = \frac{\pi}{32},\ I_y = \frac{\pi}{8}$, and $I_0 = \frac{5\pi}{32}$;
b. $R_x = \frac{1}{4},\ R_y = \frac{1}{2}$, and $R_0 = \frac{\sqrt{5}}{4}$

34. $R = \{(x,y)\,|\,9x^2 + y^2 \le 1,\ x \ge 0,\ y \ge 0\};\ \rho(x,y) = \sqrt{9x^2 + y^2}$.

35. R is the region bounded by $y = x,\ y = -x,\ y = x + 2$, and $y = -x + 2;\ \rho(x,y) = 1$.

Answer

a. $I_x = \frac{7}{3},\ I_y = \frac{1}{3}$, and $I_0 = \frac{8}{3}$;
b. $R_x = \frac{\sqrt{42}}{6},\ R_y = \frac{\sqrt{6}}{6}$, and $R_0 = \frac{2\sqrt{3}}{3}$

36. R is the region bounded by $y = \frac{1}{x},\ y = \frac{2}{x},\ y = 1$, and $y = 2;\ \rho(x,y) = 4(x + y)$.

37. Let Q be the solid unit cube. Find the mass of the solid if its density ρ is equal to the square of the distance of an arbitrary point of Q to the xy-plane.

Answer

$m = \frac{1}{3}$

38. Let Q be the solid unit hemisphere. Find the mass of the solid if its density ρ is proportional to the distance of an arbitrary point of Q to the origin.

39. The solid Q of constant density 1 is situated inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the sphere $x^2 + y^2 + z^2 = 1$. Show that the center of mass of the solid is not located within the solid.

40. Find the mass of the solid $Q = \{(x,y,z)\,|\,1 \le x^2 + z^2 \le 25,\ y \le 1 - x^2 - z^2\}$ whose density is $\rho(x,y,z) = k$, where $k > 0$.

41. [T] The solid $Q = \{(x,y,z)\,|\,x^2 + y^2 \le 9,\ 0 \le z \le 1,\ x \ge 0,\ y \ge 0\}$ has density equal to the distance to the xy-plane. Use a CAS to answer the following questions.

a. Find the mass of Q.

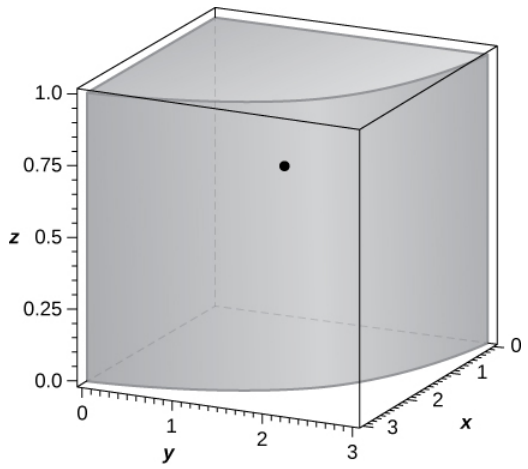
b. Find the moments $M_{xy},\ M_{xz}$ and M_{yz} about the xy-plane, xz-plane, and yz-plane, respectively.

c. Find the center of mass of Q.

d. Graph Q and locate its center of mass.

Answer

a. $m = \frac{9\pi}{4}$;
b. $M_{xy} = \frac{3\pi}{2},\ M_{xz} = \frac{81}{8},\ M_{yz} = \frac{81}{8}$;
c. $\bar{x} = \frac{9}{2\pi},\ \bar{y} = \frac{9}{2\pi},\ \bar{z} = \frac{2}{3}$;
d.



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42. Consider the solid $Q = \{(x, y, z) \mid 0 \le x \le 1,\ 0 \le y \le 2,\ 0 \le z \le 3\}$ with the density function $\rho(x, y, z) = x + y + 1$.

a. Find the mass of Q.

b. Find the moments $M_{xy},\ M_{xz}$ and M_{yz} about the xy-plane, xz-plane, and yz-plane, respectively.

c. Find the center of mass of Q.

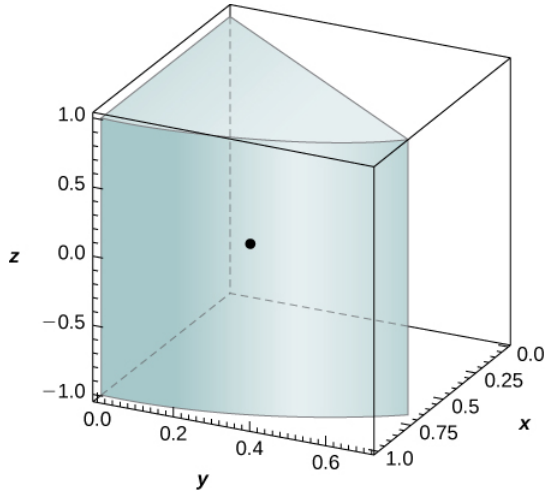
43. [T] The solid Q has the mass given by the triple integral $\int_{-1}^{1} \int_{0}^{\pi/4} \int_{0}^{1} r^2 \, dr \, d\theta \, dz$.

Use a CAS to answer the following questions.

- Show that the center of mass of Q is located in the xy-plane.
- Graph Q and locate its center of mass.

Answer

$\bar{x} = \frac{3\sqrt{2}}{2\pi}$, $\bar{y} = \frac{3(2-\sqrt{2})}{2\pi}$, $\bar{z} = 0$; 2. the solid Q and its center of mass are shown in the following figure.



44. The solid Q is bounded by the planes $x + 4y + z = 8,\ x = 0,\ y = 0$, and $z = 0$. Its density at any point is equal to the distance to the xz-plane. Find the moments of inertia of the solid about the xz-plane.

45. The solid Q is bounded by the planes $x + y + z = 3,\ x = 0,\ y = 0$, and $z = 0$. Its density is $\rho(x, y, z) = x + ay$, where $a > 0$. Show that the center of mass of the solid is located in the plane $z = \frac{3}{5}$ for any value of a.

46. Let Q be the solid situated outside the sphere $x^2 + y^2 + z^2 = z$ and inside the upper hemisphere $x^2 + y^2 + z^2 = R^2$, where $R > 1$. If the density of the solid is $\rho(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$, find R such that the mass of the solid is $\frac{7\pi}{2}$.

47. The mass of a solid Q is given by $\int_{0}^{2} \int_{0}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}} (x^2 + y^2 + z^2)^n \, dz \, dy \, dx$, where n is an integer. Determine n such the mass of the solid is $(2 - \sqrt{2})\pi$.

Answer

$n = -2$

48. Let Q be the solid bounded above the cone $x^2 + y^2 = z^2$ and below the sphere $x^2 + y^2 + z^2 - 4z = 0$. Its density is a constant $k > 0$. Find k such that the center of mass of the solid is situated 7 units from the origin.

49. The solid $Q = \{(x, y, z) \mid 0 \le x^2 + y^2 \le 16,\ x \ge 0,\ y \ge 0,\ 0 \le z \le x\}$ has the density $\rho(x, y, z) = k$. Show that the moment M_{xy} about the xy-plane is half of the moment M_{yz} about the yz-plane.

50. The solid Q is bounded by the cylinder $x^2 + y^2 = a^2$, the paraboloid $b^2 - z = x^2 + y^2$, and the xy-plane, where $0 < a < b$. Find the mass of the solid if its density is given by $\rho(x, y, z) = \sqrt{x^2 + y^2}$.

51. Let Q be a solid of constant density k, where $k > 0$, that is located in the first octant, inside the circular cone $x^2 + y^2 = 9(z - 1)^2$, and above the plane $z = 0$. Show that the moment M_{xy} about the xy-plane is the same as the moment M_{yz} about the xz-plane.

52. The solid Q has the mass given by the triple integral $\int_0^1 \int_0^{\pi/2} \int_0^{r^3} (r^4 + r)\, dz\, d\theta\, dr$.

a. Find the density of the solid in rectangular coordinates.

b. Find the moment M_{xy} about the xy-plane.

53. The solid Q has the moment of inertia I_x about the yz-plane given by the triple integral $\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{\frac{1}{2}(x^2+y^2)}^{\sqrt{x^2+y^2}} (y^2 + z^2)(x^2 + y^2)\, dz\, dx\, dy$.

a. Find the density of Q.

b. Find the moment of inertia I_z about the xy-plane.

Answer

a. $\rho(x, y, z) = x^2 + y^2$;
b. $\frac{16\pi}{7}$

54. The solid Q has the mass given by the triple integral $\int_0^{\pi/4} \int_0^{2\ sec\ \theta} \int_0^1 (r^3 \cos\theta \sin\theta + 2r)\, dz\, dr\, d\theta$.

a. Find the density of the solid in rectangular coordinates.

b. Find the moment M_{xz} about the xz-plane.

55. Let Q be the solid bounded by the xy-plane, the cylinder $x^2 + y^2 = a^2$, and the plane $z = 1$, where $a > 1$ is a real number. Find the moment M_{xy} of the solid about the xy-plane if its density given in cylindrical coordinates is $\rho(x, y, z) = \frac{d^2 f}{dr^2}(r)$, where f is a differentiable function with the first and second derivatives continuous and differentiable on $(0, a)$.

Answer

$M_{xy} = \pi(f(0) - f(a) + af'(a))$

56. A solid Q has a volume given by $\iint_D \int_a^b dA\, dz$, where D is the projection of the solid onto the xy-plane and $a < b$ are real numbers, and its density does not depend on the variable z. Show that its center of mass lies in the plane $z = \frac{a+b}{2}$.

57. Consider the solid enclosed by the cylinder $x^2 + z^2 = a^2$ and the planes $y = b$ and $y = c$, where $a > 0$ and $b < c$ are real numbers. The density of Q is given by $\rho(x, y, z) = f'(y)$, where f is a differential function whose derivative is continuous on (b, c). Show that if $f(b) = f(c)$, then the moment of inertia about the xz-plane of Q is null.

58. [T] The average density of a solid Q is defined as $\rho_{ave} = \frac{1}{V(Q)} \iiint_Q \rho(x, y, z)\, dV = \frac{m}{V(Q)}$, where $V(Q)$ and m are the volume and the mass of Q, respectively. If the density of the unit ball centered at the origin is $\rho(x, y, z) = e^{-x^2-y^2-z^2}$, use a CAS to find its average density. Round your answer to three decimal places.

59. Show that the moments of inertia $I_x,\ I_y$, and I_z about the yz-plane, xz-plane, and xy-plane, respectively, of the unit ball centered at the origin whose density is $\rho(x, y, z) = e^{-x^2-y^2-z^2}$ are the same. Round your answer to two decimal places.

Answer

$I_x = I_y = I_z \approx 0.84$

Contributors

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15.7: Change of Variables in Multiple Integrals

Learning Objectives

- Determine the image of a region under a given transformation of variables.
- Compute the Jacobian of a given transformation.
- Evaluate a double integral using a change of variables.
- Evaluate a triple integral using a change of variables.

Recall from Substitution Rule the method of integration by substitution. When evaluating an integral such as

$$\int_2^3 x(x^2-4)^5 dx,$$

we substitute $u = g(x) = x^2 - 4$. Then $du = 2x\,dx$ or $x\,dx = \frac{1}{2}du$ and the limits change to $u = g(2) = 2^2 - 4 = 0$ and $u = g(3) = 9 - 4 = 5$. Thus the integral becomes

$$\int_0^5 \frac{1}{2}u^5 du$$

and this integral is much simpler to evaluate. In other words, when solving integration problems, we make appropriate substitutions to obtain an integral that becomes much simpler than the original integral.

We also used this idea when we transformed double integrals in rectangular coordinates to polar coordinates and transformed triple integrals in rectangular coordinates to cylindrical or spherical coordinates to make the computations simpler. More generally,

$$\int_a^b f(x)dx = \int_c^d f(g(u))g'(u)du,$$

Where $x = g(u)$, $dx = g'(u)du$, and $u = c$ and $u = d$ satisfy $c = g(a)$ and $d = g(b)$.

A similar result occurs in double integrals when we substitute

- $x = f(r,\theta) = r\ \cos\ \theta$
- $y = g(r,\theta) = r\ \sin\ \theta$, and
- $dA = dx\,dy = r\,dr\,d\theta$.

Then we get

$$\iint_R f(x,y)dA = \iint_S (r\ \cos\ \theta,\ r\ \sin\ \theta)r\,dr\,d\theta$$

where the domain R is replaced by the domain S in polar coordinates. Generally, the function that we use to change the variables to make the integration simpler is called a transformation or mapping.

Planar Transformations

A planar transformation T is a function that transforms a region G in one plane into a region R in another plane by a change of variables. Both G and R are subsets of R^2. For example, Figure 15.7.1 shows a region G in the uv-plane transformed into a region R in the xy-plane by the change of variables $x = g(u,v)$ and $y = h(u,v)$, or sometimes we write $x = x(u,v)$ and $y = y(u,v)$. We shall typically assume that each of these functions has continuous first partial derivatives, which means g_u, g_v, h_u, and h_v exist and are also continuous. The need for this requirement will become clear soon.

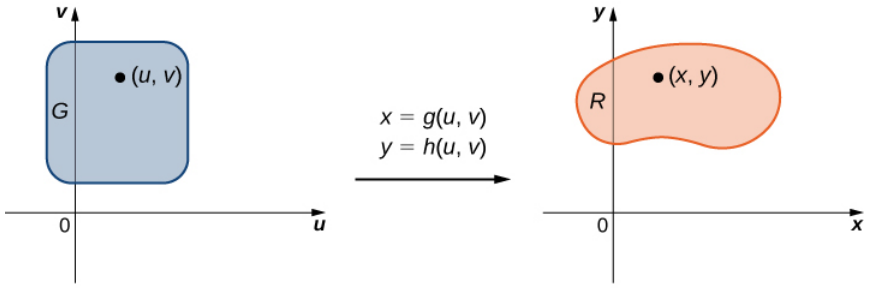


Figure 15.7.1: The transformation of a region G in the uv-plane into a region R in the xy-plane.

Definition: one-to-one transformation

A transformation $T: G \to R$, defined as $T(u, v) = (x, y)$, is said to be a one-to-one transformation if no two points map to the same image point.

To show that T s a one-to-one transformation, we assume $T(u_1, v_1) = T(u_2, v_2)$ and show that as a consequence we obtain$(u_1, v_1) = (u_2, v_2)$. If the transformation T is one-to-one in the domain G, then the inverse T^{-1} exists with the domain R such that $T^{-1} \circ T$ and $T \circ T^{-1}$ are identity functions.

Figure 15.7.2 shows the mapping $T(u, v) = (x, y)$ where x and y are related to u and v by the equations $x = g(u, v)$ and $y = h(u, v)$. The region G is the domain of T and the region R is the range of T, also known as the *image* of G under the transformation T.

Example 15.7.1A: Determining How the Transformation Works

Suppose a transformation T is defined as $T(r, \theta) = (x, y)$ where $x = r \cos \theta$, $y = r \sin \theta$. Find the image of the polar rectangle $G = \{(r, \theta) | 0 \le r \le 1,\ 0 \le \theta \le \pi/2\}$ in the $r\theta$-plane to a region R in the xy-plane. Show that T is a one-to-one transformation in G and find $T^{-1}(x, y)$.

Solution

Since r varies from 0 to 1 in the $r\theta$-plane, we have a circular disc of radius 0 to 1 in the xy-plane. Because θ varies from 0 to $\pi/2$ in the $r\theta$-plane, we end up getting a quarter circle of radius 1 in the first quadrant of the xy-plane (Figure 15.7.2). Hence R is a quarter circle bounded by $x^2 + y^2 = 1$ in the first quadrant.

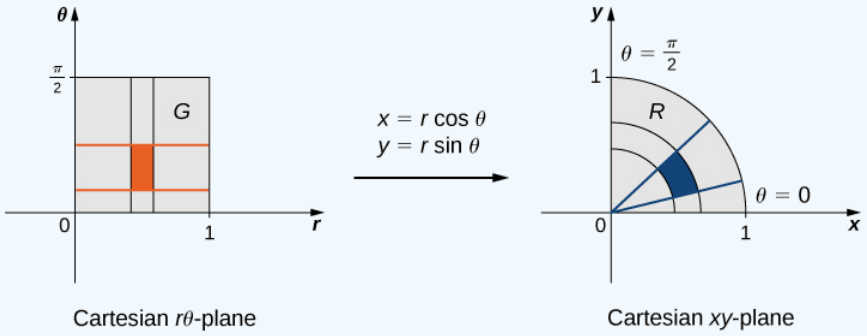


Figure 15.7.2: A rectangle in the $r\theta$-plane is mapped into a quarter circle in the xy-plane.

In order to show that T is a one-to-one transformation, assume $T(r_1, \theta_1) = T(r_2, \theta_2)$ and show as a consequence that $(r_1, \theta_1) = (r_2, \theta_2)$. In this case, we have

$$T(r_1, \theta_1) = T(r_2, \theta_2),$$

$$(x_1, y_1) = (x_1, y_1),$$

$$(r_1 \cos \theta_1, r_1 \sin \theta_1) = (r_2 \cos \theta_2, r_2 \sin \theta_2),$$

$$r_1 \cos \theta_1 = r_2 \cos \theta_2,\ r_1 \sin \theta_1 = r_2 \sin \theta_2.$$

Dividing, we obtain

$$\frac{r_1 \cos\theta_1}{r_1 \sin\theta_1} = \frac{r_2 \cos\theta_2}{r_2 \sin\theta_2}$$

$$\frac{\cos\theta_1}{\sin\theta_1} = \frac{\cos\theta_2}{\sin\theta_2}$$

$$\tan\theta_1 = \tan\theta_2$$

$$\theta_1 = \theta_2$$

since the tangent function is one-one function in the interval $0 \le \theta \le \pi/2$. Also, since $0 \le r \le 1$, we have $r_1 = r_2,\ \theta_1 = \theta_2$. Therefore, $(r_1, \theta_1) = (r_2, \theta_2)$ and T is a one-to-one transformation from G to R.

To find $T^{-1}(x, y)$ solve for r, θ in terms of x, y. We already know that $r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$. Thus $T^{-1}(x, y) = (r, \theta)$ is defined as $r = \sqrt{x^2 + y^2}$ and $\tan^{-1}(\frac{y}{x})$.

Example 15.7.1B: Finding the Image under T

Let the transformation T be defined by $T(u, v) = (x, y)$ where $x = u^2 - v^2$ and $y = uv$. Find the image of the triangle in the uv-plane with vertices $(0, 0),\ (0, 1)$, and $(1, 1)$.

Solution

The triangle and its image are shown in Figure 15.7.3. To understand how the sides of the triangle transform, call the side that joins $(0, 0)$ and $(0, 1)$ side A, the side that joins $(0, 0)$ and $(1, 1)$ side B, and the side that joins $(1, 1)$ and $(0, 1)$ side C.

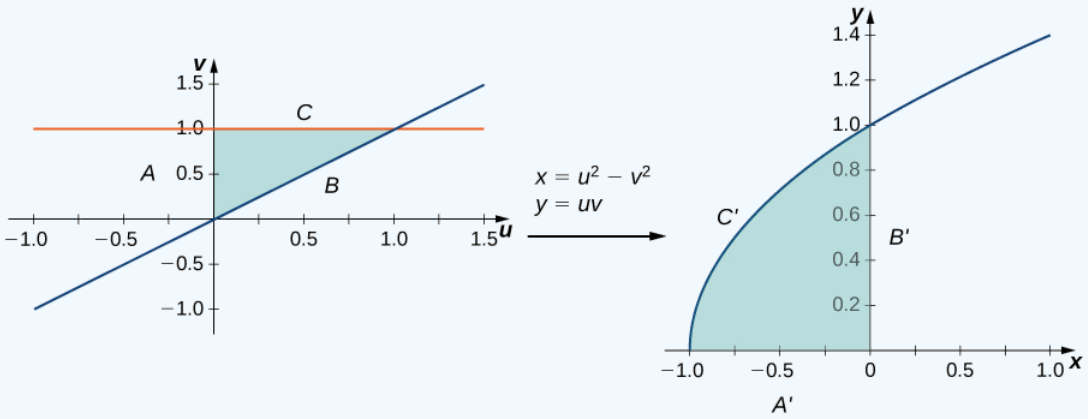


Figure 15.7.3: A triangular region in the uv-plane is transformed into an image in the xy-plane.

- For the side $A: u = 0,\ 0 \le v \le 1$ transforms to $x = -v^2,\ y = 0$ so this is the side A' that joins $(-1, 0)$ and $(0, 0)$.
- For the side $B: u = v,\ 0 \le u \le 1$ transforms to $x = 0,\ y = u^2$ so this is the side B' that joins $(0, 0)$ and $(0, 1)$.
- For the side $C: 0 \le u \le 1,\ v = 1$ transforms to $x = u^2 - 1,\ y = u$ (hence $x = y^2 - 1$ so this is the side C' that makes the upper half of the parabolic arc joining $(-1, 0)$ and $(0, 1)$.

All the points in the entire region of the triangle in the uv-plane are mapped inside the parabolic region in the xy-plane.

Exercise 15.7.1

Let a transformation T be defined as $T(u, v) = (x, y)$ where $x = u + v,\ y = 3v$. Find the image of the rectangle $G = \{(u, v): 0 \le u \le 1,\ 0 \le v \le 2\}$ from the uv-plane after the transformation into a region R in the xy-plane. Show that T is a one-to-one transformation and find $T^{-1}(x, y)$.

Hint

Follow the steps of Example 15.7.1B.

Answer

$T^{-1}(x, y) = (u, v)$ where $u = \frac{3x - y}{3}$ and $v = \frac{y}{3}$

Jacobians

Recall that we mentioned near the beginning of this section that each of the component functions must have continuous first partial derivatives, which means that g_u, g_v, h_u and h_v exist and are also continuous. A transformation that has this property is called a C^1 transformation (here C denotes continuous). Let $T(u,v) = (g(u,v),\ h(u,v))$, where $x = g(u,v)$ and $y = h(u,v)$ be a one-to-one C^1 transformation. We want to see how it transforms a small rectangular region S, Δu units by Δv units, in the uv-plane (Figure 15.7.4).

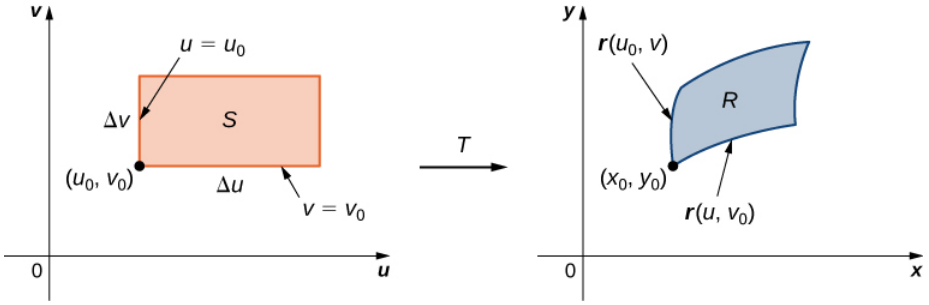


Figure 15.7.4: A small rectangle S in the uv-plane is transformed into a region R in the xy-plane.

Since $x = g(u,v)$ and $y = h(u,v)$, we have the position vector $r(u,v) = g(u,v)i + h(u,v)j$ of the image of the point (u,v). Suppose that $(u_0, v_0)$is the coordinate of the point at the lower left corner that mapped to $(x_0, y_0) = T(u_0, v_0)$ The line $v = v_0$ maps to the image curve with vector function $r(u, v_0)$, and the tangent vector at (x_0, y_0) to the image curve is

$$r_u = g_u(u_0, v_0)i + h_v(u_0, v_0)j = \frac{\partial x}{\partial u}i + \frac{\partial y}{\partial u}j.$$

Similarly, the line $u = u_0$ maps to the image curve with vector function $r(u_0, v)$, and the tangent vector at (x_0, y_0) to the image curve is

$$r_v = g_v(u_0, v_0)i + h_u(u_0, v_0)j = \frac{\partial x}{\partial v}i + \frac{\partial y}{\partial v}j.$$

Now, note that

$$r_u = \lim_{\Delta u \to 0} \frac{r(u_0 + \Delta u, v_0) - r(u_0, v_0)}{\Delta u} \; so \; r(u_0 + \Delta u, v_0) - r(u_0, v_0) \approx \Delta u r_u.$$

Similarly,

$$r_v = \lim_{\Delta v \to 0} \frac{r(u_0, v_0 + \Delta v) - r(u_0, v_0)}{\Delta v} \; so \; r(u_0, v_0 + \Delta v) - r(u_0, v_0) \approx \Delta v r_v.$$

This allows us to estimate the area ΔA of the image R by finding the area of the parallelogram formed by the sides $\Delta v r_v$ and $\Delta u r_u$. By using the cross product of these two vectors by adding the **k**th component as 0, the area ΔA of the image R (refer to The Cross Product) is approximately $|\Delta u r_u \times \Delta v r_v| = |r_u \times r_v| \Delta u \Delta v$. In determinant form, the cross product is

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} k = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right) k$$

Since $|k| = 1$, we have

$\Delta A \approx |r_u \times r_v| \Delta u \Delta v = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right) \Delta u \Delta v.$

Definition: Jacobian

The *Jacobian* of the C^1 transformation $T(u,v) = (g(u,v),\ h(u,v))$ is denoted by $J(u,v)$ and is defined by the 2×2 determinant

$$J(u,v) = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right).$$

Using the definition, we have

$$\Delta A \approx J(u,v)\Delta u\Delta v = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\Delta u\Delta v.$$

Note that the Jacobian is frequently denoted simply by

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)}.$$

Note also that

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

Hence the notation $J(u,v) = \frac{\partial(x,y)}{\partial(u,v)}$ suggests that we can write the Jacobian determinant with partials of x in the first row and partials of y in the second row.

Example 15.7.2A: Finding the Jacobian

Find the Jacobian of the transformation given in Example 15.7.1A.

Solution

The transformation in the example is $T(r,\theta) = (r\cos\theta, r\sin\theta)$ where $x = r\cos\theta$ and $y = r\sin\theta$. Thus the Jacobian is

$$J(r,\theta) = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta) = r.$$

Example 15.7.2B: Finding the Jacobian

Find the Jacobian of the transformation given in Example 15.7.1B.

Solution

The transformation in the example is $T(u,v) = (u^2 - v^2, uv)$ where $x = u^2 - v^2$ and $y = uv$. Thus the Jacobian is

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ v & u \end{vmatrix} = 2u^2 + 2v^2.$$

Exercise 15.7.2

Find the Jacobian of the transformation given in the previous checkpoint: $T(u,v) = (u+v, 2v)$.

Hint

Follow the steps in the previous two examples.

Answer

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$$

Change of Variables for Double Integrals

We have already seen that, under the change of variables $T(u,v)=(x,y)$ where $x=g(u,v)$ and $y=h(u,v)$, a small region ΔA in the xy-plane is related to the area formed by the product $\Delta u \Delta v$ in the uv-plane by the approximation

$$\Delta A \approx J(u,v)\Delta u,\ \Delta v.$$

Now let's go back to the definition of double integral for a minute:

$$\iint_R f(x,y) fA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij},y_{ij})\Delta A.$$

Referring to Figure 15.7.5, observe that we divided the region S in the uv-plane into small subrectangles S_{ij} and we let the subrectangles R_{ij} in the xy-plane be the images of S_{ij} under the transformation $T(u,v)=(x,y)$.

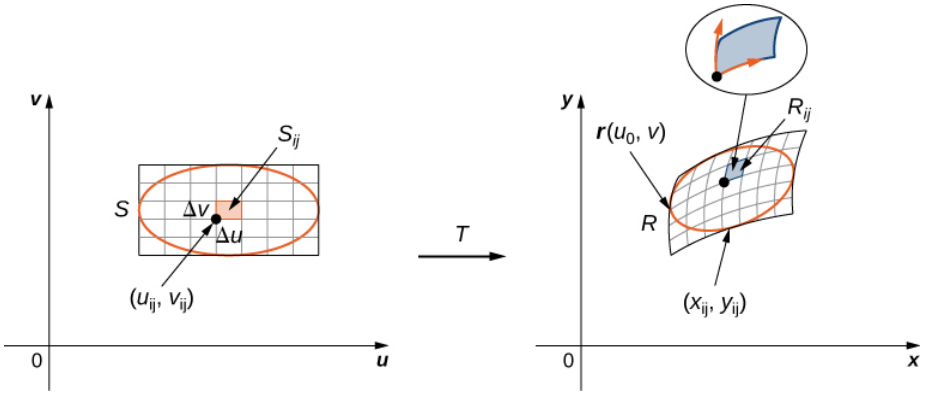


Figure 15.7.5: The subrectangles S_{ij} in the uv-plane transform into subrectangles R_{ij} in the xy-plane.

Then the double integral becomes

$$\iint_R = f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij},y_{ij})\Delta A$$

$$= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(g(u_{ij},v_{ij}),\ h(u_{ij},v_{ij}))|J(u_{ij},v_{ij})|\Delta u \Delta v.$$

Notice this is exactly the double Riemann sum for the integral

$$\iint_S f(g(u,v),\ h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv.$$

Change of Variables for Double Integrals

Let $T(u,v)=(x,y)$ where $x=g(u,v)$ and $y=h(u,v)$ be a one-to-one C^1 transformation, with a nonzero Jacobian on the interior of the region S in the uv-plane it maps S into the region R in the xy-plane. If f is continuous on R, then

$$\iint_R f(x,y)dA = \iint_S f(g(u,v),\ h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv.$$

With this theorem for double integrals, we can change the variables from (x,y) to (u,v) in a double integral simply by replacing

$$dA = dx\, dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv$$

when we use the substitutions $x=g(u,v)$ and $y=h(u,v)$ and then change the limits of integration accordingly. This change of variables often makes any computations much simpler.

Example 15.7.3: Changing Variables from Rectangular to Polar Coordinates

Consider the integral

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx.$$

Use the change of variables $x = r\cos\theta$ and $y = r\sin\theta$, and find the resulting integral.

Solution

First we need to find the region of integration. This region is bounded below by $y = 0$ and above by $y = \sqrt{2x - x^2}$ (Figure 15.7.6).

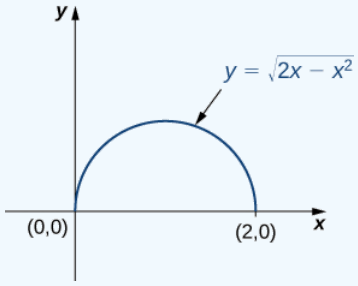


Figure 15.7.6: Changing a region from rectangular to polar coordinates.

Squaring and collecting terms, we find that the region is the upper half of the circle $x^2 + y^2 - 2x = 0$, that is $y^2 + (x-1)^2 = 1$. In polar coordinates, the circle is $r = 2\cos\theta$ so the region of integration in polar coordinates is bounded by $0 \le r \le \cos\theta$ and $0 \le \theta \le \frac{\pi}{2}$.

The Jacobian is $J(r,\theta) = r$, as shown in Example 15.7.2A. Since $r \ge 0$, we have $|J(r,\theta)| = r$.

The integrand $\sqrt{x^2+y^2}$ changes to r in polar coordinates, so the double iterated integral is

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx = \int_0^{\pi/2} \int_0^{2\cos\theta} r|j(r,\theta)|\,dr\,d\theta = \int_0^{\pi/2} \int_0^{2\cos\theta} r^2\,dr\,d\theta.$$

Exercise 15.7.3

Considering the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx$, use the change of variables $x = r\cos\theta$ and $y = r\sin\theta$ and find the resulting integral.

Hint

Follow the steps in the previous example.

Answer

$$\int_0^{\pi/2} \int_0^1 r^3\,dr\,d\theta$$

Notice in the next example that the region over which we are to integrate may suggest a suitable transformation for the integration. This is a common and important situation.

Example 15.7.4: Changing Variables

Consider the integral

$$\iint_R (x-y)\,dy\,dx,$$

where R is the parallelogram joining the points $(1,2)$, $(3,4)$, $(4,3)$, and $(6,5)$ (Figure 15.7.7). Make appropriate changes of variables, and write the resulting integral.

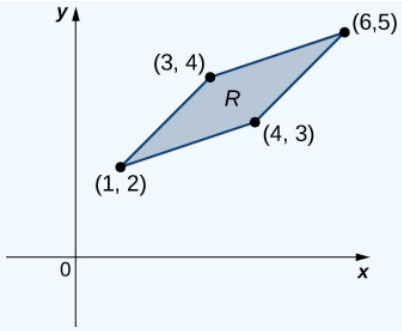


Figure 15.7.7: The region of integration for the given integral.

Solution

First, we need to understand the region over which we are to integrate. The sides of the parallelogram are $x - y + 1,\ x - y - 1 = 0,\ x - 3y + 5 = 0$ and $x - 3y + 9 = 0$ (Figure 15.7.8). Another way to look at them is $x - y = -1,\ x - y = 1,\ x - 3y = -5$, and $x - 3y = 9$.

Clearly the parallelogram is bounded by the lines $y = x + 1,\ y = x - 1,\ y = \frac{1}{3}(x + 5)$, and $y = \frac{1}{3}(x + 9)$.

Notice that if we were to make $u = x - y$ and $v = x - 3y$, then the limits on the integral would be $-1 \le u \le 1$ and $-9 \le v \le -5$.

To solve for x and y, we multiply the first equation by 3 and subtract the second equation, $3u - v = (3x - 3y) - (x - 3y) = 2x$. Then we have $x = \frac{3u-v}{2}$. Moreover, if we simply subtract the second equation from the first, we get $u - v = (x - y) - (x - 3y) = 2y$ and $y = \frac{u-v}{2}$.

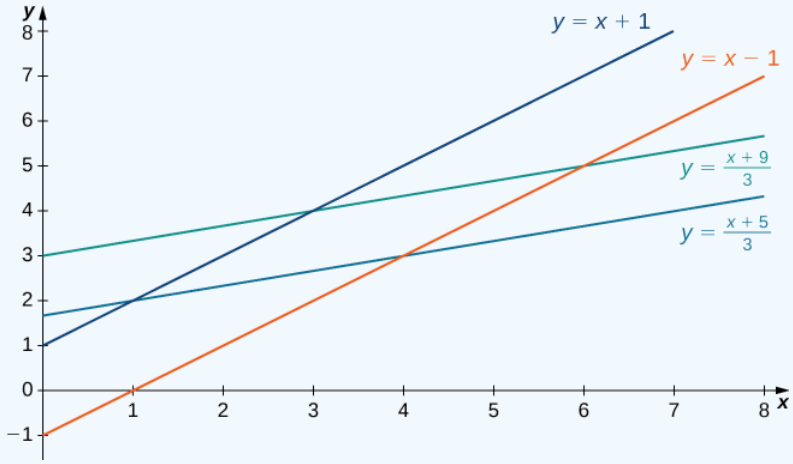


Figure 15.7.8: A parallelogram in the xy-plane that we want to transform by a change in variables.

Thus, we can choose the transformation

$$T(u, v) = \left(\frac{3u - v}{2},\ \frac{u - v}{2}\right)$$

and compute the Jacobian $J(u, v)$. We have

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 3/2 & -1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{3}{4} + \frac{1}{4} = -\frac{1}{2}$$

Therefore, $|J(u, v)| = \frac{1}{2}$. Also, the original integrand becomes

$$x - y = \frac{1}{2}[3u - v - u + v] = \frac{1}{2}[3u - u] = \frac{1}{2}[2u] = u.$$

Therefore, by the use of the transformation T, the integral changes to

$$\iint_R (x - y)dy\,dx = \int_{-9}^{-5}\int_{-1}^{1} J(u, v)u\,du\,dv = \int_{-9}^{-5}\int_{-1}^{1} \left(\frac{1}{2}\right) u\,du\,dv,$$

which is much simpler to compute.

Exercise 15.7.4

Make appropriate changes of variables in the integral

$$\iint_R \frac{4}{(x-y)^2}\,dy\,dx,$$

where R is the trapezoid bounded by the lines $x-y=2$, $x-y=4$, $x=0$, and $y=0$. Write the resulting integral.

Hint

Follow the steps in the previous example.

Answer

$x=\frac{1}{2}(v+u)$ and $y=\frac{1}{2}(v-u)$

and

$$\int_2^4 \int_{-u}^{u} \left(\frac{1}{2}\right)\cdot\frac{4}{u^2}\,dv\,du.$$

We are ready to give a problem-solving strategy for change of variables.

Problem-Solving Strategy: Change of Variables

1. Sketch the region given by the problem in the xy-plane and then write the equations of the curves that form the boundary.
2. Depending on the region or the integrand, choose the transformations $x=g(u,v)$ and $y=h(u,v)$.
3. Determine the new limits of integration in the uv-plane.
4. Find the Jacobian $J(u,v)$.
5. In the integrand, replace the variables to obtain the new integrand.
6. Replace $dy\,dx$ or $dx\,dy$, whichever occurs, by $J(u,v)du\,dv$.

In the next example, we find a substitution that makes the integrand much simpler to compute.

Example 15.7.5: Evaluating an Integral

Using the change of variables $u=x-y$ and $v=x+y$, evaluate the integral

$$\iint_R (x-y)e^{x^2-y^2}\,dA,$$

where R is the region bounded by the lines $x+y=1$ and $x+y=3$ and the curves $x^2-y^2=-1$ and $x^2-y^2=1$ (see the first region in Figure 15.7.9).

Solution

As before, first find the region R and picture the transformation so it becomes easier to obtain the limits of integration after the transformations are made (Figure 15.7.9).

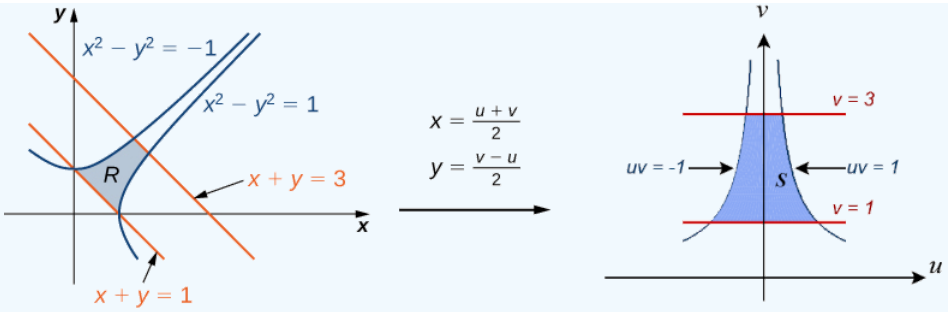


Figure 15.7.9: Transforming the region R into the region S to simplify the computation of an integral.

Given $u = x - y$ and $v = x + y$, we have $x = \frac{u+v}{2}$ and $y = \frac{v-u}{2}$ and hence the transformation to use is $T(u, v) = (\frac{u+v}{2}, \frac{v-u}{2})$. The lines $x + y = 1$ and $x + y = 3$ become $v = 1$ and $v = 3$, respectively. The curves $x^2 - y^2 = 1$ and $x^2 - y^2 = -1$ become $uv = 1$ and $uv = -1$, respectively.

Thus we can describe the region S (see the second region Figure 15.7.9) as

$$S = \left\{(u, v)|1 \le v \le 3,\ \frac{-1}{v} \le u \le \frac{1}{v}\right\}.$$

The Jacobian for this transformation is

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{vmatrix} = \frac{1}{2}.$$

Therefore, by using the transformation T, the integral changes to

$$\iint_R (x - y)e^{x^2-y^2}\,dA = \frac{1}{2}\int_1^3 \int_{-1/v}^{1/v} ue^{uv}\,du\,dv.$$

Doing the evaluation, we have

$$\frac{1}{2}\int_1^3 \int_{-1/v}^{1/v} ue^{uv}\,du\,dv = \frac{2}{3e} \approx 0.245.$$

? Exercise 15.7.5

Using the substitutions $x = v$ and $y = \sqrt{u + v}$, evaluate the integral $\iint_R y \sin(y^2 - x)\,dA$, where R is the region bounded by the lines $y = \sqrt{x}$, $x = 2$ and $y = 0$.

Hint

Sketch a picture and find the limits of integration.

Answer

$\frac{1}{2}(\sin 2 - 2)$

Change of Variables for Triple Integrals

Changing variables in triple integrals works in exactly the same way. Cylindrical and spherical coordinate substitutions are special cases of this method, which we demonstrate here.

Suppose that G is a region in uvw-space and is mapped to D in xyz-space (Figure 15.7.10) by a one-to-one C^1 transformation $T(u, v, w) = (x, y, z)$ where $x = g(u, v, w)$, $y = h(u, v, w)$, and $z = k(u, v, w)$.

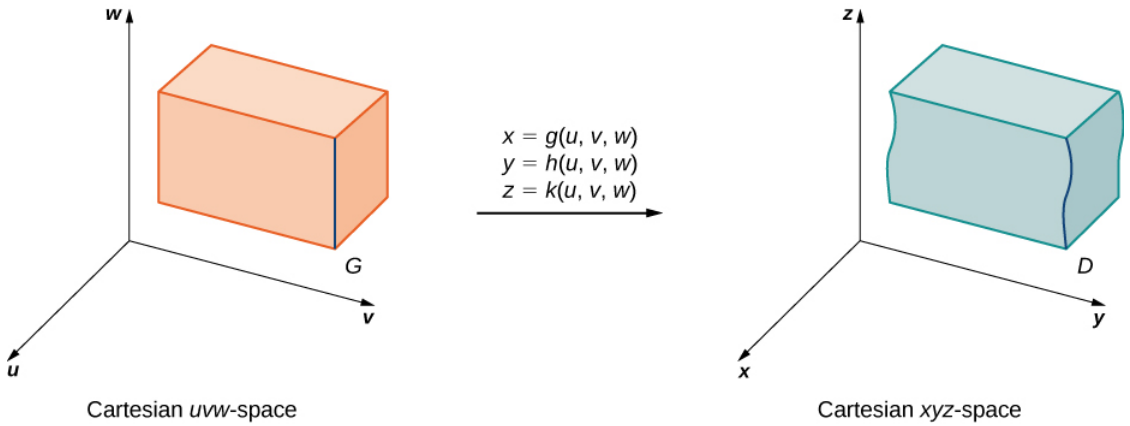


Figure 15.7.10: A region G in uvw-space mapped to a region D in xyz-space.

Then any function $F(x, y, z)$ defined on D can be thought of as another function $H(u, v, w)$ that is defined on G:

$$F(x, y, z) = F(g(u, v, w),\ h(u, v, w),\ k(u, v, w)) = H(u, v, w).$$

Now we need to define the Jacobian for three variables.

Definition: Jacobian determinant

The *Jacobian determinant* $J(u, v, w)$ in three variables is defined as follows:

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

This is also the same as

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

The Jacobian can also be simply denoted as $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.

With the transformations and the Jacobian for three variables, we are ready to establish the theorem that describes change of variables for triple integrals.

Change of Variables for Triple Integrals

Let $T(u, v, w) = (x, y, z)$ where $x = g(u, v, w),\ y = h(u, v, w)$, and $z = k(u, v, w)$, be a one-to-one C^1 transformation, with a nonzero Jacobian, that maps the region G in the uvw-space into the region D in the xyz-space. As in the two-dimensional case, if F is continuous on D, then

$$\iiint_D F(x, y, z)dV = \iiint_G f(g(u, v, w)\, h(u, v, w),\ k(u, v, w)) \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| du\, dv\, dw \tag{15.7.1}$$

$$= \iiint_G H(u, v, w)|J(u, v, w)|du\, dv\, dw. \tag{15.7.2}$$

Let us now see how changes in triple integrals for cylindrical and spherical coordinates are affected by this theorem. We expect to obtain the same formulas as in Triple Integrals in Cylindrical and Spherical Coordinates.

Example 15.7.6A: Obtaining Formulas in Triple Integrals for Cylindrical and Spherical Coordinates

Derive the formula in triple integrals for

a. cylindrical and
b. spherical coordinates.

Solution

A.

For cylindrical coordinates, the transformation is $T(r,\theta,z)=(x,y,z)$ from the Cartesian $r\theta z$-space to the Cartesian xyz-space (Figure 15.7.11). Here $x=r\cos\theta,\ y=r\sin\theta$ and $z=z$. The Jacobian for the transformation is

$$J(r,\theta,z)=\frac{\partial(x,y,z)}{\partial(r,\theta,z)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial z}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial z}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial z}\end{vmatrix}$$

$$\begin{vmatrix}\cos\theta & -r\sin\theta & 0\\ \sin\theta & r\cos\theta & 0\\ 0 & 0 & 1\end{vmatrix}=r\cos^2\theta+r\sin^2\theta=r.$$

We know that $r\geq 0$, so $|J(r,\theta,z)|=r$. Then the triple integral is

$$\iiint_D f(x,y,z)dV=\iiint_G f(r\cos\theta,\ r\sin\theta,\ z)r\,dr\,d\theta\,dz.$$

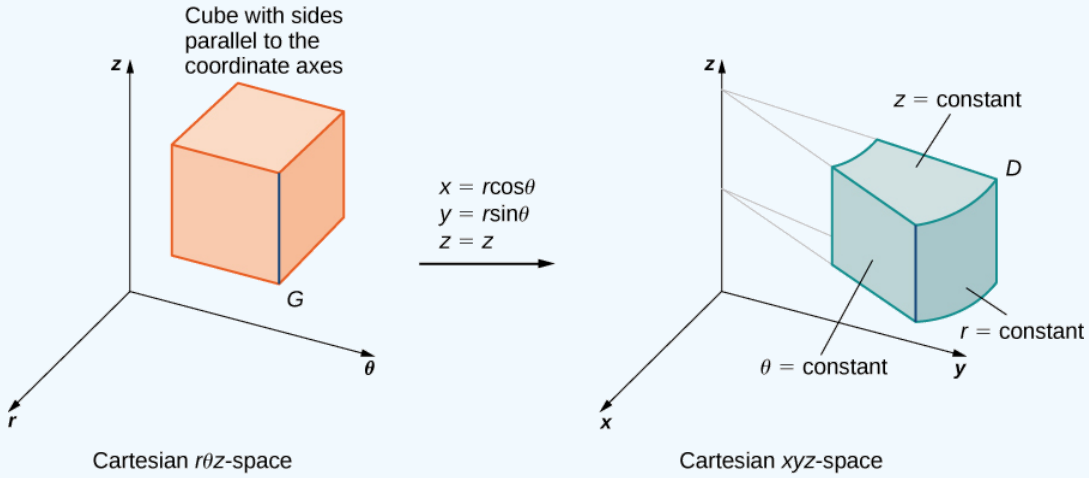


Figure 15.7.11: The transformation from rectangular coordinates to cylindrical coordinates can be treated as a change of variables from region G in $r\theta z$-space to region D in xyz-space.

B.

For spherical coordinates, the transformation is $T(\rho,\theta,\varphi)$ from the Cartesian $\rho\theta\varphi$-space to the Cartesian xyz-space (Figure 15.7.12). Here $x=\rho\sin\varphi\cos\theta,\ y=\rho\sin\varphi\sin\theta$, and $z=\rho\cos\varphi$. The Jacobian for the transformation is

$$J(\rho,\theta,\varphi)=\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}=\begin{vmatrix}\frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\varphi}\\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\varphi}\\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\varphi}\end{vmatrix}=\begin{vmatrix}\sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta\\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta\\ \cos\varphi & 0 & -\rho\sin\varphi\end{vmatrix}.$$

Expanding the determinant with respect to the third row:

$$= \cos\varphi \begin{vmatrix} -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta \end{vmatrix} - \rho\sin\varphi \begin{vmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta \end{vmatrix}$$

$$= \cos\varphi(-\rho^2\sin\varphi\,\cos\varphi\,\sin^2\theta - \rho^2\,\sin\varphi\,\cos\varphi\,\cos^2\theta)$$
$$-\rho\sin\varphi(\rho\sin^2\varphi\cos^2\theta + \rho\sin^2\varphi\sin^2\theta)$$

$$= -\rho^2\sin\varphi\cos^2\varphi(\sin^2\theta + \cos^2\theta) - \rho^2\sin\varphi\sin^2\varphi(\sin^2\theta + \cos^2\theta)$$

$$= -\rho^2\sin\varphi\cos^2\varphi - \rho^2\sin\varphi\sin^2\varphi$$

$$= -\rho\sin\varphi(\cos^2\varphi + \sin^2\varphi) = -\rho^2\sin\varphi.$$

Since $0 \le \varphi \le \pi$, we must have $\sin\varphi \ge 0$. Thus $|J(\rho,\theta,\varphi)| = |-\rho^2\sin\varphi| = \rho^2\sin\varphi$.

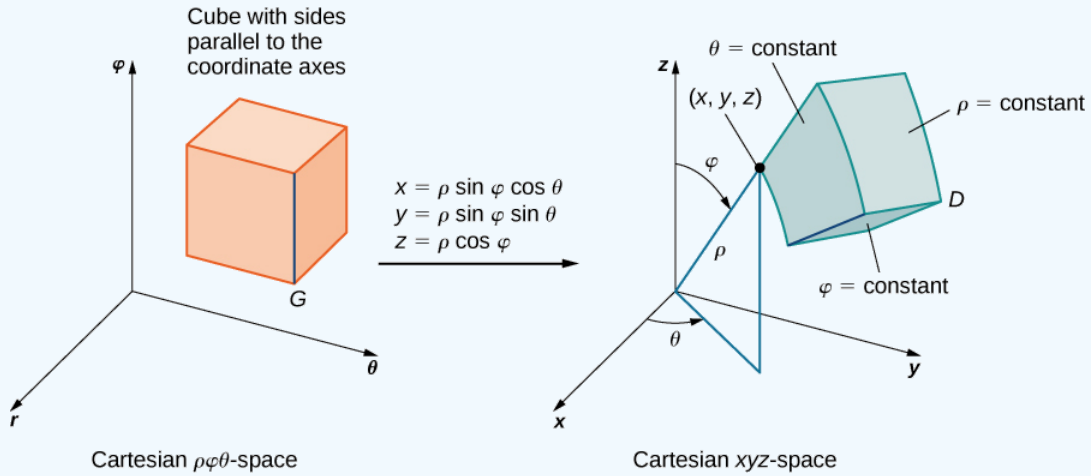


Figure 15.7.12: The transformation from rectangular coordinates to spherical coordinates can be treated as a change of variables from region G in $\rho\theta\varphi$-space to region D in xyz-space.

Then the triple integral becomes

$$\iiint_D f(x,y,z)dV = \iiint_G f(\rho\,\sin\varphi\,\cos\theta,\,\rho\,\sin\varphi\,\sin\theta,\rho\,\cos\varphi)\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta.$$

Let's try another example with a different substitution.

Example 15.7.6B: Evaluating a Triple Integral with a Change of Variables

Evaluate the triple integral

$$\int_0^3\int_0^4\int_{y/2}^{(y/2)+1}\left(x+\frac{z}{3}\right)dx\,dy\,dz$$

In xyz-space by using the transformation

$u = (2x-y)/2$, $v = y/2$, and $w = z/3$.

Then integrate over an appropriate region in uvw-space.

Solution

As before, some kind of sketch of the region G in xyz-space over which we have to perform the integration can help identify the region D in uvw-space (Figure 15.7.13). Clearly G in xyz-space is bounded by the planes $x = y/2$, $x = (y/2)+1$, $y = 0$, $y = 4$, $z = 0$, and $z = 4$. We also know that we have to use $u = (2x-y)/2$, $v = y/2$, and $w = z/3$ for the transformations. We need to solve for x, y and z. Here we find that $x = u+v$, $y = 2v$, and $z = 3w$.

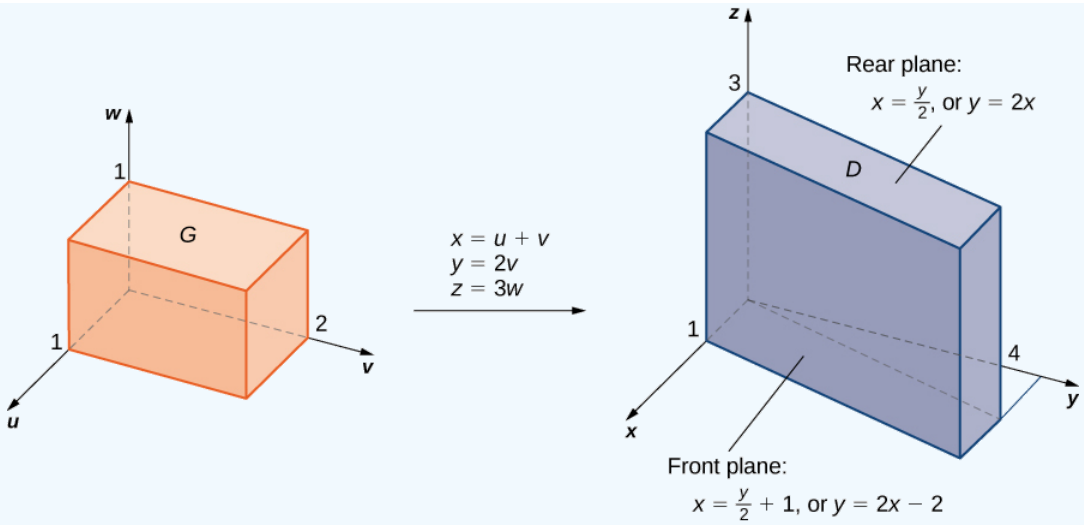


Figure 15.7.13: The region G in uvw-space is transformed to region D in xyz-space.

Using elementary algebra, we can find the corresponding surfaces for the region G and the limits of integration in uvw-space. It is convenient to list these equations in a table.

Equations in xyz for the region D	Corresponding equations in uvw for the region G	Limits for the integration in uvw
$x = y/2$	$u + v = 2v/2 = v$	$u = 0$
$x = y/2$	$u + v = (2v/2) + 1 = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$
$z = 0$	$3w = 0$	$w = 0$
$z = 3$	$3w = 3$	$w = 1$

Now we can calculate the Jacobian for the transformation:

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6.$$

The function to be integrated becomes

$$f(x,y,z) = x + \frac{z}{3} = u + v + \frac{3w}{3} = u + v + w.$$

We are now ready to put everything together and complete the problem.

$$\int_0^3 \int_0^4 \int_{y/2}^{(y/2)+1} \left(x + \frac{z}{3}\right) dx\, dy\, dz = \int_0^1 \int_0^2 \int_0^1 (u+v+w)|J(u,v,w)|du\, dv\, dw$$
$$= \int_0^1 \int_0^2 \int_0^1 (u+v+w)|6|du\, dv\, dw$$
$$= 6\int_0^1 \int_0^2 \int_0^1 (u+v+w)\, du\, dv\, dw$$
$$= 6\int_0^1 \int_0^2 \left[\frac{u^2}{2} + vu + wu\right]_0^1 dv\, dw$$
$$= 6\int_0^1 \int_0^2 \left(\frac{1}{2} + v + u\right) dv\, dw$$
$$= 6\int_0^1 \left[\frac{1}{2}v + \frac{v^2}{2} + wv\right]_0^2 dw$$
$$= 6\int_0^1 (3+2w)\, dw = 6\Big[3w + w^2\Big]_0^1 = 24.$$

? Exercise 15.7.6

Let D be the region in xyz-space defined by $1 \le x \le 2,\ 0 \le xy \le 2$, and $0 \le z \le 1$.

Evaluate $\iiint_D (x^2y + 3xyz)\, dx\, dy\, dz$ by using the transformation $u = x,\ v = xy$, and $w = 3z$.

Hint

Make a table for each surface of the regions and decide on the limits, as shown in the example.

Answer

$$\int_0^3 \int_0^2 \int_1^2 \left(\frac{v}{3} + \frac{vw}{3u}\right) du\, dv\, dw = 2 + \ln 8$$

Key Concepts

- A transformation T is a function that transforms a region G in one plane (space) into a region R. in another plane (space) by a change of variables.
- A transformation $T : G \to R$ defined as $T(u,v) = (x,y)$ (or $T(u,v,w) = (x,y,z)$)is said to be a one-to-one transformation if no two points map to the same image point.
- If f is continuous on R, then

$$\iint_R f(x,y)dA = \iint_S f(g(u,v),\ h(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv.$$

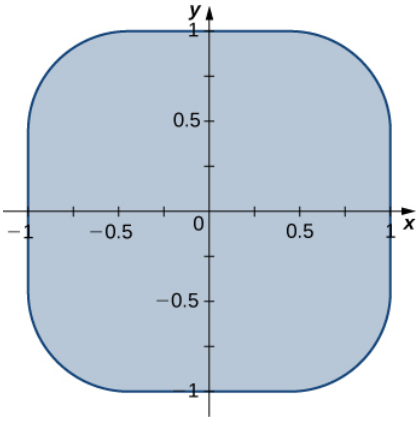
- If F is continuous on R, then

$$\iiint_R F(x,y,z)\, dV = \iiint_G F(g(u,v,w),\ h(u,v,w),\ k(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\, dv\, dw$$
$$= \iiint_G H(u,v,w)|J(u,v,w)|\, du\, dv\, dw.$$

[T] Lamé ovals (or superellipses) are plane curves of equations $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 1$, where *a*, *b*, and *n* are positive real numbers.

a. Use a CAS to graph the regions R bounded by Lamé ovals for $a = 1,\ b = 2,\ n = 4$ and $n = 6$ respectively.

b. Find the transformations that map the region R bounded by the Lamé oval $x^4 + y^4 = 1$ also called a squircle and graphed in the following figure, into the unit disk.



c. Use a CAS to find an approximation of the area $A(R)$ of the region R bounded by $x^4 + y^4 = 1$. Round your answer to two decimal places.

[T] Lamé ovals have been consistently used by designers and architects. For instance, Gerald Robinson, a Canadian architect, has designed a parking garage in a shopping center in Peterborough, Ontario, in the shape of a superellipse of the equation $(\frac{x}{a})^n + (\frac{y}{b})^n = 1$ with $\frac{a}{b} = \frac{9}{7}$ and $n = e$. Use a CAS to find an approximation of the area of the parking garage in the case $a = 900$ yards, $b = 700$ yards, and $n = 2.72$ yards.

[Hide Solution]

$A(R) \simeq 83,999.2$

Chapter Review Exercises

True or False? Justify your answer with a proof or a counterexample.

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dy\,dx$$

Fubini's theorem can be extended to three dimensions, as long as f is continuous in all variables.

[Hide solution]

True.

The integral

$$\int_0^{2\pi} \int_0^1 \int_0^1 dz\,dr\,d\theta$$

represents the volume of a right cone.

The Jacobian of the transformation for $x = u^2 - 2v$, $y = 3v - 2uv$ is given by $-4u^2 + 6u + 4v$.

[Hide Solution]

False.

Evaluate the following integrals.

$$\iint_R (5x^3y^2 - y^2)\,dA,\ R = \{(x,y) | 0 \le x \le 2,\ 1 \le y \le 4\}$$

$$\iint_D \frac{y}{3x^2+1} dA,\ D = \{(x,y) | 0 \le x \le 1,\ -x \le y \le x\}$$

[Hide Solution]

0

$$\iint_D \sin(x^2 + y^2) dA$$

where D is a disk of radius 2 centered at the origin

$$\int_0^1 \int_0^1 xye^{x^2} dx\, dy$$

[Hide Solution]

$\frac{1}{4}$

$$\int_{-1}^1 \int_0^z \int_0^{x-z} 6dy\, dx\, dz$$

$$\iiint_R 3y\, dV,$$

where $R = \{(x, y, z)|0 \leq x \leq 1,\ 0 \leq y \leq x,\ 0 \leq z \leq \sqrt{9 - y^2}\}$

[Hide Solution]

1.475

$$\int_0^2 \int_0^{2\pi} \int_r^1 r\, dz\, d\theta\, dr$$

$$\int_0^{2\pi} \int_0^{\pi/2} \int_1^3 \rho^2 \sin(\varphi) d\rho\, d\varphi,\ d\theta$$

[Hide Solution]

$\frac{52}{3}\pi$

$$\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz\, dy\, sx$$

For the following problems, find the specified area or volume.

The area of region enclosed by one petal of $r = \cos(4\theta)$.

[Hide Solution]

$\frac{\pi}{16}$

The volume of the solid that lies between the paraboloid $z = 2x^2 + 2y^2$ and the plane $z = 8$.

The volume of the solid bounded by the cylinder $x^2 + y^2 = 16$ and from $z = 1$ to $z + x = 2$.

[Hide Solution]

93.291

The volume of the intersection between two spheres of radius 1, the top whose center is $(0, 0, 0.25)$ and the bottom, which is centered at $(0, 0, 0)$.

For the following problems, find the center of mass of the region.

$\rho(x, y) = xy$ on the circle with radius 1 in the first quadrant only.

[Hide Solution]

$\left(\frac{8}{15}, \frac{8}{15}\right)$

$\rho(x, y) = (y + 1)\sqrt{x}$ in the region bounded by $y = e^x$, $y = 0$, and $x = 1$.

$\rho(x, y, z) = z$ on the inverted cone with radius 2 and height 2.

$\left(0, 0, \frac{8}{5}\right)$

The volume an ice cream cone that is given by the solid above $z = \sqrt{(x^2 + y^2)}$ and below $z^2 + x^2 + y^2 = z$.

The following problems examine Mount Holly in the state of Michigan. Mount Holly is a landfill that was converted into a ski resort. The shape of Mount Holly can be approximated by a right circular cone of height 1100 ft and radius 6000 ft.

If the compacted trash used to build Mount Holly on average has a density $400\,lb/ft^3$, find the amount of work required to build the mountain.

[Hide Solution]

$1.452\pi \times 10^{15}$ ft-lb

In reality, it is very likely that the trash at the bottom of Mount Holly has become more compacted with all the weight of the above trash. Consider a density function with respect to height: the density at the top of the mountain is still density $400\,lb/ft^3$ and the density increases. Every 100 feet deeper, the density doubles. What is the total weight of Mount Holly?

The following problems consider the temperature and density of Earth's layers.

[T] The temperature of Earth's layers is exhibited in the table below. Use your calculator to fit a polynomial of degree 3 to the temperature along the radius of the Earth. Then find the average temperature of Earth. (*Hint*: begin at 0 in the inner core and increase outward toward the surface)

Layer	Depth from center (km)	Temperature ^oC
Rocky Crust	0 to 40	0
Upper Mantle	40 to 150	870
Mantle	400 to 650	870
Inner Mantel	650 to 2700	870
Molten Outer Core	2890 to 5150	4300
Inner Core	5150 to 6378	7200

Source: http://www.enchantedlearning.com/sub...h/Inside.shtml

[Hide Solution]

$y = -1.238 \times 10^{-7}x^3 + 0.001196x^2 - 3.666x + 7208$; average temperature approximately 2800^oC

[T] The density of Earth's layers is displayed in the table below. Using your calculator or a computer program, find the best-fit quadratic equation to the density. Using this equation, find the total mass of Earth.

Layer	Depth from center (km)	Density (g/cm^3)
Inner Core	0	12.95
Outer Core	1228	11.05
Mantle	3488	5.00
Upper Mantle	6338	3.90
Crust	6378	2.55

Source: http://hyperphysics.phy-astr.gsu.edu...rthstruct.html

The following problems concern the Theorem of Pappus (see Moments and Centers of Mass for a refresher), a method for calculating volume using centroids. Assuming a region R, when you revolve around the x-axis the volume is given by $V_x = 2\pi A\bar{y}$, and when you revolve around the y-axis the volume is given by $V_y = 2\pi A\bar{x}$, where A is the area of R. Consider the region bounded by $x^2 + y^2 = 1$ and above $y = x + 1$.

Find the volume when you revolve the region around the x-axis.

[Hide Solution]

$\frac{\pi}{3}$

Find the volume when you revolve the region around the y-axis.

Glossary

Jacobian

the Jacobian $J(u, v)$ in two variables is a 2×2 determinant:

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix};$$

the Jacobian $J(u, v, w)$ in three variables is a 3×3 determinant:

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}$$

one-to-one transformation

a transformation $T : G \to R$ defined as $T(u, v) = (x, y)$ is said to be one-to-one if no two points map to the same image point

planar transformation

a function T that transforms a region G in one plane into a region R in another plane by a change of variables

transformation

a function that transforms a region GG in one plane into a region RR in another plane by a change of variables

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15.7E: EXERCISES FOR SECTION 15.7

In exercises 1 - 6, the function $T: S \to R,\ T(u,v) = (x,y)$ on the region $S = \{(u,v) \mid 0 \le u \le 1,\ 0 \le v \le 1\}$ bounded by the unit square is given, where $R \in R^2$ is the image of S under T.

a. Justify that the function T is a C^1 transformation.

b. Find the images of the vertices of the unit square S through the function T.

c. Determine the image R of the unit square S and graph it.

1. $x = 2u,\ y = 3v$

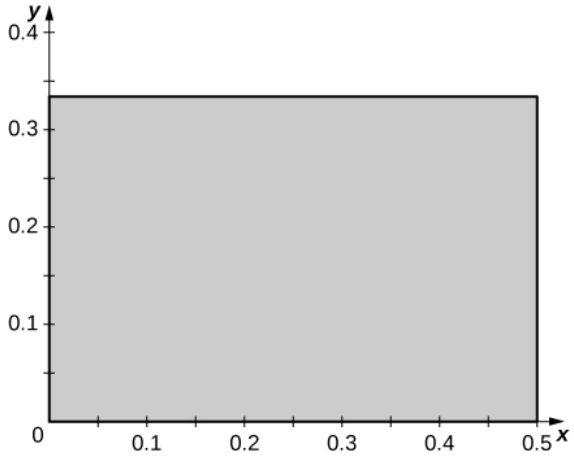
2. $x = \frac{u}{2},\ y = \frac{v}{3}$

Answer

a. $T(u,v) = (g(u,v),\ h(u,v),\ x = g(u,v) = \frac{u}{2}$ and $y = h(u,v) = \frac{v}{3}$. The functions g and h are continuous and differentiable, and the partial derivatives $g_u(u,v) = \frac{1}{2},\ g_v(u,v) = 0,\ h_u(u,v) = 0$ and $h_v(u,v) = \frac{1}{3}$ are continuous on S;

b. $T(0,0) = (0,0),\ T(1,0) = \left(\frac{1}{2}, 0\right),\ T(0,1) = \left(0, \frac{1}{3}\right)$, and $T(1,1) = \left(\frac{1}{2}, \frac{1}{3}\right)$;

c. R is the rectangle of vertices $(0,0),\ \left(0, \frac{1}{3}\right),\ \left(\frac{1}{2}, \frac{1}{3}\right)$, and $\left(0, \frac{1}{3}\right)$ in the xy-plane; the following figure.



3. $x = u - v,\ y = u + v$

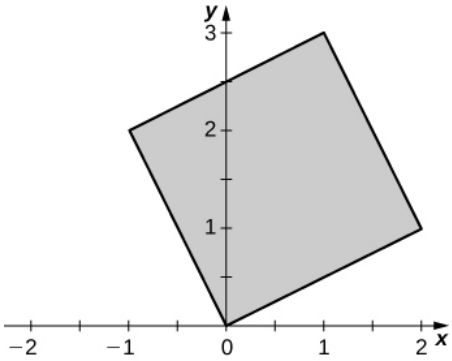
4. $x = 2u - v,\ y = u + 2v$

Answer

a. $T(u,v) = (g(u,v),\ h(u,v),\ x = g(u,v) = 2u - v$ and $y = h(u,v) = u + 2v$. The functions g and h are continuous and differentiable, and the partial derivatives $g_u(u,v) = 2,\ g_v(u,v) = -1,\ h_u(u,v) = 1$ and $h_v(u,v) = 2$ are continuous on S;

b. $T(0,0) = (0,0),\ T(1,0) = (2,1),\ T(0,1) = (-1,2)$, and $T(1,1) = (1,3)$;

c. R is the parallelogram of vertices $(0,0),\ (2,1)\ (1,3)$, and $(-1,2)$ in the xy-plane; the following figure.



5. $x = u^2,\ y = v^2$

https://math.libretexts.org/@go/page/66957

6. $x = u^3,\ y = v^3$

Answer

a. $T(u,v) = (g(u,v),\ h(u,v),\ x = g(u,v) = u^3$ and $y = h(u,v) = v^3$. The functions g and h are continuous and differentiable, and the partial derivatives $g_u(u,v) = 3u^2,\ g_v(u,v) = 0,\ h_u(u,v) = 0$ and $h_v(u,v) = 3v^2$ are continuous on S;

b. $T(0,0) = (0,0),\ T(1,0) = (1,0),\ T(0,1) = (0,1)$, and $T(1,1) = (1,1)$;

c. R is the unit square in the xy-plane see the figure in the answer to the previous exercise.

In exercises 7 - 12, determine whether the transformations $T : S \to R$ are one-to-one or not.

7. $x = u^2,\ y = v^2$, where S is the rectangle of vertices $(-1,0),\ (1,0),\ (1,1)$, and $(-1,1)$.

8. $x = u^4,\ y = u^2 + v$, where S is the triangle of vertices $(-2,0),\ (2,0)$, and $(0,2)$.

Answer

T is not one-to-one: two points of S have the same image. Indeed, $T(-2,0) = T(2,0) = (16,4)$.

9. $x = 2u,\ y = 3v$, where S is the square of vertices $(-1,1),\ (-1,-1),\ (1,-1)$, and $(1,1)$.

10. $T(u,v) = (2u - v, u)$, where S is the triangle with vertices $(-1,1),\ (-1,-1)$, and $(1,-1)$.

Answer

T is one-to-one: We argue by contradiction. $T(u_1,v_1) = T(u_2,v_2)$ implies $2u_1 - v_1 = 2u_2 - v_2$ and $u_1 = u_2$. Thus, $u_1 = u + 2$ and $v_1 = v_2$.

11. $x = u + v + w,\ y = u + v,\ z = w$, where $S = R = R^3$.

12. $x = u^2 + v + w,\ y = u^2 + v,\ z = w$, where $S = R = R^3$.

Answer

T is not one-to-one: $T(1,v,w) = (-1,v,w)$

In exercises 13 - 18, the transformations $T : R \to S$ are one-to-one. Find their related inverse transformations $T^{-1} : R \to S$.

13. $x = 4u,\ y = 5v$, where $S = R = R^2$.

14. $x = u + 2v,\ y = -u + v$, where $S = R = R^2$.

Answer

$u = \frac{x-2y}{3},\ v = \frac{x+y}{3}$

15. $x = e^{2u+v},\ y = e^{u-v}$, where $S = R^2$ and $R = \{(x,y) \mid x > 0,\ y > 0\}$

16. $x = \ln u,\ y = \ln(uv)$, where $S = \{(u,v) \mid u > 0,\ v > 0\}$ and $R = R^2$.

Answer

$u = e^x,\ v = e^{-x+y}$

17. $x = u + v + w,\ y = 3v,\ z = 2w$, where $S = R = R^3$.

18. $x = u + v,\ y = v + w,\ z = u + w$, where $S = R = R^3$.

Answer

$u = \frac{x-y+z}{2},\ v = \frac{x+y-z}{2},\ w = \frac{-x+y+z}{2}$

In exercises 19 - 22, the transformation $T : S \to R,\ T(u,v) = (x,y)$ and the region $R \subset R^2$ are given. Find the region $S \subset R^2$.

19. $x = au,\ y = bv,\ R = \{(x,y) \mid x^2 + y^2 \le a^2b^2\}$ where $a,b > 0$

20. $x = au,\ y = bc,\ R = \{(x,y) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1\}$, where $a,b > 0$

Answer

$S = \{(u,v) \mid u^2 + v^2 \le 1\}$

21. $x = \frac{u}{a},\ y = \frac{v}{b},\ z = \frac{w}{c},\ R = \{(x,y) \mid x^2 + y^2 + z^2 \le 1\}$, where $a,b,c > 0$

22. $x = au,\ y = bv,\ z = cw,\ R = \left\{(x,y)\,|\,\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} \le 1,\ z > 0\right\}$, where $a, b, c > 0$

Answer

$R = \left\{(u,v,w)\,|\,u^2 - v^2 - w^2 \le 1,\ w > 0\right\}$

In exercises 23 - 32, find the Jacobian J of the transformation.

23. $x = u + 2v,\ y = -u + v$

24. $x = \frac{u^3}{2},\ y = \frac{v}{u^2}$

Answer

$\frac{3}{2}$

25. $x = e^{2u-v},\ y = e^{u+v}$

26. $x = ue^v,\ y = e^{-v}$

Answer

-1

27. $x = u\ \cos(e^v),\ y = u\ \sin(e^v)$

28. $x = v\ \sin(u^2),\ y = v\ \cos(u^2)$

Answer

$2uv$

29. $x = u\ \cosh v,\ y = u\ \sinh v,\ z = w$

30. $x = v\ \cosh\left(\frac{1}{u}\right),\ y = v\ \sinh\left(\frac{1}{u}\right),\ z = u + w^2$

Answer

$\frac{v}{u^2}$

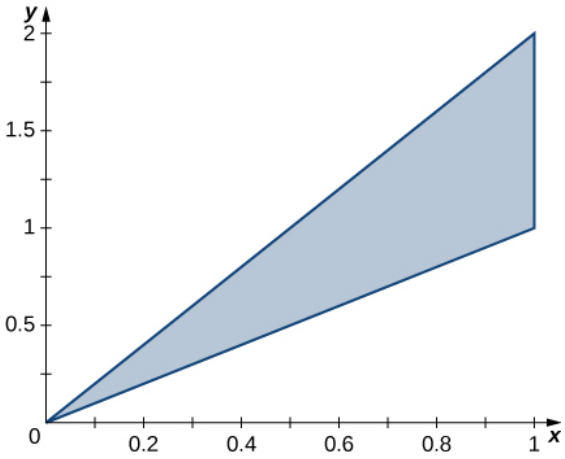
31. $x = u + v,\ y = v + w,\ z = u$

32. $x = u - v,\ y = u + v,\ z = u + v + w$

Answer

2

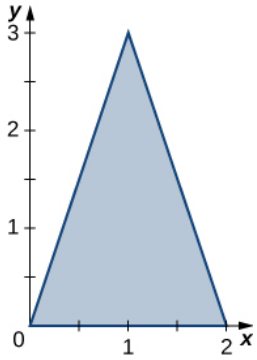
33. The triangular region R with the vertices $(0,0),\ (1,1)$, and $(1,2)$ is shown in the following figure.



a. Find a transformation $T : S \to R,\ T(u,v) = (x,y) = (au + bv + dv)$, where a, b, c, and d are real numbers with $ad - bc \neq 0$ such that $T^{-1}(0,0) = (0,0),\ T^{-1}(1,1) = (1,0)$, and $T^{-1}(1,2) = (0,1)$.

b. Use the transformation T to find the area $A(R)$ of the region R.

34. The triangular region R with the vertices $(0,0),\ (2,0)$, and $(1,3)$ is shown in the following figure.



a. Find a transformation $T: S \to R,\ T(u,v) = (x,y) = (au + bv + dv)$, where a, b, c, and d are real numbers with $ad - bc \neq 0$ such that $T^{-1}(0,0) = (0,0),\ T^{-1}(2,0) = (1,0)$, and $T^{-1}(1,3) = (0,1)$.

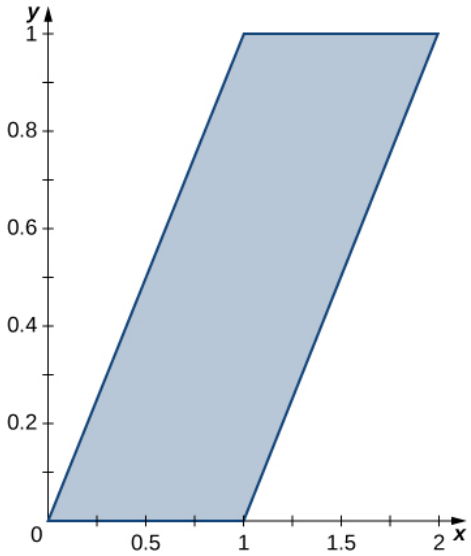
b. Use the transformation T to find the area $A(R)$ of the region R.

Answer

a. $T(u,v) = (2u + v,\ 3v)$

b. The area of R is $A(R) = \int_0^3 \int_{y/3}^{(6-y)/3} dx\, dy = \int_0^1 \int_0^{1-u} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dv\, du = \int_0^1 \int_0^{1-u} 6\, dv\, du = 3.$

In exercises 35 - 36, use the transformation $u = y - x,\ v = y$, to evaluate the integrals on the parallelogram R of vertices $(0,0)$, $(1,0)$, $(2,1)$, and $(1,1)$ shown in the following figure.



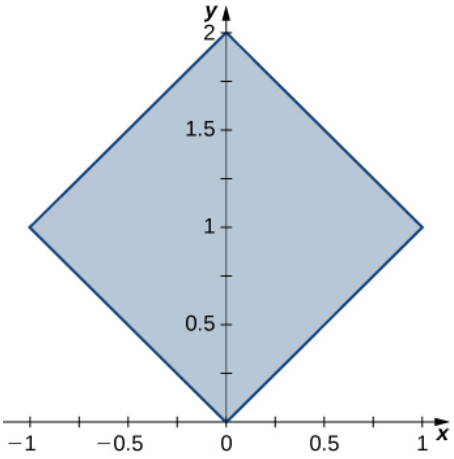
35. $\iint_R (y - x)\, dA$

36. $\iint_R (y^2 - xy)\, dA$

Answer

$-\frac{1}{4}$

In exercises 37 - 38, use the transformation $y = x = u,\ x + y = v$ to evaluate the integrals on the square R determined by the lines $y = x,\ y = -x + 2,\ y = x + 2$, and $y = -x$ shown in the following figure.



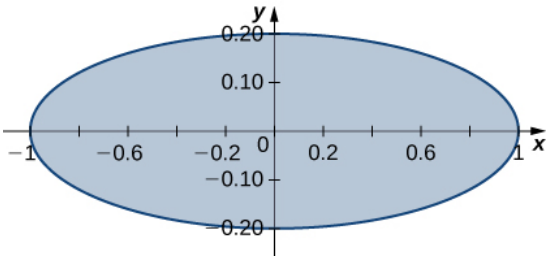
37. $\iint_R e^{x+y}\,dA$

38. $\iint_R \sin(x-y)\,dA$

Answer

$-1+\cos 2$

In exercises 39 - 40, use the transformation $x=u$, $5y=v$ to evaluate the integrals on the region R bounded by the ellipse $x^2+25y^2=1$ shown in the following figure.



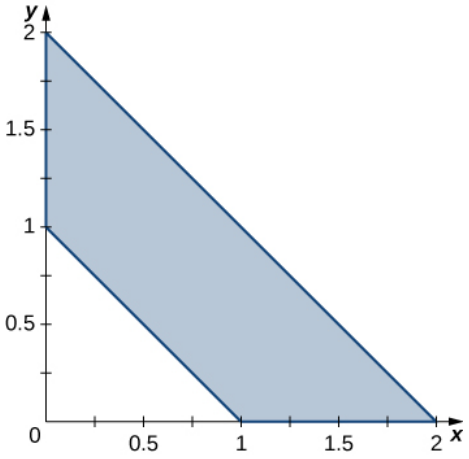
39. $\iint_R \sqrt{x^2+25y^2}\,dA$

40. $\iint_R (x^2+25y^2)^2\,dA$

Answer

$\frac{\pi}{15}$

In exercises 41 - 42, use the transformation $u=x+y$, $v=x-y$ to evaluate the integrals on the trapezoidal region R determined by the points $(1,0)$, $(2,0)$, $(0,2)$, and $(0,1)$ shown in the following figure.



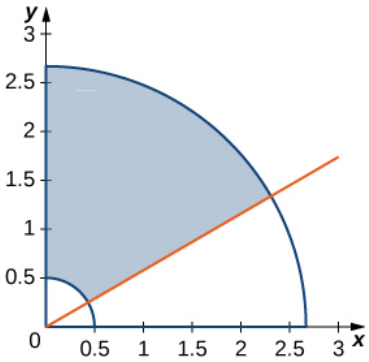
41. $\iint_R (x^2 - 2xy + y^2)\, e^{x+y}\, dA$

42. $\iint_R (x^3 + 3x^2y + 3xy^2 + y^3)\, dA$

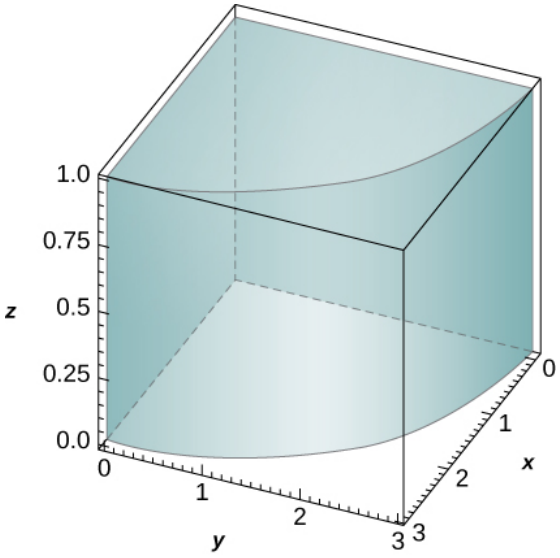
Answer

$\frac{31}{5}$

43. The circular annulus sector R bounded by the circles $4x^2 + 4y^2 = 1$ and $9x^2 + 9y^2 = 64$, the line $x = y\sqrt{3}$, and the y-axis is shown in the following figure. Find a transformation T from a rectangular region S in the $r\theta$-plane to the region R in the xy-plane. Graph S.



44. The solid R bounded by the circular cylinder $x^2 + y^2 = 9$ and the planes $z = 0,\ z = 1,\ x = 0$, and $y = 0$ is shown in the following figure. Find a transformation T from a cylindrical box S in $r\theta z$-space to the solid R in xyz-space.



https://math.libretexts.org/@go/page/66957

Answer

$T(r,\theta,z) = (r\ \cos\theta,\ r\ \sin\theta,\ z);\ S = [0,3] \times [0,\frac{\pi}{2}] \times [0,1]$ in the $r\theta z$-space

45. Show that

$$\iint_R f\left(\sqrt{\frac{x^2}{3} + \frac{y^2}{3}}\right) dA = 2\pi\sqrt{15}\int_0^1 f(\rho)\rho\ d\rho,$$

where f is a continuous function on $[0,1]$ and R is the region bounded by the ellipse $5x^2 + 3y^2 = 15$.

46. Show that

$$\iiint_R f\left(\sqrt{16x^2 + 4y^2 + z^2}\right) dV = \frac{\pi}{2}\int_0^1 f(\rho)\rho^2 d\rho,$$

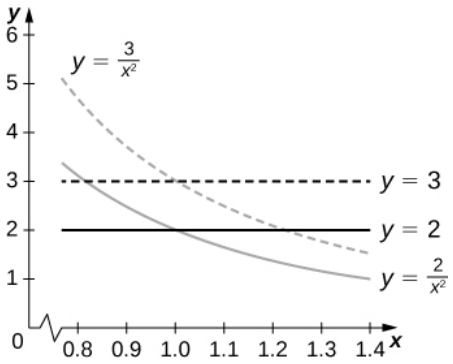
where f is a continuous function on $[0,1]$ and R is the region bounded by the ellipsoid $16x^2 + 4y^2 + z^2 = 1$.

47. [T] Find the area of the region bounded by the curves $xy = 1,\ xy = 3,\ y = 2x$, and $y = 3x$ by using the transformation $u = xy$ and $v = \frac{y}{x}$. Use a computer algebra system (CAS) to graph the boundary curves of the region R.

48. [T] Find the area of the region bounded by the curves $x^2y = 2,\ x^2y = 3,\ y = x$, and $y = 2x$ by using the transformation $u = x^2y$ and $v = \frac{y}{x}$. Use a CAS to graph the boundary curves of the region R.

Answer

The area of R is $10 - 4\sqrt{6}$; the boundary curves of R are graphed in the following figure.



49. Evaluate the triple integral

$$\int_0^1 \int_1^2 \int_z^{z+1} (y+1)\ dx\ dy\ dz$$

by using the transformation $u = x - z,\ v = 3y$, and $w = \frac{z}{2}$.

50. Evaluate the triple integral

$$\int_0^2 \int_4^6 \int_{3z}^{3z+2} (5 - 4y)\ dx\ dy\ dz$$

by using the transformation $u = x - 3z,\ v = 4y$, and $w = z$.

Answer

8

51. A transformation $T : R^2 \to R^2,\ T(u,v) = (x,y)$ of the form $x = au + bv,\ y = cu + dv$, where a, b, c, and d are real numbers, is called linear. Show that a linear transformation for which $ad - bc \neq 0$ maps parallelograms to parallelograms.

52. A transformation $T_\theta : R^2 \to R^2,\ T_\theta(u,v) = (x,y)$ of the form $x = u\ \cos\theta - v\ \sin\theta,\ y = u\ \sin\theta + v\ \cos\theta$, is called a rotation angle θ. Show that the inverse transformation of T_θ satisfies $T_\theta^{-1} = T_{-\theta}$ where $T_{-\theta}$ is the rotation of angle $-\theta$.

53. [T] Find the region S in the uv-plane whose image through a rotation of angle $\frac{\pi}{4}$ is the region R enclosed by the ellipse $x^2 + 4y^2 = 1$. Use a CAS to answer the following questions.

a. Graph the region S.

b. Evaluate the integral $\iint_S e^{-2uv}\,du\,dv$. Round your answer to two decimal places.

54. [T] The transformations $T_i : \mathbb{R}^2 \to \mathbb{R}^2$, $i = 1, ..., 4$, defined by $T_1(u,v) = (u,-v)$, $T_2(u,v) = (-u,v)$, $T_3(u,v) = (-u,-v)$, and $T_4(u,v) = (v,u)$ are called reflections about the x-axis, y-axis origin, and the line $y = x$, respectively.

a. Find the image of the region $S = \{(u,v) \mid u^2 + v^2 - 2u - 4v + 1 \le 0\}$ in the xy-plane through the transformation $T_1 \circ T_2 \circ T_3 \circ T_4$.

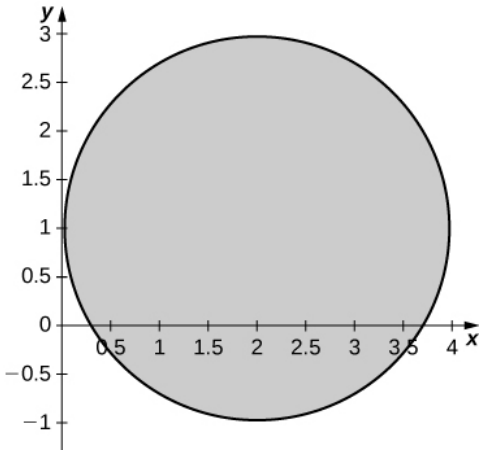
b. Use a CAS to graph R.

c. Evaluate the integral $\iint_S \sin(u^2)\,du\,dv$ by using a CAS. Round your answer to two decimal places.

Answer

a. $R = \{(x,y) \mid y^2 + x^2 - 2y - 4x + 1 \le 0\}$;

b. R is graphed in the following figure;



c. **3.16**

55. [T] The transformations $T_{k,1,1} : \mathbb{R}^3 \to \mathbb{R}^3$, $T_{k,1,1}(u,v,w) = (x,y,z)$ of the form $x = ku$, $y = v$, $z = w$, where $k \neq 1$ is a positive real number, is called a stretch if $k > 1$ and a compression if $0 < k < 1$ in the x-direction. Use a CAS to evaluate the integral $\iiint_S e^{-(4x^2+9y^2+25z^2)}\,dx\,dy\,dz$ on the solid $S = \{(x,y,z) \mid 4x^2 + 9y^2 + 25z^2 \le 1\}$ by considering the compression $T_{2,3,5}(u,v,w) = (x,y,z)$ defined by $x = \frac{u}{2}$, $y = \frac{v}{3}$, and $z = \frac{w}{5}$. Round your answer to four decimal places.

56. [T] The transformation $T_{a,0} : \mathbb{R}^2 \to \mathbb{R}^2$, $T_{a,0}(u,v) = (u + av, v)$, where $a \neq 0$ is a real number, is called a shear in the x-direction. The transformation, $T_{b,0} : R^2 \to R^2$, $T_{o,b}(u,v) = (u, bu + v)$, where $b \neq 0$ is a real number, is called a shear in the y-direction.

a. Find transformations $T_{0,2} \circ T_{3,0}$.

b. Find the image R of the trapezoidal region S bounded by $u = 0$, $v = 0$, $v = 1$, and $v = 2 - u$ through the transformation $T_{0,2} \circ T_{3,0}$.

c. Use a CAS to graph the image R in the xy-plane.

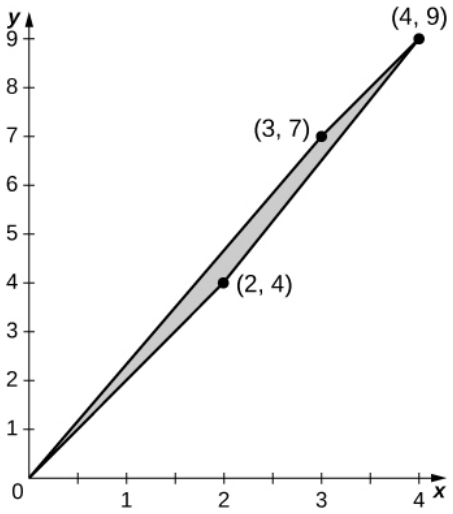
d. Find the area of the region R by using the area of region S.

Answer

a. $T_{0,2} \circ T_{3,0}(u,v) = (u + 3v, 2u + 7v)$;

b. The image S is the quadrilateral of vertices $(0,0)$, $(3,7)$, $(2,4)$, and $(4,9)$;

c. S is graphed in the following figure;



d. $\frac{3}{2}$

57. Use the transformation, $x = au,\ y = av,\ z = cw$ and spherical coordinates to show that the volume of a region bounded by the spheroid $\frac{x^2+y^2}{a^2} + \frac{z^2}{c^2} = 1$ is $\frac{4\pi a^2 c}{3}$.

58. Find the volume of a football whose shape is a spheroid $\frac{x^2+y^2}{a^2} + \frac{z^2}{c^2} = 1$ whose length from tip to tip is 11 inches and circumference at the center is 22 inches. Round your answer to two decimal places.

Answer

$\frac{2662}{3\pi} \approx 282.45\ in^3$

CONTRIBUTORS

Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

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15.R: CHAPTER 15 REVIEW EXERCISES

True or False? **Justify your answer with a proof or a counterexample.**

1. $\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dy\,dx$

2. Fubini's theorem can be extended to three dimensions, as long as f is continuous in all variables.

Answer

True

3. The integral $\int_0^{2\pi} \int_0^1 \int_r^1 dz\,dr\,d\theta$ represents the volume of a right cone.

4. The Jacobian of the transformation for $x = u^2 - 2v,\ y = 3v - 2uv$ is given by $-4u^2 + 6u + 4v$.

Answer

False

Evaluate the following integrals.

5. $\iint_R (5x^3y^2 - y^2)\,dA$, where $R = \{(x,y) \mid 0 \le x \le 2,\ 1 \le y \le 4\}$

6. $\iint_D \dfrac{y}{3x^2+1}\,dA$, where $D = \{(x,y) \mid 0 \le x \le 1,\ -x \le y \le x\}$

Answer

0

7. $\iint_D \sin(x^2+y^2)\,dA$ where D is a disk of radius 2 centered at the origin.

8. $\int_0^1 \int_y^1 xye^{x^2}\,dx\,dy$

Answer

$\frac{1}{4}$

9. $\int_{-1}^1 \int_0^z \int_0^{x-z} 6\,dy\,dx\,dz$

10. $\iiint_R 3y\,dV$, where $R = \{(x,y,z) \mid 0 \le x \le 1,\ 0 \le y \le x,\ 0 \le z \le 9 - y^2\}$

Answer

1.475

11. $\int_0^2 \int_0^{2\pi} \int_r^1 r\,dz\,d\theta\,dr$

12. $\int_0^{2\pi} \int_0^{\pi/2} \int_1^3 \rho^2 \sin(\varphi)\,d\rho\,d\varphi\,d\theta$

Answer

$\frac{52\pi}{3}$

13. $\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz\,dy\,dx$

For the following problems, find the specified area or volume.

14. The area of region enclosed by one petal of $r = \cos(4\theta)$.

Answer

$\frac{\pi}{16}$ units^3

15. The volume of the solid that lies between the paraboloid $z = 2x^2 + 2y^2$ and the plane $z = 8$.

16. The volume of the solid bounded by the cylinder $x^2 + y^2 = 16$ and from $z = 1$ to $z + x = 2$.

Answer

93.291 units^3

17. The volume of the intersection between two spheres of radius 1, the top whose center is $(0, 0, 0.25)$ and the bottom, which is centered at $(0, 0, 0)$.

For the following problems, find the center of mass of the region.

18. $\rho(x, y) = xy$ on the circle with radius 1 in the first quadrant only.

Answer

$\left(\frac{8}{15}, \frac{8}{15}\right)$

19. $\rho(x, y) = (y + 1)\sqrt{x}$ in the region bounded by $y = e^x$, $y = 0$, and $x = 1$.

20. $\rho(x, y, z) = z$ on the inverted cone with radius 2 and height 2.

Answer

$\left(0, 0, \frac{8}{5}\right)$

21. The volume an ice cream cone that is given by the solid above $z = \sqrt{x^2 + y^2}$ and below $z^2 + x^2 + y^2 = z$.

The following problems examine Mount Holly in the state of Michigan. Mount Holly is a landfill that was converted into a ski resort. The shape of Mount Holly can be approximated by a right circular cone of height 1100 ft and radius 6000 ft.

22. If the compacted trash used to build Mount Holly on average has a density 400 lb/ft^3, find the amount of work required to build the mountain.

Answer

$1.452\pi \times 10^{15}$ ft-lb

23. In reality, it is very likely that the trash at the bottom of Mount Holly has become more compacted with all the weight of the above trash. Consider a density function with respect to height: the density at the top of the mountain is still density 400 lb/ft^3, and the density increases. Every 100 feet deeper, the density doubles. What is the total weight of Mount Holly?

The following problems consider the temperature and density of Earth's layers.

24. **[T]** The temperature of Earth's layers is exhibited in the table below. Use your calculator to fit a polynomial of degree 3 to the temperature along the radius of the Earth. Then find the average temperature of Earth. (*Hint*: begin at 0 in the inner core and increase outward toward the surface)

Source: http://www.enchantedlearning.com/sub...h/Inside.shtml

Layer	Depth from center (km)	Temperature °C
Rocky Crust	0 to 40	0
Upper Mantle	40 to 150	870
Mantle	400 to 650	870
Inner Mantel	650 to 2700	870
Molten Outer Core	2890 to 5150	4300
Inner Core	5150 to 6378	7200

Answer

$y = -1.238 \times 10^{-7}x^3 + 0.001196x^2 - 3.666x + 7208$;
The average temperature is approximately 2800 °C.

25. **[T]** The density of Earth's layers is displayed in the table below. Using your calculator or a computer program, find the best-fit quadratic equation to the density. Using this equation, find the total mass of Earth.

Source: http://hyperphysics.phy-astr.gsu.edu...rthstruct.html

Layer	Depth from center (km)	Density ($\mathrm{g/cm^3}$)
Inner Core	0	12.95
Outer Core	1228	11.05
Mantle	3488	5.00
Upper Mantle	6338	3.90
Crust	6378	2.55

The following problems concern the Theorem of Pappus (see Moments and Centers of Mass for a refresher), a method for calculating volume using centroids. Assuming a region R, when you revolve around the x-axis the volume is given by $V_x = 2\pi A\overline{y}$, and when you revolve around the y-axis the volume is given by $V_y = 2\pi A\overline{x}$, where A is the area of R. Consider the region bounded by $x^2 + y^2 = 1$ and above $y = x + 1$.

26. Find the volume when you revolve the region around the x-axis.

Answer

$\frac{\pi}{3}$ units^3

27. Find the volume when you revolve the region around the y-axis.

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CHAPTER OVERVIEW

16: Vector Calculus

In this chapter, we learn to model new kinds of integrals over fields such as magnetic fields, gravitational fields, or velocity fields. We also learn how to calculate the work done on a charged particle traveling through a magnetic field, the work done on a particle with mass traveling through a gravitational field, and the volume per unit time of water flowing through a net dropped in a river. All these applications are based on the concept of a vector field.

16.0: Prelude to Vector Calculus
16.1: Vector Fields
16.1E: Exercises for Section 16.1
16.2: Line Integrals
16.2E: Exercises for Section 16.2
16.3: Conservative Vector Fields
16.3E: Exercises for Section 16.3
16.4: Green's Theorem
16.4E: Exercises for Section 16.4
16.5: Divergence and Curl
16.5E: Exercises for Section 16.5
16.6: Surface Integrals
16.6E: Exercises for Section 16.6
16.7: Stokes' Theorem
16.7E: Exercises for Section 16.7
16.8: The Divergence Theorem
16.8E: Exercises for Section 16.8
16.R: Chapter 16 Review Exercises

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16.0: Prelude to Vector Calculus

Hurricanes are huge storms that can produce tremendous amounts of damage to life and property, especially when they reach land. Predicting where and when they will strike and how strong the winds will be is of great importance for preparing for protection or evacuation. Scientists rely on studies of rotational vector fields for their forecasts.

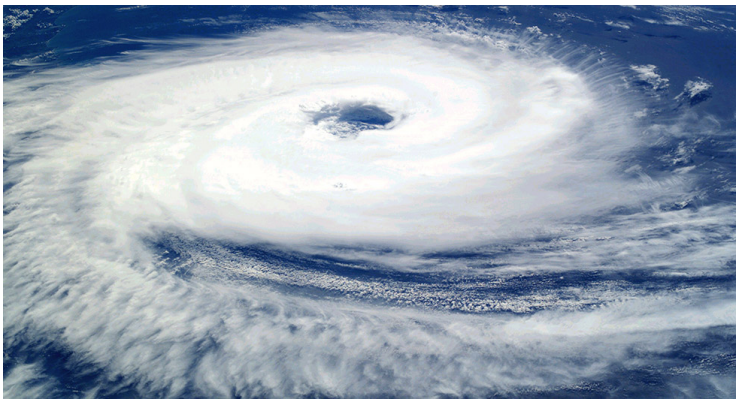

Figure 16.0.1: Hurricanes form from rotating winds driven by warm temperatures over the ocean. Meteorologists forecast the motion of hurricanes by studying the rotating vector fields of their wind velocity. Shown is Cyclone Catarina in the South Atlantic Ocean in 2004, as seen from the International Space Station. (credit: modification of work by NASA)

In this chapter, we learn to model new kinds of integrals over fields such as magnetic fields, gravitational fields, or velocity fields. We also learn how to calculate the work done on a charged particle traveling through a magnetic field, the work done on a particle with mass traveling through a gravitational field, and the volume per unit time of water flowing through a net dropped in a river.

All these applications are based on the concept of a vector field, which we explore in this chapter. Vector fields have many applications because they can be used to model real fields such as electromagnetic or gravitational fields. A deep understanding of physics or engineering is impossible without an understanding of vector fields. Furthermore, vector fields have mathematical properties that are worthy of study in their own right. In particular, vector fields can be used to develop several higher-dimensional versions of the Fundamental Theorem of Calculus.

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16.1: Vector Fields

Learning Objectives

- Recognize a vector field in a plane or in space.
- Sketch a vector field from a given equation.
- Identify a conservative field and its associated potential function.

Vector fields are an important tool for describing many physical concepts, such as gravitation and electromagnetism, which affect the behavior of objects over a large region of a plane or of space. They are also useful for dealing with large-scale behavior such as atmospheric storms or deep-sea ocean currents. In this section, we examine the basic definitions and graphs of vector fields so we can study them in more detail in the rest of this chapter.

Examples of Vector Fields

How can we model the gravitational force exerted by multiple astronomical objects? How can we model the velocity of water particles on the surface of a river? Figure 16.1.1 gives visual representations of such phenomena.

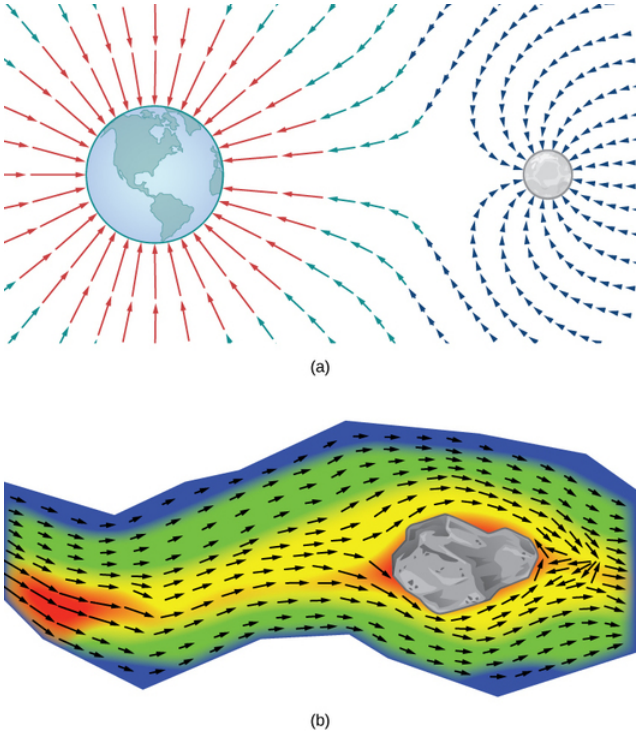


Figure 16.1.1 (a) The gravitational field exerted by two astronomical bodies on a small object. (b) The vector velocity field of water on the surface of a river shows the varied speeds of water. Red indicates that the magnitude of the vector is greater, so the water flows more quickly; blue indicates a lesser magnitude and a slower speed of water flow.

Figure 16.1.1a shows a gravitational field exerted by two astronomical objects, such as a star and a planet or a planet and a moon. At any point in the figure, the vector associated with a point gives the net gravitational force exerted by the two objects on an object of unit mass. The vectors of largest magnitude in the figure are the vectors closest to the larger object. The larger object has greater mass, so it exerts a gravitational force of greater magnitude than the smaller object.

Figure 16.1.1b shows the velocity of a river at points on its surface. The vector associated with a given point on the river's surface gives the velocity of the water at that point. Since the vectors to the left of the figure are small in magnitude, the water is flowing slowly on that part of the surface. As the water moves from left to right, it encounters some rapids around a rock. The speed of the water increases, and a whirlpool occurs in part of the rapids.

Each figure illustrates an example of a vector field. Intuitively, a vector field is a map of vectors. In this section, we study vector fields in $\mathbb{R}^2$ and $\mathbb{R}^3$.

DEFINITION: Vector Field

- A vector field $\vec{\mathbf{F}}$ in $\mathbb{R}^2$ is an assignment of a two-dimensional vector $\vec{\mathbf{F}}(x, y)$ to each point (x, y) of a subset D of $\mathbb{R}^2$. The subset D is the domain of the vector field.
- A vector field $\vec{\mathbf{F}}$ in $\mathbb{R}^3$ is an assignment of a three-dimensional vector $\vec{\mathbf{F}}(x, y, z)$ to each point (x, y, z) of a subset D of $\mathbb{R}^3$. The subset D is the domain of the vector field.

Vector Fields in $\mathbb{R}^2$

A vector field in $\mathbb{R}^2$ can be represented in either of two equivalent ways. The first way is to use a vector with components that are two-variable functions:

$$\vec{\mathbf{F}}(x, y) = \langle P(x, y), Q(x, y) \rangle$$

The second way is to use the standard unit vectors:

$$\vec{\mathbf{F}}(x, y) = P(x, y)\,\hat{\mathbf{i}} + Q(x, y)\,\hat{\mathbf{j}}.$$

A vector field is said to be *continuous* if its component functions are continuous.

Example 16.1.1: Finding a Vector Associated with a Given Point

Let $\vec{\mathbf{F}}(x, y) = (2y^2 + x - 4)\,\hat{\mathbf{i}} + \cos(x)\,\hat{\mathbf{j}}$ be a vector field in $\mathbb{R}^2$. Note that this is an example of a continuous vector field since both component functions are continuous. What vector is associated with point $(0, -1)$?

Solution

Substitute the point values for x and y:

$$\begin{aligned}\vec{\mathbf{F}}(0, -1) &= (2(-1)^2 + 0 - 4)\,\hat{\mathbf{i}} + \cos(0)\,\hat{\mathbf{j}} \\ &= -2\,\hat{\mathbf{i}} + \hat{\mathbf{j}}.\end{aligned}$$

Exercise 16.1.1

Let $\vec{\mathbf{G}}(x, y) = x^2 y\hat{\mathbf{i}} - (x + y)\,\hat{\mathbf{j}}$ be a vector field in $\mathbb{R}^2$. What vector is associated with the point $(-2, 3)$?

Hint

Substitute the point values into the vector function.

Answer

$\vec{\mathbf{G}}(-2, 3) = 12\hat{\mathbf{i}} - \hat{\mathbf{j}}$

Drawing a Vector Field

We can now represent a vector field in terms of its components of functions or unit vectors, but representing it visually by sketching it is more complex because the domain of a vector field is in $\mathbb{R}^2$, as is the range. Therefore the "graph" of a vector field in $\mathbb{R}^2$ lives in four-dimensional space. Since we cannot represent four-dimensional space visually, we instead draw vector fields in $\mathbb{R}^2$ in a plane itself. To do this, draw the vector associated with a given point at the point in a plane. For example, suppose the vector associated with point $(4, -1)$ is $\langle 3, 1 \rangle$. Then, we would draw vector $\langle 3, 1 \rangle$ at point $(4, -1)$.

We should plot enough vectors to see the general shape, but not so many that the sketch becomes a jumbled mess. If we were to plot the image vector at each point in the region, it would fill the region completely and is useless. Instead, we can choose points at the intersections of grid lines and plot a sample of several vectors from each quadrant of a rectangular coordinate system in $\mathbb{R}^2$.

There are two types of vector fields in $\mathbb{R}^2$ on which this chapter focuses: radial fields and rotational fields. Radial fields model certain gravitational fields and energy source fields, and rotational fields model the movement of a fluid in a vortex. In a radial field, all vectors either point directly toward or directly away from the origin. Furthermore, the magnitude of any vector depends

only on its distance from the origin. In a radial field, the vector located at point (x, y) is perpendicular to the circle centered at the origin that contains point (x, y), and all other vectors on this circle have the same magnitude.

Example 16.1.2: Drawing a Radial Vector Field

Sketch the vector field $\vec{\mathbf{F}}(x, y) = \frac{x}{2}\hat{\mathbf{i}} + \frac{y}{2}\hat{\mathbf{j}}$.

Solution

To sketch this vector field, choose a sample of points from each quadrant and compute the corresponding vector. The following table gives a representative sample of points in a plane and the corresponding vectors.

Table 16.1.1

(x, y)	$\vec{\mathbf{F}}(x, y)$	(x, y)	$\vec{\mathbf{F}}(x, y)$	(x, y)	$\vec{\mathbf{F}}(x, y)$
$(1, 0)$	$\langle \frac{1}{2}, 0 \rangle$	$(2, 0)$	$\langle 1, 0 \rangle$	$(1, 1)$	$\langle \frac{1}{2}, \frac{1}{2} \rangle$
$(0, 1)$	$\langle 0, \frac{1}{2} \rangle$	$(0, 2)$	$\langle 0, 1 \rangle$	$(-1, 1)$	$\langle -\frac{1}{2}, \frac{1}{2} \rangle$
$(-1, 0)$	$\langle -\frac{1}{2}, 0 \rangle$	$(-2, 0)$	$\langle -1, 0 \rangle$	$(-1, -1)$	$\langle -\frac{1}{2}, -\frac{1}{2} \rangle$
$(0, -1)$	$\langle 0, -\frac{1}{2} \rangle$	$(0, -2)$	$\langle 0, -1 \rangle$	$(1, -1)$	$\langle \frac{1}{2}, -\frac{1}{2} \rangle$

Figure 16.1.2a shows the vector field. To see that each vector is perpendicular to the corresponding circle, Figure 16.1.2b shows circles overlain on the vector field.

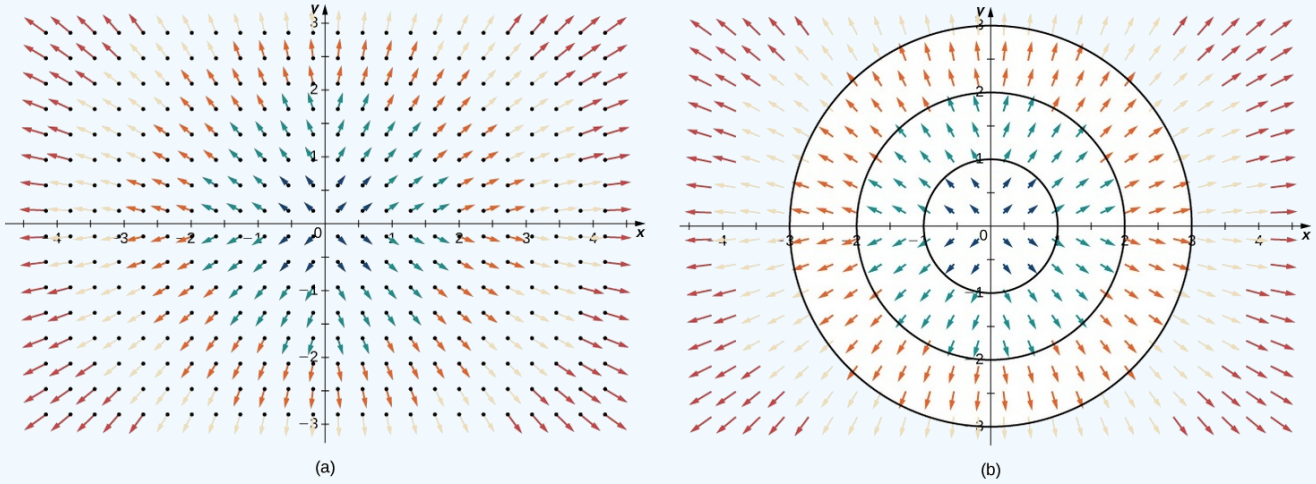


Figure 16.1.2: (a) A visual representation of the radial vector field $\vec{\mathbf{F}}(x, y) = \frac{x}{2}\hat{\mathbf{i}} + \frac{y}{2}\hat{\mathbf{j}}$. (b) The radial vector field $\vec{F}(x, y) = \frac{x}{2}\hat{\mathbf{i}} + \frac{y}{2}\hat{\mathbf{j}}$ with overlaid circles. Notice that each vector is perpendicular to the circle on which it is located.

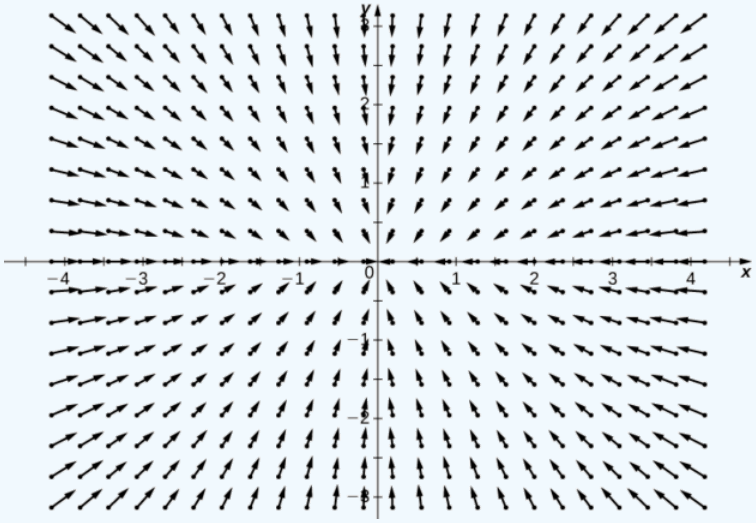
Exercise 16.1.2

Draw the radial field $\vec{\mathbf{F}}(x, y) = -\frac{x}{3}\hat{\mathbf{i}} - \frac{y}{3}\hat{\mathbf{j}}$.

Hint

Sketch enough vectors to get an idea of the shape.

Answer



In contrast to radial fields, in a *rotational field*, the vector at point (x, y) is tangent (not perpendicular) to a circle with radius $r = \sqrt{x^2 + y^2}$. In a standard rotational field, all vectors point either in a clockwise direction or in a counterclockwise direction, and the magnitude of a vector depends only on its distance from the origin. Both of the following examples are clockwise rotational fields, and we see from their visual representations that the vectors appear to rotate around the origin.

Example 16.1.3: Drawing a Rotational Vector Field

Sketch the vector field $\vec{\mathbf{F}}(x, y) = \langle y, -x \rangle$.

Solution

Create a table (see the one that follows) using a representative sample of points in a plane and their corresponding vectors. Figure 16.1.3 shows the resulting vector field.

Table 16.1.2

(x, y)	$\vec{\mathbf{F}}(x, y)$	(x, y)	$\vec{\mathbf{F}}(x, y)$	(x, y)	$\vec{\mathbf{F}}(x, y)$
$(1, 0)$	$\langle 0, -1 \rangle$	$(2, 0)$	$\langle 0, -2 \rangle$	$(1, 1)$	$\langle 1, -1 \rangle$
$(0, 1)$	$\langle 1, 0 \rangle$	$(0, 2)$	$\langle 2, 0 \rangle$	$(-1, 1)$	$\langle 1, 1 \rangle$
$(-1, 0)$	$\langle 0, 1 \rangle$	$(-2, 0)$	$\langle 0, 2 \rangle$	$(-1, -1)$	$\langle -1, 1 \rangle$
$(0, -1)$	$\langle -1, 0 \rangle$	$(0, -2)$	$\langle -2, 0 \rangle$	$(1, -1)$	$\langle -1, -1 \rangle$

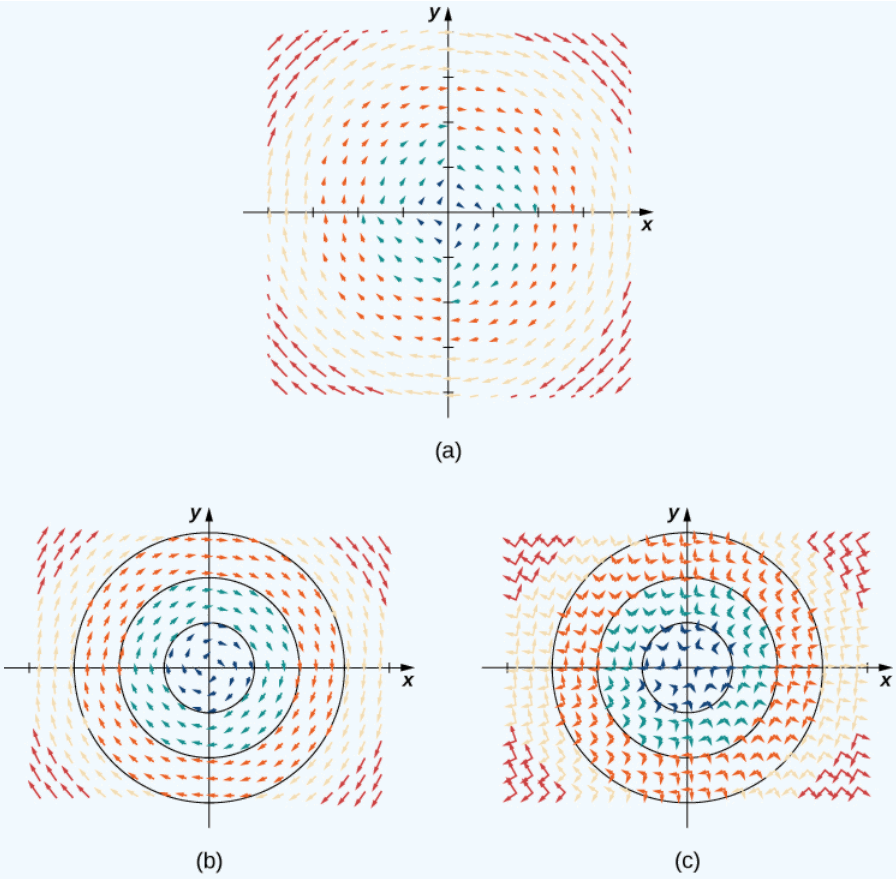


Figure 16.1.3: (a) A visual representation of vector field $\vec{\mathbf{F}}(x,y) = \langle y, -x \rangle$. (b) Vector field $\vec{\mathbf{F}}(x,y) = \langle y, -x \rangle$ with circles centered at the origin. (c) Vector $\vec{\mathbf{F}}(a,b)$ is perpendicular to radial vector $\langle a, b \rangle$ at point (a,b).

Analysis

Note that vector $\vec{\mathbf{F}}(a,b) = \langle b, -a \rangle$ points clockwise and is perpendicular to radial vector $\langle a, b \rangle$. (We can verify this assertion by computing the dot product of the two vectors: $\langle a, b \rangle \cdot \langle -b, a \rangle = -ab + ab = 0$.) Furthermore, vector $\langle b, -a \rangle$ has length $r = \sqrt{a^2 + b^2}$. Thus, we have a complete description of this rotational vector field: the vector associated with point (a,b) is the vector with length r tangent to the circle with radius r, and it points in the clockwise direction.

Sketches such as that in Figure 16.1.3 are often used to analyze major storm systems, including hurricanes and cyclones. In the northern hemisphere, storms rotate counterclockwise; in the southern hemisphere, storms rotate clockwise. (This is an effect caused by Earth's rotation about its axis and is called the Coriolis Effect.)

Figure 16.1.4: (credit: modification of work by NASA)

Example 16.1.4: Sketching a Vector Field

Sketch vector field $\vec{\mathbf{F}}(x,y) = \dfrac{y}{x^2+y^2}\hat{\mathbf{i}}, -\dfrac{x}{x^2+y^2}\hat{\mathbf{j}}$.

Solution

To visualize this vector field, first note that the dot product $\vec{\mathbf{F}}(a,b) \cdot (a\,\hat{\mathbf{i}} + b\,\hat{\mathbf{j}})$ is zero for any point (a,b). Therefore, each vector is tangent to the circle on which it is located. Also, as $(a,b) \to (0,0)$, the magnitude of $\vec{\mathbf{F}}(a,b)$ goes to infinity. To see this, note that

$$||\vec{\mathbf{F}}(a,b)|| = \sqrt{\frac{a^2+b^2}{(a^2+b^2)^2}} = \sqrt{\frac{1}{a^2+b^2}}.$$

Since $\dfrac{1}{a^2+b^2} \to \infty$ as $(a,b) \to (0,0)$, then $||\vec{\mathbf{F}}(a,b)|| \to \infty$ as $(a,b) \to (0,0)$. This vector field looks similar to the vector field in Example 16.1.3, but in this case the magnitudes of the vectors close to the origin are large. Table 16.1.3 shows a sample of points and the corresponding vectors, and Figure 16.1.5 shows the vector field. Note that this vector field models the whirlpool motion of the river in Figure 16.1.5(b). The domain of this vector field is all of $\mathbb{R}^2$ except for point $(0,0)$.

Table 16.1.3

(x,y)	$\vec{\mathbf{F}}(x,y)$	(x,y)	$\vec{\mathbf{F}}(x,y)$	(x,y)	$\vec{\mathbf{F}}(x,y)$
$(1,0)$	$\langle 0,-1\rangle$	$(2,0)$	$\langle 0,-\frac{1}{2}\rangle$	$(1,1)$	$\langle \frac{1}{2},-\frac{1}{2}\rangle$
$(0,1)$	$\langle 1,0\rangle$	$(0,2)$	$\langle \frac{1}{2},0\rangle$	$(-1,1)$	$\langle \frac{1}{2},\frac{1}{2}\rangle$
$(-1,0)$	$\langle 0,1\rangle$	$(-2,0)$	$\langle 0,\frac{1}{2}\rangle$	$(-1,-1)$	$\langle -\frac{1}{2},\frac{1}{2}\rangle$
$(0,-1)$	$\langle -1,0\rangle$	$(0,-2)$	$\langle -\frac{1}{2},0\rangle$	$(1,-1)$	$\langle -\frac{1}{2},-\frac{1}{2}\rangle$

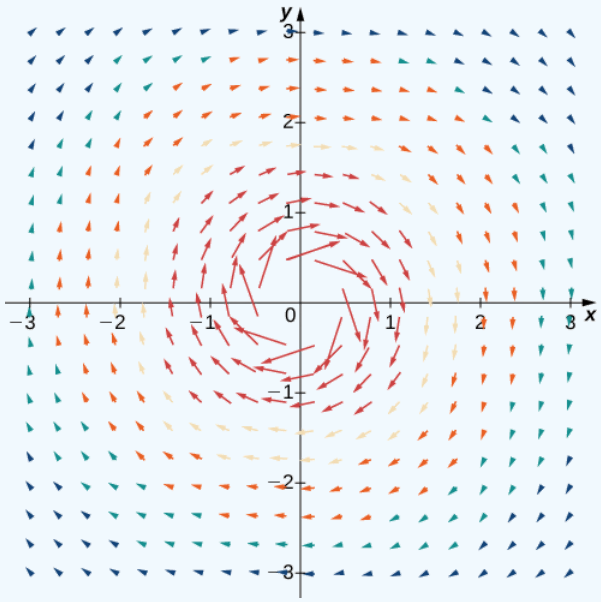


Figure 16.1.5: A visual representation of vector field $\vec{\mathbf{F}}(x,y) = \dfrac{y}{x^2+y^2}\hat{\mathbf{i}} - \dfrac{x}{x^2+y^2}\hat{\mathbf{j}}$. *This vector field could be used to model whirlpool motion of a fluid.*

? Exercise 16.1.4

Sketch vector field $\vec{\mathbf{F}}(x,y) = \langle -2y,\, 2x\rangle$. Is the vector field radial, rotational, or neither?

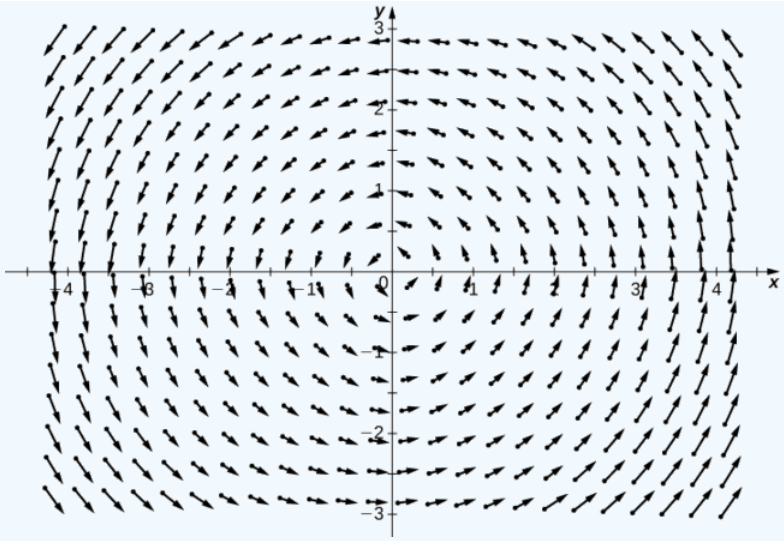
Hint

Substitute enough points into $\vec{\mathbf{F}}$ to get an idea of the shape.

Answer

Rotational

https://math.libretexts.org/@go/page/2617



Example 16.1.5: Velocity Field of a Fluid

Suppose that $\vec{\mathbf{v}}(x,y) = -\dfrac{2y}{x^2+y^2}\hat{\mathbf{i}} + \dfrac{2x}{x^2+y^2}\hat{\mathbf{j}}$ is the velocity field of a fluid. How fast is the fluid moving at point $(1,-1)$? (Assume the units of speed are meters per second.)

Solution

To find the velocity of the fluid at point $(1,-1)$, substitute the point into $\vec{\mathbf{v}}$:

$$\vec{\mathbf{v}}(1,-1) = \frac{-2(-1)}{1+1}\hat{\mathbf{i}} + \frac{2(1)}{1+1}\hat{\mathbf{j}} = \hat{\mathbf{i}} + \hat{\mathbf{j}}.$$

The speed of the fluid at $(1,-1)$ is the magnitude of this vector. Therefore, the speed is $||\hat{\mathbf{i}} + \hat{\mathbf{j}}|| = \sqrt{2}$ m/sec.

Exercise 16.1.5

Vector field $\vec{\mathbf{v}}(x,y) = \langle 4|x|,\ 1\rangle$ models the velocity of water on the surface of a river. What is the speed of the water at point $(2,3)$? Use meters per second as the units.

Hint

Remember, speed is the magnitude of velocity.

Answer

$\sqrt{65}$ m/sec

We have examined vector fields that contain vectors of various magnitudes, but just as we have unit vectors, we can also have a unit vector field. A vector field $\vec{\mathbf{F}}$ is a **unit vector field** if the magnitude of each vector in the field is 1. In a unit vector field, the only relevant information is the direction of each vector.

Example 16.1.6: A Unit Vector Field

Show that vector field $\vec{\mathbf{F}}(x,y) = \left\langle \dfrac{y}{\sqrt{x^2+y^2}}, -\dfrac{x}{\sqrt{x^2+y^2}} \right\rangle$ is a unit vector field.

Solution

To show that $\vec{\mathbf{F}}$ is a unit field, we must show that the magnitude of each vector is 1. Note that

$$\sqrt{\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2+\left(-\frac{x}{\sqrt{x^2+y^2}}\right)^2}=\sqrt{\frac{y^2}{x^2+y^2}+\frac{x^2}{x^2+y^2}}$$
$$=\sqrt{\frac{x^2+y^2}{x^2+y^2}}$$
$$=1$$

Therefore, $\vec{\mathbf{F}}$ is a unit vector field.

Exercise 16.1.6

Is vector field $\vec{\mathbf{F}}(x,y)=\langle -y,\, x\rangle$ a unit vector field?

Hint

Calculate the magnitude of $\vec{\mathbf{F}}$ at an arbitrary point (x,y).

Answer

No.

Why are unit vector fields important? Suppose we are studying the flow of a fluid, and we care only about the direction in which the fluid is flowing at a given point. In this case, the speed of the fluid (which is the magnitude of the corresponding velocity vector) is irrelevant, because all we care about is the direction of each vector. Therefore, the unit vector field associated with velocity is the field we would study.

If $\vec{\mathbf{F}}=\langle P,Q,R\rangle$ is a vector field, then the corresponding unit vector field is $\langle \frac{P}{||\vec{\mathbf{F}}||},\frac{Q}{||\vec{\mathbf{F}}||},\frac{R}{||\vec{\mathbf{F}}||}\rangle$. Notice that if $\vec{\mathbf{F}}(x,y)=\langle y,\,-x\rangle$ is the vector field from Example 16.1.6, then the magnitude of $\vec{\mathbf{F}}$ is $\sqrt{x^2+y^2}$, and therefore the corresponding unit vector field is the field $\vec{\mathbf{G}}$ from the previous example.

If $\vec{\mathbf{F}}$ is a vector field, then the process of dividing $\vec{\mathbf{F}}$ by its magnitude to form unit vector field $\vec{\mathbf{F}}/||\vec{\mathbf{F}}||$ is called normalizing the field $\vec{\mathbf{F}}$.

Vector Fields in $\mathbb{R}^3$

We have seen several examples of vector fields in $\mathbb{R}^2$; let's now turn our attention to vector fields in $\mathbb{R}^3$. These vector fields can be used to model gravitational or electromagnetic fields, and they can also be used to model fluid flow or heat flow in three dimensions. A two-dimensional vector field can really only model the movement of water on a two-dimensional slice of a river (such as the river's surface). Since a river flows through three spatial dimensions, to model the flow of the entire depth of the river, we need a vector field in three dimensions.

The extra dimension of a three-dimensional field can make vector fields in $\mathbb{R}^3$ more difficult to visualize, but the idea is the same. To visualize a vector field in $\mathbb{R}^3$, plot enough vectors to show the overall shape. We can use a similar method to visualizing a vector field in $\mathbb{R}^2$ by choosing points in each octant.

Just as with vector fields in $\mathbb{R}^2$, we can represent vector fields in $\mathbb{R}^3$ with component functions. We simply need an extra component function for the extra dimension. We write either

$$\vec{\mathbf{F}}(x,y,z)=\langle P(x,y,z),Q(x,y,z),R(x,y,z)\rangle$$

or

$$\vec{\mathbf{F}}(x,y,z)=P(x,y,z)\hat{\mathbf{i}}+Q(x,y,z)\hat{\mathbf{j}}+R(x,y,z)\hat{\mathbf{k}}.$$

Example 16.1.7: Sketching a Vector Field in Three Dimensions

Describe vector field $\vec{\mathbf{F}}(x,y,z)=\langle 1,\,1,\,z\rangle$.

Solution

For this vector field, the x- and y-components are constant, so every point in $\mathbb{R}^3$ has an associated vector with x- and y-components equal to one. To visualize $\vec{\mathbf{F}}$, we first consider what the field looks like in the xy-plane. In the xy-plane, $z = 0$. Hence, each point of the form $(a, b, 0)$ has vector $\langle 1, 1, 0\rangle$ associated with it. For points not in the xy-plane but slightly above it, the associated vector has a small but positive z-component, and therefore the associated vector points slightly upward. For points that are far above the xy-plane, the z-component is large, so the vector is almost vertical. Figure 16.1.6 shows this vector field.

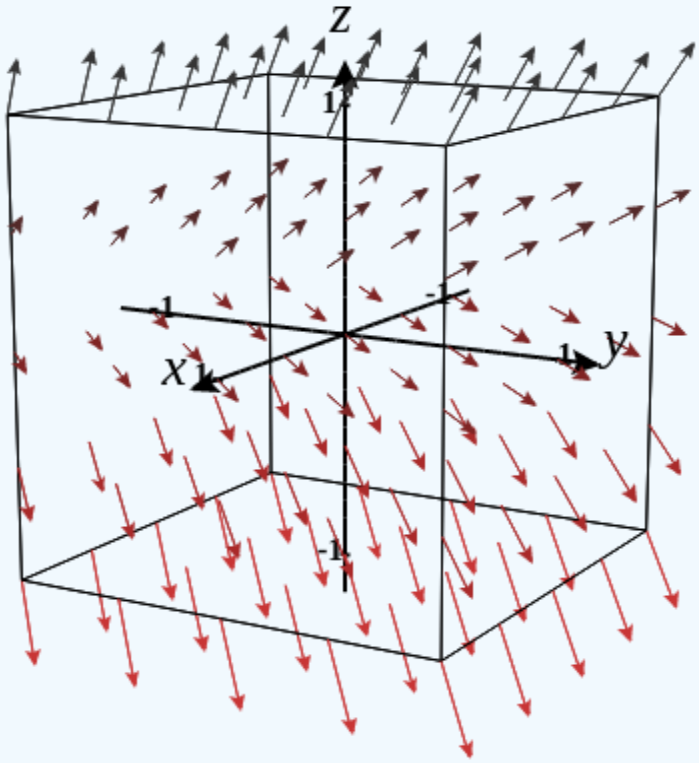


Figure 16.1.6: A visual representation of vector field $\vec{\mathbf{F}}(x, y, z) = \langle 1, 1, z\rangle$.

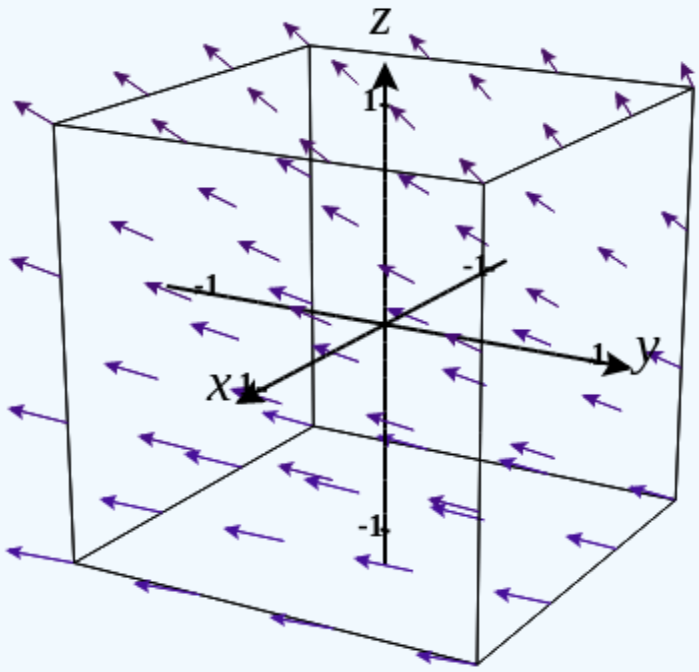
Exercise 16.1.7

Sketch vector field $\vec{\mathbf{G}}(x, y, z) = \langle 2, \frac{z}{2}, 1\rangle$.

Hint

Substitute enough points into the vector field to get an idea of the general shape.

Answer



In the next example, we explore one of the classic cases of a three-dimensional vector field: a gravitational field.

Example 16.1.8: Describing a Gravitational Vector Field

Newton's law of gravitation states that $\vec{\mathbf{F}} = -G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}}$, where G is the universal gravitational constant. It describes the gravitational field exerted by an object (object 1) of mass m_1 located at the origin on another object (object 2) of mass m_2 located at point (x, y, z). Field $\vec{\mathbf{F}}$ denotes the gravitational force that object 1 exerts on object 2, r is the distance between the two objects, and $\hat{\mathbf{r}}$ indicates the unit vector from the first object to the second. The minus sign shows that the gravitational force attracts toward the origin; that is, the force of object 1 is attractive. Sketch the vector field associated with this equation.

Solution

Since object 1 is located at the origin, the distance between the objects is given by $r = \sqrt{x^2 + y^2 + z^2}$. The unit vector from object 1 to object 2 is $\hat{\mathbf{r}} = \frac{\langle x, y, z\rangle}{||\langle x, y, z\rangle||}$, and hence $\hat{\mathbf{r}} = \langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\rangle$. Therefore, gravitational vector field $\vec{\mathbf{F}}$ exerted by object 1 on object 2 is

$$\vec{\mathbf{F}} = -Gm_1 m_2 \langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}\rangle.$$

This is an example of a radial vector field in $\mathbb{R}^3$.

Figure 16.1.7 shows what this gravitational field looks like for a large mass at the origin. Note that the magnitudes of the vectors increase as the vectors get closer to the origin.

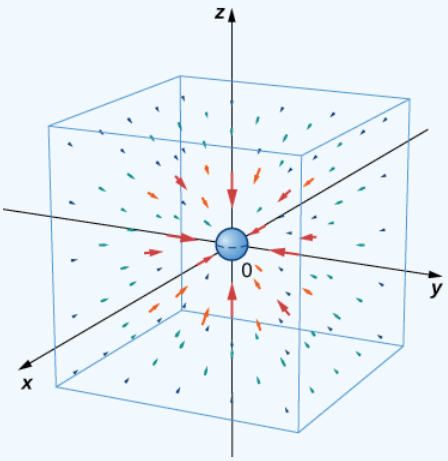


Figure 16.1.7: A visual representation of gravitational vector field $\vec{\mathbf{F}} = -Gm_1m_2\langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \rangle$ for a large mass at the origin.

? Exercise 16.1.8

The mass of asteroid 1 is 750,000 kg and the mass of asteroid 2 is 130,000 kg. Assume asteroid 1 is located at the origin, and asteroid 2 is located at $(15, -5, 10)$, measured in units of 10 to the eighth power kilometers. Given that the universal gravitational constant is $G = 6.67384 \times 10^{-11} m^3 kg^{-1} s^{-2}$, find the gravitational force vector that asteroid 1 exerts on asteroid 2.

Hint

Follow Example 16.1.8 and first compute the distance between the asteroids.

Answer

1.49063×10^{-18}, 4.96876×10^{-19}, 9.93752×10^{-19} N

Gradient Fields (Conservative Fields)

In this section, we study a special kind of vector field called a gradient field or a **conservative field**. These vector fields are extremely important in physics because they can be used to model physical systems in which energy is conserved. Gravitational fields and electric fields associated with a static charge are examples of gradient fields.

Recall that if f is a (scalar) function of x and y, then the gradient of f is

$$\text{grad } f = \vec{\nabla} f(x, y) = f_x(x, y)\hat{\mathbf{i}} + f_y(x, y)\hat{\mathbf{j}}.$$

We can see from the form in which the gradient is written that $\vec{\nabla} f$ is a vector field in $\mathbb{R}^2$. Similarly, if f is a function of x, y, and z, then the gradient of f is

$$\text{grad } f = \vec{\nabla} f(x, y, z) = f_x(x, y, z)\hat{\mathbf{i}} + f_y(x, y, z)\hat{\mathbf{j}} + f_z(x, y, z)\hat{\mathbf{k}}.$$

The gradient of a three-variable function is a vector field in $\mathbb{R}^3$. A gradient field is a vector field that can be written as the gradient of a function, and we have the following definition.

DEFINITION: Gradient Field

A vector field $\vec{\mathbf{F}}$ in $\mathbb{R}^2$ or in $\mathbb{R}^3$ is a gradient field if there exists a scalar function f such that $\vec{\nabla} f = \vec{\mathbf{F}}$.

✓ Example 16.1.9: Sketching a Gradient Vector Field

Use technology to plot the gradient vector field of $f(x, y) = x^2y^2$.

Solution

The gradient of f is $\vec{\nabla} f(x,y) = \langle 2xy^2,\ 2x^2y \rangle$. To sketch the vector field, use a computer algebra system such as Mathematica. Figure 16.1.8 shows $\vec{\nabla} f$.

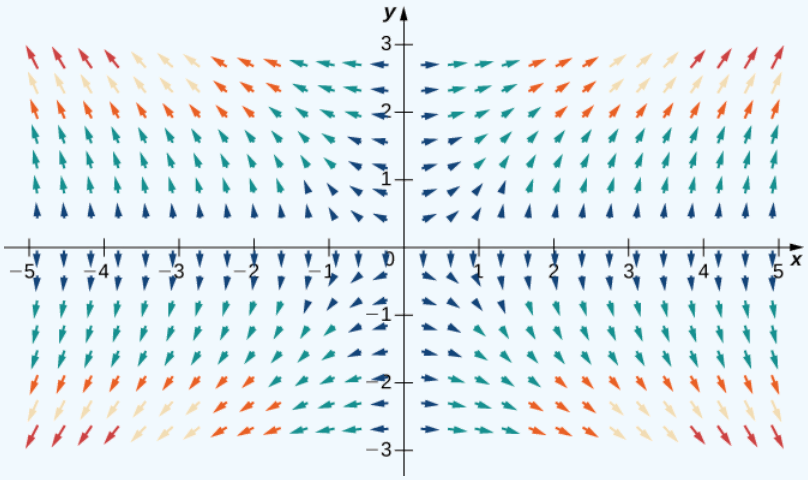


Figure 16.1.8: The gradient vector field is $\vec{\nabla} f$, where $f(x,y) = x^2y^2$.

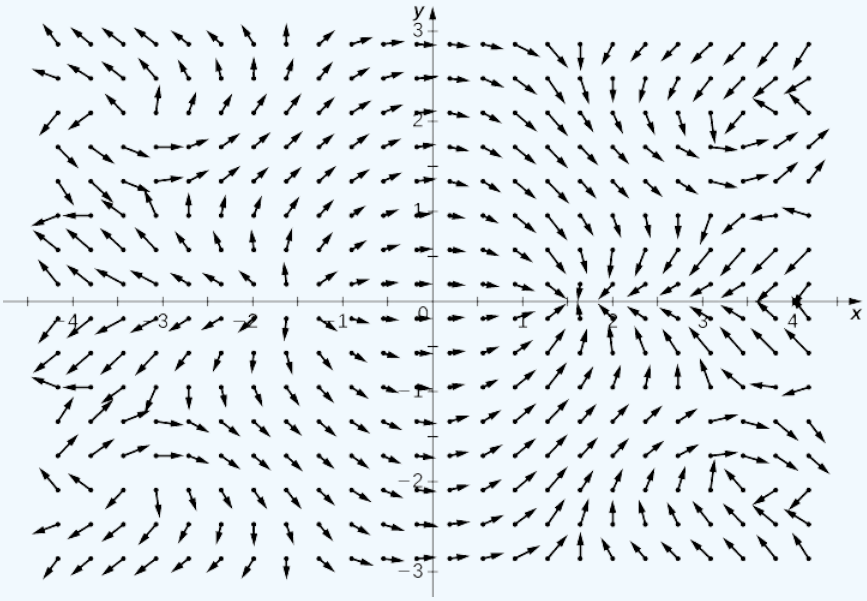
Exercise 16.1.9

Use technology to plot the gradient vector field of $f(x,y) = \sin x \cos y$.

Hint

Find the gradient of f.

Answer



Consider the function $f(x,y) = x^2y^2$ from Example 16.1.9. Figure 16.1.9 shows the level curves of this function overlaid on the function's gradient vector field. The gradient vectors are perpendicular to the level curves, and the magnitudes of the vectors get larger as the level curves get closer together, because closely grouped level curves indicate the graph is steep, and the magnitude of the gradient vector is the largest value of the directional derivative. Therefore, you can see the local steepness of a graph by investigating the corresponding function's gradient field.

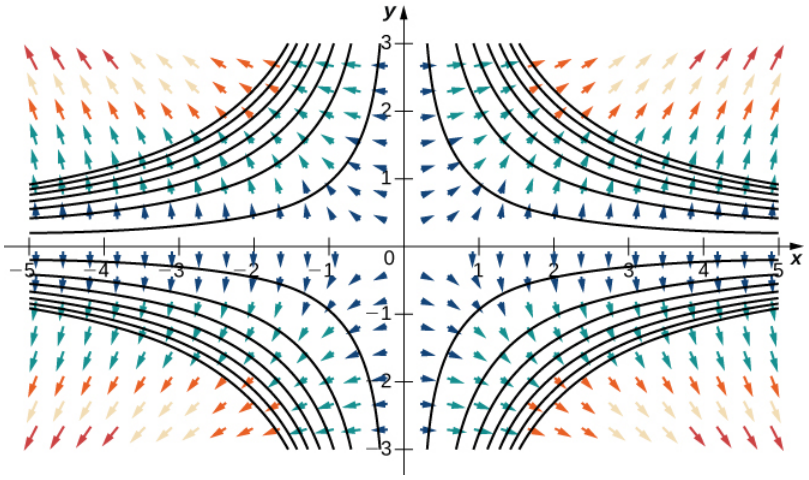


Figure 16.1.9: The gradient field of $f(x,y) = x^2y^2$ and several level curves of f. Notice that as the level curves get closer together, the magnitude of the gradient vectors increases.

As we learned earlier, a vector field $\vec{\mathbf{F}}$ is a conservative vector field, or a gradient field if there exists a scalar function f such that $\vec{\nabla} f = \vec{\mathbf{F}}$. In this situation, f is called a **potential function** for $\vec{\mathbf{F}}$. Conservative vector fields arise in many applications, particularly in physics. The reason such fields are called *conservative* is that they model forces of physical systems in which energy is conserved. We study conservative vector fields in more detail later in this chapter.

You might notice that, in some applications, a potential function f for $\vec{\mathbf{F}}$ is defined instead as a function such that $-\vec{\nabla} f = \vec{\mathbf{F}}$. This is the case for certain contexts in physics, for example.

✔ Example 16.1.10: Verifying a Potential Function

Is $f(x,y,z) = x^2yz - \sin(xy)$ a potential function for vector field

$$\vec{\mathbf{F}}(x,y,z) = \langle 2xyz - y\cos(xy), x^2z - x\cos(xy), x^2y\rangle?$$

Solution

We need to confirm whether $\vec{\nabla} f = \vec{\mathbf{F}}$. We have

$$\begin{aligned} f_x(x,y) &= 2xyz - y\cos(xy) \\ f_y(x,y) &= x^2z - x\cos(xy). \\ f_z(x,y) &= x^2y \end{aligned}$$

Therefore, $\vec{\nabla} f = \vec{\mathbf{F}}$ and f is a potential function for $\vec{\mathbf{F}}$.

? Exercise 16.1.10

Is $f(x,y,z) = x^2\cos(yz) + y^2z^2$ a potential function for $\vec{\mathbf{F}}(x,y,z) = \langle 2x\cos(yz), -x^2z\sin(yz) + 2yz^2, y^2\rangle$?

Hint

Compute the gradient of f.

Answer

No

✔ Example 16.1.11: Verifying a Potential Function

The velocity of a fluid is modeled by field $\vec{\mathbf{v}}(x,y) = \langle xy, \frac{x^2}{2} - y\rangle$. Verify that $f(x,y) = \dfrac{x^2y}{2} - \dfrac{y^2}{2}$ is a potential function for $\vec{\mathbf{v}}$.

Solution

To show that f is a potential function, we must show that $\vec{\nabla} f = \vec{v}$. Note that $f_x(x,y) = xy$ and $f_y(x,y) = \frac{x^2}{2} - y$. Therefore, $\vec{\nabla} f(x,y) = \langle xy, \frac{x^2}{2} - y \rangle$ and f is a potential function for $\vec{v}$ (Figure 16.1.10).

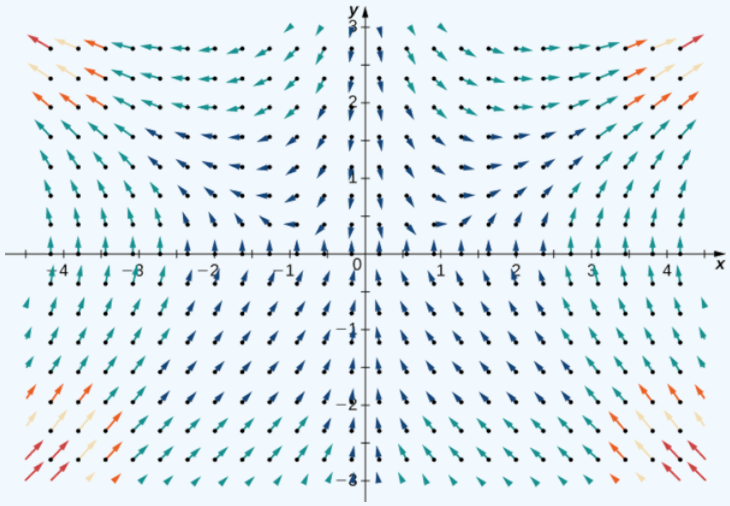


Figure 16.1.10: Velocity field $\vec{v}(x,y)$ has a potential function and is a conservative field.

? Exercise 16.1.11

Verify that $f(x,y) = x^3y^2 + x$ is a potential function for velocity field $\vec{v}(x,y) = \langle 3x^2y^2 + 1, 2x^3y \rangle$.

Hint

Calculate the gradient.

Answer

$\vec{\nabla} f(x,y) = \vec{v}(x,y)$

If $\vec{F}$ is a conservative vector field, then there is at least one potential function f such that $\vec{\nabla} f = \vec{F}$. But, could there be more than one potential function? If so, is there any relationship between two potential functions for the same vector field? Before answering these questions, let's recall some facts from single-variable calculus to guide our intuition. Recall that if $k(x)$ is an integrable function, then k has infinitely many antiderivatives. Furthermore, if $\vec{F}$ and $\vec{G}$ are both antiderivatives of k, then $\vec{F}$ and $\vec{G}$ **differ only by a constant**. That is, there is some number C such that $\vec{F}(x) = \vec{G}(x) + C$.

Now let $\vec{F}$ be a conservative vector field and let f and g be potential functions for $\vec{F}$. Since the gradient is like a derivative, $\vec{F}$ being conservative means that $\vec{F}$ is "integrable" with "antiderivatives" f and g. Therefore, if the analogy with single-variable calculus is valid, we expect there is some constant C such that $f(x) = g(x) + C$. The next theorem says that this is indeed the case.

To state the next theorem with precision, we need to assume the domain of the vector field is connected and open. To be connected means if P_1 and P_2 are any two points in the domain, then you can walk from P_1 to P_2 along a path that stays entirely inside the domain.

UNIQUENESS OF POTENTIAL FUNCTIONS

Let $\vec{F}$ be a conservative vector field on an open and connected domain and let f and g be functions such that $\vec{\nabla} f = \vec{F}$ and $\vec{\nabla} g = \vec{F}$. Then, there is a constant C such that $f = g + C$.

Proof

Since f and g are both potential functions for $\vec{F}$, then $\vec{\nabla}(f - g) = \vec{\nabla} f - \vec{\nabla} g = \vec{F} - \vec{F} = \vec{0}$. Let $h = f - g$, then we have $\vec{\nabla} h = \vec{0}$.We would like to show that h is a constant function.

Assume h is a function of x and y (the logic of this proof extends to any number of independent variables). Since $\vec{\nabla} h = \vec{0}$, we have $h_x(x,y) = 0$ and $h_y(x,y) = 0$. The expression $h_x(x,y) = 0$ implies that h is a constant function with respect to x—that is, $h(x,y) = k_1(y)$ for some function k_1. Similarly, $h_y(x,y) = 0$ implies $h(x,y) = k_2(x)$ for some function k_2. Therefore, function h depends only on y and also depends only on x. Thus, $h(x,y) = C$ for some constant C on the connected domain of $\vec{\mathbf{F}}$. Note that we really do need connectedness at this point; if the domain of $\vec{\mathbf{F}}$ came in two separate pieces, then k could be a constant C_1 on one piece but could be a different constant C_2 on the other piece. Since $f - g = h = C$, we have that $f = g + C$, as desired.

□

Conservative vector fields also have a special property called the ***cross-partial property***. This property helps test whether a given vector field is conservative.

THE CROSS-PARTIAL PROPERTY OF CONSERVATIVE VECTOR FIELDS

Let $\vec{\mathbf{F}}$ be a vector field in two or three dimensions such that the component functions of $\vec{\mathbf{F}}$ have continuous second-order mixed-partial derivatives on the domain of $\vec{\mathbf{F}}$.

If $\vec{\mathbf{F}}(x,y) = \langle P(x,y), Q(x,y) \rangle$ is a conservative vector field in $\mathbb{R}^2$, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

If $\vec{\mathbf{F}}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$ is a conservative vector field in $\mathbb{R}^3$, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$
$$\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$
$$\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}.$$

Proof

Since $\vec{\mathbf{F}}$ is conservative, there is a function $f(x,y)$ such that $\vec{\nabla} f = \vec{\mathbf{F}}$. Therefore, by the definition of the gradient, $f_x = P$ and $f_y = Q$. By Clairaut's theorem, $f_{xy} = f_{yx}$, But, $f_{xy} = P_y$ and $f_{yx} = Q_x$, and thus $P_y = Q_x$.

□

Clairaut's theorem gives a fast proof of the cross-partial property of conservative vector fields in $\mathbb{R}^3$, just as it did for vector fields in $\mathbb{R}^2$.

The Cross-Partial Property of Conservative Vector Fields shows that most vector fields are not conservative. The cross-partial property is difficult to satisfy in general, so most vector fields won't have equal cross-partials.

Show that rotational vector field $\vec{\mathbf{F}}(x,y) = \langle y, -x \rangle$ is not conservative.

Solution

Let $P(x,y) = y$ and $Q(x,y) = -x$. If $\vec{\mathbf{F}}$ is conservative, then the cross-partials would be equal—that is, P_y would equal Q_x.Therefore, to show that $\vec{\mathbf{F}}$ is not conservative, check that $P_y \neq Q_x$. Since $P_y = 1$ and $Q_x = -1$, the vector field is not conservative.

? Exercise 16.1.12

Show that vector field $\vec{\mathbf{F}}(x,y) = xy\,\hat{\mathbf{i}} - x^2 y\,\hat{\mathbf{j}}$ is not conservative.

Hint

Check the cross-partials.

Answer

$P_y(x,y) = x$ and $Q_x(x,y) = -2xy$. Since $P_y(x,y) \neq Q_x(x,y)$, $\vec{\mathbf{F}}$ is not conservative.

Example 16.1.13: Showing a Vector Field Is Not Conservative

Is vector field $\vec{\mathbf{F}}(x,y,z) = \langle 7, -2, x^3 \rangle$ conservative?

Solution

Let $P(x,y,z) = 7$, $Q(x,y,z) = -2$, and $R(x,y,z) = x^3$. If $\vec{\mathbf{F}}$ is conservative, then all three cross-partial equations will be satisfied —that is, if $\vec{\mathbf{F}}$ is conservative, then P_y would equal Q_x, Q_z would equal R_y, and R_x would equal P_z. Note that

$$P_y = Q_x = R_y = Q_z = 0$$

so the first two necessary equalities hold. However, $R_x(x,y,z) = x^3$ and $P_z(x,y,z) = 0$ so $R_x \neq P_z$. Therefore, $\vec{\mathbf{F}}$ is not conservative.

Exercise 16.1.13

Is vector field $\vec{\mathbf{G}}(x,y,z) = \langle y,\ x,\ xyz \rangle$ conservative?

Hint

Check the cross-partials.

Answer

No

We conclude this section with a word of warning: The Cross-Partial Property of Conservative Vector Fields says that if $\vec{\mathbf{F}}$ is conservative, then $\vec{\mathbf{F}}$ has the cross-partial property. The theorem does *not* say that, if $\vec{\mathbf{F}}$ has the cross-partial property, then $\vec{\mathbf{F}}$ is conservative (the converse of an implication is not logically equivalent to the original implication). In other words, The Cross-Partial Property of Conservative Vector Fields can only help determine that a field is not conservative; it does not let you conclude that a vector field is conservative.

For example, consider vector field $\vec{\mathbf{F}}(x,y) = \langle x^2 y, \frac{x^3}{3} \rangle$. This field has the cross-partial property, so it is natural to try to use The Cross-Partial Property of Conservative Vector Fields to conclude this vector field is conservative. However, this is a misapplication of the theorem. We learn later how to conclude that $\vec{\mathbf{F}}$ is conservative.

Key Concepts

- A vector field assigns a vector $\vec{\mathbf{F}}(x,y)$ to each point (x,y) in a subset D of $\mathbb{R}^2$ or $\mathbb{R}^3$. $\vec{\mathbf{F}}(x,y,z)$ to each point (x,y,z) in a subset D of $\mathbb{R}^3$.
- Vector fields can describe the distribution of vector quantities such as forces or velocities over a region of the plane or of space. They are in common use in such areas as physics, engineering, meteorology, oceanography.
- We can sketch a vector field by examining its defining equation to determine relative magnitudes in various locations and then drawing enough vectors to determine a pattern.
- A vector field $\vec{\mathbf{F}}$ is called conservative if there exists a scalar function f such that $\vec{\nabla} f = \vec{\mathbf{F}}$.

Key Equations

- **Vector field in $\mathbb{R}^2$**
  $\vec{\mathbf{F}}(x,y) = \langle P(x,y),\ Q(x,y) \rangle$
  or
  $\vec{\mathbf{F}}(x,y) = P(x,y)\,\hat{\mathbf{i}} + Q(x,y)\,\hat{\mathbf{j}}$
- **Vector field in $\mathbb{R}^3$**
  $\vec{\mathbf{F}}(x,y,z) = \langle P(x,y,z),\ Q(x,y,z),\ R(x,y,z) \rangle$
  or
  $\vec{\mathbf{F}}(x,y,z) = P(x,y,z)\,\hat{\mathbf{i}} + Q(x,y,z)\,\hat{\mathbf{j}} + R(x,y,z)\,\hat{\mathbf{k}}$

Glossary

conservative field

a vector field for which there exists a scalar function f such that $\vec{\nabla} f = \vec{\mathbf{F}}$

gradient field

a vector field $\vec{\mathbf{F}}$ for which there exists a scalar function f such that $\vec{\nabla} f = \vec{\mathbf{F}}$; in other words, a vector field that is the gradient of a function; such vector fields are also called *conservative*

potential function

a scalar function f such that $\vec{\nabla} f = \vec{\mathbf{F}}$

radial field

a vector field in which all vectors either point directly toward or directly away from the origin; the magnitude of any vector depends only on its distance from the origin

rotational field

a vector field in which the vector at point (x, y) is tangent to a circle with radius $r = \sqrt{x^2 + y^2}$; in a rotational field, all vectors flow either clockwise or counterclockwise, and the magnitude of a vector depends only on its distance from the origin

unit vector field

a vector field in which the magnitude of every vector is 1

vector field

measured in $\mathbb{R}^2$, an assignment of a vector $\vec{\mathbf{F}}(x, y)$ to each point (x, y) of a subset D of $\mathbb{R}^2$; in $\mathbb{R}^3$, an assignment of a vector $\vec{\mathbf{F}}(x, y, z)$ to each point (x, y, z) of a subset D of $\mathbb{R}^3$

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16.1E: Exercises for Section 16.1

1. The domain of vector field $\vec{\mathbf{F}} = \vec{\mathbf{F}}(x, y)$ is a set of points (x, y) in a plane, and the range of $\vec{\mathbf{F}}$ is a set of *what* in the plane?

For exercises 2 - 4, determine whether the statement is *true or false*.

2. Vector field $\vec{\mathbf{F}} = \langle 3x^2, 1\rangle$ is a gradient field for both $\phi_1(x, y) = x^3 + y$ and $\phi_2(x, y) = y + x^3 + 100$.

3. Vector field $\vec{\mathbf{F}} = \dfrac{\langle y, x\rangle}{\sqrt{x^2 + y^2}}$ is constant in direction and magnitude on a unit circle.

4. Vector field $\vec{\mathbf{F}} = \dfrac{\langle y, x\rangle}{\sqrt{x^2 + y^2}}$ is neither a radial field nor a rotation field.

For exercises 5 - 13, describe each vector field by drawing some of its vectors.

5. [T] $\vec{\mathbf{F}}(x, y) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$

6. [T] $\vec{\mathbf{F}}(x, y) = -y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$

7. [T] $\vec{\mathbf{F}}(x, y) = x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}}$

8. [T] $\vec{\mathbf{F}}(x, y) = \hat{\mathbf{i}} + \hat{\mathbf{j}}$

9. [T] $\vec{\mathbf{F}}(x, y) = 2x\,\hat{\mathbf{i}} + 3y\,\hat{\mathbf{j}}$

10. [T] $\vec{\mathbf{F}}(x, y) = 3\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$

11. [T] $\vec{\mathbf{F}}(x, y) = y\,\hat{\mathbf{i}} + \sin x\,\hat{\mathbf{j}}$

12. [T] $\vec{\mathbf{F}}(x, y, z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$

13. [T] $\vec{\mathbf{F}}(x, y, z) = 2x\,\hat{\mathbf{i}} - 2y\,\hat{\mathbf{j}} - 2z\,\hat{\mathbf{k}}$

14. [T] $\vec{\mathbf{F}}(x, y, z) = yz\,\hat{\mathbf{i}} - xz\,\hat{\mathbf{j}}$

For exercises 15 - 20, find the gradient vector field of each function f.

15. $f(x, y) = x \sin y + \cos y$

16. $f(x, y, z) = ze^{-xy}$

17. $f(x, y, z) = x^2y + xy + y^2z$

18. $f(x, y) = x^2 \sin(5y)$

19. $f(x, y) = \ln(1 + x^2 + 2y^2)$

20. $f(x, y, z) = x \cos\left(\frac{y}{z}\right)$

21. What is vector field $\vec{\mathbf{F}}(x, y)$ with a value at (x, y) that is of unit length and points toward $(1, 0)$?

For exercises 22 - 24, write formulas for the vector fields with the given properties.

22. All vectors are parallel to the x-axis and all vectors on a vertical line have the same magnitude.

23. All vectors point toward the origin and have constant length.

24. All vectors are of unit length and are perpendicular to the position vector at that point.

25. Give a formula $\vec{\mathbf{F}}(x, y) = M(x, y)\,\hat{\mathbf{i}} + N(x, y)\,\hat{\mathbf{j}}$ for the vector field in a plane that has the properties that $\vec{\mathbf{F}} = \vec{0}$ at $(0, 0)$ and that at any other point (a, b), $\vec{\mathbf{F}}$ is tangent to circle $x^2 + y^2 = a^2 + b^2$ and points in the clockwise direction with magnitude $\|\vec{\mathbf{F}}\| = \sqrt{a^2 + b^2}$.

26. Is vector field $\vec{\mathbf{F}}(x, y) = (P(x, y), Q(x, y)) = (\sin x + y)\,\hat{\mathbf{i}} + (\cos y + x)\,\hat{\mathbf{j}}$ a gradient field?

27. Find a formula for vector field $\vec{\mathbf{F}}(x, y) = M(x, y)\,\hat{\mathbf{i}} + N(x, y)\,\hat{\mathbf{j}}$ given the fact that for all points (x, y), $\vec{\mathbf{F}}$ points toward the origin and $\|\vec{\mathbf{F}}\| = \dfrac{10}{x^2 + y^2}$.

For exercises 28 - 29, assume that an electric field in the xy-plane caused by an infinite line of charge along the x-axis is a gradient field with potential function $V(x, y) = c \ln\left(\frac{r_0}{\sqrt{x^2+y^2}}\right)$, where $c > 0$ is a constant and r_0 is a reference distance at which the potential is assumed to be zero.

28. Find the components of the electric field in the x- and y-directions, where $\vec{\mathbf{E}}(x,y) = -\vec{\nabla}V(x,y)$.

29. Show that the electric field at a point in the xy-plane is directed outward from the origin and has magnitude $\|\vec{\mathbf{E}}\| = \frac{c}{r}$, where $r = \sqrt{x^2+y^2}$.

A ***flow line*** (or *streamline*) of a vector field $\vec{\mathbf{F}}$ is a curve $\vec{\mathbf{r}}(t)$ such that $d\vec{\mathbf{r}}/dt = \vec{\mathbf{F}}(\vec{\mathbf{r}}(t))$. If $\vec{\mathbf{F}}$ represents the velocity field of a moving particle, then the flow lines are paths taken by the particle. Therefore, flow lines are tangent to the vector field.

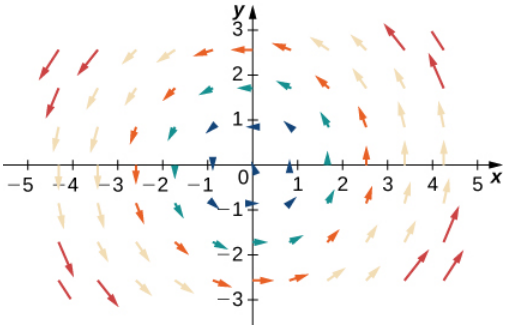
For exercises 30 and 31, show that the given curve $\vec{\mathbf{c}}(t)$ is a flow line of the given velocity vector field $\vec{\mathbf{F}}(x,y,z)$.

30. $\vec{\mathbf{c}}(t) = \langle e^{2t}, \ln|t|, \frac{1}{t}\rangle,\ t \neq 0; \quad \vec{\mathbf{F}}(x,y,z) = \langle 2x, z, -z^2\rangle$

31. $\vec{\mathbf{c}}(t) = \langle \sin t, \cos t, e^t\rangle; \quad \vec{\mathbf{F}}(x,y,z) = \langle y, -x, z\rangle$

For exercises 32 - 34, let $\vec{\mathbf{F}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$, $\vec{\mathbf{G}} = -y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$, and $\vec{\mathbf{H}} = x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}}$. Match $\vec{\mathbf{F}}$, $\vec{\mathbf{G}}$, and $\vec{\mathbf{H}}$ with their graphs.

32.



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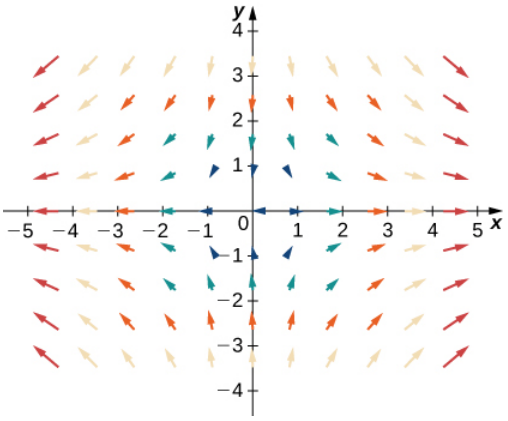

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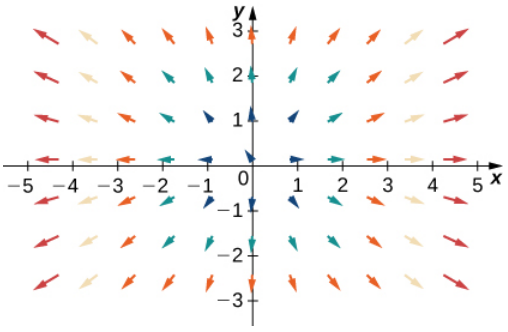


For exercises 35 - 38, let $\vec{\mathbf{F}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$, $\vec{\mathbf{G}} = -y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$, and $\vec{\mathbf{H}} = -x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$. Match the vector fields with their graphs in (I)−(IV).

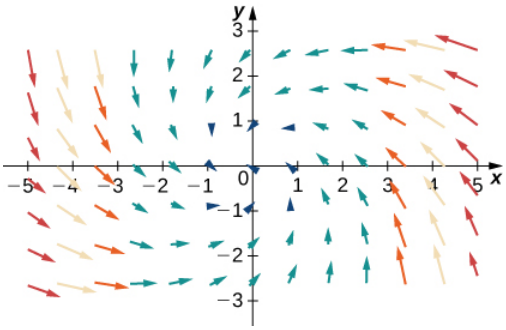
a. $\vec{\mathbf{F}} + \vec{\mathbf{G}}$

b. $\vec{\mathbf{F}} + \vec{\mathbf{H}}$

c. $\vec{\mathbf{G}} + \vec{\mathbf{H}}$

d. $-\vec{\mathbf{F}} + \vec{\mathbf{G}}$

35.



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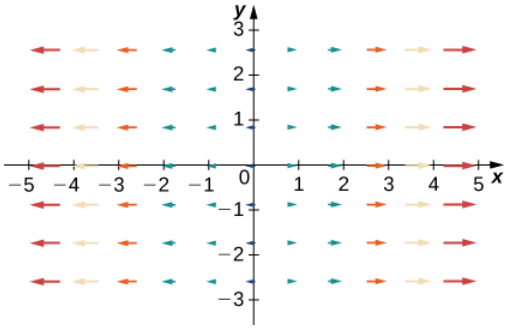

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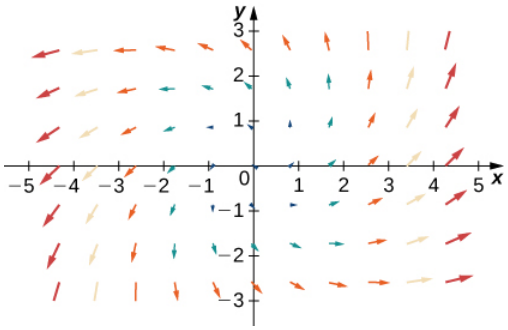

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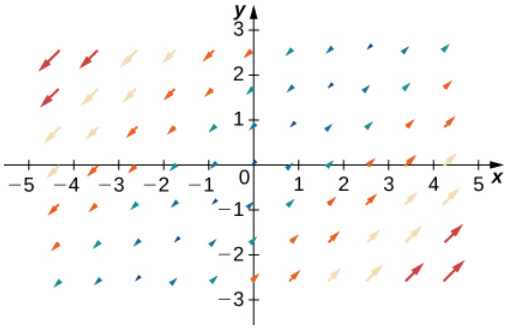

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16.2: Line Integrals

Learning Objectives

- Calculate a scalar line integral along a curve.
- Calculate a vector line integral along an oriented curve in space.
- Use a line integral to compute the work done in moving an object along a curve in a vector field.
- Describe the flux and circulation of a vector field.

We are familiar with single-variable integrals of the form $\int_a^b f(x)\,dx$, where the domain of integration is an interval $[a,b]$. Such an interval can be thought of as a curve in the xy-plane, since the interval defines a line segment with endpoints $(a,0)$ and $(b,0)$—in other words, a line segment located on the x-axis. Suppose we want to integrate over *any* curve in the plane, not just over a line segment on the x-axis. Such a task requires a new kind of integral, called a *line integral.*

Line integrals have many applications to engineering and physics. They also allow us to make several useful generalizations of the Fundamental Theorem of Calculus. And, they are closely connected to the properties of vector fields, as we shall see.

Scalar Line Integrals

A line integral gives us the ability to integrate multivariable functions and vector fields over arbitrary curves in a plane or in space. There are two types of line integrals: scalar line integrals and vector line integrals. Scalar line integrals are integrals of a scalar function over a curve in a plane or in space. Vector line integrals are integrals of a vector field over a curve in a plane or in space. Let's look at scalar line integrals first.

A scalar line integral is defined just as a single-variable integral is defined, except that for a scalar line integral, the integrand is a function of more than one variable and the domain of integration is a curve in a plane or in space, as opposed to a curve on the x-axis.

For a scalar line integral, we let C be a smooth curve in a plane or in space and let f be a function with a domain that includes C. We chop the curve into small pieces. For each piece, we choose point P in that piece and evaluate f at P. (We can do this because all the points in the curve are in the domain of f.) We multiply $f(P)$ by the arc length of the piece Δs, add the product $f(P)\Delta s$ over all the pieces, and then let the arc length of the pieces shrink to zero by taking a limit. The result is the scalar line integral of the function over the curve.

For a formal description of a scalar line integral, let C be a smooth curve in space given by the parameterization $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t)\rangle$, $a \le t \le b$. Let $f(x,y,z)$ be a function with a domain that includes curve C. To define the line integral of the function f over C, we begin as most definitions of an integral begin: we chop the curve into small pieces. Partition the parameter interval $[a,b]$ into n subintervals $[t_{i-1}, t_i]$ of equal width for $1 \le i \le n$, where $t_0 = a$ and $t_n = b$ (Figure 16.2.1). Let t_i^* be a value in the ith interval $[t_{i-1}, t_i]$. Denote the endpoints of $\vec{\mathbf{r}}(t_0)$, $\vec{\mathbf{r}}(t_1)$,…, $\vec{\mathbf{r}}(t_n)$ by P_0,…, P_n. Points P_i divide curve C into n pieces C_1, C_2,…, C_n,with lengths Δs_1, Δs_2,…, Δs_n, respectively. Let P_i^* denote the endpoint of $\vec{\mathbf{r}}(t_i^*)$ for $1 \le i \le n$. Now, we evaluate the function f at point P_i^* for $1 \le i \le n$. Note that P_i^* is in piece C_1, and therefore P_i^* is in the domain of f. Multiply $f(P_i^*)$ by the length Δs_1 of C_1, which gives the area of the "sheet" with base C_1, and height $f(P_i^*)$. This is analogous to using rectangles to approximate area in a single-variable integral. Now, we form the sum $\sum_{i=1}^{n} f(P_i^*)\,\Delta s_i$.

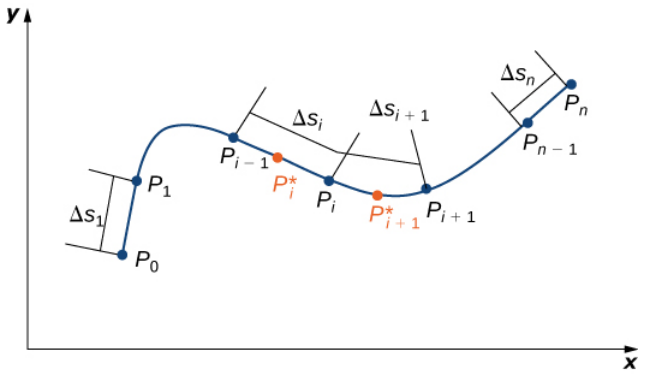

Figure 16.2.1: Curve C has been divided into n pieces, and a point inside each piece has been chosen.

Note the similarity of this sum versus a Riemann sum; in fact, this definition is a generalization of a Riemann sum to arbitrary curves in space. Just as with Riemann sums and integrals of form $\int_a^b g(x)\,dx$, we define an integral by letting the width of the pieces of the curve shrink to zero by taking a limit. The result is the scalar line integral of f along C.

You may have noticed a difference between this definition of a scalar line integral and a single-variable integral. In this definition, the arc lengths Δs_1, Δs_2,..., Δs_n aren't necessarily the same; in the definition of a single-variable integral, the curve in the x-axis is partitioned into pieces of equal length. This difference does not have any effect in the limit. As we shrink the arc lengths to zero, their values become close enough that any small difference becomes irrelevant.

DEFINITION: Scalar Line Integral

Let f be a function with a domain that includes the smooth curve C that is parameterized by $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t)\rangle$, $a \le t \le b$. The *scalar line integral* of f along C is

$$\int_C f(x,y,z)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(P_i^*)\,\Delta s_i \tag{16.2.1}$$

if this limit exists (t_i^* and Δs_i are defined as in the previous paragraphs). If C is a planar curve, then C can be represented by the parametric equations $x = x(t)$, $y = y(t)$, and $a \le t \le b$. If C is smooth and $f(x,y)$ is a function of two variables, then the scalar line integral of f along C is defined similarly as

$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(P_i^*)\,\Delta s_i, \tag{16.2.2}$$

if this limit exists.

If f is a continuous function on a smooth curve C, then $\int_C f\,ds$ always exists. Since $\int_C f\,ds$ is defined as a limit of Riemann sums, the continuity of f is enough to guarantee the existence of the limit, just as the integral $\int_a^b g(x)\,dx$ exists if g is continuous over $[a,b]$.

Before looking at how to compute a line integral, we need to examine the geometry captured by these integrals. Suppose that $f(x,y) \ge 0$ for all points (x,y) on a smooth planar curve C. Imagine taking curve C and projecting it "up" to the surface defined by $f(x,y)$, thereby creating a new curve C' that lies in the graph of $f(x,y)$ (Figure 16.2.2). Now we drop a "sheet" from C' down to the xy-plane. The area of this sheet is $\int_C f(x,y)ds$. If $f(x,y) \le 0$ for some points in C, then the value of $\int_C f(x,y)\,ds$ is the area above the xy-plane less the area below the xy-plane. (Note the similarity with integrals of the form $\int_a^b g(x)\,dx$.)

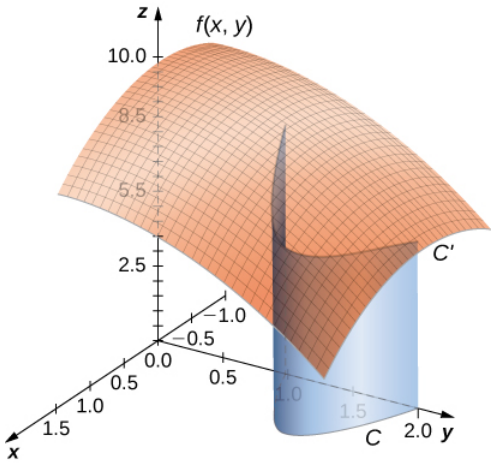

Figure 16.2.2: The area of the blue sheet is $\int_C f(x,y)\,ds$.

From this geometry, we can see that line integral $\int_C f(x,y)\,ds$ does not depend on the parameterization $\vec{\mathbf{r}}(t)$ of C. As long as the curve is traversed exactly once by the parameterization, the area of the sheet formed by the function and the curve is the same. This same kind of geometric argument can be extended to show that the line integral of a three-variable function over a curve in space does not depend on the parameterization of the curve.

Example 16.2.1: Finding the Value of a Line Integral

Find the value of integral $\int_C 2\,ds$, where C is the upper half of the unit circle.

Solution

The integrand is $f(x,y) = 2$. Figure 16.2.3 shows the graph of $f(x,y) = 2$, curve C, and the sheet formed by them. Notice that this sheet has the same area as a rectangle with width π and length 2. Therefore, $\int_C 2\,ds = 2\pi \text{ units}^2$.

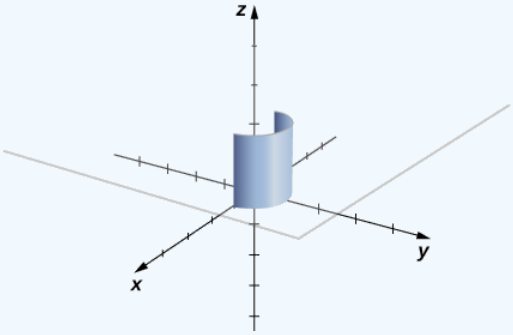

Figure 16.2.3: The sheet that is formed by the upper half of the unit circle in a plane and the graph of $f(x,y) = 2$.

To see that $\int_C 2\,ds = 2\pi$ using the definition of line integral, we let $\vec{\mathbf{r}}(t)$ be a parameterization of C. Then, $f(\vec{\mathbf{r}}(t_i)) = 2$ for any number t_i in the domain of $\vec{\mathbf{r}}$. Therefore,

$$\begin{aligned}
\int_C f\,ds &= \lim_{n\to\infty} \sum_{i=1}^{n} f(\vec{\mathbf{r}}(t_i^*))\,\Delta s_i \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} 2\,\Delta s_i \\
&= 2 \lim_{n\to\infty} \sum_{i=1}^{n} \Delta s_i \\
&= 2(\text{length of } C) \\
&= 2\pi \text{ units}^2.
\end{aligned}$$

Exercise 16.2.1

Find the value of $\int_C (x+y)\,ds$, where C is the curve parameterized by $x=t$, $y=t$, $0\le t\le 1$.

Hint

Find the shape formed by C and the graph of function $f(x,y)=x+y$.

Answer

$\sqrt{2}$

Note that in a scalar line integral, the integration is done with respect to arc length s, which can make a scalar line integral difficult to calculate. To make the calculations easier, we can translate $\int_C f\,ds$ to an integral with a variable of integration that is t.

Let $\vec{\mathbf{r}}(t)=\langle x(t),y(t),z(t)\rangle$ for $a\le t\le b$ be a parameterization of C. Since we are assuming that C is smooth, $\vec{\mathbf{r}}\,'(t)=\langle x'(t),y'(t),z'(t)\rangle$ is continuous for all t in $[a,b]$. In particular, $x'(t)$, $y'(t)$, and $z'(t)$ exist for all t in $[a,b]$. According to the arc length formula, we have

$$\text{length}(C_i)=\Delta s_i=\int_{t_{i-1}}^{t_i}\|\vec{\mathbf{r}}\,'(t)\|\,dt.$$

If width $\Delta t_i=t_i-t_{i-1}$ is small, then function $\int_{t_{i-1}}^{t_i}\|\vec{\mathbf{r}}\,'(t)\|\,dt\approx\|\vec{\mathbf{r}}\,'(t_i^*)\|\,\Delta t_i$, $\|\vec{\mathbf{r}}\,'(t)\|$ is almost constant over the interval $[t_{i-1},t_i]$.Therefore,

$$\int_{t_{i-1}}^{t_i}\|\vec{\mathbf{r}}\,'(t)\|\,dt\approx\|\vec{\mathbf{r}}\,'(t_i^*)\|\,\Delta t_i, \tag{16.2.3}$$

and we have

$$\sum_{i=1}^{n} f(\vec{\mathbf{r}}(t_i^*))\,\Delta s_i\approx\sum_{i=1}^{n} f(\vec{\mathbf{r}}(t_i^*))\|\vec{\mathbf{r}}\,'(t_i^*)\|\,\Delta t_i.$$

See Figure 16.2.4.

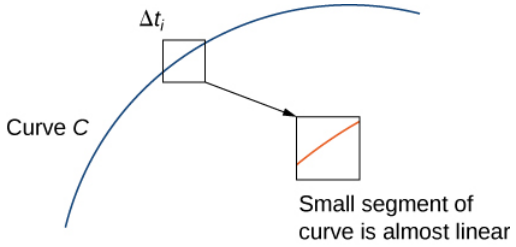

Figure 16.2.4: If we zoom in on the curve enough by making Δt_i very small, then the corresponding piece of the curve is approximately linear.

Note that

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(\vec{\mathbf{r}}(t_i^*))\|\vec{\mathbf{r}}\,'(t_i^*)\|\,\Delta t_i=\int_a^b f(\vec{\mathbf{r}}(t))\|\vec{\mathbf{r}}\,'(t)\|\,dt.$$

In other words, as the widths of intervals $[t_{i-1},t_i]$ shrink to zero, the sum $\sum_{i=1}^{n} f(\vec{\mathbf{r}}(t_i^*))\|\vec{\mathbf{r}}\,'(t_i^*)\|\,\Delta t_i$ converges to the integral $\int_a^b f(\vec{\mathbf{r}}(t))\|\vec{\mathbf{r}}\,'(t)\|\,dt$. Therefore, we have the following theorem.

Theorem: Evaluating a Scalar Line Integral

Let f be a continuous function with a domain that includes the smooth curve C with parameterization $\vec{\mathbf{r}}(t)$, $a\le t\le b$. Then

$$\int_C f\,ds=\int_a^b f(\vec{\mathbf{r}}(t))\|\vec{\mathbf{r}}\,'(t)\|\,dt. \tag{16.2.4}$$

Although we have labeled Equation 16.2.3 as an equation, it is more accurately considered an approximation because we can show that the left-hand side of Equation 16.2.3 approaches the right-hand side as $n \to \infty$. In other words, letting the widths of the pieces shrink to zero makes the right-hand sum arbitrarily close to the left-hand sum. Since

$$\|\vec{\mathbf{r}}'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2},$$

we obtain the following theorem, which we use to compute scalar line integrals.

Theorem: Scalar Line Integral Calculation

Let f be a continuous function with a domain that includes the smooth curve C with parameterization $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t)\rangle$, $a \leq t \leq b$. Then

$$\int_C f(x,y,z)\,ds = \int_a^b f(\vec{\mathbf{r}}(t))\sqrt{(x\prime(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt.$$

Similarly,

$$\int_C f(x,y)\,ds = \int_a^b f(\vec{\mathbf{r}}(t))\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

if C is a planar curve and f is a function of two variables.

Note that a consequence of this theorem is the equation $ds = \|\vec{\mathbf{r}}'(t)\|\,dt$. In other words, the change in arc length can be viewed as a change in the t-domain, scaled by the magnitude of vector $\vec{\mathbf{r}}'(t)$.

Example 16.2.2: Evaluating a Line Integral

Find the value of integral $\int_C (x^2 + y^2 + z)\,ds$, where C is part of the helix parameterized by $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t, t\rangle$, $0 \leq t \leq 2\pi$.

Solution

To compute a scalar line integral, we start by converting the variable of integration from arc length s to t. Then, we can use Equation 16.2.1 to compute the integral with respect to t. Note that

$$f(\vec{\mathbf{r}}(t)) = \cos^2 t + \sin^2 t + t = 1 + t$$

and

$$\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} = \sqrt{(-\sin(t))^2 + \cos^2(t) + 1} = \sqrt{2}.$$

Therefore,

$$\int_C (x^2 + y^2 + z)\,ds = \int_0^{2\pi} (1+t)\sqrt{2}\,dt.$$

Notice that Equation 16.2.1 translated the original difficult line integral into a manageable single-variable integral. Since

$$\begin{aligned}\int_0^{2\pi} (1+t)\sqrt{2}\,dt &= \left[\sqrt{2}t + \frac{\sqrt{2}t^2}{2}\right]_0^{2\pi} \\ &= 2\sqrt{2}\pi + 2\sqrt{2}\pi^2,\end{aligned}$$

we have

$$\int_C (x^2 + y^2 + z)\,ds = 2\sqrt{2}\pi + 2\sqrt{2}\pi^2.$$

Exercise 16.2.2

Evaluate $\int_C (x^2+y^2+z)ds$, where C is the curve with parameterization $\vec{\mathbf{r}}(t) = \langle \sin(3t), \cos(3t)\rangle$, $0 \le t \le \frac{\pi}{4}$.

Hint

Use the two-variable version of scalar line integral definition (Equation 16.2.2).

Answer

$$\frac{1}{3} + \frac{\sqrt{2}}{6} + \frac{3\pi}{4}$$

Example 16.2.3: Independence of Parameterization

Find the value of integral $\int_C (x^2+y^2+z)\,ds$, where C is part of the helix parameterized by $\vec{\mathbf{r}}(t) = \langle \cos(2t), \sin(2t), 2t\rangle$, $0 \le t \le \pi$. Notice that this function and curve are the same as in the previous example; the only difference is that the curve has been reparameterized so that time runs twice as fast.

Solution

As with the previous example, we use Equation 16.2.1 to compute the integral with respect to t. Note that $f(\vec{\mathbf{r}}(t)) = \cos^2(2t) + \sin^2(2t) + 2t = 2t + 1$ and

$$\begin{aligned}\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} &= \sqrt{(-\sin t + \cos t + 4)} \\ &= 22\end{aligned}$$

so we have

$$\begin{aligned}\int_C (x^2+y^2+z)ds &= 2\sqrt{2}\int_0^{\pi}(1+2t)dt \\ &= 2\sqrt{2}\Big[t+t^2\Big]_0^{\pi} \\ &= 2\sqrt{2}(\pi+\pi^2).\end{aligned}$$

Notice that this agrees with the answer in the previous example. Changing the parameterization did not change the value of the line integral. Scalar line integrals are independent of parameterization, as long as the curve is traversed exactly once by the parameterization.

Exercise 16.2.3

Evaluate line integral $\int_C (x^2+yz)\,ds$, where C is the line with parameterization $\vec{\mathbf{r}}(t) = \langle 2t, 5t, -t\rangle$, $0 \le t \le 10$. Reparameterize C with parameterization $s(t) = \langle 4t, 10t, -2t\rangle$, $0 \le t \le 5$, recalculate line integral $\int_C (x^2+yz)\,ds$, and notice that the change of parameterization had no effect on the value of the integral.

Hint

Use Equation 16.2.1.

Answer

Both line integrals equal $-\frac{1000\sqrt{30}}{3}$.

Now that we can evaluate line integrals, we can use them to calculate arc length. If $f(x,y,z) = 1$, then

$$\begin{aligned}\int_C f(x,y,z)\,ds &= \lim_{n\to\infty}\sum_{i=1}^{n} f(t_i^*)\,\Delta s_i \\ &= \lim_{n\to\infty}\sum_{i=1}^{n} \Delta s_i \\ &= \lim_{n\to\infty} \text{length}(C) \\ &= \text{length}(C).\end{aligned}$$

Therefore, $\int_C 1\,ds$ is the arc length of C.

Example 16.2.4: Calculating Arc Length

A wire has a shape that can be modeled with the parameterization $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t, \frac{2}{3}t^{3/2}\rangle$, $0 \le t \le 4\pi$. Find the length of the wire.

Solution

The length of the wire is given by $\int_C 1\,ds$, where C is the curve with parameterization $\vec{\mathbf{r}}$. Therefore,

$$\begin{aligned}\text{The length of the wire} &= \int_C 1\,ds \\ &= \int_0^{4\pi} ||\vec{\mathbf{r}}'(t)||\,dt \\ &= \int_0^{4\pi} \sqrt{(-\sin t)^2 + \cos^2 t + t}dt \\ &= \int_0^{4\pi} \sqrt{1+t}dt \\ &= \frac{2(1+t)^{\frac{3}{2}}}{3}\Bigg|_0^{4\pi} \\ &= \frac{2}{3}\left((1+4\pi)^{3/2} - 1\right).\end{aligned}$$

Exercise 16.2.4

Find the length of a wire with parameterization $\vec{\mathbf{r}}(t) = \langle 3t+1, 4-2t, 5+2t\rangle$, $0 \le t \le 4$.

Hint

Find the line integral of one over the corresponding curve.

Answer

$4\sqrt{17}$

Vector Line Integrals

The second type of line integrals are vector line integrals, in which we integrate along a curve through a vector field. For example, let

$$\vec{\mathbf{F}}(x,y,z) = P(x,y,z)\,\hat{\mathbf{i}} + Q(x,y,z)\,\hat{\mathbf{j}} + R(x,y,z)\,\hat{\mathbf{k}}$$

be a continuous vector field in $\mathbb{R}^3$ that represents a force on a particle, and let C be a smooth curve in $\mathbb{R}^3$ contained in the domain of $\vec{\mathbf{F}}$. How would we compute the work done by $\vec{\mathbf{F}}$ in moving a particle along C?

To answer this question, first note that a particle could travel in two directions along a curve: a forward direction and a backward direction. The work done by the vector field depends on the direction in which the particle is moving. Therefore, we must specify a direction along curve C; such a specified direction is called an orientation of a curve. The specified direction is the positive direction along C; the opposite direction is the negative direction along C. When C has been given an orientation, C is called an

oriented curve (Figure 16.2.5). The work done on the particle depends on the direction along the curve in which the particle is moving.

A closed curve is one for which there exists a parameterization $\vec{\mathbf{r}}(t)$, $a \le t \le b$, such that $\vec{\mathbf{r}}(a) = \vec{\mathbf{r}}(b)$, and the curve is traversed exactly once. In other words, the parameterization is one-to-one on the domain (a, b).

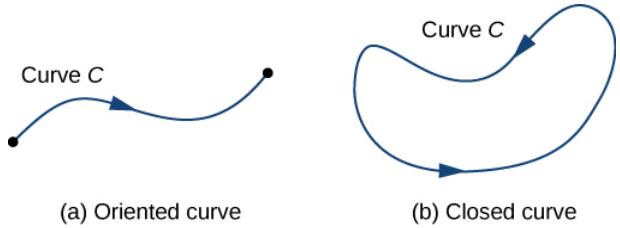

Figure 16.2.5: (a) An oriented curve between two points. (b) A closed oriented curve.

Let $\vec{\mathbf{r}}(t)$ be a parameterization of C for $a \le t \le b$ such that the curve is traversed exactly once by the particle and the particle moves in the positive direction along C. Divide the parameter interval $[a, b]$ into n subintervals $[t_{i-1}, t_i]$, $0 \le i \le n$, of equal width. Denote the endpoints of $r(t_0)$, $r(t_1)$,…, $r(t_n)$ by P_0,…,P_n. Points P_i divide C into n pieces. Denote the length of the piece from P_{i-1} to P_i by Δs_i. For each i, choose a value t_i^* in the subinterval $[t_{i-1}, t_i]$. Then, the endpoint of $\vec{\mathbf{r}}(t_i^*)$ is a point in the piece of C between P_{i-1} and P_i (Figure 16.2.6). If Δs_i is small, then as the particle moves from P_{i-1} to P_i along C, it moves approximately in the direction of $\vec{\mathbf{T}}(P_i)$, the unit tangent vector at the endpoint of $\vec{\mathbf{r}}(t_i^*)$. Let P_i^* denote the endpoint of $\vec{\mathbf{r}}(t_i^*)$. Then, the work done by the force vector field in moving the particle from P_{i-1} to P_i is $\vec{\mathbf{F}}(P_i^*) \cdot (\Delta s_i \vec{\mathbf{T}}(P_i^*))$, so the total work done along C is

$$\sum_{i=1}^{n} \vec{\mathbf{F}}(P_i^*) \cdot (\Delta s_i \vec{\mathbf{T}}(P_i^*)) = \sum_{i=1}^{n} \vec{\mathbf{F}}(P_i^*) \cdot \vec{\mathbf{T}}(P_i^*)\, \Delta s_i.$$

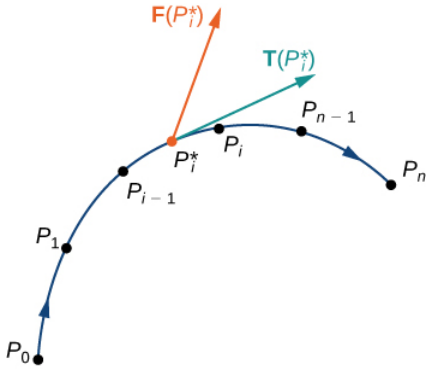

Figure 16.2.6: Curve C is divided into n pieces, and a point inside each piece is chosen. The dot product of any tangent vector in the ith piece with the corresponding vector $\vec{\mathbf{F}}$ is approximated by $\vec{\mathbf{F}}(P_i^*) \cdot \vec{\mathbf{T}}(P_i^*)$.

Letting the arc length of the pieces of C get arbitrarily small by taking a limit as $n \to \infty$ gives us the work done by the field in moving the particle along C. Therefore, the work done by $\vec{\mathbf{F}}$ in moving the particle in the positive direction along C is defined as

$$W = \int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds,$$

which gives us the concept of a vector line integral.

DEFINITION: Line Integral of a Vector Field

The vector line integral of vector field $\vec{\mathbf{F}}$ along oriented smooth curve C is

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds = \lim_{n \to \infty} \sum_{i=1}^{n} \vec{\mathbf{F}}(P_i^*) \cdot \vec{\mathbf{T}}(P_i^*) \Delta s_i$$

if that limit exists.

With scalar line integrals, neither the orientation nor the parameterization of the curve matters. As long as the curve is traversed exactly once by the parameterization, the value of the line integral is unchanged. With vector line integrals, the orientation of the

curve does matter. If we think of the line integral as computing work, then this makes sense: if you hike up a mountain, then the gravitational force of Earth does negative work on you. If you walk down the mountain by the exact same path, then Earth's gravitational force does positive work on you. In other words, reversing the path changes the work value from negative to positive in this case. Note that if C is an oriented curve, then we let $-C$ represent the same curve but with opposite orientation.

As with scalar line integrals, it is easier to compute a vector line integral if we express it in terms of the parameterization function $\vec{\mathbf{r}}$ and the variable t. To translate the integral $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}} ds$ in terms of t, note that unit tangent vector $\vec{\mathbf{T}}$ along C is given by $\vec{\mathbf{T}} = \dfrac{\vec{\mathbf{r}}'(t)}{\|\vec{\mathbf{r}}'(t)\|}$ (assuming $\|\vec{\mathbf{r}}'(t)\| \neq 0$). Since $ds = \|\vec{\mathbf{r}}'(t)\|\, dt$, as we saw when discussing scalar line integrals, we have

$$\vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds = \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \frac{\vec{\mathbf{r}}'(t)}{\|\vec{\mathbf{r}}'(t)\|} \|\vec{\mathbf{r}}'(t)\| dt = \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\, dt.$$

Thus, we have the following formula for computing vector line integrals:

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\, dt. \tag{16.2.5}$$

Because of Equation 16.2.5, we often use the notation $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ for the line integral $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$.

If $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t) \rangle$, then $\dfrac{d\vec{\mathbf{r}}}{dt}$ denotes vector $\langle x'(t), y'(t), z'(t) \rangle$, and $d\vec{\mathbf{r}} = \vec{\mathbf{r}}'(t)\, dt$.

Example 16.2.5: Evaluating a Vector Line Integral

Find the value of integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where C is the semicircle parameterized by $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t \rangle$, $0 \leq t \leq \pi$ and $\vec{\mathbf{F}} = \langle -y, x \rangle$.

Solution

We can use Equation 16.2.5 to convert the variable of integration from s to t. We then have

$$\vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) = \langle -\sin t, \cos t \rangle \text{ and } \vec{\mathbf{r}}'(t) = \langle -\sin t, \cos t \rangle.$$

Therefore,

$$\begin{aligned} \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_0^\pi \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle\, dt \\ &= \int_0^\pi \sin^2 t + \cos^2 t\, dt \\ &= \int_0^\pi 1\, dt = \pi. \end{aligned}$$

See Figure 16.2.7.

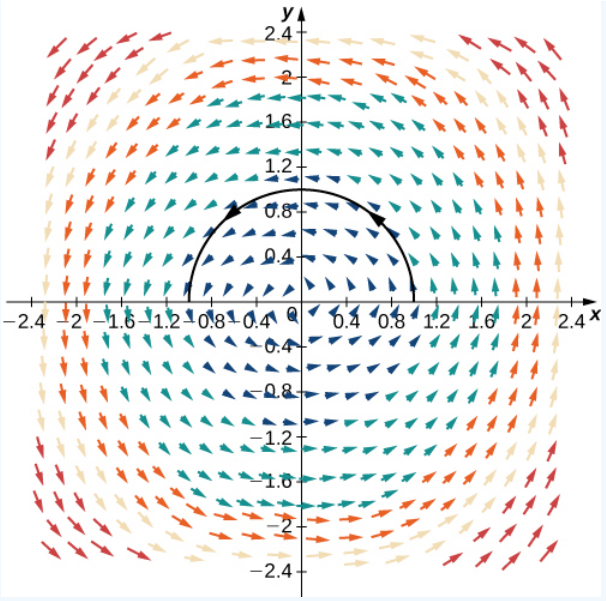

Figure 16.2.7: This figure shows curve $\vec{\mathbf{r}}(t) = \langle \cos t,\ \sin t \rangle$, $0 \le t \le \pi$ in vector field $\vec{\mathbf{F}} = \langle -y,\ x \rangle$.

Example 16.2.6: Reversing Orientation

Find the value of integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where C is the semicircle parameterized by $\vec{\mathbf{r}}(t) = \langle \cos(t + \pi), \sin t \rangle$, $0 \le t \le \pi$ and $\vec{\mathbf{F}} = \langle -y, x \rangle$.

Solution

Notice that this is the same problem as Example 16.2.5, except the orientation of the curve has been traversed. In this example, the parameterization starts at $\vec{\mathbf{r}}(0) = \langle -1, 0 \rangle$ and ends at $\vec{\mathbf{r}}(\pi) = \langle 1, 0 \rangle$. By Equation 16.2.5,

$$\begin{aligned} \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_0^\pi \langle -\sin t, \cos(t+\pi) \rangle \cdot \langle -\sin(t+\pi), \cos t \rangle dt \\ &= \int_0^\pi \langle -\sin t, -\cos t \rangle \cdot \langle \sin t, \cos t \rangle dt \\ &= \int_0^\pi (-\sin^2 t - \cos^2 t) dt \\ &= \int_0^\pi -1 dt \\ &= -\pi. \end{aligned}$$

Notice that this is the negative of the answer in Example 16.2.5. It makes sense that this answer is negative because the orientation of the curve goes against the "flow" of the vector field.

Let C be an oriented curve and let $-C$ denote the same curve but with the orientation reversed. Then, the previous two examples illustrate the following fact:

$$\int_{-C} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = -\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}.$$

That is, reversing the orientation of a curve changes the sign of a line integral.

Exercise 16.2.6

Let $\vec{\mathbf{F}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$ be a vector field and let C be the curve with parameterization $\langle t, t^2 \rangle$ for $0 \le t \le 2$. Which is greater: $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$ or $\int_{-C} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$?

Hint

Imagine moving along the path and computing the dot product $\vec{\mathbf{F}} \cdot \vec{\mathbf{T}}$ as you go.

Answer

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$$

Another standard notation for integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ is $\int_C P\,dx + Q\,dy + R\,dz$. In this notation, P, Q, and R are functions, and we think of $d\vec{\mathbf{r}}$ as vector $\langle dx, dy, dz\rangle$. To justify this convention, recall that $d\vec{\mathbf{r}} = \vec{\mathbf{T}}\, ds = \vec{\mathbf{r}}\,'(t)\, dt = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt$. Therefore,

$$\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \langle P, Q, R\rangle \cdot \langle dx, dy, dz\rangle = P\,dx + Q\,dy + R\,dz.$$

If $d\vec{\mathbf{r}} = \langle dx, dy, dz\rangle$, then $\frac{dr}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$, which implies that $d\vec{\mathbf{r}} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt$. Therefore

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C P\,dx + Q\,dy + R\,dz \tag{16.2.6}$$

$$= \int_a^b \left(P\big(\vec{\mathbf{r}}(t)\big)\frac{dx}{dt} + Q\big(\vec{\mathbf{r}}(t)\big)\frac{dy}{dt} + R\big(\vec{\mathbf{r}}(t)\big)\frac{dz}{dt} \right) dt. \tag{16.2.7}$$

Example 16.2.7: Finding the Value of an Integral of the Form $\int_C P\,dx + Q\,dy + R\,dz$

Find the value of integral $\int_C z\,dx + x\,dy + y\,dz$, where C is the curve parameterized by $\vec{\mathbf{r}}(t) = \langle t^2, \sqrt{t}, t\rangle$, $1 \le t \le 4$.

Solution

As with our previous examples, to compute this line integral we should perform a change of variables to write everything in terms of t. In this case, Equation 16.2.7 allows us to make this change:

$$\begin{aligned}\int_C z\,dx + x\,dy + y\,dz &= \int_1^4 \left(t(2t) + t^2\left(\frac{1}{2\sqrt{t}}\right) + \sqrt{t} \right) dt \\ &= \int_1^4 \left(2t^2 + \frac{t^{3/2}}{2} + \sqrt{t} \right) dt \\ &= \left[\frac{2t^3}{3} + \frac{t^{5/2}}{5} + \frac{2t^{3/2}}{3} \right]_{t=1}^{t=4} \\ &= \frac{793}{15}.\end{aligned}$$

Exercise 16.2.7

Find the value of $\int_C 4x\,dx + z\,dy + 4y^2\,dz$, where C is the curve parameterized by $\vec{\mathbf{r}}(t) = \langle 4\cos(2t), 2\sin(2t), 3\rangle$, $0 \le t \le \frac{\pi}{4}$.

Hint

Write the integral in terms of t using Equation 16.2.7.

Answer

-26

We have learned how to integrate smooth oriented curves. Now, suppose that C is an oriented curve that is not smooth, but can be written as the union of finitely many smooth curves. In this case, we say that C is a piecewise smooth curve. To be precise, curve C is piecewise smooth if C can be written as a union of n smooth curves C_1, C_2,…, C_n such that the endpoint of C_i is the starting point of C_{i+1} (Figure 16.2.8). When curves C_i satisfy the condition that the endpoint of C_i is the starting point of C_{i+1}, we write their union as $C_1 + C_2 + \cdots + C_n$.

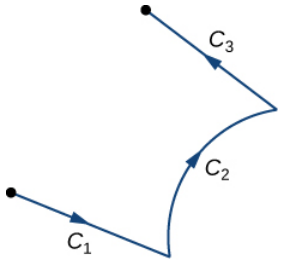

Figure 16.2.8: The union of C_1, C_2, C_3 is a piecewise smooth curve.

The next theorem summarizes several key properties of vector line integrals.

Theorem: Properties of Vector Line Integrals

Let $\vec{\mathbf{F}}$ and $\vec{\mathbf{G}}$ be continuous vector fields with domains that include the oriented smooth curve C. Then

1. $\displaystyle\int_C (\vec{\mathbf{F}} + \vec{\mathbf{G}}) \cdot d\vec{\mathbf{r}} = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_C \vec{\mathbf{G}} \cdot d\vec{\mathbf{r}}$
2. $\displaystyle\int_C k\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = k \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where k is a constant
3. $\displaystyle\int_{-C} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = -\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$
4. Suppose instead that C is a piecewise smooth curve in the domains of $\vec{\mathbf{F}}$ and $\vec{\mathbf{G}}$, where $C = C_1 + C_2 + \cdots + C_n$ and $C_1, C_2, \ldots, C_n$ are smooth curves such that the endpoint of C_i is the starting point of C_{i+1}. Then

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \cdots + \int_{C_n} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}.$$

Notice the similarities between these items and the properties of single-variable integrals. Properties i. and ii. say that line integrals are linear, which is true of single-variable integrals as well. Property iii. says that reversing the orientation of a curve changes the sign of the integral. If we think of the integral as computing the work done on a particle traveling along C, then this makes sense. If the particle moves backward rather than forward, then the value of the work done has the opposite sign. This is analogous to the equation $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$. Finally, if $[a_1, a_2]$, $[a_2, a_3]$,…, $[a_{n-1}, a_n]$ are intervals, then

$$\int_{a_1}^{a_n} f(x)\,dx = \int_{a_1}^{a_2} f(x)\,dx + \int_{a_1}^{a_3} f(x)\,dx + \cdots + \int_{a_{n-1}}^{a_n} f(x)\,dx,$$

which is analogous to property iv.

Example 16.2.8: Using Properties to Compute a Vector Line Integral

Find the value of integral $\displaystyle\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\,ds$, where C is the rectangle (oriented counterclockwise) in a plane with vertices $(0,0)$, $(2,0)$, $(2,1)$, and $(0,1)$, and where $\vec{\mathbf{F}} = \langle x - 2y, y - x \rangle$ (Figure 16.2.9).

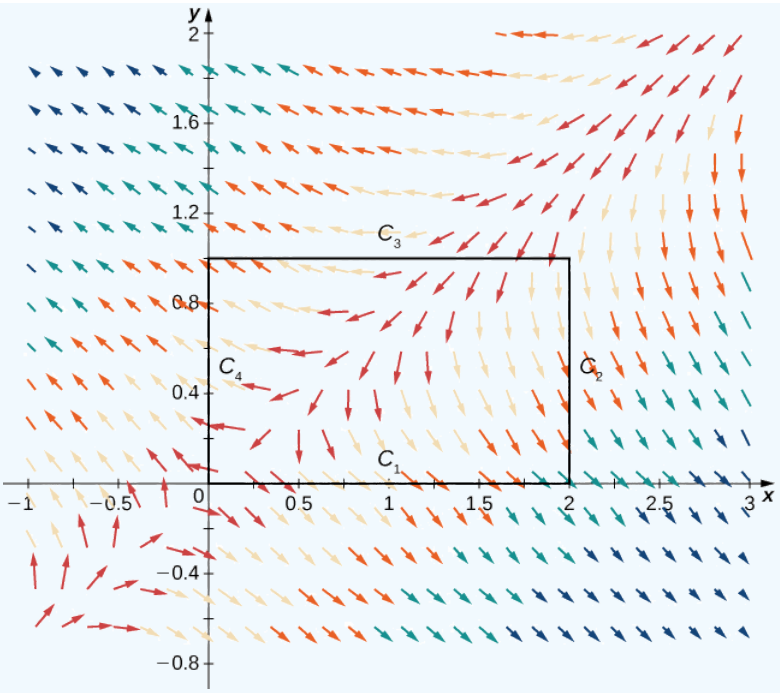

Figure 16.2.9: Rectangle and vector field for Example 16.2.8.

Solution

Note that curve C is the union of its four sides, and each side is smooth. Therefore C is piecewise smooth. Let C_1 represent the side from $(0,0)$ to $(2,0)$, let C_2 represent the side from $(2,0)$ to $(2,1)$, let C_3 represent the side from $(2,1)$ to $(0,1)$, and let C_4 represent the side from $(0,1)$ to $(0,0)$ (Figure 16.2.9). Then,

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, dr = \int_{C_1} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, dr + \int_{C_2} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, dr + \int_{C_3} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, dr + \int_{C_4} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, dr.$$

We want to compute each of the four integrals on the right-hand side using Equation 16.2.1. Before doing this, we need a parameterization of each side of the rectangle. Here are four parameterizations (note that they traverse C counterclockwise):

$$\begin{aligned} C_1 &: \langle t, 0\rangle, 0 \le t \le 2 \\ C_2 &: \langle 2, t\rangle, 0 \le t \le 1 \\ C_3 &: \langle 2-t, 1\rangle, 0 \le t \le 2 \\ C_4 &: \langle 0, 1-t\rangle, 0 \le t \le 1. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{C_1} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, dr &= \int_0^2 \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\, dt \\ &= \int_0^2 \langle t - 2(0), 0 - t\rangle \cdot \langle 1, 0\rangle\, dt = \int_0^2 t\, dt \\ &= \left[\tfrac{t^2}{2}\right]_0^2 = 2. \end{aligned}$$

Notice that the value of this integral is positive, which should not be surprising. As we move along curve C_1 from left to right, our movement flows in the general direction of the vector field itself. At any point along C_1, the tangent vector to the curve and the corresponding vector in the field form an angle that is less than 90°. Therefore, the tangent vector and the force vector have a positive dot product all along C_1, and the line integral will have positive value.

The calculations for the three other line integrals are done similarly:

$$\int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_0^1 \langle 2-2t, t-2\rangle \cdot \langle 0,1\rangle \, dt$$
$$= \int_0^1 (t-2)\, dt$$
$$= \left[\tfrac{t^2}{2} - 2t\right]_0^1 = -\frac{3}{2},$$

$$\int_{C_3} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds = \int_0^2 \langle (2-t)-2, 1-(2-t)\rangle \cdot \langle -1,0\rangle \, dt$$
$$= \int_0^2 t\, dt = 2,$$

and

$$\int_{C_4} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_0^1 \langle -2(1-t), 1-t\rangle \cdot \langle 0,-1\rangle \, dt$$
$$= \int_0^1 (t-1)\, dt$$
$$= \left[\tfrac{t^2}{2} - t\right]_0^1 = -\frac{1}{2}.$$

Thus, we have $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 2$.

? Exercise 16.2.8

Calculate line integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}$ is vector field $\langle y^2, 2xy+1\rangle$ and C is a triangle with vertices $(0,0)$, $(4,0)$, and $(0,5)$, oriented counterclockwise.

Hint

Write the triangle as a union of its three sides, then calculate three separate line integrals.

Answer

0

Applications of Line Integrals

Scalar line integrals have many applications. They can be used to calculate the length or mass of a wire, the surface area of a sheet of a given height, or the electric potential of a charged wire given a linear charge density. Vector line integrals are extremely useful in physics. They can be used to calculate the work done on a particle as it moves through a force field, or the flow rate of a fluid across a curve. Here, we calculate the mass of a wire using a scalar line integral and the work done by a force using a vector line integral.

Suppose that a piece of wire is modeled by curve C in space. The mass per unit length (the linear density) of the wire is a continuous function $\rho(x,y,z)$. We can calculate the total mass of the wire using the scalar line integral $\int_C \rho(x,y,z)\, ds$. The reason is that mass is density multiplied by length, and therefore the density of a small piece of the wire can be approximated by $\rho(x^*,y^*,z^*)\, \Delta s$ for some point (x^*,y^*,z^*) in the piece. Letting the length of the pieces shrink to zero with a limit yields the line integral $\int_C \rho(x,y,z)\, ds$.

✓ Example 16.2.9: Calculating the Mass of a Wire

Calculate the mass of a spring in the shape of a curve parameterized by $\langle t, 2\cos t, 2\sin t\rangle$, $0 \le t \le \frac{\pi}{2}$, with a density function given by $\rho(x,y,z) = e^x + yz$ kg/m (Figure 16.2.10).

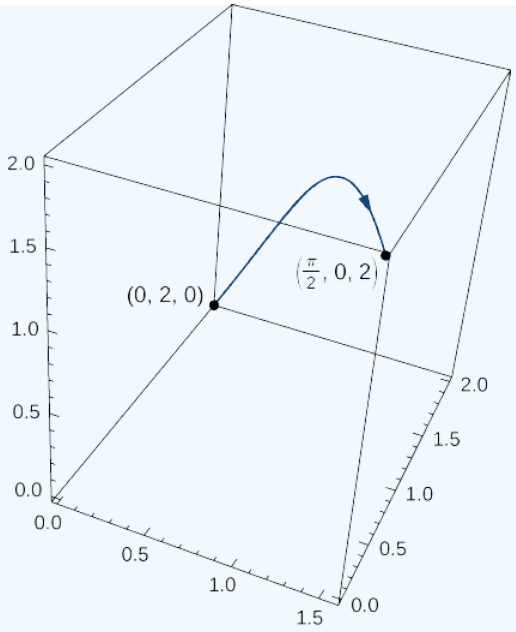

Figure 16.2.10: The wire from Example 16.2.9.

Solution

To calculate the mass of the spring, we must find the value of the scalar line integral $\int_C (e^x + yz)\, ds$, where C is the given helix. To calculate this integral, we write it in terms of t using Equation 16.2.1:

$$\begin{aligned}\int_C (e^x + yz)\, ds &= \int_0^{\frac{\pi}{2}} \left((e^t + 4\cos t \sin t)\sqrt{1 + (-2\cos t)^2 + (2\sin t)^2}\right) dt \\ &= \int_0^{\frac{\pi}{2}} \left((e^t + 4\cos t \sin t)\sqrt{5}\right) dt \\ &= \sqrt{5}\Big[e^t + 2\sin^2 t\Big]_{t=0}^{t=\pi/2} \\ &= \sqrt{5}(e^{\pi/2} + 1).\end{aligned}$$

Therefore, the mass is $\sqrt{5}(e^{\pi/2} + 1)$ kg.

Exercise 16.2.9

Calculate the mass of a spring in the shape of a helix parameterized by $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t, t\rangle$, $0 \le t \le 6\pi$, with a density function given by $\rho(x, y, z) = x + y + z$ kg/m.

Hint

Calculate the line integral of ρ over the curve with parameterization $\vec{\mathbf{r}}$.

Answer

$18\sqrt{2}\pi^2$ kg

When we first defined vector line integrals, we used the concept of work to motivate the definition. Therefore, it is not surprising that calculating the **work done by a vector field** representing a force is a standard use of vector line integrals. Recall that if an object moves along curve C in force field $\vec{\mathbf{F}}$, then the work required to move the object is given by $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$.

Example 16.2.10: Calculating Work

How much work is required to move an object in vector force field $\vec{\mathbf{F}} = \langle yz, xy, xz \rangle$ along path $\vec{\mathbf{r}}(t) = \langle t^2, t, t^4 \rangle,\ 0 \le t \le 1$? See Figure 16.2.11.

Solution

Let C denote the given path. We need to find the value of $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$. To do this, we use Equation 16.2.5 :

$$\begin{aligned} \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_0^1 (\langle t^5, t^3, t^6 \rangle \cdot \langle 2t, 1, 4t^3 \rangle)\, dt \\ &= \int_0^1 (2t^6 + t^3 + 4t^9)\, dt \\ &= \left[\frac{2t^7}{7} + \frac{t^4}{4} + \frac{2t^{10}}{5} \right]_{t=0}^{t=1} = \frac{131}{140} \text{ units of work.} \end{aligned}$$

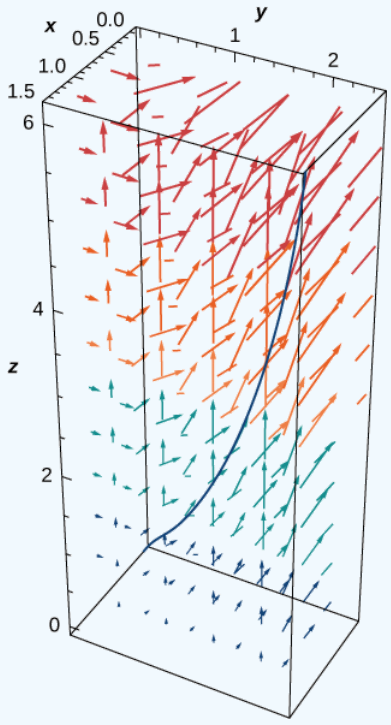

Figure 16.2.11: The curve and vector field for Example 16.2.10.

Flux

We close this section by discussing two key concepts related to line integrals: flux across a plane curve and circulation along a plane curve. Flux is used in applications to calculate fluid flow across a curve, and the concept of circulation is important for characterizing conservative gradient fields in terms of line integrals. Both these concepts are used heavily throughout the rest of this chapter. The idea of flux is especially important for Green's theorem, and in higher dimensions for Stokes' theorem and the divergence theorem.

Let C be a plane curve and let $\vec{\mathbf{F}}$ be a vector field in the plane. Imagine C is a membrane across which fluid flows, but C does not impede the flow of the fluid. In other words, C is an idealized membrane invisible to the fluid. Suppose $\vec{\mathbf{F}}$ represents the velocity field of the fluid. How could we quantify the rate at which the fluid is crossing C?

Recall that the line integral of $\vec{\mathbf{F}}$ along C is $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$—in other words, the line integral is the dot product of the vector field with the unit tangential vector with respect to arc length. If we replace the unit tangential vector with unit normal vector $\vec{\mathbf{N}}(t)$ and instead compute integral $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds$, we determine the flux across C. To be precise, the definition of integral $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds$ is the same as integral $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$, except the $\vec{\mathbf{T}}$ in the Riemann sum is replaced with $\vec{\mathbf{N}}$. Therefore, the flux across C is defined as

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds = \lim_{n\to\infty} \sum_{i=1}^{n} \vec{\mathbf{F}}(P_i^*) \cdot \vec{\mathbf{N}}(P_i^*)\, \Delta s_i,$$

where P_i^* and Δs_i are defined as they were for integral $\displaystyle\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$. Therefore, a flux integral is an integral that is *perpendicular* to a vector line integral, because $\vec{\mathbf{N}}$ and $\vec{\mathbf{T}}$ are perpendicular vectors.

If $\vec{\mathbf{F}}$ is a velocity field of a fluid and C is a curve that represents a membrane, then the flux of $\vec{\mathbf{F}}$ across C is the quantity of fluid flowing across C per unit time, or the rate of flow.

More formally, let C be a plane curve parameterized by $\vec{\mathbf{r}}(t) = \langle x(t),\, y(t)\rangle$, $a \le t \le b$. Let $\vec{\mathbf{n}}(t) = \langle y'(t),\, -x'(t)\rangle$ be the vector that is normal to C at the endpoint of $\vec{\mathbf{r}}(t)$ and points to the right as we traverse C in the positive direction (Figure 16.2.12). Then, $\vec{\mathbf{N}}(t) = \dfrac{\vec{\mathbf{n}}(t)}{\|\vec{\mathbf{n}}(t)\|}$ is the unit normal vector to C at the endpoint of $\vec{\mathbf{r}}(t)$ that points to the right as we traverse C.

DEFINITION: Flux

The *flux* of $\vec{\mathbf{F}}$ across C is line integral

$$\int_C \vec{\mathbf{F}} \cdot \frac{\vec{\mathbf{n}}(t)}{\|\vec{\mathbf{n}}(t)\|}\, ds.$$

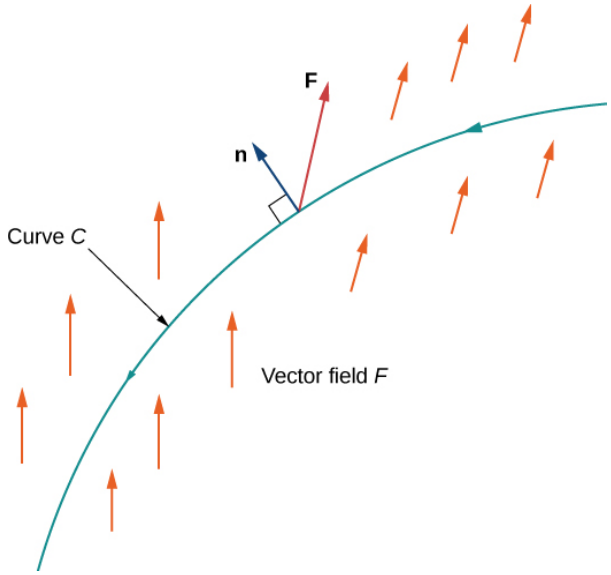

Figure 16.2.12: The flux of vector field $\vec{\mathbf{F}}$ *across curve* C *is computed by an integral similar to a vector line integral.*

We now give a formula for calculating the flux across a curve. This formula is analogous to the formula used to calculate a vector line integral (see Equation 16.2.5).

Theorem: Calculating Flux Across a Curve

Let $\vec{\mathbf{F}}$ be a vector field and let C be a smooth curve with parameterization $r(t) = \langle x(t), y(t)\rangle$, $a \le t \le b$.Let $\vec{\mathbf{n}}(t) = \langle y'(t), -x'(t)\rangle$. The flux of $\vec{\mathbf{F}}$ across C is

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{n}}(t)\, dt. \tag{16.2.8}$$

Proof

Before deriving the formula, note that

$$\|\vec{\mathbf{n}}(t)\| = \|\langle y'(t), -x'(t)\rangle\| = \sqrt{(y'(t))^2 + (x'(t))^2} = \|\vec{\mathbf{r}}'(t)\|.$$

Therefore,

$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds &= \int_C \vec{\mathbf{F}} \cdot \frac{\vec{\mathbf{n}}(t)}{\|\vec{\mathbf{n}}(t)\|}\, ds \\ &= \int_a^b \vec{\mathbf{F}} \cdot \frac{\vec{\mathbf{n}}(t)}{\|\vec{\mathbf{n}}(t)\|} \|\vec{\mathbf{r}}'(t)\|\, dt \\ &= \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{n}}(t)\, dt.\end{aligned}$$

□

Example 16.2.11: Flux across a Curve

Calculate the flux of $\vec{\mathbf{F}} = \langle 2x, 2y\rangle$ across a unit circle oriented counterclockwise (Figure 16.2.13).

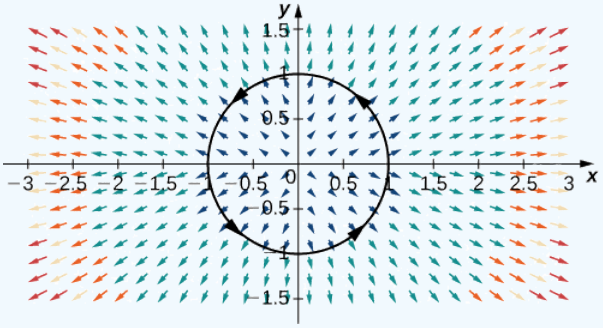

Figure 16.2.13: A unit circle in vector field $\vec{\mathbf{F}} = \langle 2x,\ 2y\rangle$.

Solution

To compute the flux, we first need a parameterization of the unit circle. We can use the standard parameterization $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t\rangle$, $0 \le t \le 2\pi$. The normal vector to a unit circle is $\langle \cos t, \sin t\rangle$. Therefore, the flux is

$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds &= \int_0^{2\pi} \langle 2\cos t, 2\sin t\rangle \cdot \langle \cos t, \sin t\rangle\, dt \\ &= \int_0^{2\pi} (2\cos^2 t + 2\sin^2 t)\, dt \\ &= 2\int_0^{2\pi} (\cos^2 t + \sin^2 t)\, dt \\ &= 2\int_0^{2\pi} dt = 4\pi.\end{aligned}$$

Exercise 16.2.11

Calculate the flux of $\vec{\mathbf{F}} = \langle x + y, 2y\rangle$ across the line segment from $(0, 0)$ to $(2, 3)$, where the curve is oriented from left to right.

Hint

Use Equation 16.2.8.

Answer

$3/2$

Let $\vec{\mathbf{F}}(x, y) = \langle P(x, y), Q(x, y)\rangle$ be a two-dimensional vector field. Recall that integral $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$ is sometimes written as $\int_C P\, dx + Q\, dy$. Analogously, flux $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds$ is sometimes written in the notation $\int_C -Q\, dx + P\, dy$, because the unit normal vector $\vec{\mathbf{N}}$ is

perpendicular to the unit tangent $\vec{\mathbf{T}}$. Rotating the vector $d\vec{\mathbf{r}} = \langle dx, dy\rangle$ by 90° results in vector $\langle dy, -dx\rangle$. Therefore, the line integral in Example 16.2.8 can be written as $\int_C -2y\,dx + 2x\,dy$.

Circulation

Now that we have defined flux, we can turn our attention to circulation. The line integral of vector field $\vec{\mathbf{F}}$ along an oriented closed curve is called the circulation of $\vec{\mathbf{F}}$ along C. Circulation line integrals have their own notation: $\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$. The circle on the integral symbol denotes that C is "circular" in that it has no endpoints. Example 16.2.5 shows a calculation of circulation.

To see where the term *circulation* comes from and what it measures, let $\vec{\mathbf{v}}$ represent the velocity field of a fluid and let C be an oriented closed curve. At a particular point P, the closer the direction of $\vec{\mathbf{v}}(P)$ is to the direction of $\vec{\mathbf{T}}(P)$, the larger the value of the dot product $\vec{\mathbf{v}}(P) \cdot \vec{\mathbf{T}}(P)$. The maximum value of $\vec{\mathbf{v}}(P) \cdot \vec{\mathbf{T}}(P)$ occurs when the two vectors are pointing in the exact same direction; the minimum value of $\vec{\mathbf{v}}(P) \cdot \vec{\mathbf{T}}(P)$ occurs when the two vectors are pointing in opposite directions. Thus, the value of the circulation $\oint_C \vec{\mathbf{v}} \cdot \vec{\mathbf{T}}\, ds$ measures the tendency of the fluid to move in the direction of C.

Example 16.2.12: Calculating Circulation

Let $\vec{\mathbf{F}} = \langle -y,\, x\rangle$ be the vector field from Example 16.2.3 and let C represent the unit circle oriented counterclockwise. Calculate the circulation of $\vec{\mathbf{F}}$ along C.

Solution

We use the standard parameterization of the unit circle: $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t\rangle$, $0 \le t \le 2\pi$. Then, $\vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) = \langle -\sin t, \cos t\rangle$ and $\vec{\mathbf{r}}\,'(t) = \langle -\sin t, \cos t\rangle$. Therefore, the circulation of $\vec{\mathbf{F}}$ along C is

$$\begin{aligned}\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds &= \int_0^{2\pi} \langle -\sin t, \cos t\rangle \cdot \langle -\sin t, \cos t\rangle\, dt \\ &= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt \\ &= \int_0^{2\pi} dt = 2\pi \text{ units of work.}\end{aligned}$$

Notice that the circulation is positive. The reason for this is that the orientation of C "flows" with the direction of $\vec{\mathbf{F}}$. At any point along the circle, the tangent vector and the vector from $\vec{\mathbf{F}}$ form an angle of less than 90°, and therefore the corresponding dot product is positive.

In Example 16.2.12, what if we had oriented the unit circle clockwise? We denote the unit circle oriented clockwise by $-C$. Then

$$\oint_{-C} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds = -\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds = -2\pi \text{ units of work.}$$

Notice that the circulation is negative in this case. The reason for this is that the orientation of the curve flows against the direction of $\vec{\mathbf{F}}$.

Exercise 16.2.12

Calculate the circulation of $\vec{\mathbf{F}}(x, y) = \langle -\frac{y}{x^2+y^2},\, \frac{x}{x^2+y^2}\rangle$ along a unit circle oriented counterclockwise.

Hint

Use Equation 16.2.8.

Answer

2π units of work

Example 16.2.13: Calculating Work

Calculate the work done on a particle that traverses circle C of radius 2 centered at the origin, oriented counterclockwise, by field $\vec{\mathbf{F}}(x, y) = \langle -2,\ y\rangle$. Assume the particle starts its movement at $(1,\ 0)$.

Solution

The work done by $\vec{\mathbf{F}}$ on the particle is the circulation of $\vec{\mathbf{F}}$ along C: $\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$. We use the parameterization $\vec{\mathbf{r}}(t) = \langle 2\cos t,\ 2\sin t\rangle$, $0 \le t \le 2\pi$ for C. Then, $\vec{\mathbf{r}}'(t) = \langle -2\sin t,\ 2\cos t\rangle$ and $\vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) = \langle -2,\ 2\sin t\rangle$. Therefore, the circulation of $\vec{\mathbf{F}}$ along C is

$$\begin{aligned}\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds &= \int_0^{2\pi} \langle -2, 2\sin t\rangle \cdot \langle -2\sin t, 2\cos t\rangle\, dt \\ &= \int_0^{2\pi} (4\sin t + 4\sin t\cos t)\, dt \\ &= \Big[-4\cos t + 4\sin^2 t\Big]_0^{2\pi} \\ &= \big(-4\cos(2\pi) + 2\sin^2(2\pi)\big) - \big(-4\cos(0) + 4\sin^2(0)\big) \\ &= -4 + 4 = 0 \text{ units of work.}\end{aligned}$$

The force field does zero work on the particle.

Notice that the circulation of $\vec{\mathbf{F}}$ along C is zero. Furthermore, notice that since $\vec{\mathbf{F}}$ is the gradient of $f(x, y) = -2x + \dfrac{y^2}{2}$, $\vec{\mathbf{F}}$ is conservative. We prove in a later section that under certain broad conditions, the circulation of a conservative vector field along a closed curve is zero.

Exercise 16.2.14

Calculate the work done by field $\vec{\mathbf{F}}(x, y) = \langle 2x,\ 3y\rangle$ on a particle that traverses the unit circle. Assume the particle begins its movement at $(-1,\ 0)$.

Hint

Use Equation 16.2.8.

Answer

0 units of work

Key Concepts

- Line integrals generalize the notion of a single-variable integral to higher dimensions. The domain of integration in a single-variable integral is a line segment along the x-axis, but the domain of integration in a line integral is a curve in a plane or in space.
- If C is a curve, then the length of C is $\displaystyle\int_C ds$.
- There are two kinds of line integral: scalar line integrals and vector line integrals. Scalar line integrals can be used to calculate the mass of a wire; vector line integrals can be used to calculate the work done on a particle traveling through a field.
- Scalar line integrals can be calculated using Equation 16.2.1; vector line integrals can be calculated using Equation 16.2.5.
- Two key concepts expressed in terms of line integrals are flux and circulation. Flux measures the rate that a field crosses a given line; circulation measures the tendency of a field to move in the same direction as a given closed curve.

Key Equations

- **Calculating a scalar line integral**
  $$\int_C f(x, y, z)\, ds = \int_a^b f(\vec{\mathbf{r}}(t))\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$$
- **Calculating a vector line integral**
  $$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\, dt$$

or

$$\int_C P\,dx + Q\,dy + R\,dz = \int_a^b \left(P\big(\vec{\mathbf{r}}(t)\big)\frac{dx}{dt} + Q\big(\vec{\mathbf{r}}(t)\big)\frac{dy}{dt} + R\big(\vec{\mathbf{r}}(t)\big)\frac{dz}{dt}\right) dt$$

- **Calculating flux**

$$\int_C \vec{\mathbf{F}} \cdot \frac{\vec{\mathbf{n}}(t)}{\|\vec{\mathbf{n}}(t)\|}\, ds = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{n}}(t)\, dt$$

Glossary

circulation

the tendency of a fluid to move in the direction of curve C. If C is a closed curve, then the circulation of $\vec{\mathbf{F}}$ along C is line integral $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$, which we also denote $\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$.

closed curve

a curve for which there exists a parameterization $\vec{\mathbf{r}}(t), a \leq t \leq b$, such that $\vec{\mathbf{r}}(a) = \vec{\mathbf{r}}(b)$, and the curve is traversed exactly once

flux

the rate of a fluid flowing across a curve in a vector field; the flux of vector field $\vec{\mathbf{F}}$ across plane curve C is line integral $\int_C \vec{\mathbf{F}} \cdot \frac{\vec{\mathbf{n}}(t)}{\|\vec{\mathbf{n}}(t)\|}\, ds$

line integral

the integral of a function along a curve in a plane or in space

orientation of a curve

the orientation of a curve C is a specified direction of C

piecewise smooth curve

an oriented curve that is not smooth, but can be written as the union of finitely many smooth curves

scalar line integral

the scalar line integral of a function f along a curve C with respect to arc length is the integral $\int_C f\, ds$, it is the integral of a scalar function f along a curve in a plane or in space; such an integral is defined in terms of a Riemann sum, as is a single-variable integral

vector line integral

the vector line integral of vector field $\vec{\mathbf{F}}$ along curve C is the integral of the dot product of $\vec{\mathbf{F}}$ with unit tangent vector $\vec{\mathbf{T}}$ of C with respect to arc length, $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$; such an integral is defined in terms of a Riemann sum, similar to a single-variable integral

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16.2E: Exercises for Section 16.2

1. *True or False?* Line integral $\int_C f(x,y)\,ds$ is equal to a definite integral if C is a smooth curve defined on $[a,b]$ and if function f is continuous on some region that contains curve C.

2. *True or False?* Vector functions $\vec{\mathbf{r}}_1 = t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}}, \quad 0 \le t \le 1,$ and $\vec{\mathbf{r}}_2 = (1-t)\,\hat{\mathbf{i}} + (1-t)^2\,\hat{\mathbf{j}}, \quad 0 \le t \le 1,$ define the same oriented curve.

3. *True or False?* $\int_{-C}(P\,dx + Q\,dy) = \int_C (P\,dx - Q\,dy)$

4. *True or False?* A piecewise smooth curve C consists of a finite number of smooth curves that are joined together end to end.

5. *True or False?* If C is given by $x(t) = t, \quad y(t) = t, \quad 0 \le t \le 1$, then $\int_C xy\,ds = \int_0^1 t^2\,dt.$

For the following exercises, use a computer algebra system (CAS) to evaluate the line integrals over the indicated path.

6. [T] $\int_C (x+y)\,ds$

$C: x = t, y = (1-t), z = 0$ from $(0,1,0)$ to $(1,0,0)$

7. [T] $\int_C (x-y)ds$

$C: \vec{\mathbf{r}}(t) = 4t\,\hat{\mathbf{i}} + 3t\,\hat{\mathbf{j}}$ when $0 \le t \le 2$

8. [T] $\int_C (x^2+y^2+z^2)\,ds$

$C: \vec{\mathbf{r}}(t) = \sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}} + 8t\,\hat{\mathbf{k}}$ when $0 \le t \le \dfrac{\pi}{2}$

9. [T] Evaluate $\int_C xy^4\,ds$, where C is the right half of circle $x^2 + y^2 = 16$ and is traversed in the clockwise direction.

10. [T] Evaluate $\int_C 4x^3 ds$, where C is the line segment from $(-2,-1)$ to $(1,2)$.

For the following exercises, find the work done.

11. Find the work done by vector field $\vec{\mathbf{F}}(x,y,z) = x\,\hat{\mathbf{i}} + 3xy\,\hat{\mathbf{j}} - (x+z)\,\hat{\mathbf{k}}$ on a particle moving along a line segment that goes from $(1,4,2)$ to $(0,5,1)$.

12. Find the work done by a person weighing 150 lb walking exactly one revolution up a circular, spiral staircase of radius 3 ft if the person rises 10 ft.

13. Find the work done by force field $\vec{\mathbf{F}}(x,y,z) = -\frac{1}{2}x\,\hat{\mathbf{i}} - \frac{1}{2}y\,\hat{\mathbf{j}} + \frac{1}{4}\,\hat{\mathbf{k}}$ on a particle as it moves along the helix $\vec{\mathbf{r}}(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$ from point $(1,0,0)$ to point $(-1,0,3\pi)$.

14. Find the work done by vector field $\vec{\mathbf{F}}(x,y) = y\,\hat{\mathbf{i}} + 2x\,\hat{\mathbf{j}}$ in moving an object along path C, which joins points $(1,0)$ and $(0,1)$.

15. Find the work done by force $\vec{\mathbf{F}}(x,y) = 2y\,\hat{\mathbf{i}} + 3x\,\hat{\mathbf{j}} + (x+y)\,\hat{\mathbf{k}}$ in moving an object along curve $\vec{\mathbf{r}}(t) = \cos(t)\,\hat{\mathbf{i}} + \sin(t)\,\hat{\mathbf{j}} + 16\,\hat{\mathbf{k}}$, where $0 \le t \le 2\pi$.

16. Find the mass of a wire in the shape of a circle of radius 2 centered at (3, 4) with linear mass density $\rho(x,y) = y^2$.

For the following exercises, evaluate the line integrals.

17. Evaluate $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x,y) = -1\,\hat{\mathbf{j}}$, and C is the part of the graph of $y = \frac{1}{2}x^3 - x$ from $(2,2)$ to $(-2,-2)$.

18. Evaluate $\int_\gamma (x^2+y^2+z^2)^{-1} ds$, where γ is the helix $x = \cos t, y = \sin t, z = t$, with $0 \le t \le T$.

19. Evaluate $\int_C yz\,dx + xz\,dy + xy\,dz$ over the line segment from $(1,1,1)$ to $(3,2,0)$.

20. Let C be the line segment from point (0, 1, 1) to point (2, 2, 3). Evaluate line integral $\int_C y\,ds$.

21. [T] Use a computer algebra system to evaluate the line integral $\int_C y^2\,dx + x\,dy$, where C is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(0, 2)$.

22. [T] Use a computer algebra system to evaluate the line integral $\int_C (x + 3y^2)\,dy$ over the path C given by $x = 2t, y = 10t$, where $0 \le t \le 1$.

23. [T] Use a CAS to evaluate line integral $\int_C xy\,dx + y\,dy$ over path C given by $x = 2t, y = 10t$, where $0 \le t \le 1$.

24. Evaluate line integral $\int_C (2x - y)\,dx + (x + 3y)\,dy$, where C lies along the x-axis from $x = 0$ to $x = 5$.

26. [T] Use a CAS to evaluate $\int_C \frac{y}{2x^2 - y^2}\,ds$, where C is defined by the parametric equations $x = t, y = t$, for $1 \le t \le 5$.

27. [T] Use a CAS to evaluate $\int_C xy\,ds$, where C is defined by the parametric equations $x = t^2, y = 4t$, for $0 \le t \le 1$.

In the following exercises, find the work done by force field $\vec{\mathbf{F}}$ on an object moving along the indicated path.

28. $\vec{\mathbf{F}}(x, y) = -x\,\hat{\mathbf{i}} - 2y\hat{\mathbf{j}}$

$C : y = x^3$ from $(0, 0)$ to $(2, 8)$

29. $\vec{\mathbf{F}}(x, y) = 2x\,\hat{\mathbf{i}} + y\hat{\mathbf{j}}$

$<C$: counterclockwise around the triangle with vertices $(0, 0), (1, 0)$, and $(1, 1)$

30. $\vec{\mathbf{F}}(x, y, z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} - 5z\,\hat{\mathbf{k}}$

$C : \vec{\mathbf{r}}(t) = 2\cos t\,\hat{\mathbf{i}} + 2\sin t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}},\ 0 \le t \le 2\pi$

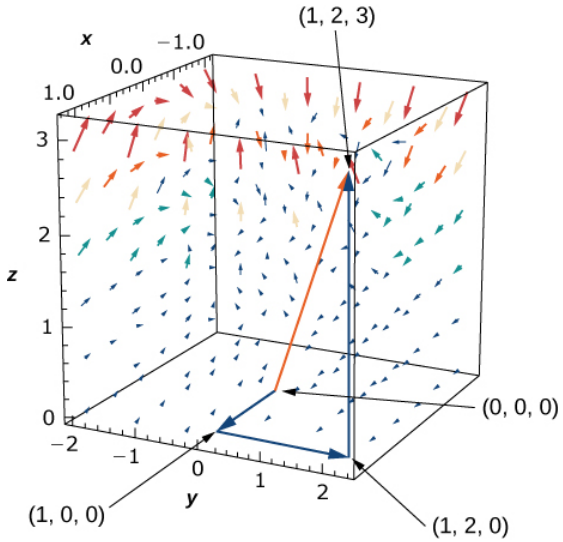
31. Let $\vec{\mathbf{F}}$ be vector field $\vec{\mathbf{F}}(x, y) = (y^2 + 2xe^y + 1)\,\hat{\mathbf{i}} + (2xy + x^2e^y + 2y)\,\hat{\mathbf{j}}$. Compute the work of integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where C is the path $\vec{\mathbf{r}}(t) = \sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}}, \quad 0 \le t \le \frac{\pi}{2}$.

32. Compute the work done by force $\vec{\mathbf{F}}(x, y, z) = 2x\,\hat{\mathbf{i}} + 3y\,\hat{\mathbf{j}} - z\,\hat{\mathbf{k}}$ along path $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}} + t^3\,\hat{\mathbf{k}}$, where $0 \le t \le 1$.

33. Evaluate $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x, y) = \frac{1}{x + y}\,\hat{\mathbf{i}} + \frac{1}{x + y}\,\hat{\mathbf{j}}$ and C is the segment of the unit circle going counterclockwise from $(1, 0)$ to $(0, 1)$.

34. Force $\vec{\mathbf{F}}(x, y, z) = zy\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}} + z^2x\,\hat{\mathbf{k}}$ acts on a particle that travels from the origin to point $(1, 2, 3)$. Calculate the work done if the particle travels:

a. along the path $(0, 0, 0) \to (1, 0, 0) \to (1, 2, 0) \to (1, 2, 3)$ along straight-line segments joining each pair of endpoints;
b. along the straight line joining the initial and final points.
c. Is the work the same along the two paths?



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35. Find the work done by vector field $\vec{\mathbf{F}}(x,y,z) = x\,\hat{\mathbf{i}} + 3xy\,\hat{\mathbf{j}} - (x+z)\,\hat{\mathbf{k}}$ on a particle moving along a line segment that goes from $(1,4,2)$ to $(0,5,1)$.

36. How much work is required to move an object in vector field $\vec{\mathbf{F}}(x,y) = y\,\hat{\mathbf{i}} + 3x\,\hat{\mathbf{j}}$ along the upper part of ellipse $\frac{x^2}{4} + y^2 = 1$ from $(2,0)$ to $(-2,0)$?

37. A vector field is given by $\vec{\mathbf{F}}(x,y) = (2x+3y)\,\hat{\mathbf{i}} + (3x+2y)\,\hat{\mathbf{j}}$. Evaluate the line integral of the field around a circle of unit radius traversed in a clockwise fashion.

38. Evaluate the line integral of scalar function xy along parabolic path $y = x^2$ connecting the origin to point $(1,1)$.

39. Find $\int_C y^2\,dx + (xy - x^2)\,dy$ along $C: y = 3x$ from $(0,0)$ to $(1,3)$.

40. Find $\int_C y^2\,dx + (xy - x^2)\,dy$ along $C: y^2 = 9x$ from $(0,0)$ to $(1,3)$.

For the following exercises, use a CAS to evaluate the given line integrals.

41. [T] Evaluate $\vec{\mathbf{F}}(x,y,z) = x^2 z\,\hat{\mathbf{i}} + 6y\,\hat{\mathbf{j}} + yz^2\,\hat{\mathbf{k}}$, where C is represented by $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}} + \ln t\,\hat{\mathbf{k}}, 1 \le t \le 3$.

42. [T] Evaluate line integral $\int_\gamma xe^y\,ds$ where, γ is the arc of curve $x = e^y$ from $(1,0)$ to $(e,1)$.

43. [T] Evaluate the integral $\int_\gamma xy^2\,ds$, where γ is a triangle with vertices $(0,1,2), (1,0,3)$, and $(0,-1,0)$.

44. [T] Evaluate line integral $\int_\gamma (y^2 - xy)\,dx$, where γ is curve $y = \ln x$ from $(1,0)$ toward $(e,1)$.

45. [T] Evaluate line integral $\int_\gamma xy^4\,ds$, where γ is the right half of circle $x^2 + y^2 = 16$.

46. [T] Evaluate $\int C\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}, \int C\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x,y,z) = x^2 y\,\hat{\mathbf{i}} + (x-z)\,\hat{\mathbf{j}} + xyz\,\hat{\mathbf{k}}$ and

$C: \vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}}, 0 \le t \le 1$.

47. Evaluate $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x,y) = 2x\sin y\,\hat{\mathbf{i}} + (x^2\cos y - 3y^2)\,\hat{\mathbf{j}}$ and

C is any path from $(-1,0)$ to $(5,1)$.

48. Find the line integral of $\vec{\mathbf{F}}(x,y,z) = 12x^2\,\hat{\mathbf{i}} - 5xy\,\hat{\mathbf{j}} + xz\,\hat{\mathbf{k}}$ over path C defined by $y = x^2, z = x^3$ from point $(0,0,0)$ to point $(2,4,8)$.

49. Find the line integral of $\int_C (1 + x^2 y)\,ds$, where C is ellipse $\vec{\mathbf{r}}(t) = 2\cos t\,\hat{\mathbf{i}} + 3\sin t\,\hat{\mathbf{j}}$ from $0 \le t \le \pi$.

For the following exercises, find the flux.

50. Compute the flux of $\vec{\mathbf{F}} = x^2\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$ across a line segment from $(0,0)$ to $(1,2)$.

51. Let $\vec{\mathbf{F}} = 5\,\hat{\mathbf{i}}$ and let C be curve $y = 0$, with $0 \le x \le 4$. Find the flux across C.

52. Let $\vec{\mathbf{F}} = 5\,\hat{\mathbf{j}}$ and let C be curve $y = 0$, with $0 \le x \le 4$. Find the flux across C.

53. Let $\vec{\mathbf{F}} = -y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$ and let $C: \vec{\mathbf{r}}(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}$ for $0 \le t \le 2\pi$. Calculate the flux across C.

54. Let $\vec{\mathbf{F}} = (x^2 + y^3)\,\hat{\mathbf{i}} + (2xy)\,\hat{\mathbf{j}}$. Calculate flux $\vec{\mathbf{F}}$ orientated counterclockwise across the curve $C: x^2 + y^2 = 9$.

Complete the rest of the exercises as stated.

55. Find the line integral of $\int_C z^2\,dx + y\,dy + 2y\,dz$, where C consists of two parts: C_1 and C_2. C_1 is the intersection of cylinder $x^2 + y^2 = 16$ and plane $z = 3$ from $(0,4,3)$ to $(-4,0,3)$. C_2 is a line segment from $(-4,0,3)$ to $(0,1,5)$.

56. A spring is made of a thin wire twisted into the shape of a circular helix $x = 2\cos t,\ y = 2\sin t,\ z = t$. Find the mass of two turns of the spring if the wire has a constant mass density of ρ grams per cm.

57. A thin wire is bent into the shape of a semicircle of radius a. If the linear mass density at point P is directly proportional to its distance from the line through the endpoints, find the mass of the wire.

58. An object moves in force field $\vec{\mathbf{F}}(x, y, z) = y^2\,\hat{\mathbf{i}} + 2(x+1)y\,\hat{\mathbf{j}}$ counterclockwise from point $(2, 0)$ along elliptical path $x^2 + 4y^2 = 4$ to $(-2, 0)$, and back to point $(2, 0)$ along the x-axis. How much work is done by the force field on the object?

59. Find the work done when an object moves in force field $\vec{\mathbf{F}}(x, y, z) = 2x\,\hat{\mathbf{i}} - (x+z)\,\hat{\mathbf{j}} + (y-x)\,\hat{\mathbf{k}}$ along the path given by $\vec{\mathbf{r}}(t) = t^2\,\hat{\mathbf{i}} + (t^2 - t)\,\hat{\mathbf{j}} + 3\,\hat{\mathbf{k}},\ 0 \le t \le 1.$

60. If an inverse force field $\vec{\mathbf{F}}$ is given by $\vec{\mathbf{F}}(x, y, z) = \dfrac{k}{\|r\|^3} r$, where k is a constant, find the work done by $\vec{\mathbf{F}}$ as its point of application moves along the x-axis from $A(1, 0, 0)$ to $B(2, 0, 0)$.

61. David and Sandra plan to evaluate line integral $\displaystyle\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ along a path in the xy-plane from $(0, 0)$ to $(1, 1)$. The force field is $\vec{\mathbf{F}}(x, y) = (x + 2y)\,\hat{\mathbf{i}} + (-x + y^2)\,\hat{\mathbf{j}}$. David chooses the path that runs along the x-axis from $(0, 0)$ to $(1, 0)$ and then runs along the vertical line $x = 1$ from $(1, 0)$ to the final point $(1, 1)$. Sandra chooses the direct path along the diagonal line $y = x$ from $(0, 0)$ to $(1, 1)$. Whose line integral is larger and by how much?

Contributors

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16.3: Conservative Vector Fields

Learning Objectives

- Describe simple and closed curves; define connected and simply connected regions.
- Explain how to find a potential function for a conservative vector field.
- Use the Fundamental Theorem for Line Integrals to evaluate a line integral in a vector field.
- Explain how to test a vector field to determine whether it is conservative.

In this section, we continue the study of conservative vector fields. We examine the Fundamental Theorem for Line Integrals, which is a useful generalization of the Fundamental Theorem of Calculus to line integrals of conservative vector fields. We also discover show how to test whether a given vector field is conservative, and determine how to build a potential function for a vector field known to be conservative.

Curves and Regions

Before continuing our study of conservative vector fields, we need some geometric definitions. The theorems in the subsequent sections all rely on integrating over certain kinds of curves and regions, so we develop the definitions of those curves and regions here. We first define two special kinds of curves: closed curves and simple curves. As we have learned, a closed curve is one that begins and ends at the same point. A simple curve is one that does not cross itself. A curve that is both closed and simple is a simple closed curve (Figure 16.3.1).

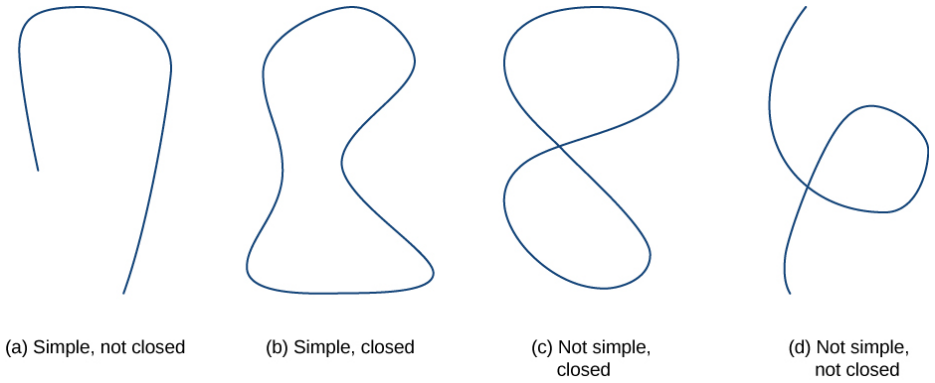


Figure 16.3.1. Types of curves that are simple or not simple and closed or not closed.

DEFINITION: Closed Curves

Curve C is a *closed curve* if there is a parameterization $\vec{\mathbf{r}}(t)$, $a \leq t \leq b$ of C such that the parameterization traverses the curve exactly once and $\vec{\mathbf{r}}(a) = \vec{\mathbf{r}}(b)$. Curve C is a simple curve if C does not cross itself. That is, C is simple if there exists a parameterization $\vec{\mathbf{r}}(t)$, $a \leq t \leq b$ of C such that $\vec{\mathbf{r}}$ is one-to-one over (a, b). It is possible for $\vec{\mathbf{r}}(a) = \vec{\mathbf{r}}(b)$, meaning that the simple curve is also closed.

Example 16.3.1: Determining Whether a Curve Is Simple and Closed

Is the curve with parameterization $\vec{\mathbf{r}}(t) = \left\langle \cos t, \frac{\sin(2t)}{2} \right\rangle$, $0 \leq t \leq 2\pi$ a simple closed curve?

Solution

Note that $\vec{\mathbf{r}}(0) = \langle 1, 0 \rangle = \vec{\mathbf{r}}(2\pi)$; therefore, the curve is closed. The curve is not simple, however. To see this, note that $\vec{\mathbf{r}}\left(\frac{\pi}{2}\right) = \langle 0, 0 \rangle = \vec{\mathbf{r}}\left(\frac{3\pi}{2}\right)$, and therefore the curve crosses itself at the origin (Figure 16.3.2).

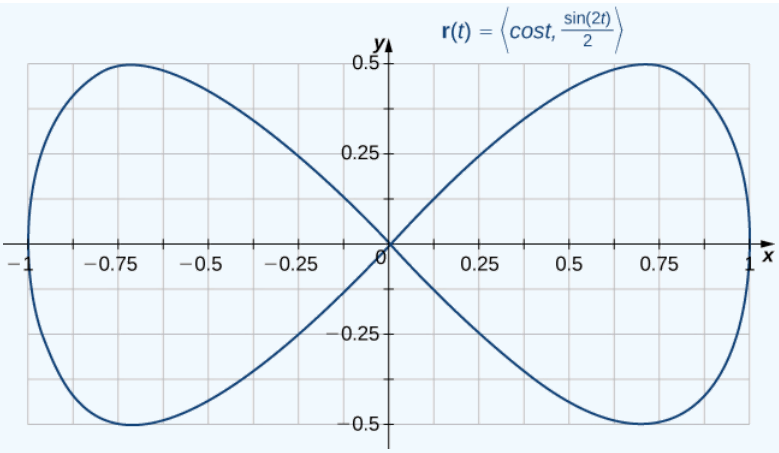


Figure 16.3.2. *A curve that is closed but not simple.*

? Exercise 16.3.1

Is the curve given by parameterization $\vec{\mathbf{r}}(t) = \langle 2\cos t, 3\sin t\rangle$, $0 \le t \le 6\pi$, a simple closed curve?

Hint

Sketch the curve.

Answer

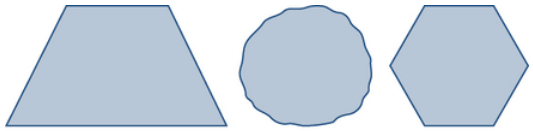
Yes

Many of the theorems in this chapter relate an integral over a region to an integral over the boundary of the region, where the region's boundary is a simple closed curve or a union of simple closed curves. To develop these theorems, we need two geometric definitions for regions: that of a connected region and that of a simply connected region. A connected region is one in which there is a path in the region that connects any two points that lie within that region. A simply connected region is a connected region that does not have any holes in it. These two notions, along with the notion of a simple closed curve, allow us to state several generalizations of the Fundamental Theorem of Calculus later in the chapter. These two definitions are valid for regions in any number of dimensions, but we are only concerned with regions in two or three dimensions.

DEFINITION: connected regions

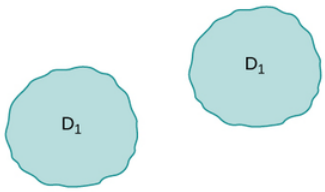
A region D is a *connected region* if, for any two points P_1 and P_2, there is a path from P_1 to P_2 with a trace contained entirely inside D. A region D is a simply connected region if D is connected for any simple closed curve C that lies inside D, and curve C can be shrunk continuously to a point while staying entirely inside D. In two dimensions, a region is simply connected if it is connected and has no holes.

All simply connected regions are connected, but not all connected regions are simply connected (Figure 16.3.3).



(a) Simply connected regions

(b) Connected regions that are not simply connected

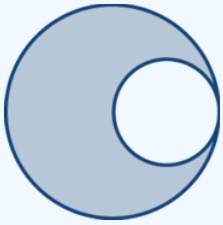


(c) A region that is not connected

Figure 16.3.3: Not all connected regions are simply connected. (a) Simply connected regions have no holes. (b) Connected regions that are not simply connected may have holes but you can still find a path in the region between any two points. (c) A region that is not connected has some points that cannot be connected by a path in the region.

? Exercise 16.3.2

Is the region in the below image connected? Is the region simply connected?



Hint

Consider the definitions.

Answer

The region in the figure is connected. The region in the figure is not simply connected.

Fundamental Theorem for Line Integrals

Now that we understand some basic curves and regions, let's generalize the Fundamental Theorem of Calculus to line integrals. Recall that the Fundamental Theorem of Calculus says that if a function f has an antiderivative F, then the integral of f from a to b depends only on the values of F at a and at b—that is,

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

If we think of the gradient as a derivative, then the same theorem holds for vector line integrals. We show how this works using a motivational example.

✓ Example 16.3.2: Evaluating a Line Integral and the Antiderivatives of the Endpoints

Let $\vec{\mathbf{F}}(x,y) = \langle 2x, 4y\rangle$. Calculate $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where C is the line segment from $(0,0)$ to $(2,2)$ (Figure 16.3.4).

Solution

We use the method from the previous section to calculate $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$. Curve C can be parameterized by $\vec{\mathbf{r}}(t) = \langle 2t, 2t \rangle$, $0 \leq t \leq 1$. Then, $\vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) = \langle 4t, 8t \rangle$ and $\vec{\mathbf{r}}'(t) = \langle 2, 2 \rangle$, which implies that

$$\begin{aligned} \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_0^1 \langle 4t, 8t \rangle \cdot \langle 2, 2 \rangle dt \\ &= \int_0^1 (8t + 16t) dt = \int_0^1 24t dt \\ &= \left[12t^2 \right]_0^1 = 12. \end{aligned}$$

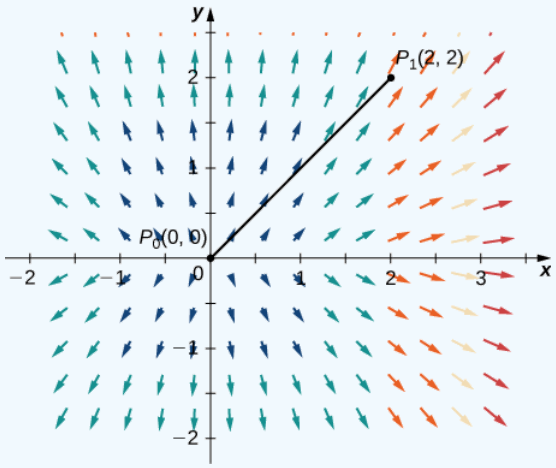


Figure 16.3.4: The value of line integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ *depends only on the value of the potential function of* $\vec{\mathbf{F}}$ *at the endpoints of the curve.*

Notice that $\vec{\mathbf{F}} = \vec{\nabla} f$, where $f(x, y) = x^2 + 2y^2$. If we think of the gradient as a derivative, then f is an "antiderivative" of $\vec{\mathbf{F}}$. In the case of single-variable integrals, the integral of derivative $g'(x)$ is $g(b) - g(a)$, where a is the start point of the interval of integration and b is the endpoint. If vector line integrals work like single-variable integrals, then we would expect integral $\vec{\mathbf{F}}$ to be $f(P_1) - f(P_0)$, where P_1 is the endpoint of the curve of integration and P_0 is the start point. Notice that this is the case for this example:

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = 12$$

and

$$f(2, 2) - f(0, 0) = 4 + 8 - 0 = 12.$$

In other words, the integral of a "derivative" can be calculated by evaluating an "antiderivative" at the endpoints of the curve and subtracting, just as for single-variable integrals.

The following theorem says that, under certain conditions, what happened in the previous example holds for any gradient field. The same theorem holds for vector line integrals, which we call the **Fundamental Theorem for Line Integrals**.

Theorem: THE FUNDAMENTAL THEOREM FOR LINE INTEGRALS

Let C be a piecewise smooth curve with parameterization $\vec{\mathbf{r}}(t)$, $a \leq t \leq b$. Let f be a function of two or three variables with first-order partial derivatives that exist and are continuous on C. Then,

$$\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a)). \tag{16.3.1}$$

Proof

First,

$$\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = \int^b \vec{\nabla} f(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\, dt.$$

By the chain rule,

$$\frac{d}{dt}(f(\vec{\mathbf{r}}(t)) = \vec{\nabla} f(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)$$

Therefore, by the Fundamental Theorem of Calculus,

$$\begin{aligned}\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} &= \int_a^b \vec{\nabla} f(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t) dt \\ &= \int_a^b \frac{d}{dt}(f(\vec{\mathbf{r}}(t)) dt \\ &= \left[f(\vec{\mathbf{r}}(t))\right]_{t=a}^{t=b} \\ &= f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a)).\end{aligned}$$

□

We know that if $\vec{\mathbf{F}}$ is a conservative vector field, there is a potential function f such that $\vec{\nabla} f = \vec{\mathbf{F}}$. Therefore

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a)).$$

In other words, just as with the Fundamental Theorem of Calculus, computing the line integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}$ is conservative, is a two-step process:

1. Find a potential function ("antiderivative") f for $\vec{\mathbf{F}}$ and
2. Compute the value of f at the endpoints of C and calculate their difference $f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a))$.

Keep in mind, however, there is one major difference between the Fundamental Theorem of Calculus and the Fundamental Theorem for Line Integrals:
A function of one variable that is continuous must have an antiderivative. However, a vector field, even if it is continuous, does not need to have a potential function.

Example 16.3.3: Applying the Fundamental Theorem

Calculate integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x, y, z) = \langle 2x \ln y, \frac{x^2}{y} + z^2, 2yz \rangle$ and C is a curve with parameterization $\vec{\mathbf{r}}(t) = \langle t^2, t, t \rangle$, $1 \leq t \leq e$

a. without using the Fundamental Theorem of Line Integrals and
b. using the Fundamental Theorem of Line Integrals.

Solution

1. First, let's calculate the integral without the Fundamental Theorem for Line Integrals and instead use the method we learned in the previous section:

$$\begin{aligned}\int_C \vec{\mathbf{F}}\cdot dr &= \int_1^e \vec{\mathbf{F}}(\vec{\mathbf{r}}(t))\cdot \vec{\mathbf{r}}\,'(t)\,dt \\ &= \int_1^e \langle 2t^2\ln t, \frac{t^4}{t}+t^2, 2t^2\rangle \cdot \langle 2t,1,1\rangle\,dt \\ &= \int_1^e (4t^3\ln t + t^3 + 3t^2)\,dt \\ &= \int_1^e 4t^3\ln t\,dt + \int_1^e (t^3+3t^2)\,dt \\ &= \int_1^e 4t^3\ln t\,dt + \left[\frac{t^4}{4}+t^3\right]_1^e \\ &= \int_1^e 4t^3\ln t\,dt + \frac{e^4}{4}+e^3-\frac{1}{4}-1 \\ &= \int_1^e 4t^3\ln t\,dt + \frac{e^4}{4}+e^3-\frac{5}{4}\end{aligned}$$

Integral $\int_1^e t^3\ln t\,dt$ requires integration by parts. Let $u=\ln t$ and $dv=t^3$. Then $u=\ln t$, $dv=t^3$ and

$$du=\frac{1}{t}\,dt,\quad v=\frac{t^4}{4}.$$

Therefore,

$$\begin{aligned}\int_1^e t^3\ln t\,dt &= \left[\frac{t^4}{4}\ln t\right]_1^e - \frac{1}{4}\int_1^e t^3\,dt \\ &= \frac{e^4}{4}-\frac{1}{4}\left(\frac{e^4}{4}-\frac{1}{4}\right).\end{aligned}$$

Thus,

$$\begin{aligned}\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} &= 4\int_1^e t^3\ln t\,dt \quad + \quad \frac{e^4}{4}+e^3-\frac{5}{4} \\ &= 4\left(\frac{e^4}{4}-\frac{1}{4}\left(\frac{e^4}{4}-\frac{1}{4}\right)\right)+\frac{e^4}{4}+e^3-\frac{5}{4} \\ &= e^4-\frac{e^4}{4}+\frac{1}{4}+\frac{e^4}{4}+e^3-\frac{5}{4} \\ &= e^4+e^3-1.\end{aligned}$$

2. Given that $f(x,y,z)=x^2\ln y+yz^2$ is a potential function for $\vec{\mathbf{F}}$, let's use the Fundamental Theorem for Line Integrals to calculate the integral. Note that

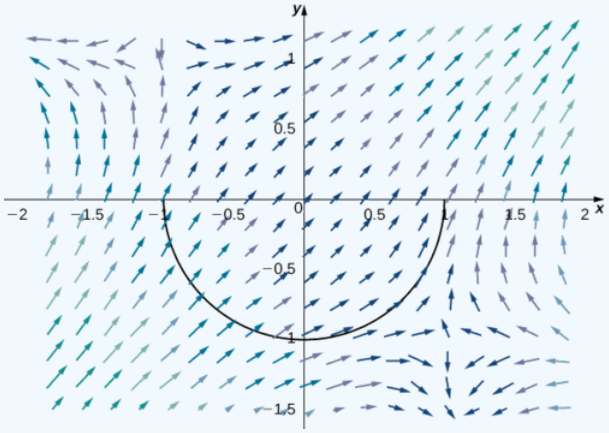
$$\begin{aligned}\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} &= \int_C \vec{\nabla} f\cdot d\vec{\mathbf{r}} \\ &= f(\vec{\mathbf{r}}(e))-f(\vec{\mathbf{r}}(1)) \\ &= f(e^2,e,e)-f(1,1,1) \\ &= e^4+e^3-1.\end{aligned}$$

This calculation is much more straightforward than the calculation we did in (a). As long as we have a potential function, calculating a line integral using the Fundamental Theorem for Line Integrals is much easier than calculating without the theorem.

Example 16.3.3 illustrates a nice feature of the Fundamental Theorem of Line Integrals: it allows us to calculate more easily many vector line integrals. As long as we have a potential function, calculating the line integral is only a matter of evaluating the potential function at the endpoints and subtracting.

Exercise 16.3.3

Given that $f(x,y) = (x-1)^2 y + (y+1)^2 x$ is a potential function for $\vec{\mathbf{F}}(x,y) = \langle 2xy - 2y + (y+1)^2, (x-1)^2 + 2yx + 2x \rangle$, calculate integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where C is the lower half of the unit circle oriented counterclockwise.



Hint

The Fundamental Theorem for Line Intervals says this integral depends only on the value of f at the endpoints of C.

Answer

2

The Fundamental Theorem for Line Integrals has two important consequences. The first consequence is that if $\vec{\mathbf{F}}$ is conservative and C is a closed curve, then the circulation of $\vec{\mathbf{F}}$ along C is zero—that is, $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0$. To see why this is true, let f be a potential function for $\vec{\mathbf{F}}$. Since C is a closed curve, the terminal point $\vec{\mathbf{r}}(b)$ of C is the same as the initial $\vec{\mathbf{r}}(a)$ of C—that is, $\vec{\mathbf{r}}(a) = \vec{\mathbf{r}}(b)$. Therefore, by the Fundamental Theorem for Line Integrals,

$$\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \oint_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} \tag{16.3.2}$$

$$= f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a)) \tag{16.3.3}$$

$$= f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(b)) \tag{16.3.4}$$

$$= 0. \tag{16.3.5}$$

Recall that the reason a conservative vector field $\vec{\mathbf{F}}$ is called "conservative" is because such vector fields model forces in which energy is conserved. We have shown gravity to be an example of such a force. If we think of vector field $\vec{\mathbf{F}}$ in integral $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ as a gravitational field, then the equation $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0$ follows. If a particle travels along a path that starts and ends at the same place, then the work done by gravity on the particle is zero.

The second important consequence of the Fundamental Theorem for Line Integrals (Equation 16.3.1) is that line integrals of conservative vector fields are independent of path—meaning, they depend only on the endpoints of the given curve, and do not depend on the path between the endpoints.

DEFINITION: Path Independence

Let $\vec{\mathbf{F}}$ be a vector field with domain D; it is independent of path (or path independent) if

$$\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

for any paths C_1 and C_2 in D with the same initial and terminal points.

The second consequence is stated formally in the following theorem.

Theorem: CONSERVATIVE FIELDS

If $\vec{\mathbf{F}}$ is a *conservative vector field*, then $\vec{\mathbf{F}}$ is independent of path.

Proof

Let D denote the domain of $\vec{\mathbf{F}}$ and let C_1 and C_2 be two paths in D with the same initial and terminal points (Figure 16.3.5). Call the initial point P_1 and the terminal point P_2. Since $\vec{\mathbf{F}}$ is conservative, there is a potential function f for $\vec{\mathbf{F}}$. By the Fundamental Theorem for Line Integrals,

$$\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = f(P_2) - f(P_1) = \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}.$$

Therefore, $\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$ is independent of path.

□

To visualize what independence of path means, imagine three hikers climbing from base camp to the top of a mountain. Hiker 1 takes a steep route directly from camp to the top. Hiker 2 takes a winding route that is not steep from camp to the top. Hiker 3 starts by taking the steep route but halfway to the top decides it is too difficult for him. Therefore he returns to camp and takes the non-steep path to the top. All three hikers are traveling along paths in a gravitational field. Since gravity is a force in which energy is conserved, the gravitational field is conservative. By independence of path, the total amount of work done by gravity on each of the hikers is the same because they all started in the same place and ended in the same place. The work done by the hikers includes other factors such as friction and muscle movement, so the total amount of energy each one expended is not the same, but the net energy expended against gravity is the same for all three hikers.

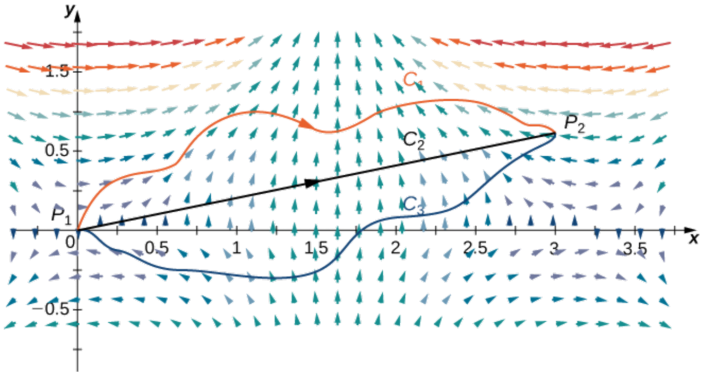


Figure 16.3.4: The vector field is conservative, and therefore independent of path.

We have shown that if $\vec{\mathbf{F}}$ is conservative, then $\vec{\mathbf{F}}$ is independent of path. It turns out that if the domain of $\vec{\mathbf{F}}$ is open and connected, then the converse is also true. That is, if $\vec{\mathbf{F}}$ is independent of path and the domain of $\vec{\mathbf{F}}$ is open and connected, then $\vec{\mathbf{F}}$ is conservative. Therefore, the set of conservative vector fields on open and connected domains is precisely the set of vector fields independent of path.

Theorem: THE PATH INDEPENDENCE TEST FOR CONSERVATIVE FIELDS

If $\vec{\mathbf{F}}$ is a continuous vector field that is independent of path and the domain D of $\vec{\mathbf{F}}$ is open and connected, then $\vec{\mathbf{F}}$ is conservative.

Proof

We prove the theorem for vector fields in $\mathbb{R}^2$. The proof for vector fields in $\mathbb{R}^3$ is similar. To show that $\vec{\mathbf{F}} = \langle P, Q \rangle$ is conservative, we must find a potential function f for $\vec{\mathbf{F}}$. To that end, let X be a fixed point in D. For any point (x, y) in D, let C be a path from X to (x, y). Define $f(x, y)$ by $f(x, y) = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$. (Note that this definition of f makes sense only because $\vec{\mathbf{F}}$ is independent of path. If $\vec{\mathbf{F}}$ was not independent of path, then it might be possible to find another path C' from X to (x, y) such that $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \neq \int_{C'} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, and in such a case $f(x, y)$ would not be a function.) We want to show that f has the property $\vec{\nabla} f = \vec{\mathbf{F}}$.

Since domain D is open, it is possible to find a disk centered at (x, y) such that the disk is contained entirely inside D. Let (a, y) with $a < x$ be a point in that disk. Let C be a path from X to (x, y) that consists of two pieces: C_1 and C_2. The first piece, C_1, is any path from C to (a, y) that stays inside D; C_2 is the horizontal line segment from (a, y) to (x, y) (Figure 16.3.6). Then

$$f(x,y) = \int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}.$$

The first integral does not depend on x, so

$$f_x(x,y) = \frac{\partial}{\partial x} \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}.$$

If we parameterize C_2 by $\vec{\mathbf{r}}(t) = \langle t, y \rangle$, $a \le t \le x$, then

$$\begin{aligned} f_x(x,y) &= \frac{\partial}{\partial x} \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \\ &= \frac{\partial}{\partial x} \int_a^x \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}\,'(t)\, dt \\ &= \frac{\partial}{\partial x} \int_a^x \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \frac{d}{dt}(\langle t, y \rangle)\, dt \\ &= \frac{\partial}{\partial x} \int_a^x \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \langle 1, 0 \rangle\, dt \\ &= \frac{\partial}{\partial x} \int_a^x P(t,y)\, dt. \end{aligned}$$

By the Fundamental Theorem of Calculus (part 1),

$$f_x(x,y) = \frac{\partial}{\partial x} \int_a^x P(t,y)\, dt = P(x,y).$$

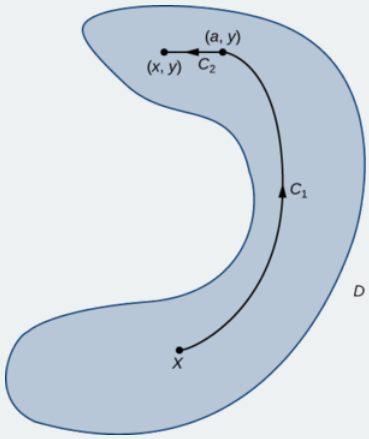


Figure 16.3.6. Here, C_1 is any path from C to (a, y) that stays inside D, and C_2 is the horizontal line segment from (a, y) to (x, y).

A similar argument using a vertical line segment rather than a horizontal line segment shows that $f_y(x, y) = Q(x, y)$.

Therefore $\vec{\nabla} f = \vec{\mathbf{F}}$ and $\vec{\mathbf{F}}$ is conservative.

□

We have spent a lot of time discussing and proving the theorems above, but we can summarize them simply: a vector field $\vec{\mathbf{F}}$ on an open and connected domain is conservative if and only if it is independent of path. This is important to know because conservative vector fields are extremely important in applications, and these theorems give us a different way of viewing what it means to be conservative using path independence.

Example 16.3.4: Showing That a Vector Field Is Not Conservative

Use path independence to show that vector field $\vec{\mathbf{F}}(x,y) = \langle x^2 y, y+5\rangle$ is not conservative.

Solution

We can indicate that $\vec{\mathbf{F}}$ is not conservative by showing that $\vec{\mathbf{F}}$ is not path independent. We do so by giving two different paths, C_1 and C_2, that both start at $(0,0)$ and end at $(1,1)$, and yet $\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \neq \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$.

Let C_1 be the curve with parameterization $\vec{\mathbf{r}}_1(t) = \langle t,\, t\rangle$, $0 \leq t \leq 1$ and let C_2 be the curve with parameterization $\vec{\mathbf{r}}_2(t) = \langle t,\, t^2\rangle$, $0 \leq t \leq 1$ (Figure 16.3.7.). Then

$$\begin{aligned}\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_0^1 \vec{\mathbf{F}}(\vec{\mathbf{r}}_1(t)) \cdot \vec{\mathbf{r}}_1{}'(t)\, dt \\ &= \int_0^1 \langle t^3, t+5\rangle \cdot \langle 1,1\rangle\, dt = \int_0^1 (t^3 + t + 5)\, dt \\ &= \left[\frac{t^4}{4} + \frac{t^2}{2} + 5t\right]_0^1 = \frac{23}{4}\end{aligned}$$

and

$$\begin{aligned}\int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_0^1 \vec{\mathbf{F}}(\vec{\mathbf{r}}_2(t)) \cdot \vec{\mathbf{r}}_2{}'(t)\, dt \\ &= \int_0^1 \langle t^4, t^2+5\rangle \cdot \langle 1, 2t\rangle\, dt = \int_0^1 (t^4 + 2t^3 + 10t)\, dt \\ &= \left[\frac{t^5}{5} + \frac{t^4}{2} + 5t^2\right]_0^1 = \frac{57}{10}.\end{aligned}$$

Since $\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \neq \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, the value of a line integral of $\vec{\mathbf{F}}$ depends on the path between two given points. Therefore, $\vec{\mathbf{F}}$ is not independent of path, and $\vec{\mathbf{F}}$ is not conservative.

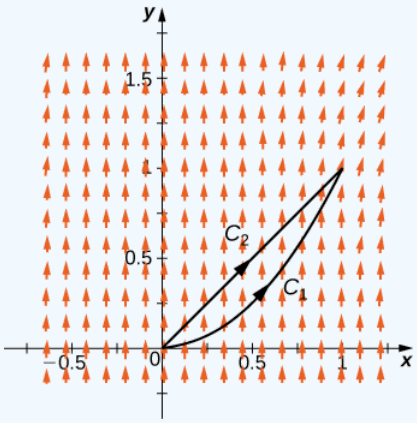


Figure 16.3.7: Curves C_1 and C_2 are both oriented from left to right.

Exercise 16.3.4

Show that $\vec{\mathbf{F}}(x,y) = \langle xy,\, x^2 y^2\rangle$ is not path independent by considering the line segment from $(0,0)$ to $(0,2)$ and the piece of the graph of $y = \dfrac{x^2}{2}$ that goes from $(0,0)$ to $(0,2)$.

Hint

Calculate the corresponding line integrals.

Answer

If C_1 and C_2 represent the two curves, then

$$\int_{C} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \neq \int_{C} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}.$$

Conservative Vector Fields and Potential Functions

As we have learned, the Fundamental Theorem for Line Integrals says that if $\vec{\mathbf{F}}$ is conservative, then calculating $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ has two steps: first, find a potential function f for $\vec{\mathbf{F}}$ and, second, calculate $f(P_1) - f(P_0)$, where P_1 is the endpoint of C and P_0 is the starting point. To use this theorem for a conservative field $\vec{\mathbf{F}}$, we must be able to find a potential function f for $\vec{\mathbf{F}}$. Therefore, we must answer the following question: Given a conservative vector field $\vec{\mathbf{F}}$, how do we find a function f such that $\vec{\nabla} f = \vec{\mathbf{F}}$? Before giving a general method for finding a potential function, let's motivate the method with an example.

Example 16.3.5: Finding a Potential Function

Find a potential function for $\vec{\mathbf{F}}(x, y) = \langle 2xy^3, 3x^2y^2 + \cos(y) \rangle$, thereby showing that $\vec{\mathbf{F}}$ is conservative.

Solution

Suppose that $f(x, y)$ is a potential function for $\vec{\mathbf{F}}$. Then, $\vec{\nabla} f = \vec{\mathbf{F}}$, and therefore

$$f_x(x, y) = 2xy^3 \quad \text{and} \quad f_y(x, y) = 3x^2y^2 + \cos y.$$

Integrating the equation $f_x(x, y) = 2xy^3$ with respect to x yields the equation

$$f(x, y) = x^2y^3 + h(y).$$

Notice that since we are integrating a two-variable function with respect to x, we must add a constant of integration that is a constant with respect to x, but may still be a function of y. The equation $f(x, y) = x^2y^3 + h(y)$ can be confirmed by taking the partial derivative with respect to x:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2y^3) + \frac{\partial}{\partial x}(h(y)) = 2xy^3 + 0 = 2xy^3.$$

Since f is a potential function for $\vec{\mathbf{F}}$,

$$f_y(x, y) = 3x^2y^2 + \cos(y),$$

and therefore

$$3x^2y^2 + g'(y) = 3x^2y^2 + \cos(y).$$

This implies that $h'(y) = \cos y$, so $h(y) = \sin y + C$. Therefore, *any* function of the form $f(x, y) = x^2y^3 + \sin(y) + C$ is a potential function. Taking, in particular, $C = 0$ gives the potential function $f(x, y) = x^2y^3 + \sin(y)$.

To verify that f is a potential function, note that $\vec{\nabla} f(x, y) = \langle 2xy^3, 3x^2y^2 + \cos y \rangle = \vec{\mathbf{F}}$.

Exercise 16.3.5

Find a potential function for $\vec{\mathbf{F}}(x, y) = \langle e^x y^3 + y, 3e^x y^2 + x \rangle$.

Hint

Follow the steps in Example 16.3.5.

Answer

$f(x, y) = e^x y^3 + xy$

The logic of the previous example extends to finding the potential function for any conservative vector field in $\mathbb{R}^2$. Thus, we have the following problem-solving strategy for finding potential functions:

PROBLEM-SOLVING STRATEGY: FINDING A POTENTIAL FUNCTION FOR A CONSERVATIVE VECTOR FIELD $\vec{\mathbf{F}}(x,y) = \langle P(x,y), Q(x,y)\rangle$

1. Integrate P with respect to x. This results in a function of the form $g(x,y) + h(y)$, where $h(y)$ is unknown.
2. Take the partial derivative of $g(x,y) + h(y)$ with respect to y, which results in the function $gy(x,y) + h'(y)$.
3. Use the equation $gy(x,y) + h'(y) = Q(x,y)$ to find $h'(y)$.
4. Integrate $h'(y)$ to find $h(y)$.
5. Any function of the form $f(x,y) = g(x,y) + h(y) + C$, where C is a constant, is a potential function for $\vec{\mathbf{F}}$.

We can adapt this strategy to find potential functions for vector fields in $\mathbb{R}^3$, as shown in the next example.

Example 16.3.6: Finding a Potential Function in $\mathbb{R}^3$

Find a potential function for $F(x,y,z) = \langle 2xy, x^2 + 2yz^3, 3y^2z^2 + 2z\rangle$, thereby showing that $\vec{\mathbf{F}}$ is conservative.

Solution

Suppose that f is a potential function. Then, $\vec{\nabla} f = \vec{\mathbf{F}}$ and therefore $f_x(x,y,z) = 2xy$. Integrating this equation with respect to x yields the equation $f(x,y,z) = x^2y + g(y,z)$ for some function g. Notice that, in this case, the constant of integration with respect to x is a function of y and z.

Since f is a potential function,

$$x^2 + 2yz^3 = f_y(x,y,z) = x^2 + g_y(y,z).$$

Therefore,

$$g_y(y,z) = 2yz^3.$$

Integrating this function with respect to y yields

$$g(y,z) = y^2z^3 + h(z)$$

for some function $h(z)$ of z alone. (Notice that, because we know that g is a function of only y and z, we do not need to write $g(y,z) = y^2z^3 + h(x,z)$.) Therefore,

$$f(x,y,z) = x^2y + g(y,z) = x^2y + y^2z^3 + h(z).$$

To find f, we now must only find h. Since f is a potential function,

$$3y^2z^2 + 2z = g_z(y,z) = 3y^2z^2 + h'(z).$$

This implies that $h'(z) = 2z$, so $h(z) = z^2 + C$. Letting $C = 0$ gives the potential function

$$f(x,y,z) = x^2y + y^2z^3 + z^2.$$

To verify that f is a potential function, note that $\vec{\nabla} f(x,y,z) = \langle 2xy, x^2 + 2yz^3, 3y^2z^2 + 2z\rangle = \vec{\mathbf{F}}(x,y,z)$.

Exercise 16.3.6

Find a potential function for $\vec{\mathbf{F}}(x,y,z) = \langle 12x^2, \cos y\cos z, 1 - \sin y\sin z\rangle$.

Hint

Following Example 16.3.6, begin by integrating with respect to x.

Answer

$f(x,y,z) = 4x^3 + \sin y\cos z + z$

We can apply the process of finding a potential function to a gravitational force. Recall that, if an object has unit mass and is located at the origin, then the gravitational force in $\mathbb{R}^2$ that the object exerts on another object of unit mass at the point (x,y) is given by vector field

$$\vec{\mathbf{F}}(x,y) = -G\left\langle \frac{x}{(x^2+y^2)^{3/2}}, \frac{y}{(x^2+y^2)^{3/2}} \right\rangle,$$

where G is the universal gravitational constant. In the next example, we build a potential function for $\vec{\mathbf{F}}$, thus confirming what we already know: that gravity is conservative.

Example 16.3.7: Finding a Potential Function

Find a potential function f for $\vec{\mathbf{F}}(x,y) = -G\left\langle \frac{x}{(x^2+y^2)^{3/2}}, \frac{y}{(x^2+y^2)^{3/2}} \right\rangle$.

Solution

Suppose that f is a potential function. Then, $\vec{\nabla} f = \vec{\mathbf{F}}$ and therefore

$$f_x(x,y) = \frac{-Gx}{(x^2+y^2)^{3/2}}.$$

To integrate this function with respect to x, we can use u-substitution. If $u = x^2+y^2$, then $\frac{du}{2} = x\,dx$, so

$$\begin{aligned}\int \frac{-Gx}{(x^2+y^2)^{3/2}}\,dx &= \int \frac{-G}{2u^{3/2}}\,du \\ &= \frac{G}{\sqrt{u}} + h(y) \\ &= \frac{G}{\sqrt{x^2+y^2}} + h(y)\end{aligned}$$

for some function $h(y)$. Therefore,

$$f(x,y) = \frac{G}{\sqrt{x^2+y^2}} + h(y).$$

Since f is a potential function for $\vec{\mathbf{F}}$,

$$f_y(x,y) = \frac{-Gy}{(x^2+y^2)^{3/2}}$$

.

Since $f(x,y) = \frac{G}{\sqrt{x^2+y^2}} + h(y)$, $f_y(x,y)$ also equals $\frac{-Gy}{(x^2+y^2)^{3/2}} + h'(y)$.

Therefore,

$$\frac{-Gy}{(x^2+y^2)^{3/2}} + h'(y) = \frac{-Gy}{(x^2+y^2)^{3/2}},$$

which implies that $h'(y) = 0$. Thus, we can take $h(y)$ to be any constant; in particular, we can let $h(y) = 0$. The function

$$f(x,y) = \frac{G}{\sqrt{x^2+y^2}}$$

is a potential function for the gravitational field $\vec{\mathbf{F}}$. To confirm that f is a potential function, note that

$$\begin{aligned}\vec{\nabla} f(x,y) &= \langle -\frac{1}{2}\frac{G}{(x^2+y^2)^{3/2}}(2x), -\frac{1}{2}\frac{G}{(x^2+y^2)^{3/2}}(2y)\rangle \\ &= \langle \frac{-Gx}{(x^2+y^2)^{3/2}}, \frac{-Gy}{(x^2+y^2)^{3/2}}\rangle \\ &= \vec{\mathbf{F}}(x,y).\end{aligned}$$

Exercise 16.3.7

Find a potential function f for the three-dimensional gravitational force $\vec{\mathbf{F}}(x,y,z) = \left\langle \frac{-Gx}{(x^2+y^2+z^2)^{3/2}}, \frac{-Gy}{(x^2+y^2+z^2)^{3/2}}, \frac{-Gz}{(x^2+y^2+z^2)^{3/2}} \right\rangle$.

Hint

Follow the Problem-Solving Strategy.

Answer

$$f(x,y,z) = \frac{G}{\sqrt{x^2+y^2+z^2}}$$

Testing a Vector Field

Until now, we have worked with vector fields that we know are conservative, but if we are not told that a vector field is conservative, we need to be able to test whether it is conservative. Recall that, if $\vec{\mathbf{F}}$ is conservative, then $\vec{\mathbf{F}}$ has the cross-partial property (see The Cross-Partial Property of Conservative Vector Fields). That is, if $\vec{\mathbf{F}} = \langle P, Q, R \rangle$ is conservative, then $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$. So, if $\vec{\mathbf{F}}$ has the cross-partial property, then is $\vec{\mathbf{F}}$ conservative? If the domain of $\vec{\mathbf{F}}$ is open and simply connected, then the answer is yes.

Theorem: THE CROSS-PARTIAL TEST FOR CONSERVATIVE FIELDS

If $\vec{\mathbf{F}} = \langle P, Q, R \rangle$ is a vector field on an open, simply connected region D and $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$ throughout D, then $\vec{\mathbf{F}}$ is conservative.

Although a proof of this theorem is beyond the scope of the text, we can discover its power with some examples. Later, we see why it is necessary for the region to be simply connected.

Combining this theorem with the cross-partial property, we can determine whether a given vector field is conservative:

Theorem: CROSS-PARTIAL PROPERTY OF CONSERVATIVE FIELDS

Let $\vec{\mathbf{F}} = \langle P, Q, R \rangle$ be a vector field on an open, simply connected region D. Then $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$ throughout D if and only if $\vec{\mathbf{F}}$ is conservative.

The version of this theorem in $\mathbb{R}^2$ is also true. If $\vec{\mathbf{F}}(x,y) = \langle P, Q \rangle$ is a vector field on an open, simply connected domain in $\mathbb{R}^2$, then $\vec{\mathbf{F}}$ is conservative if and only if $P_y = Q_x$.

Example 16.3.8: Determining Whether a Vector Field Is Conservative

Determine whether vector field $\vec{\mathbf{F}}(x,y,z) = \langle xy^2z, x^2yz, z^2 \rangle$ is conservative.

Solution

Note that the domain of $\vec{\mathbf{F}}$ is all of $\mathbb{R}^2$ and $\mathbb{R}^3$ is simply connected. Therefore, we can use **The Cross-Partial Property of Conservative Vector Fields** to determine whether $\vec{\mathbf{F}}$ is conservative. Let

$$P(x,y,z) = xy^2z$$

$$Q(x,y,z) = x^2yz$$

and

$$R(x,y,z) = z^2.$$

Since $Q_z(x,y,z) = x^2y$ and $R_y(x,y,z) = 0$, the vector field is not conservative.

Example 16.3.9: Determining Whether a Vector Field Is Conservative

Determine vector field $\vec{\mathbf{F}}(x,y) = \langle x\ln(y),\ \frac{x^2}{2y}\rangle$ is conservative.

Solution

Note that the domain of $\vec{\mathbf{F}}$ is the part of $\mathbb{R}^2$ in which $y > 0$. Thus, the domain of $\vec{\mathbf{F}}$ is part of a plane above the x-axis, and this domain is simply connected (there are no holes in this region and this region is connected). Therefore, we can use **The Cross-Partial Property of Conservative Vector Fields** to determine whether $\vec{\mathbf{F}}$ is conservative. Let

$$P(x,y) = x\ln(y) \quad \text{and} \quad Q(x,y) = \frac{x^2}{2y}.$$

Then $P_y(x,y) = \frac{x}{y} = Q_x(x,y)$ and thus $\vec{\mathbf{F}}$ is conservative.

Exercise 16.3.8

Determine whether $\vec{\mathbf{F}}(x,y) = \langle \sin x\cos y,\ \cos x\sin y\rangle$ is conservative.

Hint

Use **The Cross-Partial Property of Conservative Vector Fields** from the previous section.

Answer

It is conservative.

When using **The Cross-Partial Property of Conservative Vector Fields**, it is important to remember that a theorem is a tool, and like any tool, it can be applied only under the right conditions. In the case of **The Cross-Partial Property of Conservative Vector Fields**, the theorem can be applied only if the domain of the vector field is simply connected.

To see what can go wrong when misapplying the theorem, consider the vector field from Example 16.3.4:

$$\vec{\mathbf{F}}(x,y) = \frac{y}{x^2+y^2}\,\hat{\mathbf{i}} + \frac{-x}{x^2+y^2}\,\hat{\mathbf{j}}.$$

This vector field satisfies the cross-partial property, since

$$\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = \frac{(x^2+y^2)-y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

and

$$\frac{\partial}{\partial x}\left(\frac{-x}{x^2+y^2}\right) = \frac{-(x^2+y^2)+x(2x)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}.$$

Since $\vec{\mathbf{F}}$ satisfies the cross-partial property, we might be tempted to conclude that $\vec{\mathbf{F}}$ is conservative. However, $\vec{\mathbf{F}}$ is not conservative. To see this, let

$$\vec{\mathbf{r}}(t) = \langle \cos t, \sin t\rangle, \quad 0 \le t \le \pi$$

be a parameterization of the upper half of a unit circle oriented counterclockwise (denote this C_1) and let

$$\vec{\mathbf{s}}(t) = \langle \cos t, -\sin t\rangle, \quad 0 \le t \le \pi$$

be a parameterization of the lower half of a unit circle oriented clockwise (denote this C_2). Notice that C_1 and C_2 have the same starting point and endpoint. Since $\sin^2 t + \cos^2 t = 1$,

$$\vec{\mathbf{F}}(\vec{\mathbf{r}}(t))\cdot\vec{\mathbf{r}}\,'(t) = \langle \sin(t), -\cos(t)\rangle \cdot \langle -\sin(t), \cos(t)\rangle = -1$$

and

$$\vec{\mathbf{F}}(\vec{\mathbf{s}}(t))\cdot\vec{\mathbf{s}}\,'(t) = \langle -\sin t, -\cos t\rangle \cdot \langle -\sin t, -\cos t\rangle = \sin^2 t + \cos^2 t = 1.$$

Therefore,

$$\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_0^{\pi} -1 \, dt = -\pi$$

and

$$\int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_0^{\pi} 1 \, dt = \pi.$$

Thus, C_1 and C_2 have the same starting point and endpoint, but $\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \neq \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$. Therefore, $\vec{\mathbf{F}}$ is not independent of path and $\vec{\mathbf{F}}$ is not conservative.

To summarize: $\vec{\mathbf{F}}$ satisfies the cross-partial property and yet $\vec{\mathbf{F}}$ is not conservative. What went wrong? Does this contradict **The Cross-Partial Property of Conservative Vector Fields**? The issue is that the domain of $\vec{\mathbf{F}}$ is all of $\mathbb{R}^2$ except for the origin. In other words, the domain of $\vec{\mathbf{F}}$ has a hole at the origin, and therefore the domain is not simply connected. Since the domain is not simply connected, **The Cross-Partial Property of Conservative Vector Fields** does not apply to $\vec{\mathbf{F}}$.

We close this section by looking at an example of the usefulness of the Fundamental Theorem for Line Integrals. Now that we can test whether a vector field is conservative, we can always decide whether the Fundamental Theorem for Line Integrals can be used to calculate a vector line integral. If we are asked to calculate an integral of the form $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, then our first question should be: Is $\vec{\mathbf{F}}$ conservative? If the answer is yes, then we should find a potential function and use the Fundamental Theorem for Line Integrals to calculate the integral. If the answer is no, then the Fundamental Theorem for Line Integrals cannot help us and we have to use other methods, such as using the method from the previous section (using $\vec{\mathbf{F}}(\vec{\mathbf{r}}(t))$ and $\vec{\mathbf{r}}'(t)$).

Example 16.3.10: Using the Fundamental Theorem for Line Integrals

Calculate line integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x, y, z) = \langle 2xe^y z + e^x z,\ x^2 e^y z,\ x^2 e^y + e^x \rangle$ and C is any smooth curve that goes from the origin to $(1, 1, 1)$.

Solution

Before trying to compute the integral, we need to determine whether $\vec{\mathbf{F}}$ is conservative and whether the domain of $\vec{\mathbf{F}}$ is simply connected. The domain of $\vec{\mathbf{F}}$ is all of $\mathbb{R}^3$, which is connected and has no holes. Therefore, the domain of $\vec{\mathbf{F}}$ is simply connected. Let

$$P(x, y, z) = 2xe^y z + e^x z, \quad Q(x, y, z) = x^2 e^y z, \quad \text{and} \quad R(x, y, z) = x^2 e^y + e^x$$

so that $\vec{\mathbf{F}}(x, y, z) = \langle P, Q, R \rangle$. Since the domain of $\vec{\mathbf{F}}$ is simply connected, we can check the cross partials to determine whether $\vec{\mathbf{F}}$ is conservative. Note that

$$\begin{aligned} P_y(x, y, z) &= 2xe^y z = Q_x(x, y, z) \\ P_z(x, y, z) &= 2xe^y + e^x = R_x(x, y, z) \\ Q_z(x, y, z) &= x^2 e^y = R_y(x, y, z). \end{aligned}$$

Therefore, $\vec{\mathbf{F}}$ is conservative.

To evaluate $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ using the Fundamental Theorem for Line Integrals, we need to find a potential function f for $\vec{\mathbf{F}}$. Let f be a potential function for $\vec{\mathbf{F}}$. Then, $\vec{\nabla} f = \vec{\mathbf{F}}$, and therefore $f_x(x, y, z) = 2xe^y z + e^x z$. Integrating this equation with respect to x gives $f(x, y, z) = x^2 e^y z + e^x z + h(y, z)$ for some function h. Differentiating this equation with respect to y gives $x^2 e^y z + h_y(y, z) = Q(x, y, z) = x^2 e^y z$, which implies that $h_y(y, z) = 0$. Therefore, h is a function of z only, and $f(x, y, z) = x^2 e^y z + e^x z + h(z)$. To find h, note that $f_z = x^2 e^y + e^x + h'(z) = R = x^2 e^y + e^x$. Therefore, $h'(z) = 0$ and we can take $h(z) = 0$. A potential function for $\vec{\mathbf{F}}$ is $f(x, y, z) = x^2 e^y z + e^x z$.

Now that we have a potential function, we can use the Fundamental Theorem for Line Integrals to evaluate the integral. By the theorem,

$$\begin{aligned} \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} \\ &= f(1, 1, 1) - f(0, 0, 0) \\ &= 2e. \end{aligned}$$

Analysis

Notice that if we hadn't recognized that $\vec{\mathbf{F}}$ is conservative, we would have had to parameterize C and use the method from the previous section. Since curve C is unknown, using the Fundamental Theorem for Line Integrals is much simpler.

Exercise 16.3.9

Calculate integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x,y) = \langle \sin x \sin y, 5 - \cos x \cos y \rangle$ and C is a semicircle with starting point $(0,\pi)$ and endpoint $(0,-\pi)$.

Hint

Use the Fundamental Theorem for Line Integrals.

Answer

-10π

Example 16.3.11: Work Done on a Particle

Let $\vec{\mathbf{F}}(x,y) = \langle 2xy^2, 2x^2y \rangle$ be a force field. Suppose that a particle begins its motion at the origin and ends its movement at any point in a plane that is not on the x-axis or the y-axis. Furthermore, the particle's motion can be modeled with a smooth parameterization. Show that $\vec{\mathbf{F}}$ does positive work on the particle.

Solution

We show that $\vec{\mathbf{F}}$ does positive work on the particle by showing that $\vec{\mathbf{F}}$ is conservative and then by using the Fundamental Theorem for Line Integrals.

To show that $\vec{\mathbf{F}}$ is conservative, suppose $f(x,y)$ were a potential function for $\vec{\mathbf{F}}$. Then, $\vec{\nabla} f(x,y) = \vec{\mathbf{F}}(x,y) = \langle 2xy^2, 2x^2y \rangle$ and therefore $f_x(x,y) = 2xy^2$ and $f_y(x,y) = 2x^2y$. The equation $fx(x,y) = 2xy^2$ implies that $f(x,y) = x^2y^2 + h(y)$. Deriving both sides with respect to y yields $f_y(x,y) = 2x^2y + h'(y)$. Therefore, $h'(y) = 0$ and we can take $h(y) = 0$.

If $f(x,y) = x^2y^2$, then note that $\vec{\nabla} f(x,y) = \langle 2xy^2, 2x^2y \rangle = \vec{\mathbf{F}}$, and therefore f is a potential function for $\vec{\mathbf{F}}$.

Let (a,b) be the point at which the particle stops is motion, and let C denote the curve that models the particle's motion. The work done by $\vec{\mathbf{F}}$ on the particle is $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$. By the Fundamental Theorem for Line Integrals,

$$\begin{aligned} \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_C \nabla f \cdot d\vec{\mathbf{r}} \\ &= f(a,b) - f(0,0) \\ &= a^2b^2. \end{aligned}$$

Since $a \neq 0$ and $b \neq 0$, by assumption, $a^2b^2 > 0$. Therefore, $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} > 0$, and $\vec{\mathbf{F}}$ does positive work on the particle.

Analysis

Notice that this problem would be much more difficult without using the Fundamental Theorem for Line Integrals. To apply the tools we have learned, we would need to give a curve parameterization and use the method from the previous section. Since the path of motion C can be as exotic as we wish (as long as it is smooth), it can be very difficult to parameterize the motion of the particle.

Exercise 16.3.10

Let $\vec{\mathbf{F}}(x,y) = \langle 4x^3y^4, 4x^4y^3 \rangle$, and suppose that a particle moves from point $(4,4)$ to $(1,1)$ along any smooth curve. Is the work done by $\vec{\mathbf{F}}$ on the particle positive, negative, or zero?

Hint

Use the Fundamental Theorem for Line Integrals.

Answer

Negative

Key Concepts

- The theorems in this section require curves that are closed, simple, or both, and regions that are connected or simply connected.
- The line integral of a conservative vector field can be calculated using the Fundamental Theorem for Line Integrals. This theorem is a generalization of the Fundamental Theorem of Calculus in higher dimensions. Using this theorem usually makes the calculation of the line integral easier.
- Conservative fields are independent of path. The line integral of a conservative field depends only on the value of the potential function at the endpoints of the domain curve.
- Given vector field $\vec{\mathbf{F}}$, we can test whether $\vec{\mathbf{F}}$ is conservative by using the cross-partial property. If $\vec{\mathbf{F}}$ has the cross-partial property and the domain is simply connected, then $\vec{\mathbf{F}}$ is conservative (and thus has a potential function). If $\vec{\mathbf{F}}$ is conservative, we can find a potential function by using the Problem-Solving Strategy.
- The circulation of a conservative vector field on a simply connected domain over a closed curve is zero.

Key Equations

- **Fundamental Theorem for Line Integrals**
  $\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a))$
- **Circulation of a conservative field over curve *C* that encloses a simply connected region**
  $\oint_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = 0$

Glossary

closed curve
a curve that begins and ends at the same point

connected region
a region in which any two points can be connected by a path with a trace contained entirely inside the region

Fundamental Theorem for Line Integrals
the value of line integral $\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}}$ depends only on the value of f at the endpoints of C: $\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a))$

independence of path
a vector field $\vec{\mathbf{F}}$ has path independence if $\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ for any curves C_1 and C_2 in the domain of $\vec{\mathbf{F}}$ with the same initial points and terminal points

simple curve
a curve that does not cross itself

simply connected region
a region that is connected and has the property that any closed curve that lies entirely inside the region encompasses points that are entirely inside the region

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16.3E: Exercises for Section 16.3

1. *True* or *False?* If vector field $\vec{\mathbf{F}}$ is conservative on the open and connected region D, then line integrals of $\vec{\mathbf{F}}$ are path independent on D, regardless of the shape of D.

2. *True* or *False?* Function $\vec{\mathbf{r}}(t) = \vec{\mathbf{a}} + t(\vec{\mathbf{b}} - \vec{\mathbf{a}})$, where $0 \le t \le 1$, parameterizes the straight-line segment from $\vec{\mathbf{a}}$ to $\vec{\mathbf{b}}$.

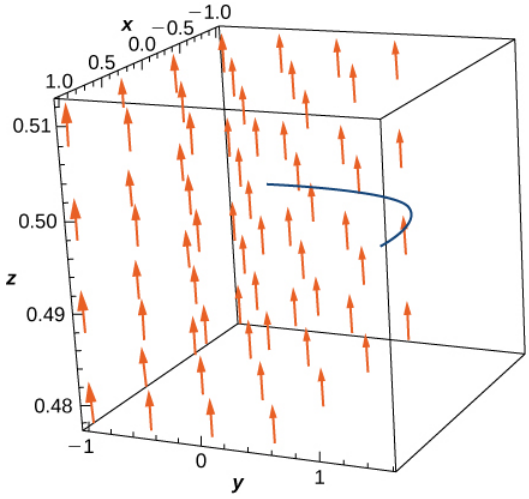
3. *True* or *False?* Vector field $\vec{\mathbf{F}}(x, y, z) = (y \sin z)\,\hat{\mathbf{i}} + (x \sin z)\,\hat{\mathbf{j}} + (xy \cos z)\,\hat{\mathbf{k}}$ is conservative.

4. *True* or *False?* Vector field $\vec{\mathbf{F}}(x, y, z) = y\,\hat{\mathbf{i}} + (x + z)\,\hat{\mathbf{j}} - y\hat{\mathbf{k}}$ is conservative.

5. Justify the Fundamental Theorem of Line Integrals for $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ in the case when $\vec{\mathbf{F}}(x, y) = (2x + 2y)\,\hat{\mathbf{i}} + (2x + 2y)\,\hat{\mathbf{j}}$ and C is a portion of the positively oriented circle $x^2 + y^2 = 25$ from $(5, 0)$ to $(3, 4)$.

6. [T] Find $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x, y) = (ye^{xy} + \cos x)\,\hat{\mathbf{i}} + \left(xe^{xy} + \frac{1}{y^2+1}\right)\hat{\mathbf{j}}$ and C is a portion of curve $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{2}$.

7. [T] Evaluate line integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x, y) = (e^x \sin y - y)\,\hat{\mathbf{i}} + (e^x \cos y - x - 2)\,\hat{\mathbf{j}}$, and C is the path given by $\vec{\mathbf{r}}(t) = (t^3 \sin \frac{\pi t}{2})\,\hat{\mathbf{i}} - (\frac{\pi}{2}\cos(\frac{\pi t}{2} + \frac{\pi}{2}))\,\hat{\mathbf{j}}$ for $0 \le t \le 1$.



For the following exercises, determine whether the vector field is conservative and, if it is, find the potential function.

8. $\vec{\mathbf{F}}(x, y) = 2xy^3\,\hat{\mathbf{i}} + 3y^2x^2\,\hat{\mathbf{j}}$

9. $\vec{\mathbf{F}}(x, y) = (-y + e^x \sin y)\,\hat{\mathbf{i}} + ((x + 2)e^x \cos y)\,\hat{\mathbf{j}}$

10. $\vec{\mathbf{F}}(x, y) = (e^{2x} \sin y)\,\hat{\mathbf{i}} + (e^{2x} \cos y)\,\hat{\mathbf{j}}$

11. $\vec{\mathbf{F}}(x, y) = (6x + 5y)\,\hat{\mathbf{i}} + (5x + 4y)\,\hat{\mathbf{j}}$

12. $\vec{\mathbf{F}}(x, y) = (2x \cos(y) - y \cos(x))\,\hat{\mathbf{i}} + (-x^2 \sin(y) - \sin(x))\,\hat{\mathbf{j}}$

13. $\vec{\mathbf{F}}(x, y) = (ye^x + \sin(y))\,\hat{\mathbf{i}} + (e^x + x \cos(y))\,\hat{\mathbf{j}}$

For the following exercises, evaluate the line integrals using the Fundamental Theorem of Line Integrals.

14. $\oint_C (y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}) \cdot d\vec{\mathbf{r}}$, where C is any path from $(0, 0)$ to $(2, 4)$

15. $\oint_C (2y\,dx + 2x\,dy)$, where C is the line segment from $(0, 0)$ to $(4, 4)$

16. [T] $\oint_C \left[\arctan\frac{y}{x} - \frac{xy}{x^2 + y^2}\right] dx + \left[\frac{x^2}{x^2 + y^2} + e^{-y}(1 - y)\right] dy$, where C is any smooth curve from $(1, 1)$ to $(-1, 2)$.

17. Find the conservative vector field for the potential function $f(x, y) = 5x^2 + 3xy + 10y^2$.

For the following exercises, determine whether the vector field is conservative and, if so, find a potential function.

18. $\vec{\mathbf{F}}(x,y) = (12xy)\,\hat{\mathbf{i}} + 6(x^2+y^2)\,\hat{\mathbf{j}}$

19. $\vec{\mathbf{F}}(x,y) = (e^x \cos y)\,\hat{\mathbf{i}} + 6(e^x \sin y)\,\hat{\mathbf{j}}$

20. $\vec{\mathbf{F}}(x,y) = (2xye^{x^2y})\,\hat{\mathbf{i}} + 6(x^2e^{x^2y})\,\hat{\mathbf{j}}$

21. $\vec{\mathbf{F}}(x,y,z) = (ye^z)\,\hat{\mathbf{i}} + (xe^z)\,\hat{\mathbf{j}} + (xye^z)\,\hat{\mathbf{k}}$

22. $\vec{\mathbf{F}}(x,y,z) = (\sin y)\,\hat{\mathbf{i}} - (x\cos y)\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$

23. $\vec{\mathbf{F}}(x,y,z) = \dfrac{1}{y}\,\hat{\mathbf{i}} - \dfrac{x}{y^2}\,\hat{\mathbf{j}} + (2z-1)\,\hat{\mathbf{k}}$

24. $\vec{\mathbf{F}}(x,y,z) = 3z^2\,\hat{\mathbf{i}} - \cos y\,\hat{\mathbf{j}} + 2xz\,\hat{\mathbf{k}}$

25. $\vec{\mathbf{F}}(x,y,z) = (2xy)\,\hat{\mathbf{i}} + (x^2+2yz)\,\hat{\mathbf{j}} + y^2\,\hat{\mathbf{k}}$

For the following exercises, determine whether the given vector field is conservative and find a potential function.

26. $\vec{\mathbf{F}}(x,y) = (e^x \cos y)\,\hat{\mathbf{i}} + 6(e^x \sin y)\,\hat{\mathbf{j}}$

27. $\vec{\mathbf{F}}(x,y) = (2xye^{x^2y})\,\hat{\mathbf{i}} + (x^2e^{x^2y})\,\hat{\mathbf{j}}$

For the following exercises, evaluate the integral using the Fundamental Theorem of Line Integrals.

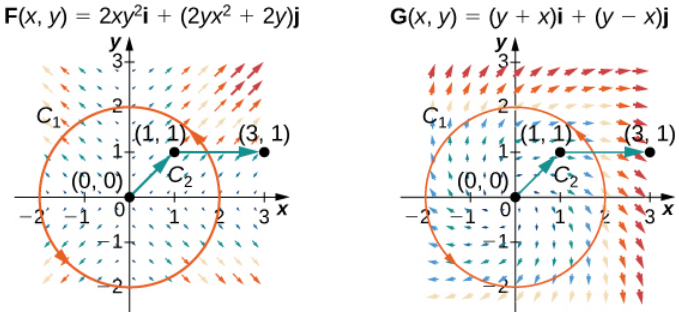
28. Evaluate $\displaystyle\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}}$, where $f(x,y,z) = \cos(\pi x) + \sin(\pi y) - xyz$ and C is any path that starts at $(1,12,2)$ and ends at $(2,1,-1)$.

29. [T] Evaluate $\displaystyle\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}}$, where $f(x,y) = xy + e^x$ and C is a straight line from $(0,0)$ to $(2,1)$.

30. [T] Evaluate $\displaystyle\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}}$, where $f(x,y) = x^2y - x$ and C is any path in a plane from (1, 2) to (3, 2).

31. Evaluate $\displaystyle\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}}$, where $f(x,y,z) = xyz^2 - yz$ and C has initial point $(1,2,3)$ and terminal point $(3,5,2)$.

For the following exercises, let $\vec{\mathbf{F}}(x,y) = 2xy^2\,\hat{\mathbf{i}} + (2yx^2+2y)\,\hat{\mathbf{j}}$ and $G(x,y) = (y+x)\,\hat{\mathbf{i}} + (y-x)\,\hat{\mathbf{j}}$, and let C_1 be the curve consisting of the circle of radius 2, centered at the origin and oriented counterclockwise, and C_2 be the curve consisting of a line segment from $(0,0)$ to $(1,1)$ followed by a line segment from $(1,1)$ to $(3,1)$.



32. Calculate the line integral of $\vec{\mathbf{F}}$ over C_1.

33. Calculate the line integral of $\vec{\mathbf{G}}$ over C_1.

34. Calculate the line integral of $\vec{\mathbf{F}}$ over C_2.

35. Calculate the line integral of $\vec{\mathbf{G}}$ over C_2.

36. [T] Let $\vec{\mathbf{F}}(x,y,z) = x^2\,\hat{\mathbf{i}} + z\sin(yz)\,\hat{\mathbf{j}} + y\sin(yz)\,\hat{\mathbf{k}}$. Calculate $\displaystyle\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where C is a path from $A = (0,0,1)$ to $B = (3,1,2)$.

37. [T] Find line integral $\displaystyle\oint_C \vec{\mathbf{F}} \cdot dr$ of vector field $\vec{\mathbf{F}}(x,y,z) = 3x^2z\,\hat{\mathbf{i}} + z^2\,\hat{\mathbf{j}} + (x^3+2yz)\,\hat{\mathbf{k}}$ along curve C parameterized by $\vec{\mathbf{r}}(t) = \left(\frac{\ln t}{\ln 2}\right)\hat{\mathbf{i}} + t^{3/2}\,\hat{\mathbf{j}} + t\cos(\pi t), 1 \le t \le 4$.

For exercises 38 - 40, show that the following vector fields are conservative. Then calculate $\displaystyle\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ for the given curve.

38. $\vec{\mathbf{F}}(x,y) = (xy^2 + 3x^2y)\,\hat{\mathbf{i}} + (x+y)x^2\,\hat{\mathbf{j}}$; C is the curve consisting of line segments from $(1,1)$ to $(0,2)$ to $(3,0)$.

39. $\vec{\mathbf{F}}(x,y) = \dfrac{2x}{y^2+1}\,\hat{\mathbf{i}} - \dfrac{2y(x^2+1)}{(y^2+1)^2}\,\hat{\mathbf{j}}$; C is parameterized by $x = t^3 - 1,\ y = t^6 - t$, for $0 \le t \le 1$.

40. [T] $\vec{\mathbf{F}}(x,y) = [\cos(xy^2) - xy^2\sin(xy^2)]\,\hat{\mathbf{i}} - 2x^2y\sin(xy^2)\,\hat{\mathbf{j}}$; C is the curve $\langle e^t, e^{t+1}\rangle$, for $-1 \le t \le 0$.

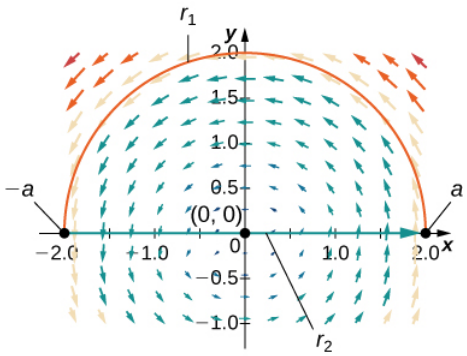
41. The mass of Earth is approximately $6 \times 10^{27} g$ and that of the Sun is 330,000 times as much. The gravitational constant is $6.7 \times 10^{-8} cm^3/s^2 \cdot g$. The distance of Earth from the Sun is about $1.5 \times 10^{12} cm$. Compute, approximately, the work necessary to increase the distance of Earth from the Sun by $1\ cm$.

42. [T] Let $\vec{\mathbf{F}}(x,y,z) = (e^x \sin y)\,\hat{\mathbf{i}} + (e^x \cos y)\,\hat{\mathbf{j}} + z^2\,\hat{\mathbf{k}}$. Evaluate the integral $\displaystyle\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{r}}(t) = \langle \sqrt{t}, t^3, e^{\sqrt{t}}\rangle$, for $0 \le t \le 1$.

43. [T] Let $C : [1,2] \to \mathbb{R}^2$ be given by $x = e^{t-1}, y = \sin\left(\frac{\pi}{t}\right)$. Use a computer to compute the integral $\displaystyle\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C 2x\cos y\,dx - x^2 \sin y\,dy$, where $\vec{\mathbf{F}}(x,y) = (2x\cos y)\,\hat{\mathbf{i}} - (x^2\sin y)\,\hat{\mathbf{j}}$.

44. [T] Use a computer algebra system to find the mass of a wire that lies along the curve $\vec{\mathbf{r}}(t) = (t^2-1)\,\hat{\mathbf{j}} + 2t\,\hat{\mathbf{k}}$, where $0 \le t \le 1$, if the density is given by $d(t) = \dfrac{3}{2}t$.

45. Find the circulation and flux of field $\vec{\mathbf{F}}(x,y) = -y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$ around and across the closed semicircular path that consists of semicircular arch $\vec{\mathbf{r}}_1(t) = (a\cos t)\,\hat{\mathbf{i}} + (a\sin t)\,\hat{\mathbf{j}},\quad 0 \le t \le \pi$, followed by line segment $\vec{\mathbf{r}}_2(t) = t\,\hat{\mathbf{i}},\quad -a \le t \le a$.



46. Compute $\displaystyle\int_C \cos x\cos y\,dx - \sin x \sin y\,dy$, where $\vec{\mathbf{r}}(t) = \langle t, t^2\rangle,\quad 0 \le t \le 1$.

47. Complete the proof of the theorem titled THE PATH INDEPENDENCE TEST FOR CONSERVATIVE FIELDS by showing that $f_y = Q(x,y)$.

Contributors

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16.4: Green's Theorem

Learning Objectives

- Apply the circulation form of Green's theorem.
- Apply the flux form of Green's theorem.
- Calculate circulation and flux on more general regions.

In this section, we examine Green's theorem, which is an extension of the Fundamental Theorem of Calculus to two dimensions. Green's theorem has two forms: a circulation form and a flux form, both of which require region D in the double integral to be simply connected. However, we will extend Green's theorem to regions that are not simply connected.

Put simply, Green's theorem relates a line integral around a simply closed plane curve C and a double integral over the region enclosed by C. The theorem is useful because it allows us to translate difficult line integrals into more simple double integrals, or difficult double integrals into more simple line integrals.

Extending the Fundamental Theorem of Calculus

Recall that the Fundamental Theorem of Calculus says that

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

As a geometric statement, this equation says that the integral over the region below the graph of $F'(x)$ and above the line segment $[a, b]$ depends only on the value of F at the endpoints a and b of that segment. Since the numbers a and b are the boundary of the line segment $[a, b]$, the theorem says we can calculate integral $\int_a^b F'(x)\,dx$ based on information about the boundary of line segment $[a, b]$ (Figure 16.4.1). The same idea is true of the Fundamental Theorem for Line Integrals:

$$\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a)).$$

When we have a potential function (an "antiderivative"), we can calculate the line integral based solely on information about the boundary of curve C.

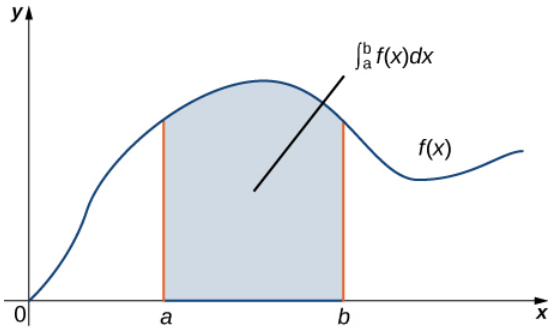


Figure 16.4.1: The Fundamental Theorem of Calculus says that the integral over line segment $[a, b]$ depends only on the values of the antiderivative at the endpoints of $[a, b]$.

Green's theorem takes this idea and extends it to calculating double integrals. Green's theorem says that we can calculate a double integral over region D based solely on information about the boundary of D. Green's theorem also says we can calculate a line integral over a simple closed curve C based solely on information about the region that C encloses. In particular, Green's theorem connects a double integral over region D to a line integral around the boundary of D.

Circulation Form of Green's Theorem

The first form of Green's theorem that we examine is the circulation form. This form of the theorem relates the vector line integral over a simple, closed plane curve C to a double integral over the region enclosed by C. Therefore, the circulation of a vector field along a simple closed curve can be transformed into a double integral and vice versa.

GREEN'S THEOREM (CIRCULATION FORM)

Let D be an open, simply connected region with a boundary curve C that is a piecewise smooth, simple closed curve oriented counterclockwise (Figure 16.4.2). Let $\vec{\mathbf{F}} = \langle P, Q \rangle$ be a vector field with component functions that have continuous partial derivatives on D. Then,

$$\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \oint_C P\,dx + Q\,dy \tag{16.4.1}$$

$$= \iint_D (Q_x - P_y)\,dA. \tag{16.4.2}$$

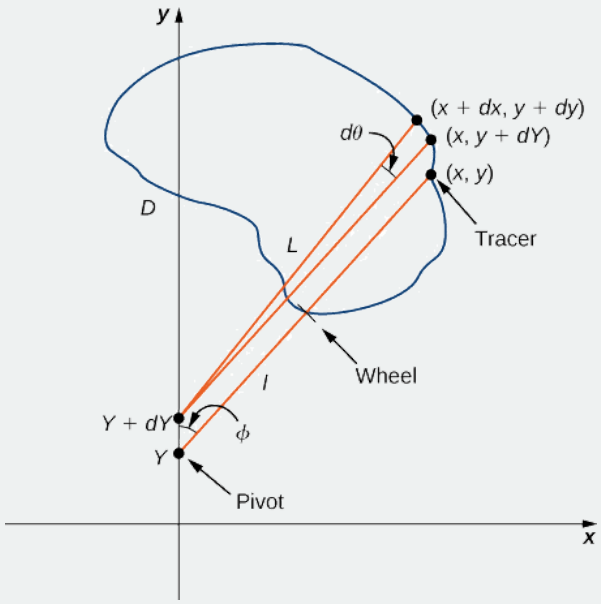


Figure 16.4.2: The circulation form of Green's theorem relates a line integral over curve C to a double integral over region D.

Notice that Green's theorem can be used only for a two-dimensional vector field $\vec{\mathbf{F}}$. If $\vec{\mathbf{F}}$ is a three-dimensional field, then Green's theorem does not apply. Since

$$\int_C P\,dx + Q\,dy = \int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\,ds$$

this version of Green's theorem is sometimes referred to as the ***tangential form* of Green's theorem.**

The proof of Green's theorem is rather technical, and beyond the scope of this text. Here we examine a proof of the theorem in the special case that D is a rectangle. For now, notice that we can quickly confirm that the theorem is true for the special case in which $\vec{\mathbf{F}} = \langle P, Q \rangle$ is conservative. In this case,

$$\oint_C P\,dx + Q\,dy = 0$$

because the circulation is zero in conservative vector fields. $\vec{\mathbf{F}}$ satisfies the cross-partial condition, so $P_y = Q_x$. Therefore,

$$\iint_D (Q_x - P_y)\,dA = \int_D 0\,dA = 0 = \oint_C P\,dx + Q\,dy$$

which confirms Green's theorem in the case of **conservative vector fields**.

Proof

Let's now prove that the circulation form of Green's theorem is true when the region D is a rectangle. Let D be the rectangle $[a, b] \times [c, d]$ oriented counterclockwise. Then, the boundary C of D consists of four piecewise smooth pieces C_1, C_2, C_3, and C_4 (Figure 16.4.3). We parameterize each side of D as follows:

$$C_1 : \vec{\mathbf{r}}_1(t) = \langle t, c \rangle,\ a \le t \le b$$

$$C_2 : \vec{\mathbf{r}}_2(t) = \langle b, t \rangle,\ c \le t \le d$$

$$-C_3 : \vec{\mathbf{r}}_3(t) = \langle t, d \rangle,\ a \le t \le b$$

$$-C_4 : \vec{\mathbf{r}}_4(t) = \langle a, t \rangle,\ c \le t \le d.$$

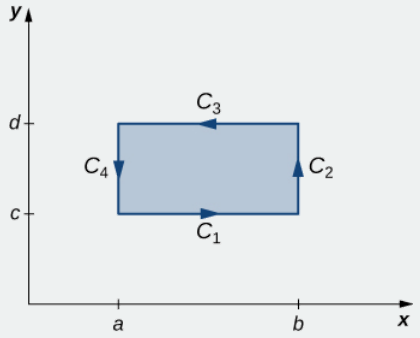


Figure 16.4.3: Rectangle D is oriented counterclockwise.

Then,

$$\begin{aligned}
\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{C_3} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{C_4} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \\
&= \int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} - \int_{-C_3} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} - \int_{-C_4} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \\
&= \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}_1(t)) \cdot \vec{\mathbf{r}}_1'(t)\, dt + \int_c^d \vec{\mathbf{F}}(\vec{\mathbf{r}}_2(t)) \cdot \vec{\mathbf{r}}_2'(t)\, dt \\
&\quad - \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}_3(t)) \cdot \vec{\mathbf{r}}_3'(t)\, dt - \int_c^d \vec{\mathbf{F}}(\vec{\mathbf{r}}_4(t)) \cdot \vec{\mathbf{r}}_4'(t)\, dt \\
&= \int_a^b P(t, c)\, dt + \int_c^d Q(b, t)\, dt - \int_a^b P(t, d)\, dt - \int_c^d Q(a, t)\, dt \\
&= \int_a^b (P(t, c) - P(t, d))\, dt + \int_c^d (Q(b, t) - Q(a, t))\, dt \\
&= -\int_a^b (P(t, d) - P(t, c))\, dt + \int_c^d (Q(b, t) - Q(a, t))\, dt.
\end{aligned}$$

By the Fundamental Theorem of Calculus,

$$P(t, d) - P(t, c) = \int_c^d \frac{\partial}{\partial y} P(t, y) dy$$

and

$$Q(b, t) - Q(a, t) = \int_a^b \frac{\partial}{\partial x} Q(x, t)\, dx.$$

Therefore,

$$\begin{aligned}
&-\int_a^b (P(t, d) - P(t, c))\, dt + \int_c^d (Q(b, t) - Q(a, t))\, dt \\
&= -\int_a^b \int_c^d \frac{\partial}{\partial y} P(t, y)\, dy\, dt + \int_c^d \int_a^b \frac{\partial}{\partial x} Q(x, t)\, dx\, dt.
\end{aligned}$$

But,

$$-\int_a^b \int_c^d \frac{\partial}{\partial y} P(t,y)\,dy\,dt + \int_c^d \int_a^b \frac{\partial}{\partial x} Q(x,t)\,dx\,dt = -\int_a^b \int_c^d \frac{\partial}{\partial y} P(x,y)\,dy\,dx + \int_c^d \int_a^b \frac{\partial}{\partial x} Q(x,y)\,dx\,dy$$
$$= \int_a^b \int_c^d (Q_x - P_y)\,dy\,dx$$
$$= \iint_D (Q_x - P_y)\,dA.$$

Therefore, $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_D (Q_x - P_y)\,dA$ and we have proved Green's theorem in the case of a rectangle.

□

To prove Green's theorem over a general region D, we can decompose D into many tiny rectangles and use the proof that the theorem works over rectangles. The details are technical, however, and beyond the scope of this text.

Example 16.4.1: Applying Green's Theorem over a Rectangle

Calculate the line integral

$$\oint_C x^2 y\,dx + (y-3)\,dy,$$

where C is a rectangle with vertices $(1,1)$, $(4,1)$, $(4,5)$, and $(1,5)$ oriented counterclockwise.

Solution

Let $\vec{\mathbf{F}}(x,y) = \langle P(x,y), Q(x,y)\rangle = \langle x^2 y, y-3\rangle$. Then, $Q_x(x,y) = 0$ and $P_y(x,y) = x^2$. Therefore, $Q_x - P_y = -x^2$.

Let D be the rectangular region enclosed by C (Figure 16.4.4). By Green's theorem,

$$\oint_C x^2 y\,dx + (y-3)\,dy = \iint_D (Q_x - P_y)\,dA$$
$$= \iint_D -x^2\,dA = \int_1^5 \int_1^4 -x^2\,dx\,dy$$
$$= \int_1^5 -\frac{x^3}{3}\Big|_1^4\,dy$$
$$= \int_1^5 \left(-\frac{64}{3} + \frac{1}{3}\right)\,dy$$
$$= \int_1^5 -21\,dy = -84.$$

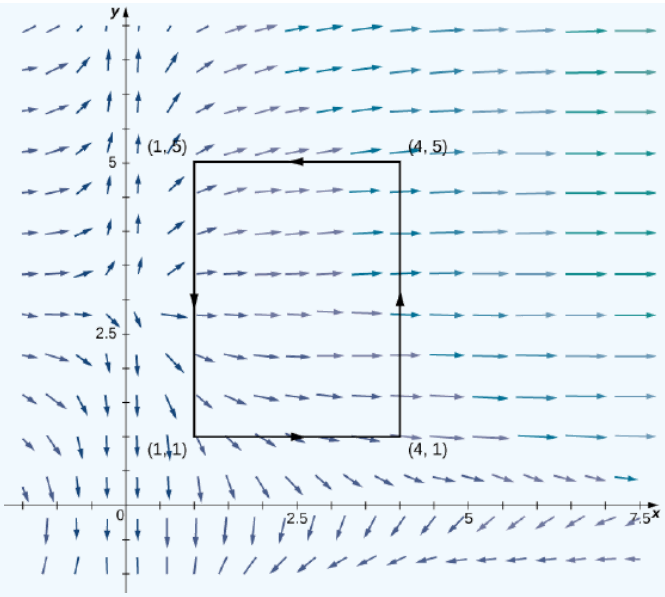


Figure 16.4.4: The line integral over the boundary of the rectangle can be transformed into a double integral over the rectangle.

Analysis

If we were to evaluate this line integral without using Green's theorem, we would need to parameterize each side of the rectangle, break the line integral into four separate line integrals, and use the methods from the section titled **Line Integrals** to evaluate each integral. Furthermore, since the vector field here is not conservative, we cannot apply the Fundamental Theorem for Line Integrals. Green's theorem makes the calculation much simpler.

Example 16.4.2: Applying Green's Theorem to Calculate Work

Calculate the work done on a particle by force field

$$\vec{\mathbf{F}}(x,y) = \langle y + \sin x, e^y - x \rangle$$

as the particle traverses circle $x^2 + y^2 = 4$ exactly once in the counterclockwise direction, starting and ending at point $(2, 0)$.

Solution

Let C denote the circle and let D be the disk enclosed by C. The work done on the particle is

$$W = \oint_C (y + \sin x)\, dx + (e^y - x)\, dy.$$

As with Example 16.4.1, this integral can be calculated using tools we have learned, but it is easier to use the double integral given by Green's theorem (Figure 16.4.5).

Let $\vec{\mathbf{F}}(x,y) = \langle P(x,y), Q(x,y) \rangle = \langle y + \sin x, e^y - x \rangle$. Then, $Q_x = -1$ and $P_y = 1$. Therefore, $Q_x - P_y = -2$.

By Green's theorem,

$$\begin{aligned} W &= \oint_C (y + \sin(x))dx + (e^y - x)\, dy \\ &= \iint_D (Q_x - P_y)\, dA \\ &= \iint_D -2\, dA \\ &= -2(area(D)) = -2\pi(2^2) = -8\pi. \end{aligned}$$

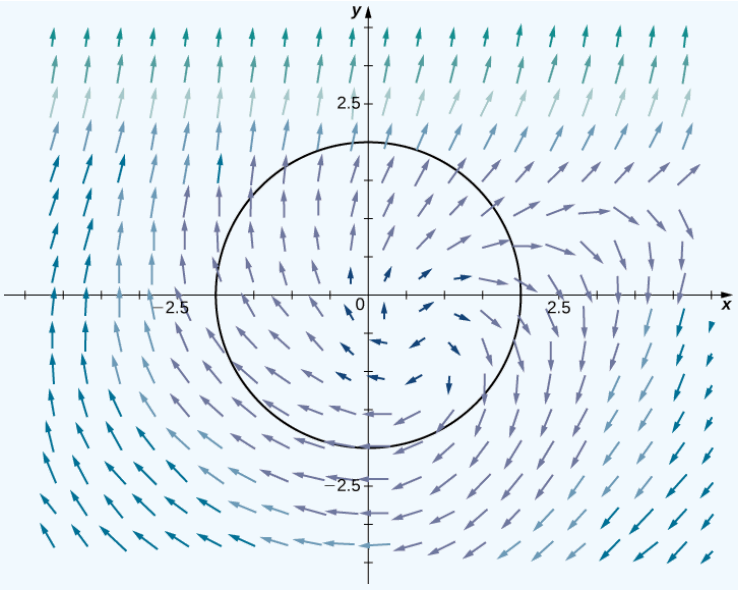


Figure 16.4.5: The line integral over the boundary circle can be transformed into a double integral over the disk enclosed by the circle.

Exercise 16.4.2

Use Green's theorem to calculate line integral

$$\oint_C \sin(x^2)\,dx + (3x - y)\,dy.$$

where C is a right triangle with vertices $(-1, 2)$, $(4, 2)$, and $(4, 5)$ oriented counterclockwise.

Hint

Transform the line integral into a double integral.

Answer

$\frac{45}{2}$

In the preceding two examples, the double integral in Green's theorem was easier to calculate than the line integral, so we used the theorem to calculate the line integral. In the next example, the double integral is more difficult to calculate than the line integral, so we use Green's theorem to translate a double integral into a line integral.

Example 16.4.3: Applying Green's Theorem over an Ellipse

Calculate the area enclosed by ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (Figure 16.4.6).

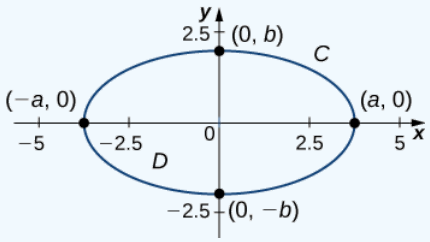


Figure 16.4.6: Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is denoted by C.

Solution

Let C denote the ellipse and let D be the region enclosed by C. Recall that ellipse C can be parameterized by

- $x = a\cos t$,
- $y = b\sin t$,
- $0 \le t \le 2\pi$.

Calculating the area of D is equivalent to computing double integral $\iint_D dA$. To calculate this integral without Green's theorem, we would need to divide D into two regions: the region above the x-axis and the region below. The area of the ellipse is

$$\int_{-a}^{a}\int_{0}^{\sqrt{b^2-(bx/a)^2}} dy\,dx + \int_{-a}^{a}\int_{-\sqrt{b^2-(bx/a)^2}}^{0} dy\,dx.$$

These two integrals are not straightforward to calculate (although when we know the value of the first integral, we know the value of the second by symmetry). Instead of trying to calculate them, we use Green's theorem to transform $\iint_D dA$ into a line integral around the boundary C.

Consider vector field

$$\vec{\mathbf{F}}(x,y) = \langle P, Q\rangle = \langle -\frac{y}{2}, \frac{x}{2}\rangle.$$

Then, $Q_x = \frac{1}{2}$ and $P_y = -\frac{1}{2}$, and therefore $Q_x - P_y = 1$. Notice that $\vec{\mathbf{F}}$ was chosen to have the property that $Q_x - P_y = 1$. Since this is the case, Green's theorem transforms the line integral of $\vec{\mathbf{F}}$ over C into the double integral of 1 over D.

By Green's theorem,

$$\begin{aligned}\iint_D dA &= \iint_D (Q_x - P_y)\,dA \\ &= \int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = \frac{1}{2}\int_C -y\,dx + x\,dy \\ &= \frac{1}{2}\int_0^{2\pi} -b\sin t(-a\sin t) + a(\cos t)b\cos t\,dt \\ &= \frac{1}{2}\int_0^{2\pi} ab\cos^2 t + ab\sin^2 t\,dt \\ &= \frac{1}{2}\int_0^{2\pi} ab\,dt = \pi ab.\end{aligned}$$

Therefore, the area of the ellipse is πab units^2.

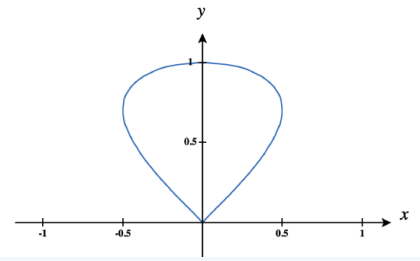
In Example 16.4.3, we used vector field $\vec{\mathbf{F}}(x,y) = \langle P, Q\rangle = \langle -\frac{y}{2}, \frac{x}{2}\rangle$ to find the area of any ellipse. The logic of the previous example can be extended to derive a formula for the area of any region D. Let D be any region with a boundary that is a simple closed curve C oriented counterclockwise. If $\vec{\mathbf{F}}(x,y) = \langle P, Q\rangle = \langle -\frac{y}{2}, \frac{x}{2}\rangle$, then $Q_x - P_y = 1$. Therefore, by the same logic as in Example 16.4.3,

$$\text{area of } D = \iint_D dA = \frac{1}{2}\oint_C -y\,dx + x\,dy. \tag{16.4.3}$$

It's worth noting that if $\vec{\mathbf{F}} = \langle P, Q\rangle$ is any vector field with $Q_x - P_y = 1$, then the logic of the previous paragraph works. So. Equation 16.4.3 is not the only equation that uses a vector field's mixed partials to get the area of a region.

? Exercise 16.4.3

Find the area of the region enclosed by the curve with parameterization $\vec{\mathbf{r}}(t) = \langle \sin t\cos t, \sin t\rangle$, $0 \le t \le \pi$.



Hint

Use Equation 16.4.3.

Answer

$\frac{2}{3}$ units^2.

Solution

For convenience, let's use the same vector field we used in Example 16.4.3, where $Q_x - P_y = 1$.

$$\vec{\mathbf{F}}(x,y) = \langle P, Q\rangle = \langle -\frac{y}{2}, \frac{x}{2}\rangle.$$

For the curve $r(t) = \langle \sin t \cos t, \sin t\rangle$ with $0 \le t \le \pi$, we have

$$\vec{\mathbf{r}}'(t) = \langle \cos^2 t - \sin^2 t, \cos t\rangle \text{ and}$$
$$\vec{\mathbf{F}}(t) = \left\langle -\frac{\sin t}{2}, \frac{\sin t \cos t}{2}\right\rangle$$

By Green's theorem,

$$\begin{aligned}
\text{Area of the region } D = \iint_D dA &= \iint_D (Q_x - P_y)\, dA \\
&= \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \frac{1}{2}\int_C -y\, dx + x\, dy \\
&= \frac{1}{2}\int_0^{\pi} \Big(-\sin t(\cos^2 t - \sin^2 t) + \sin t \cos^2 t\Big)\, dt \\
&= \frac{1}{2}\int_0^{\pi} \Big(-\sin t \cos^2 t + \sin t \sin^2 t + \sin t \cos^2 t\Big)\, dt \\
&= \frac{1}{2}\int_0^{\pi} \Big(\cancel{-\sin t \cos^2 t} + \sin t(1 - \cos^2 t) + \cancel{\sin t \cos^2 t}\Big)\, dt \\
&= \frac{1}{2}\int_0^{\pi} \Big(\sin t - \cos^2 t \sin t\Big)\, dt \\
&= \frac{1}{2}\left[-\cos t + \frac{\cos^3 t}{3}\right]\Big|_0^{\pi} \\
&= \frac{1}{2}\left[-\cos \pi + \frac{\cos^3 \pi}{3} + \cos 0 - \frac{\cos^3 0}{3}\right] \\
&= \frac{1}{2}\left[1 - \frac{1}{3} + 1 - \frac{1}{3}\right] \\
&= \frac{1}{2}\left[2 - \frac{2}{3}\right] \\
&= \frac{1}{2}\left[\frac{6}{3} - \frac{2}{3}\right] \\
&= \frac{1}{2}\left[\frac{4}{3}\right] \\
&= \frac{2}{3} \text{ units}^2
\end{aligned}$$

Therefore the area of this region is $\frac{2}{3}$ units^2.

Flux Form of Green's Theorem

The circulation form of Green's theorem relates a double integral over region D to line integral $\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}} ds$, where C is the boundary of D. The flux form of Green's theorem relates a double integral over region D to the flux across boundary C. The flux of a fluid across a curve can be difficult to calculate using the flux line integral. This form of Green's theorem allows us to translate a difficult flux integral into a double integral that is often easier to calculate.

GREEN'S THEOREM (FLUX FORM)

Let D be an open, simply connected region with a boundary curve C that is a piecewise smooth, simple closed curve that is oriented counterclockwise (Figure 16.4.7). Let $\vec{\mathbf{F}} = \langle P, Q \rangle$ be a vector field with component functions that have continuous partial derivatives on an open region containing D. Then,

$$\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}} \, ds = \iint_D P_x + Q_y \, dA. \tag{16.4.4}$$

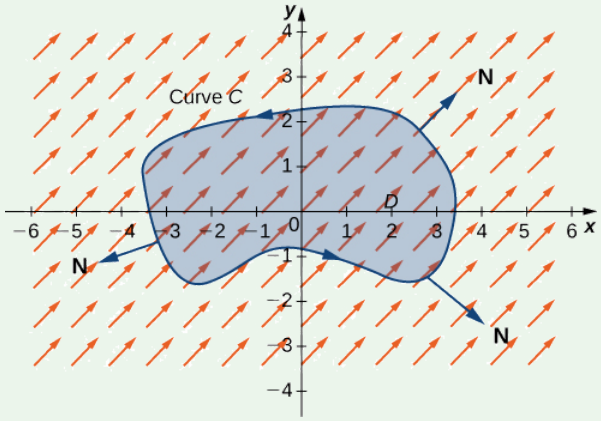


Figure 16.4.7: The flux form of Green's theorem relates a double integral over region D to the flux across curve C.

Because this form of Green's theorem contains unit normal vector $\vec{\mathbf{N}}$, it is sometimes referred to as the *normal form* of Green's theorem.

Proof

Recall that $\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}} \, ds = \oint_C -Q \, dx + P \, dy$. Let $M = -Q$ and $N = P$. By the circulation form of Green's theorem,

$$\begin{aligned} \oint_C -Q \, dx + P \, dy &= \oint_C M \, dx + N \, dy \\ &= \iint_D N_x - M_y \, dA \\ &= \iint_D P_x - (-Q)_y \, dA \\ &= \iint_D P_x + Q_y \, dA. \end{aligned}$$

□

Example 16.4.4A: Applying Green's Theorem for Flux across a Circle

Let C be a circle of radius r centered at the origin (Figure 16.4.8) and let $\vec{\mathbf{F}}(x,y) = \langle x, y\rangle$. Calculate the flux across C.

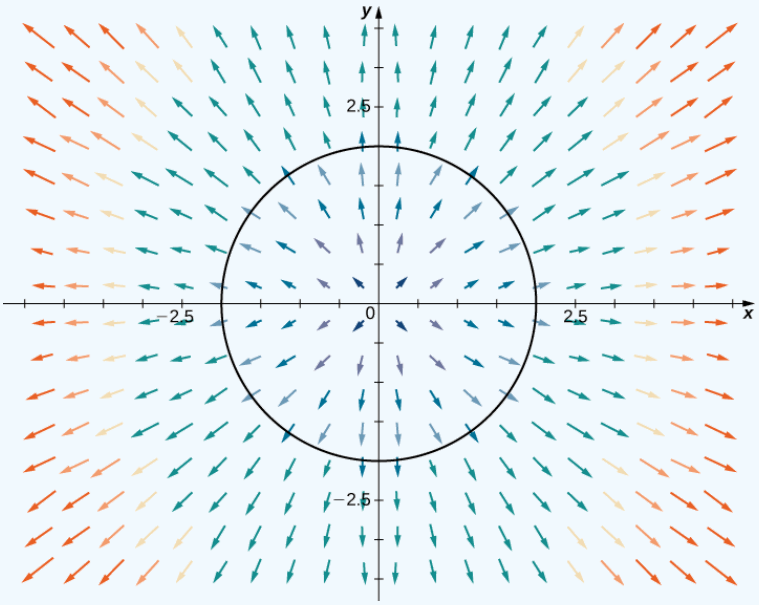


Figure 16.4.8: Curve C is a circle of radius r centered at the origin.

Solution

Let D be the disk enclosed by C. The flux across C is $\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds$. We could evaluate this integral using tools we have learned, but Green's theorem makes the calculation much more simple. Let $P(x,y) = x$ and $Q(x,y) = y$ so that $\vec{\mathbf{F}} = \langle P, Q\rangle$. Note that $P_x = 1 = Q_y$, and therefore $P_x + Q_y = 2$. By Green's theorem,

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds = \iint_D 2\, dA = 2 \iint_D dA.$$

Since $\iint_D dA$ is the area of the circle, $\iint_D dA = \pi r^2$. Therefore, the flux across C is $2\pi r^2$.

Example 16.4.4B: Applying Green's Theorem for Flux across a Triangle

Let S be the triangle with vertices $(0,0)$, $(1,0)$, and $(0,3)$ oriented clockwise (Figure 16.4.9). Calculate the flux of $\vec{\mathbf{F}}(x,y) = \langle P(x,y), Q(x,y)\rangle = \langle x^2 + e^y, x + y\rangle$ across S.

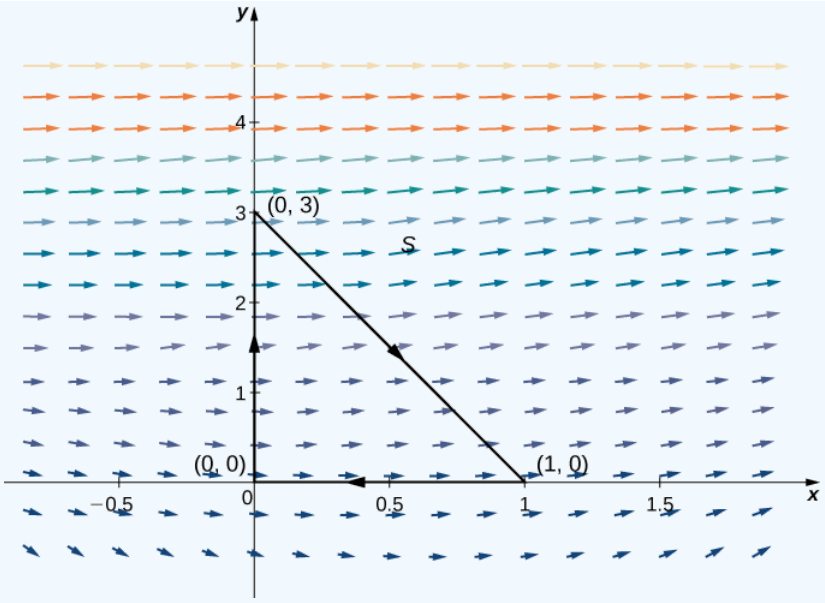


Figure 16.4.9: Curve S is a triangle with vertices $(0,0)$, $(1,0)$, and $(0,3)$ oriented clockwise.

Solution

To calculate the flux without Green's theorem, we would need to break the flux integral into three line integrals, one integral for each side of the triangle. Using Green's theorem to translate the flux line integral into a single double integral is much more simple.

Let D be the region enclosed by S. Note that $P_x = 2x$ and $Q_y = 1$; therefore, $P_x + Q_y = 2x + 1$. Green's theorem applies only to simple closed curves oriented counterclockwise, but we can still apply the theorem because $\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds = -\oint_{-S} \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds$ and $-S$ is oriented counterclockwise. By Green's theorem, the flux is

$$\begin{aligned}\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds &= \oint_{-S} \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds \\ &= -\iint_D (P_x + Q_y)\, dA \\ &= -\iint_D (2x+1)\, dA.\end{aligned}$$

Notice that the top edge of the triangle is the line $y = -3x + 3$. Therefore, in the iterated double integral, the y-values run from $y = 0$ to $y = -3x + 3$, and we have

$$\begin{aligned}-\iint_D (2x+1)\, dA &= -\int_0^1 \int_0^{-3x+3} (2x+1)\, dy\, dx \\ &= -\int_0^1 (2x+1)y\Big|_0^{-3x+3}\, dx \\ &= -\int_0^1 (2x+1)(-3x+3)\, dx \\ &= -\int_0^1 (-6x^2 + 3x + 3)\, dx \\ &= -\left[-2x^3 + \frac{3x^2}{2} + 3x\right]\Big|_0^1 \\ &= -\frac{5}{2}.\end{aligned}$$

Exercise 16.4.4

Calculate the flux of $\vec{\mathbf{F}}(x,y) = \langle x^3, y^3 \rangle$ across a unit circle oriented counterclockwise.

Hint

Apply Green's theorem and use polar coordinates.

Answer

$\frac{3\pi}{2}$

Example 16.4.5: Applying Green's Theorem for Water Flow across a Rectangle

Water flows from a spring located at the origin. The velocity of the water is modeled by vector field $\vec{\mathbf{v}}(x,y) = \langle 5x + y, x + 3y \rangle$ m/sec. Find the amount of water per second that flows across the rectangle with vertices $(-1,-2)$, $(1,-2)$, $(1,3)$,and $(-1,3)$, oriented counterclockwise (Figure 16.4.10).

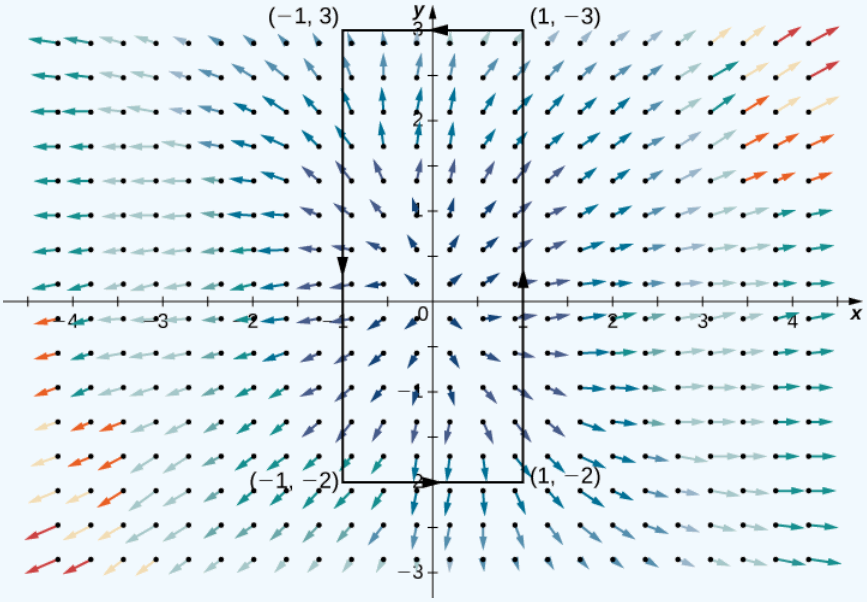


Figure 16.4.10: Water flows across the rectangle with vertices $(-1,-2)$, $(1,-2)$, $(1,3)$,and $(-1,3)$, oriented counterclockwise.

Solution

Let C represent the given rectangle and let D be the rectangular region enclosed by C. To find the amount of water flowing across C, we calculate flux $\int_C \vec{\mathbf{v}} \cdot d\vec{\mathbf{r}}$. Let $P(x,y) = 5x + y$ and $Q(x,y) = x + 3y$ so that $\vec{\mathbf{v}} = \langle P, Q \rangle$. Then, $P_x = 5$ and $Q_y = 3$. By Green's theorem,

$$\begin{aligned} \int_C \vec{\mathbf{v}} \cdot d\vec{\mathbf{r}} &= \iint_D (P_x + Q_y)\, dA \\ &= \iint_D 8\, dA \\ &= 8(area\ of\ D) = 80. \end{aligned}$$

Therefore, the water flux is 80 m^2/sec.

Recall that if vector field $\vec{\mathbf{F}}$ is conservative, then $\vec{\mathbf{F}}$ does no work around closed curves—that is, the circulation of $\vec{\mathbf{F}}$ around a closed curve is zero. In fact, if the domain of $\vec{\mathbf{F}}$ is simply connected, then $\vec{\mathbf{F}}$ is conservative if and only if the circulation of $\vec{\mathbf{F}}$ around any closed curve is zero. If we replace "circulation of $\vec{\mathbf{F}}$" with "flux of $\vec{\mathbf{F}}$," then we get a definition of a source-free vector field. The following statements are all equivalent ways of defining a source-free field $\vec{\mathbf{F}} = \langle P, Q \rangle$ on a simply connected domain (note the similarities with properties of conservative vector fields):

1. The flux $\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds$ across any closed curve C is zero.
2. If C_1 and C_2 are curves in the domain of $\vec{\mathbf{F}}$ with the same starting points and endpoints, then $\int_{C_1} \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds = \int_{C_2} \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds$. In other words, flux is independent of path.
3. There is a stream function $g(x, y)$ for $\vec{\mathbf{F}}$. A stream function for $\vec{\mathbf{F}} = \langle P, Q\rangle$ is a function g such that $P = g_y$ and $Q = -g_x$.Geometrically, $\vec{\mathbf{F}} = \langle a, b\rangle$ is tangential to the level curve of g at (a, b). Since the gradient of g is perpendicular to the level curve of g at (a, b), stream function g has the property $\vec{\mathbf{F}}(a, b) \cdot \vec{\nabla} g(a, b) = 0$ for any point (a, b) in the domain of g. (Stream functions play the same role for source-free fields that potential functions play for conservative fields.)
4. $P_x + Q_y = 0$

Example 16.4.6: Finding a Stream Function

Verify that rotation vector field $\vec{\mathbf{F}}(x, y) = \langle y, -x\rangle$ is source free, and find a stream function for $\vec{\mathbf{F}}$.

Solution

Note that the domain of $\vec{\mathbf{F}}$ is all of $\mathbb{R}^2$, which is simply connected. Therefore, to show that $\vec{\mathbf{F}}$ is source free, we can show any of items 1 through 4 from the previous list to be true. In this example, we show that item 4 is true. Let $P(x, y) = y$ and $Q(x, y) = -x$. Then $P_x = 0 = Q_y$, and therefore $P_x + Q_y = 0$. Thus, $\vec{\mathbf{F}}$ is source free.

To find a stream function for $\vec{\mathbf{F}}$, proceed in the same manner as finding a potential function for a conservative field. Let g be a stream function for $\vec{\mathbf{F}}$. Then $g_y = y$, which implies that

$$g(x, y) = \frac{y^2}{2} + h(x).$$

Since $-g_x = Q = -x$, we have $h'(x) = x$. Therefore,

$$h(x) = \frac{x^2}{2} + C.$$

Letting $C = 0$ gives stream function

$$g(x, y) = \frac{x^2}{2} + \frac{y^2}{2}.$$

To confirm that g is a stream function for $\vec{\mathbf{F}}$, note that $g_y = y = P$ and $-g_x = -x = Q$.

Notice that source-free rotation vector field $\vec{\mathbf{F}}(x, y) = \langle y, -x\rangle$ is perpendicular to conservative radial vector field $\vec{\nabla} g = \langle x, y\rangle$ (Figure 16.4.11).

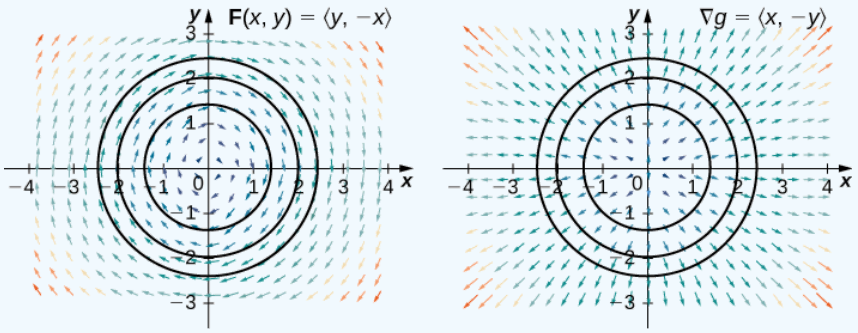


Figure 16.4.11: (a) In this image, we see the three-level curves of g and vector field $\vec{\mathbf{F}}$. Note that the $\vec{\mathbf{F}}$ vectors on a given level curve are tangent to the level curve. (b) In this image, we see the three-level curves of g and vector field $\vec{\nabla} g$. The gradient vectors are perpendicular to the corresponding level curve. Therefore, $\vec{\mathbf{F}}(a, b) \cdot \vec{\nabla} g(a, b) = 0$ for any point in the domain of g.

Exercise 16.4.6

Find a stream function for vector field $\vec{\mathbf{F}}(x, y) = \langle x \sin y, \cos y\rangle$.

Hint

Follow the outline provided in the previous example.

Answer

$g(x, y) = -x \cos y$

Vector fields that are both conservative and source free are important vector fields. One important feature of conservative and source-free vector fields on a simply connected domain is that any potential function f of such a field satisfies Laplace's equation $f_{xx} + f_{yy} = 0$. Laplace's equation is foundational in the field of partial differential equations because it models such phenomena as gravitational and magnetic potentials in space, and the velocity potential of an ideal fluid. A function that satisfies Laplace's equation is called a *harmonic* function. Therefore any potential function of a conservative and source-free vector field is harmonic.

To see that any potential function of a conservative and source-free vector field on a simply connected domain is harmonic, let f be such a potential function of vector field $\vec{\mathbf{F}} = \langle P, Q \rangle$. Then, $f_x = P$ and $f_x = Q$ because $\vec{\nabla} f = \vec{\mathbf{F}}$. Therefore, $f_{xx} = P_x$ and $f_{yy} = Q_y$. Since $\vec{\mathbf{F}}$ is source free, $f_{xx} + f_{yy} = P_x + Q_y = 0$, and we have that f is harmonic.

Example 16.4.7: Satisfying Laplace's Equation

For vector field $\vec{\mathbf{F}}(x, y) = \langle e^x \sin y, e^x \cos y \rangle$, verify that the field is both conservative and source free, find a potential function for $\vec{\mathbf{F}}$, and verify that the potential function is harmonic.

Solution

Let $P(x, y) = e^x \sin y$ and $Q(x, y) = e^x \cos y$. Notice that the domain of $\vec{\mathbf{F}}$ is all of two-space, which is simply connected. Therefore, we can check the cross-partials of $\vec{\mathbf{F}}$ to determine whether $\vec{\mathbf{F}}$ is conservative. Note that $P_y = e^x \cos y = Q_x$, so $\vec{\mathbf{F}}$ is conservative. Since $P_x = e^x \sin y$ and $Q_y = e^x \sin y$, $P_x + Q_y = 0$ and the field is source free.

To find a potential function for $\vec{\mathbf{F}}$, let f be a potential function. Then, $\vec{\nabla} f = \vec{\mathbf{F}}$, so $f_x(x, y) = e^x \sin y$. Integrating this equation with respect to x gives $f(x, y) = e^x \sin y + h(y)$. Since $f_y(x, y) = e^x \cos y$, differentiating f with respect to y gives $e^x \cos y = e^x \cos y + h'(y)$. Therefore, we can take $h(y) = 0$, and $f(x, y) = e^x \sin y$ is a potential function for f.

To verify that f is a harmonic function, note that $f_{xx}(x, y) = \dfrac{\partial}{\partial x}(e^x \sin y) = e^x \sin y$ and

$f_{yy}(x, y) = \dfrac{\partial}{\partial x}(e^x \cos y) = -e^x \sin y$. Therefore, $f_{xx} + f_{yy} = 0$, and f satisfies Laplace's equation.

Exercise 16.4.7

Is the function $f(x, y) = e^{x+5y}$ harmonic?

Hint

Determine whether the function satisfies Laplace's equation.

Answer

No

Green's Theorem on General Regions

Green's theorem, as stated, applies only to regions that are simply connected—that is, Green's theorem as stated so far cannot handle regions with holes. Here, we extend Green's theorem so that it does work on regions with finitely many holes (Figure 16.4.12).

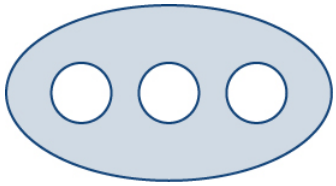


Figure 16.4.12: Green's theorem, as stated, does not apply to a nonsimply connected region with three holes like this one.

Before discussing extensions of Green's theorem, we need to go over some terminology regarding the boundary of a region. Let D be a region and let C be a component of the boundary of D. We say that C is *positively oriented* if, as we walk along C in the

direction of orientation, region D is always on our left. Therefore, the counterclockwise orientation of the boundary of a disk is a positive orientation, for example. Curve C is *negatively oriented* if, as we walk along C in the direction of orientation, region D is always on our right. The clockwise orientation of the boundary of a disk is a negative orientation, for example.

Let D be a region with finitely many holes (so that D has finitely many boundary curves), and denote the boundary of D by ∂D (Figure 16.4.13). To extend Green's theorem so it can handle D, we divide region D into two regions, D_1 and D_2 (with respective boundaries ∂D_1 and ∂D_2), in such a way that $D = D_1 \cup D_2$ and neither D_1 nor D_2 has any holes (Figure 16.4.13).

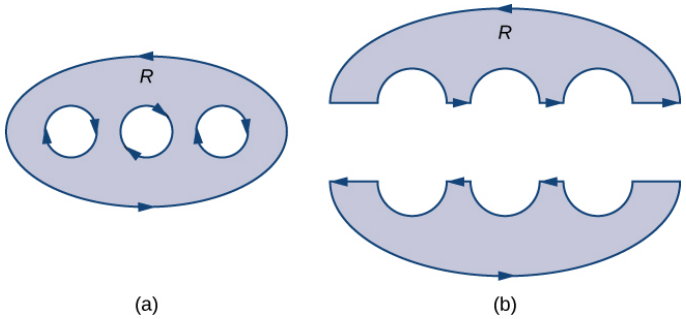


Figure 16.4.13: (a) Region D with an oriented boundary has three holes. (b) Region D split into two simply connected regions has no holes.

Assume the boundary of D is oriented as in the figure, with the inner holes given a negative orientation and the outer boundary given a positive orientation. The boundary of each simply connected region D_1 and D_2 is positively oriented. If $\vec{\mathbf{F}}$ is a vector field defined on D, then Green's theorem says that

$$\oint_{\partial D} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \oint_{\partial D_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \oint_{\partial D_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \tag{16.4.5}$$

$$= \iint_{D_1} Q_x - P_y \, dA + \iint_{D_2} Q_x - P_y \, dA \tag{16.4.6}$$

$$= \iint_{D} (Q_x - P_y) \, dA. \tag{16.4.7}$$

Therefore, Green's theorem still works on a region with holes.

To see how this works in practice, consider annulus D in Figure 16.4.14 and suppose that $F = \langle P, Q \rangle$ is a vector field defined on this annulus. Region D has a hole, so it is not simply connected. Orient the outer circle of the annulus counterclockwise and the inner circle clockwise (Figure 16.4.14) so that, when we divide the region into D_1 and D_2, we are able to keep the region on our left as we walk along a path that traverses the boundary. Let D_1 be the upper half of the annulus and D_2 be the lower half. Neither of these regions has holes, so we have divided D into two simply connected regions.

We label each piece of these new boundaries as P_i for some i, as in Figure 16.4.14. If we begin at P and travel along the oriented boundary, the first segment is P_1, then P_2, P_3, and P_4. Now we have traversed D_1 and returned to P. Next, we start at P again and traverse D_2. Since the first piece of the boundary is the same as P_4 in D_1, but oriented in the opposite direction, the first piece of D_2 is $-P_4$. Next, we have P_5, then $-P_2$, and finally P_6.

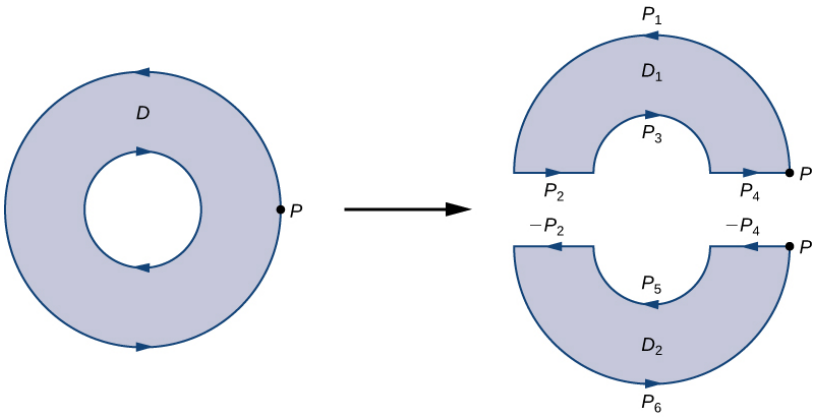


Figure 16.4.14: Breaking the annulus into two separate regions gives us two simply connected regions. The line integrals over the common boundaries cancel out.

Figure 16.4.14 shows a path that traverses the boundary of D. Notice that this path traverses the boundary of region D_1, returns to the starting point, and then traverses the boundary of region D_2. Furthermore, as we walk along the path, the region is always on our left. Notice that this traversal of the P_i paths covers the entire boundary of region D. If we had only traversed one portion of the boundary of D, then we cannot apply Green's theorem to D.

The boundary of the upper half of the annulus, therefore, is $P_1 \cup P_2 \cup P_3 \cup P_4$ and the boundary of the lower half of the annulus is $-P_4 \cup P_5 \cup -P_2 \cup P_6$. Then, Green's theorem implies

$$\begin{aligned}
\oint_{\partial D} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_{P_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_3} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_4} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{-P_4} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} && (16.4.8)\\
&\quad + \int_{P_5} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{-P_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_6} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \\
&= \int_{P_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_3} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_4} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_4} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} && (16.4.9)\\
&\quad + \int_{P_5} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{-P_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_6} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \\
&= \int_{P_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_3} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_5} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_6} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} && (16.4.10)\\
&= \oint_{\partial D_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \oint_{\partial D_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} && (16.4.11)\\
&= \iint_{D_1} (Q_x - P_y)\, dA + \iint_{D_2} (Q_x - P_y)\, dA && (16.4.12)\\
&= \iint_{D} (Q_x - P_y)\, dA. && (16.4.13)
\end{aligned}$$

Therefore, we arrive at the equation found in Green's theorem—namely,

$$\oint_{\partial D} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_{D} (Q_x - P_y)\, dA.$$

The same logic implies that the flux form of Green's theorem can also be extended to a region with finitely many holes:

$$\oint_{C} \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds = \iint_{D} (P_x + Q_y)\, dA.$$

Example 16.4.8A: Using Green's Theorem on a Region with Holes

Calculate the integral

$$\oint_{\partial D} (\sin x - \frac{y^3}{3})dx + (\frac{y^3}{3} + \sin y)dy,$$

where D is the annulus given by the polar inequalities $1 \le r \le 2$, $0 \le \theta \le 2\pi$.

Solution

Although D is not simply connected, we can use the extended form of Green's theorem to calculate the integral. Since the integration occurs over an annulus, we convert to polar coordinates:

$$\begin{aligned}
\oint_{\partial D} (\sin x - \frac{y^3}{3})\, dx + (\frac{x^3}{3} + \sin y)\, dy &= \iint_{D} (Q_x - P_y)\, dA \\
&= \iint_{D} (x^2 + y^2)\, dA \\
&= \int_0^{2\pi} \int_1^2 r^3\, dr d\theta = \int_0^{2\pi} \frac{15}{4}\, d\theta \\
&= \frac{15\pi}{2}.
\end{aligned}$$

Example 16.4.8B: Using the Extended Form of Green's Theorem

Let $\vec{\mathbf{F}} = \langle P, Q \rangle = \left\langle \frac{y}{x^2+y^2}, -\frac{x}{x^2+y^2} \right\rangle$ and let C be any simple closed curve in a plane oriented counterclockwise. What are the possible values of $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$?

Solution

We use the extended form of Green's theorem to show that $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ is either 0 or -2π—that is, no matter how crazy curve C is, the line integral of $\vec{\mathbf{F}}$ along C can have only one of two possible values. We consider two cases: the case when C encompasses the origin and the case when C does not encompass the origin.

Case 1: C Does Not Encompass the Origin

In this case, the region enclosed by C is simply connected because the only hole in the domain of $\vec{\mathbf{F}}$ is at the origin. We showed in our discussion of cross-partials that $\vec{\mathbf{F}}$ satisfies the cross-partial condition. If we restrict the domain of $\vec{\mathbf{F}}$ just to C and the region it encloses, then $\vec{\mathbf{F}}$ with this restricted domain is now defined on a simply connected domain. Since $\vec{\mathbf{F}}$ satisfies the cross-partial property on its restricted domain, the field $\vec{\mathbf{F}}$ is conservative on this simply connected region and hence the circulation $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ is zero.

Case 2: C Does Encompass the Origin

In this case, the region enclosed by C is not simply connected because this region contains a hole at the origin. Let C_1 be a circle of radius a centered at the origin so that C_1 is entirely inside the region enclosed by C (Figure 16.4.15). Give C_1 a clockwise orientation.

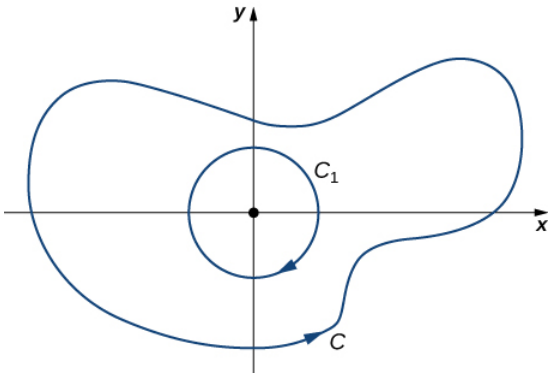


Figure 16.4.15: Choose circle C_1 centered at the origin that is contained entirely inside C.

Let D be the region between C_1 and C, and C is orientated counterclockwise. By the extended version of Green's theorem,

$$\begin{aligned} \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \iint_D Qx\_P_y \, dA \\ &= \iint_D -\frac{y^2-x^2}{(x^2+y^2)^2} + \frac{y^2-x^2}{(x^2+y^2)^2} dA \\ &= 0, \end{aligned}$$

and therefore

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = -\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}.$$

Since C_1 is a specific curve, we can evaluate $\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$. Let

$$x = a\cos t, \quad y = a\sin t, \quad 0 \le t \le 2\pi$$

be a parameterization of C_1. Then,

$$\begin{aligned}\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_0^{2\pi} \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\, dt \\ &= \int_0^{2\pi} \langle -\frac{\sin(t)}{a}, -\frac{\cos(t)}{a} \rangle \cdot \langle -a\sin(t), -a\cos(t) \rangle\, dt \\ &= \int_0^{2\pi} \sin^2(t) + \cos^2(t)\, dt \\ &= \int_0^{2\pi} dt = 2\pi.\end{aligned}$$

Therefore, $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = -2\pi$.

? Exercise 16.4.8

Calculate integral $\oint_{\partial D} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where D is the annulus given by the polar inequalities $2 \le r \le 5$, $0 \le \theta \le 2\pi$, and $\vec{\mathbf{F}}(x, y) = \langle x^3, 5x + e^y \sin y \rangle$.

Hint

Use the extended version of Green's theorem.

Answer

105π

MEASURING AREA FROM A BOUNDARY: THE PLANIMETER

Imagine you are a doctor who has just received a magnetic resonance image of your patient's brain. The brain has a tumor (Figure 16.4.16). How large is the tumor? To be precise, what is the area of the red region? The red cross-section of the tumor has an irregular shape, and therefore it is unlikely that you would be able to find a set of equations or inequalities for the region and then be able to calculate its area by conventional means. You could approximate the area by chopping the region into tiny squares (a Riemann sum approach), but this method always gives an answer with some error.

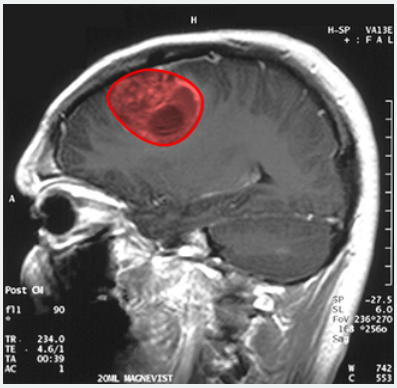


Figure 16.4.16: This magnetic resonance image of a patient's brain shows a tumor, which is highlighted in red. (credit: modification of work by Christaras A, Wikimedia Commons)

Instead of trying to measure the area of the region directly, we can use a device called a *rolling planimeter* to calculate the area of the region exactly, simply by measuring its boundary. In this project you investigate how a planimeter works, and you use Green's theorem to show the device calculates area correctly.

A rolling planimeter is a device that measures the area of a planar region by tracing out the boundary of that region (Figure 16.4.17). To measure the area of a region, we simply run the tracer of the planimeter around the boundary of the region. The planimeter measures the number of turns through which the wheel rotates as we trace the boundary; the area of the shape is proportional to this number of wheel turns. We can derive the precise proportionality equation using Green's theorem. As the tracer moves around the boundary of the region, the tracer arm rotates and the roller moves back and forth (but does not rotate).

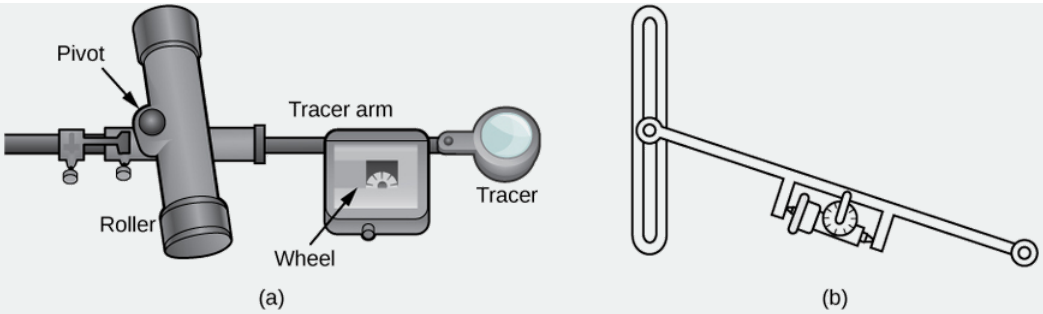


Figure 16.4.17: (a) A rolling planimeter. The pivot allows the tracer arm to rotate. The roller itself does not rotate; it only moves back and forth. (b) An interior view of a rolling planimeter. Notice that the wheel cannot turn if the planimeter is moving back and forth with the tracer arm perpendicular to the roller.

Let C denote the boundary of region D, the area to be calculated. As the tracer traverses curve C, assume the roller moves along the *y*-axis (since the roller does not rotate, one can assume it moves along a straight line). Use the coordinates (x, y) to represent points on boundary C, and coordinates $(0, Y)$ to represent the position of the pivot. As the planimeter traces C, the pivot moves along the *y*-axis while the tracer arm rotates on the pivot.

Watch a short animation of a planimeter in action.

Begin the analysis by considering the motion of the tracer as it moves from point (x, y) counterclockwise to point $(x + dx, y + dy)$ that is close to (x, y) (Figure 16.4.18). The pivot also moves, from point $(0, Y)$ to nearby point $(0, Y + dY)$. How much does the wheel turn as a result of this motion? To answer this question, break the motion into two parts. First, roll the pivot along the *y*-axis from $(0, Y)$ to $(0, Y + dY)$ without rotating the tracer arm. The tracer arm then ends up at point $(x, y + dY)$ while maintaining a constant angle ϕ with the *x*-axis. Second, rotate the tracer arm by an angle $d\theta$ without moving the roller. Now the tracer is at point $(x + dx, y + dy)$. Let ll be the distance from the pivot to the wheel and let L be the distance from the pivot to the tracer (the length of the tracer arm).

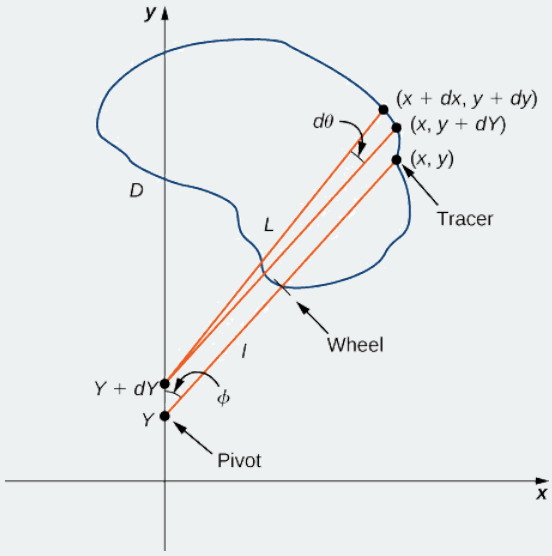


Figure 16.4.18: Mathematical analysis of the motion of the planimeter.

1. Explain why the total distance through which the wheel rolls the small motion just described is $\sin\phi dY + ld\theta = \dfrac{x}{L}dY + ld\theta$.
2. Show that $\oint_C d\theta = 0$.
3. Use step 2 to show that the total rolling distance of the wheel as the tracer traverses curve C is

   Total wheel roll $= \dfrac{1}{L}\oint_C x dY$.

   Now that you have an equation for the total rolling distance of the wheel, connect this equation to Green's theorem to calculate area D enclosed by C.
4. Show that $x^2 + (y - Y)^2 = L^2$.
5. Assume the orientation of the planimeter is as shown in Figure 16.4.18. Explain why $Y \leq y$, and use this inequality to show there is a unique value of Y for each point (x, y): $Y = y = \sqrt{L^2 - x^2}$.

6. Use step 5 to show that $dY = dy + \dfrac{x}{L^2 - x^2}\,dx$.
7. Use Green's theorem to show that $\oint_C \dfrac{x}{L^2 - x^2}\,dx = 0$.
8. Use step 7 to show that the total wheel roll is

$$\text{Total wheel roll} \quad = \quad 1L \oint_C x\,dy.$$

It took a bit of work, but this equation says that the variable of integration Y in step 3 can be replaced with y.

9. Use Green's theorem to show that the area of D is $\oint_C x\,dy$. The logic is similar to the logic used to show that the area of D $= 12 \oint_C -y\,dx + x\,dy$.
10. Conclude that the area of D equals the length of the tracer arm multiplied by the total rolling distance of the wheel.

You now know how a planimeter works and you have used Green's theorem to justify that it works. To calculate the area of a planar region D, use a planimeter to trace the boundary of the region. The area of the region is the length of the tracer arm multiplied by the distance the wheel rolled.

Key Concepts

- Green's theorem relates the integral over a connected region to an integral over the boundary of the region. Green's theorem is a version of the Fundamental Theorem of Calculus in one higher dimension.
- Green's Theorem comes in two forms: a circulation form and a flux form. In the circulation form, the integrand is $\vec{\mathbf{F}} \cdot \vec{\mathbf{T}}$. In the flux form, the integrand is $\vec{\mathbf{F}} \cdot \vec{\mathbf{N}}$.
- Green's theorem can be used to transform a difficult line integral into an easier double integral, or to transform a difficult double integral into an easier line integral.
- A vector field is source free if it has a stream function. The flux of a source-free vector field across a closed curve is zero, just as the circulation of a conservative vector field across a closed curve is zero.

Key Equations

- **Green's theorem, circulation form**
  $\oint_C P\,dx + Q\,dy = \iint_D Q_x - P_y\,dA$, where C is the boundary of D
- **Green's theorem, flux form**
  $\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\,ds = \iint_D P_x + Q_y\,dA$, where C is the boundary of D
- **Green's theorem, extended version**
  $\oint_{\partial D} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_D Q_x - P_y\,dA$

Glossary

Green's theorem

relates the integral over a connected region to an integral over the boundary of the region

stream function

if $\vec{\mathbf{F}} = \langle P, Q \rangle$ is a source-free vector field, then stream function g is a function such that $P = g_y$ and $Q = -g_x$

Contributions

Paul Seeburger created a corrected image and the worked out solution for Exercise 16.4.3.

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16.4E: Exercises for Section 16.4

For the following exercises, evaluate the line integrals by applying Green's theorem.

1. $\int_C 2xy\,dx + (x+y)\,dy$, where C is the path from $(0,0)$ to $(1,1)$ along the graph of $y = x^3$ and from $(1,1)$ to $(0,0)$ along the graph of $y = x$ oriented in the counterclockwise direction

2. $\int_C 2xy\,dx + (x+y)\,dy$, where C is the boundary of the region lying between the graphs of $y = 0$ and $y = 4 - x^2$ oriented in the counterclockwise direction

3. $\int_C 2\arctan\left(\frac{y}{x}\right)\,dx + \ln(x^2+y^2)\,dy$, where C is defined by $x = 4 + 2\cos\theta,\ y = 4\sin\theta$ oriented in the counterclockwise direction

4. $\int_C \sin x\cos y\,dx + (xy + \cos x\sin y)\,dy$, where C is the boundary of the region lying between the graphs of $y = x$ and $y = \sqrt{x}$ oriented in the counterclockwise direction

5. $\int_C xy\,dx + (x+y)\,dy$, where C is the boundary of the region lying between the graphs of $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$ oriented in the counterclockwise direction

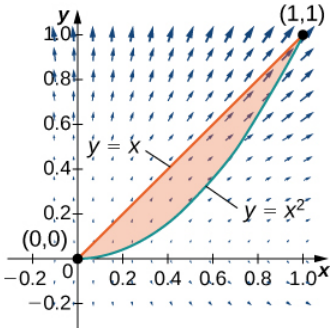
6. $\oint_C (-y\,dx + x\,dy)$, where C consists of line segment C_1 from $(-1,0)$ to $(1,0)$, followed by the semicircular arc C_2 from $(1,0)$ back to $(-1,0)$

For the following exercises, use Green's theorem.

7. Let C be the curve consisting of line segments from $(0,0)$ to $(1,1)$ to $(0,1)$ and back to $(0,0)$. Find the value of $\int_C xy\,dx + \sqrt{y^2+1}\,dy$.

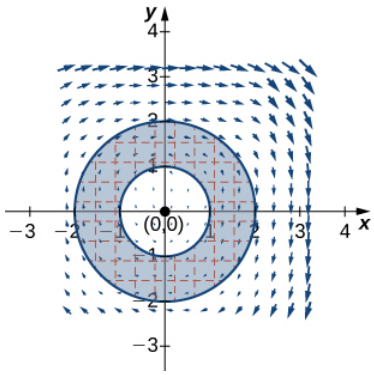
8. Evaluate line integral $\int_C xe^{-2x}\,dx + (x^4 + 2x^2y^2)\,dy$, where C is the boundary of the region between circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, and is a positively oriented curve.

9. Find the counterclockwise circulation of field $\vec{\mathbf{F}}(x,y) = xy\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}}$ around and over the boundary of the region enclosed by curves $y = x^2$ and $y = x$ in the first quadrant and oriented in the counterclockwise direction.



10. Evaluate $\oint_C y^3\,dx - x^3y^2\,dy$, where C is the positively oriented circle of radius 2 centered at the origin.

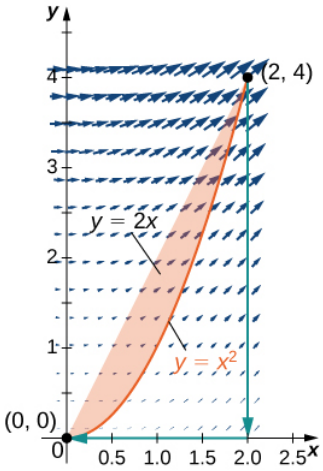
11. Evaluate $\oint_C y^3\,dx - x^3\,dy$, where C includes the two circles of radius 2 and radius 1 centered at the origin, both with positive orientation.



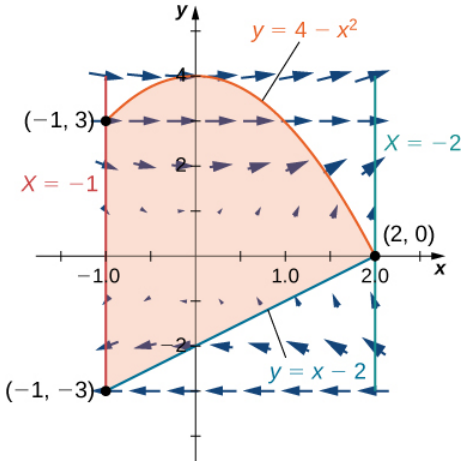
12. Calculate $\oint_C -x^2y\,dx + xy^2\,dy$, where C is a circle of radius 2 centered at the origin and oriented in the counterclockwise direction.

13. Calculate integral $\oint_C 2[y + x\sin(y)]\,dx + [x^2\cos(y) - 3y^2]\,dy$ along triangle C with vertices $(0,0)$, $(1,0)$ and $(1,1)$, oriented counterclockwise, using Green's theorem.

14. Evaluate integral $\oint_C (x^2 + y^2)\,dx + 2xy\,dy$, where C is the curve that follows parabola $y = x^2$ from $(0,0)$, $(2,4)$, then the line from $(2,4)$ to $(2,0)$, and finally the line from $(2,0)$ to $(0,0)$.



15. Evaluate line integral $\oint_C (y - \sin(y)\cos(y))\,dx + 2x\sin^2(y)\,dy$, where C is oriented in a counterclockwise path around the region bounded by $x = -1$, $x = 2$, $y = 4 - x^2$, and $y = x - 2$.



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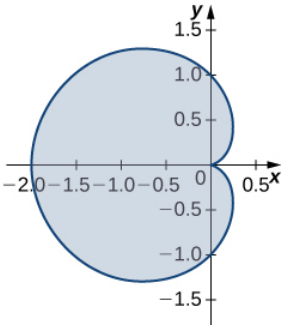
For the following exercises, use Green's theorem to find the area.

16. Find the area between ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and circle $x^2 + y^2 = 25$.

17. Find the area of the region enclosed by parametric equation

$$\vec{\mathbf{p}}(\theta) = (\cos(\theta) - \cos^2(\theta))\,\hat{\mathbf{i}} + (\sin(\theta) - \cos(\theta)\sin(\theta))\,\hat{\mathbf{j}} \text{ for } 0 \le \theta \le 2\pi.$$



18. Find the area of the region bounded by hypocycloid $\vec{\mathbf{r}}(t) = \cos^3(t)\,\hat{\mathbf{i}} + \sin^3(t)\,\hat{\mathbf{j}}$. The curve is parameterized by $t \in [0, 2\pi]$.

19. Find the area of a pentagon with vertices $(0, 4)$, $(4, 1)$, $(3, 0)$, $(-1, -1)$, and $(-2, 2)$.

20. Use Green's theorem to evaluate $\int_{C^+} (y^2 + x^3)\,dx + x^4\,dy$, where C^+ is the perimeter of square $[0, 1] \times [0, 1]$ oriented counterclockwise.

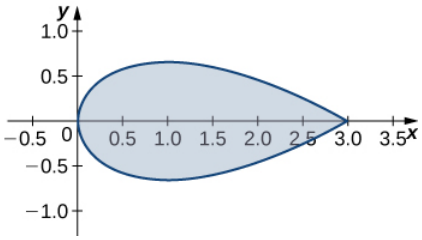
21. Use Green's theorem to prove the area of a disk with radius a is $A = \pi a^2$ units^2.

22. Use Green's theorem to find the area of one loop of a four-leaf rose $r = 3\sin 2\theta$. (*Hint*: $x\,dy - y\,dx = r^2\,d\theta$).

23. Use Green's theorem to find the area under one arch of the cycloid given by the parametric equations: $x = t - \sin t,\ y = 1 - \cos t,\ t \ge 0$.

24. Use Green's theorem to find the area of the region enclosed by curve

$$\vec{\mathbf{r}}(t) = t^2\,\hat{\mathbf{i}} + \left(\tfrac{t^3}{3} - t\right)\,\hat{\mathbf{j}}, \text{ for } -\sqrt{3} \le t \le \sqrt{3}.$$



25. [T] Evaluate Green's theorem using a computer algebra system to evaluate the integral $\int_C xe^y\,dx + e^x\,dy$, where C is the circle given by $x^2 + y^2 = 4$ and is oriented in the counterclockwise direction.

26. Evaluate $\int_C (x^2y - 2xy + y^2)\,ds$, where C is the boundary of the unit square $0 \le x \le 1,\ 0 \le y \le 1$, traversed counterclockwise.

27. Evaluate $\int_C \dfrac{-(y+2)\,dx + (x-1)\,dy}{(x-1)^2 + (y+2)^2}$, where C is any simple closed curve with an interior that does not contain point $(1, -2)$ traversed counterclockwise.

28. Evaluate $\int_C \dfrac{x\,dx + y\,dy}{x^2 + y^2}$, where C is any piecewise, smooth simple closed curve enclosing the origin, traversed counterclockwise.

For the following exercises, use Green's theorem to calculate the work done by force $\vec{\mathbf{F}}$ on a particle that is moving counterclockwise around closed path C.

29. $\vec{\mathbf{F}}(x, y) = xy\,\hat{\mathbf{i}} + (x + y)\,\hat{\mathbf{j}}, \quad C : x^2 + y^2 = 4$

30. $\vec{\mathbf{F}}(x,y) = (x^{3/2} - 3y)\,\hat{\mathbf{i}} + (6x + 5\sqrt{y})\,\hat{\mathbf{j}}$, C: boundary of a triangle with vertices $(0,0)$, $(5,0)$, and $(0,5)$

31. Evaluate $\int_C (2x^3 - y^3)\,dx + (x^3 + y^3)\,dy$, where C is a unit circle oriented in the counterclockwise direction.

32. A particle starts at point $(-2,0)$, moves along the x-axis to $(2,0)$, and then travels along semicircle $y = \sqrt{4 - x^2}$ to the starting point. Use Green's theorem to find the work done on this particle by force field $\vec{\mathbf{F}}(x,y) = x\,\hat{\mathbf{i}} + (x^3 + 3xy^2)\,\hat{\mathbf{j}}$.

33. David and Sandra are skating on a frictionless pond in the wind. David skates on the inside, going along a circle of radius 2 in a counterclockwise direction. Sandra skates once around a circle of radius 3, also in the counterclockwise direction. Suppose the force of the wind at point (x,y) is $\vec{\mathbf{F}}(x,y) = (x^2y + 10y)\,\hat{\mathbf{i}} + (x^3 + 2xy^2)\,\hat{\mathbf{j}}$. Use Green's theorem to determine who does more work.

34. Use Green's theorem to find the work done by force field $\vec{\mathbf{F}}(x,y) = (3y - 4x)\,\hat{\mathbf{i}} + (4x - y)\,\hat{\mathbf{j}}$ when an object moves once counterclockwise around ellipse $4x^2 + y^2 = 4$.

35. Use Green's theorem to evaluate line integral $\oint_C e^{2x}\sin 2y\,dx + e^{2x}\cos 2y\,dy$, where C is ellipse $9(x-1)^2 + 4(y-3)^2 = 36$ oriented counterclockwise.

36. Evaluate line integral $\oint_C y^2\,dx + x^2\,dy$, where C is the boundary of a triangle with vertices $(0,0)$, $(1,1)$, and $(1,0)$, with the counterclockwise orientation.

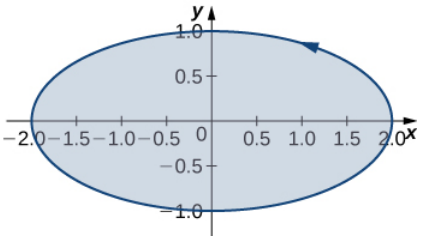
37. Use Green's theorem to evaluate line integral $\int_C \vec{\mathbf{h}} \cdot d\vec{\mathbf{r}}$ if $\vec{\mathbf{h}}(x,y) = e^y\,\hat{\mathbf{i}} - \sin \pi x\,\hat{\mathbf{j}}$, where C is a triangle with vertices $(1,0)$, $(0,1)$, and $(-1,0)$, traversed counterclockwise.

38. Use Green's theorem to evaluate line integral $\int_C \sqrt{1 + x^3}\,dx + 2xy\,dy$ where C is a triangle with vertices $(0,0)$, $(1,0)$, and $(1,3)$ oriented clockwise.

39. Use Green's theorem to evaluate line integral $\int_C x^2y\,dx - xy^2\,dy$ where C is a circle $x^2 + y^2 = 4$ oriented counterclockwise.

40. Use Green's theorem to evaluate line integral $\int_C (3y - e^{\sin x})\,dx + \left(7x + \sqrt{y^4 + 1}\right)\,dy$ where C is circle $x^2 + y^2 = 9$ oriented in the counterclockwise direction.

41. Use Green's theorem to evaluate line integral $\int_C (3x - 5y)\,dx + (x - 6y)\,dy$, where C is ellipse $\frac{x^2}{4} + y^2 = 1$ and is oriented in the counterclockwise direction.



42. Let C be a triangular closed curve from $(0,0)$ to $(1,0)$ to $(1,1)$ and finally back to $(0,0)$. Let $\vec{\mathbf{F}}(x,y) = 4y\,\hat{\mathbf{i}} + 6x^2\,\hat{\mathbf{j}}$. Use Green's theorem to evaluate $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$.

43. Use Green's theorem to evaluate line integral $\oint_C y\,dx - x\,dy$, where C is circle $x^2 + y^2 = a^2$ oriented in the clockwise direction.

44. Use Green's theorem to evaluate line integral $\oint_C (y + x)\,dx + (x + \sin y)\,dy$, where C is any smooth simple closed curve joining the origin to itself oriented in the counterclockwise direction.

45. Use Green's theorem to evaluate line integral $\oint_C \left(y - \ln(x^2 + y^2)\right)\,dx + \left(2\arctan\frac{y}{x}\right)\,dy$, where C is the positively oriented circle $(x-2)^2 + (y-3)^2 = 1$.

46. Use Green's theorem to evaluate $\oint_C xy\,dx + x^3y^3\,dy$, where C is a triangle with vertices $(0,0)$, $(1,0)$, and $(1,2)$ with positive orientation.

47. Use Green's theorem to evaluate line integral $\int_C \sin y\,dx + x\cos y\,dy$, where C is ellipse $x^2 + xy + y^2 = 1$ oriented in the counterclockwise direction.

48. Let $\vec{\mathbf{F}}(x,y) = \left(\cos(x^5) - 13y^3\right)\,\hat{\mathbf{i}} + 13x^3\,\hat{\mathbf{j}}$. Find the counterclockwise circulation $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where C is a curve consisting of the line segment joining $(-2,0)$ and $(-1,0)$, half circle $y = \sqrt{1-x^2}$, the line segment joining $(1,0)$ and $(2,0)$, and half circle $y = \sqrt{4-x^2}$.

49. Use Green's theorem to evaluate line integral $\int_C \sin(x^3)\,dx + 2ye^{x^2}\,dy$, where C is a triangular closed curve that connects the points $(0,0)$, $(2,2)$, and $(0,2)$ counterclockwise.

50. Let C be the boundary of square $0 \le x \le \pi,\ 0 \le y \le \pi$, traversed counterclockwise. Use Green's theorem to find $\int_C \sin(x + y)\,dx + \cos(x+y)\,dy$.

51. Use Green's theorem to evaluate line integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x,y) = \left(y^2 - x^2\right)\,\hat{\mathbf{i}} + \left(x^2 + y^2\right)\,\hat{\mathbf{j}}$, and C is a triangle bounded by $y = 0,\ x = 3$, and $y = x$, oriented counterclockwise.

52. Use Green's Theorem to evaluate integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x,y) = \left(xy^2\right)\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$, and C is a unit circle oriented in the counterclockwise direction.

53. Use Green's theorem in a plane to evaluate line integral $\oint_C \left(xy + y^2\right)\,dx + x^2\,dy$, where C is a closed curve of a region bounded by $y = x$ and $y = x^2$ oriented in the counterclockwise direction.

54. Calculate the outward flux of $\vec{\mathbf{F}}(x,y) = -x\,\hat{\mathbf{i}} + 2y\,\hat{\mathbf{j}}$ over a square with corners $(\pm 1, \pm 1)$, where the unit normal is outward pointing and oriented in the counterclockwise direction.

55. [T] Let C be circle $x^2 + y^2 = 4$ oriented in the counterclockwise direction. Evaluate $\oint_C \left[\left(3y - e^{\arctan x}\right)dx + \left(7x + \sqrt{y^4+1}\right)\,dy\right]$ using a computer algebra system.

56. Find the flux of field $\vec{\mathbf{F}}(x,y) = -x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$ across $x^2 + y^2 = 16$ oriented in the counterclockwise direction.

57. Let $\vec{\mathbf{F}} = \left(y^2 - x^2\right)\,\hat{\mathbf{i}} + \left(x^2 + y^2\right)\,\hat{\mathbf{j}}$, and let C be a triangle bounded by $y = 0,\ x = 3$, and $y = x$ oriented in the counterclockwise direction. Find the outward flux of $\vec{\mathbf{F}}$ through C.

58. [T] Let C be unit circle $x^2 + y^2 = 1$ traversed once counterclockwise. Evaluate $\int_C \left[-y^3 + \sin(xy) + xy\cos(xy)\right]\,dx + \left[x^3 + x^2\cos(xy)\right]\,dy$ by using a computer algebra system.

59. [T] Find the outward flux of vector field $\vec{\mathbf{F}}(x,y) = xy^2\,\hat{\mathbf{i}} + x^2y\,\hat{\mathbf{j}}$ across the boundary of annulus $R = \left\{(x,y) : 1 \le x^2 + y^2 \le 4\right\} = \left\{(r,\theta) : 1 \le r \le 2,\ 0 \le \theta \le 2\pi\right\}$ using a computer algebra system.

60. Consider region R bounded by parabolas $y = x^2$ and $x = y^2$. Let C be the boundary of R oriented counterclockwise. Use Green's theorem to evaluate $\oint_C \left(y + e^{\sqrt{x}}\right)\,dx + \left(2x + \cos(y^2)\right)\,dy$.

Contributors

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16.5: Divergence and Curl

Learning Objectives

- Determine divergence from the formula for a given vector field.
- Determine curl from the formula for a given vector field.
- Use the properties of curl and divergence to determine whether a vector field is conservative.

In this section, we examine two important operations on a vector field: divergence and curl. They are important to the field of calculus for several reasons, including the use of curl and divergence to develop some higher-dimensional versions of the Fundamental Theorem of Calculus. In addition, curl and divergence appear in mathematical descriptions of fluid mechanics, electromagnetism, and elasticity theory, which are important concepts in physics and engineering. We can also apply curl and divergence to other concepts we already explored. For example, under certain conditions, a vector field is conservative if and only if its curl is zero.

In addition to defining curl and divergence, we look at some physical interpretations of them, and show their relationship to conservative and source-free vector fields.

Divergence

Divergence is an operation on a vector field that tells us how the field behaves toward or away from a point. Locally, the divergence of a vector field $\vec{\mathbf{F}}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ at a particular point P is a measure of the "outflowing-ness" of the vector field at P. If $\vec{\mathbf{F}}$ represents the velocity of a fluid, then the divergence of $\vec{\mathbf{F}}$ at P measures the net rate of change with respect to time of the amount of fluid flowing away from P (the tendency of the fluid to flow "out of" P). In particular, if the amount of fluid flowing into P is the same as the amount flowing out, then the divergence at P is zero.

Definition: divergence in $\mathbb{R}^3$

If $\vec{\mathbf{F}} = \langle P, Q, R\rangle$ is a vector field in $\mathbb{R}^3$ and P_x, Q_y, and R_z all exist, then the divergence of $\vec{\mathbf{F}}$ is defined by

$$\operatorname{div} F = P_x + Q_y + R_z \tag{16.5.1}$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}. \tag{16.5.2}$$

Note the divergence of a vector field is not a vector field, but a scalar function. In terms of the gradient operator

$$\vec{\nabla} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle$$

divergence can be written symbolically as the dot product

$$\operatorname{div} \vec{\mathbf{F}} = \vec{\nabla} \cdot \vec{\mathbf{F}}.$$

Note this is merely helpful notation, because the dot product of a vector of operators and a vector of functions is not meaningfully defined given our current definition of dot product.

If $\vec{\mathbf{F}} = \langle P, Q\rangle$ is a vector field in $\mathbb{R}^2$, and P_x and Q_y both exist, then the divergence of $\vec{\mathbf{F}}$ is defined similarly as

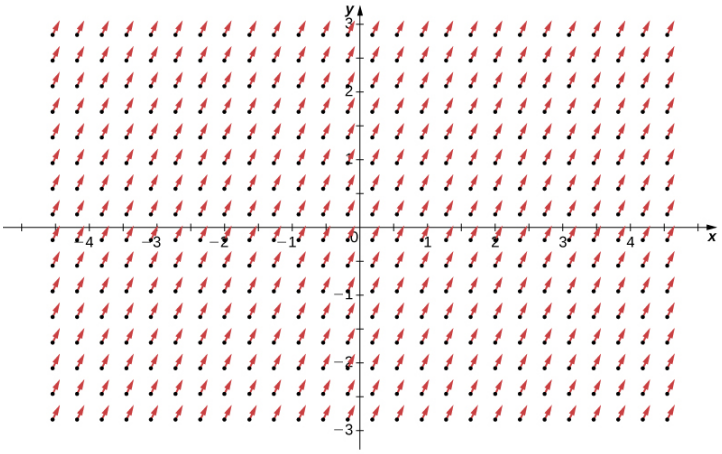
$$\begin{aligned} \operatorname{div} \vec{\mathbf{F}} &= P_x + Q_y \\ &= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \\ &= \vec{\nabla} \cdot \vec{\mathbf{F}}. \end{aligned}$$

To illustrate this point, consider the two vector fields in Figure 16.5.1. At any particular point, the amount flowing in is the same as the amount flowing out, so at every point the "outflowing-ness" of the field is zero. Therefore, we expect the divergence of both fields to be zero, and this is indeed the case, as

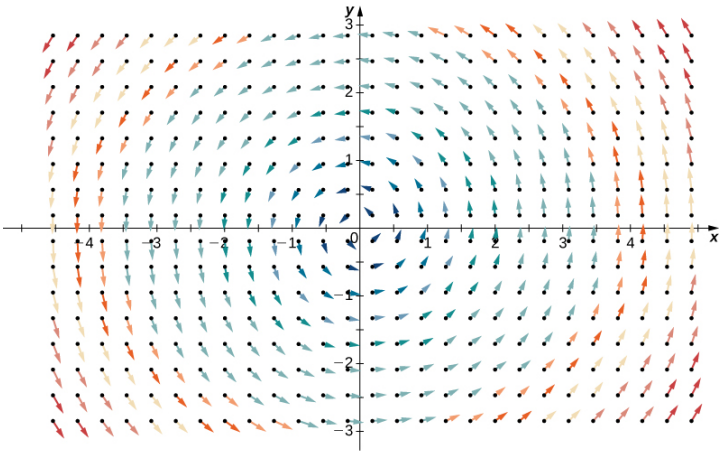
$$\operatorname{div}(\langle 1, 2\rangle) = \frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(2) = 0$$

and

$$\text{div}(\langle -y, x\rangle) = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0.$$



(a)



(b)

Figure 16.5.1: (a) Vector field $\langle 1, 2\rangle$ has zero divergence. (b) Vector field $\langle -y, x\rangle$ also has zero divergence.

By contrast, consider radial vector field $\vec{\mathbf{R}}(x, y) = \langle -x, -y\rangle$ in Figure 16.5.2. At any given point, more fluid is flowing in than is flowing out, and therefore the "outgoingness" of the field is negative. We expect the divergence of this field to be negative, and this is indeed the case, as

$$\text{div}(\vec{\mathbf{R}}) = \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(-y) = -2.$$

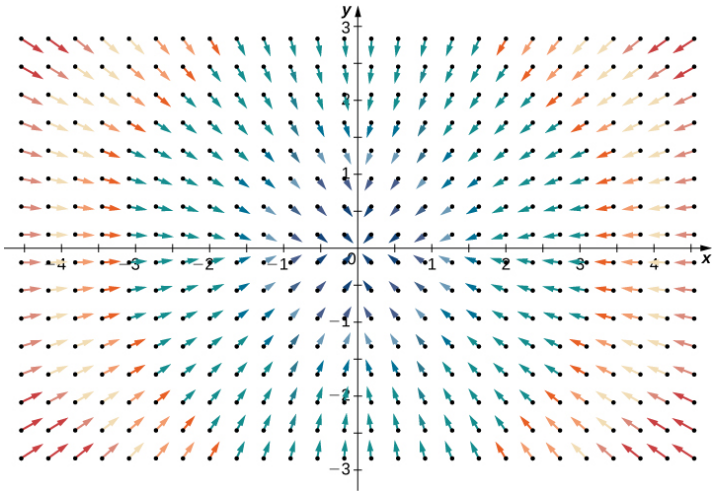


Figure 16.5.2: This vector field has negative divergence.

To get a global sense of what divergence is telling us, suppose that a vector field in $\mathbb{R}^2$ represents the velocity of a fluid. Imagine taking an elastic circle (a circle with a shape that can be changed by the vector field) and dropping it into a fluid. If the circle maintains its exact area as it flows through the fluid, then the divergence is zero. This would occur for both vector fields in Figure 16.5.1. On the other hand, if the circle's shape is distorted so that its area shrinks or expands, then the divergence is not zero. Imagine dropping such an elastic circle into the radial vector field in Figure 16.5.2 so that the center of the circle lands at point $(3, 3)$. The circle would flow toward the origin, and as it did so the front of the circle would travel more slowly than the back, causing the circle to "scrunch" and lose area. This is how you can see a negative divergence.

Example 16.5.1: Calculating Divergence at a Point

If $\vec{\mathbf{F}}(x, y, z) = e^x\hat{i} + yz\hat{j} - yz^2\hat{k}$, then find the divergence of $\vec{\mathbf{F}}$ at $(0, 2, -1)$.

Solution

The divergence of $\vec{\mathbf{F}}$ is

$$\frac{\partial}{\partial x}(e^x) + \frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(yz^2) = e^x + z - 2yz.$$

Therefore, the divergence at $(0, 2, -1)$ is $e^0 - 1 + 4 = 4$. If $\vec{\mathbf{F}}$ represents the velocity of a fluid, then more fluid is flowing out than flowing in at point $(0, 2, -1)$.

Exercise 16.5.1

Find $\operatorname{div} \vec{\mathbf{F}}$ for

$$\vec{\mathbf{F}}(x, y, z) = \langle xy,\ 5 - z^2,\ x^2 + y^2 \rangle.$$

Hint

Follow Example 16.5.1.

Answer

$\operatorname{div} \vec{\mathbf{F}} = y$

One application for divergence occurs in physics, when working with magnetic fields. A magnetic field is a vector field that models the influence of electric currents and magnetic materials. Physicists use divergence in Gauss's law for magnetism, which states that if $\vec{\mathbf{B}}$ is a magnetic field, then $\vec{\nabla} \cdot \vec{\mathbf{B}} = 0$; in other words, the divergence of a magnetic field is zero.

Example 16.5.2: Determining Whether a Field Is Magnetic

Is it possible for $\vec{\mathbf{F}}(x,y) = \langle x^2 y,\ y - xy^2 \rangle$ to be a magnetic field?

Solution

If $\vec{\mathbf{F}}$ were magnetic, then its divergence would be zero. The divergence of $\vec{\mathbf{F}}$ is

$$\frac{\partial}{\partial x}(x^2 y) + \frac{\partial}{\partial y}(y - xy^2) = 2xy + 1 - 2xy = 1$$

and therefore $\vec{\mathbf{F}}$ cannot model a magnetic field (Figure 16.5.3).

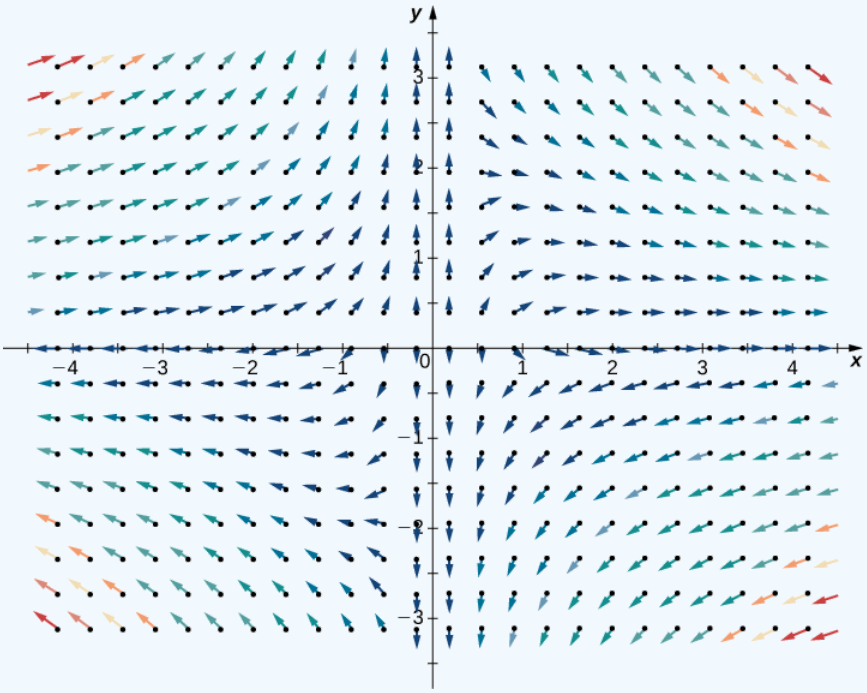


Figure 16.5.3: The divergence of vector field $\vec{\mathbf{F}}(x,y) = \langle x^2 y,\ y - xy^2 \rangle$ is one, so it cannot model a magnetic field.

Another application for divergence is detecting whether a field is source free. Recall that a source-free field is a vector field that has a stream function; equivalently, a source-free field is a field with a flux that is zero along any closed curve. The next two theorems say that, under certain conditions, source-free vector fields are precisely the vector fields with zero divergence.

Theorem: Divergence of a Source-Free Vector Field

If $\vec{\mathbf{F}} = \langle P, Q \rangle$ is a source-free continuous vector field with differentiable component functions, then $\operatorname{div} \vec{\mathbf{F}} = 0$.

Proof

Since $\vec{\mathbf{F}}$ is source free, there is a function $g(x,y)$ with $g_y = P$ and $-g_x = Q$. Therefore, $\vec{\mathbf{F}} = \langle g_y, -g_x \rangle$ and $\operatorname{div} \vec{\mathbf{F}} = g_{yx} - g_{xy} = 0$ by Clairaut's theorem.

□

The converse of Divergence of a Source-Free Vector Field is true on simply connected regions, but the proof is too technical to include here. Thus, we have the following theorem, which can test whether a vector field in $\mathbb{R}^2$ is source free.

Theorem: Divergence Test for Source-Free Vector Fields

Let $\vec{\mathbf{F}} = \langle P, Q \rangle$ be a continuous vector field with differentiable component functions with a domain that is simply connected. Then, $\operatorname{div} \vec{\mathbf{F}} = 0$ if and only if $\vec{\mathbf{F}}$ is source free.

Example 16.5.3: Determining Whether a Field Is Source Free

Is field $\vec{\mathbf{F}}(x, y) = \langle x^2 y,\ 5 - xy^2 \rangle$ source free?

Solution

Note the domain of $\vec{\mathbf{F}}$ is $\mathbb{R}^2$ which is simply connected. Furthermore, $\vec{\mathbf{F}}$ is continuous with differentiable component functions. Therefore, we can use the Divergence Test for Source-Free Vector Fields to analyze $\vec{\mathbf{F}}$. The divergence of $\vec{\mathbf{F}}$ is

$$\frac{\partial}{\partial x}(x^2 y) + \frac{\partial}{\partial y}(5 - xy^2) = 2xy - 2xy = 0.$$

Therefore, $\vec{\mathbf{F}}$ is source free by the Divergence Test for Source-Free Vector Fields.

Exercise 16.5.2

Let $\vec{\mathbf{F}}(x, y) = \langle -ay, bx \rangle$ be a rotational field where a and b are positive constants. Is $\vec{\mathbf{F}}$ source free?

Hint

Calculate the divergence.

Answer

Yes

Recall that the flux form of Green's theorem says that

$$\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds = \iint_D P_x + Q_y\, dA,$$

where C is a simple closed curve and D is the region enclosed by C. Since $P_x + Q_y = \operatorname{div} \vec{\mathbf{F}}$, Green's theorem is sometimes written as

$$\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds = \iint_D \operatorname{div} \vec{\mathbf{F}}\, dA.$$

Therefore, Green's theorem can be written in terms of divergence. If we think of divergence as a derivative of sorts, then Green's theorem says the "derivative" of $\vec{\mathbf{F}}$ on a region can be translated into a line integral of $\vec{\mathbf{F}}$ along the boundary of the region. This is analogous to the Fundamental Theorem of Calculus, in which the derivative of a function f on a line segment $[a, b]$ can be translated into a statement about f on the boundary of $[a, b]$. Using divergence, we can see that Green's theorem is a higher-dimensional analog of the Fundamental Theorem of Calculus.

We can use all of what we have learned in the application of divergence. Let $\vec{\mathbf{v}}$ be a vector field modeling the velocity of a fluid. Since the divergence of $\vec{\mathbf{v}}$ at point P measures the "outflowing-ness" of the fluid at P, $\operatorname{div} v(P) > 0$ implies that more fluid is flowing out of P than flowing in. Similarly, $\operatorname{div} v(P) < 0$ implies the more fluid is flowing in to P than is flowing out, and $\operatorname{div} \vec{\mathbf{v}}(P) = 0$ implies the same amount of fluid is flowing in as flowing out.

Example 16.5.4: Determining Flow of a Fluid

Suppose $\vec{\mathbf{v}}(x, y) = \langle -xy, y \rangle,\ y > 0$ models the flow of a fluid. Is more fluid flowing into point $(1, 4)$ than flowing out?

Solution

To determine whether more fluid is flowing into $(1, 4)$ than is flowing out, we calculate the divergence of $\vec{\mathbf{v}}$ at $(1, 4)$:

$$div(\vec{\mathbf{v}}) = \frac{\partial}{\partial x}(-xy) + \frac{\partial}{\partial y}(y) = -y + 1.$$

To find the divergence at $(1,4)$ substitute the point into the divergence: $-4+1=-3$. Since the divergence of $\vec{\mathbf{v}}$ at $(1,4)$ is negative, more fluid is flowing in than flowing out (Figure 16.5.4).

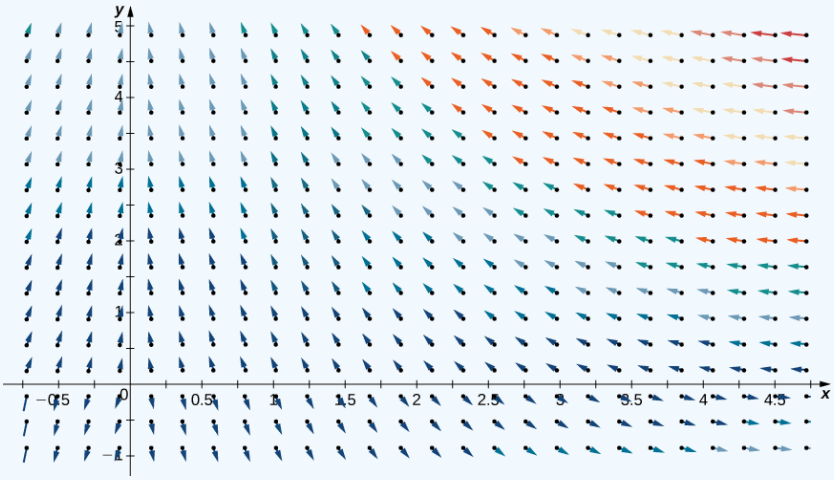


Figure 16.5.4: Vector field $\vec{\mathbf{v}}(x,y)=\langle -xy,y\rangle$ has negative divergence at $(1,4)$

? Exercise 16.5.3

For vector field $\vec{\mathbf{v}}(x,y)=\langle -xy,y\rangle$, $y>0$, find all points P such that the amount of fluid flowing in to P equals the amount of fluid flowing out of P.

Hint

Find where the divergence is zero.

Answer

All points on line $y=1$.

Curl

The second operation on a vector field that we examine is the curl, which measures the extent of rotation of the field about a point. Suppose that $\vec{\mathbf{F}}$ represents the velocity field of a fluid. Then, the curl of $\vec{\mathbf{F}}$ at point P is a vector that measures the tendency of particles near P to rotate about the axis that points in the direction of this vector. The magnitude of the curl vector at P measures how quickly the particles rotate around this axis. In other words, the curl at a point is a measure of the vector field's "spin" at that point. Visually, imagine placing a paddlewheel into a fluid at P, with the axis of the paddlewheel aligned with the curl vector (Figure 16.5.5). The curl measures the tendency of the paddlewheel to rotate.

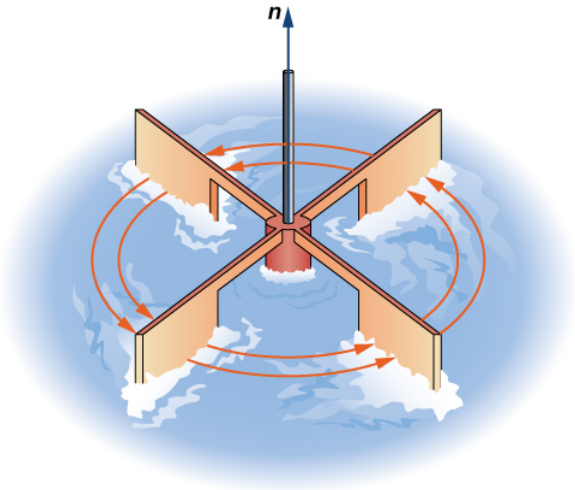


Figure 16.5.5: To visualize curl at a point, imagine placing a small paddlewheel into the vector field at a point.

Consider the vector fields in Figure 16.5.1. In part (a), the vector field is constant and there is no spin at any point. Therefore, we expect the curl of the field to be zero, and this is indeed the case. Part (b) shows a rotational field, so the field has spin. In particular, if you place a paddlewheel into a field at any point so that the axis of the wheel is perpendicular to a plane, the wheel rotates counterclockwise. Therefore, we expect the curl of the field to be nonzero, and this is indeed the case (the curl is $2\hat{\mathbf{k}}$).

To see what curl is measuring globally, imagine dropping a leaf into the fluid. As the leaf moves along with the fluid flow, the curl measures the tendency of the leaf to rotate. If the curl is zero, then the leaf doesn't rotate as it moves through the fluid.

Definition: Curl

If $\vec{\mathbf{F}} = \langle P, Q, R\rangle$ is a vector field in $\mathbb{R}^3$, and P_x, Q_y, and R_z all exist, then the curl of $\vec{\mathbf{F}}$ is defined by

$$\operatorname{curl}\vec{\mathbf{F}} = (R_y - Q_z)\,\hat{\mathbf{i}} + (P_z - R_x)\,\hat{\mathbf{j}} + (Q_x - P_y)\,\hat{\mathbf{k}} \tag{16.5.3}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{\mathbf{k}}. \tag{16.5.4}$$

Note that the curl of a vector field is a vector field, in contrast to divergence.

The definition of curl can be difficult to remember. To help with remembering, we use the notation $\vec{\nabla} \times \vec{\mathbf{F}}$ to stand for a "determinant" that gives the curl formula:

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}.$$

The determinant of this matrix is

$$(R_y - Q_z)\,\hat{\mathbf{i}} - (R_x - P_z)\,\hat{\mathbf{j}} + (Q_x - P_y)\,\hat{\mathbf{k}} = (R_y - Q_z)\,\hat{\mathbf{i}} + (P_z - R_x)\,\hat{\mathbf{j}} + (Q_x - P_y)\,\hat{\mathbf{k}} = \operatorname{curl}\vec{\mathbf{F}}.$$

Thus, this matrix is a way to help remember the formula for curl. Keep in mind, though, that the word *determinant* is used very loosely. A determinant is not really defined on a matrix with entries that are three vectors, three operators, and three functions.

If $\vec{\mathbf{F}} = \langle P, Q\rangle$ is a vector field in $\mathbb{R}^2$, then the curl of $\vec{\mathbf{F}}$, by definition, is

$$\operatorname{curl}\vec{\mathbf{F}} = (Q_x - P_y)\,\hat{\mathbf{k}} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{\mathbf{k}}.$$

Example 16.5.5: Finding the Curl of a Three-Dimensional Vector Field

Find the curl of $\vec{\mathbf{F}}(x, y, z) = \langle P, Q, R\rangle = \langle x^2 z, e^y + xz, xyz\rangle$.

Solution

The curl is

$$\begin{aligned}\operatorname{curl}\vec{\mathbf{F}} &= \vec{\nabla} \times \vec{\mathbf{F}} \\ &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} \\ &= (R_y - Q_z)\,\hat{\mathbf{i}} + (P_z - R_x)\,\hat{\mathbf{j}} + (Q_x - P_y)\,\hat{\mathbf{k}} \\ &= (xz - x)\,\hat{\mathbf{i}} + (x^2 - yz)\,\hat{\mathbf{j}} + z\hat{\mathbf{k}}.\end{aligned}$$

Exercise 16.5.4

Find the curl of $\vec{\mathbf{F}}(x, y, z) = \langle \sin x \cos z,\ \sin y \sin z,\ \cos x \cos y\rangle$ at point $\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right)$.

Hint

Find the determinant of matrix $\vec{\nabla} \times \vec{\mathbf{F}}$.

Answer

$-\hat{i}$

Example 16.5.6: Finding the Curl of a Two-Dimensional Vector Field

Find the curl of $\vec{\mathbf{F}}(x,y) = \langle P, Q\rangle = \langle y, 0\rangle$.

Solution

Notice that this vector field consists of vectors that are all parallel. In fact, each vector in the field is parallel to the *x*-axis. This fact might lead us to the conclusion that the field has no spin and that the curl is zero. To test this theory, note that

$$\operatorname{curl}\vec{\mathbf{F}} = (Q_x - P_y)\,\hat{\mathbf{k}} = -\hat{\mathbf{k}} \neq \vec{\mathbf{0}}.$$

Therefore, this vector field does have spin. To see why, imagine placing a paddlewheel at any point in the first quadrant (Figure 16.5.6). The larger magnitudes of the vectors at the top of the wheel cause the wheel to rotate. The wheel rotates in the clockwise (negative) direction, causing the coefficient of the curl to be negative.

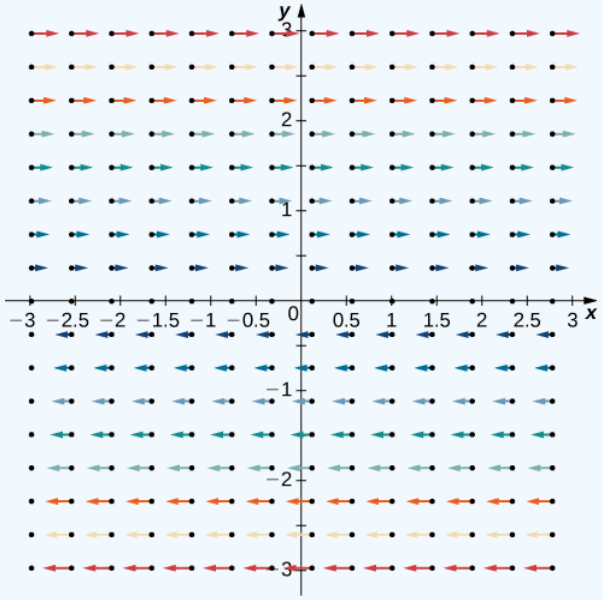


Figure 16.5.6: Vector field $\vec{\mathbf{F}}(x,y) = \langle y, 0\rangle$ consists of vectors that are all parallel.

Note that if $\vec{\mathbf{F}} = \langle P, Q\rangle$ is a vector field in a plane, then $\operatorname{curl}\vec{\mathbf{F}} \cdot \hat{\mathbf{k}} = (Q_x - P_y)\,\hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = Q_x - P_y$. Therefore, the circulation form of Green's theorem is sometimes written as

$$\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_D \operatorname{curl}\vec{\mathbf{F}} \cdot \hat{\mathbf{k}}\, dA,$$

where C is a simple closed curve and D is the region enclosed by C. Therefore, the circulation form of Green's theorem can be written in terms of the curl. If we think of curl as a derivative of sorts, then Green's theorem says that the "derivative" of $\vec{\mathbf{F}}$ on a region can be translated into a line integral of $\vec{\mathbf{F}}$ along the boundary of the region. This is analogous to the Fundamental Theorem of Calculus, in which the derivative of a function f on line segment $[a,b]$ can be translated into a statement about f on the boundary of $[a,b]$. Using curl, we can see the circulation form of Green's theorem is a higher-dimensional analog of the Fundamental Theorem of Calculus.

We can now use what we have learned about curl to show that gravitational fields have no "spin." Suppose there is an object at the origin with mass m_1 at the origin and an object with mass m_2. Recall that the gravitational force that object 1 exerts on object 2 is given by field

$$\vec{\mathbf{F}}(x,y,z) = -Gm_1m_2\left\langle \frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}}\right\rangle.$$

Example 16.5.7: Determining the Spin of a Gravitational Field

Show that a gravitational field has no spin.

Solution

To show that $\vec{\mathbf{F}}$ has no spin, we calculate its curl. Let

- $P(x,y,z) = \dfrac{x}{(x^2+y^2+z^2)^{3/2}}$,
- $Q(x,y,z) = \dfrac{y}{(x^2+y^2+z^2)^{3/2}}$, and
- $R(x,y,z) = \dfrac{z}{(x^2+y^2+z^2)^{3/2}}$.

Then,

$$\begin{aligned}\operatorname{curl}\vec{\mathbf{F}} &= -Gm_1m_2[(R_y-Q_z)\hat{\mathbf{i}}+(P_z-R_x)\hat{\mathbf{j}}+(Q_x-P_y)\hat{\mathbf{k}}] \\ &= -Gm_1m_2\left(\begin{array}{l}\left(\left(\dfrac{-3yz}{(x^2+y^2+z^2)^{5/2}}-\left(\dfrac{-3yz}{(x^2+y^2+z^2)^{5/2}}\right)\right)\right)\hat{\mathbf{i}} \\ +\left(\left(\dfrac{-3xz}{(x^2+y^2+z^2)^{5/2}}-\left(\dfrac{-3xz}{(x^2+y^2+z^2)^{5/2}}\right)\right)\right)\hat{\mathbf{j}} \\ +\left(\left(\dfrac{-3xy}{(x^2+y^2+z^2)^{5/2}}-\left(\dfrac{-3xy}{(x^2+y^2+z^2)^{5/2}}\right)\right)\right)\hat{\mathbf{k}}\end{array}\right) \\ &= \vec{\mathbf{0}}.\end{aligned}$$

Since the curl of the gravitational field is zero, the field has no spin.

Exercise 16.5.7

Field $\vec{\mathbf{v}}(x,y) = \langle -\dfrac{y}{x^2+y^2}, \dfrac{x}{x^2+y^2}\rangle$ models the flow of a fluid. Show that if you drop a leaf into this fluid, as the leaf moves over time, the leaf does not rotate.

Hint

Calculate the curl.

Answer

$\operatorname{curl}\vec{\mathbf{v}} = \vec{\mathbf{0}}$

Using Divergence and Curl

Now that we understand the basic concepts of divergence and curl, we can discuss their properties and establish relationships between them and conservative vector fields.

If $\vec{\mathbf{F}}$ is a vector field in $\mathbb{R}^3$ then the curl of $\vec{\mathbf{F}}$ is also a vector field in $\mathbb{R}^3$. Therefore, we can take the divergence of a curl. The next theorem says that the result is always zero. This result is useful because it gives us a way to show that some vector fields are not the curl of any other field. To give this result a physical interpretation, recall that divergence of a velocity field $\vec{\mathbf{v}}$ at point P measures the tendency of the corresponding fluid to flow out of P. Since $\operatorname{div}(\operatorname{curl}\vec{\mathbf{v}}) = 0$, the net rate of flow in vector field $\operatorname{curl}\vec{\mathbf{v}}$ at any point is zero. Taking the curl of vector field $\vec{\mathbf{F}}$ eliminates whatever divergence was present in $\vec{\mathbf{F}}$.

Theorem: Divergence of the Curl

Let $\vec{\mathbf{F}} = \langle P,Q,R\rangle$ be a vector field in $\mathbb{R}^3$ such that the component functions all have continuous second-order partial derivatives. Then,

$$\operatorname{div}(\operatorname{curl}\vec{\mathbf{F}}) = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{\mathbf{F}}) = 0.$$

Proof

By the definitions of divergence and curl, and by Clairaut's theorem,

$$\begin{aligned}\operatorname{div}(\operatorname{curl}\vec{\mathbf{F}}) &= \operatorname{div}[(R_y - Q_z)\,\hat{\mathbf{i}} + (P_z - R_x)\,\hat{\mathbf{j}} + (Q_x - P_y)\,\hat{\mathbf{k}}] \\ &= R_{yx} - Q_{xz} + P_{yz} - R_{yx} + Q_{zx} - P_{zy} \\ &= 0.\end{aligned}$$

□

Example 16.5.8: Showing That a Vector Field Is Not the Curl of Another

Show that $\vec{\mathbf{F}}(x, y, z) = e^x\,\hat{\mathbf{i}} + yz\hat{\mathbf{j}} + xz^2\,\hat{\mathbf{k}}$ is not the curl of another vector field. That is, show that there is no other vector $\vec{\mathbf{G}}$ with $\operatorname{curl}\vec{\mathbf{G}} = \vec{\mathbf{F}}$.

Solution

Notice that the domain of $\vec{\mathbf{F}}$ is all of $\mathbb{R}^3$ and the second-order partials of $\vec{\mathbf{F}}$ are all continuous. Therefore, we can apply the previous theorem to $\vec{\mathbf{F}}$.

The divergence of $\vec{\mathbf{F}}$ is $e^x + z + 2xz$. If $\vec{\mathbf{F}}$ were the curl of vector field $\vec{\mathbf{G}}$, then $\operatorname{div}\vec{\mathbf{F}} = \operatorname{div}(\operatorname{curl}\vec{\mathbf{G}}) = 0$. But, the divergence of $\vec{\mathbf{F}}$ is not zero, and therefore $\vec{\mathbf{F}}$ is not the curl of any other vector field.

Exercise 16.5.8

Is it possible for $\vec{\mathbf{G}}(x, y, z) = \langle \sin x,\ \cos y,\ \sin(xyz)\rangle$ to be the curl of a vector field?

Hint

Find the divergence of $\vec{\mathbf{G}}$.

Answer

No.

With the next two theorems, we show that if $\vec{\mathbf{F}}$ is a conservative vector field then its curl is zero, and if the domain of $\vec{\mathbf{F}}$ is simply connected then the converse is also true. This gives us another way to test whether a vector field is conservative.

Theorem: Curl of a Conservative Vector Field

If $\vec{\mathbf{F}} = \langle P, Q, R\rangle$ is conservative, then $\operatorname{curl}\vec{\mathbf{F}} = \vec{\mathbf{0}}$.

Proof

Since conservative vector fields satisfy the cross-partials property, all the cross-partials of $\vec{\mathbf{F}}$ are equal. Therefore,

$$\begin{aligned}\operatorname{curl}\vec{\mathbf{F}} &= (R_y - Q_z)\,\hat{\mathbf{i}} + (P_z - R_x)\,\hat{\mathbf{j}} + (Q_x - P_y)\,\hat{\mathbf{k}} \\ &= \vec{\mathbf{0}}.\end{aligned}$$

□

The same theorem is true for vector fields in a plane.

Since a conservative vector field is the gradient of a scalar function, the previous theorem says that $\operatorname{curl}(\vec{\nabla} f) = \vec{\mathbf{0}}$ for any scalar function f. In terms of our curl notation, $\vec{\nabla} \times \vec{\nabla}(f) = \vec{\mathbf{0}}$. This equation makes sense because the cross product of a vector with itself is always the zero vector. Sometimes equation $\vec{\nabla} \times \vec{\nabla}(f) = \vec{\mathbf{0}}$ is simplified as $\vec{\nabla} \times \vec{\nabla} = \vec{\mathbf{0}}$.

Theorem: Curl Test for a Conservative Field

Let $\vec{\mathbf{F}} = \langle P, Q, R \rangle$ be a vector field in space on a simply connected domain. If $\text{curl}\, \vec{\mathbf{F}} = \vec{\mathbf{0}}$, then $\vec{\mathbf{F}}$ is conservative.

Proof

Since $\text{curl}\, \vec{\mathbf{F}} = \vec{\mathbf{0}}$, we have that $R_y = Q_z$, $P_z = R_x$, and $Q_x = P_y$. Therefore, $\vec{\mathbf{F}}$ satisfies the cross-partials property on a simply connected domain, and the Cross-Partial Property of Conservative Fields implies that $\vec{\mathbf{F}}$ is conservative.

□

The same theorem is also true in a plane. Therefore, if $\vec{\mathbf{F}}$ is a vector field in a plane or in space and the domain is simply connected, then $\vec{\mathbf{F}}$ is conservative if and only if $\text{curl}\, \vec{\mathbf{F}} = \vec{\mathbf{0}}$.

Example 16.5.9: Testing Whether a Vector Field Is Conservative

Use the curl to determine whether $\vec{\mathbf{F}}(x, y, z) = \langle yz, xz, xy \rangle$ is conservative.

Solution

Note that the domain of $\vec{\mathbf{F}}$ is all of $\mathbb{R}^3$ which is simply connected (Figure 16.5.7). Therefore, we can test whether $\vec{\mathbf{F}}$ is conservative by calculating its curl.

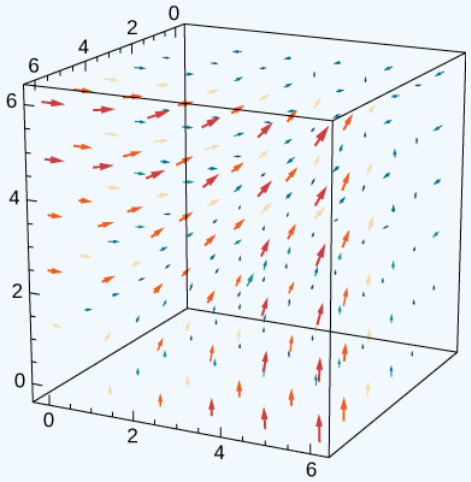


Figure 16.5.7: The curl of vector field $\vec{\mathbf{F}}(x, y, z) = \langle yz, xz, xy \rangle$ is zero.

The curl of $\vec{\mathbf{F}}$ is

$$\left(\frac{\partial}{\partial y}xy - \frac{\partial}{\partial z}xz\right)\hat{\mathbf{i}} + \left(\frac{\partial}{\partial y}yz - \frac{\partial}{\partial z}xy\right)\hat{\mathbf{j}} + \left(\frac{\partial}{\partial y}xz - \frac{\partial}{\partial z}yz\right)\hat{\mathbf{k}}$$
$$= (x - x)\hat{\mathbf{i}} + (y - y)\hat{\mathbf{j}} + (z - z)\hat{\mathbf{k}} = \vec{\mathbf{0}}.$$

Thus, $\vec{\mathbf{F}}$ is conservative.

We have seen that the curl of a gradient is zero. What is the divergence of a gradient? If f is a function of two variables, then $\text{div}(\vec{\nabla} f) = \vec{\nabla} \cdot (\vec{\nabla} f) = f_{xx} + f_{yy}$. We abbreviate this "double dot product" as $\vec{\nabla}^2$. This operator is called the *Laplace operator*, and in this notation Laplace's equation becomes $\vec{\nabla}^2 f = 0$. Therefore, a harmonic function is a function that becomes zero after taking the divergence of a gradient.

Similarly, if f is a function of three variables then

$$\text{div}(\vec{\nabla} f) = \vec{\nabla} \cdot (\vec{\nabla} f) = f_{xx} + f_{yy} + f_{zz}.$$

Using this notation we get Laplace's equation for harmonic functions of three variables:

$$\vec{\nabla}^2 f = 0.$$

Harmonic functions arise in many applications. For example, the potential function of an electrostatic field in a region of space that has no static charge is harmonic.

Example 16.5.10: Finding a Potential Function

Is it possible for $f(x,y) = x^2 + x - y$ to be the potential function of an electrostatic field that is located in a region of $\mathbb{R}^2$ free of static charge?

Solution

If f were such a potential function, then f would be harmonic. Note that $f_{xx} = 2$ and $f_{yy} = 0$, and so $f_{xx} + f_{yy} \neq 0$. Therefore, f is not harmonic and f cannot represent an electrostatic potential.

Exercise 16.5.10

Is it possible for function $f(x,y) = x^2 - y^2 + x$ to be the potential function of an electrostatic field located in a region of $\mathbb{R}^2$ free of static charge?

Hint

Determine whether the function is harmonic.

Answer

Yes.

Key Concepts

- The divergence of a vector field is a scalar function. Divergence measures the "outflowing-ness" of a vector field. If $\vec{\mathbf{v}}$ is the velocity field of a fluid, then the divergence of $\vec{\mathbf{v}}$ at a point is the outflow of the fluid less the inflow at the point.
- The curl of a vector field is a vector field. The curl of a vector field at point P measures the tendency of particles at P to rotate about the axis that points in the direction of the curl at P.
- A vector field with a simply connected domain is conservative if and only if its curl is zero.

Key Equations

- **Curl**

$$\vec{\nabla} \times \vec{\mathbf{F}} = (R_y - Q_z)\,\hat{\mathbf{i}} + (P_z - R_x)\,\hat{\mathbf{j}} + (Q_x - P_y)\,\hat{\mathbf{k}}$$

- **Divergence**

$$\vec{\nabla} \cdot \vec{\mathbf{F}} = P_x + Q_y + R_z$$

- **Divergence of curl is zero**

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{\mathbf{F}}) = 0$$

- **Curl of a gradient is the zero vector**

$$\vec{\nabla} \times (\vec{\nabla} f) = 0$$

Glossary

curl

the curl of vector field $\vec{\mathbf{F}} = \langle P, Q, R \rangle$, denoted $\vec{\nabla} \times \vec{\mathbf{F}}$ is the "determinant" of the matrix

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}.$$

and is given by the expression $(R_y - Q_z)\,\hat{\mathbf{i}} + (P_z - R_x)\,\hat{\mathbf{j}} + (Q_x - P_y)\,\hat{\mathbf{k}}$; it measures the tendency of particles at a point to rotate about the axis that points in the direction of the curl at the point

divergence

the divergence of a vector field $\vec{\mathbf{F}} = \langle P, Q, R \rangle$, denoted $\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{F}}$, is $P_x + Q_y + R_z$; it measures the "outflowing-ness" of a vector field

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16.5E: Exercises for Section 16.5

For the following exercises, determine whether the statement is *True* or *False*.

1. If the coordinate functions of $\vec{\mathbf{F}} : \mathbb{R}^3 \to \mathbb{R}^3$ have continuous second partial derivatives, then $\text{curl}\,(\text{div}\,\vec{\mathbf{F}})$ equals zero.

2. $\vec{\nabla} \cdot (x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}) = 1$.

Answer

False

3. All vector fields of the form $\vec{\mathbf{F}}(x, y, z) = f(x)\,\hat{\mathbf{i}} + g(y)\,\hat{\mathbf{j}} + h(z)\,\hat{\mathbf{k}}$ are conservative.

4. If $\text{curl}\,\vec{\mathbf{F}} = \vec{\mathbf{0}}$, then $\vec{\mathbf{F}}$ is conservative.

Answer

True

5. If $\vec{\mathbf{F}}$ is a constant vector field then $\text{div}\,\vec{\mathbf{F}} = 0$.

6. If $\vec{\mathbf{F}}$ is a constant vector field then $\text{curl}\,\vec{\mathbf{F}} = \vec{\mathbf{0}}$.

Answer

True

For the following exercises, find the curl of $\vec{\mathbf{F}}$.

7. $\vec{\mathbf{F}}(x, y, z) = xy^2z^4\,\hat{\mathbf{i}} + (2x^2y + z)\,\hat{\mathbf{j}} + y^3z^2\,\hat{\mathbf{k}}$

8. $\vec{\mathbf{F}}(x, y, z) = x^2z\,\hat{\mathbf{i}} + y^2x\,\hat{\mathbf{j}} + (y + 2z)\,\hat{\mathbf{k}}$

Answer

$\text{curl}\,\vec{\mathbf{F}} = \hat{\mathbf{i}} + x^2\,\hat{\mathbf{j}} + y^2\,\hat{\mathbf{k}}$

9. $\vec{\mathbf{F}}(x, y, z) = 3xyz^2\,\hat{\mathbf{i}} + y^2 \sin z\,\hat{\mathbf{j}} + xe^{2z}\,\hat{\mathbf{k}}$

10. $\vec{\mathbf{F}}(x, y, z) = x^2yz\,\hat{\mathbf{i}} + xy^2z\,\hat{\mathbf{j}} + xyz^2\,\hat{\mathbf{k}}$

Answer

$\text{curl}\,\vec{\mathbf{F}} = (xz^2 - xy^2)\,\hat{\mathbf{i}} + (x^2y - yz^2)\,\hat{\mathbf{j}} + (y^2z - x^2z)\,\hat{\mathbf{k}}$

11. $\vec{\mathbf{F}}(x, y, z) = (x\,\cos y)\,\hat{\mathbf{i}} + xy^2\,\hat{\mathbf{j}}$

12. $\vec{\mathbf{F}}(x, y, z) = (x - y)\,\hat{\mathbf{i}} + (y - z)\,\hat{\mathbf{j}} + (z - x)\,\hat{\mathbf{k}}$

Answer

$\text{curl}\ \vec{\mathbf{F}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$

13. $\vec{\mathbf{F}}(x, y, z) = xyz\,\hat{\mathbf{i}} + x^2y^2z^2\,\hat{\mathbf{j}} + y^2z^3\,\hat{\mathbf{k}}$

14. $\vec{\mathbf{F}}(x, y, z) = xy\,\hat{\mathbf{i}} + yz\,\hat{\mathbf{j}} + xz\,\hat{\mathbf{k}}$

Answer

$\text{curl}\ \vec{\mathbf{F}} = -y\,\hat{\mathbf{i}} - z\,\hat{\mathbf{j}} - x\,\hat{\mathbf{k}}$

15. $\vec{\mathbf{F}}(x, y, z) = x^2\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}} + z^2\,\hat{\mathbf{k}}$

16. $\vec{\mathbf{F}}(x, y, z) = ax\,\hat{\mathbf{i}} + by\,\hat{\mathbf{j}} + c\,\hat{\mathbf{k}}$ for constants a, b, c.

Answer

$\text{curl}\ \vec{\mathbf{F}} = \vec{\mathbf{0}}$

For the following exercises, find the divergence of $\vec{\mathbf{F}}$.

17. $\vec{\mathbf{F}}(x,y,z) = x^2 z\,\hat{\mathbf{i}} + y^2 x\,\hat{\mathbf{j}} + (y+2z)\,\hat{\mathbf{k}}$

18. $\vec{\mathbf{F}}(x,y,z) = 3xyz^2\,\hat{\mathbf{i}} + y^2 \sin z\,\hat{\mathbf{j}} + xe^2 z\,\hat{\mathbf{k}}$

Answer

$\operatorname{div}\vec{\mathbf{F}} = 3yz^2 + 2y\sin z + 2xe^{2z}$

19. $\vec{\mathbf{F}}(x,y) = (\sin x)\,\hat{\mathbf{i}} + (\cos y)\,\hat{\mathbf{j}}$

20. $\vec{\mathbf{F}}(x,y,z) = x^2\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}} + z^2\,\hat{\mathbf{k}}$

Answer

$\operatorname{div}\vec{\mathbf{F}} = 2(x+y+z)$

21. $\vec{\mathbf{F}}(x,y,z) = (x-y)\,\hat{\mathbf{i}} + (y-z)\,\hat{\mathbf{j}} + (z-x)\,\hat{\mathbf{k}}$

22. $\vec{\mathbf{F}}(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}\,\hat{\mathbf{i}} + \dfrac{y}{\sqrt{x^2+y^2}}\,\hat{\mathbf{j}}$

Answer

$\operatorname{div}\vec{\mathbf{F}} = \dfrac{1}{\sqrt{x^2+y^2}}$

23. $\vec{\mathbf{F}}(x,y) = x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}}$

24. $\vec{\mathbf{F}}(x,y,z) = ax\,\hat{\mathbf{i}} + by\,\hat{\mathbf{j}} + c\,\hat{\mathbf{k}}$ for constants a, b, c.

Answer

$\operatorname{div}\vec{\mathbf{F}} = a + b$

25. $\vec{\mathbf{F}}(x,y,z) = xyz\,\hat{\mathbf{i}} + x^2y^2z^2\,\hat{\mathbf{j}} + y^2z^3\,\hat{\mathbf{k}}$

26. $\vec{\mathbf{F}}(x,y,z) = xy\,\hat{\mathbf{i}} + yz\,\hat{\mathbf{j}} + xz\,\hat{\mathbf{k}}$

Answer

$\operatorname{div}\vec{\mathbf{F}} = x + y + z$

For exercises 27 & 28, determine whether each of the given scalar functions is harmonic.

27. $u(x,y,z) = e^{-x}(\cos y - \sin y)$

28. $w(x,y,z) = (x^2+y^2+z^2)^{-1/2}$

Answer

Harmonic

29. If $\vec{\mathbf{F}}(x,y,z) = 2\,\hat{\mathbf{i}} + 2xj + 3yk$ and $\vec{\mathbf{G}}(x,y,z) = x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$, find $\operatorname{curl}(\vec{\mathbf{F}} \times \vec{\mathbf{G}})$.

30. If $\vec{\mathbf{F}}(x,y,z) = 2\,\hat{\mathbf{i}} + 2xj + 3yk$ and $\vec{\mathbf{G}}(x,y,z) = x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$, find $\operatorname{div}(\vec{\mathbf{F}} \times \vec{\mathbf{G}})$.

Answer

$\operatorname{div}(\vec{\mathbf{F}} \times \vec{\mathbf{G}}) = 2z + 3x$

31. Find $\operatorname{div}\vec{\mathbf{F}}$, given that $\vec{\mathbf{F}} = \vec{\nabla} f$, where $f(x,y,z) = xy^3z^2$.

32. Find the divergence of $\vec{\mathbf{F}}$ for vector field $\vec{\mathbf{F}}(x,y,z) = (y^2+z^2)(x+y)\,\hat{\mathbf{i}} + (z^2+x^2)(y+z)\,\hat{\mathbf{j}} + (x^2+y^2)(z+x)\,\hat{\mathbf{k}}$.

Answer

$\operatorname{div}\vec{\mathbf{F}} = 2r^2$

33. Find the divergence of $\vec{\mathbf{F}}$ for vector field $\vec{\mathbf{F}}(x,y,z) = f_1(y,z)\,\hat{\mathbf{i}} + f_2(x,z)\,\hat{\mathbf{j}} + f_3(x,y)\,\hat{\mathbf{k}}$.

For exercises 34 - 36, use $r = |\vec{\mathbf{r}}|$ and $\vec{\mathbf{r}}(x,y,z) = \langle x,y,z\rangle$.

34. Find the $\text{curl}\ \vec{\mathbf{r}}$

Answer

$\text{curl}\ \vec{\mathbf{r}} = \vec{\mathbf{0}}$

35. Find the $\text{curl}\ \dfrac{\vec{\mathbf{r}}}{r}$.

36. Find the $\text{curl}\ \dfrac{\vec{\mathbf{r}}}{r^3}$.

Answer

$\text{curl}\ \dfrac{\vec{\mathbf{r}}}{r^3} = \vec{\mathbf{0}}$

37. Let $\vec{\mathbf{F}}(x,y) = \dfrac{-y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}}{x^2+y^2}$, where $\vec{\mathbf{F}}$ is defined on $\{(x,y) \in \mathbb{R} | (x,y) \neq (0,0)\}$. Find $\text{curl}\,\vec{\mathbf{F}}$.

For the following exercises, use a computer algebra system to find the curl of the given vector fields.

38. [T] $\vec{\mathbf{F}}(x,y,z) = \arctan\left(\dfrac{x}{y}\right)\,\hat{\mathbf{i}} + \ln\sqrt{x^2+y^2}\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$

Answer

$\text{curl}\ \vec{\mathbf{F}} = \dfrac{2x}{x^2+y^2}\,\hat{\mathbf{k}}$

39. [T] $\vec{\mathbf{F}}(x,y,z) = \sin(x-y)\,\hat{\mathbf{i}} + \sin(y-z)\,\hat{\mathbf{j}} + \sin(z-x)\,\hat{\mathbf{k}}$

For the following exercises, find the divergence of $\vec{\mathbf{F}}$ at the given point.

40. $\vec{\mathbf{F}}(x,y,z) = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ at $(2,-1,3)$

Answer

$\text{div}\,\vec{\mathbf{F}} = 0$

41. $\vec{\mathbf{F}}(x,y,z) = xyz\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ at $(1,2,3)$

42. $\vec{\mathbf{F}}(x,y,z) = e^{-xy}\,\hat{\mathbf{i}} + e^{xz}\,\hat{\mathbf{j}} + e^{yz}\,\hat{\mathbf{k}}$ at $(3,2,0)$

Answer

$\text{div}\,\vec{\mathbf{F}} = 2 - 2e^{-6}$

43. $\vec{\mathbf{F}}(x,y,z) = xyz\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ at $(1,2,1)$

44. $\vec{\mathbf{F}}(x,y,z) = e^x \sin y\,\hat{\mathbf{i}} - e^x \cos y\,\hat{\mathbf{j}}$ at $(0,0,3)$

Answer

$\text{div}\,\vec{\mathbf{F}} = 0$

For exercises 45- 49, find the curl of $\vec{\mathbf{F}}$ at the given point.

45. $\vec{\mathbf{F}}(x,y,z) = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ at $(2,-1,3)$

46. $\vec{\mathbf{F}}(x,y,z) = xyz\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + x\,\hat{\mathbf{k}}$ at $(1,2,3)$

Answer

$\text{curl}\ \vec{\mathbf{F}} = \hat{\mathbf{j}} - 3\,\hat{\mathbf{k}}$

47. $\vec{\mathbf{F}}(x,y,z) = e^{-xy}\,\hat{\mathbf{i}} + e^{xz}\,\hat{\mathbf{j}} + e^{yz}\,\hat{\mathbf{k}}$ at $(3,2,0)$

48. $\vec{\mathbf{F}}(x,y,z) = xyz\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ at $(1,2,1)$

Answer

$\text{curl}\ \vec{\mathbf{F}} = 2\,\hat{\mathbf{j}} - \hat{\mathbf{k}}$

49. $\vec{\mathbf{F}}(x,y,z) = e^x \sin y\,\hat{\mathbf{i}} - e^x \cos y\,\hat{\mathbf{j}}$ at $(0,0,3)$

50. Let $\vec{\mathbf{F}}(x,y,z) = (3x^2y + az)\,\hat{\mathbf{i}} + x^3\,\hat{\mathbf{j}} + (3x + 3z^2)\,\hat{\mathbf{k}}$. For what value of a is $\vec{\mathbf{F}}$ conservative?

Answer

$a = 3$

51. Given vector field $\vec{\mathbf{F}}(x,y) = \dfrac{1}{x^2+y^2}\langle -y, x\rangle$ on domain $D = \dfrac{\mathbb{R}^2}{\{(0,0)\}} = \{(x,y) \in \mathbb{R}^2 | (x,y) \neq (0,0)\}$, is $\vec{\mathbf{F}}$ conservative?

52. Given vector field $\vec{\mathbf{F}}(x,y) = \dfrac{1}{x^2+y^2}\langle x, y\rangle$ on domain $D = \dfrac{\mathbb{R}^2}{\{(0,0)\}}$, is $\vec{\mathbf{F}}$ conservative?

Answer

$\vec{\mathbf{F}}$ is conservative.

53. Find the work done by force field $\vec{\mathbf{F}}(x,y) = e^{-y}\,\hat{\mathbf{i}} - xe^{-y}\,\hat{\mathbf{j}}$ in moving an object from $P(0,1)$ to $Q(2,0)$. Is the force field conservative?

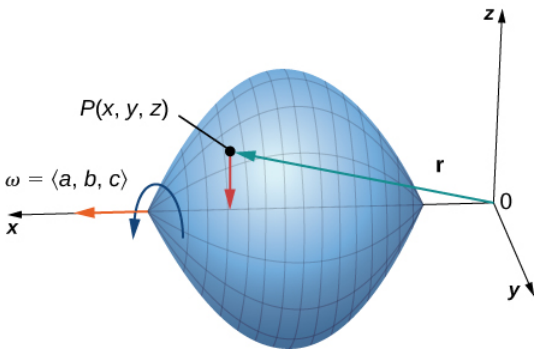
54. Compute divergence $\vec{\mathbf{F}}(x,y,z) = (\sinh x)\,\hat{\mathbf{i}} + (\cosh y)\,\hat{\mathbf{j}} - xyz\,\hat{\mathbf{k}}$.

Answer

$\text{div}\,\vec{\mathbf{F}} = \cosh x + \sinh y - xy$

55. Compute $\text{curl}\ \vec{\mathbf{F}} = (\sinh x)\,\hat{\mathbf{i}} + (\cosh y)\,\hat{\mathbf{j}} - xyz\,\hat{\mathbf{k}}$.

For the following exercises, consider a rigid body that is rotating about the x-axis counterclockwise with constant angular velocity $\vec{\omega} = \langle a, b, c\rangle$. If P is a point in the body located at $\vec{\mathbf{r}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$, the velocity at P is given by vector field $\vec{\mathbf{F}} = \vec{\omega} \times \vec{\mathbf{r}}$.



56. Express $\vec{\mathbf{F}}$ in terms of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ vectors.

Answer

$\vec{\mathbf{F}} = (bz - cy)\,\hat{\mathbf{i}} + (cx - az)\,\hat{\mathbf{j}} + (ay - bx)\,\hat{\mathbf{k}}$

57. Find $\text{div}\,F$.

58. Find $\text{curl}\,F$

Answer

$\text{curl}\ \vec{\mathbf{F}} = 2\vec{\omega}$

In the following exercises, suppose that $\vec{\nabla} \cdot \vec{\mathbf{F}} = 0$ and $\vec{\nabla} \cdot \vec{\mathbf{G}} = 0$.

59. Does $\vec{\mathbf{F}} + \vec{\mathbf{G}}$ necessarily have zero divergence?

60. Does $\vec{\mathbf{F}} \times \vec{\mathbf{G}}$ necessarily have zero divergence?

Answer

$\vec{\mathbf{F}} \times \vec{\mathbf{G}}$ does not have zero divergence.

In the following exercises, suppose a solid object in $\mathbb{R}^3$ has a temperature distribution given by $T(x, y, z)$. The heat flow vector field in the object is $\vec{\mathbf{F}} = -k\vec{\nabla}T$, where $k > 0$ is a property of the material. The heat flow vector points in the direction opposite to that of the gradient, which is the direction of greatest temperature decrease. The divergence of the heat flow vector is $\vec{\nabla} \cdot \vec{\mathbf{F}} = -k\vec{\nabla} \cdot \vec{\nabla}T = -k\vec{\nabla}^2 T$.

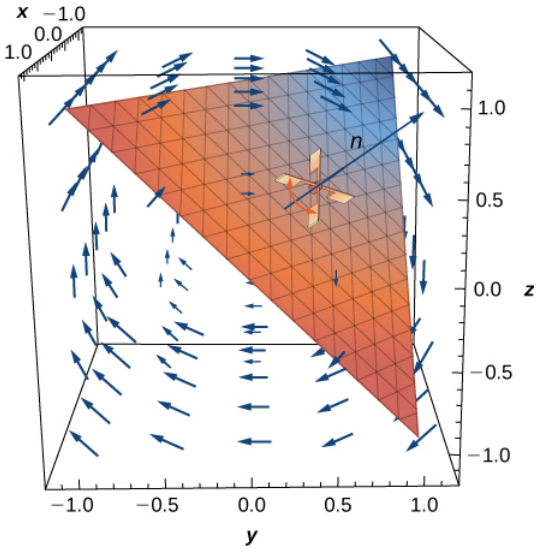
61. Compute the heat flow vector field.

62. Compute the divergence.

Answer

$\vec{\nabla} \cdot \vec{\mathbf{F}} = -200k[1 + 2(x^2 + y^2 + z^2)]e^{-x^2+y^2+z^2}$

63. [T] Consider rotational velocity field $\vec{\mathbf{v}} = \langle 0, 10z, -10y \rangle$. If a paddlewheel is placed in plane $x + y + z = 1$ with its axis normal to this plane, using a computer algebra system, calculate how fast the paddlewheel spins in revolutions per unit time.



Contributors

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16.6: Surface Integrals

Learning Objectives

- Find the parametric representations of a cylinder, a cone, and a sphere.
- Describe the surface integral of a scalar-valued function over a parametric surface.
- Use a surface integral to calculate the area of a given surface.
- Explain the meaning of an oriented surface, giving an example.
- Describe the surface integral of a vector field.
- Use surface integrals to solve applied problems.

We have seen that a line integral is an integral over a path in a plane or in space. However, if we wish to integrate over a surface (a two-dimensional object) rather than a path (a one-dimensional object) in space, then we need a new kind of integral that can handle integration over objects in higher dimensions. We can extend the concept of a line integral to a surface integral to allow us to perform this integration.

Surface integrals are important for the same reasons that line integrals are important. They have many applications to physics and engineering, and they allow us to develop higher dimensional versions of the Fundamental Theorem of Calculus. In particular, surface integrals allow us to generalize Green's theorem to higher dimensions, and they appear in some important theorems we discuss in later sections.

Parametric Surfaces

A surface integral is similar to a line integral, except the integration is done over a surface rather than a path. In this sense, surface integrals expand on our study of line integrals. Just as with line integrals, there are two kinds of surface integrals: a surface integral of a scalar-valued function and a surface integral of a vector field.

However, before we can integrate over a surface, we need to consider the surface itself. Recall that to calculate a scalar or vector line integral over curve C, we first need to parameterize C. In a similar way, to calculate a surface integral over surface S, we need to parameterize S. That is, we need a working concept of a *parameterized surface* (or a *parametric surface*), in the same way that we already have a concept of a parameterized curve.

A parameterized surface is given by a description of the form

$$\vec{\mathbf{r}}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v)\rangle.$$

Notice that this parameterization involves two parameters, u and v, because a surface is two-dimensional, and therefore two variables are needed to trace out the surface. The parameters u and v vary over a region called the parameter domain, or parameter space—the set of points in the uv-plane that can be substituted into $\vec{\mathbf{r}}$. Each choice of u and v in the parameter domain gives a point on the surface, just as each choice of a parameter t gives a point on a parameterized curve. The entire surface is created by making all possible choices of u and v over the parameter domain.

Definition: Parameter Domain

Given a parameterization of surface

$$\vec{\mathbf{r}}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v)\rangle.$$

the *parameter domain* of the parameterization is the set of points in the uv-plane that can be substituted into $\vec{\mathbf{r}}$.

Example 16.6.1: Parameterizing a Cylinder

Describe surface S parameterized by

$$\vec{\mathbf{r}}(u,v) = \langle \cos u,\, \sin u,\, v\rangle,\ -\infty < u < \infty,\ -\infty < v < \infty.$$

Solution

To get an idea of the shape of the surface, we first plot some points. Since the parameter domain is all of $\mathbb{R}^2$, we can choose any value for u and v and plot the corresponding point. If $u = v = 0$, then $\vec{\mathbf{r}}(0,0) = \langle 1,0,0\rangle$, so point (1, 0, 0) is on S. Similarly, points $\vec{\mathbf{r}}(\pi,2) = (-1,0,2)$ and $\vec{\mathbf{r}}\left(\frac{\pi}{2},4\right) = (0,1,4)$ are on S.

Although plotting points may give us an idea of the shape of the surface, we usually need quite a few points to see the shape. Since it is time-consuming to plot dozens or hundreds of points, we use another strategy. To visualize S, we visualize two families of curves that lie on S. In the first family of curves we hold u constant; in the second family of curves we hold v constant. This allows us to build a "skeleton" of the surface, thereby getting an idea of its shape.

- Suppose that u is a constant K. Then the curve traced out by the parameterization is $\langle \cos K,\ \sin K,\ v\rangle$, which gives a vertical line that goes through point $(\cos K, \sin K, v\rangle$ in the xy-plane.
- Suppose that v is a constant K. Then the curve traced out by the parameterization is $\langle \cos u,\ \sin u,\ K\rangle$, which gives a circle in plane $z = K$ with radius 1 and center $(0,0,K)$.

If u is held constant, then we get vertical lines; if v is held constant, then we get circles of radius 1 centered around the vertical line that goes through the origin. Therefore the surface traced out by the parameterization is cylinder $x^2 + y^2 = 1$ (Figure 16.6.1).

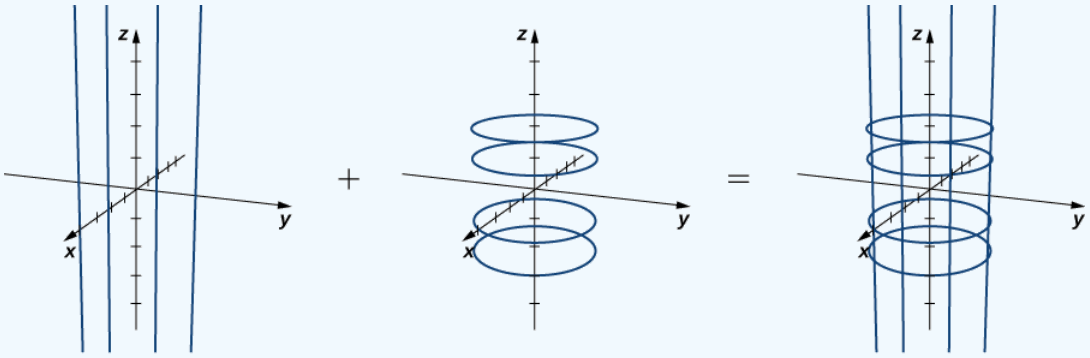


Figure 16.6.1: (a) Lines $\langle \cos K,\ \sin K, v\rangle$ for $K = 0,\ \frac{\pi}{2},\ \pi$, and $\frac{3\pi}{2}$. (b) Circles $\langle \cos u,\ \sin u,\ K\rangle$ for $K = -2, -1, 1$, and 2 (c) The lines and circles together. As u and v vary, they describe a cylinder.

Notice that if $x = \cos u$ and $y = \sin u$, then $x^2 + y^2 = 1$, so points from S do indeed lie on the cylinder. Conversely, each point on the cylinder is contained in some circle $\langle \cos u,\ \sin u,\ k\rangle$ for some k, and therefore each point on the cylinder is contained in the parameterized surface (Figure 16.6.2).

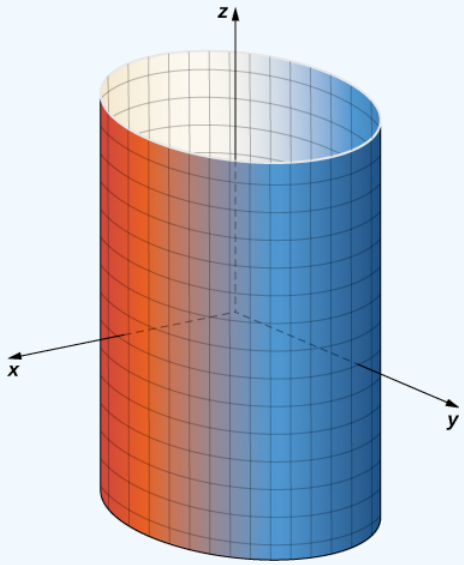


Figure 16.6.2: Cylinder $x^2 + y^2 = r^2$ has parameterization $\vec{\mathbf{r}}(u,v) = \langle r\ \cos u,\ r\ \sin u,\ v\rangle$, $0 \le u \le 2\pi,\ -\infty < v < \infty$.

Analysis

Notice that if we change the parameter domain, we could get a different surface. For example, if we restricted the domain to $0 \leq u \leq \pi,\ -\infty < v < 6$, then the surface would be a half-cylinder of height 6.

? Exercise 16.6.1

Describe the surface with parameterization

$$\vec{\mathbf{r}}(u,v) = \langle 2\cos u,\ 2\sin u,\ v\rangle,\ 0 \leq u \leq 2\pi,\ -\infty < v < \infty$$

Hint

Hold u and v constant, and see what kind of curves result.

Answer

Cylinder $x^2 + y^2 = 4$

It follows from Example 16.6.1 that we can parameterize all cylinders of the form $x^2 + y^2 = R^2$. If S is a cylinder given by equation $x^2 + y^2 = R^2$, then a parameterization of S is $\vec{\mathbf{r}}(u,v) = \langle R\cos u,\ R\sin u,\ v\rangle,\ 0 \leq u \leq 2\pi,\ -\infty < v < \infty$.

We can also find different types of surfaces given their parameterization, or we can find a parameterization when we are given a surface.

✓ Example 16.6.2: Describing a Surface

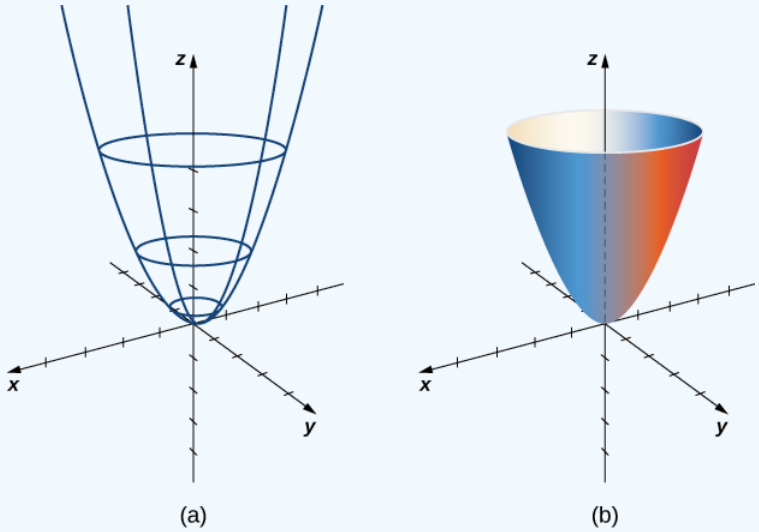
Describe surface S parameterized by $\vec{\mathbf{r}}(u,v) = \langle u\cos v,\ u\sin v,\ u^2\rangle,\ 0 \leq u < \infty,\ 0 \leq v < 2\pi$.

Solution

Notice that if u is held constant, then the resulting curve is a circle of radius u in plane $z = u$. Therefore, as u increases, the radius of the resulting circle increases. If v is held constant, then the resulting curve is a vertical parabola. Therefore, we expect the surface to be an elliptic paraboloid. To confirm this, notice that

$$\begin{aligned} x^2 + y^2 &= (u\cos v)^2 + (u\sin v)^2 \\ &= u^2\cos^2 v + u^2 sin^2 v \\ &= u^2 \\ &= z \end{aligned}$$

Therefore, the surface is the elliptic paraboloid $x^2 + y^2 = z$ (Figure 16.6.3).



(a) (b)

Figure 16.6.3: (a) Circles arise from holding u constant; the vertical parabolas arise from holding v constant. (b) An elliptic paraboloid results from all choices of u and v in the parameter domain.

Exercise 16.6.2

Describe the surface parameterized by $\vec{\mathbf{r}}(u,v) = \langle u\cos v,\ u\sin v,\ u\rangle,\ -\infty < u < \infty,\ 0 \le v < 2\pi$.

Hint

Hold u constant and see what kind of curves result. Imagine what happens as u increases or decreases.

Answer

Cone $x^2 + y^2 = z^2$

Example 16.6.3: Finding a Parameterization

Give a parameterization of the cone $x^2 + y^2 = z^2$ lying on or above the plane $z = -2$.

Solution

The horizontal cross-section of the cone at height $z = u$ is circle $x^2 + y^2 = u^2$. Therefore, a point on the cone at height u has coordinates $(u\cos v,\ u\sin v,\ u)$ for angle v. Hence, a parameterization of the cone is $\vec{\mathbf{r}}(u,v) = \langle u\cos v,\ u\sin v,\ u\rangle$. Since we are not interested in the entire cone, only the portion on or above plane $z = -2$, the parameter domain is given by $-2 < u < \infty,\ 0 \le v < 2\pi$ (Figure 16.6.4).

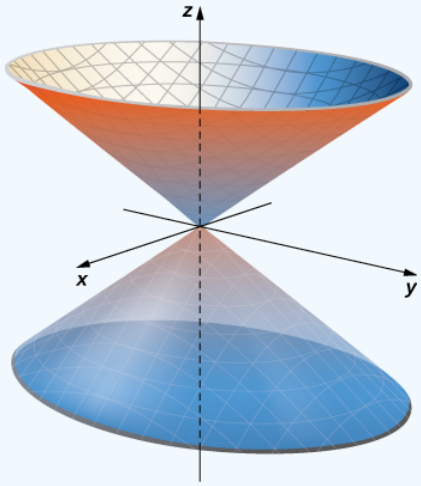


Figure 16.6.4: Cone $x^2 + y^2 = z^2$ has parametrization $\vec{\mathbf{r}}(u,v) = \langle u\cos v,\ u\sin v\rangle,\ -\infty < u < \infty,\ 0 \le v \le 2\pi$.

Exercise 16.6.3

Give a parameterization for the portion of cone $x^2 + y^2 = z^2$ lying in the first octant.

Hint

Consider the parameter domain for this surface.

Answer

$\vec{\mathbf{r}}(u,v) = \langle u\cos v,\ u\sin v,\ u\rangle,\ 0 < u < \infty,\ 0 \le v < \dfrac{\pi}{2}$

We have discussed parameterizations of various surfaces, but two important types of surfaces need a separate discussion: spheres and graphs of two-variable functions. To parameterize a sphere, it is easiest to use spherical coordinates. The sphere of radius ρ centered at the origin is given by the parameterization

$$\vec{\mathbf{r}}(\phi,\theta) = \langle \rho\cos\theta\sin\phi,\ \rho\sin\theta\sin\phi,\ \rho\cos\phi\rangle,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi.$$

The idea of this parameterization is that as ϕ sweeps downward from the positive z-axis, a circle of radius $\rho\sin\phi$ is traced out by letting θ run from 0 to 2π. To see this, let ϕ be fixed. Then

$$\begin{aligned} x^2 + y^2 &= (\rho \cos\theta \sin\phi)^2 + (\rho \sin\theta \sin\phi)^2 \\ &= \rho^2 \sin^2\phi(\cos^2\theta + \sin^2\theta) \\ &= \rho^2 \sin^2\phi \\ &= (\rho \sin\phi)^2. \end{aligned}$$

This results in the desired circle (Figure 16.6.5).

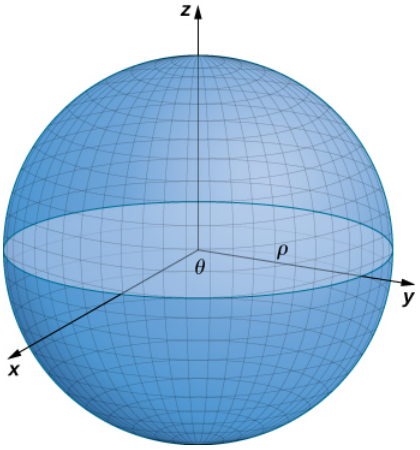


Figure 16.6.5: The sphere of radius ρ has parameterization $\vec{\mathbf{r}}(\phi, \theta) = \langle \rho \cos\theta \sin\phi,\ \rho \sin\theta \sin\phi,\ \rho \cos\phi \rangle,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi$.

Finally, to parameterize the graph of a two-variable function, we first let $z = f(x, y)$ be a function of two variables. The simplest parameterization of the graph of f is $\vec{\mathbf{r}}(x, y) = \langle x, y, f(x, y) \rangle$, where x and y vary over the domain of f (Figure 16.6.6). For example, the graph of $f(x, y) = x^2 y$ can be parameterized by $\vec{\mathbf{r}}(x, y) = \langle x, y, x^2 y \rangle$, where the parameters x and y vary over the domain of f. If we only care about a piece of the graph of f - say, the piece of the graph over rectangle $[1, 3] \times [2, 5]$ - then we can restrict the parameter domain to give this piece of the surface:

$$\vec{\mathbf{r}}(x, y) = \langle x, y, x^2 y \rangle,\ 1 \le x \le 3,\ 2 \le y \le 5.$$

Similarly, if S is a surface given by equation $x = g(y, z)$ or equation $y = h(x, z)$, then a parameterization of S is $\vec{\mathbf{r}}(y, z) = \langle g(y, z),\ y, z \rangle$ or $\vec{\mathbf{r}}(x, z) = \langle x, h(x, z), z \rangle$, respectively. For example, the graph of paraboloid $2y = x^2 + z^2$ can be parameterized by $\vec{\mathbf{r}}(x, y) = \left\langle x, \dfrac{x^2 + z^2}{2}, z \right\rangle,\ 0 \le x < \infty,\ 0 \le z < \infty$. Notice that we do not need to vary over the entire domain of y because x and z are squared.

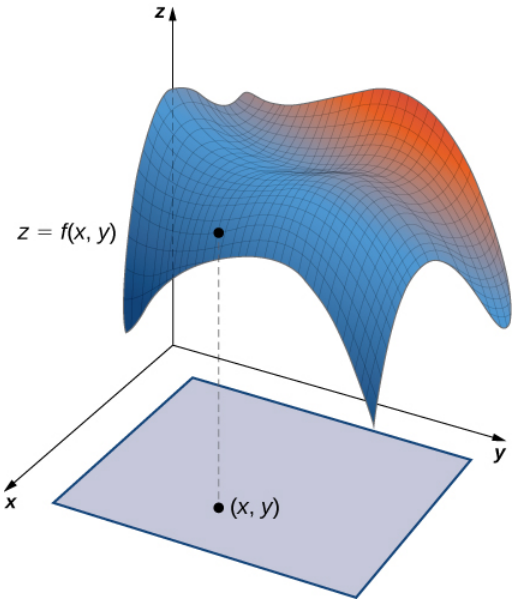


Figure 16.6.6: The simplest parameterization of the graph of a function is $\vec{\mathbf{r}}(x, y) = \langle x, y, f(x, y) \rangle$.

Let's now generalize the notions of smoothness and regularity to a parametric surface. Recall that curve parameterization $\vec{\mathbf{r}}(t),\ a \le t \le b$ is regular (or **smooth**) if $\vec{\mathbf{r}}'(t) \neq \vec{\mathbf{0}}$ for all t in $[a,b]$. For a curve, this condition ensures that the image of $\vec{\mathbf{r}}$ really is a curve, and not just a point. For example, consider curve parameterization $\vec{\mathbf{r}}(t) = \langle 1, 2\rangle,\ 0 \le t \le 5$. The image of this parameterization is simply point $(1,2)$, which is not a curve. Notice also that $\vec{\mathbf{r}}'(t) = \vec{\mathbf{0}}$. The fact that the derivative is the zero vector indicates we are not actually looking at a curve.

Analogously, we would like a notion of regularity (or smoothness) for surfaces so that a surface parameterization really does trace out a surface. To motivate the definition of regularity of a surface parameterization, consider the parameterization

$$\vec{\mathbf{r}}(u,v) = \langle 0,\ \cos v,\ 1\rangle,\ 0 \le u \le 1,\ 0 \le v \le \pi.$$

Although this parameterization appears to be the parameterization of a surface, notice that the image is actually a line (Figure 16.6.7). How could we avoid parameterizations such as this? Parameterizations that do not give an actual surface? Notice that $\vec{\mathbf{r}}_u = \langle 0,0,0\rangle$ and $\vec{\mathbf{r}}_v = \langle 0, -\sin v, 0\rangle$, and the corresponding cross product is zero. The analog of the condition $\vec{\mathbf{r}}'(t) = \vec{\mathbf{0}}$ is that $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$ is not zero for point (u,v) in the parameter domain, which is a regular parameterization.

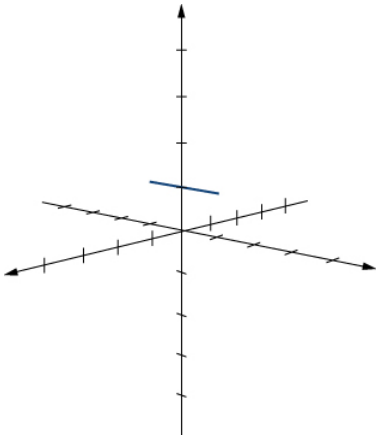


Figure 16.6.7: The image of parameterization $\vec{\mathbf{r}}(u,v) = \langle 0, \cos v, 1\rangle,\ 0 \le u \le 1,\ 0 \le v \le \pi$ is a line.

Definition: Regular Parameterization

Parameterization $\vec{\mathbf{r}}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ is a **regular parameterization** if $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$ is not zero for point (u,v) in the parameter domain.

If parameterization $\vec{r}$ is regular, then the image of $\vec{r}$ is a two-dimensional object, as a surface should be. Throughout this chapter, parameterizations $\vec{\mathbf{r}}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$are assumed to be regular.

Recall that curve parameterization $\vec{\mathbf{r}}(t),\ a \le t \le b$ is smooth if $\vec{\mathbf{r}}'(t)$ is continuous and $\vec{\mathbf{r}}'(t) \neq \vec{\mathbf{0}}$ for all t in $[a,b]$. Informally, a curve parameterization is smooth if the resulting curve has no sharp corners. The definition of a smooth surface parameterization is similar. Informally, a surface parameterization is *smooth* if the resulting surface has no sharp corners.

Definition: Smooth Surfaces

A surface parameterization $\vec{\mathbf{r}}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ is *smooth* if vector $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$ is not zero for any choice of u and v in the parameter domain.

A surface may also be *piecewise smooth* if it has smooth faces but also has locations where the directional derivatives do not exist.

Example 16.6.4: Identifying Smooth and Nonsmooth Surfaces

Which of the figures in Figure 16.6.8 is smooth?

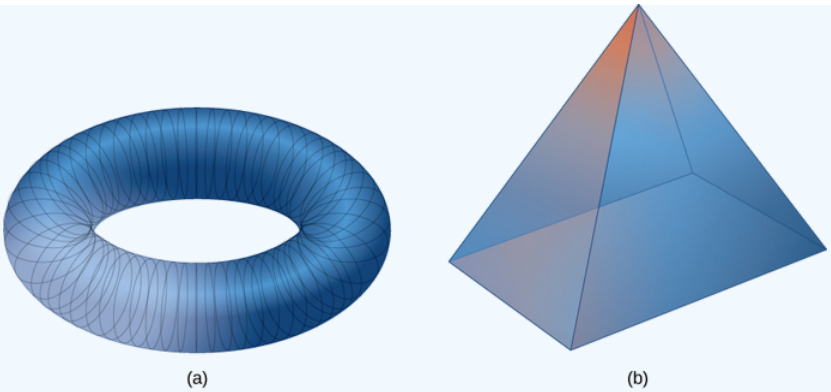


Figure 16.6.8: (a) This surface is smooth. (b) This surface is piecewise smooth.

Solution

The surface in Figure 16.6.8a can be parameterized by

$$\vec{\mathbf{r}}(u,v) = \langle (2+\cos v)\cos u,\ (2+\cos v)\sin u,\ \sin v\rangle,\ 0 \le u < 2\pi,\ 0 \le v < 2\pi$$

(we can use technology to verify). Notice that vectors

$$\vec{\mathbf{r}}_u = \langle -(2+\cos v)\sin u,\ (2+\cos v)\cos u, 0\rangle$$

and

$$\vec{\mathbf{r}}_v = \langle -\sin v\cos u,\ -\sin v\sin u,\ \cos v\rangle$$

exist for any choice of u and v in the parameter domain, and

$$\begin{aligned}\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -(2+\cos v)\sin u & (2+\cos v)\cos u & 0 \\ -\sin v\cos u & -\sin v\sin u & \cos v \end{vmatrix} \\ &= [(2+\cos v)\cos u\cos v]\hat{\mathbf{i}} + [2+\cos v)\sin u\cos v]\hat{\mathbf{j}} \\ &\quad + [(2+\cos v)\sin v\sin^2 u + (2+\cos v)\sin v\cos^2 u]\hat{\mathbf{k}} \\ &= [(2+\cos v)\cos u\cos v]\hat{\mathbf{i}} + [(2+\cos v)\sin u\cos v]\hat{\mathbf{j}} + [(2+\cos v)\sin v]\hat{\mathbf{k}}.\end{aligned}$$

The $\hat{\mathbf{k}}$ component of this vector is zero only if $v = 0$ or $v = \pi$. If $v = 0$ or $v = \pi$, then the only choices for u that make the $\hat{\mathbf{j}}$ component zero are $u = 0$ or $u = \pi$. But, these choices of u do not make the $\hat{\mathbf{i}}$ component zero. Therefore, $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$ is not zero for any choice of u and v in the parameter domain, and the parameterization is smooth. Notice that the corresponding surface has no sharp corners.

In the pyramid in Figure 16.6.8b, the sharpness of the corners ensures that directional derivatives do not exist at those locations. Therefore, the pyramid has no smooth parameterization. However, the pyramid consists of four smooth faces, and thus this surface is piecewise smooth.

? Exercise 16.6.4

Is the surface parameterization $\vec{\mathbf{r}}(u,v) = \langle u^{2v}, v+1, \sin u\rangle$, $0 \le u \le 2$, $0 \le v \le 3$ smooth?

Hint

Investigate the cross product $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$.

Answer

Yes

Surface Area of a Parametric Surface

Our goal is to define a surface integral, and as a first step we have examined how to parameterize a surface. The second step is to define the surface area of a parametric surface. The notation needed to develop this definition is used throughout the rest of this chapter.

Let S be a surface with parameterization $\vec{\mathbf{r}}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$ over some parameter domain D. We assume here and throughout that the surface parameterization $\vec{\mathbf{r}}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$ is continuously differentiable—meaning, each component function has continuous partial derivatives. Assume for the sake of simplicity that D is a rectangle (although the following material can be extended to handle nonrectangular parameter domains). Divide rectangle D into subrectangles D_{ij} with horizontal width Δu and vertical length Δv. Suppose that *i* ranges from 1 to *m* and *j* ranges from 1 to *n* so that D is subdivided into *mn* rectangles. This division of D into subrectangles gives a corresponding division of surface S into pieces S_{ij}. Choose point P_{ij} in each piece S_{ij}. Point P_{ij} corresponds to point (u_i, v_j) in the parameter domain.

Note that we can form a grid with lines that are parallel to the u-axis and the v-axis in the uv-plane. These grid lines correspond to a set of grid curves on surface S that is parameterized by $\vec{\mathbf{r}}(u,v)$. Without loss of generality, we assume that P_{ij} is located at the corner of two grid curves, as in Figure 16.6.9. If we think of $\vec{\mathbf{r}}$ as a mapping from the uv-plane to $\mathbb{R}^3$, the grid curves are the image of the grid lines under $\vec{\mathbf{r}}$. To be precise, consider the grid lines that go through point (u_i, v_j). One line is given by $x = u_i,\ y = v$; the other is given by $x = u,\ y = v_j$. In the first grid line, the horizontal component is held constant, yielding a vertical line through (u_i, v_j). In the second grid line, the vertical component is held constant, yielding a horizontal line through (u_i, v_j). The corresponding grid curves are $\vec{\mathbf{r}}(u_i, v)$ and (u, v_j) and these curves intersect at point P_{ij}.

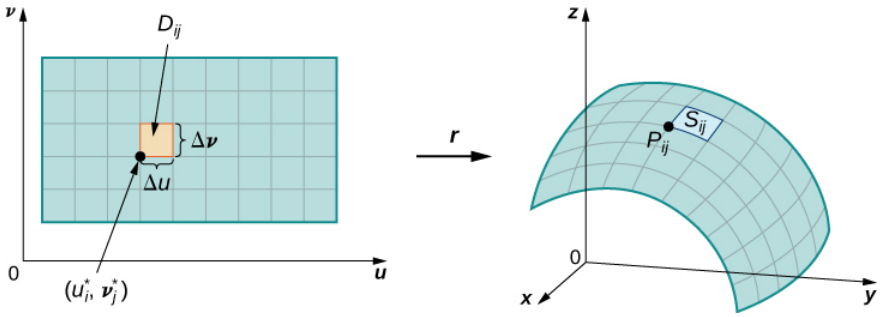


Figure 16.6.9: Grid lines on a parameter domain correspond to grid curves on a surface.

Now consider the vectors that are tangent to these grid curves. For grid curve $\vec{\mathbf{r}}(u_i, v)$, the tangent vector at P_{ij} is

$$\vec{\mathbf{t}}_v(P_{ij}) = \vec{\mathbf{r}}_v(u_i, v_j) = \langle x_v(u_i, v_j),\ y_v(u_i, v_j),\ z_v(u_i, v_j)\rangle.$$

For grid curve $\vec{\mathbf{r}}(u, v_j)$, the tangent vector at P_{ij} is

$$\vec{\mathbf{t}}_u(P_{ij}) = \vec{\mathbf{r}}_u(u_i, v_j) = \langle x_u(u_i, v_j),\ y_u(u_i, v_j),\ z_u(u_i, v_j)\rangle.$$

If vector $\vec{\mathbf{N}} = \vec{\mathbf{t}}_u(P_{ij}) \times \vec{\mathbf{t}}_v(P_{ij})$ exists and is not zero, then the tangent plane at P_{ij} exists (Figure 16.6.10). If piece S_{ij} is small enough, then the tangent plane at point P_{ij} is a good approximation of piece S_{ij}.

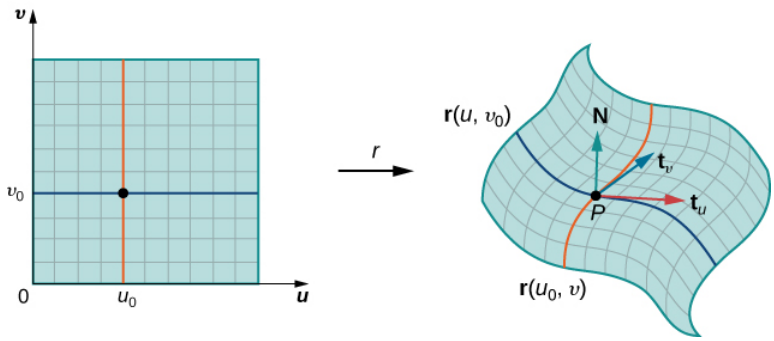


Figure 16.6.10: If the cross product of vectors $\vec{\mathbf{t}}_u$ and $\vec{\mathbf{t}}_v$ exists, then there is a tangent plane.

The tangent plane at P_{ij} contains vectors $\vec{\mathbf{t}}_u(P_{ij})$ and $\vec{\mathbf{t}}_v(P_{ij})$ and therefore the parallelogram spanned by $\vec{\mathbf{t}}_u(P_{ij})$ and $\vec{\mathbf{t}}_v(P_{ij})$ is in the tangent plane. Since the original rectangle in the uv-plane corresponding to S_{ij} has width Δu and length Δv, the parallelogram

that we use to approximate S_{ij} is the parallelogram spanned by $\Delta u \vec{\mathbf{t}}_u(P_{ij})$ and $\Delta v \vec{\mathbf{t}}_v(P_{ij})$. In other words, we scale the tangent vectors by the constants Δu and Δv to match the scale of the original division of rectangles in the parameter domain. Therefore, the area of the parallelogram used to approximate the area of S_{ij} is

$$\Delta S_{ij} \approx ||(\Delta u \vec{\mathbf{t}}_u(P_{ij})) \times (\Delta v \vec{\mathbf{t}}_v(P_{ij}))|| = ||\vec{\mathbf{t}}_u(P_{ij}) \times \vec{\mathbf{t}}_v(P_{ij})|| \Delta u \, \Delta v.$$

Varying point P_{ij} over all pieces S_{ij} and the previous approximation leads to the following definition of surface area of a parametric surface (Figure 16.6.11).

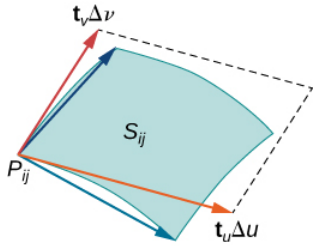


Figure 16.6.11: The parallelogram spanned by $\vec{\mathbf{t}}_u$ *and* $\vec{\mathbf{t}}_v$ *approximates the piece of surface* S_{ij}.

Definition: Smooth Parameterization of Surface

Let $\vec{\mathbf{r}}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$ with parameter domain D be a **smooth parameterization of surface** S. Furthermore, assume that S is traced out only once as (u,v) varies over D. The surface area of S is

$$\iint_D ||\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v|| \, dA, \tag{16.6.1}$$

where $\vec{\mathbf{t}}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$

and

$$\vec{\mathbf{t}}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle.$$

Example 16.6.5: Calculating Surface Area

Calculate the lateral surface area (the area of the "side," not including the base) of the right circular cone with height h and radius r.

Solution

Before calculating the surface area of this cone using Equation 16.6.1, we need a parameterization. We assume this cone is in $\mathbb{R}^3$ with its vertex at the origin (Figure 16.6.12). To obtain a parameterization, let α be the angle that is swept out by starting at the positive z-axis and ending at the cone, and let $k = \tan\alpha$. For a height value v with $0 \le v \le h$, the radius of the circle formed by intersecting the cone with plane $z = v$ is kv. Therefore, a parameterization of this cone is

$$\vec{\mathbf{s}}(u,v) = \langle kv\cos u,\ kv\sin u,\ v\rangle,\ 0 \le u < 2\pi,\ 0 \le v \le h.$$

The idea behind this parameterization is that for a fixed v-value, the circle swept out by letting u vary is the circle at height v and radius kv. As v increases, the parameterization sweeps out a "stack" of circles, resulting in the desired cone.

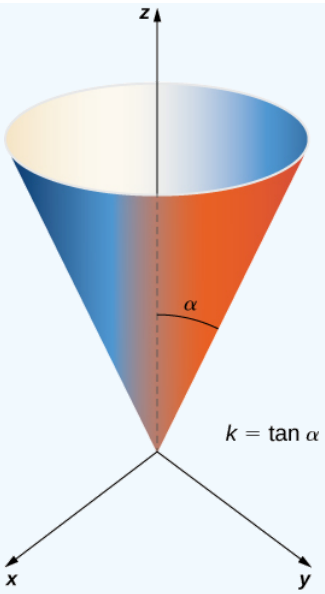


Figure 16.6.12: The right circular cone with radius $r = kh$ and height h has parameterization $\vec{\mathbf{s}}(u, v) = \langle kv\ \cos u,\ kv\ \sin u,\ v\rangle,\ 0 \le u < 2\pi,\ 0 \le v \le h$.

With a parameterization in hand, we can calculate the surface area of the cone using Equation 16.6.1. The tangent vectors are $\vec{\mathbf{t}}_u = \langle -kv\ \sin u,\ kv\ \cos u,\ 0\rangle$ and $\vec{\mathbf{t}}_v = \langle k\ \cos u,\ k\ \sin u,\ 1\rangle$. Therefore,

$$\begin{aligned}\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -kv\sin u & kv\cos u & 0 \\ k\cos u & k\sin u & 1 \end{vmatrix} \\ &= \langle kv\ \cos u,\ kv\ \sin u,\ -k^2 v\ \sin^2 u - k^2 v\ \cos^2 u\rangle \\ &= \langle kv\ \cos u,\ kv\ \sin u,\ -k^2 v\rangle.\end{aligned}$$

The magnitude of this vector is

$$\begin{aligned}||\langle kv\ \cos u,\ kv\ \sin u,\ -k^2 v\rangle|| &= \sqrt{k^2v^2\cos^2 u + k^2v^2\sin^2 u + k^4v^2} \\ &= \sqrt{k^2v^2 + k^4v^2} \\ &= kv\sqrt{1+k^2}.\end{aligned}$$

By Equation 16.6.1, the surface area of the cone is

$$\begin{aligned}\iint_D ||\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v||\, dA &= \int_0^h \int_0^{2\pi} kv\sqrt{1+k^2}\, du\, dv \\ &= 2\pi k\sqrt{1+k^2}\int_0^h v\, dv \\ &= 2\pi k\sqrt{1+k^2}\left[\frac{v^2}{2}\right]_0^h \\ &= \pi k h^2\sqrt{1+k^2}.\end{aligned}$$

Since $k = \tan\alpha = r/h$,

$$\pi k h^2 \sqrt{1+k^2} = \pi \frac{r}{h} h^2 \sqrt{1+\frac{r^2}{h^2}}$$
$$= \pi r h \sqrt{1+\frac{r^2}{h^2}}$$
$$= \pi r \sqrt{h^2 + h^2\left(\frac{r^2}{h^2}\right)}$$
$$= \pi r \sqrt{h^2+r^2}.$$

Therefore, the lateral surface area of the cone is $\pi r\sqrt{h^2+r^2}$.

Analysis

The surface area of a right circular cone with radius r and height h is usually given as $\pi r^2 + \pi r\sqrt{h^2+r^2}$. The reason for this is that the circular base is included as part of the cone, and therefore the area of the base πr^2 is added to the lateral surface area $\pi r\sqrt{h^2+r^2}$ that we found.

Exercise 16.6.5

Find the surface area of the surface with parameterization $\vec{\mathbf{r}}(u,v) = \langle u+v,\, u^2,\, 2v\rangle,\ 0 \le u \le 3,\ 0 \le v \le 2$.

Hint

Use Equation 16.6.1.

Answer

≈ 43.02

Example 16.6.6: Calculating Surface Area

Show that the surface area of the sphere $x^2+y^2+z^2 = r^2$ is $4\pi r^2$.

Solution

The sphere has parameterization

$$r\cos\theta\sin\phi,\ r\sin\theta\sin\phi,\ r\cos\phi\rangle,\ 0 \le \theta < 2\pi,\ 0 \le \phi \le \pi.$$

The tangent vectors are

$$\vec{\mathbf{t}}_\theta = \langle -r\sin\theta\sin\phi,\ r\cos\theta\sin\phi,\ 0\rangle$$

and

$$\vec{\mathbf{t}}_\phi = \langle r\cos\theta\cos\phi,\ r\sin\theta\cos\phi,\ -r\sin\phi\rangle.$$

Therefore,

$$\vec{\mathbf{t}}_\phi \times \vec{\mathbf{t}}_\theta = \langle r^2\cos\theta\sin^2\phi,\ r^2\sin\theta\sin^2\phi,\ r^2\sin^2\theta\sin\phi\cos\phi + r^2\cos^2\theta\sin\phi\cos\phi\rangle$$
$$= \langle r^2\cos\theta\sin^2\phi,\ r^2\sin\theta\sin^2\phi,\ r^2\sin\phi\cos\phi\rangle.$$

Now,

$$||\vec{\mathbf{t}}_\phi \times \vec{\mathbf{t}}_\theta|| = \sqrt{r^4\sin^4\phi\cos^2\theta + r^4\sin^4\phi\sin^2\theta + r^4\sin^2\phi\cos^2\phi}$$
$$= \sqrt{r^4\sin^4\phi + r^4\sin^2\phi\cos^2\phi}$$
$$= r^2\sqrt{\sin^2\phi}$$
$$= r\sin\phi.$$

Notice that $\sin\phi \geq 0$ on the parameter domain because $0 \leq \phi < \pi$, and this justifies equation $\sqrt{\sin^2\phi} = \sin\phi$. The surface area of the sphere is

$$\int_0^{2\pi}\int_0^{\pi} r^2 \sin\phi \, d\phi \, d\theta = r^2 \int_0^{2\pi} 2 \, d\theta = 4\pi r^2.$$

We have derived the familiar formula for the surface area of a sphere using surface integrals.

Exercise 16.6.6

Show that the surface area of cylinder $x^2 + y^2 = r^2$, $0 \leq z \leq h$ is $2\pi rh$. Notice that this cylinder does not include the top and bottom circles.

Hint

Use the standard parameterization of a cylinder and follow the previous example.

Answer

With the standard parameterization of a cylinder, Equation 16.6.1 shows that the surface area is $2\pi rh$.

In addition to parameterizing surfaces given by equations or standard geometric shapes such as cones and spheres, we can also parameterize surfaces of revolution. Therefore, we can calculate the surface area of a surface of revolution by using the same techniques. Let $y = f(x) \geq 0$ be a positive single-variable function on the domain $a \leq x \leq b$ and let S be the surface obtained by rotating f about the x-axis (Figure 16.6.13). Let θ be the angle of rotation. Then, S can be parameterized with parameters x and θ by

$$\vec{\mathbf{r}}(x,\theta) = \langle x, f(x)\cos\theta, f(x)\sin\theta\rangle, \; a \leq x \leq b, \; 0 \leq x \leq 2\pi.$$

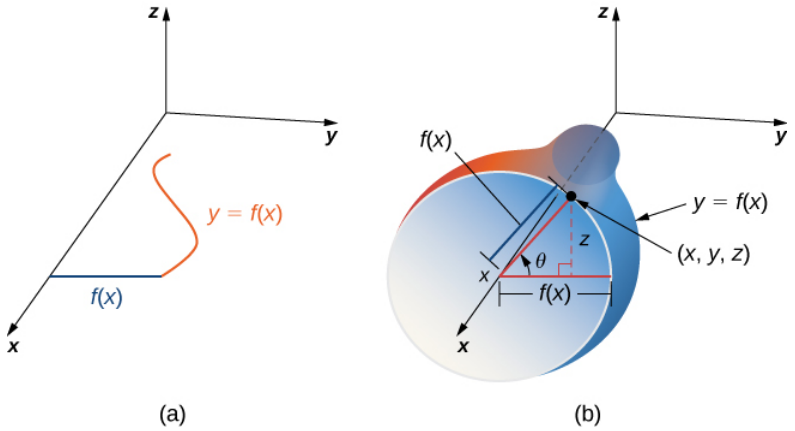


Figure 16.6.13: We can parameterize a surface of revolution by $\vec{\mathbf{r}}(x,\theta) = \langle x, f(x)\cos\theta, f(x)\sin\theta\rangle$, $a \leq x \leq b$, $0 \leq x < 2\pi$.

Example 16.6.7: Calculating Surface Area

Find the area of the surface of revolution obtained by rotating $y = x^2$, $0 \leq x \leq b$ about the x-axis (Figure 16.6.14).

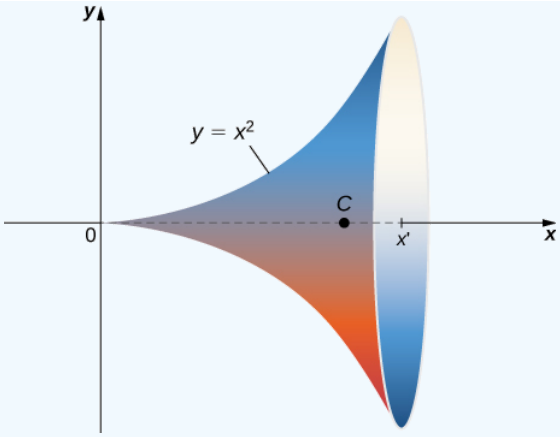


Figure 16.6.14: A surface integral can be used to calculate the surface area of this solid of revolution.

Solution

This surface has parameterization $\vec{\mathbf{r}}(x,\theta) = \langle x,\, x^2\cos\theta,\, x^2\sin\theta\rangle,\ 0 \le x \le b,\ 0 \le x < 2\pi.$

The tangent vectors are $\vec{\mathbf{t}}_x = \langle 1,\, 2x\,\cos\theta,\, 2x\,\sin\theta\rangle$ and $\vec{\mathbf{t}}_\theta = \langle 0,\, -x^2\sin\theta,\, -x^2\cos\theta\rangle$.

Therefore,

$$\begin{aligned}\vec{\mathbf{t}}_x \times \vec{\mathbf{t}}_\theta &= \langle 2x^3\cos^2\theta + 2x^3\sin^2\theta,\, -x^2\cos\theta,\, -x^2\sin\theta\rangle \\ &= \langle 2x^3,\, -x^2\cos\theta,\, -x^2\sin\theta\rangle\end{aligned}$$

and

$$\begin{aligned}\vec{\mathbf{t}}_x \times \vec{\mathbf{t}}_\theta &= \sqrt{4x^6 + x^4\cos^2\theta + x^4\sin^2\theta} \\ &= \sqrt{4x^6 + x^4} \\ &= x^2\sqrt{4x^2+1}\end{aligned}$$

The area of the surface of revolution is

$$\begin{aligned}\int_0^b\int_0^{2\pi} x^2\sqrt{4x^2+1}\,d\theta\,dx &= 2\pi\int_0^b x^2\sqrt{4x^2+1}\,dx \\ &= 2\pi\left[\frac{1}{64}\left(2\sqrt{4x^2+1}(8x^3+x)\,\sinh^{-1}(2x)\right)\right]_0^b \\ &= 2\pi\left[\frac{1}{64}\left(2\sqrt{4b^2+1}(8b^3+b)\,\sinh^{-1}(2b)\right)\right].\end{aligned}$$

? Exercise 16.6.7

Use Equation 16.6.1 to find the area of the surface of revolution obtained by rotating curve $y = \sin x,\ 0 \le x \le \pi$ about the x-axis.

Hint

Use the parameterization of surfaces of revolution given before Example 16.6.7.

Answer

$2\pi(\sqrt{2} + \sinh^{-1}(1))$

Surface Integral of a Scalar-Valued Function

Now that we can parameterize surfaces and we can calculate their surface areas, we are able to define surface integrals. First, let's look at the surface integral of a scalar-valued function. Informally, the surface integral of a scalar-valued function is an analog of a scalar line integral in one higher dimension. The domain of integration of a scalar line integral is a parameterized curve (a one-dimensional object); the domain of integration of a scalar surface integral is a parameterized surface (a two-dimensional object).

Therefore, the definition of a surface integral follows the definition of a line integral quite closely. For scalar line integrals, we chopped the domain curve into tiny pieces, chose a point in each piece, computed the function at that point, and took a limit of the corresponding Riemann sum. For scalar surface integrals, we chop the domain *region* (no longer a curve) into tiny pieces and proceed in the same fashion.

Let S be a piecewise smooth surface with parameterization $\vec{\mathbf{r}}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$ with parameter domain D and let $f(x,y,z)$ be a function with a domain that contains S. For now, assume the parameter domain D is a rectangle, but we can extend the basic logic of how we proceed to any parameter domain (the choice of a rectangle is simply to make the notation more manageable). Divide rectangle D into subrectangles D_{ij} with horizontal width Δu and vertical length Δv. Suppose that i ranges from 1 to m and j ranges from 1 to n so that D is subdivided into mn rectangles. This division of D into subrectangles gives a corresponding division of S into pieces S_{ij}. Choose point P_{ij} in each piece S_{ij} evaluate P_{ij} at f, and multiply by area S_{ij} to form the Riemann sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij})\,\Delta S_{ij}.$$

To define a surface integral of a scalar-valued function, we let the areas of the pieces of S shrink to zero by taking a limit.

Definition: Surface Integral of a Scalar-Valued Function

The **surface integral of a scalar-valued function** of f over a piecewise smooth surface S is

$$\iint_S f(x,y,z)dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij})\Delta S_{ij}.$$

Again, notice the similarities between this definition and the definition of a scalar line integral. In the definition of a line integral we chop a curve into pieces, evaluate a function at a point in each piece, and let the length of the pieces shrink to zero by taking the limit of the corresponding Riemann sum. In the definition of a surface integral, we chop a surface into pieces, evaluate a function at a point in each piece, and let the area of the pieces shrink to zero by taking the limit of the corresponding Riemann sum. Thus, a surface integral is similar to a line integral but in one higher dimension.

The definition of a scalar line integral can be extended to parameter domains that are not rectangles by using the same logic used earlier. The basic idea is to chop the parameter domain into small pieces, choose a sample point in each piece, and so on. The exact shape of each piece in the sample domain becomes irrelevant as the areas of the pieces shrink to zero.

Scalar surface integrals are difficult to compute from the definition, just as scalar line integrals are. To develop a method that makes surface integrals easier to compute, we approximate surface areas ΔS_{ij} with small pieces of a tangent plane, just as we did in the previous subsection. Recall the definition of vectors $\vec{\mathbf{t}}_u$ and $\vec{\mathbf{t}}_v$:

$$\vec{\mathbf{t}}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right\rangle \text{ and } \vec{\mathbf{t}}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right\rangle.$$

From the material we have already studied, we know that

$$\Delta S_{ij} \approx ||\vec{\mathbf{t}}_u(P_{ij}) \times \vec{\mathbf{t}}_v(P_{ij})||\,\Delta u\,\Delta v.$$

Therefore,

$$\iint_S f(x,y,z)\,dS \approx \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij})||\vec{\mathbf{t}}_u(P_{ij}) \times \vec{\mathbf{t}}_v(P_{ij})||\,\Delta u\,\Delta v.$$

This approximation becomes arbitrarily close to $\lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij})\Delta S_{ij}$ as we increase the number of pieces S_{ij} by letting m and n go to infinity. Therefore, we have the following equation to calculate **scalar surface integrals**:

$$\iint_S f(x,y,z)\,dS = \iint_D f(\vec{\mathbf{r}}(u,v))||\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v||\,dA. \tag{16.6.2}$$

Equation 16.6.2 allows us to calculate a surface integral by transforming it into a double integral. This equation for surface integrals is analogous to the equation for line integrals:

$$\iint_C f(x,y,z)\,ds = \int_a^b f(\vec{\mathbf{r}}(t))||\vec{\mathbf{r}}'(t)||\,dt.$$

In this case, vector $\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v$ is perpendicular to the surface, whereas vector $\vec{\mathbf{r}}'(t)$ is tangent to the curve.

Example 16.6.8: Calculating a Surface Integral

Calculate surface integral

$$\iint_S 5\,dS,$$

where S is the surface with parameterization $\vec{\mathbf{r}}(u,v) = \langle u, u^2, v\rangle$ for $0 \le u \le 2$ and $0 \le v \le u$.

Solution

Notice that this parameter domain D is a triangle, and therefore the parameter domain is not rectangular. This is not an issue though, because Equation 16.6.2 does not place any restrictions on the shape of the parameter domain.

To use Equation 16.6.2 to calculate the surface integral, we first find vectors $\vec{\mathbf{t}}_u$ and $\vec{\mathbf{t}}_v$. Note that $\vec{\mathbf{t}}_u = \langle 1, 2u, 0\rangle$ and $\vec{\mathbf{t}}_v = \langle 0, 0, 1\rangle$. Therefore,

$$\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 2u, -1, 0\rangle$$

and

$$||\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v|| = \sqrt{1+4u^2}.$$

By Equation 16.6.2,

$$\begin{aligned} \iint_S 5\,dS &= 5\iint_D \sqrt{1+4u^2}\,dA \\ &= 5\int_0^2 \int_0^u \sqrt{1+4u^2}\,dv\,du = 5\int_0^2 u\sqrt{1+4u^2}\,du \\ &= 5\left[\frac{(1+4u^2)^{3/2}}{3}\right]_0^2 \\ &= \frac{5(17^{3/2}-1)}{3} \approx 115.15. \end{aligned}$$

Example 16.6.9: Calculating the Surface Integral of a Cylinder

Calculate surface integral

$$\iint_S (x+y^2)\,dS,$$

where S is cylinder $x^2+y^2=4$, $0 \le z \le 3$ (Figure 16.6.15).

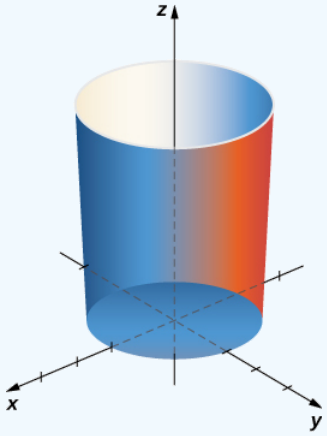


Figure 16.6.15: Integrating function $f(x, y, z) = x + y^2$ over a cylinder.

Solution

To calculate the surface integral, we first need a parameterization of the cylinder. A parameterization is $\vec{\mathbf{r}}(u, v) = \langle \cos u,\ \sin u,\ v \rangle, 0 \le u \le 2\pi,\ 0 \le v \le 3$.

The tangent vectors are $\vec{\mathbf{t}}_u = \langle \sin u,\ \cos u,\ 0 \rangle$ and $\vec{\mathbf{t}}_v = \langle 0, 0, 1 \rangle$. Then,

$$\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle \cos u,\ \sin u,\ 0 \rangle$$

and $||\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v|| = \sqrt{\cos^2 u + \sin^2 u} = 1$. By Equation 16.6.2,

$$\begin{aligned} \iint_S f(x, y, z) dS &= \iint_D f(\vec{\mathbf{r}}(u, v)) ||\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v|| \, dA \\ &= \int_0^3 \int_0^{2\pi} (\cos u + \sin^2 u) \, du \, dv \\ &= \int_0^3 \left[\sin u + \frac{u}{2} - \frac{\sin(2u)}{4} \right]_0^{2\pi} dv \\ &= \int_0^3 \pi \, dv = 3\pi. \end{aligned}$$

? Exercise 16.6.8

Calculate

$$\iint_S (x^2 - z) \, dS,$$

where S is the surface with parameterization $\vec{\mathbf{r}}(u, v) = \langle v,\ u^2 + v^2,\ 1 \rangle,\ 0 \le u \le 2,\ 0 \le v \le 3$.

Hint

Use Equation 16.6.2.

Answer

24

✓ Example 16.6.10: Calculating the Surface Integral of a Piece of a Sphere

Calculate surface integral

$$\iint_S f(x, y, z) \, dS,$$

where $f(x, y, z) = z^2$ and S is the surface that consists of the piece of sphere $x^2 + y^2 + z^2 = 4$ that lies on or above plane $z = 1$ and the disk that is enclosed by intersection plane $z = 1$ and the given sphere (Figure 16.6.16).

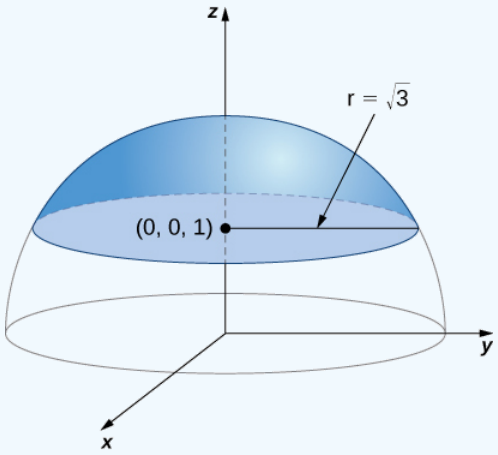


Figure 16.6.16: Calculating a surface integral over surface S.

Solution

Notice that S is not smooth but is piecewise smooth; S can be written as the union of its base S_1 and its spherical top S_2, and both S_1 and S_2 are smooth. Therefore, to calculate

$$\iint_S z^2 dS,$$

we write this integral as

$$\iint_{S_1} z^2\, dS + \iint_{S_2} z^2\, dS$$

and we calculate integrals

$$\iint_{S_1} z^2\, dS$$

and

$$\iint_{S_2} Z^2\, dS.$$

First, we calculate $\iint_{S_1} z^2\, dS$. To calculate this integral we need a parameterization of S_1. This surface is a disk in plane $z = 1$ centered at $(0, 0, 1)$. To parameterize this disk, we need to know its radius. Since the disk is formed where plane $z = 1$ intersects sphere $x^2 + y^2 + z^2 = 4$, we can substitute $z = 1$ into equation $x^2 + y^2 + z^2 = 4$:

$$x^2 + y^2 + 1 = 4 \Rightarrow x^2 + y^2 = 3.$$

Therefore, the radius of the disk is $\sqrt{3}$ and a parameterization of S_1 is $\vec{\mathbf{r}}(u, v) = \langle u\, \cos v,\ u\, \sin v,\ 1\rangle$, $0 \le u \le \sqrt{3}$, $0 \le v \le 2\pi$. The tangent vectors are $\vec{\mathbf{t}}_u = \langle \cos v,\ \sin v,\ 0\rangle$ and $\vec{\mathbf{t}}_v = \langle -u\, \sin v,\ u\, \cos v,\ 0\rangle$, and thus

$$\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 0 \end{vmatrix} = \langle 0,\ 0,\ u\, \cos^2 v + u\, \sin^2 v\rangle = \langle 0, 0, u\rangle.$$

The magnitude of this vector is u. Therefore,

$$\begin{aligned}\iint_{S_1} z^2\,dS &= \int_0^{\sqrt{3}}\int_0^{2\pi} f(r(u,v))||t_u \times t_v||\,dv\,du \\ &= \int_0^{\sqrt{3}}\int_0^{2\pi} u\,dv\,du \\ &= 2\pi\int_0^{\sqrt{3}} u\,du \\ &= 2\pi\sqrt{3}.\end{aligned}$$

Now we calculate

$$\iint_{S_2} dS.$$

To calculate this integral, we need a parameterization of S_2. The parameterization of full sphere $x^2+y^2+z^2=4$ is

$$\vec{\mathbf{r}}(\phi,\theta) = \langle 2\cos\theta\sin\phi,\ 2\sin\theta\sin\phi,\ 2\cos\phi\rangle,\ 0\le\theta\le 2\pi, 0\le\phi\le\pi.$$

Since we are only taking the piece of the sphere on or above plane $z=1$, we have to restrict the domain of ϕ. To see how far this angle sweeps, notice that the angle can be located in a right triangle, as shown in Figure 16.6.17 (the $\sqrt{3}$ comes from the fact that the base of S is a disk with radius $\sqrt{3}$). Therefore, the tangent of ϕ is $\sqrt{3}$, which implies that ϕ is $\pi/6$. We now have a parameterization of S_2:

$\vec{\mathbf{r}}(\phi,\theta) = \langle 2\cos\theta\sin\phi,\ 2\sin\theta\sin\phi,\ 2\cos\phi\rangle,\ 0\le\theta\le 2\pi,\ 0\le\phi\le\pi/3.$

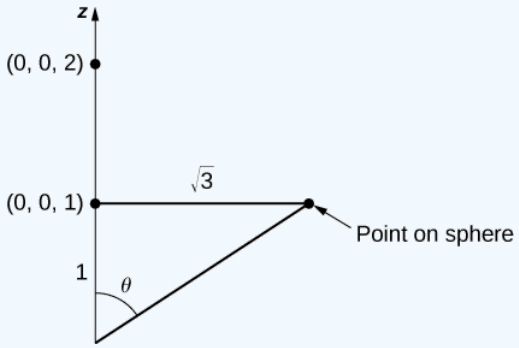


Figure 16.6.17: The maximum value of ϕ has a tangent value of $\sqrt{3}$.

The tangent vectors are $\vec{\mathbf{t}}_\phi = \langle 2\cos\theta\cos\phi,\ 2\sin\theta\cos\phi,\ -2\sin\phi\rangle$ and $\vec{\mathbf{t}}_\theta = \langle -2\sin\theta\sin\phi,\ u\cos\theta\sin\phi,\ 0\rangle$, and thus

$$\begin{aligned}\vec{\mathbf{t}}_\phi \times \vec{\mathbf{t}}_\theta &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2\cos\theta\cos\phi & 2\sin\theta\cos\phi & -2\sin\phi \\ -2\sin\theta\sin\phi & 2\cos\theta\sin\phi & 0 \end{vmatrix} \\ &= \langle 4\cos\theta\sin^2\phi,\ 4\sin\theta\sin^2\phi,\ 4\cos^2\theta\cos\phi\sin\phi + 4\sin^2\theta\cos\phi\sin\phi\rangle \\ &= \langle 4\cos\theta\sin^2\phi,\ 4\sin\theta\sin^2\phi,\ 4\cos\phi\sin\phi\rangle.\end{aligned}$$

The magnitude of this vector is

$$\begin{aligned}\vec{\mathbf{t}}_\phi \times \vec{\mathbf{t}}_\theta &= \sqrt{16\cos^2\theta\sin^4\phi + 16\sin^2\theta\sin^4\phi + 16\cos^2\phi\sin^2\phi} \\ &= 4\sqrt{\sin^4\phi + \cos^2\phi\sin^2\phi}.\end{aligned}$$

Therefore,

$$\begin{aligned}\iint_{S_2} z\,dS &= \int_0^{\pi/6}\int_0^{2\pi} f(\vec{\mathbf{r}}(\phi,\theta))||\vec{\mathbf{t}}_\phi \times \vec{\mathbf{t}}_\theta||\,d\theta\,d\phi \\ &= \int_0^{\pi/6}\int_0^{2\pi} 16\cos^2\phi\sqrt{\sin^4\phi+\cos^2\phi\sin^2\phi}\,d\theta\,d\phi \\ &= 32\pi\int_0^{\pi/6}\cos^2\phi\sqrt{\sin^4\phi+\cos^2\phi\sin^2\phi}\,d\phi \\ &= 32\pi\int_0^{\pi/6}\cos^2\phi\sin\phi\sqrt{\sin^2\phi+\cos^2\phi}\,d\phi \\ &= 32\pi\int_0^{\pi/6}\cos^2\phi\sin\phi\,d\phi \\ &= 32\pi\left[-\frac{\cos^3\phi}{3}\right]_0^{\pi/6} \\ &= 32\pi\left[\frac{1}{3}-\frac{\sqrt{3}}{8}\right] = \frac{32\pi}{3}-4\sqrt{3}.\end{aligned}$$

Since

$$\iint_S z^2\,dS = \iint_{S_1} z^2\,dS + \iint_{S_2} z^2\,dS,$$

we have

$$\iint_S z^2\,dS = (2\pi-4)\sqrt{3}+\frac{32\pi}{3}.$$

Analysis

In this example we broke a surface integral over a piecewise surface into the addition of surface integrals over smooth subsurfaces. There were only two smooth subsurfaces in this example, but this technique extends to finitely many smooth subsurfaces.

? Exercise 16.6.9

Calculate line integral $\iint_S (x-y)\,dS$, where S is cylinder $x^2+y^2=1,\ 0\le z\le 2$, including the circular top and bottom.

Hint

Break the integral into three separate surface integrals.

Answer

0

Scalar surface integrals have several real-world applications. Recall that scalar line integrals can be used to compute the mass of a wire given its density function. In a similar fashion, we can use scalar surface integrals to compute the mass of a sheet given its density function. If a thin sheet of metal has the shape of surface S and the density of the sheet at point (x,y,z) is $\rho(x,y,z)$ then mass m of the sheet is

$$m = \iint_S \rho(x,y,z)\,dS. \tag{16.6.3}$$

✓ Example 16.6.11: Calculating the Mass of a Sheet

A flat sheet of metal has the shape of surface $z=1+x+2y$ that lies above rectangle $0\le x\le 4$ and $0\le y\le 2$. If the density of the sheet is given by $\rho(x,y,z)=x^2yz$, what is the mass of the sheet?

Solution

Let S be the surface that describes the sheet. Then, the mass of the sheet is given by $m = \iint_S x^2yx\,dS$. To compute this surface integral, we first need a parameterization of S. Since S is given by the function $f(x,y) = 1 + x + 2y$, a parameterization of S is $\vec{\mathbf{r}}(x,y) = \langle x,\ y,\ 1 + x + 2y\rangle,\ 0 \le x \le 4,\ 0 \le y \le 2$.

The tangent vectors are $\vec{\mathbf{t}}_x = \langle 1, 0, 1\rangle$ and $\vec{\mathbf{t}}_y = \langle 1, 0, 2\rangle$. Therefore, $\vec{\mathbf{t}}_x + \vec{\mathbf{t}}_y = \langle -1, -2, 1\rangle$ and $||\vec{\mathbf{t}}_x \times \vec{\mathbf{t}}_y|| = \sqrt{6}$.

By the definition of the line integral (Section 16.2),

$$\begin{aligned} m &= \iint_S x^2yz\,dS \\ &= \sqrt{6}\int_0^4 \int_0^2 x^2y(1 + x + 2y)\,dy\,dx \\ &= \sqrt{6}\int_0^4 \frac{22x^2}{3} + 2x^3\,dx \\ &= \frac{2560\sqrt{6}}{9} \approx 696.74. \end{aligned}$$

Exercise 16.6.10

A piece of metal has a shape that is modeled by paraboloid $z = x^2 + y^2,\ 0 \le z \le 4$, and the density of the metal is given by $\rho(x,y,z) = z + 1$. Find the mass of the piece of metal.

Hint

The mass of a sheet is given by Equation 16.6.3. A useful parameterization of a paraboloid was given in a previous example.

Answer

$38.401\pi \approx 120.640$

Orientation of a Surface

Recall that when we defined a scalar line integral, we did not need to worry about an orientation of the curve of integration. The same was true for scalar surface integrals: we did not need to worry about an "orientation" of the surface of integration.

On the other hand, when we defined vector line integrals, the curve of integration needed an orientation. That is, we needed the notion of an oriented curve to define a vector line integral without ambiguity. Similarly, when we define a surface integral of a vector field, we need the notion of an oriented surface. An oriented surface is given an "upward" or "downward" orientation or, in the case of surfaces such as a sphere or cylinder, an "outward" or "inward" orientation.

Let S be a smooth surface. For any point (x,y,z) on S, we can identify two unit normal vectors $\vec{\mathbf{N}}$ and $-\vec{\mathbf{N}}$. If it is possible to choose a unit normal vector $\vec{\mathbf{N}}$ at every point (x,y,z) on S so that $\vec{\mathbf{N}}$ varies continuously over S, then S is "*orientable*." Such a choice of unit normal vector at each point gives the **orientation of a surface** S. If you think of the normal field as describing water flow, then the side of the surface that water flows toward is the "negative" side and the side of the surface at which the water flows away is the "positive" side. Informally, a choice of orientation gives S an "outer" side and an "inner" side (or an "upward" side and a "downward" side), just as a choice of orientation of a curve gives the curve "forward" and "backward" directions.

Closed surfaces such as spheres are orientable: if we choose the outward normal vector at each point on the surface of the sphere, then the unit normal vectors vary continuously. This is called the *positive orientation of the closed surface* (Figure 16.6.18). We also could choose the inward normal vector at each point to give an "inward" orientation, which is the negative orientation of the surface.

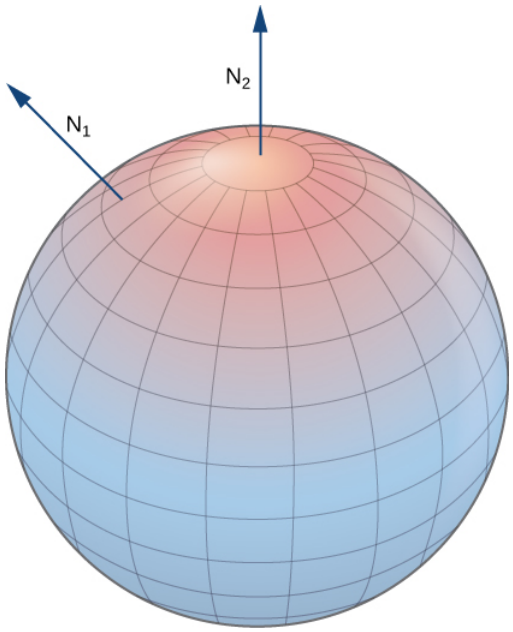


Figure 16.6.18: An oriented sphere with positive orientation.

A portion of the graph of any smooth function $z = f(x, y)$ is also orientable. If we choose the unit normal vector that points "above" the surface at each point, then the unit normal vectors vary continuously over the surface. We could also choose the unit normal vector that points "below" the surface at each point. To get such an orientation, we parameterize the graph of f in the standard way: $\vec{\mathbf{r}}(x, y) = \langle x,\ y,\ f(x, y)\rangle$, where x and y vary over the domain of f. Then, $\vec{\mathbf{t}}_x = \langle 1, 0, f_x\rangle$ and $\vec{\mathbf{t}}_y = \langle 0, 1, f_y\rangle$, and therefore the cross product $\vec{\mathbf{t}}_x \times \vec{\mathbf{t}}_y$ (which is normal to the surface at any point on the surface) is $\langle -f_x,\ -f_y,\ 1\rangle$Since the z-component of this vector is one, the corresponding unit normal vector points "upward," and the upward side of the surface is chosen to be the "positive" side.

Let S be a smooth orientable surface with parameterization $\vec{\mathbf{r}}(u, v)$. For each point $\vec{\mathbf{r}}(a, b)$ on the surface, vectors $\vec{\mathbf{t}}_u$ and $\vec{\mathbf{t}}_v$ lie in the tangent plane at that point. Vector $\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v$ is normal to the tangent plane at $\vec{\mathbf{r}}(a, b)$ and is therefore normal to S at that point. Therefore, the choice of unit normal vector

$$\vec{\mathbf{N}} = \frac{\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v}{||\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v||}$$

gives an orientation of surface S.

Example 16.6.12: Choosing an Orientation

Give an orientation of cylinder $x^2 + y^2 = r^2,\ 0 \le z \le h$.

Solution

This surface has parameterization $\vec{\mathbf{r}}(u, v) = \langle r\cos u,\ r\sin u,\ v\rangle,\ 0 \le u < 2\pi,\ 0 \le v \le h$.

The tangent vectors are $\vec{\mathbf{t}}_u = \langle -r\sin u,\ r\cos u,\ 0\rangle$ and $\vec{\mathbf{t}}_v = \langle 0, 0, 1\rangle$. To get an orientation of the surface, we compute the unit normal vector

$$\vec{\mathbf{N}} = \frac{\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v}{||\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v||}$$

In this case, $\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v = \langle r\cos u,\ r\sin u,\ 0\rangle$ and therefore

$$||\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v|| = \sqrt{r^2\cos^2 u + r^2\sin^2 u} = r.$$

An orientation of the cylinder is

$$\vec{\mathbf{N}}(u, v) = \frac{\langle r\cos u,\ r\sin u,\ 0\rangle}{r} = \langle \cos u,\ \sin u,\ 0\rangle.$$

Notice that all vectors are parallel to the xy-plane, which should be the case with vectors that are normal to the cylinder. Furthermore, all the vectors point outward, and therefore this is an outward orientation of the cylinder (Figure 16.6.19).

Figure 16.6.19: If all the vectors normal to a cylinder point outward, then this is an outward orientation of the cylinder.

? Exercise 16.6.11

Give the "upward" orientation of the graph of $f(x,y) = xy$.

Hint

Parameterize the surface and use the fact that the surface is the graph of a function.

Answer

$$\vec{\mathbf{N}}(x,y) = \left\langle \frac{-y}{\sqrt{1+x^2+y^2}}, \frac{-x}{\sqrt{1+x^2+y^2}}, \frac{1}{\sqrt{1+x^2+y^2}} \right\rangle$$

Since every curve has a "forward" and "backward" direction (or, in the case of a closed curve, a clockwise and counterclockwise direction), it is possible to give an orientation to any curve. Hence, it is possible to think of every curve as an oriented curve. This is not the case with surfaces, however. Some surfaces cannot be oriented; such surfaces are called *nonorientable*. Essentially, a surface can be oriented if the surface has an "inner" side and an "outer" side, or an "upward" side and a "downward" side. Some surfaces are twisted in such a fashion that there is no well-defined notion of an "inner" or "outer" side.

The classic example of a nonorientable surface is the Möbius strip. To create a Möbius strip, take a rectangular strip of paper, give the piece of paper a half-twist, and the glue the ends together (Figure 16.6.20). Because of the half-twist in the strip, the surface has no "outer" side or "inner" side. If you imagine placing a normal vector at a point on the strip and having the vector travel all the way around the band, then (because of the half-twist) the vector points in the opposite direction when it gets back to its original position. Therefore, the strip really only has one side.

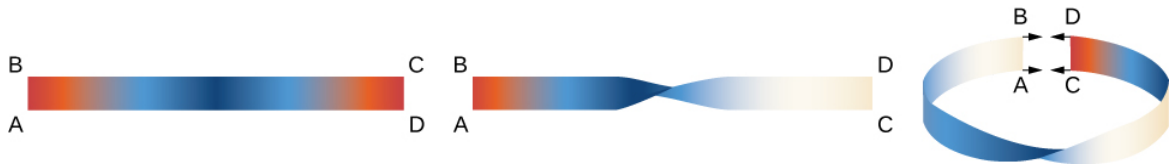


Figure 16.6.20: The construction of a Möbius strip.

Since some surfaces are *nonorientable*, it is not possible to define a vector surface integral on all piecewise smooth surfaces. This is in contrast to vector line integrals, which can be defined on any piecewise smooth curve.

Surface Integral of a Vector Field

With the idea of orientable surfaces in place, we are now ready to define a surface integral of a vector field. The definition is analogous to the definition of the flux of a vector field along a plane curve. Recall that if $\vec{\mathbf{F}}$ is a two-dimensional vector field and C is a plane curve, then the definition of the flux of $\vec{\mathbf{F}}$ along C involved chopping C into small pieces, choosing a point inside each piece, and calculating $\vec{\mathbf{F}} \cdot \vec{\mathbf{N}}$ at the point (where $\vec{\mathbf{N}}$ is the unit normal vector at the point). The definition of a surface integral of a vector field proceeds in the same fashion, except now we chop surface S into small pieces, choose a point in the small (two-dimensional) piece, and calculate $\vec{\mathbf{F}} \cdot \vec{\mathbf{N}}$ at the point.

To place this definition in a real-world setting, let S be an oriented surface with unit normal vector $\vec{\mathbf{N}}$. Let $\vec{\mathbf{v}}$ be a velocity field of a fluid flowing through S, and suppose the fluid has density $\rho(x,y,z)$ Imagine the fluid flows through S, but S is completely permeable so that it does not impede the fluid flow (Figure 16.6.21). The mass flux of the fluid is the rate of mass flow per unit area. The mass flux is measured in mass per unit time per unit area. How could we calculate the mass flux of the fluid across S?

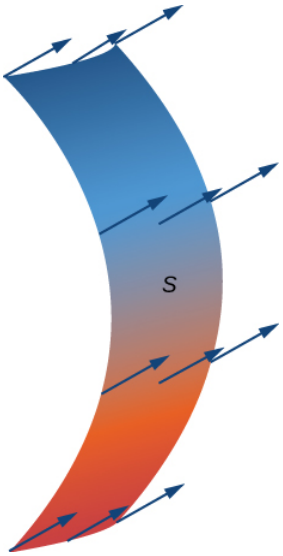


Figure 16.6.21: Fluid flows across a completely permeable surface S.

The rate of flow, measured in mass per unit time per unit area, is $\rho\vec{\mathbf{N}}$. To calculate the mass flux across S, chop S into small pieces S_{ij}. If S_{ij} is small enough, then it can be approximated by a tangent plane at some point P in S_{ij}. Therefore, the unit normal vector at P can be used to approximate $\vec{\mathbf{N}}(x,y,z)$ across the entire piece S_{ij} because the normal vector to a plane does not change as we move across the plane. The component of the vector ρv at P in the direction of $\vec{\mathbf{N}}$ is $\rho\vec{\mathbf{v}}\cdot\vec{\mathbf{N}}$ at P. Since S_{ij} is small, the dot product $\rho v \cdot N$ changes very little as we vary across S_{ij} and therefore $\rho\vec{\mathbf{v}}\cdot\vec{\mathbf{N}}$ can be taken as approximately constant across S_{ij}. To approximate the mass of fluid per unit time flowing across S_{ij} (and not just locally at point P), we need to multiply $(\rho\vec{\mathbf{v}}\cdot\vec{\mathbf{N}})(P)$ by the area of S_{ij}. Therefore, the mass of fluid per unit time flowing across S_{ij} in the direction of $\vec{\mathbf{N}}$ can be approximated by $(\rho\vec{\mathbf{v}}\cdot\vec{\mathbf{N}})\Delta S_{ij}$ where $\vec{\mathbf{N}}$, ρ and $\vec{\mathbf{v}}$ are all evaluated at P (Figure 16.6.22). This is analogous to the flux of two-dimensional vector field $\vec{\mathbf{F}}$ across plane curve C, in which we approximated flux across a small piece of C with the expression $(\vec{\mathbf{F}}\cdot\vec{\mathbf{N}})\,\Delta s$. To approximate the mass flux across S, form the sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(\rho\vec{\mathbf{v}}\cdot\vec{\mathbf{N}})\Delta S_{ij}.$$

As pieces S_{ij} get smaller, the sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(\rho\vec{\mathbf{v}}\cdot\vec{\mathbf{N}})\Delta S_{ij}$$

gets arbitrarily close to the mass flux. Therefore, the mass flux is

$$\iint_S \rho\vec{\mathbf{v}}\cdot\vec{\mathbf{N}}\,dS=\lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}(\rho\vec{\mathbf{v}}\cdot\vec{\mathbf{N}})\Delta S_{ij}.$$

This is a surface integral of a vector field. Letting the vector field $\rho\vec{\mathbf{v}}$ be an arbitrary vector field $\vec{\mathbf{F}}$ leads to the following definition.

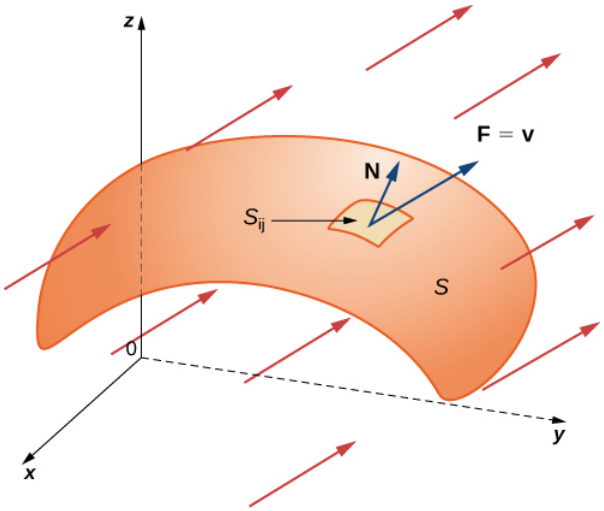


Figure 16.6.22: The mass of fluid per unit time flowing across S_{ij} in the direction of $\vec{\mathbf{N}}$ can be approximated by $(\rho\vec{\mathbf{v}}\cdot\vec{\mathbf{N}})\Delta S_{ij}$.

Definition: Surface Integrals

Let $\vec{\mathbf{F}}$ be a continuous vector field with a domain that contains oriented surface S with unit normal vector $\vec{\mathbf{N}}$. The surface integral of $\vec{\mathbf{F}}$ over S is

$$\iint_S \vec{\mathbf{F}}\cdot d\vec{\mathbf{S}} = \iint_S \vec{\mathbf{F}}\cdot\vec{\mathbf{N}}\,dS. \tag{16.6.4}$$

Notice the parallel between this definition and the definition of vector line integral $\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{N}}\,dS$. A surface integral of a vector field is defined in a similar way to a flux line integral across a curve, except the domain of integration is a surface (a two-dimensional object) rather than a curve (a one-dimensional object). Integral $\iint_S \vec{\mathbf{F}}\cdot\vec{\mathbf{N}}\,dS$ is called the *flux of* $\vec{\mathbf{F}}$ *across* S, just as integral $\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{N}}\,dS$ is the flux of $\vec{\mathbf{F}}$ across curve C. A surface integral over a vector field is also called a **flux integral**.

Just as with vector line integrals, surface integral $\iint_S \vec{\mathbf{F}}\cdot\vec{\mathbf{N}}\,dS$ is easier to compute after surface S has been parameterized. Let $\vec{\mathbf{r}}(u,v)$ be a parameterization of S with parameter domain D. Then, the unit normal vector is given by $\vec{\mathbf{N}} = \dfrac{\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v}{||\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v||}$ and, from Equation 16.6.4, we have

$$\begin{aligned}\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{N}}\,dS &= \iint_S \vec{\mathbf{F}}\cdot\frac{\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v}{||\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v||}\,dS\\ &= \iint_D \left(\vec{\mathbf{F}}(\vec{\mathbf{r}}(u,v))\cdot\frac{\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v}{||\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v||}\right)||\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v||\,dA\\ &= \iint_D (\vec{\mathbf{F}}(\vec{\mathbf{r}}(u,v))\cdot(\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v))\,dA.\end{aligned}$$

Therefore, to compute a surface integral over a vector field we can use the equation

$$\iint_S \vec{\mathbf{F}}\cdot\vec{\mathbf{N}}\,dS = \iint_D (\vec{\mathbf{F}}(\vec{\mathbf{r}}(u,v))\cdot(\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v))\,dA. \tag{16.6.5}$$

Example 16.6.13: Calculating a Surface Integral

Calculate the surface integral

$$\iint_S \vec{\mathbf{F}}\cdot\vec{\mathbf{N}}\,dS,$$

where $\vec{\mathbf{F}} = \langle -y, x, 0\rangle$ and S is the surface with parameterization

$$\vec{\mathbf{r}}(x,y) = \langle x, y^2, \ldots\rangle,\ 0\le x\le 3,\ 0\le y\le 4$$

$\vec{\mathbf{r}}(u,v) = \langle u, v^2 - u, u + v \rangle$, $0 \le u \le 3$, $0 \le v \le 4$.

Solution

The tangent vectors are $\vec{\mathbf{t}}_u = \langle 1, -1, 1 \rangle$ and $\vec{\mathbf{t}}_v = \langle 0, 2v, 1 \rangle$. Therefore,

$$\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v = \langle -1 - 2v, -1, 2v \rangle.$$

By Equation 16.6.5,

$$\begin{aligned}
\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \int_0^4 \int_0^3 F(\vec{\mathbf{r}}(u,v)) \cdot (\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v)\, du\, dv \\
&= \int_0^4 \int_0^3 \langle u - v^2,\ u,\ 0 \rangle \cdot \langle -1 - 2v,\ -1,\ 2v \rangle\, du\, dv \\
&= \int_0^4 \int_0^3 [(u - v^2)(-1 - 2v) - u]\, du\, dv \\
&= \int_0^4 \int_0^3 (2v^3 + v^2 - 2uv - 2u)\, du\, dv \\
&= \int_0^4 [2v^3 u + v^2 u - v u^2 - u^2]\Big|_0^3\, dv \\
&= \int_0^4 (6v^3 + 3v^2 - 9v - 9)\, dv \\
&= \left[\frac{3v^4}{2} + v^3 - \frac{9v^2}{2} - 9v \right]_0^4 \\
&= 340.
\end{aligned}$$

Therefore, the flux of $\vec{\mathbf{F}}$ across S is 340.

? Exercise 16.6.12

Calculate surface integral

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, dS,$$

where $\vec{\mathbf{F}} = \langle 0, -z, y \rangle$ and S is the portion of the unit sphere in the first octant with outward orientation.

Hint

Use Equation 16.6.5.

Answer

0

✓ Example 16.6.14: Calculating Mass Flow Rate

Let $\vec{\mathbf{v}}(x,y,z) = \langle 2x,\ 2y,\ z \rangle$ represent a velocity field (with units of meters per second) of a fluid with constant density 80 kg/m^3. Let S be hemisphere $x^2 + y^2 + z^2 = 9$ with $z \le 0$ such that S is oriented outward. Find the mass flow rate of the fluid across S.

Solution

A parameterization of the surface is

$$\vec{\mathbf{r}}(\phi, \theta) = \langle 3\cos\theta\sin\phi,\ 3\sin\theta\sin\phi,\ 3\cos\phi \rangle,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi/2.$$

As in Example 16.6.7, the tangent vectors are $\vec{\mathbf{t}}_\theta = \langle -3\sin\theta\sin\phi,\ 3\cos\theta\sin\phi,\ 0 \rangle$ and $\vec{\mathbf{t}}_\phi = \langle 3\cos\theta\cos\phi,\ 3\sin\theta\cos\phi,\ -3\sin\phi \rangle$, and their cross product is

$$\vec{\mathbf{t}}_\phi \times \vec{\mathbf{t}}_\theta = \langle 9\cos\theta\sin^2\phi,\ 9\sin\theta\sin^2\phi,\ 9\sin\phi\cos\phi \rangle.$$

Notice that each component of the cross product is positive, and therefore this vector gives the outward orientation. Therefore we use the orientation

$\vec{\mathbf{N}} = \langle 9\cos\theta\sin^2\phi,\ 9\sin\theta\sin^2\phi,\ 9\sin\phi\cos\phi\rangle$

for the sphere.

By Equation 16.6.4,

$$\begin{aligned}\iint_S \rho v\cdot dS &= 80\int_0^{2\pi}\int_0^{\pi/2} v(r(\phi,\theta))\cdot(t_\phi\times t_\theta)\,d\phi\,d\theta\\ &= 80\int_0^{2\pi}\int_0^{\pi/2} \langle 6\cos\theta\sin\phi,\ 6\sin\theta\sin\phi,\ 3\cos\phi\rangle\\ &\quad\cdot\langle 9\cos\theta\sin^2\phi,\ 9\sin\theta\sin^2\phi,\ 9\sin\phi\cos\phi\rangle\,d\phi\,d\theta\\ &= 80\int_0^{2\pi}\int_0^{\pi/2} 54\sin^3\phi + 27\cos^2\phi\sin\phi\,d\phi\,d\theta\\ &= 80\int_0^{2\pi}\int_0^{\pi/2} 54(1-\cos^2\phi)\sin\phi + 27\cos^2\phi\sin\phi\,d\phi\,d\theta\\ &= 80\int_0^{2\pi}\int_0^{\pi/2} 54\sin\phi - 27\cos^2\phi\sin\phi\,d\phi\,d\theta\\ &= 80\int_0^{2\pi}\Big[-54\cos\phi + 9\cos^3\phi\Big]_{\phi=0}^{\phi=2\pi} d\theta\\ &= 80\int_0^{2\pi} 45\,d\theta\\ &= 7200\pi.\end{aligned}$$

Therefore, the mass flow rate is $7200\pi\,\text{kg/sec/m}^2$.

? Exercise 16.6.13

Let $\vec{\mathbf{v}}(x,y,z) = \langle x^2+y^2,\ z,\ 4y\rangle$ m/sec represent a velocity field of a fluid with constant density 100 kg/m^3. Let S be the half-cylinder $\vec{\mathbf{r}}(u,v) = \langle\cos u,\ \sin u,\ v\rangle$, $0\le u\le\pi$, $0\le v\le 2$ oriented outward. Calculate the mass flux of the fluid across S.

Hint

Use Equation 16.6.4.

Answer

400 kg/sec/m

In Example 16.6.14, we computed the mass flux, which is the rate of mass flow per unit area. If we want to find the flow rate (measured in volume per time) instead, we can use flux integral

$$\iint_S \vec{\mathbf{v}}\cdot\vec{\mathbf{N}}\,dS,$$

which leaves out the density. Since the flow rate of a fluid is measured in volume per unit time, flow rate does not take mass into account. Therefore, we have the following characterization of the flow rate of a fluid with velocity $\vec{\mathbf{v}}$ across a surface S:

$$\text{Flow rate of fluid across S} = \iint_S \vec{\mathbf{v}}\cdot d\vec{\mathbf{S}}.$$

To compute the flow rate of the fluid in Example 16.6.14, we simply remove the density constant, which gives a flow rate of 90π m^3/sec.

Both mass flux and flow rate are important in physics and engineering. Mass flux measures how much mass is flowing across a surface; flow rate measures how much volume of fluid is flowing across a surface.

In addition to modeling fluid flow, surface integrals can be used to model heat flow. Suppose that the temperature at point (x,y,z) in an object is $T(x,y,z)$. Then the heat flow is a vector field proportional to the negative temperature gradient in the object. To be precise, the heat flow is defined as vector field $F = -k\nabla T$, where the constant *k* is the *thermal conductivity* of the substance from

which the object is made (this constant is determined experimentally). The rate of heat flow across surface S in the object is given by the flux integral

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_S -k\vec{\nabla}T \cdot d\vec{\mathbf{S}}.$$

Example 16.6.15: Calculating Heat Flow

A cast-iron solid cylinder is given by inequalities $x^2 + y^2 \leq 1,\ 1 \leq z \leq 4$. The temperature at point (x, y, z) in a region containing the cylinder is $T(x, y, z) = (x^2 + y^2)z$. Given that the thermal conductivity of cast iron is 55, find the heat flow across the boundary of the solid if this boundary is oriented outward.

Solution

Let S denote the boundary of the object. To find the heat flow, we need to calculate flux integral

$$\iint_S -k\vec{\nabla}T \cdot d\vec{\mathbf{S}}.$$

Notice that S is not a smooth surface but is piecewise smooth, since S is the union of three smooth surfaces (the circular top and bottom, and the cylindrical side). Therefore, we calculate three separate integrals, one for each smooth piece of S. Before calculating any integrals, note that the gradient of the temperature is $\vec{\nabla}T = \langle 2xz,\ 2yz,\ x^2 + y^2 \rangle$.

First we consider the circular bottom of the object, which we denote S_1. We can see that S_1 is a circle of radius 1 centered at point $(0, 0, 1)$ sitting in plane $z = 1$. This surface has parameterization $\vec{\mathbf{r}}(u, v) = \langle v\cos u,\ v\sin u,\ 1 \rangle,\ 0 \leq u < 2\pi,\ 0 \leq v \leq 1$.

Therefore,

$\vec{\mathbf{t}}_u = \langle -v\sin u,\ v\cos u,\ 0 \rangle$ and $\vec{\mathbf{t}}_v = \langle \cos u,\ v\sin u,\ 0 \rangle$, and $\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v = \langle 0,\ 0, -v\sin^2 u - v\cos^2 u \rangle = \langle 0,\ 0, -v \rangle$.

Since the surface is oriented outward and S_1 is the bottom of the object, it makes sense that this vector points downward. By Equation 16.6.5 the heat flow across S_1 is

$$\begin{aligned}
\iint_{S_1} -k\vec{\nabla}T \cdot d\vec{\mathbf{S}} &= -55\int_0^{2\pi}\int_0^1 \vec{\nabla}T(u, v) \cdot (\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v)\, dv\, du \\
&= -55\int_0^{2\pi}\int_0^1 \langle 2v\cos u,\ 2v\sin u,\ v^2\cos^2 u + v^2\sin^2 u \rangle \cdot \langle 0, 0, -v \rangle\, dv\, du \\
&= -55\int_0^{2\pi}\int_0^1 \langle 2v\cos u,\ 2v\sin u,\ v^2 \rangle \cdot \langle 0, 0, -v \rangle\, dv\, du \\
&= -55\int_0^{2\pi}\int_0^1 -v^3\, dv\, du \\
&= -55\int_0^{2\pi} -\frac{1}{4}du \\
&= \frac{55\pi}{2}.
\end{aligned}$$

Now let's consider the circular top of the object, which we denote S_2. We see that S_2 is a circle of radius 1 centered at point $(0, 0, 4)$, sitting in plane $z = 4$. This surface has parameterization $\vec{\mathbf{r}}(u, v) = \langle v\cos u,\ v\sin u,\ 4 \rangle,\ 0 \leq u < 2\pi,\ 0 \leq v \leq 1$.

Therefore, $\vec{\mathbf{t}}_u = \langle -v\sin u,\ v\cos u,\ 0 \rangle$ and $\vec{\mathbf{t}}_v = \langle \cos u,\ v\sin u,\ 0 \rangle$, and $\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v = \langle 0,\ 0, -v\sin^2 u - v\cos^2 u \rangle = \langle 0, 0, -v \rangle$.

Since the surface is oriented outward and S_1 is the top of the object, we instead take vector $\vec{\mathbf{t}}_v \times \vec{\mathbf{t}}_u = \langle 0, 0, v \rangle$. By Equation 16.6.5, the heat flow across S_1 is

$$\begin{aligned}\iint_{S_2} -k\vec{\nabla}T \cdot d\vec{\mathbf{S}} &= -55\int_0^{2\pi}\int_0^1 \vec{\nabla}T(u,v)\cdot(\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v)\,dv\,du \\ &= -55\int_0^{2\pi}\int_0^1 \langle 8v\cos u,\ 8v\sin u,\ v^2\cos^2 u + v^2\sin^2 u\rangle \cdot \langle 0,0,-v\rangle\,dv\,du \\ &= -55\int_0^{2\pi}\int_0^1 \langle 8v\cos u,\ 8v\sin u,\ v^2\rangle \cdot \langle 0,0,-v\rangle\ dv\,du \\ &= -55\int_0^{2\pi}\int_0^1 -v^3\,dv\,du = -55\int_0^{2\pi} -\frac{1}{4}\,du = -\frac{55\pi}{2}.\end{aligned}$$

Last, let's consider the cylindrical side of the object. This surface has parameterization $\vec{\mathbf{r}}(u,v) = \langle \cos u,\ \sin u,\ v\rangle,\ 0 \le u < 2\pi,\ 1 \le v \le 4$. By Example 16.6.9, we know that $\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v = \langle \cos u,\ \sin u,\ 0\rangle$. By Equation 16.6.5,

$$\begin{aligned}\iint_{S_3} -k\vec{\nabla}T \cdot d\vec{\mathbf{S}} &= -55\int_0^{2\pi}\int_1^4 \vec{\nabla}T(u,v)\cdot(\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v)\,dv\,du \\ &= -55\int_0^{2\pi}\int_1^4 \langle 2v\cos u,\ 2v\sin u,\ \cos^2 u + \sin^2 u\rangle \cdot \langle \cos u,\ \sin u,\ 0\rangle\,dv\,du \\ &= -55\int_0^{2\pi}\int_0^1 \langle 2v\cos^2 u,\ 2v\sin u,\ 1\rangle \cdot \langle \cos u,\ \sin u,\ 0\rangle\,dv\ du \\ &= -55\int_0^{2\pi}\int_0^1 (2v\cos^2 u + 2v\sin^2 u)\,dv\,du \\ &= -55\int_0^{2\pi}\int_0^1 2v\,dv\,du \\ &= -55\int_0^{2\pi} du \\ &= -110\pi.\end{aligned}$$

Therefore, the rate of heat flow across S is

$$\frac{55\pi}{2} - \frac{55\pi}{2} - 110\pi = -110\pi.$$

? Exercise 16.6.14

A cast-iron solid ball is given by inequality $x^2 + y^2 + z^2 \le 1$. The temperature at a point in a region containing the ball is $T(x,y,z) = \frac{1}{3}(x^2 + y^2 + z^2)$. Find the heat flow across the boundary of the solid if this boundary is oriented outward.

Hint

Follow the steps of Example 16.6.15.

Answer

$$-\frac{440\pi}{3}$$

Key Concepts

- Surfaces can be parameterized, just as curves can be parameterized. In general, surfaces must be parameterized with two parameters.
- Surfaces can sometimes be oriented, just as curves can be oriented. Some surfaces, such as a Möbius strip, cannot be oriented.
- A surface integral is like a line integral in one higher dimension. The domain of integration of a surface integral is a surface in a plane or space, rather than a curve in a plane or space.
  - The integrand of a surface integral can be a scalar function or a vector field. To calculate a surface integral with an integrand that is a function, use Equation 16.6.2. To calculate a surface integral with an integrand that is a vector field, use Equation 16.6.5.
- If S is a surface, then the area of S is

$$\iint_S dS.$$

Key Equations

- **Scalar surface integral**

$$\iint_S f(x,y,z)\,dS = \iint_D f(\vec{\mathbf{r}}(u,v))||\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v||\,dA$$

- **Flux integral**

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\,dS = \iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_D \vec{\mathbf{F}}(\vec{\mathbf{r}}(u,v)) \cdot (\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v)\,dA$$

Glossary

flux integral
another name for a surface integral of a vector field; the preferred term in physics and engineering

grid curves
curves on a surface that are parallel to grid lines in a coordinate plane

heat flow
a vector field proportional to the negative temperature gradient in an object

mass flux
the rate of mass flow of a fluid per unit area, measured in mass per unit time per unit area

orientation of a surface
if a surface has an "inner" side and an "outer" side, then an orientation is a choice of the inner or the outer side; the surface could also have "upward" and "downward" orientations

parameter domain (parameter space)
the region of the uv-plane over which the parameters u and v vary for parameterization $\vec{\mathbf{r}}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$

parameterized surface (parametric surface)
a surface given by a description of the form $\vec{\mathbf{r}}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$, where the parameters u and v vary over a parameter domain in the uv-plane

regular parameterization
parameterization $\vec{\mathbf{r}}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$ such that $r_u \times r_v$ is not zero for point (u,v) in the parameter domain

surface area
the area of surface S given by the surface integral

$$\iint_S dS$$

surface integral
an integral of a function over a surface

surface integral of a scalar-valued function
a surface integral in which the integrand is a scalar function

surface integral of a vector field
a surface integral in which the integrand is a vector field

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16.6E: Exercises for Section 16.6

In exercises 1 - 4, determine whether the statements are *true or false*.

1. If surface S is given by $\{(x, y, z) : 0 \le x \le 1,\ 0 \le y \le 1,\ z = 10\}$, then $\iint_S f(x, y, z)\, dS = \int_0^1 \int_0^1 f(x, y, 10)\, dx\, dy$.

Answer

True

2. If surface S is given by $\{(x, y, z) : 0 \le x \le 1,\ 0 \le y \le 1,\ z = x\}$, then $\iint_S f(x, y, z)\, dS = \int_0^1 \int_0^1 f(x, y, x)\, dx\, dy$.

3. Surface $\vec{\mathbf{r}} = \langle v\ \cos u,\ v\ \sin u,\ v^2 \rangle$, for $0 \le u \le \pi,\ 0 \le v \le 2$ is the same surface $\vec{\mathbf{r}} = \langle \sqrt{v}\ \cos 2u,\ \sqrt{v}\ \sin 2u,\ v \rangle$, for $0 \le u \le \frac{\pi}{2},\ 0 \le v \le 4$.

Answer

True

4. Given the standard parameterization of a sphere, normal vectors $\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v$ are outward normal vectors.

In exercises 5 - 10, find parametric descriptions for the following surfaces.

5. Plane $3x - 2y + z = 2$

Answer

$\vec{\mathbf{r}}(u, v) = \langle u,\ v,\ 2 - 3u + 2v \rangle$ for $-\infty \le u < \infty$ and $-\infty \le v < \infty$.

6. Paraboloid $z = x^2 + y^2$, for $0 \le z \le 9$.

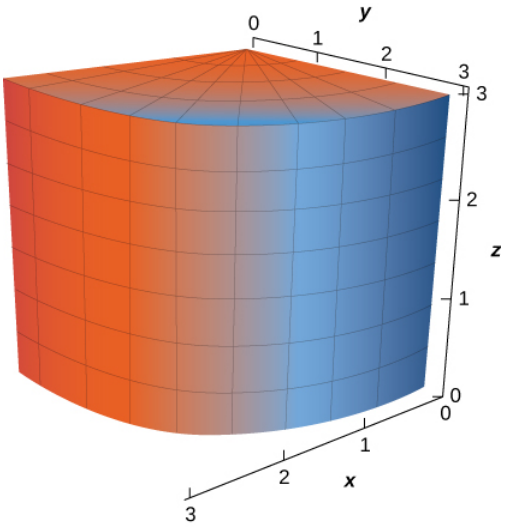
7. Plane $2x - 4y + 3z = 16$

Answer

$\vec{\mathbf{r}}(u, v) = \langle u,\ v,\ \frac{1}{3}(16 - 2u + 4v) \rangle$ for $|u| < \infty$ and $|v| < \infty$.

8. The frustum of cone $z^2 = x^2 + y^2$, for $2 \le z \le 8$

9. The portion of cylinder $x^2 + y^2 = 9$ in the first octant, for $0 \le z \le 3$



Answer

$\vec{\mathbf{r}}(u, v) = \langle 3\ \cos u,\ 3\ \sin u,\ v \rangle$ for $0 \le u \le \frac{\pi}{2},\ 0 \le v \le 3$

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10. A cone with base radius r and height h, where r and h are positive constants.

For exercises 11 - 12, use a computer algebra system to approximate the area of the following surfaces using a parametric description of the surface.

11. [T] Half cylinder $\{(r, \theta, z) : r = 4,\ 0 \le \theta \le \pi,\ 0 \le z \le 7\}$

Answer

$A = 87.9646$

12. [T] Plane $z = 10 - z - y$ above square $|x| \le 2,\ |y| \le 2$

In exercises 13 - 15, let S be the hemisphere $x^2 + y^2 + z^2 = 4$, with $z \ge 0$, and evaluate each surface integral, in the counterclockwise direction.

13. $\iint_S z\, dS$

Answer

$\iint_S z\, dS = 8\pi$

14. $\iint_S (x - 2y)\, dS$

15. $\iint_S (x^2 + y^2)\, dS$

Answer

$\iint_S (x^2 + y^2)\, dS = 16\pi$

In exercises 16 - 18, evaluate $\int \int_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, dS$ for vector field $\vec{\mathbf{F}}$ where $\vec{\mathbf{N}}$ is an outward normal vector to surface S.

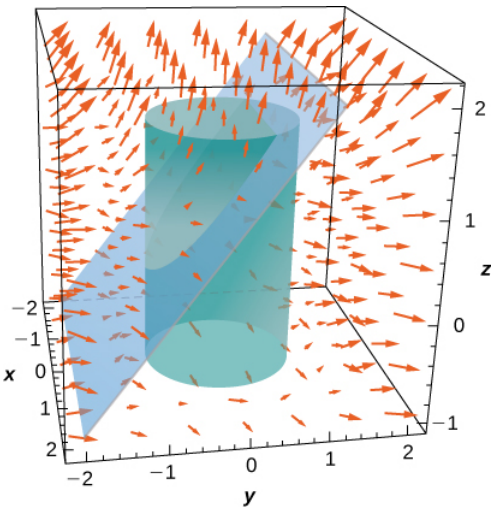
16. $\vec{\mathbf{F}}(x, y, z) = x\,\hat{\mathbf{i}} + 2y\,\hat{\mathbf{j}} = 3z\,\hat{\mathbf{k}}$, and S is that part of plane $15x - 12y + 3z = 6$ that lies above unit square $0 \le x \le 1,\ 0 \le y \le 1$.

17. $\vec{\mathbf{F}}(x, y) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$, and S is hemisphere $z = \sqrt{1 - x^2 - y^2}$.

Answer

$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, dS = \frac{4\pi}{3}$

18. $\vec{\mathbf{F}}(x, y, z) = x^2\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}} + z^2\,\hat{\mathbf{k}}$, and S is the portion of plane $z = y + 1$ that lies inside cylinder $x^2 + y^2 = 1$.



In exercises 19 - 20, approximate the mass of the homogeneous lamina that has the shape of given surface S. Round to four decimal places.

19. [T] S is surface $z = 4 - x - 2y$, with $z \geq 0,\ x \geq 0,\ y \geq 0$; $\xi = x$.

Answer

$m \approx 13.0639$

20. [T] S is surface $z = x^2 + y^2$, with $z \leq 1$; $\xi = z$.

21. [T] S is surface $x^2 + y^2 + x^2 = 5$, with $z \geq 1$; $\xi = \theta^2$.

Answer

$m \approx 228.5313$

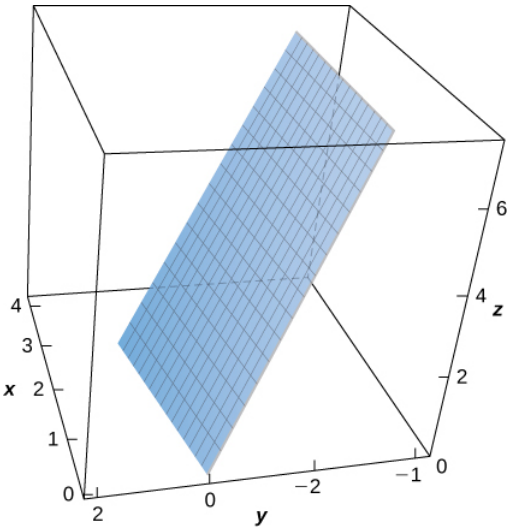
22. Evaluate $\iint_S (y^2 z\,\hat{\mathbf{i}} + y^3\,\hat{\mathbf{j}} + xz\hat{\mathbf{k}}) \cdot dS$, where S is the surface of cube $-1 \leq x \leq 1,\ -1 \leq y \leq 1$, and $0 \leq z \leq 2$ in a counterclockwise direction.

23. Evaluate surface integral $\iint_S g\,dS$, where $g(x, y, z) = xz + 2x^2 - 3xy$ and S is the portion of plane $2x - 3y + z = 6$ that lies over unit square $R : 0 \leq x \leq 1,\ 0 \leq y \leq 1$.

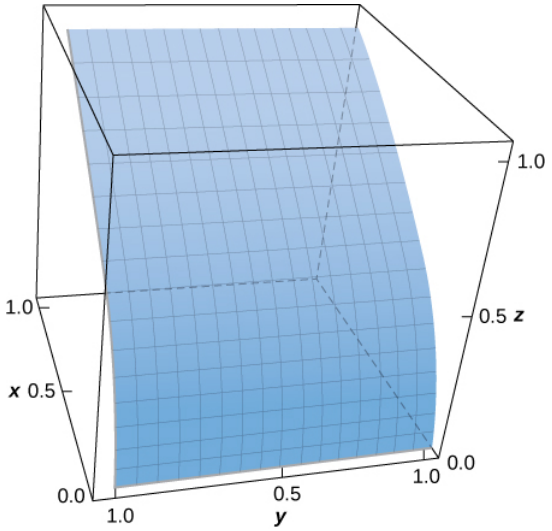
Answer

$\iint_S g\,dS = 3\sqrt{4}$

24. Evaluate $\iint_S (x + y + z)\,dS$, where S is the surface defined parametrically by $\vec{\mathbf{R}}(u, v) = (2u + v)\,\hat{\mathbf{i}} + (u - 2v)\,\hat{\mathbf{j}} + (u + 3v)\,\hat{\mathbf{k}}$ for $0 \leq u \leq 1$, and $0 \leq v \leq 2$.



25. [T] Evaluate $\iint_S (x - y^2 + z)\,dS$, where S is the surface defined parametrically by $\vec{\mathbf{R}}(u, v) = u^2\,\hat{\mathbf{i}} + v\,\hat{\mathbf{j}} + u\,\hat{\mathbf{k}}$ for $0 \leq u \leq 1,\ 0 \leq v \leq 1$.



Answer

$$\iint_S (x^2 + y - z)\, dS \approx 0.9617$$

26. [T] Evaluate where S is the surface defined by $\vec{\mathbf{R}}(u, v) = u\,\hat{\mathbf{i}} - u^2\,\hat{\mathbf{j}} + v\,\hat{\mathbf{k}},\ 0 \le u \le 2,\ 0 \le v \le 1$ for $0 \le u \le 1,\ 0 \le v \le 2$.

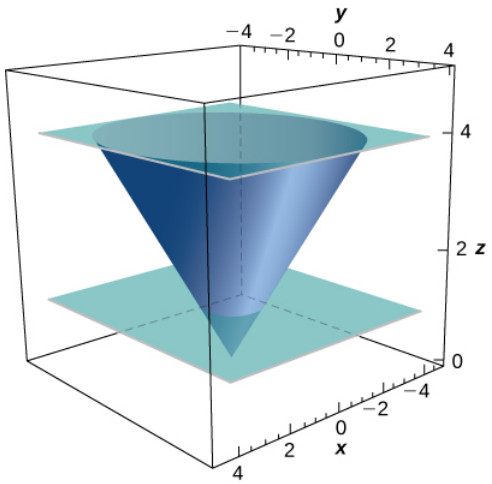
27. Evaluate $\iint_S (x^2 + y^2)\, dS$, where S is the surface bounded above hemisphere $z = \sqrt{1 - x^2 - y^2}$, and below by plane $z = 0$.

Answer

$$\iint_S (x^2 + y^2)\, dS = \frac{4\pi}{3}$$

28. Evaluate $\iint_S (x^2 + y^2 + z^2)\, dS$, where S is the portion of plane that lies inside cylinder $x^2 + y^2 = 1$.

29. [T] Evaluate $\iint_S x^2 z\, dS$, where S is the portion of cone $z^2 = x^2 + y^2$ that lies between planes $z = 1$ and $z = 4$.



Answer

$$\operatorname{div} \vec{\mathbf{F}} = a + b$$

$$\iint_S x^2 z dS = \frac{1023\sqrt{2\pi}}{5}$$

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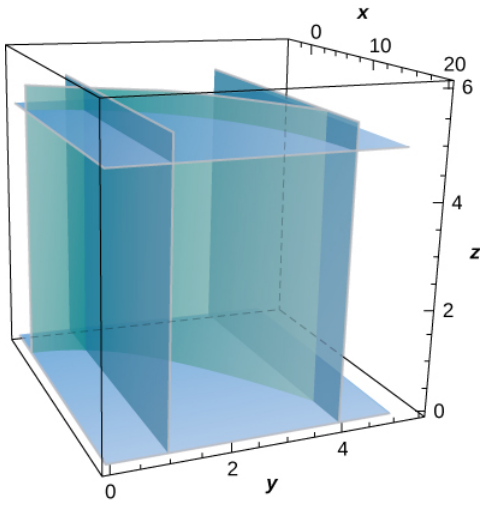
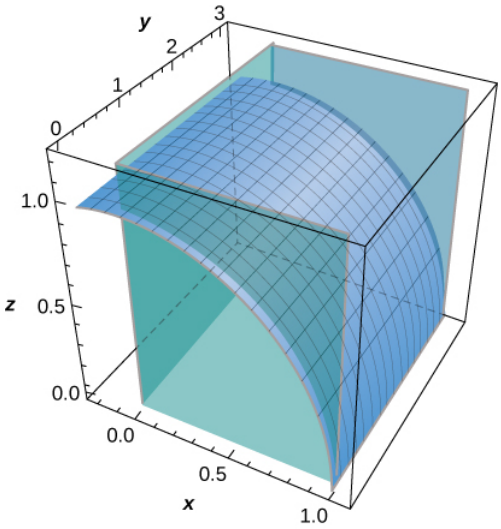
30. [T] Evaluate $\iint_S \frac{xz}{y}\,dS$, where S is the portion of cylinder $x = y^2$ that lies in the first octant between planes $z = 0$, $z = 5$, and $y = 4$.



31. [T] Evaluate $\iint_S (z + y)\,dS$, where S is the part of the graph of $z = \sqrt{1 - x^2}$ in the first octant between the xy-plane and plane $y = 3$.



Answer

$\iint_S (z + y)\,dS \approx 10.1$

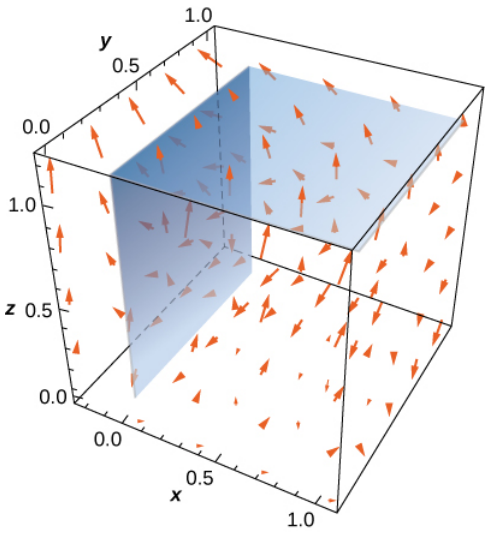
32. Evaluate $\iint_S xyz\,dS$ if S is the part of plane $z = x + y$ that lies over the triangular region in the xy-plane with vertices (0, 0, 0), (1, 0, 0), and (0, 2, 0).

33. Find the mass of a lamina of density $\xi(x, y, z) = z$ in the shape of hemisphere $z = (a^2 - x^2 - y^2)^{1/2}$.

Answer

$m = \pi a^3$

34. Compute $\int\int_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\,dS$, where $\vec{\mathbf{F}}(x, y, z) = x\,\hat{\mathbf{i}} - 5y\,\hat{\mathbf{j}} + 4z\,\hat{\mathbf{k}}$ and $\vec{\mathbf{N}}$ is an outward normal vector S, where S is the union of two squares S_1 : $x = 0$, $0 \le y \le 1$, $0 \le z \le 1$ and S_2 : $x = 0$, $0 \le x \le 1$, $0 \le y \le 1$.



35. Compute $\int\int_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, dS$, where $\vec{\mathbf{F}}(x,y,z) = xy\,\hat{\mathbf{i}} + z\,\hat{\mathbf{j}} + (x+y)\,\hat{\mathbf{k}}$ and $\vec{\mathbf{N}}$ is an outward normal vector S, where S is the triangular region cut off from plane $x+y+z=1$ by the positive coordinate axes.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, dS = \frac{13}{24}$$

36. Compute $\int\int_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, dS$, where $\vec{\mathbf{F}}(x,y,z) = 2yz\,\hat{\mathbf{i}} + (\tan^{-1} xz)\,\hat{\mathbf{j}} + e^{xy}\,\hat{\mathbf{k}}$ and $\vec{\mathbf{N}}$ is an outward normal vector S, where S is the surface of sphere $x^2+y^2+z^2=1$.

37. Compute $\int\int_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, dS$, where $\vec{\mathbf{F}}(x,y,z) = xyz\,\hat{\mathbf{i}} + xyz\,\hat{\mathbf{j}} + xyz\,\hat{\mathbf{k}}$ and $\vec{\mathbf{N}}$ is an outward normal vector S, where S is the surface of the five faces of the unit cube $0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1$ missing $z=0$.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, dS = \frac{3}{4}$$

For exercises 38 - 39, express the surface integral as an iterated double integral by using a projection on S on the yz-plane.

38. $\iint_S xy^2z^3\, dS$; S is the first-octant portion of plane $2x+3y+4z=12$.

39. $\iint_S (x^2-2y+z)\, dS$; S is the portion of the graph of $4x+y=8$ bounded by the coordinate planes and plane $z=6$.

Answer

$$\int_0^8 \int_0^6 \left(4 - 3y + \frac{1}{16}y^2 + z\right)\left(\frac{1}{4}\sqrt{17}\right)\, dz\, dy$$

For exercises 40 - 41, express the surface integral as an iterated double integral by using a projection on S on the xz-plane.

40. $\iint_S xy^2z^3\, dS$; S is the first-octant portion of plane $2x+3y+4z=12$.

41. $\iint_S (x^2-2y+z)\, dS$; is the portion of the graph of $4x+y=8$ bounded by the coordinate planes and plane $z=6$.

Answer

$$\int_0^2 \int_0^6 \left[x^2 - 2(8-4x) + z\right]\sqrt{17}\, dz\, dx$$

42. Evaluate surface integral $\iint_S yz\, dS$, where S is the first-octant part of plane $x+y+z=\lambda$, where λ is a positive constant.

43. Evaluate surface integral $\iint_S (x^2z + y^2z)\, dS$, where S is hemisphere $x^2 + y^2 + z^2 = a^2,\ z \geq 0$.

Answer

$$\iint_S (x^2z + y^2z)\, dS = \frac{\pi a^5}{2}$$

44. Evaluate surface integral $\iint_S z\, dA$, where S is surface $z = \sqrt{x^2 + y^2},\ 0 \leq z \leq 2$.

45. Evaluate surface integral $\iint_S x^2yz\, dS$, where S is the part of plane $z = 1 + 2x + 3y$ that lies above rectangle $0 \leq x \leq 3$ and $0 \leq y \leq 2$.

Answer

$$\iint_S x^2yz\, dS = 171\sqrt{14}$$

46. Evaluate surface integral $\iint_S yz\, dS$, where S is plane $x + y + z = 1$ that lies in the first octant.

47. Evaluate surface integral $\iint_S yz\, dS$, where S is the part of plane $z = y + 3$ that lies inside cylinder $x^2 + y^2 = 1$.

Answer

$$\iint_S yz\, dS = \frac{\sqrt{2}\pi}{4}$$

For exercises 48 - 50, use geometric reasoning to evaluate the given surface integrals.

48. $\iint_S \sqrt{x^2 + y^2 + z^2}\, dS$, where S is surface $x^2 + y^2 + z^2 = 4,\ z \geq 0$

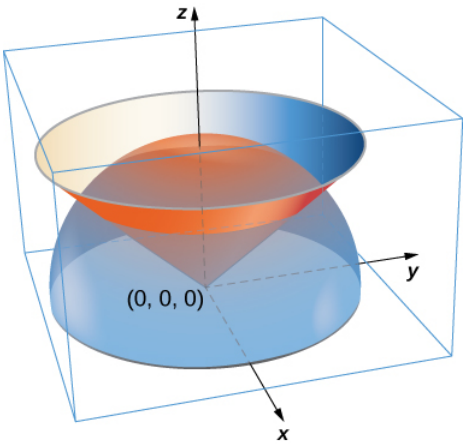
49. $\iint_S (x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}) \cdot dS$, where S is surface $x^2 + y^2 = 4,\ 1 \leq z \leq 3$, oriented with unit normal vectors pointing outward

Answer

$$\iint_S (x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}) \cdot dS = 16\pi$$

50. $\iint_S (z\,\hat{\mathbf{k}}) \cdot dS$, where S is disc $x^2 + y^2 \leq 9$ on plane $z = 4$ oriented with unit normal vectors pointing upward

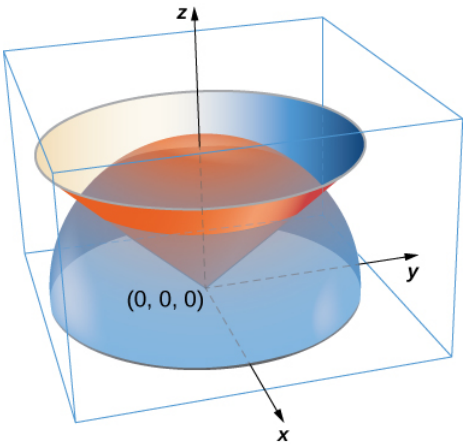
51. A lamina has the shape of a portion of sphere $x^2 + y^2 + z^2 = a^2$ that lies within cone $z = \sqrt{x^2 + y^2}$. Let S be the spherical shell centered at the origin with radius a, and let C be the right circular cone with a vertex at the origin and an axis of symmetry that coincides with the z-axis. Determine the mass of the lamina if $\delta(x, y, z) = x^2y^2z$.



Answer

$$m = \frac{\pi a^7}{192}$$

52. A lamina has the shape of a portion of sphere $x^2 + y^2 + z^2 = a^2$ that lies within cone $z = \sqrt{x^2 + y^2}$. Let S be the spherical shell centered at the origin with radius a, and let C be the right circular cone with a vertex at the origin and an axis of symmetry that coincides with the z-axis. Suppose the vertex angle of the cone is ϕ_0, with $0 \leq \phi_0 < \frac{\pi}{2}$. Determine the mass of that portion of the shape enclosed in the intersection of S and C. Assume $\delta(x, y, z) = x^2 y^2 z$.



53. A paper cup has the shape of an inverted right circular cone of height 6 in. and radius of top 3 in. If the cup is full of water weighing $62.5\, lb/ft^3$, find the magnitude of the total force exerted by the water on the inside surface of the cup.

Answer

$F \approx 4.57\, lb$

For exercises 54 - 55, the heat flow vector field for conducting objects is $\vec{\mathbf{F}} = -k\vec{\nabla}T$, where $T(x, y, z)$ is the temperature in the object and $k > 0$ is a constant that depends on the material. Find the outward flux of $\vec{\mathbf{F}}$ across the following surfaces S for the given temperature distributions and assume $k = 1$.

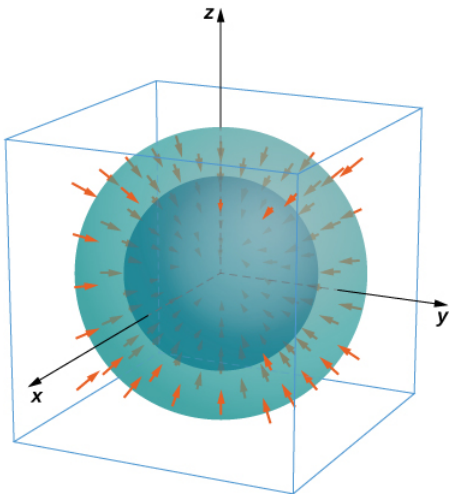
54. $T(x, y, z) = 100e^{-x-y}$; S consists of the faces of cube $|x| \leq 1$, $|y| \leq 1$, $|z| \leq 1$.

55. $T(x, y, z) = -\ln(x^2 + y^2 + z^2)$; S is sphere $x^2 + y^2 + z^2 = a^2$.

Answer

$8\pi a$

For exercises 56 - 57, consider the radial fields $\vec{\mathbf{F}} = \dfrac{\langle x, y, z\rangle}{(x^2 + y^2 + z^2)^{p/2}} = \dfrac{r}{|r|^p}$, where p is a real number. Let S consist of spheres A and B centered at the origin with radii $0 < a < b$. The total outward flux across S consists of the outward flux across the outer sphere B less the flux into S across inner sphere A.



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56. Find the total flux across S with $p = 0$.

57. Show that for $p = 3$ the flux across S is independent of a and b.

Answer

The net flux is zero.

Contributors

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16.7: Stokes' Theorem

Learning Objectives

- Explain the meaning of Stokes' theorem.
- Use Stokes' theorem to evaluate a line integral.
- Use Stokes' theorem to calculate a surface integral.
- Use Stokes' theorem to calculate a curl.

In this section, we study Stokes' theorem, a higher-dimensional generalization of Green's theorem. This theorem, like the Fundamental Theorem for Line Integrals and Green's theorem, is a generalization of the Fundamental Theorem of Calculus to higher dimensions. Stokes' theorem relates a vector surface integral over surface S in space to a line integral around the boundary of S. Therefore, just as the theorems before it, Stokes' theorem can be used to reduce an integral over a geometric object S to an integral over the boundary of S. In addition to allowing us to translate between line integrals and surface integrals, Stokes' theorem connects the concepts of curl and circulation. Furthermore, the theorem has applications in fluid mechanics and electromagnetism. We use Stokes' theorem to derive Faraday's law, an important result involving electric fields.

Stokes' Theorem

Stokes' theorem says we can calculate the flux of $curl\,\vec{\mathbf{F}}$ across surface S by knowing information only about the values of $\vec{\mathbf{F}}$ along the boundary of S. Conversely, we can calculate the line integral of vector field $\vec{\mathbf{F}}$ along the boundary of surface S by translating to a double integral of the curl of $\vec{\mathbf{F}}$ over S.

Let S be an oriented smooth surface with unit normal vector $\vec{\mathbf{N}}$. Furthermore, suppose the boundary of S is a simple closed curve C. The orientation of S induces the positive orientation of C if, as you walk in the positive direction around C with your head pointing in the direction of $\vec{\mathbf{N}}$, the surface is always on your left. With this definition in place, we can state *Stokes' theorem*.

Theorem 16.7.1: Stokes' Theorem

Let S be a piecewise smooth oriented surface with a boundary that is a simple closed curve C with positive orientation (Figure 16.7.1). If $\vec{\mathbf{F}}$ is a vector field with component functions that have continuous partial derivatives on an open region containing S, then

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_S curl\,\vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}. \tag{16.7.1}$$

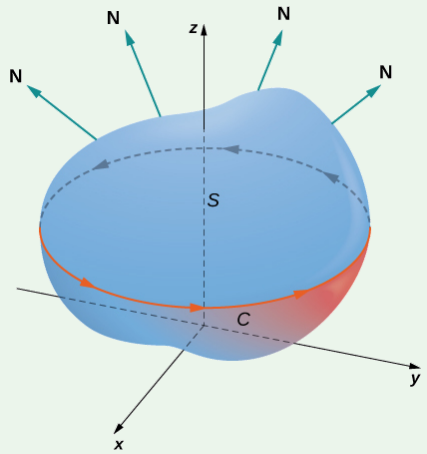


Figure 16.7.1: Stokes' theorem relates the flux integral over the surface to a line integral around the boundary of the surface. Note that the orientation of the curve is positive.

Suppose surface S is a flat region in the xy-plane with upward orientation. Then the unit normal vector is $\vec{\mathbf{k}}$ and surface integral

$$\iint_S curl\,\vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$$

is actually the double integral

$$\iint_S curl\,\vec{\mathbf{F}} \cdot \vec{\mathbf{k}}\, dA.$$

In this special case, Stokes' theorem gives

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_S curl\,\vec{\mathbf{F}} \cdot \vec{\mathbf{k}}\, dA.$$

However, this is the flux form of Green's theorem, which shows us that Green's theorem is a special case of Stokes' theorem. Green's theorem can only handle surfaces in a plane, but Stokes' theorem can handle surfaces in a plane or in space.

The complete proof of Stokes' theorem is beyond the scope of this text. We look at an intuitive explanation for the truth of the theorem and then see proof of the theorem in the special case that surface S is a portion of a graph of a function, and S, the boundary of S, and $\vec{\mathbf{F}}$ are all fairly tame.

Proof

First, we look at an informal proof of the theorem. This proof is not rigorous, but it is meant to give a general feeling for why the theorem is true. Let S be a surface and let D be a small piece of the surface so that D does not share any points with the boundary of S. We choose D to be small enough so that it can be approximated by an oriented square E. Let D inherit its orientation from S, and give E the same orientation. This square has four sides; denote them E_l, E_r, E_u, and E_d for the left, right, up, and down sides, respectively. On the square, we can use the flux form of Green's theorem:

$$\int_{E_l+E_d+E_r+E_u} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_E curl\,\vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, d\vec{\mathbf{S}} = \iint_E curl\,\vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}.$$

To approximate the flux over the entire surface, we add the values of the flux on the small squares approximating small pieces of the surface (Figure 16.7.2).

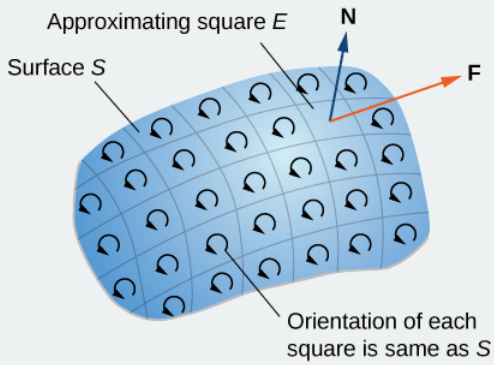


Figure 16.7.2: Chop the surface into small pieces. The pieces should be small enough that they can be approximated by a square.

By Green's theorem, the flux across each approximating square is a line integral over its boundary. Let F be an approximating square with an orientation inherited from S and with a right side E_l (so F is to the left of E). Let F_r denote the right side of F; then, $E_l = -F_r$. In other words, the right side of F is the same curve as the left side of E, just oriented in the opposite direction. Therefore,

$$\int_{E_l} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = -\int_{F_r} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}.$$

As we add up all the fluxes over all the squares approximating surface S, line integrals

$$\int_{E_l} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

and

$$\int_{F_r} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

cancel each other out. The same goes for the line integrals over the other three sides of E. These three line integrals cancel out with the line integral of the lower side of the square above E, the line integral over the left side of the square to the right of E, and the line integral over the upper side of the square below E (Figure 16.7.3). After all this cancelation occurs over all the approximating squares, the only line integrals that survive are the line integrals over sides approximating the boundary of S. Therefore, the sum of all the fluxes (which, by Green's theorem, is the sum of all the line integrals around the boundaries of approximating squares) can be approximated by a line integral over the boundary of S. In the limit, as the areas of the approximating squares go to zero, this approximation gets arbitrarily close to the flux.

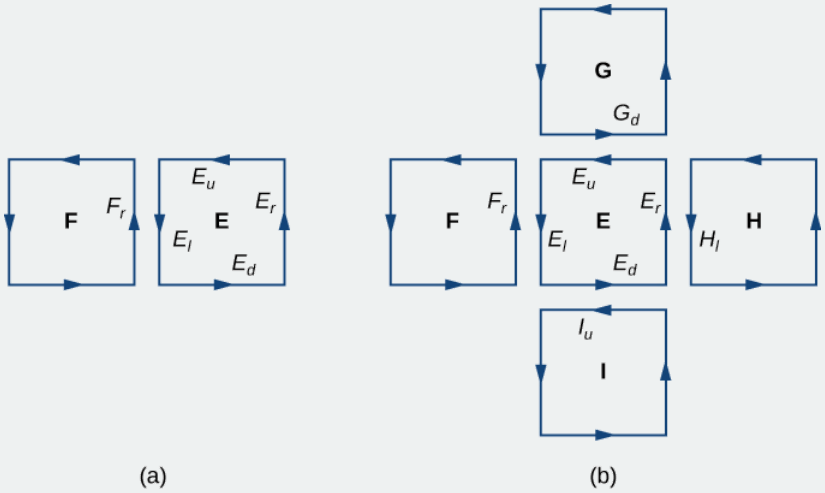


Figure 16.7.3: (a) The line integral along E_l cancels out the line integral along F_r because $E_l = -F_r$. (b) The line integral along any of the sides of E cancels out with the line integral along a side of an adjacent approximating square.

Let's now look at a rigorous proof of the theorem in the special case that S is the graph of function $z = f(x, y)$, where x and y vary over a bounded, simply connected region D of finite area (Figure 16.7.4). Furthermore, assume that f has continuous second-order partial derivatives. Let C denote the boundary of S and let C' denote the boundary of D. Then, D is the "shadow" of S in the plane and C' is the "shadow" of C. Suppose that S is oriented upward. The counterclockwise orientation of C is positive, as is the counterclockwise orientation of C'. Let $\vec{\mathbf{F}}(x, y, z) = \langle P, Q, R\rangle$ be a vector field with component functions that have continuous partial derivatives.

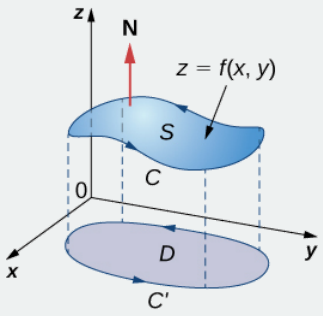


Figure 16.7.4: D is the "shadow," or projection, of S in the plane and C' is the projection of C.

We take the standard parameterization of $S : x = x,\ y = y,\ z = g(x, y)$. The tangent vectors are $\vec{\mathbf{t}}_x = \langle 1, 0, g_x\rangle$ and $\vec{\mathbf{t}}_y = \langle 0, 1, g_y\rangle$, and therefore $\vec{\mathbf{t}}_x \times \vec{\mathbf{t}}_y = \langle -g_x, -g_y, 1\rangle$.

$$\iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_D [-(R_y - Q_z)z_x - (P_z - R_x)z_y + (Q_x - P_y)]\, dA,$$

where the partial derivatives are all evaluated at $(x, y, g(x, y))$, making the integrand depend on x and y only. Suppose $\langle x(t),\ y(t)\rangle,\ a \le t \le b$ is a parameterization of C'. Then, a parameterization of C is $\langle x(t),\ y(t),\ g(x(t),\ y(t))\rangle,\ a \le t \le b$. Armed with these parameterizations, the Chain rule, and Green's theorem, and keeping in mind that P, Q and R are all functions of x and y, we can evaluate line integral

$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_a^b (Px'(t) + Qy'(t) + Rz'(t))\, dt \\ &= \int_a^b \left[Px'(t) + Qy'(t) + R\left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\right)\right] dt \\ &= \int_a^b \left[\left(P + R\frac{\partial z}{\partial x}\right)x'(t) + \left(Q + R\frac{\partial z}{\partial y}\right)y'(t)\right] dt \\ &= \int_{C'} \left(P + R\frac{\partial z}{\partial x}\right) dx + \left(Q + R\frac{\partial z}{\partial y}\right) dy \\ &= \iint_D \left[\frac{\partial}{\partial x}\left(Q + R\frac{\partial z}{\partial y}\right) - \frac{\partial}{\partial y}\left(P + R\frac{\partial z}{\partial x}\right)\right] dA \\ &= \iint_D \left(\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial R}{\partial x}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + R\frac{\partial^2 z}{\partial x \partial y}\right) \\ &\quad - \left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x} + R\frac{\partial^2 z}{\partial y \partial x}\right)\end{aligned}$$

By Clairaut's theorem,

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

Therefore, four of the terms disappear from this double integral, and we are left with

$$\iint_D [-(R_y - Q_z)Z_x - (P_z - R_x)z_y + (Q_x - P_y)]\, dA,$$

which equals

$$\iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}.$$

□

We have shown that Stokes' theorem is true in the case of a function with a domain that is a simply connected region of finite area. We can quickly confirm this theorem for another important case: when vector field $\vec{\mathbf{F}}$ is a conservative field. If $\vec{\mathbf{F}}$ is conservative, the curl of $\vec{\mathbf{F}}$ is zero, so

$$\iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = 0.$$

Since the boundary of S is a closed curve, the integral

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}.$$

is also zero.

Example 16.7.1: Verifying Stokes' Theorem for a Specific Case

Verify that Stokes' theorem is true for vector field $\vec{\mathbf{F}}(x, y) = \langle -z, x, 0\rangle$ and surface S, where S is the hemisphere, oriented outward, with parameterization $\vec{\mathbf{r}}(\phi, \theta) = \langle \sin\phi\, \cos\theta,\, \sin\phi\, \sin\theta,\, \cos\phi\rangle$, $0 \le \theta \le \pi$, $0 \le \phi \le \pi$ as shown in Figure 16.7.5.

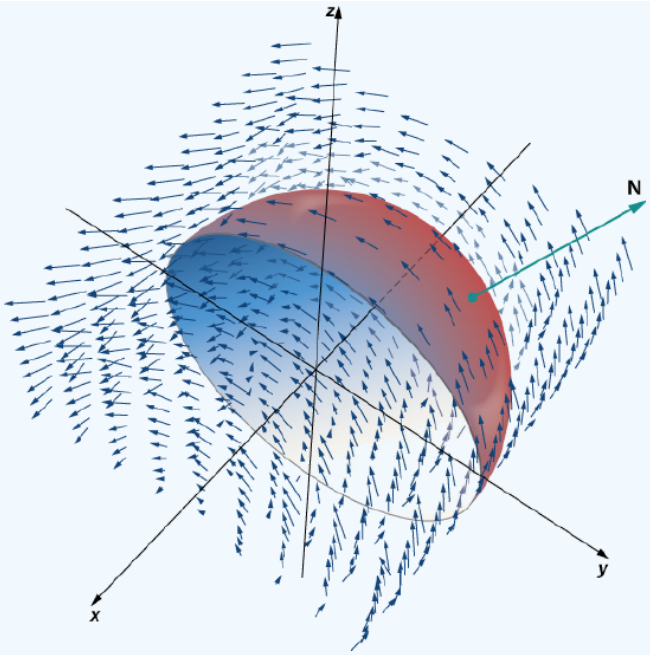


Figure 16.7.5: Verifying Stokes' theorem for a hemisphere in a vector field.

Solution

Let C be the boundary of S. Note that C is a circle of radius 1, centered at the origin, sitting in plane $y = 0$. This circle has parameterization $\langle \cos t,\ 0,\ \sin t \rangle$, $0 \le t \le 2\pi$. the equation for scalar surface integrals

$$\begin{aligned} \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_0^{2\pi} \langle -\sin t,\ \cos t,\ 0 \rangle \cdot \langle -\sin t,\ 0,\ \cos t \rangle \, dt \\ &= \int_0^{2\pi} \sin^2 t \, dt \\ &= \pi. \end{aligned}$$

By the equation for vector line integrals,

$$\begin{aligned} \iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \iint_D curl\, \vec{\mathbf{F}}(\vec{\mathbf{r}}(\phi,\theta)) \cdot (\vec{\mathbf{t}}_\phi \times \vec{\mathbf{t}}_\theta) \, dA \\ &= \iint_D \langle 0, -1, 1 \rangle \cdot \langle \cos\theta \sin^2\phi,\ \sin\theta \sin^2\phi,\ \sin\phi \cos\phi \rangle \, dA \\ &= \int_0^{\pi} \int_0^{\pi} (\sin\phi \cos\phi - \sin\theta \sin^2\phi) \, d\phi d\theta \\ &= \frac{\pi}{2} \int_0^{\pi} \sin\theta \, d\theta \\ &= \pi. \end{aligned}$$

Therefore, we have verified Stokes' theorem for this example.

? Exercise 16.7.1

Verify that Stokes' theorem is true for vector field $\vec{\mathbf{F}}(x,y,z) = \langle y, x, -z \rangle$ and surface S, where S is the upwardly oriented portion of the graph of $f(x,y) = x^2 y$ over a triangle in the xy-plane with vertices $(0,0)$, $(2,0)$, and $(0,2)$.

Hint

Calculate the double integral and line integral separately.

Answer

Both integrals give $-\frac{136}{45}$:

Applying Stokes' Theorem

Stokes' theorem translates between the flux integral of surface S to a line integral around the boundary of S. Therefore, the theorem allows us to compute surface integrals or line integrals that would ordinarily be quite difficult by translating the line integral into a surface integral or vice versa. We now study some examples of each kind of translation.

Example 16.7.2: Calculating a Surface Integral

Calculate surface integral

$$\iint_S curl\,\vec{\mathbf{F}} \cdot d\vec{\mathbf{S}},$$

where S is the surface, oriented outward, in Figure 16.7.6 and $\vec{\mathbf{F}} = \langle z,\, 2xy,\, x+y\rangle$.

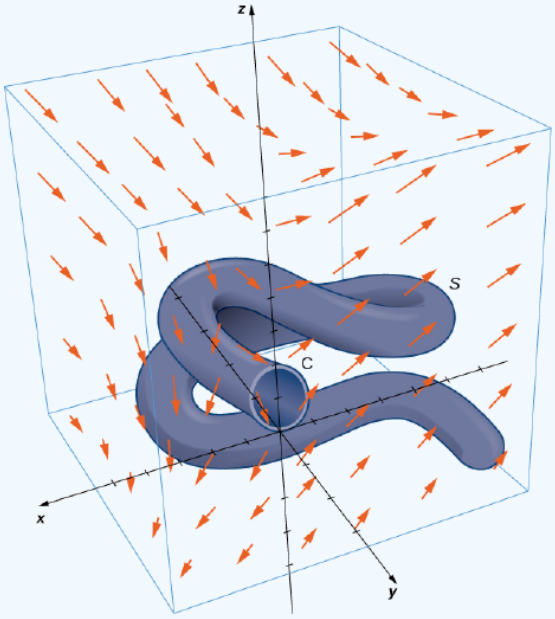


Figure 16.7.6: A complicated surface in a vector field.

Solution

Note that to calculate

$$\iint_S curl\,\vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$$

without using Stokes' theorem, we would need the equation for scalar surface integrals. Use of this equation requires a parameterization of S. Surface S is complicated enough that it would be extremely difficult to find a parameterization. Therefore, the methods we have learned in previous sections are not useful for this problem. Instead, we use Stokes' theorem, noting that the boundary C of the surface is merely a single circle with radius 1.

The curl of $\vec{\mathbf{F}}$ is $\langle 1, 1, 2y\rangle$. By Stokes' theorem,

$$\iint_S curl\,\vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}},$$

where C has parameterization $\langle \cos t,\, \sin t,\, 1\rangle, 0 \le t \le 2\pi$. By the equation for vector line integrals,

$$\begin{aligned}\iint_S curl\, F \cdot d\vec{\mathbf{S}} &= \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \\ &= \int_0^2 \langle 1,\, \sin t\, \cos t,\, \cos t + \sin t \rangle \cdot \langle -\sin t,\, \cos t,\, 0 \rangle\, dt \\ &= \int_0^{2\pi} (-\sin t + 2\, \sin t\, \cos^2 t)\, dt \\ &= \left[\cos t - \frac{2\, \cos^3 t}{3}\right]_0^{2\pi} \\ &= \cos(2\pi) - \frac{2\, \cos^3(2\pi)}{3} - \left(\cos(0) - \frac{2\, \cos^3(0)}{3}\right) \\ &= 0.\end{aligned}$$

An amazing consequence of Stokes' theorem is that if S' is any other smooth surface with boundary C and the same orientation as S, then

$$\iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0$$

because Stokes' theorem says the surface integral depends on the line integral around the boundary only.

In Example 16.7.2, we calculated a surface integral simply by using information about the boundary of the surface. In general, let S_1 and S_2 be smooth surfaces with the same boundary C and the same orientation. By Stokes' theorem,

$$\iint_{S_1} curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_{S_2} curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}. \tag{16.7.2}$$

Therefore, if

$$\iint_{S_1} curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$$

is difficult to calculate but

$$\iint_{S_2} curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$$

is easy to calculate, Stokes' theorem allows us to calculate the easier surface integral. In Example 16.7.2, we could have calculated

$$\iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$$

by calculating

$$\iint_{S'} curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}},$$

where $\vec{\mathbf{S}}'$ is the disk enclosed by boundary curve C (a much more simple surface with which to work).

Equation 16.7.2 shows that flux integrals of curl vector fields are surface independent in the same way that line integrals of gradient fields are path independent. Recall that if $\vec{\mathbf{F}}$ is a two-dimensional conservative vector field defined on a simply connected domain, f is a *potential function* for $\vec{\mathbf{F}}$, and C is a curve in the domain of $\vec{\mathbf{F}}$, then

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

depends only on the endpoints of C. Therefore if C' is any other curve with the same starting point and endpoint as C (that is, C' has the same orientation as C), then

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{C'} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

In other words, the value of the integral depends on the boundary of the path only; it does not really depend on the path itself.

Analogously, suppose that S and S' are surfaces with the same boundary and same orientation, and suppose that $\vec{\mathbf{G}}$ is a three-dimensional vector field that can be written as the curl of another vector field $\vec{\mathbf{F}}$ (so that $\vec{\mathbf{F}}$ is like a "potential field" of $\vec{\mathbf{G}}$). By Equation 16.7.2,

$$\iint_S \vec{\mathbf{G}} \cdot d\vec{\mathbf{S}} = \iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_{S'} curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_{S'} \vec{\mathbf{G}} \cdot d\vec{\mathbf{S}}.$$

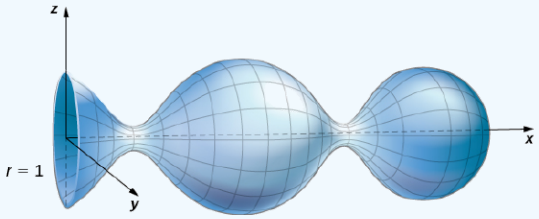
Therefore, the flux integral of $\vec{\mathbf{G}}$ does not depend on the surface, only on the boundary of the surface. Flux integrals of vector fields that can be written as the curl of a vector field are surface independent in the same way that line integrals of vector fields that can be written as the gradient of a scalar function are path independent.

? Exercise 16.7.2

Use Stokes' theorem to calculate surface integral

$$\iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}},$$

where $\vec{\mathbf{F}}(x, y, z) = \langle z, x, y \rangle$ and S is the surface as shown in the following figure. The boundary curve, C is oriented clockwise when looking along the positive y-axis.



Hint

Parameterize the boundary of S and translate to a line integral.

Answer

$-\pi$

✓ Example 16.7.3: Calculating a Line Integral

Calculate the line integral

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}},$$

where $\vec{\mathbf{F}} = \langle xy,\ x^2 + y^2 + z^2,\ yz \rangle$ and C is the boundary of the parallelogram with vertices $(0, 0, 1)$, $(0, 1, 0)$, $(2, 0, -1)$, and $(2, 1, -2)$.

Solution

To calculate the line integral directly, we need to parameterize each side of the parallelogram separately, calculate four separate line integrals, and add the result. This is not overly complicated, but it is time-consuming.

By contrast, let's calculate the line integral using Stokes' theorem. Let S denote the surface of the parallelogram. Note that S is the portion of the graph of $z = 1 - x - y$ for (x, y) varying over the rectangular region with vertices $(0, 0)$, $(0, 1)$, $(2, 0)$, and $(2, 1)$ in the xy-plane. Therefore, a parameterization of S is $\langle x, y,\ 1 - x - y \rangle$, $0 \le x \le 2$, $0 \le y \le 1$. The curl of $\vec{\mathbf{F}}$ is $\langle -z,\ 0,\ x \rangle$,and Stokes' theorem and the equation for scalar surface integrals

$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} \\ &= \int_0^2 \int_0^1 curl\, \vec{\mathbf{F}}(x,y) \cdot (\vec{\mathbf{t}}_x \times \vec{\mathbf{t}}_y)\, dy\, dx \\ &= \int_0^2 \int_0^1 \langle -(1-x-y),\, 0,\, x\rangle \cdot (\langle 1,\, 0, -1\rangle \times \langle 0,\, 1,\, -1\rangle)\, dy\, dx \\ &= \int_0^2 \int_0^1 \langle x+y-1,\, 0,\, x\rangle \cdot \langle 1,1,1\rangle\, dy\, dx \\ &= \int_0^2 \int_0^1 2x+y-1\, dy\, dx \\ &= 3.\end{aligned}$$

? Exercise 16.7.3

Use Stokes' theorem to calculate line integral

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}},$$

where $\vec{\mathbf{F}} = \langle z, x, y\rangle$ and C is the boundary of a triangle with vertices $(0,0,1)$, $(3,0,-2)$, and $(0,1,2)$.

Hint

This triangle lies in plane $z = 1 - x + y$.

Answer

$\frac{3}{2}$

Interpretation of Curl

In addition to translating between line integrals and flux integrals, Stokes' theorem can be used to justify the physical interpretation of curl that we have learned. Here we investigate the relationship between curl and circulation, and we use Stokes' theorem to state Faraday's law—an important law in electricity and magnetism that relates the curl of an electric field to the rate of change of a magnetic field.

Recall that if C is a closed curve and $\vec{\mathbf{F}}$ is a vector field defined on C, then the circulation of $\vec{\mathbf{F}}$ around C is line integral

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}.$$

If $\vec{\mathbf{F}}$ represents the velocity field of a fluid in space, then the circulation measures the tendency of the fluid to move in the direction of C.

Let $\vec{\mathbf{F}}$ be a continuous vector field and let D_τ be a small disk of radius r with center P_0 (Figure 16.7.7). If D_τ is small enough, then $(curl\, \vec{\mathbf{F}})(P) \approx (curl\, \vec{\mathbf{F}})(P_0)$ for all points P in D_τ because the curl is continuous. Let C_τ be the boundary circle of D_τ: By Stokes' theorem,

$$\int_{C_\tau} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_{D_\tau} curl\, \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, d\vec{\mathbf{S}} \approx \iint_{D_\tau} (curl\, \vec{\mathbf{F}})(P_0) \cdot \vec{\mathbf{N}}(P_0)\, d\vec{\mathbf{S}}.$$

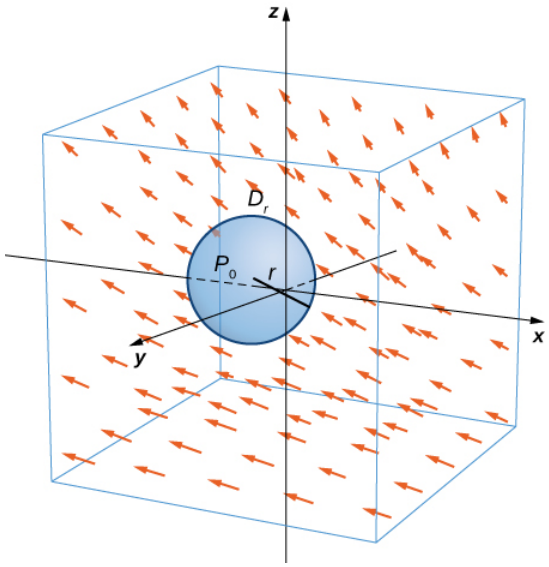


Figure 16.7.7: Disk D_τ is a small disk in a continuous vector field.

The quantity $(curl\,\vec{\mathbf{F}})(P_0)\cdot\vec{\mathbf{N}}(P_0)$ is constant, and therefore

$$\iint_{D_\tau}(curl\,\vec{\mathbf{F}})(P_0)\cdot\vec{\mathbf{N}}(P_0)\,d\vec{\mathbf{S}}=\pi r^2[(curl\,\vec{\mathbf{F}})(P_0)\cdot\vec{\mathbf{N}}(P_0)].$$

Thus

$$\int_{C_\tau}\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}\approx\pi r^2[(curl\,\vec{\mathbf{F}})(P_0)\cdot\vec{\mathbf{N}}(P_0)],$$

and the approximation gets arbitrarily close as the radius shrinks to zero. Therefore Stokes' theorem implies that

$$(curl\,\vec{\mathbf{F}})(P_0)\cdot\vec{\mathbf{N}}(P_0)=\lim_{r\to0^+}\frac{1}{\pi r^2}\int_{C_\tau}\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}.$$

This equation relates the curl of a vector field to the circulation. Since the area of the disk is πr^2, this equation says we can view the curl (in the limit) as the circulation per unit area. Recall that if $\vec{\mathbf{F}}$ is the velocity field of a fluid, then circulation

$$\oint_{C_\tau}\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}=\oint_{C_\tau}\vec{\mathbf{F}}\cdot\vec{\mathbf{T}}\,ds$$

is a measure of the tendency of the fluid to move around C_τ: The reason for this is that $\vec{\mathbf{F}}\cdot\vec{\mathbf{T}}$ is a component of $\vec{\mathbf{F}}$ in the direction of $\vec{\mathbf{T}}$, and the closer the direction of $\vec{\mathbf{F}}$ is to $\vec{\mathbf{T}}$, the larger the value of $\vec{\mathbf{F}}\cdot\vec{\mathbf{T}}$ (remember that if $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are vectors and $\vec{\mathbf{b}}$ is fixed, then the dot product $\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}$ is maximal when $\vec{\mathbf{a}}$ points in the same direction as $\vec{\mathbf{b}}$). Therefore, if $\vec{\mathbf{F}}$ is the velocity field of a fluid, then $curl\,\vec{\mathbf{F}}\cdot\vec{\mathbf{N}}$ is a measure of how the fluid rotates about axis $\vec{\mathbf{N}}$. The effect of the curl is largest about the axis that points in the direction of $\vec{\mathbf{N}}$, because in this case $curl\,\vec{\mathbf{F}}\cdot\vec{\mathbf{N}}$ is as large as possible.

To see this effect in a more concrete fashion, imagine placing a tiny paddlewheel at point P_0 (Figure 16.7.8). The paddlewheel achieves its maximum speed when the axis of the wheel points in the direction of curl $\vec{\mathbf{F}}$. This justifies the interpretation of the curl we have learned: curl is a measure of the rotation in the vector field about the axis that points in the direction of the normal vector $\vec{\mathbf{N}}$, and Stokes' theorem justifies this interpretation.

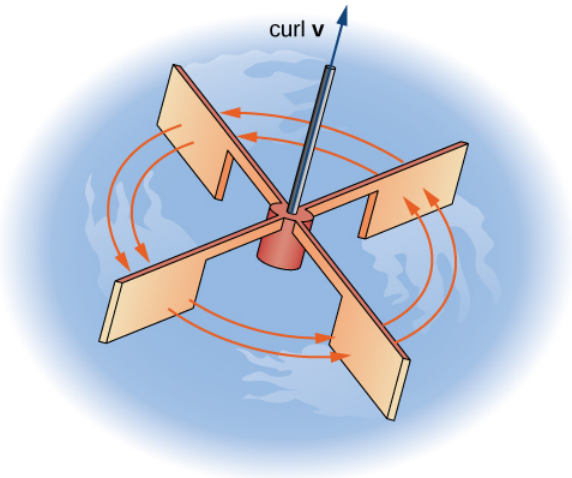


Figure 16.7.8: To visualize curl at a point, imagine placing a tiny paddlewheel at that point in the vector field.

Now that we have learned about Stokes' theorem, we can discuss applications in the area of electromagnetism. In particular, we examine how we can use Stokes' theorem to translate between two equivalent forms of Faraday's law. Before stating the two forms of Faraday's law, we need some background terminology.

Let C be a closed curve that models a thin wire. In the context of electric fields, the wire may be moving over time, so we write $C(t)$ to represent the wire. At a given time t, curve $C(t)$ may be different from original curve C because of the movement of the wire, but we assume that $C(t)$ is a closed curve for all times t. Let $D(t)$ be a surface with $C(t)$ as its boundary, and orient $C(t)$ so that $D(t)$ has positive orientation. Suppose that $C(t)$is in a magnetic field $\vec{\mathbf{B}}(t)$ that can also change over time. In other words, $\vec{\mathbf{B}}$ has the form

$$\vec{\mathbf{B}}(x,y,z) = \langle P(x,y,z),\, Q(x,y,z),\, R(x,y,z)\rangle,$$

where P, Q, and R can all vary continuously over time. We can produce current along the wire by changing field $\vec{\mathbf{B}}(t)$ (this is a consequence of Ampere's law). Flux $\phi(t) = \iint_{D(t)} \vec{\mathbf{B}}(t)\cdot d\vec{\mathbf{S}}$ creates electric field $\vec{\mathbf{E}}(t)$ that does work. The integral form of Faraday's law states that

$$Work = \int_{C(t)} \vec{\mathbf{E}}(t)\cdot d\vec{\mathbf{r}} = -\frac{\partial\phi}{\partial t}.$$

In other words, the work done by $\vec{\mathbf{E}}$ is the line integral around the boundary, which is also equal to the rate of change of the flux with respect to time. The differential form of Faraday's law states that

$$curl\,\vec{\mathbf{E}} = -\frac{\partial\vec{\mathbf{B}}}{\partial t}.$$

Using Stokes' theorem, we can show that the differential form of Faraday's law is a consequence of the integral form. By Stokes' theorem, we can convert the line integral in the integral form into surface integral

$$-\frac{\partial\phi}{\partial t} = \int_{C(t)} \vec{\mathbf{E}}(t)\cdot d\vec{\mathbf{r}} = \iint_{D(t)} curl\,\vec{\mathbf{E}}(t)\cdot d\vec{\mathbf{S}}.$$

Since

$$\phi(t) = \iint_{D(t)} B(t)\cdot d\vec{\mathbf{S}},$$

then as long as the integration of the surface does not vary with time we also have

$$-\frac{\partial\phi}{\partial t} = \iint_{D(t)} -\frac{\partial\vec{\mathbf{B}}}{\partial t}\cdot d\vec{\mathbf{S}}.$$

Therefore,

$$\iint \quad \frac{\partial\vec{\mathbf{B}}}{} \quad \vec{} \quad \iint \quad \vec{} \quad \vec{}$$

$$\iint_{D(t)} -\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = \iint_{D(t)} curl\, \mathbf{E} \cdot d\mathbf{S}.$$

To derive the differential form of Faraday's law, we would like to conclude that $curl\, \vec{\mathbf{E}} = -\dfrac{\partial \vec{\mathbf{B}}}{\partial t}$: In general, the equation

$$\iint_{D(t)} -\frac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{S}} = \iint_{D(t)} curl\, \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}}$$

is not enough to conclude that $curl\, \vec{\mathbf{E}} = -\dfrac{\partial \vec{\mathbf{B}}}{\partial t}$: The integral symbols do not simply "cancel out," leaving equality of the integrands. To see why the integral symbol does not just cancel out in general, consider the two single-variable integrals $\int_0^1 x\, dx$ and $\int_0^1 f(x)\, dx$, where

$$f(x) = \begin{cases} 1, & \text{if } 0 \le x \le 1/2 \\ 0, & \text{if } 1/2 \le x \le 1. \end{cases}$$

Both of these integrals equal $\dfrac{1}{2}$, so $\int_0^1 x\, dx = \int_0^1 f(x)\, dx$.

However, $x \neq f(x)$. Analogously, with our equation

$$\iint_{D(t)} -\frac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{S}} = \iint_{D(t)} curl\, \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}},$$

we cannot simply conclude that $curl\, \vec{\mathbf{E}} = -\dfrac{\partial \vec{\mathbf{B}}}{\partial t}$ just because their integrals are equal. However, in our context, equation

$$\iint_{D(t)} -\frac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{S}} = \iint_{D(t)} curl\, \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}}$$

is true for *any* region, however small (this is in contrast to the single-variable integrals just discussed). If $\vec{\mathbf{F}}$ and $\vec{\mathbf{G}}$ are three-dimensional vector fields such that

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_S \vec{\mathbf{G}} \cdot d\vec{\mathbf{S}}$$

for any surface S, then it is possible to show that $\vec{\mathbf{F}} = \vec{\mathbf{G}}$ by shrinking the area of S to zero by taking a limit (the smaller the area of S, the closer the value of $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$ to the value of $\vec{\mathbf{F}}$ at a point inside S). Therefore, we can let area $D(t)$ shrink to zero by taking a limit and obtain the differential form of Faraday's law:

$$curl\, \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}.$$

In the context of electric fields, the curl of the electric field can be interpreted as the negative of the rate of change of the corresponding magnetic field with respect to time.

Example 16.7.4: Using Faraday's Law

Calculate the curl of electric field $\vec{\mathbf{E}}$ if the corresponding magnetic field is constant field $\vec{\mathbf{B}}(t) = \langle 1, -4, 2 \rangle$.

Solution

Since the magnetic field does not change with respect to time, $-\dfrac{\partial \vec{\mathbf{B}}}{\partial t} = \vec{\mathbf{0}}$. By Faraday's law, the curl of the electric field is therefore also zero.

Analysis

A consequence of Faraday's law is that the curl of the electric field corresponding to a constant magnetic field is always zero.

Exercise 16.7.4

Calculate the curl of electric field $\vec{\mathbf{E}}$ if the corresponding magnetic field is $\vec{\mathbf{B}}(t) = \langle tx,\, ty,\, -2tz\rangle,\, 0 \le t < \infty$.

Hint

- Use the differential form of Faraday's law.
- Notice that the curl of the electric field does not change over time, although the magnetic field does change over time.

Answer

$curl\, \vec{\mathbf{E}} = \langle x,\, y,\, -2z\rangle$

Key Concepts

- Stokes' theorem relates a flux integral over a surface to a line integral around the boundary of the surface. Stokes' theorem is a higher dimensional version of Green's theorem, and therefore is another version of the Fundamental Theorem of Calculus in higher dimensions.
- Stokes' theorem can be used to transform a difficult surface integral into an easier line integral, or a difficult line integral into an easier surface integral.
- Through Stokes' theorem, line integrals can be evaluated using the simplest surface with boundary C.
- Faraday's law relates the curl of an electric field to the rate of change of the corresponding magnetic field. Stokes' theorem can be used to derive Faraday's law.

Key Equations

- **Stokes' theorem**

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$$

Glossary

Stokes' theorem

relates the flux integral over a surface S to a line integral around the boundary C of the surface S

surface independent

flux integrals of curl vector fields are surface independent if their evaluation does not depend on the surface but only on the boundary of the surface

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16.7E: Exercises for Section 16.7

In exercises 1 - 6, without using Stokes' theorem, calculate directly both the flux of $curl\,\vec{\mathbf{F}}\cdot\vec{\mathbf{N}}$ over the given surface and the circulation integral around its boundary, assuming all are oriented clockwise.

1. $\vec{\mathbf{F}}(x,y,z)=y^2\,\hat{\mathbf{i}}+z^2\,\hat{\mathbf{j}}+x^2\,\hat{\mathbf{k}}$; S is the first-octant portion of plane $x+y+z=1$.

2. $\vec{\mathbf{F}}(x,y,z)=z\,\hat{\mathbf{i}}+x\,\hat{\mathbf{j}}+y\,\hat{\mathbf{k}}$; S is hemisphere $z=(a^2-x^2-y^2)^{1/2}$.

Answer

$$\iint_S (curl\,\vec{\mathbf{F}}\cdot\vec{\mathbf{N}})\,dS=\pi a^2$$

3. $\vec{\mathbf{F}}(x,y,z)=y^2\,\hat{\mathbf{i}}+2x\,\hat{\mathbf{j}}+5\,\hat{\mathbf{k}}$; S is hemisphere $z=(4-x^2-y^2)^{1/2}$.

4. $\vec{\mathbf{F}}(x,y,z)=z\,\hat{\mathbf{i}}+2x\,\hat{\mathbf{j}}+3y\,\hat{\mathbf{k}}$; S is upper hemisphere $z=\sqrt{9-x^2-y^2}$.

Answer

$$\iint_S (curl\,\vec{\mathbf{F}}\cdot\vec{\mathbf{N}})\,dS=18\pi$$

5. $\vec{\mathbf{F}}(x,y,z)=(x+2z)\,\hat{\mathbf{i}}+(y-x)\,\hat{\mathbf{j}}+(z-y)\,\hat{\mathbf{k}}$; S is a triangular region with vertices $(3,0,0)$, $(0,3/2,0)$, and $(0,0,3)$.

6. $\vec{\mathbf{F}}(x,y,z)=2y\,\hat{\mathbf{i}}+6z\,\hat{\mathbf{j}}+3x\,\hat{\mathbf{k}}$; S is a portion of paraboloid $z=4-x^2-y^2$ and is above the xy-plane.

Answer

$$\iint_S (curl\,\vec{\mathbf{F}}\cdot\vec{\mathbf{N}})\,dS=-8\pi$$

In exercises 7 - 9, use Stokes' theorem to evaluate $\iint_S (curl\,\vec{\mathbf{F}}\cdot\vec{\mathbf{N}})\,dS$ for the vector fields and surface.

7. $\vec{\mathbf{F}}(x,y,z)=xy\,\hat{\mathbf{i}}-z\,\hat{\mathbf{j}}$ and S is the surface of the cube $0\le x\le 1,\ 0\le y\le 1,\ 0\le z\le 1$, except for the face where $z=0$ and using the outward unit normal vector.

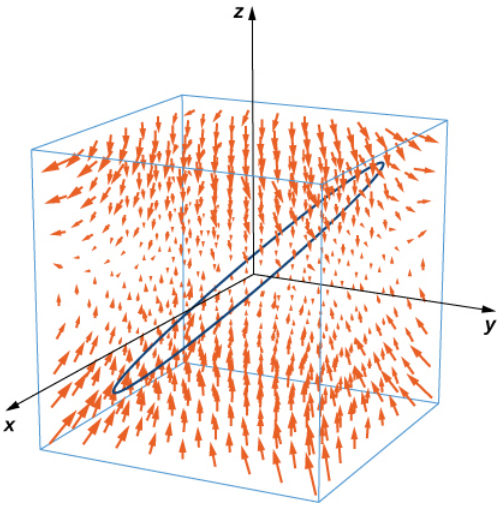
8. $\vec{\mathbf{F}}(x,y,z)=xy\,\hat{\mathbf{i}}+x^2\,\hat{\mathbf{j}}+z^2\,\hat{\mathbf{k}}$; and C is the intersection of paraboloid $z=x^2+y^2$ and plane $z=y$, and using the outward normal vector.

Answer

$$\iint_S (curl\,\vec{\mathbf{F}}\cdot\vec{\mathbf{N}})\,dS=0$$

9. $\vec{\mathbf{F}}(x,y,z)=4y\,\hat{\mathbf{i}}+z\,\hat{\mathbf{j}}+2y\,\hat{\mathbf{k}}$; and C is the intersection of sphere $x^2+y^2+z^2=4$ with plane $z=0$, and using the outward normal vector.

10. Use Stokes' theorem to evaluate $\int_C \left[2xy^2z\,dx+2x^2yz\,dy+(x^2y^2-2z)\,dz\right]$, where C is the curve given by $x=\cos t,\ y=\sin t,\ 0\le t\le 2\pi$, traversed in the direction of increasing t.



Answer

$$\int_C \vec{\mathbf{F}} \cdot dS = 0$$

11. [T] Use a computer algebraic system (CAS) and Stokes' theorem to approximate line integral $\int_C (y\,dx + z\,dy + x\,dz)$, where C is the intersection of plane $x + y = 2$ and surface $x^2 + y^2 + z^2 = 2(x + y)$, traversed counterclockwise viewed from the origin.

12. [T] Use a CAS and Stokes' theorem to approximate line integral $\int_C (3y\,dx + 2z\,dy - 5x\,dz)$, where C is the intersection of the xy-plane and hemisphere $z = \sqrt{1 - x^2 - y^2}$, traversed counterclockwise viewed from the top—that is, from the positive z-axis toward the xy-plane.

Answer

$$\int_C \vec{\mathbf{F}} \cdot dS = -9.4248$$

13. [T] Use a CAS and Stokes' theorem to approximate line integral $\int_C [(1 + y)\,z\,dx + (1 + z)x\,dy + (1 + x)y\,dz]$, where C is a triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ oriented counterclockwise.

14. Use Stokes' theorem to evaluate $\iint_S curl\ \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x, y, z) = e^{xy} cos\ z\ \hat{\mathbf{i}} + x^2 z\ \hat{\mathbf{j}} + xy\ \hat{\mathbf{k}}$, and S is half of sphere $x = \sqrt{1 - y^2 - z^2}$, oriented out toward the positive x-axis.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot dS = 0$$

15. [T] Use a CAS and Stokes' theorem to evaluate $\iint_S (curl\,\vec{\mathbf{F}} \cdot \vec{\mathbf{N}})\,dS$, where $\vec{\mathbf{F}}(x, y, z) = x^2 y\,\hat{\mathbf{i}} + xy^2\,\hat{\mathbf{j}} + z^3\,\hat{\mathbf{k}}$ and C is the curve of the intersection of plane $3x + 2y + z = 6$ and cylinder $x^2 + y^2 = 4$, oriented clockwise when viewed from above.

16. [T] Use a CAS and Stokes' theorem to evaluate $\iint_S curl\,\vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x, y, z) = \left(\sin(y + z) - yx^2 - \frac{y^3}{3}\right)\hat{\mathbf{i}} + x\ \cos(y\ + z)\,\hat{\mathbf{j}} + \cos(2y)\,\hat{\mathbf{k}}$ and S consists of the top and the four sides but not the bottom of the cube with vertices $(\pm 1, \pm 1, \pm 1)$, oriented outward.

Answer

$$\iint_S curl\,\vec{\mathbf{F}} \cdot dS = 2.6667$$

17. [T] Use a CAS and Stokes' theorem to evaluate $\iint_S curl\,\vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x, y, z) = z^2\,\hat{\mathbf{i}} + 3xy\hat{\mathbf{j}} + x^3 y^3\,\hat{\mathbf{k}}$ and S is the top part of $z = 5 - x^2 - y^2$ above plane $z = 1$ and S is oriented upward.

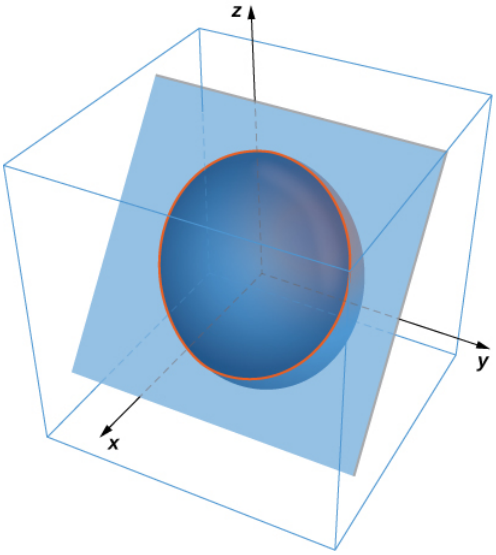
18. Use Stokes' theorem to evaluate $\iint_S (curl\,\vec{\mathbf{F}} \cdot \vec{\mathbf{N}})dS$, where $\vec{\mathbf{F}}(x,y,z) = z^2\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}} + x\,\hat{\mathbf{k}}$ and S is a triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ with counterclockwise orientation.

Answer

$$\iint_S (curl\,\vec{\mathbf{F}} \cdot \vec{\mathbf{N}})dS = -\frac{1}{6}$$

19. Use Stokes' theorem to evaluate line integral $\int_C (z\,dx + x\,dy + y\,dz)$, where C is a triangle with vertices $(3,0,0)$, $(0,0,2)$, and $(0,6,0)$ traversed in the given order.

20. Use Stokes' theorem to evaluate $\int_C \left(\frac{1}{2}y^2\,dx + z\,dy + x\,dz\right)$, where C is the curve of intersection of plane $x + z = 1$ and ellipsoid $x^2 + 2y^2 + z^2 = 1$, oriented clockwise from the origin.



Answer

$$\int_C \left(\frac{1}{2}y^2\,dx + z\,dy + x\,dz\right) = -\frac{\pi}{4}$$

21. Use Stokes' theorem to evaluate $\iint_S (curl\,\vec{\mathbf{F}} \cdot \vec{\mathbf{N}})dS$, where $\vec{\mathbf{F}}(x,y,z) = x\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}} + ze^{xy}\,\hat{\mathbf{k}}$ and S is the part of surface $z = 1 - x^2 - 2y^2$ with $z \geq 0$, oriented counterclockwise.

22. Use Stokes' theorem for vector field $\vec{\mathbf{F}}(x,y,z) = z\,\hat{\mathbf{i}} + 3x\,\hat{\mathbf{j}} + 2z\,\hat{\mathbf{k}}$ where S is surface $z = 1 - x^2 - 2y^2$, $z \geq 0$, C is boundary circle $x^2 + y^2 = 1$, and S is oriented in the positive z-direction.

Answer

$$\iint_S (curl\,\vec{\mathbf{F}} \cdot \vec{\mathbf{N}})dS = -3\pi$$

23. Use Stokes' theorem for vector field $\vec{\mathbf{F}}(x,y,z) = -\frac{3}{2}y^2\,\hat{\mathbf{i}} - 2xy\hat{\mathbf{j}} + yz\,\hat{\mathbf{k}}$, where S is that part of the surface of plane $x + y + z = 1$ contained within triangle C with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, traversed counterclockwise as viewed from above.

24. A certain closed path C in plane $2x + 2y + z = 1$ is known to project onto unit circle $x^2 + y^2 = 1$ in the xy-plane. Let C be a constant and let $\vec{\mathbf{R}}(x,y,z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$. Use Stokes' theorem to evaluate $\int_C (c\,\hat{\mathbf{k}} \times \vec{\mathbf{R}}) \cdot dS$.

Answer

$$\int_C (c\,\hat{\mathbf{k}} \times \vec{\mathbf{R}}) \cdot dS = 2\pi c$$

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25. Use Stokes' theorem and let C be the boundary of surface $z = x^2 + y^2$ with $0 \le x \le 2$ and $0 \le y \le 1$ oriented with upward facing normal. Define $\vec{\mathbf{F}}(x, y, z) = \left(\sin(x^3) + xz\right)\hat{\mathbf{i}} + (x - yz)\hat{\mathbf{j}} + \cos(z^4)\hat{\mathbf{k}}$ and evaluate $\int_C \vec{\mathbf{F}} \cdot dS$.

26. Let S be hemisphere $x^2 + y^2 + z^2 = 4$ with $z \ge 0$, oriented upward. Let $\vec{\mathbf{F}}(x, y, z) = x^2 e^{yz}\,\hat{\mathbf{i}} + y^2 e^{xz}\,\hat{\mathbf{j}} + z^2 e^{xy}\,\hat{\mathbf{k}}$ be a vector field. Use Stokes' theorem to evaluate $\iint_S curl\, \vec{\mathbf{F}} \cdot dS$.

Answer

$\iint_S curl\, \vec{\mathbf{F}} \cdot dS = 0$

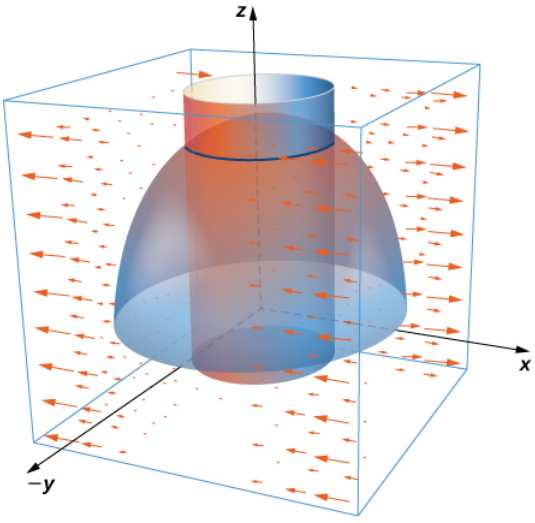
27. Let $\vec{\mathbf{F}}(x, y, z) = xy\,\hat{\mathbf{i}} + (e^{z^2} + y)\,\hat{\mathbf{j}} + (x + y)\,\hat{\mathbf{k}}$ and let S be the graph of function $y = \frac{x^2}{9} + \frac{z^2}{9} - 1$ with $z \le 0$ oriented so that the normal vector S has a positive y component. Use Stokes' theorem to compute integral $\iint_S curl\, \vec{\mathbf{F}} \cdot dS$.

28. Use Stokes' theorem to evaluate $\oint \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x, y, z) = y\,\hat{\mathbf{i}} + z\,\hat{\mathbf{j}} + x\,\hat{\mathbf{k}}$ and C is a triangle with vertices $(0, 0, 0)$, $(2, 0, 0)$ and $0, -2, 2)$ oriented counterclockwise when viewed from above.

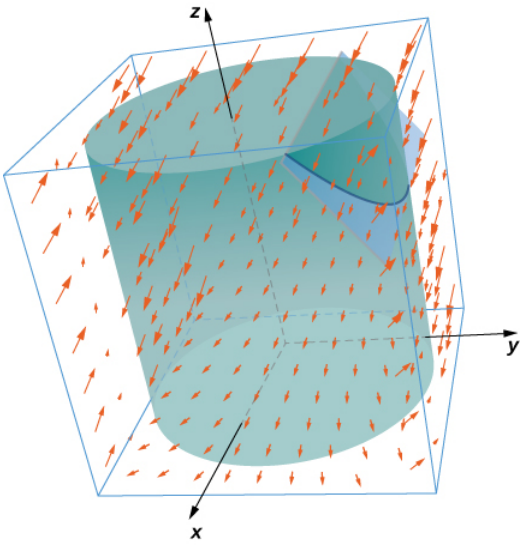
Answer

$\oint \vec{\mathbf{F}} \cdot dS = -4$

29. Use the surface integral in Stokes' theorem to calculate the circulation of field $\vec{\mathbf{F}}$, $\vec{\mathbf{F}}(x, y, z) = x^2 y^3\,\hat{\mathbf{i}} + \hat{\mathbf{j}} + z\hat{\mathbf{k}}$ around C, which is the intersection of cylinder $x^2 + y^2 = 4$ and hemisphere $x^2 + y^2 + z^2 = 16$, $z \ge 0$, oriented counterclockwise when viewed from above.



30. Use Stokes' theorem to compute $\iint_S curl\, \vec{\mathbf{F}} \cdot dS$. where $\vec{\mathbf{F}}(x, y, z) = \hat{\mathbf{i}} + xy^2\,\hat{\mathbf{j}} + xy^2\,\hat{\mathbf{k}}$ and S is a part of plane $y + z = 2$ inside cylinder $x^2 + y^2 = 1$ and oriented counterclockwise.



Answer

$$\iint_S curl\ \vec{\mathbf{F}} \cdot dS = 0$$

31. Use Stokes' theorem to evaluate $\iint_S curl\ \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x, y, z) = -y^2\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}} + z^2\,\hat{\mathbf{k}}$ and S is the part of plane $x + y + z = 1$ in the positive octant and oriented counterclockwise $x \geq 0,\ y \geq 0,\ z \geq 0$.

32. Let $\vec{\mathbf{F}}(x, y, z) = xy\,\hat{\mathbf{i}} + 2z\,\hat{\mathbf{j}} - 2y\,\hat{\mathbf{k}}$ and let C be the intersection of plane $x + z = 5$ and cylinder $x^2 + y^2 = 9$, which is oriented counterclockwise when viewed from the top. Compute the line integral of $\vec{\mathbf{F}}$ over C using Stokes' theorem.

Answer

$$\iint_S curl\ \vec{\mathbf{F}} \cdot dS = -36\pi$$

33. [T] Use a CAS and let $\vec{\mathbf{F}}(x, y, z) = xy^2\,\hat{\mathbf{i}} + (yz - x)\,\hat{\mathbf{j}} + e^{yxz}\,\hat{\mathbf{k}}$. Use Stokes' theorem to compute the surface integral of curl $\vec{\mathbf{F}}$ over surface S with inward orientation consisting of cube $[0, 1] \times [0, 1] \times [0, 1]$ with the right side missing.

34. Let S be ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$ oriented counterclockwise and let $\vec{\mathbf{F}}$ be a vector field with component functions that have continuous partial derivatives.

Answer

$$\iint_S curl\ \vec{\mathbf{F}} \cdot \vec{\mathbf{N}} = 0$$

35. Let S be the part of paraboloid $z = 9 - x^2 - y^2$ with $z \geq 0$. Verify Stokes' theorem for vector field $\vec{\mathbf{F}}(x, y, z) = 3z\,\hat{\mathbf{i}} + 4x\,\hat{\mathbf{j}} + 2y\,\hat{\mathbf{k}}$.

36. [T] Use a CAS and Stokes' theorem to evaluate $\oint \vec{\mathbf{F}} \cdot dS$, if $\vec{\mathbf{F}}(x, y, z) = (3z - \sin x)\,\hat{\mathbf{i}} + (x^2 + e^y)\,\hat{\mathbf{j}} + (y^3 - \cos z)\,\hat{\mathbf{k}}$, where C is the curve given by $x = \cos t,\ y = \sin t,\ z = 1;\ 0 \leq t \leq 2\pi$.

Answer

$$\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0$$

37. [T] Use a CAS and Stokes' theorem to evaluate $\vec{\mathbf{F}}(x, y, z) = 2y\,\hat{\mathbf{i}} + e^z\,\hat{\mathbf{j}} - \arctan x\,\hat{\mathbf{k}}$ with S as a portion of paraboloid $z = 4 - x^2 - y^2$ cut off by the xy-plane oriented counterclockwise.

38. [T] Use a CAS to evaluate $\iint_S curl(F) \cdot dS$, where $\vec{\mathbf{F}}(x, y, z) = 2z\,\hat{\mathbf{i}} + 3x\,\hat{\mathbf{j}} + 5y\,\hat{\mathbf{k}}$ and S is the surface parametrically by $\vec{\mathbf{r}}(r, \theta) = r\cos\theta\,\hat{\mathbf{i}} + r\sin\theta\,\hat{\mathbf{j}} + (4 - r^2)\,\hat{\mathbf{k}}\,(0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 3)$.

Answer

$$\iint_S curl(F) \cdot dS = 84.8230$$

39. Let S be paraboloid $z = a(1 - x^2 - y^2)$, for $z \geq 0$, where $a > 0$ is a real number. Let $\vec{\mathbf{F}}(x, y, z) = \langle x - y,\ y + z,\ z - x \rangle$. For what value(s) of a (if any) does $\iint_S (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \vec{\mathbf{n}}\, dS$ have its maximum value?

For application exercises 40 - 41, the goal is to evaluate $A = \iint_S (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \vec{\mathbf{n}}\, dS$, where $\vec{\mathbf{F}} = \langle xz,\ -xz,\ xy \rangle$ and S is the upper half of ellipsoid $x^2 + y^2 + 8z^2 = 1$, where $z \geq 0$.

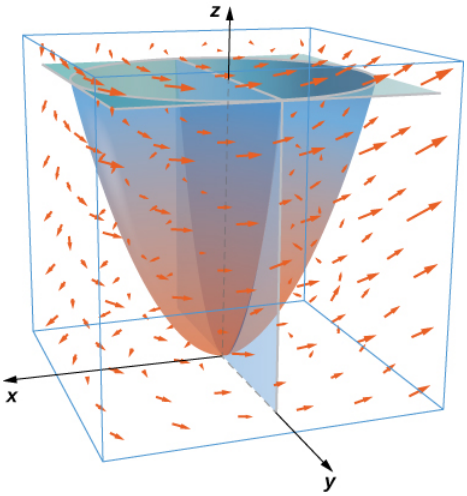
40. Evaluate a surface integral over a more convenient surface to find the value of A.

Answer

$$A = \iint_S (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \vec{\mathbf{n}}\, dS = 0$$

41. Evaluate A using a line integral.

42. Take paraboloid $z = x^2 + y^2$, for $0 \leq z \leq 4$, and slice it with plane $y = 0$. Let S be the surface that remains for $y \geq 0$, including the planar surface in the xz-plane. Let C be the semicircle and line segment that bounded the cap of S in plane $z = 4$ with counterclockwise orientation. Let $\vec{\mathbf{F}} = \langle 2z + y,\ 2x + z,\ 2y + x \rangle$. Evaluate $\iint_S (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \vec{\mathbf{n}}\, dS$.



Answer

$$\iint_S (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \vec{\mathbf{n}}\, dS = 2\pi$$

For exercises 43 - 45, let S be the disk enclosed by curve C : $\vec{\mathbf{r}}(t) = \langle \cos\varphi\ \cos t,\ \sin t,\ \sin\varphi\ \cos t \rangle$, for $0 \leq t \leq 2\pi$, where $0 \leq \varphi \leq \frac{\pi}{2}$ is a fixed angle.

43. What is the length of C in terms of φ?

44. What is the circulation of C of vector field $\vec{\mathbf{F}} = \langle -y,\ -z,\ x \rangle$ as a function of φ?

Answer

$$C = \pi(\cos\varphi - \sin\varphi)$$

45. For what value of φ is the circulation a maximum?

46. Circle C in plane $x + y + z = 8$ has radius 4 and center $(2, 3, 3)$. Evaluate $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ for $\vec{\mathbf{F}} = \langle 0,\ -z,\ 2y \rangle$, where C has a counterclockwise orientation when viewed from above.

Answer

$$\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 48\pi$$

47. Velocity field $v = \langle 0,\ 1 - x^2,\ 0 \rangle$, for $|x| \leq 1$ and $|z| \leq 1$, represents a horizontal flow in the y-direction. Compute the curl of $\vec{\mathbf{v}}$ in a clockwise rotation.

48. Evaluate integral $\iint_S (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \vec{\mathbf{n}}\, dS$, where $\vec{\mathbf{F}} = -xz\hat{\mathbf{i}} + yz\hat{\mathbf{j}} + xye^z\hat{\mathbf{k}}$ and S is the cap of paraboloid $z = 5 - x^2 - y^2$ above plane $z = 3$, and $\vec{\mathbf{n}}$ points in the positive z-direction on S.

Answer

$$\iint_S (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \vec{\mathbf{n}} = 0$$

In exercises 49 - 50, use Stokes' theorem to find the circulation of the following vector fields around any smooth, simple closed curve C.

49. $\vec{\mathbf{F}} = \vec{\nabla}(x\ \sin ye^z)$

50. $\vec{\mathbf{F}} = \langle y^2z^3,\ z2xyz^3, 3xy^2z^2 \rangle$

Answer

0

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16.8: The Divergence Theorem

Learning Objectives

- Explain the meaning of the divergence theorem.
- Use the divergence theorem to calculate the flux of a vector field.
- Apply the divergence theorem to an electrostatic field.

We have examined several versions of the Fundamental Theorem of Calculus in higher dimensions that relate the integral around an oriented boundary of a domain to a "derivative" of that entity on the oriented domain. In this section, we state the divergence theorem, which is the final theorem of this type that we will study. The divergence theorem has many uses in physics; in particular, the divergence theorem is used in the field of partial differential equations to derive equations modeling heat flow and conservation of mass. We use the theorem to calculate flux integrals and apply it to electrostatic fields.

Overview of Theorems

Before examining the divergence theorem, it is helpful to begin with an overview of the versions of the Fundamental Theorem of Calculus we have discussed:

1. The Fundamental Theorem of Calculus:

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

This theorem relates the integral of derivative f' over line segment $[a, b]$ along the x-axis to a difference of f evaluated on the boundary.

2. The Fundamental Theorem for Line Integrals:

$$\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = f(P_1) - f(P_0),$$

where P_0 is the initial point of C and P_1 is the terminal point of C. The Fundamental Theorem for Line Integrals allows path C to be a path in a plane or in space, not just a line segment on the x-axis. If we think of the gradient as a derivative, then this theorem relates an integral of derivative $\vec{\nabla} f$ over path C to a difference of f evaluated on the boundary of C.

3. Green's theorem, circulation form:

$$\iint_D (Q_x - P_y)\,dA = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}.$$

Since $Q_x - P_y = \text{curl } \vec{\mathbf{F}} \cdot \hat{\mathbf{k}}$ and curl is a derivative of sorts, Green's theorem relates the integral of derivative curl $\vec{\mathbf{F}}$ over planar region D to an integral of $\vec{\mathbf{F}}$ over the boundary of D.

4. Green's theorem, flux form:

$$\iint_D (P_x + Q_y)\,dA = \int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\,dS.$$

Since $P_x + Q_y = \text{div } \vec{\mathbf{F}}$ and divergence is a derivative of sorts, the flux form of Green's theorem relates the integral of derivative div $\vec{\mathbf{F}}$ over planar region D to an integral of $\vec{\mathbf{F}}$ over the boundary of D.

5. Stokes' theorem:

$$\iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}.$$

If we think of the curl as a derivative of sorts, then Stokes' theorem relates the integral of derivative curl $\vec{\mathbf{F}}$ over surface S (not necessarily planar) to an integral of $\vec{\mathbf{F}}$ over the boundary of S.

Stating the Divergence Theorem

The divergence theorem follows the general pattern of these other theorems. If we think of divergence as a derivative of sorts, then the divergence theorem relates a triple integral of derivative div $\vec{\mathbf{F}}$ over a solid to a flux integral of $\vec{\mathbf{F}}$ over the boundary of the solid.

More specifically, the divergence theorem relates a flux integral of vector field $\vec{\mathbf{F}}$ over a closed surface S to a triple integral of the divergence of $\vec{\mathbf{F}}$ over the solid enclosed by S.

The Divergence Theorem

Let S be a piecewise, smooth closed surface that encloses solid E in space. Assume that S is oriented outward, and let $\vec{\mathbf{F}}$ be a vector field with continuous partial derivatives on an open region containing E (Figure 16.8.1). Then

$$\iiint_E \operatorname{div} \vec{\mathbf{F}}\, dV = \iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}. \tag{16.8.1}$$

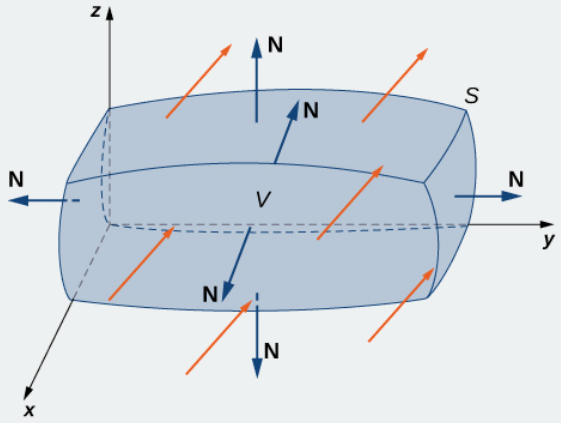


Figure 16.8.1: The divergence theorem relates a flux integral across a closed surface S to a triple integral over solid E enclosed by the surface.

Recall that the flux form of Green's theorem states that

$$\iint_D \operatorname{div} \vec{\mathbf{F}}\, dA = \int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, dS.$$

Therefore, the divergence theorem is a version of Green's theorem in one higher dimension.

The proof of the divergence theorem is beyond the scope of this text. However, we look at an informal proof that gives a general feel for why the theorem is true, but does not prove the theorem with full rigor. This explanation follows the informal explanation given for why Stokes' theorem is true.

Proof

Let B be a small box with sides parallel to the coordinate planes inside E (Figure 16.8.2a). Let the center of B have coordinates (x, y, z) and suppose the edge lengths are Δx, Δy, and Δz. (Figure 16.8.1b). The normal vector out of the top of the box is $\hat{\mathbf{k}}$ and the normal vector out of the bottom of the box is $-\hat{\mathbf{k}}$. The dot product of $\vec{\mathbf{F}} = \langle P, Q, R \rangle$ with $\hat{\mathbf{k}}$ is R and the dot product with $-\hat{\mathbf{k}}$ is $-R$. The area of the top of the box (and the bottom of the box) ΔS is $\Delta x \Delta y$.

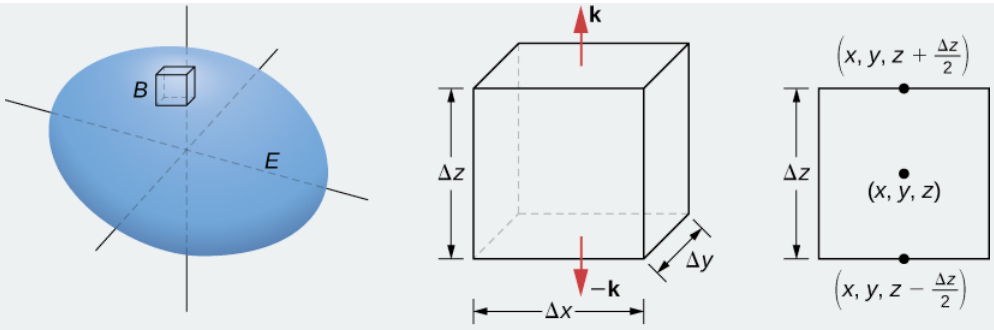


Figure 16.8.2: (a) A small box B inside surface E has sides parallel to the coordinate planes. (b) Box B has side lengthsΔx, Δy, and Δz (c) If we look at the side view of B, we see that, since (x, y, z) is the center of the box, to get to the top of the box we must travel a vertical distance of $\frac{\Delta z}{2}$ up from (x, y, z). Similarly, to get to the bottom of the box we must travel a distance $\frac{\Delta z}{2}$ down from (x, y, z).

The flux out of the top of the box can be approximated by $R\left(x,\, y,\, z+\frac{\Delta z}{2}\right)\,\Delta x\,\Delta y$ (Figure 16.8.2*c*) and the flux out of the bottom of the box is $-R\left(x,\, y,\, z-\frac{\Delta z}{2}\right)\,\Delta x\,\Delta y$. If we denote the difference between these values as ΔR, then the net flux in the vertical direction can be approximated by $\Delta R\,\Delta x\,\Delta y$. However,

$$\Delta R\,\Delta x\,\Delta y = \left(\frac{\Delta R}{\Delta z}\right)\,\Delta x\,\Delta y\Delta z \approx \left(\frac{\partial R}{\partial z}\right)\,\Delta V.$$

Therefore, the net flux in the vertical direction can be approximated by $\left(\frac{\partial R}{\partial z}\right)\Delta V$. Similarly, the net flux in the x-direction can be approximated by $\left(\frac{\partial P}{\partial x}\right)\,\Delta V$ and the net flux in the y-direction can be approximated by $\left(\frac{\partial Q}{\partial y}\right)\,\Delta V$. Adding the fluxes in all three directions gives an approximation of the total flux out of the box:

$$\text{Total flux } \approx \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)\Delta V = \text{div } \vec{\mathbf{F}}\,\Delta V.$$

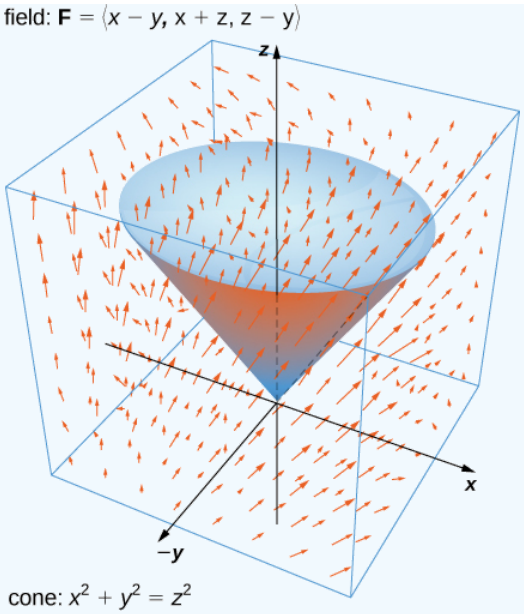
This approximation becomes arbitrarily close to the value of the total flux as the volume of the box shrinks to zero.

The sum of $\text{div } \vec{\mathbf{F}}\,\Delta V$ over all the small boxes approximating E is approximately $\iiint_E \text{div } \vec{\mathbf{F}}\,dV$. On the other hand, the sum of $\text{div } \vec{\mathbf{F}}\,\Delta V$ over all the small boxes approximating E is the sum of the fluxes over all these boxes. Just as in the informal proof of Stokes' theorem, adding these fluxes over all the boxes results in the cancelation of a lot of the terms. If an approximating box shares a face with another approximating box, then the flux over one face is the negative of the flux over the shared face of the adjacent box. These two integrals cancel out. When adding up all the fluxes, the only flux integrals that survive are the integrals over the faces approximating the boundary of E. As the volumes of the approximating boxes shrink to zero, this approximation becomes arbitrarily close to the flux over S.

□

Example 16.8.1: Verifying the Divergence Theorem

Verify the divergence theorem for vector field $\vec{\mathbf{F}} = \langle x-y,\, x+z,\, z-y\rangle$ and surface S that consists of cone $x^2+y^2=z^2$, $0 \le z \le 1$, and the circular top of the cone (see the following figure). Assume this surface is positively oriented.



Solution

Let E be the solid cone enclosed by S. To verify the theorem for this example, we show that

$$\iiint_E \text{div } \vec{\mathbf{F}}\, dV = \iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$$

by calculating each integral separately.

To compute the triple integral, note that $\text{div } \vec{\mathbf{F}} = P_x + Q_y + R_z = 2$, and therefore the triple integral is

$$\begin{aligned}\iiint_E \text{div } \vec{\mathbf{F}}\, dV &= 2 \iiint_E dV \\ &= 2\,(volume\, of\, E).\end{aligned}$$

The volume of a right circular cone is given by $\pi r^2 \frac{h}{3}$. In this case, $h = r = 1$. Therefore,

$$\iiint_E \text{div } \vec{\mathbf{F}}\, dV = 2\,(volume\, of\, E) = \frac{2\pi}{3}.$$

To compute the flux integral, first note that S is piecewise smooth; S can be written as a union of smooth surfaces. Therefore, we break the flux integral into two pieces: one flux integral across the circular top of the cone and one flux integral across the remaining portion of the cone. Call the circular top S_1 and the portion under the top S_2. We start by calculating the flux across the circular top of the cone. Notice that S_1 has parameterization

$$\vec{\mathbf{r}}(u, v) = \langle u\, \cos v,\, u\, \sin v,\, 1\rangle,\, 0 \le u \le 1,\, 0 \le v \le 2\pi.$$

Then, the tangent vectors are $\vec{\mathbf{t}}_u = \langle \cos v,\, \sin v,\, 0\rangle$ and $\vec{\mathbf{t}}_v = \langle -u\, \sin v,\, u\, \cos v, 0\rangle$. Therefore, the flux across S_1 is

$$\begin{aligned}\iint_{S_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \int_0^1 \int_0^{2\pi} \vec{\mathbf{F}}(\vec{\mathbf{r}}(u, v)) \cdot (\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v)\, dA \\ &= \int_0^1 \int_0^{2\pi} \langle u\, \cos v - u\, \sin v,\, u\, \cos v + 1,\, 1 - u\, \sin v\rangle \cdot \langle 0, 0, u\rangle\, dv\, du \\ &= \int_0^1 \int_0^{2\pi} u - u^2 \sin v\, dv du \\ &= \pi.\end{aligned}$$

We now calculate the flux over S_2. A parameterization of this surface is

$$\vec{\mathbf{r}}(u, v) = \langle u\, \cos v,\, u\, \sin v,\, u\rangle,\, 0 \le u \le 1,\, 0 \le v \le 2\pi.$$

The tangent vectors are $\vec{\mathbf{t}}_u = \langle \cos v,\ \sin v,\ 1\rangle$ and $\vec{\mathbf{t}}_v = \langle -u\ \sin v,\ u\ \cos v, 0\rangle$, so the cross product is

$$\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v = \langle -u\ \cos v,\ -u\ \sin v,\ u\rangle.$$

Notice that the negative signs on the x and y components induce the negative (or inward) orientation of the cone. Since the surface is positively oriented, we use vector $\vec{\mathbf{t}}_v \times \vec{\mathbf{t}}_u = \langle u\ \cos v,\ u\ \sin v,\ -u\rangle$ in the flux integral. The flux across S_2 is then

$$\begin{aligned}\iint_{S_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \int_0^1 \int_0^{2\pi} \vec{\mathbf{F}}(\vec{\mathbf{r}}(u,v)) \cdot (\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v)\, dA \\ &= \int_0^1 \int_0^{2\pi} \langle u\ \cos v - u\ \sin v,\ u\ \cos v + u,\ u\ - u \sin v\rangle \cdot \langle u\ \cos v,\ u\ \sin v,\ -u\rangle\, dv\, du \\ &= \int_0^1 \int_0^{2\pi} u^2 \cos^2 v + 2u^2 \sin v - u^2\, dv\, du \\ &= -\frac{\pi}{3}\end{aligned}$$

The total flux across S is

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_{S_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} + \iint_{S_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \frac{2\pi}{3} = \iiint_E \text{div}\ \vec{\mathbf{F}}\, dV,$$

and we have verified the divergence theorem for this example.

? Exercise 16.8.1

Verify the divergence theorem for vector field $\vec{\mathbf{F}}(x,y,z) = \langle x + y + z,\ y,\ 2x - y\rangle$ and surface S given by the cylinder $x^2 + y^2 = 1$, $0 \le z \le 3$ plus the circular top and bottom of the cylinder. Assume that S is positively oriented.

Hint

Calculate both the flux integral and the triple integral with the divergence theorem and verify they are equal.

Answer

Both integrals equal 6π.

Recall that the divergence of continuous field $\vec{\mathbf{F}}$ at point P is a measure of the "outflowing-ness" of the field at P. If $\vec{\mathbf{F}}$ represents the velocity field of a fluid, then the divergence can be thought of as the rate per unit volume of the fluid flowing out less the rate per unit volume flowing in. The divergence theorem confirms this interpretation. To see this, let P be a point and let B_τ be a ball of small radius r centered at P (Figure 16.8.3). Let S_τ be the boundary sphere of B_τ. Since the radius is small and $\vec{\mathbf{F}}$ is continuous, $\text{div}\ \vec{\mathbf{F}}(Q) \approx \text{div}\ \vec{\mathbf{F}}(P)$ for all other points Q in the ball. Therefore, the flux across S_τ can be approximated using the divergence theorem:

$$\iint_{S_\tau} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iiint_{B_\tau} \text{div}\ \vec{\mathbf{F}}\, dV \approx \iiint_{B_\tau} \text{div}\ \vec{\mathbf{F}}(P)\, dV.$$

Since $divF(P)$ is a constant,

$$\iiint_{B_\tau} \text{div}\ \vec{\mathbf{F}}(P)\, dV = \text{div}\ \vec{\mathbf{F}}(P)\, V(B_\tau).$$

Therefore, flux

$$\iint_{S_\tau} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$$

can be approximated by $\vec{\mathbf{F}}(P)\, V(B_\tau)$. This approximation gets better as the radius shrinks to zero, and therefore

$$\text{div}\ \vec{\mathbf{F}}(P) = \lim_{\tau \to 0} \frac{1}{V(B_\tau)} \iint_{S_\tau} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}.$$

This equation says that the divergence at P is the net rate of outward flux of the fluid per unit volume.

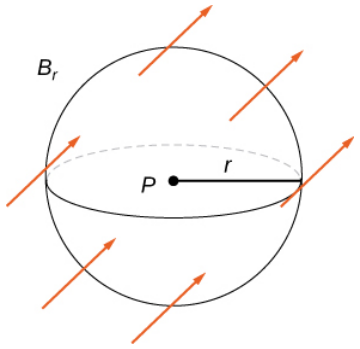


Figure 16.8.3: Ball B_r of small radius r centered at P.

Using the Divergence Theorem

The divergence theorem translates between the flux integral of closed surface S and a triple integral over the solid enclosed by S. Therefore, the theorem allows us to compute flux integrals or triple integrals that would ordinarily be difficult to compute by translating the flux integral into a triple integral and vice versa.

Example 16.8.2: Applying the Divergence Theorem

Calculate the surface integral

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}},$$

where S is cylinder $x^2 + y^2 = 1$, $0 \le z \le 2$, including the circular top and bottom, and $\vec{\mathbf{F}} = \left\langle \frac{x^3}{3} + yz,\ \frac{y^3}{3} - \sin(xz),\ z - x - y \right\rangle$.

Solution

We could calculate this integral without the divergence theorem, but the calculation is not straightforward because we would have to break the flux integral into three separate integrals: one for the top of the cylinder, one for the bottom, and one for the side. Furthermore, each integral would require parameterizing the corresponding surface, calculating tangent vectors and their cross product..

By contrast, the divergence theorem allows us to calculate the single triple integral

$$\iiint_E \operatorname{div} \vec{\mathbf{F}}\, dV,$$

where E is the solid enclosed by the cylinder. Using the divergence theorem (Equation 16.8.1) and converting to cylindrical coordinates, we have

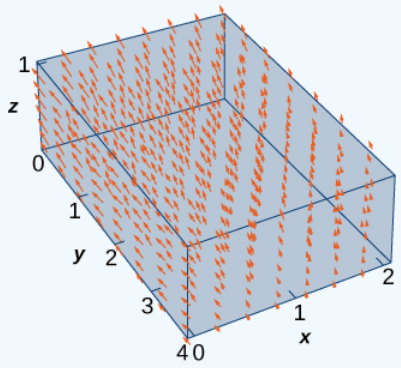
$$\begin{aligned}
\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \iiint_E \operatorname{div} \vec{\mathbf{F}}\, dV, \\
&= \iiint_E (x^2 + y^2 + 1)\, dV \\
&= \int_0^{2\pi} \int_0^1 \int_0^2 (r^2 + 1)\, r\, dz\, dr\, d\theta \\
&= \frac{3}{2} \int_0^{2\pi} d\theta \\
&= 3\pi.
\end{aligned}$$

Exercise 16.8.2

Use the divergence theorem to calculate flux integral

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}},$$

where S is the boundary of the box given by $0 \le x \le 2,\ 0 \le y \le 4,\ 0 \le z \le 1$ and $\vec{\mathbf{F}} = \langle x^2 + yz,\ y - z,\ 2x + 2y + 2z \rangle$ (see the following figure).



Hint

Calculate the corresponding triple integral.

Answer

40

Example 16.8.3: Applying the Divergence Theorem

Let $\vec{\mathrm{v}} = \langle -\frac{y}{z},\ \frac{x}{z},\ 0 \rangle$ be the velocity field of a fluid. Let C be the solid cube given by $1 \le x \le 4,\ 2 \le y \le 5,\ 1 \le z \le 4$, and let S be the boundary of this cube (see the following figure). Find the flow rate of the fluid across S.

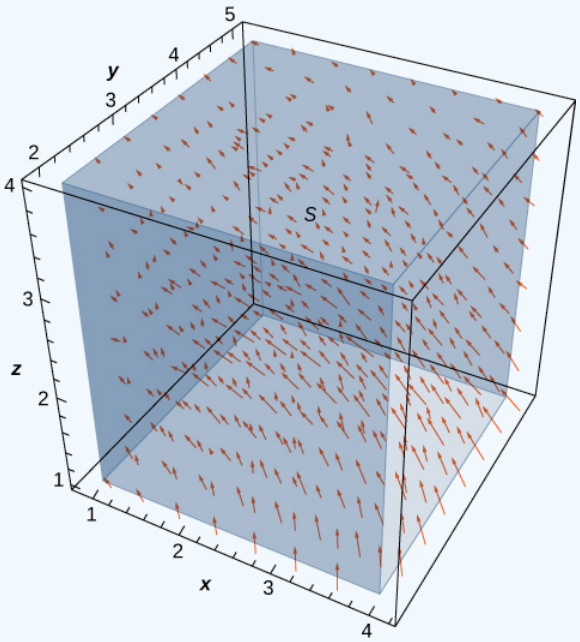


Figure 16.8.4: Vector field $\vec{\mathrm{v}} = \langle -\frac{y}{z},\ \frac{x}{z},\ 0 \rangle$.

Solution

The flow rate of the fluid across S is $\iint_S \vec{\mathrm{v}} \cdot d\vec{\mathbf{S}}$. Before calculating this flux integral, let's discuss what the value of the integral should be. Based on Figure 16.8.4, we see that if we place this cube in the fluid (as long as the cube doesn't encompass the origin), then the rate of fluid entering the cube is the same as the rate of fluid exiting the cube. The field is rotational in nature and, for a given circle parallel to the xy-plane that has a center on the z-axis, the vectors along that circle are all the same magnitude. That is how we can see that the flow rate is the same entering and exiting the cube. The flow into the cube cancels with the flow out of the cube, and therefore the flow rate of the fluid across the cube should be zero.

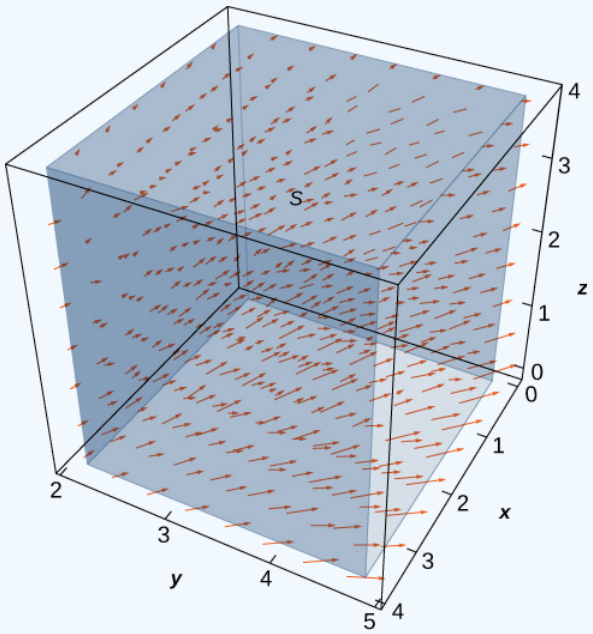
To verify this intuition, we need to calculate the flux integral. Calculating the flux integral directly requires breaking the flux integral into six separate flux integrals, one for each face of the cube. We also need to find tangent vectors, compute their cross product. However, using the divergence theorem makes this calculation go much more quickly:

$$\begin{aligned}\iint_S \vec{\mathbf{v}} \cdot d\vec{\mathbf{S}} &= \iiint_C \operatorname{div} \vec{\mathbf{v}}\, dV \\ &= \iiint_C 0\, dV = 0.\end{aligned}$$

Therefore the flux is zero, as expected.

Exercise 16.8.3

Let $\vec{\mathbf{v}} = \langle \frac{x}{z}, \frac{y}{z}, 0 \rangle$ be the velocity field of a fluid. Let C be the solid cube given by $1 \le x \le 4$, $2 \le y \le 5$, $1 \le z \le 4$, and let S be the boundary of this cube (see the following figure). Find the flow rate of the fluid across S.



Hint

Use the divergence theorem and calculate a triple integral

Answer

$9 \ln(16)$

Example illustrates a remarkable consequence of the divergence theorem. Let S be a piecewise, smooth closed surface and let $\vec{\mathbf{F}}$ be a vector field defined on an open region containing the surface enclosed by S. If $\vec{\mathbf{F}}$ has the form $F = \langle f(y,z),\ g(x,z),\ h(x,y) \rangle$, then the divergence of $\vec{\mathbf{F}}$ is zero. By the divergence theorem, the flux of $\vec{\mathbf{F}}$ across S is also zero. This makes certain flux integrals incredibly easy to calculate. For example, suppose we wanted to calculate the flux integral $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$ where S is a cube and

$$\vec{\mathbf{F}} = \langle \sin(y)\, e^{yz},\ x^2 z^2,\ \cos(xy)\, e^{\sin x} \rangle.$$

Calculating the flux integral directly would be difficult, if not impossible, using techniques we studied previously. At the very least, we would have to break the flux integral into six integrals, one for each face of the cube. But, because the divergence of this field is zero, the divergence theorem immediately shows that the flux integral is zero.

We can now use the divergence theorem to justify the physical interpretation of divergence that we discussed earlier. Recall that if $\vec{\mathbf{F}}$ is a continuous three-dimensional vector field and P is a point in the domain of $\vec{\mathbf{F}}$, then the divergence of $\vec{\mathbf{F}}$ at P is a measure of

the "outflowing-ness" of $\vec{\mathbf{F}}$ at P. If $\vec{\mathbf{F}}$ represents the velocity field of a fluid, then the divergence of $\vec{\mathbf{F}}$ at P is a measure of the net flow rate out of point P (the flow of fluid out of P less the flow of fluid in to P). To see how the divergence theorem justifies this interpretation, let B_τ be a ball of very small radius r with center P, and assume that B_τ is in the domain of $\vec{\mathbf{F}}$. Furthermore, assume that B_τ has a positive, outward orientation. Since the radius of B_τ is small and $\vec{\mathbf{F}}$ is continuous, the divergence of $\vec{\mathbf{F}}$ is approximately constant on B_τ. That is, ifv P' is any point in B_τ, then $\text{div } \vec{\mathbf{F}}(P) \approx \text{div } \vec{\mathbf{F}}(P')$. Let S_τ denote the boundary sphere of B_τ. We can approximate the flux across S_τ using the divergence theorem as follows:

$$\begin{aligned} \iint_{S_\tau} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \iiint_{B_\tau} \text{div } \vec{\mathbf{F}}\, dV \\ &\approx \iiint_{B_\tau} \text{div } \vec{\mathbf{F}}(P)\, dV \\ &= \text{div } \vec{\mathbf{F}}(P)\, V(B_\tau). \end{aligned}$$

As we shrink the radius r to zero via a limit, the quantity $\text{div } \vec{\mathbf{F}}(P)\, V(B_\tau)$ gets arbitrarily close to the flux. Therefore,

$$\text{div } \vec{\mathbf{F}}(P) = \lim_{\tau \to 0} \frac{1}{V(B_\tau)} \iint_{S_\tau} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$$

and we can consider the divergence at P as measuring the net rate of outward flux per unit volume at P. Since "outflowing-ness" is an informal term for the net rate of outward flux per unit volume, we have justified the physical interpretation of divergence we discussed earlier, and we have used the divergence theorem to give this justification.

Application to Electrostatic Fields

The divergence theorem has many applications in physics and engineering. It allows us to write many physical laws in both an integral form and a differential form (in much the same way that Stokes' theorem allowed us to translate between an integral and differential form of Faraday's law). Areas of study such as fluid dynamics, electromagnetism, and quantum mechanics have equations that describe the conservation of mass, momentum, or energy, and the divergence theorem allows us to give these equations in both integral and differential forms.

One of the most common applications of the divergence theorem is to **electrostatic fields**. An important result in this subject is **Gauss' law**. This law states that if S is a closed surface in electrostatic field $\vec{\mathbf{E}}$, then the flux of $\vec{\mathbf{E}}$ across S is the total charge enclosed by S (divided by an electric constant). We now use the divergence theorem to justify the special case of this law in which the electrostatic field is generated by a stationary point charge at the origin.

If (x, y, z) is a point in space, then the distance from the point to the origin is $r = \sqrt{x^2 + y^2 + z^2}$. Let $\vec{\mathbf{F}}_\tau$ denote radial vector field $\vec{\mathbf{F}}_\tau = \frac{1}{\tau^2} \left\langle \frac{x}{\tau}, \frac{y}{\tau}, \frac{z}{\tau} \right\rangle$.The vector at a given position in space points in the direction of unit radial vector $\left\langle \frac{x}{\tau}, \frac{y}{\tau}, \frac{z}{\tau} \right\rangle$ and is scaled by the quantity $1/\tau^2$. Therefore, the magnitude of a vector at a given point is inversely proportional to the square of the vector's distance from the origin. Suppose we have a stationary charge of q Coulombs at the origin, existing in a vacuum. The charge generates electrostatic field $\vec{\mathbf{E}}$ given by

$$\vec{\mathbf{E}} = \frac{q}{4\pi\epsilon_0} \vec{\mathbf{F}}_\tau,$$

where the approximation $\epsilon_0 = 8.854 \times 10^{-12}$ farad (F)/m is an electric constant. (The constant ϵ_0 is a measure of the resistance encountered when forming an electric field in a vacuum.) Notice that $\vec{\mathbf{E}}$ is a radial vector field similar to the gravitational field described in [link]. The difference is that this field points outward whereas the gravitational field points inward. Because

$$\vec{\mathbf{E}} = \frac{q}{4\pi\epsilon_0} \vec{\mathbf{F}}_\tau = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{\tau^2} \left\langle \frac{x}{\tau}, \frac{y}{\tau}, \frac{z}{\tau} \right\rangle \right),$$

we say that electrostatic fields obey an inverse-square law. That is, the electrostatic force at a given point is inversely proportional to the square of the distance from the source of the charge (which in this case is at the origin). Given this vector field, we show that the flux across closed surface S is zero if the charge is outside of S, and that the flux is $q/epsilon_0$ if the charge is inside of S. In other words, the flux across S is the charge inside the surface divided by constant ϵ_0. This is a special case of Gauss' law, and here we use the divergence theorem to justify this special case.

To show that the flux across S is the charge inside the surface divided by constant ϵ_0, we need two intermediate steps. First we show that the divergence of $\vec{\mathbf{F}}_\tau$ is zero and then we show that the flux of $\vec{\mathbf{F}}_\tau$ across any smooth surface S is either zero or 4π. We

can then justify this special case of Gauss' law.

Example 16.8.4: The Divergence of F_τ is Zero

Verify that the divergence of $\vec{\mathbf{F}}_\tau$ is zero where $\vec{\mathbf{F}}_\tau$ is defined (away from the origin).

Solution

Since $\tau = \sqrt{x^2+y^2+z^2}$, the quotient rule gives us

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{x}{\tau^3}\right) &= \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right)\\ &= \frac{(x^2+y^2+z^2)^{3/2} - x\left[\frac{3}{2}(x^2+y^2+z^2)^{1/2}2x\right]}{(x^2+y^2+z^2)^3}\\ &= \frac{\tau^3 - 3x^2\tau}{\tau^6} = \frac{\tau^2-3x^2}{\tau^5}.\end{aligned}$$

Similarly,

$$\frac{\partial}{\partial y}\left(\frac{y}{\tau^3}\right) = \frac{\tau^2-3y^2}{\tau^5} \text{ and } \frac{\partial}{\partial z}\left(\frac{z}{\tau^3}\right) = \frac{\tau^2-3z^2}{\tau^5}.$$

Therefore,

$$\begin{aligned}\text{div } \vec{\mathbf{F}}_\tau &= \frac{\tau^2-3x^2}{\tau^5} + \frac{\tau^2-3y^2}{\tau^5} + \frac{\tau^2-3z^2}{\tau^5}\\ &= \frac{3\tau^2 - 3(x^2+y^2+z^2)}{\tau^5}\\ &= \frac{3\tau^2-3\tau^2}{\tau^5} = 0.\end{aligned}$$

Notice that since the divergence of $\vec{\mathbf{F}}_\tau$ is zero and $\vec{\mathbf{E}}$ is $\vec{\mathbf{F}}_\tau$ scaled by a constant, the divergence of electrostatic field $\vec{\mathbf{E}}$ is also zero (except at the origin).

Flux across a Smooth Surface

Let S be a connected, piecewise smooth closed surface and let $\vec{\mathbf{F}}_\tau = \frac{1}{\tau^2}\left\langle \frac{x}{\tau}, \frac{y}{\tau}, \frac{z}{\tau}\right\rangle$. Then,

$$\iint_S \vec{\mathbf{F}}_\tau \cdot d\vec{\mathbf{S}} = \begin{cases} 0, & \text{if } S \text{ does not encompass the origin} \\ 4\pi, & \text{if } S \text{ encompasses the origin.}\end{cases}$$

In other words, this theorem says that the flux of $\vec{\mathbf{F}}_\tau$ across any piecewise smooth closed surface S depends only on whether the origin is inside of S.

Proof

The logic of this proof follows the logic of [link], only we use the divergence theorem rather than Green's theorem.

First, suppose that S does not encompass the origin. In this case, the solid enclosed by S is in the domain of $\vec{\mathbf{F}}_\tau$, and since the divergence of $\vec{\mathbf{F}}_\tau$ is zero, we can immediately apply the divergence theorem and find that

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$$

is zero.

Now suppose that S does encompass the origin. We cannot just use the divergence theorem to calculate the flux, because the field is not defined at the origin. Let S_a be a sphere of radius a inside of S centered at the origin. The outward normal vector field on the sphere, in spherical coordinates, is

$$\vec{\mathbf{t}}_\phi \times \vec{\mathbf{t}}_\theta = \langle a^2 \cos\theta \sin^2\phi,\ a^2 \sin\theta \sin^2\phi,\ a^2 \sin\phi \cos\phi\rangle$$

(see [link]). Therefore, on the surface of the sphere, the dot product $\vec{\mathbf{F}}_\tau \cdot \vec{\mathbf{N}}$ (in spherical coordinates) is

$$\begin{aligned}\vec{\mathbf{F}}_\tau \cdot \vec{\mathbf{N}} &= \left\langle \frac{\sin\phi\cos\theta}{a^2}, \frac{\sin\phi\sin\theta}{a^2}, \frac{\cos\phi}{a^2} \right\rangle \cdot \langle a^2\cos\theta\sin^2\phi, a^2\sin\theta\sin^2\phi, a^2\sin\phi\cos\phi\rangle \\ &= \sin\phi(\langle\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle \cdot \langle\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle) \\ &= \sin\phi.\end{aligned}$$

The flux of $\vec{\mathbf{F}}_\tau$ across S_a is

$$\iint_{S_a} \vec{\mathbf{F}}_\tau \cdot \vec{\mathbf{N}} dS = \int_0^{2\pi}\int_0^{\pi} \sin\phi\, d\phi\, d\theta = 4\pi.$$

Now, remember that we are interested in the flux across S, not necessarily the flux across S_a. To calculate the flux across S, let E be the solid between surfaces S_a and S. Then, the boundary of E consists of S_a and S. Denote this boundary by $S - S_a$ to indicate that S is oriented outward but now S_a is oriented inward. We would like to apply the divergence theorem to solid E. Notice that the divergence theorem, as stated, can't handle a solid such as E because E has a hole. However, the divergence theorem can be extended to handle solids with holes, just as Green's theorem can be extended to handle regions with holes. This allows us to use the divergence theorem in the following way. By the divergence theorem,

$$\begin{aligned}\iint_{S-S_a} \vec{\mathbf{F}}_\tau \cdot d\vec{\mathbf{S}} &= \iint_S \vec{\mathbf{F}}_\tau \cdot d\vec{\mathbf{S}} - \iint_{S_a} \vec{\mathbf{F}}_\tau \cdot d\vec{\mathbf{S}} \\ &= \iiint_E \operatorname{div} \vec{\mathbf{F}}_\tau\, dV \\ &= \iiint_E 0\, dV = 0.\end{aligned}$$

Therefore,

$$\iint_S \vec{\mathbf{F}}_\tau \cdot d\vec{\mathbf{S}} = \iint_{S_a} \vec{\mathbf{F}}_\tau \cdot d\vec{\mathbf{S}} = 4\pi,$$

and we have our desired result.

□

Now we return to calculating the flux across a smooth surface in the context of electrostatic field $\vec{\mathbf{E}} = \frac{q}{4\pi\epsilon_0}\vec{\mathbf{F}}_\tau$ of a point charge at the origin. Let S be a piecewise smooth closed surface that encompasses the origin. Then

$$\begin{aligned}\iint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} &= \iint_S \frac{q}{4\pi\epsilon_0}\vec{\mathbf{F}}_\tau \cdot d\vec{\mathbf{S}} \\ &= \frac{q}{4\pi\epsilon_0}\iint_S \vec{\mathbf{F}}_\tau \cdot d\vec{\mathbf{S}} \\ &= \frac{q}{\epsilon_0}.\end{aligned}$$

If S does not encompass the origin, then

$$\iint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} = \frac{q}{4\pi\epsilon_0}\iint_S \vec{\mathbf{F}}_\tau \cdot d\vec{\mathbf{S}} = 0.$$

Therefore, we have justified the claim that we set out to justify: the flux across closed surface S is zero if the charge is outside of S, and the flux is q/ϵ_0 if the charge is inside of S.

This analysis works only if there is a single point charge at the origin. In this case, Gauss' law says that the flux of $\vec{\mathbf{E}}$ across S is the total charge enclosed by S. Gauss' law can be extended to handle multiple charged solids in space, not just a single point charge at the origin. The logic is similar to the previous analysis, but beyond the scope of this text. In full generality, Gauss' law states that if S is a piecewise smooth closed surface and Q is the total amount of charge inside of S, then the flux of $\vec{\mathbf{E}}$ across S is Q/ϵ_0.

Example 16.8.5: Using Gauss' law

Suppose we have four stationary point charges in space, all with a charge of 0.002 Coulombs (C). The charges are located at $(0,0,1)$, $(1,1,4)$, $(-1,0,0)$, and $(-2,-2,2)$. Let $\vec{\mathbf{E}}$ denote the electrostatic field generated by these point charges. If S is the sphere of radius 2 oriented outward and centered at the origin, then find

$$\iint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}}.$$

Solution

According to Gauss' law, the flux of $\vec{\mathbf{E}}$ across S is the total charge inside of S divided by the electric constant. Since S has radius 2, notice that only two of the charges are inside of S: the charge at $(0,1,1)$ and the charge at $(-1,0,0)$. Therefore, the total charge encompassed by S is 0.004 and, by Gauss' law,

$$\iint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} = \frac{0.004}{8.854 \times 10^{-12}} \approx 4.418 \times 10^9\, V - m.$$

Exercise 16.8.4

Work the previous example for surface S that is a sphere of radius 4 centered at the origin, oriented outward.

Hint

Use Gauss' law.

Answer

$\approx 6.777 \times 10^9$

Key Concepts

- The divergence theorem relates a surface integral across closed surface S to a triple integral over the solid enclosed by S. The divergence theorem is a higher dimensional version of the flux form of Green's theorem, and is therefore a higher dimensional version of the Fundamental Theorem of Calculus.
- The divergence theorem can be used to transform a difficult flux integral into an easier triple integral and vice versa.
- The divergence theorem can be used to derive Gauss' law, a fundamental law in electrostatics.

Key Equations

- **Divergence theorem**

$$\iiint_E \operatorname{div} \vec{\mathbf{F}}\, dV = \iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$$

Glossary

divergence theorem
a theorem used to transform a difficult flux integral into an easier triple integral and vice versa

Gauss' law
if S is a piecewise, smooth closed surface in a vacuum and Q is the total stationary charge inside of S, then the flux of electrostatic field $\vec{\mathbf{E}}$ across S is Q/ϵ_0

inverse-square law
the electrostatic force at a given point is inversely proportional to the square of the distance from the source of the charge

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16.8E: Exercises for Section 16.8

For exercises 1 - 9, use a computer algebraic system (CAS) and the divergence theorem to evaluate surface integral $\int_S \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, ds$ for the given choice of $\vec{\mathbf{F}}$ and the boundary surface S. For each closed surface, assume $\vec{\mathbf{N}}$ is the outward unit normal vector.

1. [T] $\vec{\mathbf{F}}(x,y,z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$; S is the surface of cube $0 \le x \le 1,\ 0 \le y \le 1,\ 0 < z \le 1$.

2. [T] $\vec{\mathbf{F}}(x,y,z) = (\cos yz)\,\hat{\mathbf{i}} + e^{xz}\,\hat{\mathbf{j}} + 3z^2\,\hat{\mathbf{k}}$; S is the surface of hemisphere $z = \sqrt{4 - x^2 - y^2}$ together with disk $x^2 + y^2 \le 4$ in the xy-plane.

Answer

$$\int_S \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, ds = 75.3982$$

3. [T] $\vec{\mathbf{F}}(x,y,z) = (x^2 + y^2 - x^2)\,\hat{\mathbf{i}} + x^2 y\,\hat{\mathbf{j}} + 3z\,\hat{\mathbf{k}}$; S is the surface of the five faces of unit cube $0 \le x \le 1,\ 0 \le y \le 1,\ 0 < z \le 1$.

4. [T] $\vec{\mathbf{F}}(x,y,z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$; S is the surface of paraboloid $z = x^2 + y^2$ for $0 \le z \le 9$.

Answer

$$\int_S \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, ds = 127.2345$$

5. [T] $\vec{\mathbf{F}}(x,y,z) = x^2\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}} + z^2\,\hat{\mathbf{k}}$; S is the surface of sphere $x^2 + y^2 + z^2 = 4$.

6. [T] $\vec{\mathbf{F}}(x,y,z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + (z^2 - 1)\,\hat{\mathbf{k}}$; S is the surface of the solid bounded by cylinder $x^2 + y^2 = 4$ and planes $z = 0$ and $z = 1$.

Answer

$$\int_S \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, ds = 37.699$$

7. [T] $\vec{\mathbf{F}}(x,y,z) = x^2\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}} + z^2\,\hat{\mathbf{k}}$; S is the surface bounded above by sphere $\rho = 2$ and below by cone $\varphi = \dfrac{\pi}{4}$ in spherical coordinates. (Think of S as the surface of an "ice cream cone.")

8. [T] $\vec{\mathbf{F}}(x,y,z) = x^3\,\hat{\mathbf{i}} + y^3\,\hat{\mathbf{j}} + 3a^2 z\,\hat{\mathbf{k}}\ (constant\ a > 0)$; S is the surface bounded by cylinder $x^2 + y^2 = a^2$ and planes $z = 0$ and $z = 1$.

Answer

$$\int_S \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, ds = \frac{9\pi a^4}{2}$$

9. [T] Surface integral $\iint_S \vec{\mathbf{F}} \cdot dS$, where S is the solid bounded by paraboloid $z = x^2 + y^2$ and plane $z = 4$, and $\vec{\mathbf{F}}(x,y,z) = (x + y^2 z^2)\,\hat{\mathbf{i}} + (y + z^2 x^2)\,\hat{\mathbf{j}} + (z + x^2 y^2)\,\hat{\mathbf{k}}$

10. Use the divergence theorem to calculate surface integral $\iint_S \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x,y,z) = (e^{y^2}\,\hat{\mathbf{i}} + (y + \sin(z^2))\,\hat{\mathbf{j}} + (z - 1)\,\hat{\mathbf{k}}$ and S is upper hemisphere $x^2 + y^2 + z^2 = 1,\ z \ge 0$, oriented upward.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot dS = \frac{\pi}{3}$$

11. Use the divergence theorem to calculate surface integral $\iint_S \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x,y,z) = x^4\,\hat{\mathbf{i}} - x^3 z^2\,\hat{\mathbf{j}} + 4xy^2 z\,\hat{\mathbf{k}}$ and S is the surface bounded by cylinder $x^2 + y^2 = 1$ and planes $z = x + 2$ and $z = 0$.

12. Use the divergence theorem to calculate surface integral $\iint_S \vec{\mathbf{F}} \cdot dS$, when $\vec{\mathbf{F}}(x,y,z) = x^2 z^3\,\hat{\mathbf{i}} + 2xyz^3\,\hat{\mathbf{j}} + xz^4\,\hat{\mathbf{k}}$ and S is the surface of the box with vertices $(\pm 1,\ \pm 2,\ \pm 3)$.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot dS = 0$$

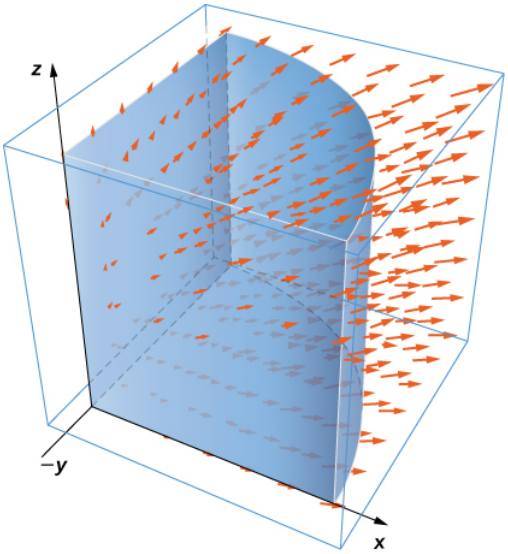
13. Use the divergence theorem to calculate surface integral $\iint_S \vec{\mathbf{F}} \cdot dS$, when $\vec{\mathbf{F}}(x, y, z) = z \tan^{-1}(y^2)\,\hat{\mathbf{i}} + z^3 \ln(x^2 + 1)\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ and S is a part of paraboloid $x^2 + y^2 + z = 2$ that lies above plane $z = 1$ and is oriented upward.

14. [T] Use a CAS and the divergence theorem to calculate flux $\iint_S \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x, y, z) = (x^3 + y^3)\,\hat{\mathbf{i}} + (y^3 + z^3)\,\hat{\mathbf{j}} + (z^3 + x^3)\,\hat{\mathbf{k}}$ and S is a sphere with center $(0, 0)$ and radius 2.

Answer

$\iint_S \vec{\mathbf{F}} \cdot dS = 241.2743$

15. Use the divergence theorem to compute the value of flux integral $\iint_S \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x, y, z) = (y^3 + 3x)\,\hat{\mathbf{i}} + (xz + y)\,\hat{\mathbf{j}} + (z + x^4 \cos(x^2 y))\,\hat{\mathbf{k}}$ and S is the area of the region bounded by $x^2 + y^2 = 1$, $x \geq 0$, $y \geq 0$, and $0 \leq z \leq 1$.



16. Use the divergence theorem to compute flux integral $\iint_S \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x, y, z) = y\,\hat{\mathbf{j}} - z\,\hat{\mathbf{k}}$ and S consists of the union of paraboloid $y = x^2 + z^2$, $0 \leq y \leq 1$, and disk $x^2 + z^2 \leq 1$, $y = 1$, oriented outward. What is the flux through just the paraboloid?

Answer

$\iint_S \vec{\mathbf{F}} \cdot dS = -\pi$

17. Use the divergence theorem to compute flux integral $\iint_S \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x, y, z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z^4\,\hat{\mathbf{k}}$ and S is a part of cone $z = \sqrt{x^2 + y^2}$ beneath top plane $z = 1$ oriented downward.

18. Use the divergence theorem to calculate surface integral $\iint_S \vec{\mathbf{F}} \cdot dS$ for $\vec{\mathbf{F}}(x, y, z) = x^4\,\hat{\mathbf{i}} - x^3 z^2\,\hat{\mathbf{j}} + 4xy^2 z\,\hat{\mathbf{k}}$, where S is the surface bounded by cylinder $x^2 + y^2 = 1$ and planes $z = x + 2$ and $z = 0$.

Answer

$\iint_S \vec{\mathbf{F}} \cdot dS = \dfrac{2\pi}{3}$

19. Consider $\vec{\mathbf{F}}(x, y, z) = x^2\,\hat{\mathbf{i}} + xy\,\hat{\mathbf{j}} + (z + 1)\,\hat{\mathbf{k}}$. Let E be the solid enclosed by paraboloid $z = 4 - x^2 - y^2$ and plane $z = 0$ with normal vectors pointing outside E. Compute flux $\vec{\mathbf{F}}$ across the boundary of E using the divergence theorem.

In exercises 20 - 23, use a CAS along with the divergence theorem to compute the net outward flux for the fields across the given surfaces S.

20. [T] $\vec{\mathbf{F}} = \langle x, -2y, 3z \rangle$; S is sphere $\{(x, y, z) : x^2 + y^2 + z^2 = 6\}$.

Answer

$15\sqrt{6}\pi$

21. [T] $\vec{\mathbf{F}} = \langle x,\ 2y,\ z\rangle$; S is the boundary of the tetrahedron in the first octant formed by plane $x + y + z = 1$.

22. [T] $\vec{\mathbf{F}} = \langle y - 2x,\ x^3 - y,\ y^2 - z\rangle$; S is sphere $\{(x, y, z)\ :\ x^2 + y^2 + z^2 = 4\}$.

Answer

$-\dfrac{128}{3}\pi$

23. [T] $\vec{\mathbf{F}} = \langle x, y, z\rangle$; S is the surface of paraboloid $z = 4 - x^2 - y^2$, for $z \geq 0$, plus its base in the xy-plane.

For exercises 24 - 26, use a CAS and the divergence theorem to compute the net outward flux for the vector fields across the boundary of the given regions D.

24. [T] $\vec{\mathbf{F}} = \langle z - x,\ x - y,\ 2y - z\rangle$; D is the region between spheres of radius 2 and 4 centered at the origin.

Answer

-703.7168

25. [T] $\vec{\mathbf{F}} = \dfrac{\vec{\mathbf{r}}}{\|\vec{\mathbf{r}}\|} = \dfrac{\langle x, y, z\rangle}{\sqrt{x^2 + y^2 + z^2}}$; D is the region between spheres of radius 1 and 2 centered at the origin.

26. [T] $\vec{\mathbf{F}} = \langle x^2,\ -y^2,\ z^2\rangle$; D is the region in the first octant between planes $z = 4 - x - y$ and $z = 2 - x - y$.

Answer

20

27. Let $\vec{\mathbf{F}}(x, y, z) = 2x\,\hat{\mathbf{i}} - 3xy\hat{\mathbf{j}} + xz^2\,\hat{\mathbf{k}}$. Use the divergence theorem to calculate $\iint_S \vec{\mathbf{F}} \cdot dS$, where S is the surface of the cube with corners at $(0, 0, 0),\ (1, 0, 0),\ (0, 1, 0),\ (1, 1, 0),\ (0, 0, 1),\ (1, 0, 1),\ (0, 1, 1)$, and $(1, 1, 1)$, oriented outward.

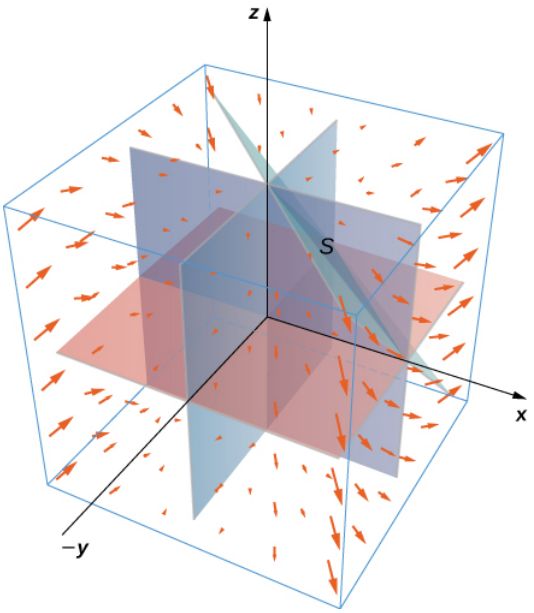
28. Use the divergence theorem to find the outward flux of field $\vec{\mathbf{F}}(x, y, z) = (x^3 - 3y)\,\hat{\mathbf{i}} + (2yz + 1)\,\hat{\mathbf{j}} + xyz\,\hat{\mathbf{k}}$ through the cube bounded by planes $x = \pm 1$, $y = \pm 1$, and $z = \pm 1$.

Answer

$\iint_S \vec{\mathbf{F}} \cdot dS = 8$

29. Let $\vec{\mathbf{F}}(x, y, z) = 2x\,\hat{\mathbf{i}} - 3y\hat{\mathbf{j}} + 5z\,\hat{\mathbf{k}}$ and let S be hemisphere $z = \sqrt{9 - x^2 - y^2}$ together with disk $x^2 + y^2 \leq 9$ in the xy-plane. Use the divergence theorem.

30. Evaluate $\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, dS$, where $\vec{\mathbf{F}}(x, y, z) = x^2\,\hat{\mathbf{i}} + xy\,\hat{\mathbf{j}} + x^3y^3\,\hat{\mathbf{k}}$ and S is the surface consisting of all faces except the tetrahedron bounded by plane $x + y + z = 1$ and the coordinate planes, with outward unit normal vector $\vec{\mathbf{N}}$.



Answer

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, dS = \frac{1}{8}$$

31. Find the net outward flux of field $\vec{\mathbf{F}} = \langle bz - cy,\ cx - az,\ ay - bx \rangle$ across any smooth closed surface in R^3 where $a,\ b,$ and c are constants.

32. Use the divergence theorem to evaluate $\iint_S ||\vec{\mathbf{R}}||\vec{\mathbf{R}} \cdot \vec{\mathbf{n}}\, ds$, where $\vec{\mathbf{R}}(x, y, z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ and S is sphere $x^2 + y^2 + z^2 = a^2$, with constant $a > 0$.

Answer

$$\iint_S ||\vec{\mathbf{R}}||\vec{\mathbf{R}} \cdot \vec{\mathbf{n}}\, ds = 4\pi a^4$$

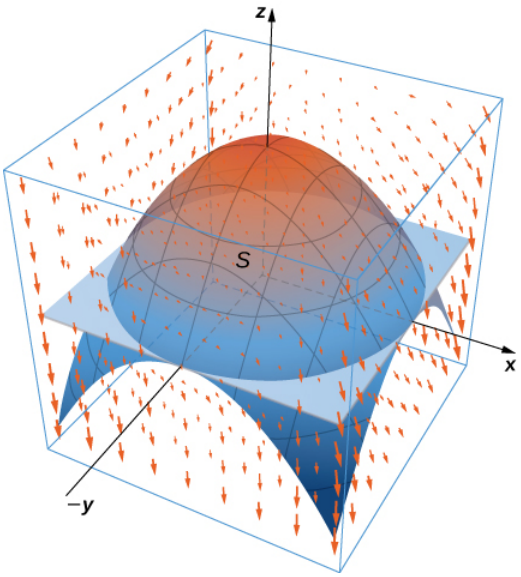
33. Use the divergence theorem to evaluate $\iint_S \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x, y, z) = y^2 z\,\hat{\mathbf{i}} + y^3\,\hat{\mathbf{j}} + xz\,\hat{\mathbf{k}}$ and S is the boundary of the cube defined by $-1 \le x \le 1,\ -1 \le y \le 1$, and $0 \le z \le 2$.

34. Let R be the region defined by $x^2 + y^2 + z^2 \le 1$. Use the divergence theorem to find $\iiint_R z^2\, dV$.

Answer

$$\iiint_R z^2 dV = \frac{4\pi}{15}$$

35. Let E be the solid bounded by the xy-plane and paraboloid $z = 4 - x^2 - y^2$ so that S is the surface of the paraboloid piece together with the disk in the xy-plane that forms its bottom. If $\vec{\mathbf{F}}(x, y, z) = (xz\,\sin(yz) + x^3)\,\hat{\mathbf{i}} + \cos(yz)\,\hat{\mathbf{j}} + (3zy^2 - e^{x^2+y^2})\,\hat{\mathbf{k}}$, find $\iint_S \vec{\mathbf{F}} \cdot dS$ using the divergence theorem.



36. Let E be the solid unit cube with diagonally opposite corners at the origin and $(1, 1, 1)$, and faces parallel to the coordinate planes. Let S be the surface of E, oriented with the outward-pointing normal. Use a CAS to find $\iint_S \vec{\mathbf{F}} \cdot dS$ using the divergence theorem if $\vec{\mathbf{F}}(x, y, z) = 2xy\,\hat{\mathbf{i}} + 3ye^z\,\hat{\mathbf{j}} + x \sin z\,\hat{\mathbf{k}}$.

Answer

$\iint_S \vec{\mathbf{F}} \cdot dS = 6.5759$

37. Use the divergence theorem to calculate the flux of $\vec{\mathbf{F}}(x, y, z) = x^3\,\hat{\mathbf{i}} + y^3\,\hat{\mathbf{j}} + z^3\,\hat{\mathbf{k}}$ through sphere $x^2 + y^2 + z^2 = 1$.

38. Find $\iint_S \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x, y, z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ and S is the outwardly oriented surface obtained by removing cube $[1, 2] \times [1, 2] \times [1, 2]$ from cube $[0, 2] \times [0, 2] \times [0, 2]$.

Answer

$\iint_S \vec{\mathbf{F}} \cdot dS = 21$

39. Consider radial vector field $\vec{\mathbf{F}} = \dfrac{\vec{\mathbf{r}}}{\|\vec{\mathbf{r}}\|} = \dfrac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{1/2}}$. Compute the surface integral, where S is the surface of a sphere of radius *a* centered at the origin.

40. Compute the flux of water through parabolic cylinder $S : y = x^2$, from $0 \le x \le 2$, $0 \le z \le 3$, if the velocity vector is $\vec{\mathbf{F}}(x, y, z) = 3z^2\,\hat{\mathbf{i}} + 6\,\hat{\mathbf{j}} + 6xz\,\hat{\mathbf{k}}$.

Answer

$\iint_S \vec{\mathbf{F}} \cdot dS = 72$

41. [T] Use a CAS to find the flux of vector field $\vec{\mathbf{F}}(x, y, z) = z\,\hat{\mathbf{i}} + z\,\hat{\mathbf{j}} + \sqrt{x^2 + y^2}\,\hat{\mathbf{k}}$ across the portion of hyperboloid $x^2 + y^2 = z^2 + 1$ between planes $z = 0$ and $z = \dfrac{\sqrt{3}}{3}$, oriented so the unit normal vector points away from the z-axis.

42. Use a CAS to find the flux of vector field $\vec{\mathbf{F}}(x, y, z) = (e^y + x)\,\hat{\mathbf{i}} + (3\cos(xz) - y)\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ through surface S, where S is given by $z^2 = 4x^2 + 4y^2$ from $0 \le z \le 4$, oriented so the unit normal vector points downward.

Answer

$\iint_S \vec{\mathbf{F}} \cdot dS = -33.5103$

43. [T] Use a CAS to compute $\iint_S \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x,y,z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + 2z\,\hat{\mathbf{k}}$ and S is a part of sphere $x^2 + y^2 + z^2 = 2$ with $0 \le z \le 1$.

44. Evaluate $\iint_S \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x,y,z) = bxy^2\,\hat{\mathbf{i}} + bx^2y\,\hat{\mathbf{j}} + (x^2+y^2)z^2\,\hat{\mathbf{k}}$ and S is a closed surface bounding the region and consisting of solid cylinder $x^2 + y^2 \le a^2$ and $0 \le z \le b$.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot dS = \pi a^4 b^2$$

45. [T] Use a CAS to calculate the flux of $\vec{\mathbf{F}}(x,y,z) = (x^3 + y\,\sin z)\,\hat{\mathbf{i}} + (y^3 + z\,\sin x)\,\hat{\mathbf{j}} + 3z\,\hat{\mathbf{k}}$ across surface S, where S is the boundary of the solid bounded by hemispheres $z = \sqrt{4 - x^2 - y^2}$ and $z = \sqrt{1 - x^2 - y^2}$, and plane $z = 0$.

46. Use the divergence theorem to evaluate $\iint_S \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x,y,z) = xy\,\hat{\mathbf{i}} - \frac{1}{2}y^2\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ and S is the surface consisting of three pieces: $z = 4 - 3x^2 - 3y^2$, $1 \le z \le 4$ on the top; $x^2 + y^2 = 1$, $0 \le z \le 1$ on the sides; and $z = 0$ on the bottom.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot dS = \frac{5}{2}\pi$$

47. [T] Use a CAS and the divergence theorem to evaluate $\iint_S \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x,y,z) = (2x + y\,\cos z)\,\hat{\mathbf{i}} + (x^2 - y)\,\hat{\mathbf{j}} + y^2 z\,\hat{\mathbf{k}}$ and S is sphere $x^2 + y^2 + z^2 = 4$ orientated outward.

48. Use the divergence theorem to evaluate $\iint_S \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x,y,z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ and S is the boundary of the solid enclosed by paraboloid $y = x^2 + z^2 - 2$, cylinder $x^2 + z^2 = 1$, and plane $x + y = 2$, and S is oriented outward.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot dS = \frac{21\pi}{2}$$

For the following exercises, **Fourier's law of heat transfer** states that the heat flow vector $\vec{\mathbf{F}}$ at a point is proportional to the negative gradient of the temperature; that is, $\vec{\mathbf{F}} = -k\vec{\nabla}T$, which means that heat energy flows hot regions to cold regions. The constant $k > 0$ is called the *conductivity*, which has metric units of joules per meter per second-kelvin or watts per meter-kelvin. A temperature function for region D s given. Use the divergence theorem to find net outward heat flux $\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\,dS = -k \iint_S \vec{\nabla}T \cdot N\,dS$ across the boundary S of D, where $k = 1$.

49. $T(x,y,z) = 100 + x + 2y + z$;

$D = \{(x,y,z) : 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$

50. $T(x,y,z) = 100 + e^{-z}$;

$D = \{(x,y,z) : 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$

Answer

$-(1 - e^{-1})$

51. $T(x,y,z) = 100e^{-x^2-y^2-z^2}$; D is the sphere of radius a centered at the origin.

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16.R: Chapter 16 Review Exercises

***True or False?* Justify your answer with a proof or a counterexample.**

1. The vector field $\vec{\mathbf{F}}(x,y) = x^2y\,\hat{\mathbf{i}} + y^2x\,\hat{\mathbf{j}}$ is conservative.

Answer

False

2. For vector field $\vec{\mathbf{F}}(x,y) = P(x,y)\,\hat{\mathbf{i}} + Q(x,y)\,\hat{\mathbf{j}}$, if $P_y(x,y) = Q_z(x,y)$ in open region D, then $\int_{\partial D} P\,dx + Q\,dy = 0$.

3. The divergence of a vector field is a vector field.

Answer

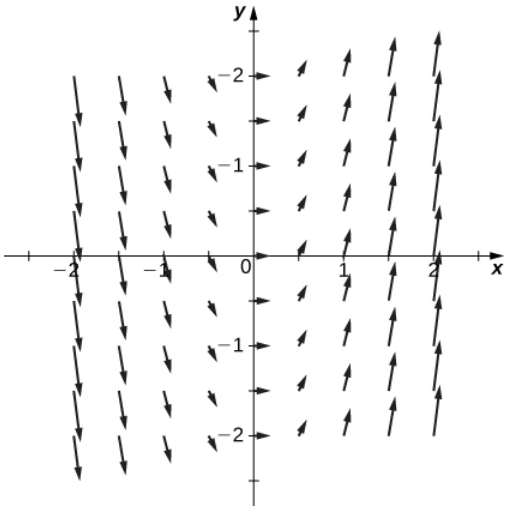
False

4. If $curl\,\vec{\mathbf{F}} = \vec{0}$, then $\vec{\mathbf{F}}$ is a conservative vector field.

Draw the following vector fields.

5. $\vec{\mathbf{F}}(x,y) = \frac{1}{2}\,\hat{\mathbf{i}} + 2x\,\hat{\mathbf{j}}$

Answer



6. $\vec{\mathbf{F}}(x,y) = \sqrt{\frac{y\,\hat{\mathbf{i}} + 3x\,\hat{\mathbf{j}}}{x^2+y^2}}$

Are the following the vector fields conservative? If so, find the potential function $\vec{\mathbf{F}}$ such that $\vec{\mathbf{F}} = \vec{\nabla} f$.

7. $\vec{\mathbf{F}}(x,y) = y\,\hat{\mathbf{i}} + (x - 2e^y)\,\hat{\mathbf{j}}$

Answer

Conservative, $f(x,y) = xy - 2e^y$

8. $\vec{\mathbf{F}}(x,y) = (6xy)\,\hat{\mathbf{i}} + (3x^2 - ye^y)\,\hat{\mathbf{j}}$

9. $\vec{\mathbf{F}}(x,y) = (2xy + z^2)\,\hat{\mathbf{i}} + (x^2 + 2yz)\,\hat{\mathbf{j}} + (2xz + y^2)\,\hat{\mathbf{k}}$

Answer

Conservative, $f(x,y,z) = x^2y + y^2z + z^2x$

10. $\vec{\mathbf{F}}(x,y,z) = (e^xy)\,\hat{\mathbf{i}} + (e^x + z)\,\hat{\mathbf{j}} + (e^x + y^2)\,\hat{\mathbf{k}}$

Evaluate the following integrals.

11. $\int_C x^2\,dy + (2x - 3xy)\,dx$, along $C : y = \frac{1}{2}x$ from $(0,0)$ to $(4,2)$

Answer

$-\frac{16}{3}$

12. $\int_C y\,dx + xy^2\,dy$, where $C : x = \sqrt{t},\ y = t - 1,\ 0 \le t \le 1$

13. $\iint_S xy^2\,dS$, where S is the surface $z = x^2 - y,\ 0 \le x \le 1,\ 0 \le y \le 4$

Answer

$\frac{32\sqrt{2}}{9}(3\sqrt{3} - 1)$

Find the divergence and curl for the following vector fields.

14. $\vec{\mathbf{F}}(x,y,z) = 3xyz\,\hat{\mathbf{i}} + xye^x\,\hat{\mathbf{j}} - 3xy\hat{\mathbf{k}}$

15. $\vec{\mathbf{F}}(x,y,z) = e^x\,\hat{\mathbf{i}} + e^{xy}\,\hat{\mathbf{j}} - e^{xyz}\,\hat{\mathbf{k}}$

Answer

Divergence: $e^x + x\,e^{xy} + xy\,e^{xyz}$
Curl: $xze^{xyz}\,\hat{\mathbf{i}} - yze^{xyz}\,\hat{\mathbf{j}} + ye^{xy}\,\hat{\mathbf{k}}$

Use Green's theorem to evaluate the following integrals.

16. $\int_C 3xy\,dx + 2xy^2\,dy$, where C is a square with vertices $(0,0)$, $(0,2)$, $(2,2)$ and $(2,0)$.

17. $\oint_C 3y\,dx + (x + e^y)\,dy$, where C is a circle centered at the origin with radius 3.

Answer

-2π

Use Stokes' theorem to evaluate $\iint_S curl\,\vec{\mathbf{F}} \cdot dS$.

18. $\vec{\mathbf{F}}(x,y,z) = y\,\hat{\mathbf{i}} - x\,\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, where S is the upper half of the unit sphere

19. $\vec{\mathbf{F}}(x,y,z) = y\,\hat{\mathbf{i}} + xyz\,\hat{\mathbf{j}} - 2zx\,\hat{\mathbf{k}}$, where S is the upward-facing paraboloid $z = x^2 + y^2$ lying in cylinder $x^2 + y^2 = 1$

Answer

$-\pi$

Use the divergence theorem to evaluate $\iint_S \vec{\mathbf{F}} \cdot dS$.

20. $\vec{\mathbf{F}}(x,y,z) = (x^3y)\,\hat{\mathbf{i}} + (3y - e^x)\,\hat{\mathbf{j}} + (z + x)\,\hat{\mathbf{k}}$, over cube S defined by $-1 \le x \le 1,\ 0 \le y \le 2,\ 0 \le z \le 2$

21. $\vec{\mathbf{F}}(x,y,z) = (2xy)\,\hat{\mathbf{i}} + (-y^2)\,\hat{\mathbf{j}} + (2z^3)\,\hat{\mathbf{k}}$, where S is bounded by paraboloid $z = x^2 + y^2$ and plane $z = 2$

Answer

$31\pi/2$

22. Find the amount of work performed by a 50-kg woman ascending a helical staircase with radius 2 m and height 100 m. The woman completes five revolutions during the climb.

23. Find the total mass of a thin wire in the shape of a semicircle with radius $\sqrt{2}$, and a density function of $\rho(x,y) = y + x^2$.

Answer

$\sqrt{2}(2 + \pi)$

24. Find the total mass of a thin sheet in the shape of a hemisphere with radius 2 for $z \geq 0$ with a density function $\rho(x, y, z) = x + y + z$.

25. Use the divergence theorem to compute the value of the flux integral over the unit sphere with $\vec{\mathbf{F}}(x, y, z) = 3z\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} + 2x\hat{\mathbf{k}}$.

Answer

$2\pi/3$

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CHAPTER OVERVIEW

17: Second-Order Differential Equations

We have already studied the basics of differential equations, including separable first-order equations. In this chapter, we go a little further and look at second-order equations, which are equations containing second derivatives of the dependent variable. The solution methods we examine are different from those discussed earlier, and the solutions tend to involve trigonometric functions as well as exponential functions. Here we concentrate primarily on second-order equations with constant coefficients.

17.0: Prelude to Second-Order Differential Equations

17.1: Second-Order Linear Equations

17.1E: Exercises for Section 17.1

17.2: Nonhomogeneous Linear Equations

17.2E: Exercises for Section 17.2

17.3: Applications of Second-Order Differential Equations

17.3E: Exercises for Section 17.3

17.4: Series Solutions of Differential Equations

17.4E: Exercises for Section 17.4

17.R: Chapter 17 Review Exercises

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17.0: Prelude to Second-Order Differential Equations

We have already studied the basics of differential equations, including separable first-order equations. In this chapter, we go a little further and look at second-order equations, which are equations containing second derivatives of the dependent variable. The solution methods we examine are different from those discussed earlier, and the solutions tend to involve trigonometric functions as well as exponential functions. Here we concentrate primarily on second-order equations with constant coefficients.

Figure 17.0.1: A motorcycle suspension system is an example of a damped spring-mass system. The spring absorbs bumps and keeps the tire in contact with the road. The shock absorber damps the motion so the motorcycle does not continue to bounce after going over each bump. (credit: nSeika, Flickr)

Such equations have many practical applications. The operation of certain electrical circuits, known as resistor–inductor–capacitor (*RLC*) circuits, can be described by second-order differential equations with constant coefficients. These circuits are found in all kinds of modern electronic devices—from computers to smartphones to televisions. Such circuits can be used to select a range of frequencies from the entire radio wave spectrum, and are they commonly used for tuning AM/FM radios. We look at these circuits more closely in Applications.

Spring-mass systems, such as motorcycle shock absorbers, are a second common application of second-order differential equations. For motocross riders, the suspension systems on their motorcycles are very important. The off-road courses on which they ride often include jumps, and losing control of the motorcycle when landing could cost them the race. The movement of the shock absorber depends on the amount of damping in the system. In this chapter, we model forced and unforced spring-mass systems with varying amounts of damping.

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17.1: Second-Order Linear Equations

Learning Objectives

- Recognize homogeneous and nonhomogeneous linear differential equations.
- Determine the characteristic equation of a homogeneous linear equation.
- Use the roots of the characteristic equation to find the solution to a homogeneous linear equation.
- Solve initial-value and boundary-value problems involving linear differential equations.

When working with differential equations, usually the goal is to find a solution. In other words, we want to find a function (or functions) that satisfies the differential equation. The technique we use to find these solutions varies, depending on the form of the differential equation with which we are working. Second-order differential equations have several important characteristics that can help us determine which solution method to use. In this section, we examine some of these characteristics and the associated terminology.

Homogeneous Linear Equations

Consider the second-order differential equation

$$xy'' + 2x^2y' + 5x^3y = 0.$$

Notice that y and its derivatives appear in a relatively simple form. They are multiplied by functions of x, but are not raised to any powers themselves, nor are they multiplied together. As discussed in previously, first-order equations with similar characteristics are said to be linear. The same is true of second-order equations. Also note that all the terms in this differential equation involve either y or one of its derivatives. There are no terms involving only functions of x. Equations like this, in which every term contains y or one of its derivatives, are called **homogeneous**.

Not all differential equations are homogeneous. Consider the differential equation

$$xy'' + 2x^2y' + 5x^3y = x^2.$$

The x^2 term on the right side of the equal sign does not contain y or any of its derivatives. Therefore, this differential equation is *nonhomogeneous*.

Definition: Homogeneous and Nonhomogeneous Linear Equations

A second-order differential equation is linear if it can be written in the form

$$a_2(x)y'' + a)1(x)y' + a_0(x)y = r(x), \tag{17.1.1}$$

where $a_2(x), a_1(x), a_0(x)$, and $r(x)$ are real-valued functions and $a_2(x)$ is not identically zero. If $r(x) \equiv 0$—in other words, if $r(x) = 0$ for every value of x—the equation is said to be a *homogeneous linear equation*. If $r(x) \neq 0$ for some value of x, the equation is said to be a *nonhomogeneous linear equation*.

In linear differential equations, y and its derivatives can be raised only to the first power and they may not be multiplied by one another. Terms involving y^2 or $\sqrt{y'}$ make the equation nonlinear. Functions of y and its derivatives, such as $\sin y$ or $e^{y'}$, are similarly prohibited in linear differential equations.

Note that equations may not always be given in standard form (the form shown in the definition). It can be helpful to rewrite them in that form to decide whether they are linear, or whether a linear equation is homogeneous.

Example 17.1.1: Classifying Second-Order Equations

Classify each of the following equations as linear or nonlinear. If the equation is linear, determine further whether it is homogeneous or nonhomogeneous.

a. $y'' + 3x^4y' + x^2y^2 = x^3$

b. $(\sin x)y'' + (\cos x)y' + 3y = 0$
c. $4t^2x'' + 3txx' + 4x = 0$
d. $5y'' + y = 4x^5$
e. $(\cos x)y'' - \sin y' + (\sin x)y - \cos x = 0$
f. $8ty'' - 6t^2y' + 4ty - 3t^2 = 0$
g. $\sin(x^2)y'' - (\cos x)y' + x^2y = y' - 3$
h. $y'' + 5xy' - 3y = \cos y$

Solution

a. This equation is nonlinear because of the y^2 term.
b. This equation is linear. There is no term involving a power or function of y, and the coefficients are all functions of x.The equation is already written in standard form, and $r(x)$ is identically zero, so the equation is homogeneous.
c. This equation is nonlinear. Note that, in this case, x is the dependent variable and $t$is the independent variable. The second term involves the product of x and x', so the equation is nonlinear.
d. This equation is linear. Since $r(x) = 4x^5$, the equation is nonhomogeneous.
e. This equation is nonlinear, because of the $\sin y'$ term.
f. This equation is linear. Rewriting it in standard form gives

$$8t^2y'' - 6t^2y' + 4ty = 3t^2.$$

With the equation in standard form, we can see that $r(t) = 3t^2$,so the equation is nonhomogeneous.
g. This equation looks like it's linear, but we should rewrite it in standard form to be sure. We get

$$\sin(x^2)y'' - (\cos x + 1)y' + x^2y = -3.$$

This equation is, indeed, linear. With $r(x) = -3$, it is nonhomogeneous.
h. This equation is nonlinear because of the $\cos y$ term.

? Exercise 17.1.1

Classify each of the following equations as linear or nonlinear. If the equation is linear, determine further whether it is homogeneous or nonhomogeneous.

a. $(y'')2 - y' + 8x^3y = 0$
b. $(\sin t)y'' + \cos t - 3ty' = 0$

Hint

Write the equation in standard form (Equation 17.1.1) if necessary. Check for powers or functions of y and its derivatives.

Answer a

Nonlinear Linear

Answer b

nonhomogeneous

Later in this section, we will see some techniques for solving specific types of differential equations. Before we get to that, however, let's get a feel for how solutions to linear differential equations behave. In many cases, solving differential equations depends on making educated guesses about what the solution might look like. Knowing how various types of solutions behave will be helpful.

✓ Example 17.1.2: Verifying a Solution

Consider the linear, homogeneous differential equation

$$x^2y'' - xy' - 3y = 0.$$

Looking at this equation, notice that the coefficient functions are polynomials, with higher powers of x associated with higher-order derivatives of y. Show that $y = x^3$ is a solution to this differential equation.

Solution

Let $y = x^3$. Then $y' = 3x^2$ and $y'' = 6x$. Substituting into the differential equation, we see that

$$\begin{aligned} x^2y'' - xy' - 3y &= x^2(6x) - x(3x^2) - 3(x^3) \\ &= 6x^3 - 3x^3 - 3x^3 \\ &= 0. \end{aligned}$$

Exercise 17.1.2

Show that $y = 2x^2$ is a solution to the differential equation

$$\frac{1}{2}x^2y'' - xy' + y = 0. \tag{17.1.2}$$

Hint

Calculate the derivatives and substitute them into the differential equation.

Answer

This requires calculating y' and y''.

$$y' = \frac{dy}{dx} = 4x$$

and

$$y'' = \frac{dy'}{dx} = 4$$

Inserting these derivatives along with $y = 2x^2$ into Equation 17.1.2.

$$\begin{aligned} \frac{1}{2}x^2y'' - xy' + y &\stackrel{?}{=} 0 \\ \frac{1}{2}x^2(4) - x(4x) + 2x^2 &\stackrel{?}{=} 0 \\ 2x^2 - 4x^2 + 2x^2 &\stackrel{\checkmark}{=} 0 \end{aligned}$$

Yes, this is a solution to the differential equation in Equation 17.1.2.

Although simply finding any solution to a differential equation is important, mathematicians and engineers often want to go beyond finding *one* solution to a differential equation to finding *all* solutions to a differential equation. In other words, we want to find a general solution**.** Just as with first-order differential equations, a general solution (or family of solutions) gives the entire set of solutions to a differential equation. An important difference between first-order and second-order equations is that, with second-order equations, we typically need to find two different solutions to the equation to find the general solution. If we find two solutions, then any linear combination of these solutions is also a solution. We state this fact as the following theorem.

Theorem: SUPERPOSITION PRINCIPLE

If $y_1(x)$ and $y_2(x)$ are solutions to a linear homogeneous differential equation, then the function

$$y(x) = c_1y_1(x) + c_2y_2(x), \tag{17.1.3}$$

where c_1 and c_2 are constants, is also a solution.

The proof of this superposition principle theorem is left as an exercise.

Example 17.1.3: Verifying the Superposition Principle

Consider the differential equation

$$y'' - 4y' - 5y = 0.$$

Given that e^{-x} and e^{5x} are solutions to this differential equation, show that $4e^{-x} + e^{5x}$ is a solution.

Solution

Although this can be done through a simple application of the Superposition principle (Equation 17.1.3), but we can also confirm it is a solution via an approach like in Example 17.1.2. We have

$$\begin{aligned} y(x) &= 4e^{-x} + e^{5x} \\ y'(x) &= -4e^{-x} + 5e^{5x} \\ y''(x) &= 4e^{-x} + 25e^{5x}. \end{aligned}$$

Then

$$\begin{aligned} y'' - 4y' - 5y &\stackrel{?}{=} (4e^{-x} + 25e^{5x}) - 4(-4e^{-x} + 5e^{5x}) - 5(4e^{-x} + e^{5x}) \\ &\stackrel{?}{=} 4e^{-x} + 25e^{5x} + 16e^{-x} - 20e^{5x} - 20e^{-x} - 5e^{5x} \\ &\stackrel{\checkmark}{=} 0. \end{aligned}$$

Thus, $y(x) = 4e^{-x} + e^{5x}$ is a solution.

Exercise 17.1.3

Consider the differential equation

$$y'' + 5y' + 6y = 0.$$

Given that e^{-2x} and e^{-3x} are solutions to this differential equation, show that $3e^{-2x} + 6e^{-3x}$ is a solution.

Hint

Differentiate the function and substitute into the differential equation.

Answer

Although this can be a simple application of the Superposition principle (Equation 17.1.3), we can also set through it like in Example 17.1.2. We have

$$\begin{aligned} y(x) &= 3e^{-2x} + 6e^{-3x} \\ y'(x) &= -6e^{-2x} - 18e^{-3x} \\ y''(x) &= 12e^{-2x} + 54e^{3x}. \end{aligned}$$

Then

$$\begin{aligned} y'' + 5y' + 6y &= (12e^{-2x} + 54e^{3x}) + 5(-6e^{-2x} - 18e^{-3x}) + 6(3e^{-2x} + 6e^{3x}) \\ &\stackrel{?}{=} \cancel{12e^{-2x}} + \cancel{54e^{3x}} - \cancel{30e^{-2x}} - \cancel{90e^{3x}} + \cancel{18e^{-2x}} + \cancel{36e^{3x}} \\ &\stackrel{\checkmark}{=} 0. \end{aligned}$$

Thus, $3e^{-2x} + 6e^{-3x}$ is a solution to the differential equation

Unfortunately, to find the general solution to a second-order differential equation, it is not enough to find any two solutions and then combine them. Consider the differential equation

$$x'' + 7x' + 12x = 0.$$

Both e^{-3t} and $2e^{-3t}$ are solutions (you can check this). However,

$$x(t) = c_1 e^{-3t} + c_2(2e^{-3t})$$

is *not* the general solution. This expression does not account for all solutions to the differential equation. In particular, it fails to account for the function e^{-4t}, which is also a solution to the differential equation. It turns out that to find the general solution to a second-order differential equation, we must find two *linearly independent solutions*. We define that terminology here.

Definition: Linearly Dependent functions

A set of functions $f_1(x),\ f_2(x), \ldots, f_n(x)$ is said to be *linearly dependent* if there are constants $c_1,\ c_2, \ldots c_n,$, not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for all x over the interval of interest. A set of functions that is not linearly dependent is said to be *linearly independent*.

In this chapter, we usually test sets of only two functions for linear independence, which allows us to simplify this definition. From a practical perspective, we see that two functions are linearly dependent if either one of them is identically zero or if they are constant multiples of each other.

First we show that if the functions meet the conditions given previously, then they are linearly dependent. If one of the functions is identically zero—say, $f_2(x) \equiv 0$—then choose $c_1 = 0$ and $c_2 = 1$, and the condition for linear dependence is satisfied. If, on the other hand, neither $f_1(x)$ nor $f_2(x)$ is identically zero, but $f_1(x) = Cf_2(x)$ for some constant C, then choose $c_1 = C$ and $c_2 = -1$, and again, the condition is satisfied.

Next, we show that if two functions are linearly dependent, then either one is identically zero or they are constant multiples of one another. Assume $f_1(x)$ and $f_2(x)$ are linearly independent. Then, there are constants, c_1 and c_2, not both zero, such that

$$c_1 f_1(x) + c_2 f_2(x) = 0$$

for all x over the interval of interest. Then,

$$c_1 f_1(x) = -c_2 f_2(x).$$

Now, since we stated that c_1 and c_2 can't both be zero, assume $c_2 \neq 0$. Then, there are two cases: either $c_1 = 0$ or $c_1 \neq 0$. If $c_1 = 0$, then

$$\begin{aligned} 0 &= -c_2 f_2(x) \\ 0 &= f_2(x), \end{aligned}$$

so one of the functions is identically zero. Now suppose $c_1 \neq 0$. Then,

$$f_1(x) = \left(-\frac{c_2}{c_1}\right) f_2(x)$$

and we see that the functions are constant multiples of one another.

Theorem: Linear Dependence of Two Functions

Two functions, $f_1(x)$ and $f_2(x)$, are said to be linearly dependent if either one of them is identically zero or if $f_1(x) = Cf_2(x)$ for some constant C and for all x over the interval of interest. Functions that are not linearly dependent are said to be *linearly independent*.

Example 17.1.4: Testing for Linear Dependence

Determine whether the following pairs of functions are linearly dependent or linearly independent.

a. $f_1(x) = x^2$ and $f_2(x) = 5x^2$
b. $f_1(x) = \sin x$ and $f_2(x) = \cos x$
c. $f_1(x) = e^{3x}$ and $f_2(x) = e^{-3x}$

d. $f_1(x) = 3x$ and $f_2(x) = 3x + 1$

Solution

a. $f_2(x) = 5f_1(x)$, so the functions are linearly dependent.
b. There is no constant C such that $f_1(x) = Cf_2(x)$, so the functions are linearly independent.
c. There is no constant C such that $f_1(x) = Cf_2(x)$, so the functions are linearly independent. Don't get confused by the fact that the exponents are constant multiples of each other. With two exponential functions, unless the exponents are equal, the functions are linearly independent.
d. There is no constant C such that $f_1(x) = Cf_2(x)$, so the functions are linearly independent.

Exercise 17.1.4

Determine whether the following pairs of functions are linearly dependent or linearly independent: $f_1(x) = e^x$ and $f_2(x) = 3e^{3x}$.

Hint

Are the functions constant multiples of one another?

Answer

Linearly independent

If we are able to find two linearly independent solutions to a second-order differential equation, then we can combine them to find the general solution. This result is formally stated in the following theorem.

Theorem: General Solution to a Homogeneous Equation

If $y_1(x)$ and $y_2(x)$ are linearly independent solutions to a second-order, linear, homogeneous differential equation, then the general solution is given by

$$y(x) = c_1y_1(x) + c_2y_2(x),$$

where c_1 and c_2 are constants.

When we say a family of functions is the general solution to a differential equation, we mean that

1. every expression of that form is a solution and
2. every solution to the differential equation can be written in that form, which makes this theorem extremely powerful.

If we can find two linearly independent solutions to a second order differential equation, we have, effectively, found *all* solutions to the second order differential equation—quite a remarkable statement. The proof of this theorem is beyond the scope of this text.

Example 17.1.5: Writing the General Solution

If $y_1(t) = e^{3t}$ and $y_2(t) = e^{-3t}$ are solutions to $y'' - 9y = 0$, what is the general solution?

Solution

Note that y_1 and y_2 are not constant multiples of one another, so they are linearly independent. Then, the general solution to the differential equation is

$y(t) = c_1e^{3t} + c_2e^{-3t}$.

Exercise 17.1.5

If $y_1(x) = e^{3x}$ and $y_2(x) = xe^{3x}$ are solutions to $y'' - 6y' + 9y = 0$, what is the general solution?

Hint

Check for linear independence first.

Answer

$y(x) = c_1 e^{3x} + c_2 x e^{3x}$

Second-Order Equations with Constant Coefficients

Now that we have a better feel for linear differential equations, we are going to concentrate on solving second-order equations of the form

$$ay'' + by' + cy = 0, \tag{17.2}$$

where $a, b,$ and c are constants.

Since all the coefficients are constants, the solutions are probably going to be functions with derivatives that are constant multiples of themselves. We need all the terms to cancel out, and if taking a derivative introduces a term that is not a constant multiple of the original function, it is difficult to see how that term cancels out. Exponential functions have derivatives that are constant multiples of the original function, so let's see what happens when we try a solution of the form $y(x) = e^{\lambda x}$, where λ (the lowercase Greek letter lambda) is some constant.

If $y(x) = e^{\lambda x}$, then $y'(x) = \lambda e^{\lambda x}$ and $y'' = \lambda^2 e^{\lambda x}$. Substituting these expressions into Equation 17.1.1, we get

$$\begin{aligned} ay'' + by' + cy &= a(\lambda^2 e^{\lambda x}) + b(\lambda e^{\lambda x}) + ce^{\lambda x} \\ &= e^{\lambda x}(a\lambda^2 + b\lambda + c). \end{aligned}$$

Since $e^{\lambda x}$ is never zero, this expression can be equal to zero for all x only if

$$a\lambda^2 + b\lambda + c = 0.$$

We call this the *characteristic equation* of the differential equation.

Definition: characteristic equation

The *characteristic equation* of the second order differential equation $ay'' + by' + cy = 0$ is

$$a\lambda^2 + b\lambda + c = 0.$$

The characteristic equation is very important in finding solutions to differential equations of this form. We can solve the characteristic equation either by factoring or by using the quadratic formula

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This gives three cases. The characteristic equation has

1. distinct real roots;
2. a single, repeated real root; or
3. complex conjugate roots.

We consider each of these cases separately.

Case 1: Distinct Real Roots

If the characteristic equation has distinct real roots λ_1 and λ_2, then $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ are linearly independent solutions to Example 17.1.1, and the general solution is given by

$$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x},$$

where c_1 and c_2 are constants.

For example, the differential equation $y'' + 9y' + 14y = 0$ has the associated characteristic equation $\lambda^2 + 9\lambda + 14 = 0$. This factors into $(\lambda + 2)(\lambda + 7) = 0$, which has roots $\lambda_1 = -2$ and $\lambda_2 = -7$. Therefore, the general solution to this differential equation is

$$y(x) = c_1 e^{-2x} + c_2 e^{-7x}.$$

Case 2: Single Repeated Real Root

Things are a little more complicated if the characteristic equation has a repeated real root, λ. In this case, we know $e^{\lambda x}$ is a solution to Equation 17.1.1, but it is only one solution and we need two linearly independent solutions to determine the general solution. We might be tempted to try a function of the form $ke^{\lambda x}$, where k is some constant, but it would not be linearly independent of $e^{\lambda x}$. Therefore, let's try $xe^{\lambda x}$ as the second solution. First, note that by the quadratic formula,

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

But, λ is a repeated root, so the discriminate $(b^2 - 4ac)$ is zero and $\lambda = \frac{-b}{2a}$. Thus, if $y = xe^{\lambda x}$, we have

$$\begin{aligned} y' &= e^{\lambda x} + \lambda x e^{\lambda x} \\ y'' &= 2\lambda e^{\lambda x} + \lambda^2 x e^{\lambda x}. \end{aligned}$$

Substituting both expressions into Equation 17.1.1, we see that

$$\begin{aligned} ay'' + by' + cy &= a(2\lambda e^{\lambda x} + \lambda^2 x e^{\lambda x}) + b(e^{\lambda x} + \lambda x e^{\lambda x}) + cxe^{\lambda x} \\ &= xe^{\lambda x}(a\lambda^2 + b\lambda + c) + e^{\lambda x}(2a\lambda + b) \\ &= xe^{\lambda x}(0) + e^{\lambda x}(2a(-b2a) + b) \\ &= 0 + e^{\lambda x}(0) \\ &\overset{\checkmark}{=} 0. \end{aligned}$$

This shows that $xe^{\lambda x}$ is a solution to Equation 17.1.1. Since $e^{\lambda x}$ and $xe^{\lambda x}$ are linearly independent, when the characteristic equation has a repeated root λ, the general solution to Equation 17.1.1 is given by

$$y(x) = c_1 e^{\lambda x} + c_2 x e^{\lambda x},$$

where c_1 and c_2 are constants.

For example, the differential equation $y'' + 12y' + 36y = 0$ has the associated characteristic equation

$$\lambda^2 + 12\lambda + 36 = 0.$$

This factors into $(\lambda + 6)^2 = 0$, which has a repeated root $\lambda = -6$. Therefore, the general solution to this differential equation is

$$y(x) = c_1 e^{-6x} + c_2 x e^{-6x}.$$

Case 3: Complex Conjugate Roots

The third case we must consider is when $b^2 - 4ac < 0$. In this case, when we apply the quadratic formula, we are taking the square root of a negative number. We must use the imaginary number $i = \sqrt{-1}$ to find the roots, which take the form $\lambda_1 = \alpha + \beta i$ and $\lambda_2 = \alpha - \beta i$. The **complex number** $\alpha + \beta i$ is called the *conjugate* of $\alpha - \beta i$. Thus, we see that when the discriminate $b^2 - 4ac$ is negative, the roots of our characteristic equation are always **complex conjugates**.

This creates a little bit of a problem for us. If we follow the same process we used for distinct real roots—using the roots of the characteristic equation as the coefficients in the exponents of exponential functions—we get the functions $e^{(\alpha+\beta i)x}$ and $e^{(\alpha-\beta i)x}$ as our solutions. However, there are problems with this approach. First, these functions take on complex (imaginary) values, and a complete discussion of such functions is beyond the scope of this text. Second, even if we were comfortable with complex-value functions, in this course we do not address the idea of a derivative for such functions. So, if possible, we'd like to find two linearly independent *real-value* solutions to the differential equation. For purposes of this development, we are going to manipulate and differentiate the functions $e^{(\alpha+\beta i)x}$ and $e^{(\alpha-\beta i)x}$ as if they were real-value functions. For these particular functions, this approach is valid mathematically, but be aware that there are other instances when complex-value functions do not follow the same rules as real-value functions. Those of you interested in a more in-depth discussion of complex-value functions should consult a complex analysis text.

Based on the roots $\alpha \pm \beta i$ of the characteristic equation, the functions $e^{(\alpha+\beta i)x}$ and $e^{(\alpha-\beta i)x}$ are linearly independent solutions to the differential equation and the general solution is given by

$$y(x) = c_1 e^{(\alpha+\beta i)x} + c_2 e^{(\alpha-\beta i)x}.$$

Using some smart choices for c_1 and c_2, and a little bit of algebraic manipulation, we can find two linearly independent, real-value solutions to Equation 17.1.1 and express our general solution in those terms.

We encountered exponential functions with complex exponents earlier. One of the key tools we used to express these exponential functions in terms of sines and cosines was Euler's formula, which tells us that

$$\underbrace{e^{i\theta} = \cos\theta + i\sin\theta}_{\text{Euler's formula}} \tag{17.1.4}$$

for all real numbers θ.

Going back to the general solution, we have

$$\begin{aligned} y(x) &= c_1 e^{(\alpha+\beta i)x} + c_2 e^{(\alpha-\beta i)x} \\ &= c_1 e^{\alpha x} e^{\beta i x} + c_2 e^{\alpha x} e^{-\beta i x} \\ &= e^{\alpha x}(c_1 e^{\beta i x} + c_2 e^{-\beta i x}). \end{aligned}$$

Applying Euler's formula (Equation 17.1.4) together with the identities $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$, we get

$$\begin{aligned} y(x) &= e^{\alpha x}[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos(-\beta x) + i\sin(-\beta x))] \\ &= e^{\alpha x}[(c_1 + c_2)\cos\beta x + (c_1 - c_2)i\sin\beta x]. \end{aligned} \tag{17.1.5}$$

Now, if we choose $c_1 = c_2 = \frac{1}{2}$, the second term is zero and we get

$$y(x) = e^{\alpha x}\cos\beta x$$

as a real-value solution to Equation 17.1.1. Similarly, if we choose $c_1 = -\frac{i}{2}$ and $c_2 = \frac{i}{2}$, the first term of Equation 17.1.5 is zero and we get

$$y(x) = e^{\alpha x}\sin\beta x$$

as a second, linearly independent, real-value solution to Equation 17.1.1.

Based on this, we see that if the characteristic equation has complex conjugate roots $\alpha \pm \beta i$, then the general solution to Equation 17.1.1 is given by

$$\begin{aligned} y(x) &= c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x \\ &= e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x), \end{aligned}$$

where c_1 and c_2 are constants.

For example, the differential equation $y'' - 2y' + 5y = 0$ has the associated characteristic equation $\lambda^2 - 2\lambda + 5 = 0$. By the quadratic formula, the roots of the characteristic equation are $1 \pm 2i$. Therefore, the general solution to this differential equation is

$$y(x) = e^x(c_1\cos 2x + c_2\sin 2x).$$

Summary of Results

We can solve second-order, linear, homogeneous differential equations with constant coefficients by finding the roots of the associated characteristic equation. The form of the general solution varies, depending on whether the characteristic equation has distinct, real roots; a single, repeated real root; or complex conjugate roots. The three cases are summarized in Table 17.1.1.

Table 17.1.1: Summary of Characteristic Equation Cases

Characteristic Equation Roots	General Solution to the Differential Equation
Distinct real roots, λ_1 and λ_2	$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
A repeated real root, λ	$y(x) = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$
Complex conjugate roots $\alpha \pm \beta i$	$y(x) = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$

PROBLEM-SOLVING STRATEGY: USING THE CHARACTERISTIC EQUATION TO SOLVE SECOND-ORDER DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

1. Write the differential equation in the form $a'' + by' + cy = 0$.
2. Find the corresponding characteristic equation $a\lambda^2 + b\lambda + c = 0$.
3. Either factor the characteristic equation or use the quadratic formula to find the roots.
4. Determine the form of the general solution based on whether the characteristic equation has distinct, real roots; a single, repeated real root; or complex conjugate roots.

Example 17.1.6: Solving Second-Order Equations with Constant Coefficients

Find the general solution to the following differential equations. Give your answers as functions of x.

a. $y'' + 3y' - 4y = 0$
b. $y'' + 6y' + 13y = 0$
c. $y'' + 2y' + y = 0$
d. $y'' - 5y' = 0$
e. $y'' - 16y = 0$
f. $y'' + 16y = 0$

Solution

Note that all these equations are already given in standard form (step 1).

1. The characteristic equation is $\lambda^2 + 3\lambda - 4 = 0$ (step 2). This factors into $(\lambda + 4)(\lambda - 1) = 0$, so the roots of the characteristic equation are $\lambda_1 = -4$ and $\lambda_2 = 1$ (step 3). Then the general solution to the differential equation is

$$y(x) = c_1 e^{-4x} + c_2 e^{x}. \qquad \text{(step 1)}$$

2. The characteristic equation is $\lambda^2 + 6\lambda + 13 = 0$ (step 2). Applying the quadratic formula, we see this equation has complex conjugate roots $-3 \pm 2i$ (step 3). Then the general solution to the differential equation is

$$y(t) = e^{-3t}(c_1 \cos 2t + c_2 \sin 2t). \qquad \text{(step 2)}$$

3. The characteristic equation is $\lambda^2 + 2\lambda + 1 = 0$ (step 2). This factors into $(\lambda + 1)2 = 0$, so the characteristic equation has a repeated real root $\lambda = -1$ (step 3). Then the general solution to the differential equation is

$$y(t) = c_1 e^{-t} + c_2 t e^{-t}. \qquad \text{(step 3)}$$

4. The characteristic equation is $\lambda^2 - 5\lambda$ (step 2). This factors into $\lambda(\lambda - 5) = 0$, so the roots of the characteristic equation are $\lambda_1 = 0$ and $\lambda_2 = 5$ (step 3). Note that $e^{0x} = e^0 = 1$, so our first solution is just a constant. Then the general solution to the differential equation is

$$y(x) = c_1 + c_2 e^{5x}. \qquad \text{(step 4)}$$

5. The characteristic equation is $\lambda^2 - 16 = 0$ (step 2). This factors into $(\lambda + 4)(\lambda - 4) = 0$, so the roots of the characteristic equation are $\lambda_1 = 4$ and $\lambda_2 = -4$ (step 3). Then the general solution to the differential equation is

$$y(x) = c_1 e^{4x} + c_2 e^{-4x}. \qquad \text{(step 5)}$$

6. The characteristic equation is $\lambda^2 + 16 = 0$ (step 2). This has complex conjugate roots $\pm 4i$ (step 3). Note that $e^{0x} = e^0 = 1$, so the exponential term in our solution is just a constant. Then the general solution to the differential equation is

$$y(t) = c_1 \cos 4t + c_2 \sin 4t. \qquad \text{(step 6)}$$

Exercise 17.1.6

Find the general solution to the following differential equations:

a. $y'' - 2y' + 10y = 0$

b. $y'' + 14y' + 49y = 0$

Hint

Find the roots of the characteristic equation.

Answer a

$y(x) = e^x(c_1 \cos 3x + c_2 \sin 3x)$

Answer b

$y(x) = c_1 e^{-7x} + c_2 x e^{-7x}$

Initial-Value Problems and Boundary-Value Problems

So far, we have been finding general solutions to differential equations. However, differential equations are often used to describe physical systems, and the person studying that physical system usually knows something about the state of that system at one or more points in time. For example, if a constant-coefficient differential equation is representing how far a motorcycle shock absorber is compressed, we might know that the rider is sitting still on his motorcycle at the start of a race, time $t = t_0$. This means the system is at equilibrium, so $y(t_0) = 0$, and the compression of the shock absorber is not changing, so $y'(t_0) = 0$. With these two initial conditions and the general solution to the differential equation, we can find the *specific* solution to the differential equation that satisfies both initial conditions. This process is known as *solving an initial-value problem*. (Recall that we discussed initial-value problems in Introduction to Differential Equations.) Note that second-order equations have two arbitrary constants in the general solution, and therefore we require two initial conditions to find the solution to the initial-value problem.

Sometimes we know the condition of the system at two different times. For example, we might know $y(t_0) = y_0$ and $y(t_1) = y_1$. These conditions are called boundary conditions, and finding the solution to the differential equation that satisfies the boundary conditions is called solving a boundary-value problem.

Mathematicians, scientists, and engineers are interested in understanding the conditions under which an initial-value problem or a boundary-value problem has a unique solution. Although a complete treatment of this topic is beyond the scope of this text, it is useful to know that, within the context of constant-coefficient, second-order equations, initial-value problems are guaranteed to have a unique solution as long as two initial conditions are provided. Boundary-value problems, however, are not as well behaved. Even when two boundary conditions are known, we may encounter boundary-value problems with unique solutions, many solutions, or no solution at all.

Example 17.1.7: Solving an Initial-Value Problem

Solve the following initial-value problem: $y'' + 3y' - 4y = 0,\ y(0) = 1,\ y'(0) = -9.$

Solution

We already solved this differential equation in Example 17.6a. and found the general solution to be

$$y(x) = c_1 e^{-4x} + c_2 e^x.$$

Then

$$y'(x) = -4c_1 e^{-4x} + c_2 e^x.$$

When $x = 0$, we have $y(0) = c_1 + c_2$ and $y'(0) = -4c_1 + c_2$. Applying the initial conditions, we have

$$c_1 + c_2 = 1$$
$$-4c_1 + c_2 = -9.$$

Then $c_1 = 1 - c_2$. Substituting this expression into the second equation, we see that

$$-4(1-c_2)+c_2=-9$$
$$-4+4c_2+c_2=-9$$
$$5c_2=-5$$
$$c_2=-1.$$

So, $c_1=2$ and the solution to the initial-value problem is

$$y(x)=2e^{-4x}-e^{x}.$$

Exercise 17.1.7

Solve the initial-value problem $y''-3y'-10y=0, \quad y(0)=0,\ y'(0)=7.$

Hint

Use the initial conditions to determine values for c_1 and c_2.

Answer

$$y(x)=-e^{-2x}+e^{5x}$$

Example 17.1.8: Solving an Initial-Value Problem and Graphing the Solution

Solve the following initial-value problem and graph the solution:

$$y''+6y'+13y=0, \quad y(0)=0,\ y'(0)=2$$

Solution

We already solved this differential equation in Example 17.1.6*b*. and found the general solution to be

$$y(x)=e^{-3x}(c_1\cos 2x+c_2\sin 2x).$$

Then

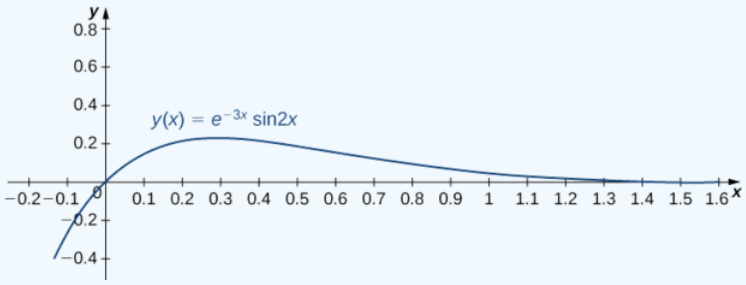
$$y'(x)=e^{-3x}(-2c_1\sin 2x+2c_2\cos 2x)-3e^{-3x}(c_1\cos 2x+c_2\sin 2x).$$

When $x=0$, we have $y(0)=c_1$ and $y'(0)=2c_2-3c_1$. Applying the initial conditions, we obtain

$$c_1=0$$
$$-3c_1+2c_2=2.$$

Therefore, $c_1=0$, $c_2=1$, and the solution to the initial value problem is shown in the following graph.

$$y=e^{-3x}\sin 2x.$$



Exercise 17.1.8

Solve the following initial-value problem and graph the solution: $y''-2y'+10y=0, \quad y(0)=2,\ y'(0)=-1$

Hint

Use the initial conditions to determine values for c_1 and c_2.

Answer

$$y(x) = e^x(2\cos 3x - \sin 3x)$$

Example 17.1.9: Initial-Value Problem Representing a Spring-Mass System

The following initial-value problem models the position of an object with mass attached to a spring. Spring-mass systems are examined in detail in Applications. The solution to the differential equation gives the position of the mass with respect to a neutral (equilibrium) position (in meters) at any given time. (Note that for spring-mass systems of this type, it is customary to define the downward direction as positive.)

$$y'' + 2y' + y = 0, \quad y(0) = 1,\ y'(0) = 0$$

Solve the initial-value problem and graph the solution. What is the position of the mass at time $t = 2$ sec? How fast is the mass moving at time $t = 1$ sec? In what direction?

Solution

In Example Example 17.1.6c. we found the general solution to this differential equation to be

$$y(t) = c_1e^{-t} + c_2te^{-t}.$$

Then

$$y'(t) = -c_1e^{-t} + c_2(-te^{-t} + e^{-t}).$$

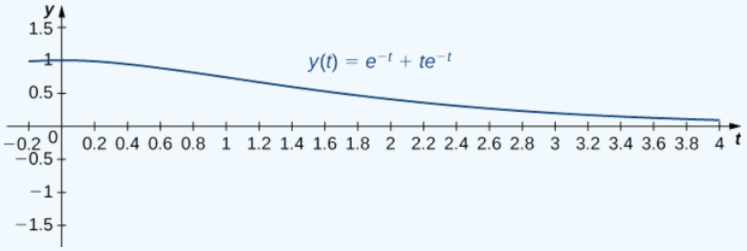
When $t = 0$, we have $y(0) = c_1$ and $y'(0) = c_1 + c_2$. Applying the initial conditions, we obtain

$$\begin{aligned} c_1 &= 1 \\ -c_1 + c_2 &= 0. \end{aligned}$$

Thus, $c_1 = 1, c_2 = 1$, and the solution to the initial value problem is

$$y(t) = e^{-t} + te^{-t}.$$

This solution is represented in the following graph. At time $t = 2$, the mass is at position $y(2) = e^{-2} + 2e^{-2} = 3e^{-2} \approx 0.406$ m below equilibrium.



To calculate the velocity at time $t = 1$, we need to find the derivative. We have $y(t) = e^{-t} + te^{-t}$, so

$$y'(t) = -e^{-t} + e^{-t} - te^{-t} = -te^{-t}.$$

Then $y'(1) = -e^{-1} \approx -0.3679$. At time $t = 1$, the mass is moving upward at 0.3679 m/sec.

Exercise 17.1.9

Suppose the following initial-value problem models the position (in feet) of a mass in a spring-mass system at any given time. Solve the initial-value problem and graph the solution. What is the position of the mass at time $t = 0.3$ sec? How fast is it moving at time $t = 0.1$ sec? In what direction?

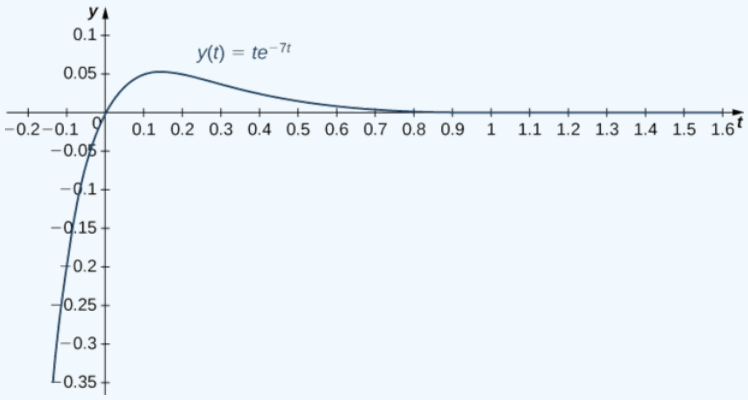
$$y'' + 14y' + 49y = 0, \quad y(0) = 0,\ y'(0) = 1$$

Hint

Use the initial conditions to determine values for c_1 and c_2.

Answer

$$y(t) = te^{-7t}$$



At time $t = 0.3,\ y(0.3) = 0.3e^{(-7*0.3)} = 0.3e^{-2.1} \approx 0.0367$. The mass is 0.0367ft below equilibrium. At time $t = 0.1,\ y'(0.1) = 0.3e^{-0.7} \approx 0.1490$. The mass is moving downward at a speed of 0.1490 ft/sec.

Example 17.1.10: Solving a Boundary-Value Problem

In Example 17.6f. we solved the differential equation $y'' + 16y = 0$ and found the general solution to be $y(t) = c_1 \cos 4t + c_2 \sin 4t$. If possible, solve the boundary-value problem if the boundary conditions are the following:

a. $y(0) = 0, y(\frac{\pi}{4}) = 0$
b. $y(0) = 1, y(0) = 1, y(\frac{\pi}{8}) = 0$
c. $y(\frac{\pi}{8}) = 0, y(\frac{3\pi}{8}) = 2$

Solution

We have

$$y(x) = c_1 \cos 4t + c_2 \sin 4t.$$

1. Applying the first boundary condition given here, we get $y(0) = c_1 = 0$. So the solution is of the form $y(t) = c_2 \sin 4t$. When we apply the second boundary condition, though, we get $y(\frac{\pi}{4}) = c_2 \sin(4(\frac{\pi}{4})) = c_2 \sin \pi = 0$ for all values of c_2. The boundary conditions are not sufficient to determine a value for c_2, so this boundary-value problem has infinitely many solutions. Thus, $y(t) = c_2 \sin 4t$ is a solution for any value of c_2.
2. Applying the first boundary condition given here, we get $y(0) = c_1 = 1$. Applying the second boundary condition gives $y(\frac{\pi}{8}) = c_2 = 0$, so $c_2 = 0$. In this case, we have a unique solution: $y(t) = \cos 4t$.
3. Applying the first boundary condition given here, we get $y(\frac{\pi}{8}) = c_2 = 0$. However, applying the second boundary condition gives $y(\frac{3\pi}{8}) = -c_2 = 2$, so $c_2 = -2$. We cannot have $c_2 = 0 = -2$, so this boundary value problem has no solution.

Key Concepts

- Second-order differential equations can be classified as linear or nonlinear, homogeneous or nonhomogeneous.
- To find a general solution for a homogeneous second-order differential equation, we must find two linearly independent solutions. If $y_1(x)$ and $y_2(x)$ are linearly independent solutions to a second-order, linear, homogeneous differential equation, then the general solution is given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

- To solve homogeneous second-order differential equations with constant coefficients, find the roots of the characteristic equation. The form of the general solution varies depending on whether the characteristic equation has distinct, real roots; a

single, repeated real root; or complex conjugate roots.
- Initial conditions or boundary conditions can then be used to find the specific solution to a differential equation that satisfies those conditions, except when there is no solution or infinitely many solutions.

Key Equations

- **Linear second-order differential equation**

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = r(x)$$

- **Second-order equation with constant coefficients**

$$ay'' + by' + cy = 0$$

Glossary

boundary conditions
the conditions that give the state of a system at different times, such as the position of a spring-mass system at two different times

boundary-value problem
a differential equation with associated boundary conditions

characteristic equation
the equation $a\lambda^2 + b\lambda + c = 0$ for the differential equation $ay''+by'+cy = 0$

homogeneous linear equation
a second-order differential equation that can be written in the form $a_2(x)y''+a_1(x)y'+a_0(x)y = r(x)$, but $r(x) = 0$ for every value of x

nonhomogeneous linear equation
a second-order differential equation that can be written in the form $a_2(x)y''+a_1(x)y'+a_0(x)y = r(x)$, but $r(x) \neq 0$ for some value of x

linearly dependent
a set of functions $f_1(x),\ f_2(x),\ \ldots,\ f_n(x)$ for whichthere are constants $c_1,\ c_2,\ \ldots,\ c_n$, not all zero, such that $c_1f_1(x) + c_2f_2(x) + \cdots + c_nf_n(x) = 0$ for all x in the interval of interest

linearly independent
a set of functions $f_1(x),\ f_2(x),\ \ldots,\ f_n(x)$ for which there are no constants $c_1,\ c_2,\ \ldots,\ c_n$, such that $c_1f_1(x) + c_2f_2(x) + \cdots + c_nf_n(x) = 0$ for all x in the interval of interest

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17.1E: Exercises for Section 17.1

In exercises 1 - 6, classify each of the following equations as linear or nonlinear. If the equation is linear, determine whether it is homogeneous or nonhomogeneous.

1. $x^3y'' + (x-1)y' - 8y = 0$

Answer

linear, homogenous

2. $(1+y^2)y'' + xy' - 3y = \cos x$

3. $xy'' + e^y y' = x$

Answer

nonlinear

4. $y'' + \dfrac{4}{x}y' - 8xy = 5x^2 + 1$

5. $y'' + (\sin x)y' - xy = 4y$

Answer

linear, homogenous

6. $y'' + \left(\dfrac{x+3}{y}\right)y' = 0$

In exercises 7 - 10, verify that the given function is a solution to the differential equation. Use a graphing utility to graph the particular solutions for several values of c_1 and c_2. What do the solutions have in common?

7. **[T]** $y'' + 2y' - 3y = 0; \quad y(x) = c_1e^x + c_2e^{-3x}$

8. **[T]** $x^2y'' - 2y - 3x^2 + 1 = 0; \quad y(x) = c_1x^2 + c_2x^{-1} + x^2\ln(x) + \frac{1}{2}$

9. **[T]** $y'' + 14y + 49y = 0; \quad y(x) = c_1e^{-7x} + c_2xe^{-7x}$

10. **[T]** $6y'' - 49y' + 8y = 0; \quad y(x) = c_1e^{x/6} + c_2e^{8x}$

In exercises 11 - 30, find the general solution to the linear differential equation.

11. $y'' - 3y' - 10y = 0$

Answer

$y = c_1e^{5x} + c_2e^{-2x}$

12. $y'' - 7y' + 12y = 0$

13. $y'' + 4y' + 4y = 0$

Answer

$y = c_1e^{-2x} + c_2xe^{-2x}$

14. $4y'' - 12y' + 9y = 0$

15. $2y'' - 3y' - 5y = 0$

Answer

$y = c_1e^{5x/2} + c_2e^{-x}$

16. $3y'' - 14y' + 8y = 0$

17. $y'' + y' + y = 0$

Answer

$y = e^{-x/2}\left(c_1 \cos \frac{\sqrt{3}x}{2} + c_2 \sin \frac{\sqrt{3}x}{2}\right)$

18. $5y'' + 2y' + 4y = 0$

19. $y'' - 121y = 0$

Answer

$y = c_1 e^{-11x} + c_2 e^{11x}$

20. $8y'' + 14y' - 15y = 0$

21. $y'' + 81y = 0$

Answer

$y = c_1 \cos 9x + c_2 \sin 9x$

22. $y'' - y' + 11y = 0$

23. $2y'' = 0$

Answer

$y = c_1 + c_2 x$

24. $y'' - 6y' + 9y = 0$

25. $3y'' - 2y' - 7y = 0$

Answer

$y = c_1 e^{\left((1+\sqrt{22})/3\right)x} + c_2 e^{\left((1-\sqrt{22})/3\right)x}$

26. $4y'' - 10y' = 0$

27. $36\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + y = 0$

Answer

$y = c_1 e^{-x/6} + c_2 x e^{-x/6}$

28. $25\frac{d^2y}{dx^2} - 80\frac{dy}{dx} + 64y = 0$

29. $\frac{d^2y}{dx^2} - 9\frac{dy}{dx} = 0$

Answer

$y = c_1 + c_2 e^{9x}$

30. $4\frac{d^2y}{dx^2} + 8y = 0$

In exercises 31 - 38, solve the initial-value problem.

31. $y'' + 5y' + 6y = 0, \quad y(0) = 0,\ y'(0) = -2$

Answer

$y = -2e^{-2x} + 2e^{-3x}$

32. $y'' + 2y' - 8y = 0, \quad y(0) = 5,\ y'(0) = 4$

33. $y'' + 4y = 0, \quad y(0) = 3,\ y'(0) = 10$

Answer

$y = 3\cos(2x) + 5\sin(2x)$

34. $y'' - 18y' + 81y = 0, \quad y(0) = 1,\ y'(0) = 5$

35. $y'' - y' - 30y = 0, \quad y(0) = 1,\ y'(0) = -16$

Answer

$y = -e^{6x} + 2e^{-5x}$

36. $4y'' + 4y' - 8y = 0, \quad y(0) = 2,\ y'(0) = 1$

37. $25y'' + 10y' + y = 0, \quad y(0) = 2,\ y'(0) = 1$

Answer

$y = 2e^{-x/5} + \frac{7}{5}xe^{-x/5}$

38. $y'' + y = 0, \quad y(\pi) = 1,\ y'(\pi) = -5$

In exercises 39 - 46, solve the boundary-value problem, if possible.

39. $y'' + y' - 42y = 0, \quad y(0) = 0,\ y(1) = 2$

Answer

$y = \left(\frac{2}{e^6 - e^{-7}}\right)e^{6x} - \left(\frac{2}{e^6 - e^{-7}}\right)e^{-7x}$

40. $9y'' + y = 0, \quad y(3\pi^2) = 6,\ y(0) = -8$

41. $y'' + 10y' + 34y = 0, \quad y(0) = 6,\ y(\pi) = 2$

Answer

No solutions exist.

42. $y'' + 7y' - 60y = 0, \quad y(0) = 4,\ y(2) = 0$

43. $y'' - 4y' + 4y = 0, \quad y(0) = 2,\ y(1) = -1$

Answer

$y = 2e^{2x} - \left(\frac{2e^2 + 1}{e^2}\right)xe^{2x}$

44. $y'' - 5y' = 0, \quad y(0) = 3,\ y(-1) = 2$

45. $y'' + 9y = 0, \quad y(0) = 4,\ y(\pi^3) = -4$

Answer

$y = 4\cos 3x + c_2 \sin 3x$, infinitely many solutions

46. $4y'' + 25y = 0, \quad y(0) = 2,\ y(2\pi) = -2$

47. Find a differential equation with a general solution that is $y = c_1e^{x/5} + c_2e^{-4x}$.

Answer

$5y'' + 19y' - 4y = 0$

48. Find a differential equation with a general solution that is $y = c_1e^{x} + c_2e^{-4x/3}$.

For each differential equation in exercises 49 - 51:

a. **Solve the initial value problem.**

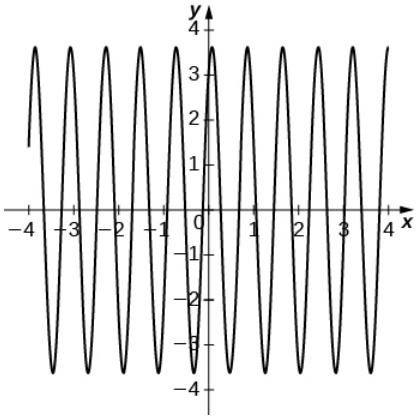
b. **[T] Use a graphing utility to graph the particular solution.**

49. $y'' + 64y = 0; \quad y(0) = 3,\ y'(0) = 16$

Answer

a. $y = 3\cos 8x + 2\sin 8x$

b.



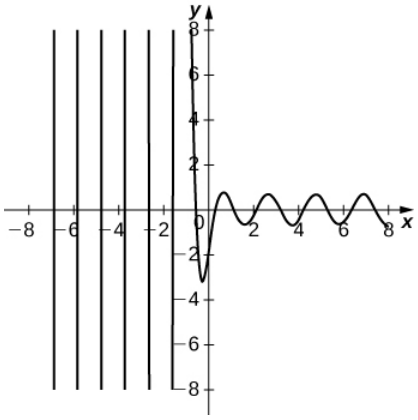
50. $y'' - 2y' + 10y = 0; \quad y(0) = 1,\ y'(0) = 13$

51. $y'' + 5y' + 15y = 0; \quad y(0) = -2,\ y'(0) = 7$

Answer

a. $y = e^{-5/2}\left[-2\cos\left(\frac{\sqrt{35}}{2}x\right) + \frac{4\sqrt{35}}{35}\sin\left(\frac{\sqrt{35}}{2}x\right)\right]$

b.



52. (Principle of superposition) Prove that if $y_1(x)$ and $y_2(x)$ are solutions to a linear homogeneous differential equation, $y'' + p(x)y' + q(x)y = 0$, then the function $y(x) = c_1y_1(x) + c_2y_2(x)$, where c_1 and c_2 are constants, is also a solution.

53. Prove that if a, b and c are positive constants, then all solutions to the second-order linear differential equation $ay'' + by' + cy = 0$ approach zero as $x \to \infty$. (*Hint:* Consider three cases: two distinct roots, repeated real roots, and complex conjugate roots.)

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17.2: Nonhomogeneous Linear Equations

Learning Objectives

- Write the general solution to a nonhomogeneous differential equation.
- Solve a nonhomogeneous differential equation by the method of undetermined coefficients.
- Solve a nonhomogeneous differential equation by the method of variation of parameters.

In this section, we examine how to solve nonhomogeneous differential equations. The terminology and methods are different from those we used for homogeneous equations, so let's start by defining some new terms.

General Solution to a Nonhomogeneous Linear Equation

Consider the nonhomogeneous linear differential equation

$$a_2(x)y''+a_1(x)y'+a_0(x)y = r(x).$$

The associated homogeneous equation

$$a_2(x)y''+a_1(x)y'+a_0(x)y = 0$$

is called the *complementary equation*. We will see that solving the complementary equation is an important step in solving a nonhomogeneous differential equation.

Definition: Particular Solution

A solution $y_p(x)$ of a differential equation that contains no arbitrary constants is called a *particular solution* to the equation.

GENERAL Solution TO A NONHOMOGENEOUS EQUATION

Let $y_p(x)$ be any particular solution to the nonhomogeneous linear differential equation

$$a_2(x)y''+a_1(x)y'+a_0(x)y = r(x).$$

Also, let $c_1y_1(x) + c_2y_2(x)$ denote the general solution to the complementary equation. Then, the *general solution* to the nonhomogeneous equation is given by

$$y(x) = c_1y_1(x) + c_2y_2(x) + y_p(x).$$

Proof

To prove $y(x)$ is the general solution, we must first show that it solves the differential equation and, second, that any solution to the differential equation can be written in that form. Substituting $y(x)$ into the differential equation, we have

$$\begin{aligned} a_2(x)y''+a_1(x)y'+a_0(x)y &= a_2(x)(c_1y_1 + c_2y_2 + y_p)''+a_1(x)(c_1y_1 + c_2y_2 + y_p)' \\ &\quad +a_0(x)(c_1y_1 + c_2y_2 + y_p) \\ &= [a_2(x)(c_1y_1 + c_2y_2)''+a_1(x)(c_1y_1 + c_2y_2)'+a_0(x)(c_1y_1 + c_2y_2)] \\ &\quad +a_2(x)y_p''+a_1(x)y_p'+a_0(x)y_p \\ &= 0 + r(x) \\ &= r(x). \end{aligned}$$

So $y(x)$ is a solution.

Now, let $z(x)$ be any solution to $a_2(x)y'' + a_1(x)y'+a_0(x)y = r(x)$. Then

$$\begin{aligned} a_2(x)(z - y_p)''+a_1(x)(z - y_p)'+a_0(x)(z - y_p) &= (a_2(x)z''+a_1(x)z'+a_0(x)z) \\ &\quad -(a_2(x)y_p''+a_1(x)y_p'+a_0(x)y_p) \\ &= r(x) - r(x) \\ &= 0, \end{aligned}$$

so $z(x) - y_p(x)$ is a solution to the complementary equation. But, $c_1y_1(x) + c_2y_2(x)$ is the general solution to the complementary equation, so there are constants c_1 and c_2 such that

$$z(x) - y_p(x) = c_1y_1(x) + c_2y_2(x).$$

Hence, we see that

$$z(x) = c_1y_1(x) + c_2y_2(x) + y_p(x).$$

Example 17.2.1: Verifying the General Solution

Given that $y_p(x) = x$ is a particular solution to the differential equation $y''+y = x$, write the general solution and check by verifying that the solution satisfies the equation.

Solution

The complementary equation is $y''+y = 0$, which has the general solution $c_1 \cos x + c_2 \sin x$. So, the general solution to the nonhomogeneous equation is

$$y(x) = c_1 \cos x + c_2 \sin x + x.$$

To verify that this is a solution, substitute it into the differential equation. We have

$$y'(x) = -c_1 \sin x + c_2 \cos x + 1$$

and

$$y''(x) = -c_1 \cos x - c_2 \sin x.$$

Then

$$\begin{aligned} y''(x) + y(x) &= -c_1 \cos x - c_2 \sin x + c_1 \cos x + c_2 \sin x + x \\ &= x. \end{aligned}$$

So, $y(x)$ is a solution to $y''+y = x$.

Exercise 17.2.1

Given that $y_p(x) = -2$ is a particular solution to $y''-3y'-4y = 8$, write the general solution and verify that the general solution satisfies the equation.

Hint

Find the general solution to the complementary equation.

Answer

$y(x) = c_1e^{-x} + c_2e^{4x} - 2$

In the preceding section, we learned how to solve homogeneous equations with constant coefficients. Therefore, for nonhomogeneous equations of the form $ay''+by'+cy = r(x)$, we already know how to solve the complementary equation, and the problem boils down to finding a particular solution for the nonhomogeneous equation. We now examine two techniques for this: the method of undetermined coefficients and the method of variation of parameters.

Undetermined Coefficients

The method of *undetermined coefficients* involves making educated guesses about the form of the particular solution based on the form of $r(x)$. When we take derivatives of polynomials, exponential functions, sines, and cosines, we get polynomials, exponential functions, sines, and cosines. So when $r(x)$ has one of these forms, it is possible that the solution to the nonhomogeneous differential equation might take that same form. Let's look at some examples to see how this works.

Example 17.2.2: Undetermined Coefficients When $r(x)$ Is a Polynomial

Find the general solution to $y''+4y'+3y=3x$.

Solution

The complementary equation is $y''+4y'+3y=0$, with general solution $c_1e^{-x}+c_2e^{-3x}$. Since $r(x)=3x$, the particular solution might have the form $y_p(x)=Ax+B$. If this is the case, then we have $y_p'(x)=A$ and $y_p''(x)=0$. For y_p to be a solution to the differential equation, we must find values for A and B such that

$$\begin{aligned} y''+4y'+3y &= 3x \\ 0+4(A)+3(Ax+B) &= 3x \\ 3Ax+(4A+3B) &= 3x. \end{aligned}$$

Setting coefficients of like terms equal, we have

$$\begin{aligned} 3A &= 3 \\ 4A+3B &= 0. \end{aligned}$$

Then, $A=1$ and $B=-\frac{4}{3}$, so $y_p(x)=x-\frac{4}{3}$ and the general solution is

$$y(x)=c_1e^{-x}+c_2e^{-3x}+x-\frac{4}{3}.$$

In Example 17.2.2, notice that even though $r(x)$ did not include a constant term, it was necessary for us to include the constant term in our guess. If we had assumed a solution of the form $y_p=Ax$ (with no constant term), we would not have been able to find a solution. (Verify this!) If the function $r(x)$ is a polynomial, our guess for the particular solution should be a polynomial of the same degree, and it must include all lower-order terms, regardless of whether they are present in $r(x)$.

Example 17.2.3: Undetermined Coefficients When $r(x)$ Is an Exponential

Find the general solution to $y''-y'-2y=2e^{3x}$.

Solution

The complementary equation is $y''-y'-2y=0$, with the general solution $c_1e^{-x}+c_2e^{2x}$. Since $r(x)=2e^{3x}$, the particular solution might have the form $y_p(x)=Ae^{3x}$. Then, we have $yp'(x)=3Ae^{3x}$ and $y_p''(x)=9Ae^{3x}$. For y_p to be a solution to the differential equation, we must find a value for A such that

$$\begin{aligned} y''-y'-2y &= 2e^{3x} \\ 9Ae^{3x}-3Ae^{3x}-2Ae^{3x} &= 2e^{3x} \\ 4Ae^{3x} &= 2e^{3x}. \end{aligned}$$

So, $4A=2$ and $A=1/2$. Then, $y_p(x)=(\frac{1}{2})e^{3x}$, and the general solution is

$$y(x)=c_1e^{-x}+c_2e^{2x}+\frac{1}{2}e^{3x}.$$

Exercise 17.2.2

Find the general solution to $y''-4y'+4y=7\sin t-\cos t$.

Hint

Use $y_p(t)=A\sin t+B\cos t$ as a guess for the particular solution.

Answer

$y(t)=c_1e^{2t}+c_2te^{2t}+\sin t+\cos t$

In the previous checkpoint, $r(x)$ included both sine and cosine terms. However, even if $r(x)$ included a sine term only or a cosine term only, both terms must be present in the guess. The method of undetermined coefficients also works with products of

polynomials, exponentials, sines, and cosines. Some of the key forms of $r(x)$ and the associated guesses for $y_p(x)$ are summarized in Table 17.2.1.

Table 17.2.1: Key Forms for the Method of Undetermined Coefficients

$r(x)$	**Initial guess for** $y_p(x)$
k (a constant)	A (a constant)
$ax+b$	$Ax+B$ (*Note*: The guess must include both terms even if $b=0$.)
ax^2+bx+c	Ax^2+Bx+C (*Note*: The guess must include all three terms even if b or c are zero.)
Higher-order polynomials	Polynomial of the same order as $r(x)$
$ae^{\lambda x}$	$Ae^{\lambda x}$
$a\cos\beta x+b\sin\beta x$	$A\cos\beta x+B\sin\beta x$ (*Note*: The guess must include both terms even if either $a=0$ or $b=0$.)
$ae^{\alpha x}\cos\beta x+be^{\alpha x}\sin\beta x$	$Ae^{\alpha x}\cos\beta x+Be^{\alpha x}\sin\beta x$
$(ax^2+bx+c)e^{\lambda x}$	$(Ax^2+Bx+C)e^{\lambda x}$
$(a_2x^2+a_1x+a0)\cos\beta x$ $+(b_2x^2+b_1x+b_0)\sin\beta x$	$(A_2x^2+A_1x+A_0)\cos\beta x$ $+(B_2x^2+B_1x+B_0)\sin\beta x$
$(a_2x^2+a_1x+a_0)e^{\alpha x}\cos\beta x$ $+(b_2x^2+b_1x+b_0)e^{\alpha x}\sin\beta x$	$(A_2x^2+A_1x+A_0)e^{\alpha x}\cos\beta x$ $+(B_2x^2+B_1x+B_0)e^{\alpha x}\sin\beta x$

Keep in mind that there is a key pitfall to this method. Consider the differential equation $y''+5y'+6y=3e^{-2x}$. Based on the form of $r(x)$, we guess a particular solution of the form $y_p(x)=Ae^{-2x}$. But when we substitute this expression into the differential equation to find a value for A,we run into a problem. We have

$$y_p{}'(x)=-2Ae^{-2x}$$

and

$$y_p''=4Ae^{-2x},$$

so we want

$$\begin{aligned} y''+5y'+6y &= 3e^{-2x} \\ 4Ae^{-2x}+5(-2Ae^{-2x})+6Ae^{-2x} &= 3e^{-2x} \\ 4Ae^{-2x}-10Ae^{-2x}+6Ae^{-2x} &= 3e^{-2x} \\ 0 &= 3e^{-2x}, \end{aligned}$$

which is not possible.

Looking closely, we see that, in this case, the general solution to the complementary equation is $c_1e^{-2x}+c_2e^{-3x}$. The exponential function in $r(x)$ is actually a solution to the complementary equation, so, as we just saw, all the terms on the left side of the equation cancel out. We can still use the method of undetermined coefficients in this case, but we have to alter our guess by multiplying it by x. Using the new guess, $y_p(x)=Axe^{-2x}$, we have

$$y_p{}'(x)=A(e^{-2x}-2xe^{-2x}$$

and

$$y_p''(x)=-4Ae^{-2x}+4Axe^{-2x}.$$

Substitution gives

$$y''+5y'+6y = 3e^{-2x}$$
$$(-4Ae^{-2x} + 4Axe^{-2x}) + 5(Ae^{-2x} - 2Axe^{-2x}) + 6Axe^{-2x} = 3e^{-2x}$$
$$-4Ae^{-2x} + 4Axe^{-2x} + 5Ae^{-2x} - 10Axe^{-2x} + 6Axe^{-2x} = 3e^{-2x}$$
$$Ae^{-2x} = 3e^{-2x}.$$

So, $A = 3$ and $y_p(x) = 3xe^{-2x}$. This gives us the following general solution

$$y(x) = c_1e^{-2x} + c_2e^{-3x} + 3xe^{-2x}.$$

Note that if xe^{-2x} were also a solution to the complementary equation, we would have to multiply by x again, and we would try $y_p(x) = Ax^2e^{-2x}$.

PROBLEM-SOLVING STRATEGY: METHOD OF UNDETERMINED COEFFICIENTS

1. Solve the complementary equation and write down the general solution.
2. Based on the form of $r(x)$, make an initial guess for $y_p(x)$.
3. Check whether any term in the guess for$y_p(x)$ is a solution to the complementary equation. If so, multiply the guess by x. Repeat this step until there are no terms in $y_p(x)$ that solve the complementary equation.
4. Substitute $y_p(x)$ into the differential equation and equate like terms to find values for the unknown coefficients in $y_p(x)$.
5. Add the general solution to the complementary equation and the particular solution you just found to obtain the general solution to the nonhomogeneous equation.

Example 17.2.4: Solving Nonhomogeneous Equations

Find the general solutions to the following differential equations.

a. $y'' - 9y = -6\cos 3x$
b. $x''+2x'+x = 4e^{-t}$
c. $y''-2y'+5y = 10x^2 - 3x - 3$
d. $y'' - 3y' = -12t$

Solution

a. The complementary equation is $y''-9y = 0$, which has the general solution $c_1e^{3x} + c_2e^{-3x}$(step 1). Based on the form of $r(x) = -6\cos 3x$, our initial guess for the particular solution is $y_p(x) = A\cos 3x + B\sin 3x$ (step 2). None of the terms in $y_p(x)$ solve the complementary equation, so this is a valid guess (step 3).
Now we want to find values for A and B, so substitute y_p into the differential equation. We have

$$y_p'(x) = -3A\sin 3x + 3B\cos 3x \text{ and } y_p''(x) = -9A\cos 3x - 9B\sin 3x,$$

so we want to find values of A and B such that

$$y''-9y = -6\cos 3x$$
$$-9A\cos 3x - 9B\sin 3x - 9(A\cos 3x + B\sin 3x) = -6\cos 3x$$
$$-18A\cos 3x - 18B\sin 3x = -6\cos 3x.$$

Therefore,

$$-18A = -6$$
$$-18B = 0.$$

This gives $A = \frac{1}{3}$ and $B = 0$, so $y_p(x) = (\frac{1}{3})\cos 3x$ (step 4).
Putting everything together, we have the general solution

$$y(x) = c_1e^{3x} + c_2e^{-3x} + \frac{1}{3}\cos 3x.$$

b. The complementary equation is $x'' + 2x'+x = 0$, which has the general solution $c_1e^{-t} + c_2te^{-t}$ (step 1). Based on the form $r(t) = 4e^{-t}$, our initial guess for the particular solution is $x_p(t) = Ae^{-t}$ (step 2). However, we see that this guess solves the complementary equation, so we must multiply by t, which gives a new guess: $x_p(t) = Ate^{-t}$ (step 3). Checking this new

guess, we see that it, too, solves the complementary equation, so we must multiply by t again, which gives $x_p(t) = At^2e^{-t}$ (step 3 again). Now, checking this guess, we see that $x_p(t)$ does not solve the complementary equation, so this is a valid guess (step 3 yet again).
We now want to find a value for A, so we substitute x_p into the differential equation. We have

$$x_p(t) = At^2e^{-t}, \text{ so}$$
$$x_p{}'(t) = 2Ate^{-t} - At^2e^{-t}$$

and

$$x_p{}''(t) = 2Ae^{-t} - 2Ate^{-t} - (2Ate^{-t} - At^2e^{-t}) = 2Ae^{-t} - 4Ate^{-t} + At^2e^{-t}.$$

Substituting into the differential equation, we want to find a value of A so that

$$\begin{aligned} x''+2x'+x &= 4e^{-t} \\ 2Ae^{-t} - 4Ate^{-t} + At^2e^{-t} + 2(2Ate^{-t} - At^2e^{-t}) + At^2e^{-t} &= 4e^{-t} \\ 2Ae^{-t} &= 4e^{-t}. \end{aligned}$$

This gives $A = 2$, so $x_p(t) = 2t^2e^{-t}$ (step 4). Putting everything together, we have the general solution

$$x(t) = c_1e^{-t} + c_2te^{-t} + 2t^2e^{-t}.$$

c. The complementary equation is $y''-2y'+5y = 0$, which has the general solution $c_1e^x \cos 2x + c_2e^x \sin 2x$ (step 1). Based on the form $r(x) = 10x^2 - 3x - 3$, our initial guess for the particular solution is $y_p(x) = Ax^2 + Bx + C$ (step 2). None of the terms in $y_p(x)$ solve the complementary equation, so this is a valid guess (step 3). We now want to find values for A, B, and C, so we substitute y_p into the differential equation. We have $y_p{}'(x) = 2Ax + B$ and $y_p{}''(x) = 2A$, so we want to find values of A, B, and C such that

$$\begin{aligned} y''-2y'+5y &= 10x^2 - 3x - 3 \\ 2A - 2(2Ax + B) + 5(Ax^2 + Bx + C) &= 10x^2 - 3x - 3 \\ 5Ax^2 + (5B - 4A)x + (5C - 2B + 2A) &= 10x^2 - 3x - 3. \end{aligned}$$

Therefore,

$$\begin{aligned} 5A &= 10 \\ 5B - 4A &= -3 \\ 5C - 2B + 2A &= -3. \end{aligned}$$

This gives $A = 2$, $B = 1$, and $C = -1$, so $y_p(x) = 2x^2 + x - 1$ (step 4). Putting everything together, we have the general solution

$$y(x) = c_1e^x \cos 2x + c_2e^x \sin 2x + 2x^2 + x - 1.$$

d. The complementary equation is $y''-3y'= 0$, which has the general solution $c_1e^{3t} + c_2$ (step 1). Based on the form r(t)=−12t,r(t)=−12t, our initial guess for the particular solution is $y_p(t) = At + B$ (step 2). However, we see that the constant term in this guess solves the complementary equation, so we must multiply by t, which gives a new guess: $y_p(t) = At^2 + Bt$ (step 3). Checking this new guess, we see that none of the terms in $y_p(t)$ solve the complementary equation, so this is a valid guess (step 3 again). We now want to find values for A and B, so we substitute y_p into the differential equation. We have $y_p{}'(t) = 2At + B$ and $y_p{}''(t) = 2A$, so we want to find values of AA and BB such that

$$\begin{aligned} y''-3y' &= -12t \\ 2A - 3(2At + B) &= -12t \\ -6At + (2A - 3B) &= -12t. \end{aligned}$$

Therefore,

$$\begin{aligned} -6A &= -12 \\ 2A - 3B &= 0. \end{aligned}$$

This gives $A = 2$ and $B = 4/3$, so $y_p(t) = 2t^2 + (4/3)t$ (step 4). Putting everything together, we have the general solution

$$y(t) = c_1 e^{3t} + c_2 + 2t^2 + \frac{4}{3}t.$$

? Exercise 17.2.3

Find the general solution to the following differential equations.

a. $y'' - 5y' + 4y = 3e^x$
b. $y'' + y' - 6y = 52\cos 2t$

Hint

Use the problem-solving strategy.

Answer a

$y(x) = c_1 e^{4x} + c_2 e^x - xe^x$

Answer b

$y(t) = c_1 e^{-3t} + c_2 e^{2t} - 5\cos 2t + \sin 2t$

Variation of Parameters

Sometimes, $r(x)$ is not a combination of polynomials, exponentials, or sines and cosines. When this is the case, the method of undetermined coefficients does not work, and we have to use another approach to find a particular solution to the differential equation. We use an approach called the *method of variation of parameters*.

To simplify our calculations a little, we are going to divide the differential equation through by a, so we have a leading coefficient of 1. Then the differential equation has the form

$$y'' + py' + qy = r(x),$$

where p and q are constants.

If the general solution to the complementary equation is given by $c_1 y_1(x) + c_2 y_2(x)$, we are going to look for a particular solution of the form

$$y_p(x) = u(x)y_1(x) + v(x)y_2(x).$$

In this case, we use the two linearly independent solutions to the complementary equation to form our particular solution. However, we are assuming the coefficients are functions of x, rather than constants. We want to find functions $u(x)$ and $v(x)$ such that $y_p(x)$ satisfies the differential equation. We have

$$\begin{aligned} y_p &= uy_1 + vy_2 \\ y_p' &= u'y_1 + uy_1' + v'y_2 + vy_2' \\ y_p'' &= (u'y_1 + v'y_2)' + u'y_1' + uy_1'' + v'y_2' + vy_2''. \end{aligned}$$

Substituting into the differential equation, we obtain

$$\begin{aligned} y_p'' + py_p' + qy_p &= [(u'y_1 + v'y_2)' + u'y_1' + uy_1'' + v'y_2' + vy_2''] \\ &\quad + p[u'y_1 + uy_1' + v'y_2 + vy_2'] + q[uy_1 + vy_2] \\ &= u[y_1'' + py_1' + qy_1] + v[y_2'' + py_2' + qy_2] \\ &\quad + (u'y_1 + v'y_2)' + p(u'y_1 + v'y_2) + (u'y_1' + v'y_2'). \end{aligned}$$

Note that y_1 and y_2 are solutions to the complementary equation, so the first two terms are zero. Thus, we have

$$(u'y_1 + v'y_2)' + p(u'y_1 + v'y_2) + (u'y_1' + v'y_2') = r(x).$$

If we simplify this equation by imposing the additional condition $u'y_1 + v'y_2 = 0$, the first two terms are zero, and this reduces to $u'y_1' + v'y_2' = r(x)$. So, with this additional condition, we have a system of two equations in two unknowns:

$$u'y_1 + v'y_2 = 0$$
$$u'y_1{}' + v'y_2{}' = r(x).$$

Solving this system gives us u' and v', which we can integrate to find u and v.

Then, $y_p(x) = u(x)y_1(x) + v(x)y_2(x)$ is a particular solution to the differential equation. Solving this system of equations is sometimes challenging, so let's take this opportunity to review Cramer's rule, which allows us to solve the system of equations using determinants.

RULE: CRAMER'S RULE

The system of equations

$$a_1z_1 + b_1z_2 = r_1$$
$$a_2z_1 + b_2z_2 = r_2$$

has a unique solution if and only if the determinant of the coefficients is not zero. In this case, the solution is given by

$$z_1 = \frac{\begin{vmatrix} r_1 b_1 \\ r_2 b_2 \end{vmatrix}}{\begin{vmatrix} a_1 b_1 \\ a_2 b_2 \end{vmatrix}} \quad \text{and} \quad z_2 = \frac{\begin{vmatrix} a_1 r_1 \\ a_2 r_2 \end{vmatrix}}{\begin{vmatrix} a_1 b_1 \\ a_2 b_2 \end{vmatrix}}. \tag{17.2.1}$$

Example 17.2.5: Using Cramer's Rule

Use Cramer's rule to solve the following system of equations.

$$x^2z_1 + 2xz_2 = 0$$
$$z_1 - 3x^2z_2 = 2x$$

Solution

We have

$$a_1(x) = x^2$$
$$a_2(x) = 1$$
$$b_1(x) = 2x$$
$$b_2(x) = -3x^2$$
$$r_1(x) = 0$$
$$r_2(x) = 2x.$$

Then,

$$\begin{vmatrix} a_1 b_1 \\ a_2 b_2 \end{vmatrix} = \begin{vmatrix} x^2 2x \\ 1 - 3x^2 \end{vmatrix} = -3x^4 - 2x$$

and

$$\begin{vmatrix} r_1 b_1 \\ r_2 b_2 \end{vmatrix} = \begin{vmatrix} 02x \\ 2x - 3x^2 \end{vmatrix} = 0 - 4x^2 = -4x^2.$$

Thus,

$$z1 = \frac{\begin{vmatrix} r_1 b_1 \\ r_2 b_2 \end{vmatrix}}{\begin{vmatrix} a_1 b_1 \\ a_2 b_2 \end{vmatrix}} = \frac{-4x^2}{-3x^4 - 2x} = \frac{4x}{3x^3 + 2}.$$

In addition,

$$\begin{vmatrix} a_1 r_1 \end{vmatrix} \quad \begin{vmatrix} x^2 0 \end{vmatrix}$$

$$\begin{vmatrix} a_1r_1 \\ a_2r_2 \end{vmatrix} = \begin{vmatrix} x^3 & 0 \\ 12x \end{vmatrix} = 2x^3 - 0 = 2x^3.$$

Thus,

$$z2 = \frac{\begin{vmatrix} a_1r_1 \\ a_2r_2 \end{vmatrix}}{\begin{vmatrix} a_1b_1 \\ a_2b_2 \end{vmatrix}} = \frac{2x^3}{-3x^4 - 2x} = \frac{-2x^2}{3x^3 + 2}.$$

Exercise 17.2.4

Use Cramer's rule to solve the following system of equations.

$$2xz_1 - 3z_2 = 0$$
$$x^2z_1 + 4xz_2 = x + 1$$

Hint

Use the process from the previous example.

Answer

$z_1 = \frac{3x+3}{11x^2}, z_2 = \frac{2x+2}{11x}$

PROBLEM-SOLVING STRATEGY: METHOD OF VARIATION OF PARAMETERS

1. Solve the complementary equation and write down the general solution
$$c_1y_1(x) + c_2y_2(x).$$
2. Use Cramer's rule or another suitable technique to find functions $u'(x)$ and $v'(x)$ satisfying
$$u'y_1 + v'y_2 = 0$$
$$u'y_1' + v'y_2' = r(x).$$
3. Integrate u' and v' to find $u(x)$ and $v(x)$. Then, $y_p(x) = u(x)y_1(x) + v(x)y_2(x)$ is a particular solution to the equation.
4. Add the general solution to the complementary equation and the particular solution found in step 3 to obtain the general solution to the nonhomogeneous equation.

Example 17.2.6: Using the Method of Variation of Parameters

Find the general solution to the following differential equations.

a. $y'' - 2y' + y = \dfrac{e^t}{t^2}$
b. $y'' + y = 3\sin^2 x$

Solution

a. The complementary equation is $y'' - 2y' + y = 0$ with associated general solution $c_1e^t + c_2te^t$. Therefore, $y_1(t) = e^t$ and $y_2(t) = te^t$. Calculating the derivatives, we get $y_1'(t) = e^t$ and $y_2'(t) = e^t + te^t$ (step 1). Then, we want to find functions $u'(t)$ and $v'(t)$ so that

$$u'e^t + v'te^t = 0$$
$$u'e^t + v'(e^t + te^t) = \frac{e^t}{t^2}.$$

Applying Cramer's rule (Equation 17.2.1), we have

$$u' = \frac{\begin{vmatrix} 0 & te^t \\ \frac{e^t}{t^2}e^t + te^t \end{vmatrix}}{\begin{vmatrix} e^t & te^t \end{vmatrix}} = \frac{0 - te^t(\frac{e^t}{t^2})}{e^t(e^t + te^t) - e^tte^t} = \frac{-\frac{e^{2t}}{t}}{e^{2t}} = -\frac{1}{t}$$

$$\begin{vmatrix} & \\ e^t e^t + te^t & \end{vmatrix}$$

and

$$v' = \frac{\begin{vmatrix} e^t 0 \\ e^t \frac{e^t}{t^2} \end{vmatrix}}{\begin{vmatrix} e^t te^t \\ e^t e^t + te^t \end{vmatrix}} = \frac{e^t(\frac{e^t}{t^2})}{e^{2t}} = \frac{1}{t^2} \quad \text{(step 2)}.$$

Integrating, we get

$$u = -\int \frac{1}{t} dt = -\ln|t|$$
$$v = \int \frac{1}{t^2} dt = -\frac{1}{t} \quad \text{(step 3)}$$

Then we have

$$y_p = -e^t \ln|t| - \frac{1}{t} te^t$$
$$= -e^t \ln|t| - e^t. \quad \text{(step 4)}$$

The e^t term is a solution to the complementary equation, so we don't need to carry that term into our general solution explicitly. The general solution is

$$y(t) = c_1 e^t + c_2 te^t - e^t \ln|t| \quad \text{(step 5)}$$

b. The complementary equation is $y'' + y = 0$ with associated general solution $c_1 \cos x + c_2 \sin x$. So, $y_1(x) = \cos x$ and $y_2(x) = \sin x$ (step 1). Then, we want to find functions $u'(x)$ and $v'(x)$ such that

$$u' \cos x + v' \sin x = 0$$
$$-u' \sin x + v' \cos x = 3 \sin_2 x.$$

Applying Cramer's rule, we have

$$u' = \frac{\begin{vmatrix} 0 \sin x \\ 3\sin^2 x \cos x \end{vmatrix}}{\begin{vmatrix} \cos x \sin x \\ -\sin x \cos x \end{vmatrix}} = \frac{0 - 3\sin^3 x}{\cos^2 x + \sin^2 x} = -3\sin^3 x$$

and

$$v' = \frac{\begin{vmatrix} \cos x 0 \\ -\sin x 3\sin^2 x \end{vmatrix}}{\begin{vmatrix} \cos x \sin x \\ -\sin x \cos x \end{vmatrix}} = \frac{3\sin^2 x \cos x}{1} = 3\sin^2 x \cos x (\text{step 2}).$$

Integrating first to find u, we get

$$u = \int -3\sin^3 x dx = -3\left[-\frac{1}{3}\sin^2 x \cos x + \frac{2}{3}\int \sin x dx\right] = \sin^2 x \cos x + 2\cos x.$$

Now, we integrate to find v. Using substitution (with $w = \sin x$), we get

$$v = \int 3\sin^2 x \cos x dx = \int 3w^2 dw = w^3 = sin^3 x.$$

Then,

$$\begin{aligned} y_p &= (\sin^2 x \cos x + 2\cos x)\cos x + (\sin^3 x)\sin x \\ &= \sin_2 x \cos_2 x + 2\cos_2 x + \sin_4 x \\ &= 2\cos_2 x + \sin_2 x(\cos^2 x + \sin^2 x) \qquad \text{(step 4)}. \\ &= 2\cos_2 x + \sin_2 x \\ &= \cos_2 x + 1 \end{aligned}$$

The general solution is

$$y(x) = c_1 \cos x + c_2 \sin x + 1 + \cos^2 x \text{(step 5)}.$$

? Exercise 17.2.5

Find the general solution to the following differential equations.

a. $y''+y = \sec x$

b. $x''-2x'+x = \dfrac{e^t}{t}$

Hint

Follow the problem-solving strategy.

Answer a

$y(x) = c_1 \cos x + c_2 \sin x + \cos x \ln|\cos x| + x \sin x$

Answer b

$x(t) = c_1 e^t + c_2 t e^t + t e^t \ln|t|$

Key Concepts

- To solve a nonhomogeneous linear second-order differential equation, first find the general solution to the complementary equation, then find a particular solution to the nonhomogeneous equation.
- Let $y_p(x)$ be any particular solution to the nonhomogeneous linear differential equation

  $$a_2(x)y'' + a_1(x)y'+a_0(x)y = r(x),$$

  and let $c_1y_1(x) + c_2y_2(x)$ denote the general solution to the complementary equation. Then, the general solution to the nonhomogeneous equation is given by

  $$y(x) = c_1y_1(x) + c_2y_2(x) + y_p(x).$$

- When $r(x)$ is a combination of polynomials, exponential functions, sines, and cosines, use the method of undetermined coefficients to find the particular solution. To use this method, assume a solution in the same form as $r(x)$, multiplying by x as necessary until the assumed solution is linearly independent of the general solution to the complementary equation. Then, substitute the assumed solution into the differential equation to find values for the coefficients.
- When $r(x)$ is *not* a combination of polynomials, exponential functions, or sines and cosines, use the method of variation of parameters to find the particular solution. This method involves using Cramer's rule or another suitable technique to find functions and $v'(x)$ satisfying

  $$\begin{aligned} u'y_1 + v'y_2 &= 0 \\ u'y_1'+v'y_2' &= r(x). \end{aligned}$$

  Then, $y_p(x) = u(x)y_1(x) + v(x)y_2(x)$ is a particular solution to the differential equation.

Key Equations

- **Complementary equation**
  $a_2(x)y''+a_1(x)y'+a_0(x)y = 0$
- **General solution to a nonhomogeneous linear differential equation**
  $y(x) = c_1y_1(x) + c_2y_2(x) + y_p(x)$

Glossary

complementary equation

for the nonhomogeneous linear differential equation $a + 2(x)y'' + a_1(x)y' + a_0(x)y = r(x)$, the associated homogeneous equation, called the *complementary equation*, is $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$

method of undetermined coefficients

a method that involves making a guess about the form of the particular solution, then solving for the coefficients in the guess

method of variation of parameters

a method that involves looking for particular solutions in the form $y_p(x) = u(x)y_1(x) + v(x)y_2(x)$, where y_1 and y_2 are linearly independent solutions to the complementary equations, and then solving a system of equations to find $u(x)$ and $v(x)$

particular solution

a solution $y_p(x)$ of a differential equation that contains no arbitrary constants

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17.2E: Exercises for Section 17.2

Solve the following equations using the method of undetermined coefficients.

1. $2y'' - 5y' - 12y = 6$

2. $3y'' + y' - 4y = 8$

Answer

$y = c_1 e^{-4x/3} + c_2 e^{x} - 2$

3. $y'' - 6y' + 5y = e^{-x}$

4. $y'' + 16y = e^{-2x}$

Answer

$y = c_1 \cos 4x + c_2 \sin 4x + \frac{1}{20} e^{-2x}$

5. $y'' - 4y = x^2 + 1$

6. $y'' - 4y' + 4y = 8x^2 + 4x$

Answer

$y = c_1 e^{2x} + c_2 x e^{2x} + 2x^2 + 5x$

7. $y'' - 2y' - 3y = \sin 2x$

8. $y'' + 2y' + y = \sin x + \cos x$

Answer

$y = c_1 e^{-x} + c_2 x e^{-x} + \frac{1}{2} \sin x - \frac{1}{2} \cos x$

9. $y'' + 9y = e^{x} \cos x$

10. $y'' + y = 3 \sin 2x + x \cos 2x$

Answer

$y = c_1 \cos x + c_2 \sin x - \frac{1}{3} x \cos 2x - \frac{5}{9} \sin 2x$

11. $y'' + 3y' - 28y = 10e^{4x}$

12. $y'' + 10y' + 25y = xe^{-5x} + 4$

Answer

$y = c_1 e^{-5x} + c_2 x e^{-5x} + \frac{1}{6} x^3 e^{-5x} + \frac{4}{25}$

In exercises 13 - 18,

a. **Write the form for the particular solution $y_p(x)$ for the method of undetermined coefficients.**
b. **[T]** Use a computer algebra system to find a particular solution to the given equation.

13. $y'' - y' - y = x + e^{-x}$

14. $y'' - 3y = x^2 - 4x + 11$

Answer

a. $y_p(x) = Ax^2 + Bx + C$

b. $y_p(x) = -\frac{1}{3} x^2 + \frac{4}{3} x - \frac{35}{9}$

15. $y'' - y' - 4y = e^x \cos 3x$

16. $2y'' - y' + y = (x^2 - 5x)e^{-x}$

Answer

a. $y_p(x) = (Ax^2 + Bx + C)e^{-x}$

b. $y_p(x) = (\frac{1}{4}x^2 - \frac{5}{8}x - \frac{33}{32})e^{-x}$

17. $4y'' + 5y' - 2y = e^{2x} + x \sin x$

18. $y'' - y' - 2y = x^2 e^x \sin x$

Answer

a. $y_p(x) = (Ax^2 + Bx + C)e^x \cos x + (Dx^2 + Ex + F)e^x \sin x$

b. $y_p(x) = (-\frac{1}{10}x^2 - \frac{11}{25}x - \frac{27}{250})e^x \cos x + (-\frac{3}{10}x^2 + \frac{2}{25}x + \frac{39}{250})e^x \sin x$

Solve the differential equation using either the method of undetermined coefficients or the variation of parameters.

19. $y'' + 3y' - 4y = 2e^x$

20. $y'' + 2y' = e^{3x}$

Answer

$y = c_1 + c_2 e^{-2x} + \frac{1}{15}e^{3x}$

21. $y'' + 6y' + 9y = e^{-x}$

22. $y'' + 2y' - 8y = 6e^{2x}$

Answer

$y = c_1 e^{2x} + c_2 e^{-4x} + xe^{2x}$

Solve the differential equation using the method of variation of parameters.

23. $4y'' + y = 2 \sin x$

24. $y'' - 9y = 8x$

Answer

$y = c_1 e^{3x} + c_2 e^{-3x} - \frac{8x}{9}$

25. $y'' + y = \sec x, \quad 0 < x < \pi/2$

26. $y'' + 4y = 3 \csc 2x, \quad 0 < x < \pi/2$

Answer

$y = c_1 \cos 2x + c_2 \sin 2x - \frac{3}{2}x \cos 2x + \frac{3}{4}\sin 2x \ln(\sin 2x)$

Find the unique solution satisfying the differential equation and the initial conditions given, where $y_p(x)$ is the particular solution.

27. $y'' - 2y' + y = 12e^x, \quad y_p(x) = 6x^2 e^x, \; y(0) = 6, \; y'(0) = 0$

28. $y'' - 7y' = 4xe^{7x}, \quad y_p(x) = \frac{2}{7}x^2 e^{7x} - \frac{4}{49}xe^{7x}, \; y(0) = -1, \; y'(0) = 0$

Answer

$y = -\frac{347}{343} + \frac{4}{343}e^{7x} + \frac{2}{7}x^2 e^{7x} - \frac{4}{49}xe^{7x}$

29. $y'' + y = \cos x - 4 \sin x, \quad y_p(x) = 2x \cos x + \frac{1}{2}x \sin x, \; y(0) = 8, \; y'(0) = -4$

30. $y''-5y'=e^{5x}+8e^{-5x}, \quad y_p(x)=\frac{1}{5}xe^{5x}+\frac{4}{25}e^{-5x},\ y(0)=-2,\ y'(0)=0$

Answer

$y=-\frac{57}{25}+\frac{3}{25}e^{5x}+\frac{1}{5}xe^{5x}+\frac{4}{25}e^{-5x}$

In problems 31 - 32, two linearly independent solutions—y_1 and y_2—are given that satisfy the corresponding homogeneous equation. Use the method of variation of parameters to find a particular solution to the given nonhomogeneous equation. Assume $x>0$ in each exercise.

31. $x^2y''+2xy'-2y=3x, \quad y_1(x)=x,\ y2(x)=x^{-2}$

32. $x^2y''-2y=10x^2-1, \quad y_1(x)=x^2,\ y_2(x)=x^{-1}$

Answer

$y_p=\frac{1}{2}+\frac{10}{3}x^2\ln x$

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17.3: Applications of Second-Order Differential Equations

Learning Objectives

- Solve a second-order differential equation representing simple harmonic motion.
- Solve a second-order differential equation representing damped simple harmonic motion.
- Solve a second-order differential equation representing forced simple harmonic motion.
- Solve a second-order differential equation representing charge and current in an RLC series circuit.

We saw in the chapter introduction that second-order linear differential equations are used to model many situations in physics and engineering. In this section, we look at how this works for systems of an object with mass attached to a vertical spring and an electric circuit containing a resistor, an inductor, and a capacitor connected in series. Models such as these can be used to approximate other more complicated situations; for example, bonds between atoms or molecules are often modeled as springs that vibrate, as described by these same differential equations.

Simple Harmonic Motion

Consider a mass suspended from a spring attached to a rigid support. (This is commonly called a **spring-mass system**.) Gravity is pulling the mass downward and the restoring force of the spring is pulling the mass upward. As shown in Figure 17.3.1, when these two forces are equal, the mass is said to be at the equilibrium position. If the mass is displaced from equilibrium, it oscillates up and down. This behavior can be modeled by a second-order constant-coefficient differential equation.

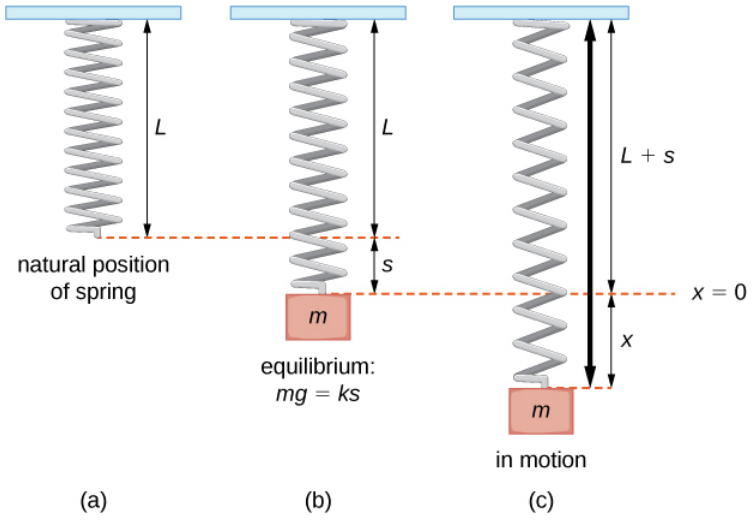

Figure 17.3.1: A spring in its natural position (a), at equilibrium with a mass *m* attached (b), and in oscillatory motion (c).

Let $x(t)$ denote the displacement of the mass from equilibrium. Note that for spring-mass systems of this type, it is customary to adopt the convention that down is positive. Thus, a positive displacement indicates the mass is *below* the equilibrium point, whereas a negative displacement indicates the mass is *above* equilibrium. Displacement is usually given in feet in the English system or meters in the metric system.

Consider the forces acting on the mass. The force of gravity is given by mg.mg. In the English system, mass is in slugs and the acceleration resulting from gravity is in feet per second squared. The acceleration resulting from gravity is constant, so in the English system, $g = 32\ ft/sec^2$. Recall that 1 slug-foot/sec^2 is a pound, so the expression *mg* can be expressed in pounds. Metric system units are kilograms for mass and m/sec^2 for gravitational acceleration. In the metric system, we have $g = 9.8$ m/sec^2.

According to **Hooke's law**, the restoring force of the spring is proportional to the displacement and acts in the opposite direction from the displacement, so the restoring force is given by $-k(s + x)$. The spring constant is given in pounds per foot in the English system and in newtons per meter in the metric system.

Now, by Newton's second law, the sum of the forces on the system (gravity plus the restoring force) is equal to mass times acceleration, so we have

$$\begin{aligned} mx'' &= -k(s+x)+mg \\ &= -ks-kx+mg. \end{aligned}$$

However, by the way we have defined our equilibrium position, $mg = ks$, the differential equation becomes

$$mx''+kx = 0.$$

It is convenient to rearrange this equation and introduce a new variable, called the **angular frequency**, ω. Letting $\omega = \sqrt{k/m}$, we can write the equation as

$$x'' + \omega^2 x = 0.$$

This differential equation has the general solution

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t, \tag{17.3.1}$$

which gives the position of the mass at any point in time. The motion of the mass is called simple **harmonic motion**. The period of this motion (the time it takes to complete one oscillation) is $T = \dfrac{2\pi}{\omega}$ and the frequency is $f = \dfrac{1}{T} = \dfrac{\omega}{2\pi}$ (Figure 17.3.2).

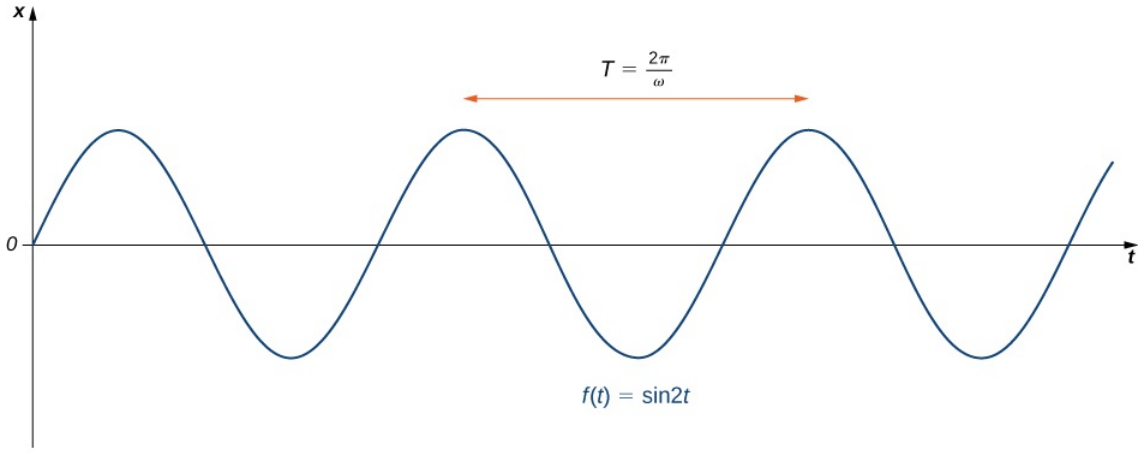

Figure 17.3.2: A graph of vertical displacement versus time for simple harmonic motion.

Example 17.3.1: Simple Harmonic Motion

Assume an object weighing 2 lb stretches a spring 6 in. Find the equation of motion if the spring is released from the equilibrium position with an upward velocity of 16 ft/sec. What is the period of the motion?

Solution

We first need to find the spring constant. We have

$$\begin{aligned} mg &= ks \\ 2 &= k\left(\frac{1}{2}\right) \\ k &= 4. \end{aligned}$$

We also know that weight W equals the product of mass m and the acceleration due to gravity g. In English units, the acceleration due to gravity is 32 ft/sec^2.

$$\begin{aligned} W &= mg \\ 2 &= m(32) \\ m &= \frac{1}{16} \end{aligned}$$

Thus, the differential equation representing this system is

$$\frac{1}{16}x''+4x = 0.$$

Multiplying through by 16, we get $x'' + 64x = 0$, which can also be written in the form $x'' + (8^2)x = 0$. This equation has the general solution

$$x(t) = c_1 \cos(8t) + c_2 \sin(8t).$$

The mass was released from the equilibrium position, so $x(0) = 0$, and it had an initial upward velocity of 16 ft/sec, so $x'(0) = -16$. Applying these initial conditions to solve for c_1 and c_2. gives

$$x(t) = -2 \sin 8t.$$

The period of this motion is $\frac{2\pi}{8} = \frac{\pi}{4}$ sec.

? Exercise 17.3.1

A 200-g mass stretches a spring 5 cm. Find the equation of motion of the mass if it is released from rest from a position 10 cm below the equilibrium position. What is the frequency of this motion?

Hint

First find the spring constant.

Answer

$x(t) = 0.1 \cos(14t)$ (in meters); frequency is $\frac{14}{2\pi}$ Hz.

Writing the general solution in the form $x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$ (Equation 17.3.1) has some advantages. It is easy to see the link between the differential equation and the solution, and the period and frequency of motion are evident. This form of the function tells us very little about the amplitude of the motion, however. In some situations, we may prefer to write the solution in the form

$$x(t) = A \sin(\omega t + \phi).$$

Although the link to the differential equation is not as explicit in this case, the period and frequency of motion are still evident. Furthermore, the amplitude of the motion, A, is obvious in this form of the function. The constant ϕ is called a **phase shift** and has the effect of shifting the graph of the function to the left or right.

To convert the solution to this form, we want to find the values of A and ϕ such that

$$c_1 \cos(\omega t) + c_2 \sin(\omega t) = A \sin(\omega t + \phi).$$

We first apply the trigonometric identity

$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$$

to get

$$\begin{aligned} c_1 \cos(\omega t) + c_2 \sin(\omega t) &= A(\sin(\omega t)\cos\phi + \cos(\omega t)\sin\phi) \\ &= A\sin\phi(\cos(\omega t)) + A\cos\phi(\sin(\omega t)). \end{aligned}$$

Thus,

$$c_1 = A\sin\phi \text{ and } c_2 = A\cos\phi.$$

If we square both of these equations and add them together, we get

$$\begin{aligned} c_1^2 + c_2^2 &= A^2 \sin_2\phi + A^2 \cos_2\phi \\ &= A^2(\sin^2\phi + \cos^2\phi) \\ &= A^2. \end{aligned}$$

Thus,

$$A = \sqrt{c_1^2 + c_2^2}.$$

Now, to find ϕ, go back to the equations for c_1 and c_2, but this time, divide the first equation by the second equation to get

$$\begin{aligned}\frac{c_1}{c_2} &= \frac{A\sin\phi}{A\cos\phi} \\ &= \tan\phi.\end{aligned}$$

Then,

$$\tan\phi = \frac{c_1}{c_2}.$$

We summarize this finding in the following theorem.

SOLUTION TO THE EQUATION FOR SIMPLE HARMONIC MOTION

The function $x(t) = c_1\cos(\omega t) + c_2\sin(\omega t)$ can be written in the form $x(t) = A\sin(\omega t + \phi)$, where $A = \sqrt{c_1^2 + c_2^2}$ and $\tan\phi = \frac{c_1}{c_2}$.

Note that when using the formula $\tan\phi = \frac{c_1}{c_2}$ to find ϕ, we must take care to ensure ϕ is in the right quadrant (Figure 17.3.3).

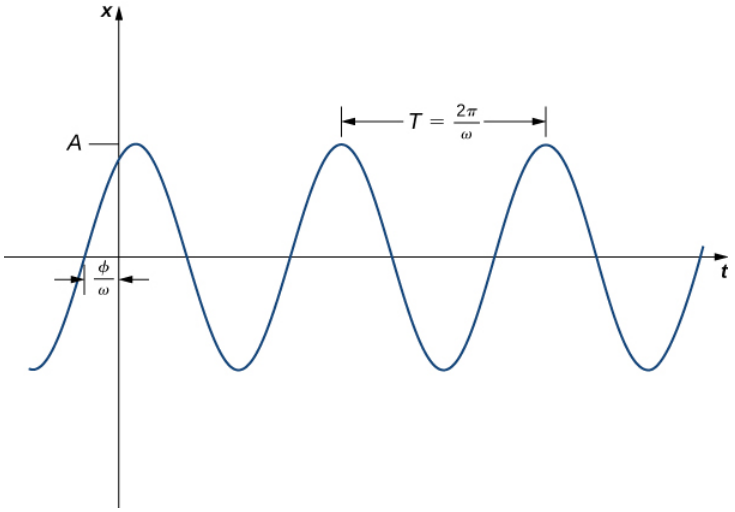

Figure 17.3.3: A graph of vertical displacement versus time for simple harmonic motion with a phase change.

Example 17.3.2: Expressing the Solution with a Phase Shift

Express the following functions in the form $A\sin(\omega t + \phi)$. What is the frequency of motion? The amplitude?

a. $x(t) = 2\cos(3t) + \sin(3t)$
b. $x(t) = 3\cos(2t) - 2\sin(2t)$

Solution

We have

$$A = \sqrt{c_1^2 + c_2^2} = \sqrt{2^2 + 1^2} = \sqrt{5}$$

and

$$\tan\phi = \frac{c_1}{c_2} = \frac{2}{1} = 2.$$

Note that both c_1 and c_2 are positive, so ϕ is in the first quadrant. Thus,

$$\phi \approx 1.107 \text{ rad},$$

so we have

$$x(t) = 2\cos(3t) + \sin(3t) = 5\sin(3t + 1.107).$$

The frequency is $\frac{\omega}{2\pi} = \frac{3}{2\pi} \approx 0.477$. The amplitude is $\sqrt{5}$.

1. We have

$$A = \sqrt{c_1^2 + c_2^2} = \sqrt{3^2 + 2^2} = \sqrt{13}$$

and

$$\tan\phi = \frac{c_1}{c_2} = \frac{3}{-2} = -\frac{3}{2}.$$

Note that c_1 is positive but c_2 is negative, so ϕ is in the fourth quadrant. Thus,

$$\phi \approx -0.983 \text{ rad},$$

so we have

$$\begin{aligned} x(t) &= 3\cos(2t) - 2\sin(2t) \\ &= \sqrt{13}\sin(2t - 0.983). \end{aligned}$$

The frequency is $\frac{\omega}{2\pi} = \frac{2}{2\pi} \approx 0.318$. The amplitude is $\sqrt{13}$.

? Exercise 17.3.2

Express the function $x(t) = \cos(4t) + 4\sin(4t)$ in the form $A\sin(\omega t + \phi)$. What is the frequency of motion? The amplitude?

Hint

Use the process from the Example 17.3.2.

Answer

$x(t) = \sqrt{17}\sin(4t + 0.245)$, frequency $= \frac{4}{2\pi} \approx 0.637$, $A = \sqrt{17}$

Damped Vibrations

With the model just described, the motion of the mass continues indefinitely. Clearly, this doesn't happen in the real world. In the real world, there is almost always some friction in the system, which causes the oscillations to die off slowly—an effect called *damping*. So now let's look at how to incorporate that damping force into our differential equation.

Physical spring-mass systems almost always have some damping as a result of friction, air resistance, or a physical damper, called a *dashpot* (a pneumatic cylinder; Figure 17.3.4).

Figure 17.3.4: A dashpot is a pneumatic cylinder that dampens the motion of an oscillating system.

Because damping is primarily a friction force, we assume it is proportional to the velocity of the mass and acts in the opposite direction. So the damping force is given by $-bx'$ for some constant $b > 0$. Again applying Newton's second law, the differential equation becomes

$$mx''+bx'+kx = 0.$$

Then the associated characteristic equation is

$$m\lambda^2 + b\lambda + k = 0.$$

Applying the quadratic formula, we have

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}.$$

Just as in Second-Order Linear Equations we consider three cases, based on whether the characteristic equation has distinct real roots, a repeated real root, or complex conjugate roots.

Case 1: Overdamped Vibrations

When $b^2 > 4mk$, we say the system is ***overdamped***. The general solution has the form

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t},$$

where both λ_1 and λ_2 are less than zero. Because the exponents are negative, the displacement decays to zero over time, usually quite quickly. Overdamped systems do not oscillate (no more than one change of direction), but simply move back toward the equilibrium position. Figure 17.3.5 shows what typical critically damped behavior looks like.

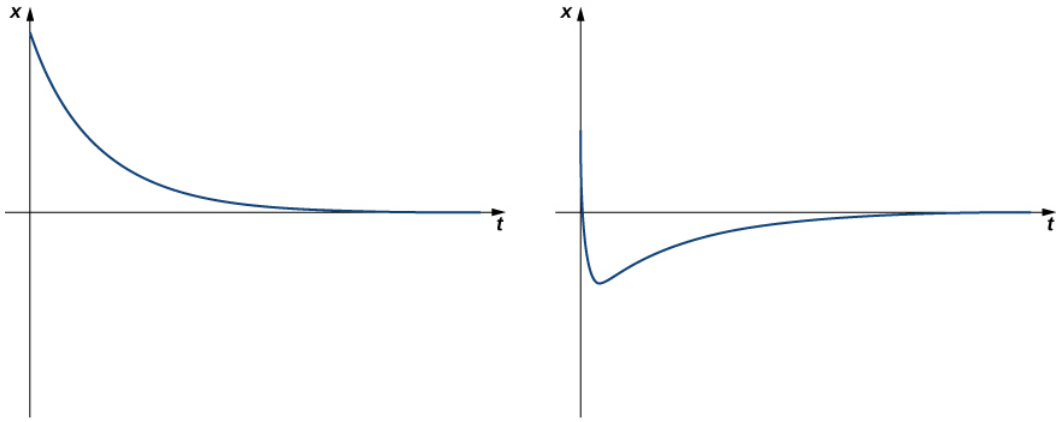

Figure 17.3.5: Behavior of an overdamped spring-mass system, with no change in direction (a) and only one change in direction (b).

Example 17.3.3: Overdamped Spring-Mass System

A 16-lb mass is attached to a 10-ft spring. When the mass comes to rest in the equilibrium position, the spring measures 15 ft 4 in. The system is immersed in a medium that imparts a damping force equal to 5252 times the instantaneous velocity of the mass. Find the equation of motion if the mass is pushed upward from the equilibrium position with an initial upward velocity of 5 ft/sec. What is the position of the mass after 10 sec? Its velocity?

Solution

The mass stretches the spring 5 ft 4 in., or $\frac{16}{3}$ ft. Thus, $16 = \left(\frac{16}{3}\right)k$, so $k = 3$. We also have $m = \frac{16}{32} = \frac{1}{2}$, so the differential equation is

$$\frac{5}{2}x'+3x = 0.$$

Multiplying through by 2 gives $x''+5x'+6x = 0$, which has the general solution

$$x(t) = c_1 e^{-2t} + c_2 e^{-3t}.$$

Applying the initial conditions, $x(0) = 0$ and $x'(0) = -5$, we get

$$x(t) = -5e^{-2t} + 5e^{-3t}.$$

After 10 sec the mass is at position

$$x(10) = -5e^{-20} + 5e^{-30} \approx -1.0305 \times 10^{-8} \approx 0,$$

so it is, effectively, at the equilibrium position. We have $x'(t) = 10e^{-2t} - 15e^{-3t}$, so after 10 sec the mass is moving at a velocity of

$$x'(10) = 10e^{-20} - 15e^{-30} \approx 2.061 \times 10^{-8} \approx 0.$$

After only 10 sec, the mass is barely moving.

? Exercise 17.3.3

A 2-kg mass is attached to a spring with spring constant 24 N/m. The system is then immersed in a medium imparting a damping force equal to 16 times the instantaneous velocity of the mass. Find the equation of motion if it is released from rest at a point 40 cm below equilibrium.

Hint

Follow the process from the previous example.

Answer

$x(t) = 0.6e^{-2t} - 0.2e^{-6t}$

Case 2: Critically Damped Vibrations

When $b^2 = 4mk$, we say the system is *critically damped*. The general solution has the form

$$x(t) = c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t},$$

where λ_1 is less than zero. The motion of a critically damped system is very similar to that of an overdamped system. It does not oscillate. However, with a critically damped system, if the damping is reduced even a little, oscillatory behavior results. From a practical perspective, physical systems are almost always either overdamped or underdamped (case 3, which we consider next). It is impossible to fine-tune the characteristics of a physical system so that b^2 and $4mk$ are exactly equal. Figure 17.3.6 shows what typical critically damped behavior looks like.

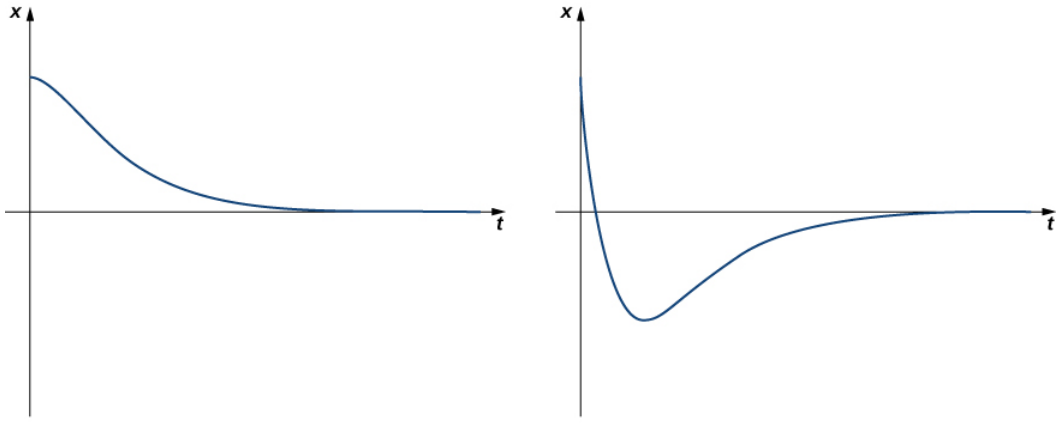

Figure 17.3.6: Behavior of a critically damped spring-mass system. The system graphed in part (a) has more damping than the system graphed in part (b).

Example 17.3.4: Critically Damped Spring-Mass System

A 1-kg mass stretches a spring 20 cm. The system is attached to a dashpot that imparts a damping force equal to 14 times the instantaneous velocity of the mass. Find the equation of motion if the mass is released from equilibrium with an upward velocity of 3 m/sec.

Solution

We have $mg = 1(9.8) = 0.2k$, so $k = 49$. Then, the differential equation is

$$x''+14x'+49x = 0,$$

which has general solution

$$x(t) = c_1e^{-7t} + c_2te^{-7t}.$$

Applying the initial conditions $x(0) = 0$ and $x'(0) = -3$ gives

$$x(t) = -3te^{-7t}.$$

Exercise 17.3.4

A 1-lb weight stretches a spring 6 in., and the system is attached to a dashpot that imparts a damping force equal to half the instantaneous velocity of the mass. Find the equation of motion if the mass is released from rest at a point 6 in. below equilibrium.

Hint

First find the spring constant.

Answer

$$x(t) = \frac{1}{2}e^{-8t} + 4te^{-8t}$$

Case 3: Undamped Vibrations

When $b^2 < 4mk$, we say the system is *underdamped*. The general solution has the form

$$x(t) = e^{\alpha t}(c_1\cos(\beta t) + c_2\sin(\beta t)),$$

where α is less than zero. Underdamped systems do oscillate because of the sine and cosine terms in the solution. However, the exponential term dominates eventually, so the amplitude of the oscillations decreases over time. Figure 17.3.7 shows what typical underdamped behavior looks like.

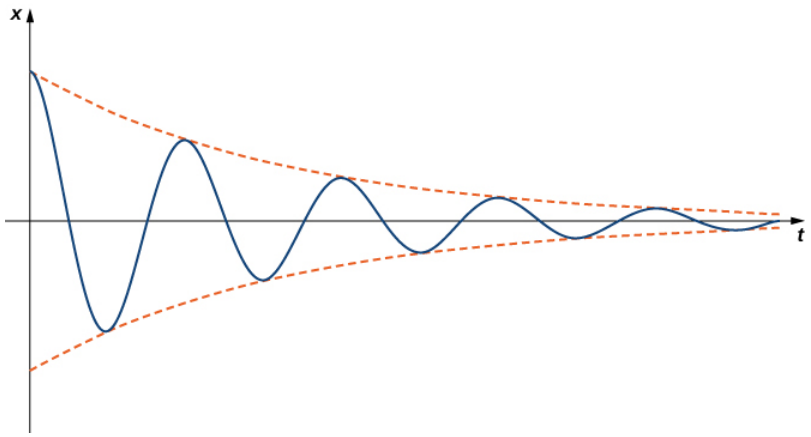

Figure 17.3.7: Behavior of an underdamped spring-mass system.

Note that for all damped systems, $\lim_{t\to\infty} x(t) = 0$. The system always approaches the equilibrium position over time.

Example 17.3.5: Underdamped Spring-Mass System

A 16-lb weight stretches a spring 3.2 ft. Assume the damping force on the system is equal to the instantaneous velocity of the mass. Find the equation of motion if the mass is released from rest at a point 9 in. below equilibrium.

Solution

We have $k = \frac{16}{3.2} = 5$ and $m = \frac{16}{32} = \frac{1}{2}$, so the differential equation is

$$\frac{1}{2}x'' + x' + 5x = 0, \text{ or } x'' + 2x' + 10x = 0.$$

This equation has the general solution

$$x(t) = e^{-t}(c_1 \cos(3t) + c_2 \sin(3t)).$$

Applying the initial conditions, $x(0) = \frac{3}{4}$ and $x'(0) = 0$, we get

$$x(t) = e^{-t}\left(\frac{3}{4}\cos(3t) + \frac{1}{4}\sin(3t)\right).$$

Exercise 17.3.5

A 1-kg mass stretches a spring 49 cm. The system is immersed in a medium that imparts a damping force equal to four times the instantaneous velocity of the mass. Find the equation of motion if the mass is released from rest at a point 24 cm above equilibrium.

Hint

First find the spring constant.

Answer

$x(t) = -0.24e^{-2t}\cos(4t) - 0.12e^{-2t}\sin(4t)$

Example 17.3.6: Chapter Opener: Modeling a Motorcycle Suspension System

For motocross riders, the suspension systems on their motorcycles are very important. The off-road courses on which they ride often include jumps, and losing control of the motorcycle when they land could cost them the race.

Figure 17.3.8: (credit: modification of work by nSeika, Flickr)

This suspension system can be modeled as a damped spring-mass system. We define our frame of reference with respect to the frame of the motorcycle. Assume the end of the shock absorber attached to the motorcycle frame is fixed. Then, the "mass" in our spring-mass system is the motorcycle wheel. We measure the position of the wheel with respect to the motorcycle frame. This may seem counterintuitive, since, in many cases, it is actually the motorcycle frame that moves, but this frame of reference preserves the development of the differential equation that was done earlier. As with earlier development, we define the downward direction to be positive.

When the motorcycle is lifted by its frame, the wheel hangs freely and the spring is uncompressed. This is the spring's natural position. When the motorcycle is placed on the ground and the rider mounts the motorcycle, the spring compresses and the system is in the equilibrium position (Figure 17.3.9).

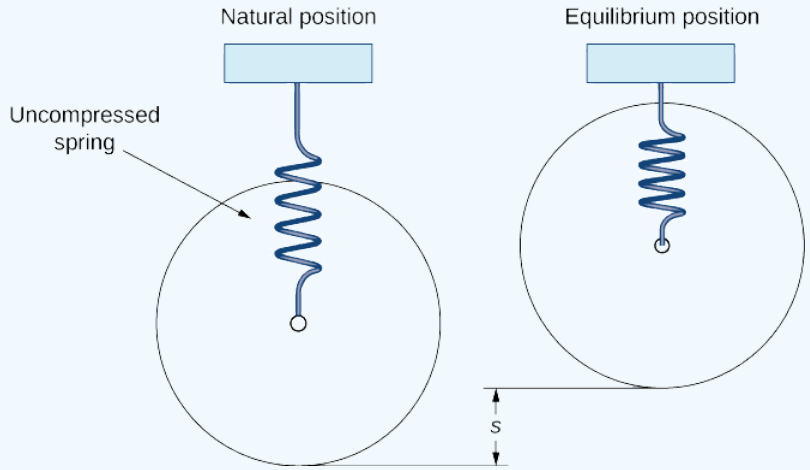

Figure 17.3.9: We can use a spring-mass system to model a motorcycle suspension.

This system can be modeled using the same differential equation we used before:

$$mx''+bx'+kx=0.$$

A motocross motorcycle weighs 204 lb, and we assume a rider weight of 180 lb. When the rider mounts the motorcycle, the suspension compresses 4 in., then comes to rest at equilibrium. The suspension system provides damping equal to 240 times the instantaneous vertical velocity of the motorcycle (and rider).

1. Set up the differential equation that models the behavior of the motorcycle suspension system.
2. We are interested in what happens when the motorcycle lands after taking a jump. Let time

   $$t=0$$

   denote the time when the motorcycle first contacts the ground. If the motorcycle hits the ground with a velocity of 10 ft/sec downward, find the equation of motion of the motorcycle after the jump.
3. Graph the equation of motion over the first second after the motorcycle hits the ground.

Solution

1. We have defined equilibrium to be the point where $mg=ks$, so we have

   $$\begin{aligned} mg &= ks \\ 384 &= k\left(\frac{1}{3}\right) \\ k &= 1152. \end{aligned}$$

   We also have

   $$\begin{aligned} W &= mg \\ 384 &= m(32) \\ m &= 12. \end{aligned}$$

   Therefore, the differential equation that models the behavior of the motorcycle suspension is

   $$12x''+240x'+1152x=0.$$

Dividing through by 12, we get

$$x'' + 20x' + 96x = 0.$$

2. The differential equation found in part a. has the general solution

$$x(t) = c_1 e^{-8t} + c_2 e^{-12t}.$$

Now, to determine our initial conditions, we consider the position and velocity of the motorcycle wheel when the wheel first contacts the ground. Since the motorcycle was in the air prior to contacting the ground, the wheel was hanging freely and the spring was uncompressed. Therefore the wheel is 4 in. $\left(\frac{1}{3}\text{ ft}\right)$ below the equilibrium position (with respect to the motorcycle frame), and we have $x(0) = \frac{1}{3}$. According to the problem statement, the motorcycle has a velocity of 10 ft/sec downward when the motorcycle contacts the ground, so the spring will be given an equal and opposite velocity of 10 ft/sec upward, giving us $x'(0) = -10$. Applying these initial conditions, we get $c_1 = -\frac{3}{2}$ and $c_2 = \frac{11}{6}$, so the equation of motion is

$$x(t) = -\frac{3}{2}e^{-8t} + \frac{11}{6}e^{-12t}.$$

3. The graph is shown in Figure 17.3.10.

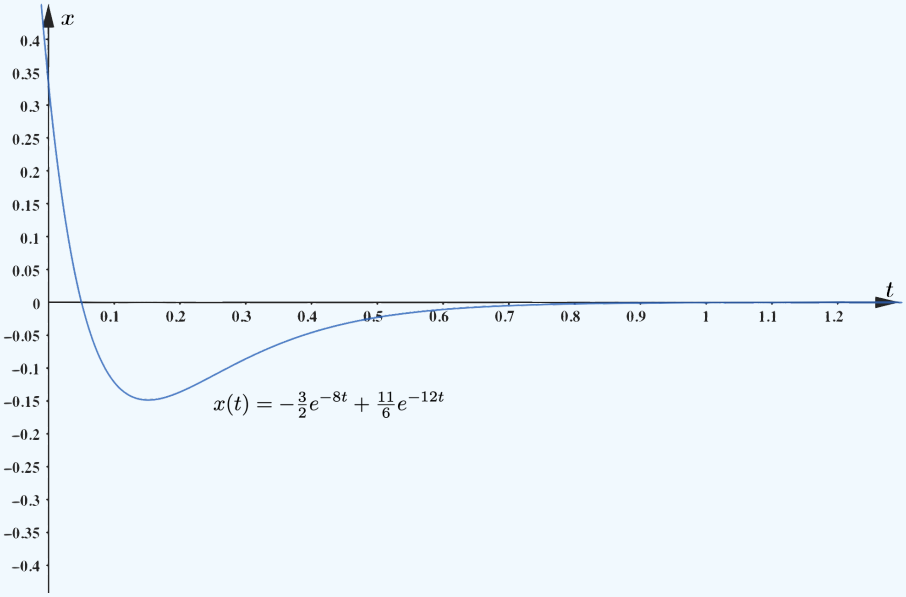

Figure 17.3.10: Graph of the equation of motion over a time of one second.

LANDING VEHICLE

NASA is planning a mission to Mars. To save money, engineers have decided to adapt one of the moon landing vehicles for the new mission. However, they are concerned about how the different gravitational forces will affect the suspension system that cushions the craft when it touches down. The acceleration resulting from gravity on the moon is 1.6 m/sec^2, whereas on Mars it is 3.7 m/sec^2.

The suspension system on the craft can be modeled as a damped spring-mass system. In this case, the spring is below the moon lander, so the spring is slightly compressed at equilibrium, as shown in Figure 17.3.11.

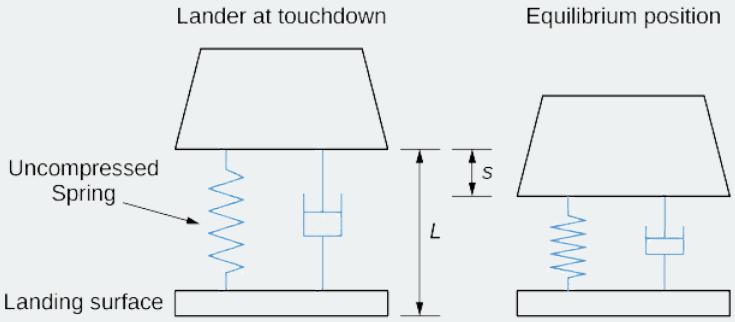

Figure 17.3.11: The landing craft suspension can be represented as a damped spring-mass system. (credit "lander": NASA)

We retain the convention that down is positive. Despite the new orientation, an examination of the forces affecting the lander shows that the same differential equation can be used to model the position of the landing craft relative to equilibrium:

$$mx'' + bx' + kx = 0,$$

where m is the mass of the lander, b is the damping coefficient, and k is the spring constant.

1. The lander has a mass of 15,000 kg and the spring is 2 m long when uncompressed. The lander is designed to compress the spring 0.5 m to reach the equilibrium position under lunar gravity. The dashpot imparts a damping force equal to 48,000 times the instantaneous velocity of the lander. Set up the differential equation that models the motion of the lander when the craft lands on the moon.
2. Let time $t = 0$ denote the instant the lander touches down. The rate of descent of the lander can be controlled by the crew, so that it is descending at a rate of 2 m/sec when it touches down. Find the equation of motion of the lander on the moon.
3. If the lander is traveling too fast when it touches down, it could fully compress the spring and "bottom out." Bottoming out could damage the landing craft and must be avoided at all costs. Graph the equation of motion found in part 2. If the spring is 0.5 m long when fully compressed, will the lander be in danger of bottoming out?
4. Assuming NASA engineers make no adjustments to the spring or the damper, how far does the lander compress the spring to reach the equilibrium position under Martian gravity?
5. If the lander crew uses the same procedures on Mars as on the moon, and keeps the rate of descent to 2 m/sec, will the lander bottom out when it lands on Mars?
6. What adjustments, if any, should the NASA engineers make to use the lander safely on Mars?

Forced Vibrations

The last case we consider is when an external force acts on the system. In the case of the motorcycle suspension system, for example, the bumps in the road act as an external force acting on the system. Another example is a spring hanging from a support; if the support is set in motion, that motion would be considered an external force on the system. We model these forced systems with the nonhomogeneous differential equation

$$mx'' + bx' + kx = f(t),$$

where the external force is represented by the $f(t)$ term. As we saw in Nonhomogenous Linear Equations, differential equations such as this have solutions of the form

$$x(t) = c_1x_1(t) + c_2x_2(t) + x_p(t),$$

where $c_1x_1(t) + c_2x_2(t)$ is the general solution to the complementary equation and $x_p(t)$ is a particular solution to the nonhomogeneous equation. If the system is damped, $\lim_{t\to\infty} c_1x_1(t) + c_2x_2(t) = 0$. Since these terms do not affect the long-term behavior of the system, we call this part of the solution the ***transient solution.*** The long-term behavior of the system is determined by $x_p(t)$, so we call this part of the solution the **steady-state solution**.

Example 17.3.7: Forced Vibrations

A mass of 1 slug stretches a spring 2 ft and comes to rest at equilibrium. The system is attached to a dashpot that imparts a damping force equal to eight times the instantaneous velocity of the mass. Find the equation of motion if an external force

equal to $f(t) = 8\sin(4t)$ is applied to the system beginning at time $t = 0$. What is the transient solution? What is the steady-state solution?

Solution

We have $mg = 1(32) = 2k$, so $k = 16$ and the differential equation is

$$x''+8x'+16x = 8\sin(4t).$$

The general solution to the complementary equation is

$$c_1e^{-4t} + c_2te^{-4t}.$$

Assuming a particular solution of the form $x_p(t) = A\cos(4t) + B\sin(4t)$ and using the method of undetermined coefficients, we find $x_p(t) = -\frac{1}{4}\cos(4t)$, so

$$x(t) = c_1e^{-4t} + c_2te^{-4t} - \frac{1}{4}\cos(4t).$$

At $t = 0$, the mass is at rest in the equilibrium position, so $x(0) = x'(0) = 0$. Applying these initial conditions to solve for c_1 and c_2, we get

$$x(t) = \frac{1}{4}e^{-4t} + te^{-4t} - \frac{1}{4}\cos(4t).$$

The transient solution is $\frac{1}{4}e^{-4t} + te^{-4t}$. The steady-state solution is $-\frac{1}{4}\cos(4t)$.

Exercise 17.3.6

A mass of 2 kg is attached to a spring with constant 32 N/m and comes to rest in the equilibrium position. Beginning at time $t = 0$, an external force equal to $f(t) = 68e^{-2t}\cos(4t)$ is applied to the system. Find the equation of motion if there is no damping. What is the transient solution? What is the steady-state solution?

Hint

Find the particular solution *before* applying the initial conditions.

Answer

$x(t) = -\frac{1}{2}\cos(4t) + \frac{9}{4}\sin(4t) + \frac{1}{2}e^{-2t}\cos(4t) - 2e^{-2t}\sin(4t)$

Transient solution: $\frac{1}{2}e^{-2t}\cos(4t) - 2e^{-2t}\sin(4t)$

Steady-state solution: $-\frac{1}{2}\cos(4t) + \frac{9}{4}\sin(4t)$

RESONANCE

Consider an undamped system exhibiting simple harmonic motion. In the real world, we never truly have an undamped system; –some damping always occurs. For theoretical purposes, however, we could imagine a spring-mass system contained in a vacuum chamber. With no air resistance, the mass would continue to move up and down indefinitely.

The frequency of the resulting motion, given by $f = \frac{1}{T} = \frac{\omega}{2\pi}$, is called the *natural frequency of the system*. If an external force acting on the system has a frequency close to the natural frequency of the system, a phenomenon called ***resonance*** results. The external force reinforces and amplifies the natural motion of the system.

1. Consider the differential equation $x''+x = 0$. Find the general solution. What is the natural frequency of the system?
2. Now suppose this system is subjected to an external force given by $f(t) = 5\cos t$. Solve the initial-value problem $x''+x = 5\cos t$, $x(0) = 0$, $x'(0) = 1$.
3. Graph the solution. What happens to the behavior of the system over time?

4. In the real world, there is always some damping. However, if the damping force is weak, and the external force is strong enough, real-world systems can still exhibit resonance. One of the most famous examples of resonance is the collapse of the **Tacoma Narrows Bridge** on November 7, 1940. The bridge had exhibited strange behavior ever since it was built. The roadway had a strange “bounce” to it. On the day it collapsed, a strong windstorm caused the roadway to twist and ripple violently. The bridge was unable to withstand these forces and it ultimately collapsed. Experts believe the windstorm exerted forces on the bridge that were very close to its natural frequency, and the resulting resonance ultimately shook the bridge apart.

   This website contains more information about the collapse of the Tacoma Narrows Bridge.

   During the short time the Tacoma Narrows Bridge stood, it became quite a tourist attraction. Several people were on site the day the bridge collapsed, and one of them caught the collapse on film.

Watch the video to see the collapse of the Tacoma Narrows Bridge "Gallopin' Gertie". https://www.youtube.com/watch?v=j-zczJXSxnw

1. Another real-world example of resonance is a singer shattering a crystal wineglass when she sings just the right note. When someone taps a crystal wineglass or wets a finger and runs it around the rim, a tone can be heard. That note is created by the wineglass vibrating at its natural frequency. If a singer then sings that same note at a high enough volume, the glass shatters as a result of resonance.

   The TV show *Mythbusters* aired an episode on this phenomenon. Visit this website to learn more about it. Adam Savage also described the experience. Watch this video for his account.

The *RLC* Series Circuit

Consider an electrical circuit containing a resistor, an inductor, and a capacitor, as shown in Figure 17.3.12. Such a circuit is called an ***RLC* series circuit**. *RLC* circuits are used in many electronic systems, most notably as tuners in AM/FM radios. The tuning knob varies the capacitance of the capacitor, which in turn tunes the radio. Such circuits can be modeled by second-order, constant-coefficient differential equations.

Let $I(t)$ denote the current in the *RLC* circuit and $q(t)$ denote the charge on the capacitor. Furthermore, let L denote inductance in henrys (H), R denote resistance in ohms (Ω), and C denote capacitance in farads (F). Last, let $E(t)$ denote electric potential in volts (V).

Kirchhoff’s voltage rule states that the sum of the voltage drops around any closed loop must be zero. So, we need to consider the voltage drops across the inductor (denoted E_L), the resistor (denoted E_R), and the capacitor (denoted E_C). Because the *RLC* circuit shown in Figure 17.3.12 includes a voltage source, $E(t)$, which adds voltage to the circuit, we have $E_L + E_R + E_C = E(t)$.

We present the formulas below without further development and those of you interested in the derivation of these formulas can review the links. Using Faraday's law and Lenz's law, the voltage drop across an inductor can be shown to be proportional to the instantaneous rate of change of current, with proportionality constant L. Thus,

$$E_L = L\frac{dI}{dt}.$$

Next, according to Ohm's law, the voltage drop across a resistor is proportional to the current passing through the resistor, with proportionality constant R. Therefore,

$$E_R = RI.$$

Last, the voltage drop across a capacitor is proportional to the charge, q, on the capacitor, with proportionality constant $1/C$. Thus,

$$E_C = \frac{1}{C}q.$$

Adding these terms together, we get

$$L\frac{dI}{dt} + RI + \frac{1}{C}q = E(t).$$

Noting that $I = (dq)/(dt)$, this becomes

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t).$$

Mathematically, this system is analogous to the spring-mass systems we have been examining in this section.

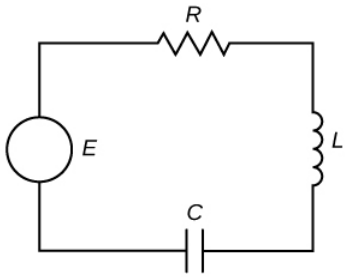

Figure 17.3.12: An *RLC* series circuit can be modeled by the same differential equation as a mass-spring system.

Series Circuit

Find the charge on the capacitor in an *RLC* series circuit where $L = 5/3$ H, $R = 10\Omega$, $C = 1/30$ F, and $E(t) = 300$ V. Assume the initial charge on the capacitor is 0 C and the initial current is 9 A. What happens to the charge on the capacitor over time?

Solution

We have

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$
$$\frac{5}{3}\frac{d^2q}{dt^2} + 10\frac{dq}{dt} + 30q = 300$$
$$\frac{d^2q}{dt^2} + 6\frac{dq}{dt} + 18q = 180.$$

The general solution to the complementary equation is

$$e^{-3t}(c_1\cos(3t) + c_2\sin(3t)).$$

Assume a particular solution of the form $q_p = A$, where A is a constant. Using the method of undetermined coefficients, we find $A = 10$. So,

$$q(t) = e^{-3t}(c_1\cos(3t) + c_2\sin(3t)) + 10.$$

Applying the initial conditions $q(0) = 0$ and $i(0) = ((dq)/(dt))(0) = 9$, we find $c_1 = -10$ and $c_2 = -7$. So the charge on the capacitor is

$$q(t) = -10e^{-3t}\cos(3t) - 7e^{-3t}\sin(3t) + 10.$$

Looking closely at this function, we see the first two terms will decay over time (as a result of the negative exponent in the exponential function). Therefore, the capacitor eventually approaches a steady-state charge of 10 C.

? Exercise 17.3.7

Find the charge on the capacitor in an *RLC* series circuit where $L = 1/5$ H, $R = 2/5\Omega$, $C = 1/2$ F, and $E(t) = 50$ V. Assume the initial charge on the capacitor is 0 C and the initial current is 4 A.

Hint

Remember, $E_L = L((dI)/(dt))$.

Answer

$$q(t) = -25e^{-t}\cos(3t) - 7e^{-t}\sin(3t) + 25$$

Key Concepts

- Second-order constant-coefficient differential equations can be used to model spring-mass systems.
- An examination of the forces on a spring-mass system results in a differential equation of the form

  $$mx''+bx'+kx = f(t),$$

  where m represents the mass, b is the coefficient of the damping force, k is the spring constant, and $f(t)$ represents any net external forces on the system.
- If $b = 0$, there is no damping force acting on the system, and simple harmonic motion results.
- If $b \neq 0$,the behavior of the system depends on whether $b^2 - 4mk > 0, b^2 - 4mk = 0$, or $b^2 - 4mk < 0$.
  - If $b^2 - 4mk > 0$, the system is overdamped and does not exhibit oscillatory behavior.
  - If $b^2 - 4mk = 0$, the system is critically damped. It does not exhibit oscillatory behavior, but any slight reduction in the damping would result in oscillatory behavior.
  - If $b^2 - 4mk < 0$, the system is underdamped. It exhibits oscillatory behavior, but the amplitude of the oscillations decreases over time.
- If$f(t) \neq 0$, the solution to the differential equation is the sum of a transient solution and a steady-state solution. The steady-state solution governs the long-term behavior of the system.
- The charge on the capacitor in an *RLC* series circuit can also be modeled with a second-order constant-coefficient differential equation of the form

  $$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t),$$

  where L is the inductance, R is the resistance, C is the capacitance, and $E(t)$ is the voltage source.

Key Equations

- Equation of simple harmonic motion

  $$x''+\omega^2 x = 0$$

- Solution for simple harmonic motion

  $$x(t) = c_1\cos(\omega t) + c_2\sin(\omega t)$$

- Alternative form of solution for SHM

  $$x(t) = A\sin(\omega t + \phi)$$

- Forced harmonic motion

$$mx''+bx'+kx = f(t)$$

- Charge in a RLC series circuit

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t),$$

Glossary

***RLC* series circuit**

a complete electrical path consisting of a resistor, an inductor, and a capacitor; a second-order, constant-coefficient differential equation can be used to model the charge on the capacitor in an *RLC* series circuit

simple harmonic motion

motion described by the equation $x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$, as exhibited by an undamped spring-mass system in which the mass continues to oscillate indefinitely

steady-state solution

a solution to a nonhomogeneous differential equation related to the forcing function; in the long term, the solution approaches the steady-state solution

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17.3E: Exercises for Section 17.3

1. A mass weighing 4 lb stretches a spring 8 in. Find the equation of motion if the spring is released from the equilibrium position with a downward velocity of 12 ft/sec. What is the period and frequency of the motion?

2. A mass weighing 2 lb stretches a spring 2 ft. Find the equation of motion if the spring is released from 2 in. below the equilibrium position with an upward velocity of 8 ft/sec. What is the period and frequency of the motion?

Answer

differential equation: $x''+16x=0$,
equation of motion: $x(t)=\frac{1}{6}\cos(4t)-2\sin(4t)$,
period $=\frac{\pi}{2}$ sec,
frequency $=\frac{2}{\pi}$ Hz

3. A 100-g mass stretches a spring 0.1 m. Find the equation of motion of the mass if it is released from rest from a position 20 cm below the equilibrium position. What is the frequency of this motion?

4. A 400-g mass stretches a spring 5 cm. Find the equation of motion of the mass if it is released from rest from a position 15 cm below the equilibrium position. What is the frequency of this motion?

Answer

differential equation: $x''+196x=0$,
equation of motion: $x(t)=0.15\cos(14t)$,
period $=\frac{\pi}{7}$ sec,
frequency $=\frac{7}{\pi}$ Hz

5. A block has a mass of 9 kg and is attached to a vertical spring with a spring constant of 0.25 N/m. The block is stretched 0.75 m below its equilibrium position and released.

a. Find the position function $x(t)$ of the block.
b. Find the period and frequency of the vibration.
c. Sketch a graph of $x(t)$.
d. At what time does the block first pass through the equilibrium position?

6. A block has a mass of 5 kg and is attached to a vertical spring with a spring constant of 20 N/m. The block is released from the equilibrium position with a downward velocity of 10 m/sec.

a. Find the position function $x(t)$ of the block.
b. Find the period and frequency of the vibration.
c. Sketch a graph of $x(t)$.
d. At what time does the block first pass through the equilibrium position?

Answer

a. $x(t)=5\sin(2t)$
b. period $=\pi$ sec,
frequency $=\frac{1}{\pi}$ Hz
c.

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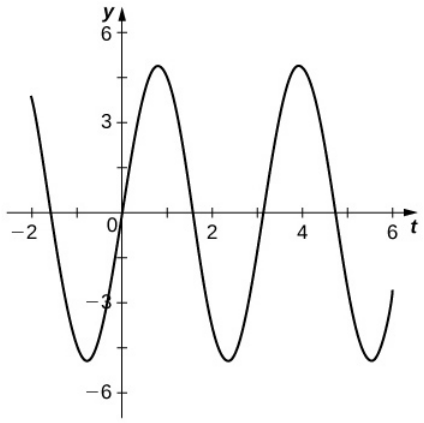

d. $t = \frac{\pi}{2}$ sec

7. A 1-kg mass is attached to a vertical spring with a spring constant of 21 N/m. The resistance in the spring-mass system is equal to 10 times the instantaneous velocity of the mass.

a. Find the equation of motion if the mass is released from a position 2 m below its equilibrium position with a downward velocity of 2 m/sec.
b. Graph the solution and determine whether the motion is overdamped, critically damped, or underdamped.

8. An 800-lb weight (25 slugs) is attached to a vertical spring with a spring constant of 226 lb/ft. The system is immersed in a medium that imparts a damping force equal to 10 times the instantaneous velocity of the mass.

a. Find the equation of motion if it is released from a position 20 ft below its equilibrium position with a downward velocity of 41 ft/sec.
b. Graph the solution and determine whether the motion is overdamped, critically damped, or underdamped.

Answer

a. $x(t) = e^{-t/5}(20\cos(3t) + 15\sin(3t))$

b. underdamped

9. A 9-kg mass is attached to a vertical spring with a spring constant of 16 N/m. The system is immersed in a medium that imparts a damping force equal to 24 times the instantaneous velocity of the mass.

a. Find the equation of motion if it is released from its equilibrium position with an upward velocity of 4 m/sec.
b. Graph the solution and determine whether the motion is overdamped, critically damped, or underdamped.

10. A 1-kg mass stretches a spring 6.25 cm. The resistance in the spring-mass system is equal to eight times the instantaneous velocity of the mass.

a. Find the equation of motion if the mass is released from a position 5 m below its equilibrium position with an upward velocity of 10 m/sec.
b. Determine whether the motion is overdamped, critically damped, or underdamped.

Answer

a. $x(t) = 5e^{-4t} + 10te^{-4t}$

b. critically damped

11. A 32-lb weight (1 slug) stretches a vertical spring 4 in. The resistance in the spring-mass system is equal to four times the instantaneous velocity of the mass.

a. Find the equation of motion if it is released from its equilibrium position with a downward velocity of 12 ft/sec.
b. Determine whether the motion is overdamped, critically damped, or underdamped.

12. A 64-lb weight is attached to a vertical spring with a spring constant of 4.625 lb/ft. The resistance in the spring-mass system is equal to the instantaneous velocity. The weight is set in motion from a position 1 ft below its equilibrium position with an upward velocity of 2 ft/sec. Is the mass above or below the equation position at the end of π sec? By what distance?

Answer

$x(\pi) = \frac{7e^{-\pi/4}}{6}$ ft below

13. A mass that weighs 8 lb stretches a spring 6 inches. The system is acted on by an external force of $8\sin 8t$lb. If the mass is pulled down 3 inches and then released, determine the position of the mass at any time.

14. A mass that weighs 6 lb stretches a spring 3 in. The system is acted on by an external force of $8\sin(4t)$ lb. If the mass is pulled down 1 inch and then released, determine the position of the mass at any time.

Answer

$x(t) = \frac{32}{9}\sin(4t) + \cos(\sqrt{128}t) - \frac{16}{9\sqrt{2}}\sin(\sqrt{128}t)$

15. Find the charge on the capacitor in an *RLC* series circuit where $L = 40$ H, $R = 30\,\Omega$, $C = 1/200$ F, and $E(t) = 200$ V. Assume the initial charge on the capacitor is 7 C and the initial current is 0 A.

16. Find the charge on the capacitor in an *RLC* series circuit where $L = 2$ H, $R = 24\,\Omega$, $C = 0.005$ F, and $E(t) = 12\sin 10t$ V. Assume the initial charge on the capacitor is 0.001 C and the initial current is 0 A.

Answer

$q(t) = e^{-6t}(0.051\cos(8t) + 0.03825\sin(8t)) - \frac{1}{20}\cos(10t)$

17. A series circuit consists of a device where$L = 1$ H, $R = 20\,\Omega$, $C = 0.002$ F, and $E(t) = 12$ V. If the initial charge and current are both zero, find the charge and current at time t.

18. A series circuit consists of a device where $L = 12$ H, $R = 10\,\Omega$, $C = \frac{1}{50}$ F, and $E(t) = 250$ V. If the initial charge on the capacitor is 0 C and the initial current is 18 A, find the charge and current at time t.

Answer

$q(t) = e^{-10t}(-32t - 5) + 5, I(t) = 2e^{-10t}(160t + 9)$

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17.4: Series Solutions of Differential Equations

Learning Objectives

- Use power series to solve first-order and second-order differential equations.

Previously, we studied how functions can be represented as power series, $y(x) = \sum_{n=0}^{\infty} a_n x^n$. We also saw that we can find series representations of the derivatives of such functions by differentiating the power series term by term. This gives

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

and

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

In some cases, these power series representations can be used to find solutions to differential equations.

The examples and exercises in this section were chosen for which power solutions exist. However, it is not always the case that power solutions exist. Those of you interested in a more rigorous treatment of this topic should review the differential equations section of the LibreTexts.

Problem-Solving Strategy: Finding Power Series Solutions to Differential Equations

1. Assume the differential equation has a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

2. Differentiate the power series term by term to get

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

and

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

3. Substitute the power series expressions into the differential equation.
4. Re-index sums as necessary to combine terms and simplify the expression.
5. Equate coefficients of like powers of x to determine values for the coefficients a_n in the power series.
6. Substitute the coefficients back into the power series and write the solution.

Example 17.4.1: Series Solutions to Differential Equations

Find a power series solution for the following differential equations.

a. $y'' - y = 0$
b. $(x^2 - 1)y'' + 6xy' + 4y = -4$

Solution

Part a

Assume

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \qquad \text{(step 1)}$$

Then,

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad \text{(step 2A)}$$

and

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \qquad \text{(step 2B)}$$

We want to find values for the coefficients a_n such that

$$y'' - y = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0. \qquad \text{(step 3)}$$

We want the indices on our sums to match so that we can express them using a single summation. That is, we want to rewrite the first summation so that it starts with $n = 0$.

To re-index the first term, replace n with $n + 2$ inside the sum, and change the lower summation limit to $n = 0$. We get

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n.$$

This gives

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x_n = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} - a_n] x^n = 0. \qquad \text{(step 4)}$$

Because power series expansions of functions are unique, this equation can be true only if the coefficients of each power of x are zero. So we have

$$(n+2)(n+1) a_{n+2} - a_n = 0 \text{ for } n = 0, 1, 2, \ldots .$$

This recurrence relationship allows us to express each coefficient a_n in terms of the coefficient two terms earlier. This yields one expression for even values of n and another expression for odd values of n. Looking first at the equations involving even values of n, we see that

$$a_2 = \frac{a_0}{2}$$

$$a_4 = \frac{a_2}{4 \cdot 3} = \frac{a_0}{4!}$$

$$a_6 = \frac{a_4}{6 \cdot 5} = \frac{a_0}{6!}$$

$$\vdots$$

Thus, in general, when n is even,

$$a_n = \frac{a_0}{n!}. \qquad \text{(step 5)}$$

For the equations involving odd values of n, we see that

$$a_3 = \frac{a_1}{3 \cdot 2} = \frac{a_1}{3!}$$

$$a_5 = \frac{a_3}{5 \cdot 4} = \frac{a_1}{5!}$$

$$a_7 = \frac{a_5}{7 \cdot 6} = \frac{a_1}{7!}$$

$$\vdots$$

Therefore, in general, when n is odd,

$$a_n = \frac{a_1}{n!}. \qquad \text{(step 5)}$$

Putting this together, we have

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + a_1 x + \frac{a_0}{2}x^2 + \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 + \frac{a_1}{5!}x^5 + \cdots. \end{aligned}$$

Re-indexing the sums to account for the even and odd values of n separately, we obtain

$$y(x) = a_0 \sum_{k=0}^{\infty} \frac{1}{(2k)!}x^{2k} + a_1 \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}x^{2k+1}. \qquad \text{(step 6)}$$

Analysis for part a.

As expected for a second-order differential equation, this solution depends on two arbitrary constants. However, note that our differential equation is a constant-coefficient differential equation, yet the power series solution does not appear to have the familiar form (containing exponential functions) that we are used to seeing. Furthermore, since $y(x) = c_1 e^x + c_2 e^{-x}$ is the general solution to this equation, we must be able to write any solution in this form, and it is not clear whether the power series solution we just found can, in fact, be written in that form.

Fortunately, after writing the power series representations of e^x and e^{-x}, and doing some algebra, we find that if we choose

$$c_0 = \frac{(a_0 + a_1)}{2}, c_1 = \frac{(a_0 - a_1)}{2},$$

we then have $a_0 = c_0 + c_1$ and $a_1 = c_0 - c_1$, and

$$\begin{aligned} y(x) &= a_0 + a_1 x + \frac{a_0}{2}x^2 + \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 + \frac{a_1}{5!}x^5 + \cdots \\ &= (c_0 + c_1) + (c_0 - c_1)x + \frac{(c_0 + c_1)}{2}x^2 + \frac{(c_0 - c_1)}{3!}x^3 + \frac{(c_0 + c_1)}{4!}x^4 + \frac{(c_0 - c_1)}{5!}x^5 + \cdots \\ &= c_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} + c_1 \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \\ &= c_0 e^x + c_1 e^{-x}. \end{aligned}$$

So we have, in fact, found the same general solution. Note that this choice of c_1 and c_2 is not obvious. This is a case when we know what the answer should be, and have essentially "reverse-engineered" our choice of coefficients.

Part b

Assume

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \qquad \text{(step 1)}$$

Then,

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad \text{(step 2)}$$

and

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \qquad \text{(step 2)}$$

We want to find values for the coefficients a_n such that

$$(x^2-1)y''+6xy'+4y=-4$$
$$(x^2-1)\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}+6x\sum_{n=1}^{\infty}na_nx^{n-1}+4\sum_{n=0}^{\infty}a_nx^n=-4$$
$$x^2\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}-\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}+6x\sum_{n=1}^{\infty}na_nx^{n-1}+4\sum_{n=0}^{\infty}a_nx^n=-4.$$

Taking the external factors inside the summations, we get

$$\sum_{n=2}^{\infty}n(n-1)a_nx^n-\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}+\sum_{n=1}^{\infty}6na_nx^n+\sum_{n=0}^{\infty}4a_nx^n=-4. \quad \text{(step 3)}$$

Now, in the first summation, we see that when $n=0$ or $n=1$, the term evaluates to zero, so we can add these terms back into our sum to get

$$\sum_{n=2}^{\infty}n(n-1)a_nx^n=\sum_{n=0}^{\infty}n(n-1)a_nx^n.$$

Similarly, in the third term, we see that when $n=0$, the expression evaluates to zero, so we can add that term back in as well. We have

$$\sum_{n=1}^{\infty}6na_nx^n=\sum_{n=0}^{\infty}6na_nx^n.$$

Then, we need only shift the indices in our second term. We get

$$\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n.$$

Thus, we have

$$\sum_{n=0}^{\infty}n(n-1)a_nx^n-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}6na_nx^n+\sum_{n=0}^{\infty}4a_nx^n=-4 \quad \text{(step 4)}$$
$$\sum_{n=0}^{\infty}[n(n-1)a_n-(n+2)(n+1)a_{n+2}+6na_n+4a_n]x^n=-4$$
$$\sum_{n=0}^{\infty}[(n^2-n)a_n+6na_n+4a_n-(n+2)(n+1)a_{n+2}]x^n=-4$$
$$\sum_{n=0}^{\infty}[n^2a_n+5na_n+4a_n-(n+2)(n+1)a_{n+2}]x^n=-4$$
$$\sum_{n=0}^{\infty}[(n^2+5n+4)a_n-(n+2)(n+1)a_{n+2}]x^n=-4$$
$$\sum_{n=0}^{\infty}[(n+4)(n+1)a_n-(n+2)(n+1)a_{n+2}]x^n=-4$$

Looking at the coefficients of each power of x, we see that the constant term must be equal to -4, and the coefficients of all other powers of x must be zero. Then, looking first at the constant term,

$$\begin{aligned}4a_0-2a_2&=-4\\ a_2&=2a_0+2\end{aligned} \quad \text{(step 3)}$$

For $n\geq 1$, we have

$$\begin{aligned}(n+4)(n+1)a_n-(n+2)(n+1)a_{n+2}&=0\\ (n+1)[(n+4)a_n-(n+2)a_{n+2}]&=0.\end{aligned}$$

Since $n\geq 1,\ n+1\neq 0$, we see that

$$(n+4)a_n-(n+2)a_{n+2}=0$$

and thus

$$a_{n+2} = \frac{n+4}{n+2} a_n.$$

For even values of n, we have

$$\begin{aligned} a_4 &= \frac{6}{4}(2a_0 + 2) = 3a_0 + 3 \\ a_6 &= \frac{8}{6}(3a_0 + 3) = 4a_0 + 4 \\ &\vdots \end{aligned}$$

In general,

$$a_{2k} = (k+1)(a_0 + 1). \qquad \text{(step 5)}$$

For odd values of n, we have

$$\begin{aligned} a_3 &= \frac{5}{3} a_1 \\ a_5 &= \frac{7}{5} a_3 = \frac{7}{3} a_1 \\ a_7 &= \frac{9}{7} a_5 = \frac{9}{3} a_1 = 3a_1 \\ &\vdots \end{aligned}$$

In general,

$$a_{2k+1} = \frac{2k+3}{3} a_1. \qquad \text{(step 5 continued)}$$

Putting this together, we have

$$y(x) = \sum_{k=0}^{\infty} (k+1)(a_0+1)x^{2k} + \sum_{k=0}^{\infty} (\frac{2k+3}{3}) a_1 x^{2k+1}. \qquad \text{(step 6)}$$

? Exercise 17.4.1

Find a power series solution for the following differential equations.

a. $y' + 2xy = 0$
b. $(x+1)y' = 3y$

Hint

Follow the problem-solving strategy.

Answer a

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n} = a_0 e^{-x^2}$$

Answer b

$$y(x) = a_0 (x+1)^3$$

Bessel functions

We close this section with a brief introduction to *Bessel functions*. Complete treatment of Bessel functions is well beyond the scope of this course, but we get a little taste of the topic here so we can see how series solutions to differential equations are used in real-world applications. The Bessel equation of order n is given by

$$x^2 y'' + xy' + (x^2 - n^2) y = 0.$$

This equation arises in many physical applications, particularly those involving cylindrical coordinates, such as the vibration of a circular drum head and transient heating or cooling of a cylinder. In the next example, we find a power series solution to the Bessel equation of order 0.

Example 17.4.2: Power Series Solution to the Bessel Equation

Find a power series solution to the Bessel equation of order 0 and graph the solution.

Solution

The Bessel equation of order 0 is given by

$$x^2y''+xy'+x^2y=0.$$

We assume a solution of the form $y=\sum_{n=0}^{\infty}a_nx^n$. Then $y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}$ and $y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}$.Substituting this into the differential equation, we get

$$\begin{aligned}
&x^2\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}+x\sum_{n=1}^{\infty}na_nx^{n-1}\\
&\qquad+x^2\sum_{n=0}^{\infty}a_nx^n=0 && \text{Substitution.}\\
&\sum_{n=2}^{\infty}n(n-1)a_nx^n+\sum_{n=1}^{\infty}na_nx^n+\sum_{n=0}^{\infty}a_nx^{n+2}=0 && \text{Bring external factors within sums.}\\
&\sum_{n=2}^{\infty}n(n-1)a_nx^n+\sum_{n=1}^{\infty}na_nx^n+\sum_{n=2}^{\infty}a_{n-2}x^n=0 && \text{Re-index third sum.}\\
&\sum_{n=2}^{\infty}n(n-1)a_nx^n+a_1x\\
&\qquad+\sum_{n=2}^{\infty}na_nx^n+\sum_{n=2}^{\infty}a_{n-2}x^n=0 && \text{Separate } n=1 \text{ term from second sum.}\\
&a_1x+\sum_{n=2}^{\infty}[n(n-1)a_n+na_n+a_{n-2}]x^n=0 && \text{Collect summation terms.}\\
&a_1x+\sum_{n=2}^{\infty}[(n^2-n)a_n+na_n+a_{n-2}]x^n=0 && \text{Multiply through in first term.}\\
&a_1x+\sum_{n=2}^{\infty}[n^2a_n+a_{n-2}]x^n=0. && \text{Simplify.}
\end{aligned}$$

Then, $a_1=0$, and for $n\geq 2$,

$$\begin{aligned}
n^2a_n+a_{n-2}&=0\\
a_n&=-\frac{1}{n^2}a_{n-2}.
\end{aligned}$$

Because $a_1=0$, all odd terms are zero. Then, for even values of n, we have

$$\begin{aligned}
a_2&=-\frac{1}{2^2}a_0\\
a_4&=-\frac{1}{4^2}a_2=\frac{1}{4^2\cdot 2^2}a_0.\\
a_6&=-\frac{1}{6^2}a_4=-\frac{1}{6^2\cdot 4^2\cdot 2^2}a_0
\end{aligned}$$

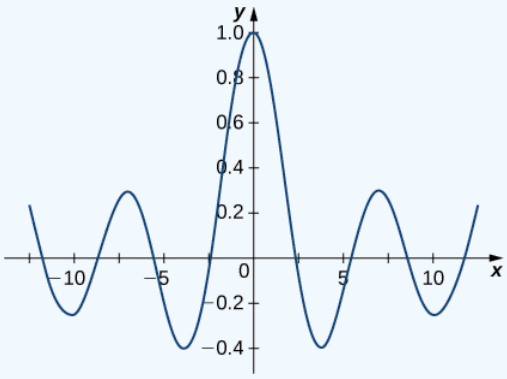
In general,

$$a_{2k}=\frac{(-1)^k}{(2)^{2k}(k!)^2}a_0.$$

Thus, we have

$$y(x) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2)^{2k}(k!)^2} x^{2k}.$$

The graph appears below.



? Exercise 17.4.2

Verify that the expression found in Example 17.4.2 is a solution to the Bessel equation of order 0.

Hint

Differentiate the power series term by term and substitute it into the differential equation.

Key Concepts

- Power series representations of functions can sometimes be used to find solutions to differential equations.
- Differentiate the power series term by term and substitute into the differential equation to find relationships between the power series coefficients.

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17.4E: Exercises for Section 17.4

Find a power series solution for each of the following differential equations.

1. $y''+6y'=0$

2. $5y''+y'=0$

Answer

$$y=c_0+5c_1\sum_{n=1}^{\infty}\frac{(-x/5)^n}{n!}=c_0+5c_1e^{-x/5}$$

3. $y''+25y=0$

4. $y''-y=0$

Answer

$$y=c_0\sum_{n=0}^{\infty}\frac{(x)^{2n}}{(2n)!}+c_1\sum_{n=0}^{\infty}\frac{(x)^{2n+1}}{(2n+1)!}$$

5. $2y'+y=0$

6. $y'-2xy=0$

Answer

$$y=c_0\sum_{n=0}^{\infty}\frac{x^{2n}}{n!}=c_0e^{x^2}$$

7. $(x-7)y'+2y=0$

8. $y''-xy'-y=0$

Answer

$$y=c_0\sum_{n=0}^{\infty}\frac{x^{2n}}{2^n n!}+c_1\sum_{n=0}^{\infty}\frac{x^{2n+1}}{1\cdot 3\cdot 5\cdot 7\cdots(2n+1)}$$

9. $(1+x^2)y''-4xy'+6y=0$

10. $x^2y''-xy'-3y=0$

Answer

$$y=c_1x^3+\frac{c_2}{x}$$

11. $y''-8y'=0,\quad y(0)=-2,\ y'(0)=10$

12. $y''-2xy=0,\quad y(0)=1,\ y'(0)=-3$

Answer

$$y=1-3x+\frac{2x^3}{3!}-\frac{12x^4}{4!}+\frac{16x^6}{6!}-\frac{120x^7}{7!}+\cdots$$

13. The differential equation $x^2y''+xy'+(x^2-1)y=0$ is a Bessel equation of order 1. Use a power series of the form $y=\sum_{n=0}^{\infty}a_nx^n$ to find the solution.

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17.R: Chapter 17 Review Exercises

***True or False*? Justify your answer with a proof or a counterexample.**

1. If y and z are both solutions to $y''+2y'+y=0$, then $y+z$ is also a solution.

Answer

True

2. The following system of algebraic equations has a unique solution:

$6z_1+3z_2=8$
$4z_1+2z_2=4.$

3. $y=e^x\cos(3x)+e^x\sin(2x)$ is a solution to the second-order differential equation $y''+2y'+10=0.$

Answer

False

4. To find the particular solution to a second-order differential equation, you need one initial condition.

In problems 5 - 8, classify the differential equations. Determine the order, whether it is linear and, if linear, whether the differential equation is homogeneous or nonhomogeneous. If the equation is second-order homogeneous and linear, find the characteristic equation.

5. $y''-2y=0$

Answer

second order, linear, homogeneous, $\lambda^2-2=0$

6. $y''-3y+2y=\cos(t)$

7. $\left(\frac{dy}{dt}\right)^2+yy'=1$

Answer

first order, nonlinear, nonhomogeneous

8. $\frac{d^2y}{dt^2}+t\frac{dy}{dt}+\sin^2(t)y=e^t$

In problems 9 - 16, find the general solution.

9. $y''+9y=0$

Answer

$y=c_1\sin(3x)+c_2\cos(3x)$

10. $y''+2y'+y=0$

11. $y''-2y'+10y=4x$

Answer

$y=c_1e^x\sin(3x)+c_2e^x\cos(3x)+\frac{2}{5}x+\frac{2}{25}$

12. $y''=\cos(x)+2y'+y$

13. $y''+5y+y=x+e^{2x}$

Answer

$y=c_1e^{-x}+c_2e^{-4x}+\frac{x}{4}+\frac{e^{2x}}{18}-\frac{5}{16}$

14. $y''=3y'+xe^{-x}$

15. $y'' - x^2 = -3y' - \frac{9}{4}y + 3x$

Answer

$y = c_1 e^{(-3/2)x} + c_2 x e^{(-3/2)x} + \frac{4}{9}x^2 + \frac{4}{27}x - \frac{16}{27}$

16. $y'' = 2\cos x + y' - y$

In problems 17 - 18, find the solution to the initial-value problem, if possible.

17. $y'' + 4y' + 6y = 0,\ y(0) = 0,\ y'(0) = \sqrt{2}$

Answer

$y = e^{-2x}\sin(\sqrt{2}x)$

18. $y'' = 3y - \cos(x),\ y(0) = \frac{9}{4},\ y'(0) = 0$

In problems 19 - 20, find the solution to the boundary-value problem.

19. $4y' = -6y + 2y'',\ y(0) = 0,\ y(1) = 1$

Answer

$y = \frac{e^{1-x}}{e^4 - 1}(e^{4x} - 1)$

20. $y'' = 3x - y - y',\ y(0) = -3,\ y(1) = 0$

For the following problem, set up and solve the differential equation.

21. The motion of a swinging pendulum for small angles θ can be approximated by $\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$, where θ is the angle the pendulum makes with respect to a vertical line, g is the acceleration resulting from gravity, and L is the length of the pendulum. Find the equation describing the angle of the pendulum at time t, assuming an initial displacement of θ_0 and an initial velocity of zero.

Answer

$\theta(t) = \theta_0 \cos\left(\sqrt{\frac{g}{l}}t\right)$

In problems 22 - 23, consider the "beats" that occur when the forcing term of a differential equation causes "slow" and "fast" amplitudes. Consider the general differential equation $ay'' + by = \cos(\omega t)$ that governs undamped motion. Assume that $\sqrt{\frac{b}{a}} \neq \omega$.

22. Find the general solution to this equation (*Hint:* call $\omega_0 = \sqrt{b/a}$).

23. Assuming the system starts from rest, show that the particular solution can be written as $y = \frac{2}{a(\omega_0^2 - \omega^2)}\sin\left(\frac{\omega_0 - \omega t}{2}\right)\sin\left(\frac{\omega_0 + \omega t}{2}\right)$.

24. [T] Using your solutions derived earlier, plot the solution to the system $2y'' + 9y = \cos(2t)$ over the interval $t = [-50, 50]$. Find, analytically, the period of the fast and slow amplitudes.

For the following problem, set up and solve the differential equations.

25. An opera singer is attempting to shatter a glass by singing a particular note. The vibrations of the glass can be modeled by $y'' + ay = \cos(bt)$, where $y'' + ay = 0$ represents the natural frequency of the glass and the singer is forcing the vibrations at $\cos(bt)$. For what value b would the singer be able to break that glass? (*Note*: in order for the glass to break, the oscillations would need to get higher and higher.)

Answer

$b = \sqrt{a}$

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CHAPTER OVERVIEW

Appendices

Appendix A: Table of Derivatives
Appendix B: Table of Integrals

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Appendix A: Table of Derivatives

General Formulas

1. $\frac{d}{dx}(c) = 0$
2. $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$
3. $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$
4. $\frac{d}{dx}(x^n) = nx^{n-1}, \quad \text{for real numbers } n$
5. $\frac{d}{dx}(cf(x)) = cf'(x)$
6. $\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$
7. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$
8. $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

Trigonometric Functions

9. $\frac{d}{dx}(\sin x) = \cos x$
10. $\frac{d}{dx}(\tan x) = \sec^2 x$
11. $\frac{d}{dx}(\sec x) = \sec x \tan x$
12. $\frac{d}{dx}(\cos x) = -\sin x$
13. $\frac{d}{dx}(\cot x) = -\csc^2 x$
14. $\frac{d}{dx}(\csc x) = -\csc x \cot x$

Inverse Trigonometric Functions

15. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
16. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
17. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$
18. $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$
19. $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$
20. $\frac{d}{dx}(\csc^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$

Exponential and Logarithmic Functions

21. $\frac{d}{dx}(e^x) = e^x$
22. $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$
23. $\frac{d}{dx}(b^x) = b^x \ln b$
24. $\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$

Hyperbolic Functions

25. $\frac{d}{dx}(\sinh x) = \cosh x$
26. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$
27. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$
28. $\frac{d}{dx}(\cosh x) = \sinh x$
29. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$
30. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

Inverse Hyperbolic Functions

31. $\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{x^2+1}}$
32. $\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2} \quad (|x| < 1)$
33. $\frac{d}{dx}\left(\operatorname{sech}^{-1} x\right) = \frac{-1}{x\sqrt{1-x^2}} \quad (0 < x < 1)$
34. $\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2-1}} \quad (x > 1)$
35. $\frac{d}{dx}\left(\coth^{-1} x\right) = \frac{1}{1-x^2} \quad (|x| > 1)$
36. $\frac{d}{dx}\left(\operatorname{csch}^{-1} x\right) = \frac{-1}{|x|\sqrt{1+x^2}} \quad (x \neq 0)$

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Appendix B: Table of Integrals

Basic Integrals

1. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$
2. $\int \frac{du}{u} = \ln|u| + C$
3. $\int e^u\,du = e^u + C$
4. $\int a^u\,du = \frac{a^u}{\ln a} + C$
5. $\int \sin u\,du = -\cos u + C$
6. $\int \cos u\,du = \sin u + C$
7. $\int \sec^2 u\,du = \tan u + C$
8. $\int \csc^2 u\,du = -\cot u + C$
9. $\int \sec u \tan u\,du = \sec u + C$
10. $\int \csc u \cot u\,du = -\csc u + C$
11. $\int \tan u\,du = \ln|\sec u| + C$
12. $\int \cot u\,du = \ln|\sin u| + C$
13. $\int \sec u\,du = \ln|\sec u + \tan u| + C$
14. $\int \csc u\,du = \ln|\csc u - \cot u| + C$
15. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$
16. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
17. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{|u|}{a} + C$

Trigonometric Integrals

18. $\int \sin^2 u\,du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$
19. $\int \cos^2 u\,du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$
20. $\int \tan^2 u\,du = \tan u - u + C$
21. $\int \cot^2 u\,du = -\cot u - u + C$
22. $\int \sin^3 u\,du = -\frac{1}{3}(2 + \sin^2 u)\cos u + C$
23. $\int \cos^3 u\,du = \frac{1}{3}(2 + \cos^2 u)\sin u + C$
24. $\int \tan^3 u\,du = \frac{1}{2}\tan^2 u + \ln|\cos u| + C$
25. $\int \cot^3 u\,du = -\frac{1}{2}\cot^2 u - \ln|\sin u| + C$

26. $\int \sec^3 u\,du = \frac{1}{2}\sec u \tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$

27. $\int \csc^3 u\,du = -\frac{1}{2}\csc u \cot u + \frac{1}{2}\ln|\csc u - \cot u| + C$

28. $\int \sin^n u\,du = \frac{-1}{n}\sin^{n-1} u \cos u + \frac{n-1}{n}\int \sin^{n-2} u\,du$

29. $\int \cos^n u\,du = \frac{1}{n}\cos^{n-1} u \sin u + \frac{n-1}{n}\int \cos^{n-2} u\,du$

30. $\int \tan^n u\,du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u\,du$

31. $\int \cot^n u\,du = \frac{-1}{n-1}\cot^{n-1} u - \int \cot^{n-2} u\,du$

32. $\int \sec^n u\,du = \frac{1}{n-1}\tan u \sec^{n-2} u + \frac{n-2}{n-1}\int \sec^{n-2} u\,du$

33. $\int \csc^n u\,du = \frac{-1}{n-1}\cot u \csc^{n-2} u + \frac{n-2}{n-1}\int \csc^{n-2} u\,du$

34. $\int \sin au \sin bu\,du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C$

35. $\int \cos au \cos bu\,du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$

36. $\int \sin au \cos bu\,du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$

37. $\int u \sin u\,du = \sin u - u\cos u + C$

38. $\int u \cos u\,du = \cos u + u\sin u + C$

39. $\int u^n \sin u\,du = -u^n \cos u + n\int u^{n-1}\cos u\,du$

40. $\int u^n \cos u\,du = u^n \sin u - n\int u^{n-1}\sin u\,du$

41. Erroneous nesting of equation structures

Exponential and Logarithmic Integrals

42. $\int ue^{au}\,du = \frac{1}{a^2}(au-1)e^{au} + C$

43. $\int u^n e^{au}\,du = \frac{1}{a}u^n e^{au} - \frac{n}{a}\int u^{n-1}e^{au}\,du$

44. $\int e^{au}\sin bu\,du = \frac{e^{au}}{a^2+b^2}(a\sin bu - b\cos bu) + C$

45. $\int e^{au}\cos bu\,du = \frac{e^{au}}{a^2+b^2}(a\cos bu + b\sin bu) + C$

46. $\int \ln u\,du = u\ln u - u + C$

47. $\int u^n \ln u\,du = \frac{u^{n+1}}{(n+1)^2}[(n+1)\ln u - 1] + C$

48. $\int \frac{1}{u\ln u}\,du = \ln|\ln u| + C$

Hyperbolic Integrals

49. $\int \sinh u\,du = \cosh u + C$

50. $\int \cosh u\,du = \sinh u + C$

51. $\int \tanh u\,du = \ln\cosh u + C$

52. $\int \coth u\,du = \ln|\sinh u| + C$

53. $\int \operatorname{sech} u\,du = \tan^{-1}|\sinh u| + C$

54. $\int \operatorname{csch} u\,du = \ln|\tanh\frac{1}{2}u| + C$

55. $\int \operatorname{sech}^2 u\,du = \tanh u + C$

56. $\int \operatorname{csch}^2 u\,du = -\coth u + C$

57. $\int \operatorname{sech} u\tanh u\,du = -\operatorname{sech} u + C$

58. $\int \operatorname{csch} u\coth u\,du = -\operatorname{csch} u + C$

Inverse Trigonometric Integrals

59. $\int \sin^{-1} u\,du = u\sin^{-1} u + \sqrt{1-u^2} + C$

60. $\int \cos^{-1} u\,du = u\cos^{-1} u - \sqrt{1-u^2} + C$

61. $\int \tan^{-1} u\,du = u\tan^{-1} u - \frac{1}{2}\ln(1+u^2) + C$

62. $\int u\sin^{-1} u\,du = \frac{2u^2-1}{4}\sin^{-1} u + \frac{u\sqrt{1-u^2}}{4} + C$

63. $\int u\cos^{-1} u\,du = \frac{2u^2-1}{4}\cos^{-1} u - \frac{u\sqrt{1-u^2}}{4} + C$

64. $\int u\tan^{-1} u\,du = \frac{u^2+1}{2}\tan^{-1} u - \frac{u}{2} + C$

65. $\int u^n\sin^{-1} u\,du = \frac{1}{n+1}\left[u^{n+1}\sin^{-1} u - \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], \quad n \neq -1$

66. $\int u^n\cos^{-1} u\,du = \frac{1}{n+1}\left[u^{n+1}\cos^{-1} u + \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], \quad n \neq -1$

67. $\int u^n\tan^{-1} u\,du = \frac{1}{n+1}\left[u^{n+1}\tan^{-1} u - \int \frac{u^{n+1}\,du}{1+u^2}\right], \quad n \neq -1$

Integrals Involving $a^2 + u^2$, $a > 0$

68. $\int \sqrt{a^2+u^2}\,du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

69. $\int u^2\sqrt{a^2+u^2}\,du = \frac{u}{8}(a^2+2u^2)\sqrt{a^2+u^2} - \frac{a^4}{8}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

70. $\int \frac{\sqrt{a^2+u^2}}{u}\,du = \sqrt{a^2+u^2} - a\ln\left|\frac{a+\sqrt{a^2+u^2}}{u}\right| + C$

71. $\int \frac{\sqrt{a^2+u^2}}{u^2}\,du = -\frac{\sqrt{a^2+u^2}}{u} + \ln\left(u+\sqrt{a^2+u^2}\right) + C$

72. $\int \frac{du}{\sqrt{a^2+u^2}} = \ln\left(u+\sqrt{a^2+u^2}\right) + C$

73. $\int \frac{u^2}{\sqrt{a^2+u^2}}\,du = \frac{u}{2}\left(\sqrt{a^2+u^2}\right) - \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

74. $\int \frac{du}{u\sqrt{a^2+u^2}} = \frac{-1}{a}\ln\left|\frac{\sqrt{a^2+u^2}+a}{u}\right| + C$

75. $\int \frac{du}{u^2\sqrt{a^2+u^2}} = -\frac{\sqrt{a^2+u^2}}{a^2u} + C$

76. $\int \frac{du}{(a^2+u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2+u^2}} + C$

Integrals Involving $u^2 - a^2$, $a > 0$

77. $\displaystyle\int \sqrt{u^2-a^2}\,du = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

78. $\displaystyle\int u^2\sqrt{u^2-a^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{u^2-a^2} - \frac{a^4}{8}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

79. $\displaystyle\int \frac{\sqrt{u^2-a^2}}{u}\,du = \sqrt{u^2-a^2} - a\cos^{-1}\frac{a}{|u|} + C$

80. $\displaystyle\int \frac{\sqrt{u^2-a^2}}{u^2}\,du = -\frac{\sqrt{u^2-a^2}}{u} + \ln\left|u+\sqrt{u^2-a^2}\right| + C$

81. $\displaystyle\int \frac{du}{\sqrt{u^2-a^2}} = \ln\left|u+\sqrt{u^2-a^2}\right| + C$

82. $\displaystyle\int \frac{u^2}{\sqrt{u^2-a^2}}\,du = \frac{u}{2}\sqrt{u^2-a^2} + \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

83. $\displaystyle\int \frac{du}{u^2\sqrt{u^2-a^2}} = \frac{\sqrt{u^2-a^2}}{a^2u} + C$

84. $\displaystyle\int \frac{du}{(u^2-a^2)^{3/2}} = -\frac{u}{a^2\sqrt{u^2-a^2}} + C$

Integrals Involving $a^2 - u^2$, $a > 0$

85. $\displaystyle\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

86. $\displaystyle\int u^2\sqrt{a^2-u^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2} + \frac{a^4}{8}\sin^{-1}\frac{u}{a} + C$

87. $\displaystyle\int \frac{\sqrt{a^2-u^2}}{u}\,du = \sqrt{a^2-u^2} - a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$

88. $\displaystyle\int \frac{\sqrt{a^2-u^2}}{u^2}\,du = \frac{-1}{u}\sqrt{a^2-u^2} - \sin^{-1}\frac{u}{a} + C$

89. $\displaystyle\int \frac{u^2}{\sqrt{a^2-u^2}}\,du = \frac{1}{2}\left(-u\sqrt{a^2-u^2} + a^2\sin^{-1}\frac{u}{a}\right) + C$

90. $\displaystyle\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$

91. $\displaystyle\int \frac{du}{u^2\sqrt{a^2-u^2}} = -\frac{1}{a^2u}\sqrt{a^2-u^2} + C$

92. $\displaystyle\int \left(a^2-u^2\right)^{3/2}\,du = -\frac{u}{8}\left(2u^2-5a^2\right)\sqrt{a^2-u^2} + \frac{3a^4}{8}\sin^{-1}\frac{u}{a} + C$

93. $\displaystyle\int \frac{du}{(a^2-u^2)^{3/2}} = -\frac{u}{a^2\sqrt{a^2-u^2}} + C$

Integrals Involving $2au - u^2$, $a > 0$

94. $\displaystyle\int \sqrt{2au-u^2}\,du = \frac{u-a}{2}\sqrt{2au-u^2} + \frac{a^2}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + C$

95. $\displaystyle\int \frac{du}{\sqrt{2au-u^2}} = \cos^{-1}\left(\frac{a-u}{a}\right) + C$

96. $\displaystyle\int u\sqrt{2au-u^2}\,du = \frac{2u^2-au-3a^2}{6}\sqrt{2au-u^2} + \frac{a^3}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + C$

97. $\displaystyle\int \frac{du}{u\sqrt{2au-u^2}} = -\frac{\sqrt{2au-u^2}}{au} + C$

Integrals Involving $a + bu$, $a \neq 0$

98. $\displaystyle\int \frac{u}{a+bu}\,du = \frac{1}{b^2}(a+bu-a\ln|a+bu|) + C$

99. $\int \frac{u^2}{a+bu}\,du = \frac{1}{2b^3}\left[(a+bu)^2 - 4a(a+bu) + 2a^2\ln|a+bu|\right] + C$

100. $\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$

101. $\int \frac{du}{u^2(a+bu)} = -\frac{1}{au} + \frac{b}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$

102. $\int \frac{u}{(a+bu)^2}\,du = \frac{a}{b^2(a+bu)} + \frac{1}{b^2}\ln|a+bu| + C$

103. $\int \frac{u}{u(a+bu)^2}\,du = \frac{1}{a(a+bu)} - \frac{1}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$

104. $\int \frac{u^2}{(a+bu)^2}\,du = \frac{1}{b^3}\left(a+bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right) + C$

105. $\int u\sqrt{a+bu}\,du = \frac{2}{15b^2}(3bu-2a)(a+bu)^{3/2} + C$

106. $\int \frac{u}{\sqrt{a+bu}}\,du = \frac{2}{3b^2}(bu-2a)\sqrt{a+bu} + C$

107. $\int \frac{u^2}{\sqrt{a+bu}}\,du = \frac{2}{15b^3}(8a^2+3b^2u^2-4abu)\sqrt{a+bu} + C$

108. $\int \frac{du}{u\sqrt{a+bu}} = \begin{cases} \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + C, & \text{if } a > 0 \\ \frac{\sqrt{2}}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a+bu}{-a}} + C, & \text{if } a < 0 \end{cases}$

109. $\int \frac{\sqrt{a+bu}}{u}\,du = 2\sqrt{a+bu} + a\int \frac{du}{u\sqrt{a+bu}}$

110. $\int \frac{\sqrt{a+bu}}{u^2}\,du = -\frac{\sqrt{a+bu}}{u} + \frac{b}{2}\int \frac{du}{u\sqrt{a+bu}}$

111. $\int u^n\sqrt{a+bu}\,du = \frac{2}{b(2n+3)}\left[u^n(a+bu)^{3/2} - na\int u^{n-1}\sqrt{a+bu}\,du\right]$

112. $\int \frac{u^n}{\sqrt{a+bu}}\,du = \frac{2u^n\sqrt{a+bu}}{b(2n+1)} - \frac{2na}{b(2n+1)}\int \frac{u^{n-1}}{\sqrt{a+bu}}\,du$

113. $\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{du}{u^{n-1}\sqrt{a+bu}}$

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Index

A

absolute convergence
9.5: Alternating Series
absolute error
7.6: Numerical Integration
absolute extremum
4.3: Maxima and Minima
absolute maximum
4.3: Maxima and Minima
absolute minimum
4.3: Maxima and Minima
Absolute Value Function
1.1: Review of Functions
acceleration
3.4: Derivatives as Rates of Change
acceleration vector
13.4: Motion in Space
air resistance
8.5: First-order Linear Equations
Airy's equation
10.4: Working with Taylor Series
algebraic function
1.2: Basic Classes of Functions
Alternating Harmonic Series
9.5: Alternating Series
alternating series
9.5: Alternating Series
Alternating Series Test
9.5: Alternating Series
amount of change
3.4: Derivatives as Rates of Change
angular coordinate
11.3: Polar Coordinates
angular frequency
17.3: Applications of Second-Order Differential Equations
antiderivative
4.10: Antiderivatives
arc length
6.4: Arc Length of a Curve and Surface Area
Arc length of a parametric curve
11.2: Calculus of Parametric Curves
Arc length of a polar curve
11.4: Area and Arc Length in Polar Coordinates
Archimedean spiral
11.3: Polar Coordinates
AREA BETWEEN TWO CURVES
6.1: Areas between Curves
AREA OF A PARALLELOGRAM
12.4: The Cross Product
Area of a region bounded by a polar curve
11.4: Area and Arc Length in Polar Coordinates
Area under a parametric curve
11.2: Calculus of Parametric Curves
Areas of Compound Regions
6.1: Areas between Curves
arithmetic sequence
9.1: Sequences
asymptotically stable solution
8.2: Direction Fields and Numerical Methods
asymptotically unstable solution
8.2: Direction Fields and Numerical Methods
autonomous differential equation
8.3: Separable Equations
average rate of change
3.4: Derivatives as Rates of Change
average value of a function
5.2: The Definite Integral
average velocity
2.1: A Preview of Calculus
3.1: Defining the Derivative

B

bessel functions
17.4: Series Solutions of Differential Equations
binomial series
10.4: Working with Taylor Series
binormalbinormal vector
13.3: Arc Length and Curvature
boundary point
14.2: Limits and Continuity

C

Calculating flux
16.2: Line Integrals
cardioid
11.3: Polar Coordinates
carrying capacity
8.4: The Logistic Equation
catenary
6.9: Calculus of the Hyperbolic Functions
center of gravity
15.6: Calculating Centers of Mass and Moments of Inertia
center of mass
6.6: Moments and Centers of Mass
15.6: Calculating Centers of Mass and Moments of Inertia
chain rule
3.6: The Chain Rule
change of variables
5.5: Substitution
characteristic equation
17.1: Second-Order Linear Equations
circulation
16.4: Green's Theorem
Clairaut's theorem
14.3: Partial Derivatives
16.1: Vector Fields
closed curve
14.2: Limits and Continuity
16.3: Conservative Vector Fields
closed set
14.2: Limits and Continuity
Closeness
2.5: The Precise Definition of a Limit
Combining Vectors
12.1: Vectors in the Plane
comparison test
9.4: Comparison Tests
complementary equation
17.2: Nonhomogeneous Linear Equations
component
12.1: Vectors in the Plane
component functions
13.1: Vector-Valued Functions and Space Curves
composite function
1.1: Review of Functions
compound interest
6.8: Exponential Growth and Decay
computer algebra system
7.5: Other Strategies for Integration
concave down
4.5: Derivatives and the Shape of a Graph
concave up
4.5: Derivatives and the Shape of a Graph
concavity
4.5: Derivatives and the Shape of a Graph
concavity test
4.5: Derivatives and the Shape of a Graph
conditional convergence
9.5: Alternating Series
conic sections
11.5: Conic Sections
12.6: Quadric Surfaces
connected region
14.2: Limits and Continuity
16.3: Conservative Vector Fields
connected set
14.2: Limits and Continuity
conservative field
16.1: Vector Fields
Conservative Vector Fields
16.3: Conservative Vector Fields
constant multiple law for limits
2.3: The Limit Laws
constant multiple rule
3.3: Differentiation Rules
Constant Rule
3.3: Differentiation Rules
constraint
14.8: Lagrange Multipliers
continuity
2.4: Continuity
continuity at a point
2.4: Continuity
continuity from the left
2.4: Continuity
continuity from the right
2.4: Continuity
continuity over an interval
2.4: Continuity
contour map
14.1: Functions of Several Variables
14.3: Partial Derivatives
convergence of a series
9.2: Infinite Series
convergent sequence
9.1: Sequences
coordinate plane
12.2: Vectors in Three Dimensions
Cramer's Rule
17.2: Nonhomogeneous Linear Equations
critical point
4.3: Maxima and Minima
critical point of a function of two variables
14.7: Maxima/Minima Problems
critical points
14.7: Maxima/Minima Problems

cross product
12.4: The Cross Product
cubic function
1.2: Basic Classes of Functions
curl
16.5: Divergence and Curl
curl vector fields
16.7: Stokes' Theorem
curtate cycloid
11.1: Parametric Equations
curvature
13.3: Arc Length and Curvature
cusp
11.1: Parametric Equations
cusps
11.1: Parametric Equations
cycloid
11.1: Parametric Equations
Cycloids
11.1: Parametric Equations
cylinder
12.6: Quadric Surfaces
cylinders
12.6: Quadric Surfaces
cylindrical coordinate system
12.7: Cylindrical and Spherical Coordinates

D

decreasing on the interval I
1.1: Review of Functions
definite integral
5.2: The Definite Integral
degree
1.2: Basic Classes of Functions
delta ball
14.2: Limits and Continuity
delta disk
14.2: Limits and Continuity
DENSITY FUNCTION
6.5: Physical Applications of Integration
dependent variable
1.1: Review of Functions
derivative
3.1: Defining the Derivative
derivative function
3.2: The Derivative as a Function
Derivative of cosecant function
3.5: Derivatives of Trigonometric Functions
Derivative of cosine function
3.5: Derivatives of Trigonometric Functions
Derivative of cotangent function
3.5: Derivatives of Trigonometric Functions
Derivative of exponential functions
3.9: Derivatives of Exponential and Logarithmic Functions
Derivative of Hyperbolic Functions
6.9: Calculus of the Hyperbolic Functions
Derivative of inverse cosecant function
3.7: Derivatives of Inverse Functions
Derivative of inverse cosine function
3.7: Derivatives of Inverse Functions
Derivative of inverse cotangent function
3.7: Derivatives of Inverse Functions
Derivative of inverse secant function
3.7: Derivatives of Inverse Functions
Derivative of inverse tangent function
3.7: Derivatives of Inverse Functions
Derivative of logarithmic functions
3.9: Derivatives of Exponential and Logarithmic Functions
Derivative of parametric equations
11.2: Calculus of Parametric Curves
Derivative of secant function
3.5: Derivatives of Trigonometric Functions
Derivative of sine function
3.5: Derivatives of Trigonometric Functions
Derivative of tangent function
3.5: Derivatives of Trigonometric Functions
Derivative of the general exponential function
3.9: Derivatives of Exponential and Logarithmic Functions
Derivative of the general logarithmic function
3.9: Derivatives of Exponential and Logarithmic Functions
Derivative of the natural exponential function
3.9: Derivatives of Exponential and Logarithmic Functions
Derivative of the natural logarithmic function
3.9: Derivatives of Exponential and Logarithmic Functions
determinant
12.4: The Cross Product
difference law for limits
2.3: The Limit Laws
difference quotient
3.1: Defining the Derivative
Difference Rule
3.3: Differentiation Rules
Differentiability (three variables)
14.4: Tangent Planes and Linear Approximations
Differentiability (two variables)
14.4: Tangent Planes and Linear Approximations
differentiable
14.4: Tangent Planes and Linear Approximations
differentiable function
3.2: The Derivative as a Function
differentiable on S
3.2: The Derivative as a Function
differential
4.2: Linear Approximations and Differentials
differential calculus
2.1: A Preview of Calculus
differential equation
8.1: Basics of Differential Equations
differential form
4.2: Linear Approximations and Differentials
direction angles
12.3: The Dot Product
direction cosines
12.3: The Dot Product
direction vector
12.5: Equations of Lines and Planes in Space
DIRECTIONAL COSINES
14.6: Directional Derivatives and the Gradient
directional derivative
14.6: Directional Derivatives and the Gradient
directrix
11.5: Conic Sections
discontinuity at a point
2.4: Continuity
discriminant
11.5: Conic Sections
14.7: Maxima/Minima Problems
disk method
6.2: Determining Volumes by Slicing
Distance between a Plane and a Point
12.5: Equations of Lines and Planes in Space
divergence
16.5: Divergence and Curl
divergence of a series
9.2: Infinite Series
DIVERGENCE OF THE CURL
16.5: Divergence and Curl
Divergence Test
9.3: The Divergence and Integral Tests
divergence theorem
16.8: The Divergence Theorem
divergent sequence
9.1: Sequences
domain
1.1: Review of Functions
dot product
12.3: The Dot Product
double integral
15.1: Double Integrals over Rectangular Regions
double Riemann sum
15.1: Double Integrals over Rectangular Regions
doubling time
6.8: Exponential Growth and Decay

E

eccentricity
11.5: Conic Sections
ellipses
11.5: Conic Sections
ellipsoid
12.6: Quadric Surfaces
Elliptic cone
12.6: Quadric Surfaces
Elliptic paraboloid
12.6: Quadric Surfaces
end behavior
4.6: Limits at Infinity and Asymptotes
equilibrium solution
8.2: Direction Fields and Numerical Methods
equivalent vectors
12.1: Vectors in the Plane
Euler's Method
8.2: Direction Fields and Numerical Methods
even function
1.1: Review of Functions
exponent
1.5: Exponential and Logarithmic Functions
exponential function
6.7: Integrals, Exponential Functions, and Logarithms
extreme value theorem
4.3: Maxima and Minima

F

family of solutions
8.1: Basics of Differential Equations
Faraday's Law
16.7: Stokes' Theorem

Fermat’s theorem
4.3: Maxima and Minima
Fermat’s Theorem for Functions of Two Variables
14.7: Maxima/Minima Problems
Fibonacci Numbers
9.1: Sequences
First Derivative Test
4.5: Derivatives and the Shape of a Graph
first moment
6.6: Moments and Centers of Mass
flux
16.2: Line Integrals
16.4: Green’s Theorem
16.8: The Divergence Theorem
flux integral
16.6: Surface Integrals
focal parameter
11.5: Conic Sections
focus
11.5: Conic Sections
Forced harmonic motion
17.3: Applications of Second-Order Differential Equations
Frenet frame of reference
13.3: Arc Length and Curvature
frustum
6.4: Arc Length of a Curve and Surface Area
FUBINI’S THEOREM
15.5: Triple Integrals in Cylindrical and Spherical Coordinates
Fubini’s thereom
15.4: Triple Integrals
Fubini's theorem
15.1: Double Integrals over Rectangular Regions
function
1.1: Review of Functions
Function of Two Variables
14.1: Functions of Several Variables
FUNDAMENTAL THEOREM FOR LINE INTEGRALS
16.3: Conservative Vector Fields
16.8: The Divergence Theorem
fundamental theorem of calculus
5.3: The Fundamental Theorem of Calculus
16.8: The Divergence Theorem
fundamental theorem of calculus, part 1
5.3: The Fundamental Theorem of Calculus
fundamental theorem of calculus, part 2
5.3: The Fundamental Theorem of Calculus

G

Gauss' Law
16.8: The Divergence Theorem
general form
11.5: Conic Sections
general form of the equation of a plane
12.5: Equations of Lines and Planes in Space
general solution
8.1: Basics of Differential Equations
generalized chain rule
14.5: The Chain Rule for Multivariable Functions
Geometric Sequences
9.1: Sequences
geometric series
9.2: Infinite Series
9.5: Alternating Series
gradient
14.6: Directional Derivatives and the Gradient
gradient field
16.1: Vector Fields
graph of a function
1.1: Review of Functions
graph of a function of two variables
14.1: Functions of Several Variables
Green's Theorem
16.4: Green’s Theorem
16.8: The Divergence Theorem
grid curves
16.6: Surface Integrals
growth rate
8.4: The Logistic Equation

H

harmonic series
9.2: Infinite Series
heat flow
16.6: Surface Integrals
helix
13.1: Vector-Valued Functions and Space Curves
Hooke’s law
6.5: Physical Applications of Integration
17.3: Applications of Second-Order Differential Equations
horizontal asymptote
4.6: Limits at Infinity and Asymptotes
horizontal line test
1.4: Inverse Functions
hydrostatic pressure
6.5: Physical Applications of Integration
hyperbolas
11.5: Conic Sections
hyperbolic functions
1.5: Exponential and Logarithmic Functions
hyperboloid
12.6: Quadric Surfaces
Hyperboloid of one sheet
12.6: Quadric Surfaces
Hyperboloid of two sheets
12.6: Quadric Surfaces
hypocycloid
11.1: Parametric Equations

I

implicit differentiation
3.8: Implicit Differentiation
improper double integral
15.2: Double Integrals over General Regions
improper integral
7.7: Improper Integrals
increasing on the interval I
1.1: Review of Functions
indefinite integral
4.10: Antiderivatives
independent variable
1.1: Review of Functions
indeterminate forms
4.8: L’Hôpital’s Rule
index variable
9.1: Sequences
infinite discontinuity
2.4: Continuity
infinite limit
2.2: The Limit of a Function
infinite limit at infinity
4.6: Limits at Infinity and Asymptotes
infinite series
9.2: Infinite Series
inflection point
4.5: Derivatives and the Shape of a Graph
initial point
12.1: Vectors in the Plane
initial population
8.4: The Logistic Equation
initial value
8.1: Basics of Differential Equations
initial value problem
4.10: Antiderivatives
initial velocity
8.1: Basics of Differential Equations
inner product
12.3: The Dot Product
instantaneous rate of change
3.1: Defining the Derivative
instantaneous velocity
2.1: A Preview of Calculus
3.1: Defining the Derivative
integrable function
5.2: The Definite Integral
integral calculus
2.1: A Preview of Calculus
Integral Test
9.3: The Divergence and Integral Tests
Integrals of Exponential Functions
5.6: Integrals Involving Exponential and Logarithmic Functions
Integrals That Produce Inverse Trigonometric Functions
5.7: Integrals Resulting in Inverse Trigonometric Functions
5.7: Integrals Resulting in Inverse Trigonometric Functions
integrand
5.2: The Definite Integral
integrating factor
8.5: First-order Linear Equations
integration by parts
7.1: Integration by Parts
integration by substitution
5.5: Substitution
Integration Formulas Involving Logarithmic Functions
5.6: Integrals Involving Exponential and Logarithmic Functions
integration table
7.5: Other Strategies for Integration
interior point
14.2: Limits and Continuity
intermediate value theorem
2.4: Continuity
intermediate variable
14.5: The Chain Rule for Multivariable Functions
Interpretation of Curl
16.7: Stokes’ Theorem
interval of convergence
10.1: Power Series and Functions
intuitive definition of the limit
2.2: The Limit of a Function
inverse function
1.4: Inverse Functions

inverse function theorem
3.7: Derivatives of Inverse Functions
inverse hyperbolic functions
1.5: Exponential and Logarithmic Functions
inverse trigonometric functions
1.4: Inverse Functions
iterated integral
15.1: Double Integrals over Rectangular Regions
iterative process
4.9: Newton's Method

J
Jacobian
15.7: Change of Variables in Multiple Integrals
jump discontinuity
2.4: Continuity

K
Kepler's laws of planetary motion
13.4: Motion in Space

L
L'Hôpital's rule
4.8: L'Hôpital's Rule
Lagrange multiplier
14.8: Lagrange Multipliers
level curve of a function of two variables
14.1: Functions of Several Variables
level surface of a function of three variables
14.1: Functions of Several Variables
limacon
11.3: Polar Coordinates
limit
2.1: A Preview of Calculus
limit at infinity
4.6: Limits at Infinity and Asymptotes
Limit Comparison Test
9.4: Comparison Tests
limit laws
2.3: The Limit Laws
LIMIT LAWS FOR FUNCTIONS OF TWO VARIABLES
14.2: Limits and Continuity
limit of a sequence
9.1: Sequences
line integrals
16.2: Line Integrals
linear
8.5: First-order Linear Equations
linear approximation
4.2: Linear Approximations and Differentials
14.4: Tangent Planes and Linear Approximations
linear function
1.2: Basic Classes of Functions
linearization
4.2: Linear Approximations and Differentials
linearly dependent
17.1: Second-Order Linear Equations
linearly independent
17.1: Second-Order Linear Equations
local extremum
4.3: Maxima and Minima
local maximum
4.3: Maxima and Minima
local minimum
4.3: Maxima and Minima
logarithmic function
1.2: Basic Classes of Functions
logistic differential equation
8.4: The Logistic Equation
lower sum
5.1: Approximating Areas

M
Möbius strip
16.6: Surface Integrals
Maclaurin polynomial
10.3: Taylor and Maclaurin Series
Maclaurin series
10.3: Taylor and Maclaurin Series
magnitude
12.1: Vectors in the Plane
major axis
11.5: Conic Sections
marginal cost
3.4: Derivatives as Rates of Change
marginal profit
3.4: Derivatives as Rates of Change
marginal revenue
3.4: Derivatives as Rates of Change
mass flux
16.6: Surface Integrals
mathematical model
1.2: Basic Classes of Functions
mean value theorem
4.4: The Mean Value Theorem
Mean Value Theorem for Integrals
5.3: The Fundamental Theorem of Calculus
method of cylindrical shells
6.3: Volumes of Revolution - Cylindrical Shells
method of equating coefficients
7.4: Partial Fractions
method of exhaustion
5.1: Approximating Areas
method of Lagrange multipliers
14.8: Lagrange Multipliers
method of strategic substitution
7.4: Partial Fractions
method of undetermined coefficients
17.2: Nonhomogeneous Linear Equations
method of variation of parameters
17.2: Nonhomogeneous Linear Equations
midpoint rule
7.6: Numerical Integration
minor axis
11.5: Conic Sections
mixed partial derivatives
14.3: Partial Derivatives
Moments of Inertia
6.6: Moments and Centers of Mass
15.6: Calculating Centers of Mass and Moments of Inertia
Monotone Convergence Theorem
9.1: Sequences
monotone sequence
9.1: Sequences
Multiple Integration
15: Multiple Integration
multivariable calculus
2.1: A Preview of Calculus

N
nappe
11.5: Conic Sections
nappes
11.5: Conic Sections
natural exponential function
1.5: Exponential and Logarithmic Functions
natural logarithm
1.5: Exponential and Logarithmic Functions
Natural logarithm function
6.7: Integrals, Exponential Functions, and Logarithms
net change theorem
5.4: Integration Formulas and the Net Change Theorem
net signed area
5.2: The Definite Integral
Newton's Law of Cooling
6.8: Exponential Growth and Decay
8.3: Separable Equations
Newton's method
4.9: Newton's Method
nonelementary integral
10.4: Working with Taylor Series
nonhomogeneous linear equation
17.1: Second-Order Linear Equations
normal component of acceleration
13.4: Motion in Space
normal plane
13.3: Arc Length and Curvature
normal vector
12.5: Equations of Lines and Planes in Space
Normalization
12.1: Vectors in the Plane
number e
1.5: Exponential and Logarithmic Functions
numerical integration
7.6: Numerical Integration

O
objective function
14.8: Lagrange Multipliers
oblique asymptote
4.6: Limits at Infinity and Asymptotes
octants
12.2: Vectors in Three Dimensions
odd function
1.1: Review of Functions
open set
14.2: Limits and Continuity
optimization problem
14.8: Lagrange Multipliers
optimization problems
4.7: Applied Optimization Problems
order of a differential equation
8.1: Basics of Differential Equations
ordinary point
17.4: Series Solutions of Differential Equations
orientation
11.1: Parametric Equations
orientation of a surface
16.6: Surface Integrals
orthogonal vectors
12.3: The Dot Product
osculating circle
13.3: Arc Length and Curvature

osculating plane
13.3: Arc Length and Curvature

P

parabola
11.5: Conic Sections
paraboloid
12.6: Quadric Surfaces
parallelepiped
12.4: The Cross Product
parallelogram method
12.1: Vectors in the Plane
parameter
11.1: Parametric Equations
parameter domain
16.6: Surface Integrals
parameter space
16.6: Surface Integrals
parameterization of a curve
11.1: Parametric Equations
parameterized surface
16.6: Surface Integrals
parametric curve
11.1: Parametric Equations
parametric equations
11.1: Parametric Equations
parametric equations of a line
12.5: Equations of Lines and Planes in Space
parametric surface
16.6: Surface Integrals
Parametric Surfaces
16.6: Surface Integrals
partial derivative
14.3: Partial Derivatives
partial differential equation
14.3: Partial Derivatives
Partial Differentiation
14.3: Partial Derivatives
partial fraction decomposition
7.4: Partial Fractions
partial sum
9.2: Infinite Series
particular solution
8.1: Basics of Differential Equations
17.2: Nonhomogeneous Linear Equations
partition
5.1: Approximating Areas
percentage error
4.2: Linear Approximations and Differentials
periodic function
1.3: Trigonometric Functions
phase line
8.4: The Logistic Equation
planar transformation
15.7: Change of Variables in Multiple Integrals
plane curve
13.1: Vector-Valued Functions and Space Curves
points of inflection
4.5: Derivatives and the Shape of a Graph
Polar Areas
15.3: Double Integrals in Polar Coordinates
polar axis
11.3: Polar Coordinates
polar coordinate system
11.3: Polar Coordinates
polar equation
11.3: Polar Coordinates
polar rectangle
15.3: Double Integrals in Polar Coordinates
Polar Volumes
15.3: Double Integrals in Polar Coordinates
pole
11.3: Polar Coordinates
polynomial function
1.2: Basic Classes of Functions
population growth
6.8: Exponential Growth and Decay
population growth rate
3.4: Derivatives as Rates of Change
power function
1.2: Basic Classes of Functions
Power Reduction Formula
7.2: Trigonometric Integrals
power rule
3.3: Differentiation Rules
Power rule with rational exponents
3.7: Derivatives of Inverse Functions
power series
10: Power Series
10.1: Power Series and Functions
principal unit normal vector
13.3: Arc Length and Curvature
principal unit tangent vector
13.2: Calculus of Vector-Valued Functions
product law for limits
2.3: The Limit Laws
product rule
3.3: Differentiation Rules
projectile motion
13.4: Motion in Space
propagated error
4.2: Linear Approximations and Differentials

Q

quadratic function
1.2: Basic Classes of Functions
Quadric Surfaces
12.6: Quadric Surfaces
quotient law for limits
2.3: The Limit Laws
quotient rule
3.3: Differentiation Rules

R

radial coordinate
11.3: Polar Coordinates
radial density
6.5: Physical Applications of Integration
radians
1.3: Trigonometric Functions
radius of convergence
10.1: Power Series and Functions
radius of curvature
13.3: Arc Length and Curvature
radius of gyration
15.6: Calculating Centers of Mass and Moments of Inertia
range
1.1: Review of Functions
ratio test
9.6: Ratio and Root Tests
rational function
1.2: Basic Classes of Functions
recurrence relation
9.1: Sequences
region
14.2: Limits and Continuity
regular parameterization
16.6: Surface Integrals
regular partition
5.1: Approximating Areas
related rates
4.1: Related Rates
relative error
4.2: Linear Approximations and Differentials
7.6: Numerical Integration
Remainder Estimate
9.3: The Divergence and Integral Tests
removable discontinuity
2.4: Continuity
reparameterization
13.1: Vector-Valued Functions and Space Curves
repeated linear factors
7.4: Partial Fractions
restricted domain
1.4: Inverse Functions
Riemann Rearrangement Theorem
9.5: Alternating Series
Riemann sum
5.1: Approximating Areas
RLC series circuit
17.3: Applications of Second-Order Differential Equations
Rolle's Theorem
4.4: The Mean Value Theorem
root function
1.2: Basic Classes of Functions
root law for limits
2.3: The Limit Laws
root test
9.6: Ratio and Root Tests
rose
11.3: Polar Coordinates
rotational field
16.1: Vector Fields
rulings
12.6: Quadric Surfaces

S

saddle point
14.7: Maxima/Minima Problems
scalar
12.1: Vectors in the Plane
Scalar Equation of a Plane
12.5: Equations of Lines and Planes in Space
scalar line integral
16.2: Line Integrals
scalar multiplication
12.1: Vectors in the Plane
scalar product
12.3: The Dot Product
scalar projection
12.3: The Dot Product
secant line
2.1: A Preview of Calculus
second derivative test
4.5: Derivatives and the Shape of a Graph

separation of variables
8.3: Separable Equations
sequence
9.1: Sequences
sequences
9.1: Sequences
sigma notation
5.1: Approximating Areas
simple harmonic motion
17.3: Applications of Second-Order Differential Equations
singular point
17.4: Series Solutions of Differential Equations
skew lines
12.5: Equations of Lines and Planes in Space
slicing method
6.2: Determining Volumes by Slicing
slope
1.2: Basic Classes of Functions
slope field
8.2: Direction Fields and Numerical Methods
smooth
13.3: Arc Length and Curvature
solid of revolution
6.2: Determining Volumes by Slicing
6.3: Volumes of Revolution - Cylindrical Shells
solution curve
8.2: Direction Fields and Numerical Methods
solution to a differential equation
8.1: Basics of Differential Equations
space curve
13.1: Vector-Valued Functions and Space Curves
speed
3.4: Derivatives as Rates of Change
sphere
12.2: Vectors in Three Dimensions
spherical coordinate system
12.7: Cylindrical and Spherical Coordinates
spherical coordinates
12.7: Cylindrical and Spherical Coordinates
squeeze theorem
2.3: The Limit Laws
Squeeze Theorem for Sequences
9.1: Sequences
standard equation of a sphere
12.2: Vectors in Three Dimensions
standard form
11.5: Conic Sections
standard unit vectors
12.1: Vectors in the Plane
step size
8.2: Direction Fields and Numerical Methods
Stokes' Theorem
16.7: Stokes' Theorem
16.8: The Divergence Theorem
sum law for limits
2.3: The Limit Laws
Sum Rule
3.3: Differentiation Rules
summation notation
5.1: Approximating Areas
superposition principle
17.1: Second-Order Linear Equations
surface
14.1: Functions of Several Variables
surface area
6.4: Arc Length of a Curve and Surface Area
16.6: Surface Integrals
Surface area generated by a parametric curve
11.2: Calculus of Parametric Curves
surface independent
16.7: Stokes' Theorem
surface integral
16.6: Surface Integrals
surface integral of a vector field
16.6: Surface Integrals
surface integrals
16.6: Surface Integrals
surface of revolution
6.4: Arc Length of a Curve and Surface Area
Symmetric Equations of a Line
12.5: Equations of Lines and Planes in Space
symmetry about the origin
1.1: Review of Functions

T

table of values
1.1: Review of Functions
tangent
2.1: A Preview of Calculus
tangent line
3.1: Defining the Derivative
tangent line approximation
4.2: Linear Approximations and Differentials
tangent plane
14.4: Tangent Planes and Linear Approximations
tangent vector
13.2: Calculus of Vector-Valued Functions
tangential component of acceleration
13.4: Motion in Space
Taylor polynomials
10.3: Taylor and Maclaurin Series
Taylor series
10.3: Taylor and Maclaurin Series
Taylor's theorem with remainder
10.3: Taylor and Maclaurin Series
telescoping series
9.2: Infinite Series
term
9.1: Sequences
terminal point
12.1: Vectors in the Plane
The Dot Product
12.3: The Dot Product
The Logistic Equation
8.4: The Logistic Equation
The power rule for functions
3.6: The Chain Rule
Theorem of Pappus
6.6: Moments and Centers of Mass
threshold population
8.4: The Logistic Equation
torque
12.4: The Cross Product
total area
5.2: The Definite Integral
total differential
14.4: Tangent Planes and Linear Approximations
trace
12.6: Quadric Surfaces
transcendental function
1.2: Basic Classes of Functions
transformation
15.7: Change of Variables in Multiple Integrals
transformation of a function
1.2: Basic Classes of Functions
triangle inequality
12.1: Vectors in the Plane
triangle method
12.1: Vectors in the Plane
trigonometric functions
1.3: Trigonometric Functions
trigonometric identity
1.3: Trigonometric Functions
trigonometric integral
7.2: Trigonometric Integrals
trigonometric substitution
7.3: Trigonometric Substitution
triple integral
15.4: Triple Integrals
triple integral in cylindrical coordinates
15.5: Triple Integrals in Cylindrical and Spherical Coordinates
triple integral in spherical coordinates
15.5: Triple Integrals in Cylindrical and Spherical Coordinates
Triple Scalar Product
12.4: The Cross Product
Type I
15.2: Double Integrals over General Regions
Type II
15.2: Double Integrals over General Regions

U

unit circle
1.3: Trigonometric Functions
unit vector
12.1: Vectors in the Plane
upper sum
5.1: Approximating Areas

V

variable of integration
5.2: The Definite Integral
vector
12.1: Vectors in the Plane
vector addition
12.1: Vectors in the Plane
vector difference
12.1: Vectors in the Plane
vector equation of a line
12.5: Equations of Lines and Planes in Space
Vector Equation of a Plane
12.5: Equations of Lines and Planes in Space
vector fields
16.1: Vector Fields
VECTOR LINE INTEGRALS
16.2: Line Integrals
vector parameterization
13.1: Vector-Valued Functions and Space Curves
vector product
12.4: The Cross Product
vector projection
12.3: The Dot Product
vector sum
12.1: Vectors in the Plane

Vectors in R3
12.2: Vectors in Three Dimensions
velocity vector
13.4: Motion in Space
vertex
11.5: Conic Sections
vertical asymptote
2.2: The Limit of a Function
vertical line test
1.1: Review of Functions
vertical trace
14.1: Functions of Several Variables
Volume by Shells
6.3: Volumes of Revolution - Cylindrical Shells
12.7: Cylindrical and Spherical Coordinates

W

washer method
6.2: Determining Volumes by Slicing
Witch of Agnesi
11.1: Parametric Equations
work
6.5: Physical Applications of Integration
work done by a force
12.3: The Dot Product

Z

zero vector
12.1: Vectors in the Plane
zeros of a function
1.1: Review of Functions

Glossary

absolute convergence | if the series $\sum_{n=1}^{\infty}|a_n|$ converges, the series $\sum_{n=1}^{\infty}a_n$ is said to converge absolutely

absolute error | if B is an estimate of some quantity having an actual value of A, then the absolute error is given by $|A-B|$

absolute extremum | if f has an absolute maximum or absolute minimum at c, we say f has an absolute extremum at c

absolute maximum | if $f(c)\geq f(x)$ for all x in the domain of f, we say f has an absolute maximum at c

absolute minimum | if $f(c)\leq f(x)$ for all x in the domain of f, we say f has an absolute minimum at c

absolute value function | $f(x)=\{-x$, if $x<0$ \if $x\geq 0$

acceleration | is the rate of change of the velocity, that is, the derivative of velocity

acceleration vector | the second derivative of the position vector

algebraic function | a function involving any combination of only the basic operations of addition, subtraction, multiplication, division, powers, and roots applied to an input variable x

alternating series | a series of the form $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ or $\sum_{n=1}^{\infty}(-1)^{n}b_n$, where $b_n\geq 0$, is called an alternating series

alternating series test | for an alternating series of either form, if $b_{n+1}\leq b_n$ for all integers $n\geq 1$ and $b_n\to 0$, then an alternating series converges

amount of change | the amount of a function $f(x)$ over an interval $[x,x+h]isf(x+h)-f(x)$

angular coordinate | θ the angle formed by a line segment connecting the origin to a point in the polar coordinate system with the positive radial (x) axis, measured counterclockwise

antiderivative | a function F such that $F'(x)=f(x)$ for all x in the domain of f is an antiderivative of f

arc length | the arc length of a curve can be thought of as the distance a person would travel along the path of the curve

arc-length function | a function $s(t)$ that describes the arc length of curve C as a function of t

arc-length parameterization | a reparameterization of a vector-valued function in which the parameter is equal to the arc length

arithmetic sequence | a sequence in which the difference between every pair of consecutive terms is the same is called an arithmetic sequence

asymptotically semi-stable solution | $y=k$ if it is neither asymptotically stable nor asymptotically unstable

asymptotically stable solution | $y=k$ if there exists $\varepsilon>0$ such that for any value $c\in(k-\varepsilon,k+\varepsilon)$ the solution to the initial-value problem $y'=f(x,y),y(x_0)=c$ approaches k as x approaches infinity

asymptotically unstable solution | $y=k$ if there exists $\varepsilon>0$ such that for any value $c\in(k-\varepsilon,k+\varepsilon)$ the solution to the initial-value problem $y'=f(x,y),y(x_0)=c$ never approaches k as $x$approaches infinity

autonomous differential equation | an equation in which the right-hand side is a function of y alone

average rate of change | is a function $f(x)$ over an interval $[x,x+h]$ is $\frac{f(x+h)-f(a)}{b-a}$

average value of a function | (or f_{ave}) the average value of a function on an interval can be found by calculating the definite integral of the function and dividing that value by the length of the interval

average velocity | the change in an object's position divided by the length of a time period; the average velocity of an object over a time interval $[t,a]$ (if $t<a$ or $[a,t]$ if $t>a$), with a position given by $s(t)$, that is $v_{ave}=\dfrac{s(t)-s(a)}{t-a}$

base | the number b in the exponential function $f(x)=b^x$ and the logarithmic function $f(x)=\log_b x$

binomial series | the Maclaurin series for $f(x)=(1+x)^r$; it is given by $(1+x)^r=\sum_{n=0}^{\infty}\binom{r}{n}x^n=1+rx+\dfrac{r(r-1)}{2!}x^2+\cdots+\dfrac{r(r-1)\cdots(r-n+1)}{n!}x^n+\cdots$ for $|x|<1$

binormal vector | a unit vector orthogonal to the unit tangent vector and the unit normal vector

boundary conditions | the conditions that give the state of a system at different times, such as the position of a spring-mass system at two different times

boundary point | a point P_0 of R is a boundary point if every δ disk centered around P_0 contains points both inside and outside R

boundary-value problem | a differential equation with associated boundary conditions

bounded above | a sequence a_n is bounded above if there exists a constant M such that $a_n\leq M$ for all positive integers n

bounded below | a sequence a_n is bounded below if there exists a constant M such that $M\leq a_n$ for all positive integers n

bounded sequence | a sequence a_n is bounded if there exists a constant M such that $|a_n|\leq M$ for all positive integers n

cardioid | a plane curve traced by a point on the perimeter of a circle that is rolling around a fixed circle of the same radius; the equation of a cardioid is $r=a(1+\sin\theta)$ or $r=a(1+\cos\theta)$

carrying capacity | the maximum population of an organism that the environment can sustain indefinitely

catenary | a curve in the shape of the function $y=a\cdot\cosh(x/a)$ is a catenary; a cable of uniform density suspended between two supports assumes the shape of a catenary

center of mass | the point at which the total mass of the system could be concentrated without changing the moment

centroid | the centroid of a region is the geometric center of the region; laminas are often represented by regions in the plane; if the lamina has a constant density, the center of mass of the lamina depends only on the shape of the corresponding planar region; in this case, the center of mass of the lamina corresponds to the centroid of the representative region

chain rule | the chain rule defines the derivative of a composite function as the derivative of the outer function evaluated at the inner function times the derivative of the inner function

change of variables | the substitution of a variable, such as u, for an expression in the integrand

characteristic equation | the equation $a\lambda^2 + b\lambda + c = 0$ for the differential equation $ay'' + by' + cy = 0$

circulation | the tendency of a fluid to move in the direction of curve C. If C is a closed curve, then the circulation of $\vec{\mathbf{F}}$ along C is line integral $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$, which we also denote $\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$.

closed curve | a curve for which there exists a parameterization $\vec{\mathbf{r}}(t), a \leq t \leq b$, such that $\vec{\mathbf{r}}(a) = \vec{\mathbf{r}}(b)$, and the curve is traversed exactly once

closed curve | a curve that begins and ends at the same point

closed set | a set S that contains all its boundary points

comparison test | If $0 \leq a_n \leq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges; if $a_n \geq b_n \geq 0$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

complementary equation | for the nonhomogeneous linear differential equation

$$a + 2(x)y'' + a_1(x)y' + a_0(x)y = r(x),$$

the associated homogeneous equation, called the *complementary equation*, is

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

component | a scalar that describes either the vertical or horizontal direction of a vector

component functions | the component functions of the vector-valued function $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}}$ are $f(t)$ and $g(t)$, and the component functions of the vector-valued function $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$ are $f(t)$, $g(t)$ and $h(t)$

composite function | given two functions f and g, a new function, denoted $g \circ f$, such that $(g \circ f)(x) = g(f(x))$

computer algebra system (CAS) | technology used to perform many mathematical tasks, including integration

concave down | if f is differentiable over an interval I and f' is decreasing over I, then f is concave down over I

concave up | if f is differentiable over an interval I and f' is increasing over I, then f is concave up over I

concavity | the upward or downward curve of the graph of a function

concavity test | suppose f is twice differentiable over an interval I; if $f'' > 0$ over I, then f is concave up over I; if $f'' <$ over I, then f is concave down over I

conditional convergence | if the series $\sum_{n=1}^{\infty} a_n$ converges, but the series $\sum_{n=1}^{\infty} |a_n|$ diverges, the series $\sum_{n=1}^{\infty} a_n$ is said to converge conditionally

conic section | a conic section is any curve formed by the intersection of a plane with a cone of two nappes

connected region | a region in which any two points can be connected by a path with a trace contained entirely inside the region

connected set | an open set S that cannot be represented as the union of two or more disjoint, nonempty open subsets

conservative field | a vector field for which there exists a scalar function f such that $\vec{\nabla} f = \vec{\mathbf{F}}$

constant multiple law for limits | the limit law

$$\lim_{x \to a} cf(x) = c \cdot \lim_{x \to a} f(x) = cL$$

constant multiple rule | the derivative of a constant c multiplied by a function f is the same as the constant multiplied by the derivative: $\frac{d}{dx}\big(cf(x)\big) = cf'(x)$

constant rule | the derivative of a constant function is zero: $\frac{d}{dx}(c) = 0$, where c is a constant

constraint | an inequality or equation involving one or more variables that is used in an optimization problem; the constraint enforces a limit on the possible solutions for the problem

continuity at a point | A function $f(x)$ is continuous at a point a if and only if the following three conditions are satisfied: (1) $f(a)$ is defined, (2) $\lim_{x \to a} f(x)$ exists, and (3) $\lim x \to af(x) = f(a)$

continuity from the left | A function is continuous from the left at b if $\lim_{x \to b^-} f(x) = f(b)$

continuity from the right | A function is continuous from the right at a if $\lim_{x \to a^+} f(x) = f(a)$

continuity over an interval | a function that can be traced with a pencil without lifting the pencil; a function is continuous over an open interval if it is continuous at every point in the interval; a function $f(x)$ is continuous over a closed interval of the form $[a, b]$ if it is continuous at every point in (a, b), and it is continuous from the right at a and from the left at b

contour map | a plot of the various level curves of a given function $f(x, y)$

convergence of a series | a series converges if the sequence of partial sums for that series converges

convergent sequence | a convergent sequence is a sequence a_n for which there exists a real number L such that a_n is arbitrarily close to L as long as n is sufficiently large

coordinate plane | a plane containing two of the three coordinate axes in the three-dimensional coordinate system, named by the axes it contains: the xy-plane, xz-plane, or the yz-plane

critical point | if $f'(c) = 0$ or $f'(c)$ is undefined, we say that c is a critical point of f

critical point of a function of two variables | the point (x_0, y_0) is called a critical point of $f(x, y)$ if one of the two following conditions holds: 1. $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$ 2. At least one of $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ do not exist

cross product | $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (u_2v_3 - u_3v_2)\hat{\mathbf{i}} - (u_1v_3 - u_3v_1)\hat{\mathbf{j}} + (u_1v_2 - u_2v_1)\hat{\mathbf{k}}$, where $\vec{\mathbf{u}} = \langle u_1, u_2, u_3\rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3\rangle$ determinant a real number associated with a square matrix parallelepiped a three-dimensional prism with six faces that are parallelograms torque the effect of a force that causes an object to rotate triple scalar product the dot product of a vector with the cross product of two other vectors: $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})$ vector product the cross product of two vectors.

cross-section | the intersection of a plane and a solid object

cubic function | a polynomial of degree 3; that is, a function of the form $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$

curl | the curl of vector field $\vec{\mathbf{F}} = \langle P, Q, R\rangle$, denoted $\vec{\nabla} \times \vec{\mathbf{F}}$ is the "determinant" of the matrix

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

and is given by the expression $(R_y - Q_z)\,\hat{\mathbf{i}} + (P_z - R_x)\,\hat{\mathbf{j}} + (Q_x - P_y)\,\hat{\mathbf{k}}$; it measures the tendency of particles at a point to rotate about the axis that points in the direction of the curl at the point

curvature | the derivative of the unit tangent vector with respect to the arc-length parameter

cusp | a pointed end or part where two curves meet

cycloid | the curve traced by a point on the rim of a circular wheel as the wheel rolls along a straight line without slippage

cylinder | a set of lines parallel to a given line passing through a given curve

cylindrical coordinate system | a way to describe a location in space with an ordered triple (r, θ, z), where (r, θ) represents the polar coordinates of the point's projection in the xy-plane, and z represents the point's projection onto the z-axis

decreasing on the interval

definite integral | a primary operation of calculus; the area between the curve and the x-axis over a given interval is a definite integral

definite integral of a vector-valued function | the vector obtained by calculating the definite integral of each of the component functions of a given vector-valued function, then using the results as the components of the resulting function

degree | for a polynomial function, the value of the largest exponent of any term

density function | a density function describes how mass is distributed throughout an object; it can be a linear density, expressed in terms of mass per unit length; an area density, expressed in terms of mass per unit area; or a volume density, expressed in terms of mass per unit volume; weight-density is also used to describe weight (rather than mass) per unit volume

dependent variable | the output variable for a function

derivative | the slope of the tangent line to a function at a point, calculated by taking the limit of the difference quotient, is the derivative

derivative function | gives the derivative of a function at each point in the domain of the original function for which the derivative is defined

derivative of a vector-valued function | the derivative of a vector-valued function $\vec{\mathbf{r}}(t)$ is $\vec{\mathbf{r}}'(t) = \lim_{\Delta t \to 0} \frac{\vec{\mathbf{r}}(t+\Delta t) - \vec{\mathbf{r}}(t)}{\Delta t}$, provided the limit exists

difference law for limits | the limit law

$$\begin{aligned} &\lim_{x \to a}(f(x) \\ &- g(x)) \\ &= \lim_{x \to a} f(x) \\ &- \lim_{x \to a} g(x) \\ &= L - M \end{aligned}$$

difference quotient | of a function $f(x)$ at a is given by $\frac{f(a+h) - f(a)}{h}$ or $\frac{f(x) - f(a)}{x - a}$

difference rule | the derivative of the difference of a function f and a function g is the same as the difference of the derivative of f and the derivative of g: $\frac{d}{dx}\big(f(x) - g(x)\big) = f'(x) - g'(x)$

differentiable | a function $f(x, y)$ is differentiable at (x_0, y_0) if $f(x, y)$ can be expressed in the form $f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + E(x, y)$, where the error term $E(x, y)$ satisfies $\lim_{(x,y)\to(x_0,y_0)} \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0$

differentiable at

differentiable function | a function for which $f'(x)$ exists is a differentiable function

differentiable on

differential | the differential dx is an independent variable that can be assigned any nonzero real number; the differential dy is defined to be $dy = f'(x)\,dx$

differential calculus | the field of calculus concerned with the study of derivatives and their applications

differential equation | an equation involving a function $y = y(x)$ and one or more of its derivatives

differential form | given a differentiable function $y = f'(x)$, the equation $dy = f'(x)\,dx$ is the differential form of the derivative of y with respect to x

differentiation | the process of taking a derivative

direction angles | the angles formed by a nonzero vector and the coordinate axes

direction cosines | the cosines of the angles formed by a nonzero vector and the coordinate axes

direction field (slope field) | a mathematical object used to graphically represent solutions to a first-order differential equation; at each point in a direction field, a line segment appears whose slope is equal to the slope of a solution to the differential equation passing through that point

direction vector | a vector parallel to a line that is used to describe the direction, or orientation, of the line in space

directional derivative | the derivative of a function in the direction of a given unit vector

directrix | a directrix (plural: directrices) is a line used to construct and define a conic section; a parabola has one directrix; ellipses and hyperbolas have two

discontinuity at a point | A function is discontinuous at a point or has a discontinuity at a point if it is not continuous at the point

discriminant | the value $4AC - B^2$, which is used to identify a conic when the equation contains a term involving xy, is called a discriminant

discriminant | the discriminant of the function $f(x, y)$ is given by the formula $D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$

disk method | a special case of the slicing method used with solids of revolution when the slices are disks

divergence | the divergence of a vector field $\vec{\mathbf{F}} = \langle P, Q, R\rangle$, denoted $\vec{\nabla} \times \vec{\mathbf{F}}$, is $P_x + Q_y + R_z$; it measures the "outflowing-ness" of a vector field

divergence of a series | a series diverges if the sequence of partial sums for that series diverges

divergence test | if $\lim\limits_{n\to\infty} a_n \neq 0$, then the series $\sum\limits_{n=1}^{\infty} a_n$ diverges

divergent sequence | a sequence that is not convergent is divergent

domain | the set of inputs for a function

dot product or scalar product | $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_1v_1 + u_2v_2 + u_3v_3$ where $\vec{\mathbf{u}} = \langle u_1, u_2, u_3\rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3\rangle$

double integral | of the function $f(x, y)$ over the region R in the xy-plane is defined as the limit of a double Riemann sum,

$$\iint_R f(x, y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A.$$

double Riemann sum | of the function $f(x, y)$ over a rectangular region R is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A,$$

where R is divided into smaller subrectangles R_{ij} and (x_{ij}^*, y_{ij}^*) is an arbitrary point in R_{ij}

doubling time | if a quantity grows exponentially, the doubling time is the amount of time it takes the quantity to double, and is given by $(\ln 2)/k$

eccentricity | the eccentricity is defined as the distance from any point on the conic section to its focus divided by the perpendicular distance from that point to the nearest directrix

ellipsoid | a three-dimensional surface described by an equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$; all traces of this surface are ellipses

elliptic cone | a three-dimensional surface described by an equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$; traces of this surface include ellipses and intersecting lines

elliptic paraboloid | a three-dimensional surface described by an equation of the form $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$; traces of this surface include ellipses and parabolas

end behavior | the behavior of a function as $x \to \infty$ and $x \to -\infty$

epsilon-delta definition of the limit | $\lim\limits_{x\to a} f(x) = L$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$

equilibrium solution | any solution to the differential equation of the form $y = c$, where c is a constant

equivalent vectors | vectors that have the same magnitude and the same direction

Euler's Method | a numerical technique used to approximate solutions to an initial-value problem

even function | a function is even if $f(-x) = f(x)$ for all x in the domain of f

explicit formula | a sequence may be defined by an explicit formula such that $a_n = f(n)$

exponent | the value x in the expression b^x

exponential decay | systems that exhibit exponential decay follow a model of the form $y = y_0e^{-kt}$

exponential growth | systems that exhibit exponential growth follow a model of the form $y = y_0e^{kt}$

extreme value theorem | if f is a continuous function over a finite, closed interval, then f has an absolute maximum and an absolute minimum

Fermat's theorem | if f has a local extremum at c, then c is a critical point of f

first derivative test | let f be a continuous function over an interval I containing a critical point c such that f is differentiable over I except possibly at c; if f' changes sign from positive to negative as x increases through c, then f has a local maximum at c; if f' changes sign from negative to positive as x increases through c, then f has a local minimum at c; if f' does not change sign as x increases through c, then f does not have a local extremum at c

flux | the rate of a fluid flowing across a curve in a vector field; the flux of vector field $\vec{\mathbf{F}}$ across plane curve C is line integral $\int_C \vec{\mathbf{F}} \cdot \frac{\vec{\mathbf{n}}(t)}{\|\vec{\mathbf{n}}(t)\|}\, ds$

flux integral | another name for a surface integral of a vector field; the preferred term in physics and engineering

focal parameter | the focal parameter is the distance from a focus of a conic section to the nearest directrix

focus | a focus (plural: foci) is a point used to construct and define a conic section; a parabola has one focus; an ellipse and a hyperbola have two

formal definition of an infinite limit | $\lim\limits_{x\to a} f(x) = \infty$ if for every $M > 0$, there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $f(x) > M$ $\lim\limits_{x\to a} f(x) = -\infty$ if for every $M > 0$, there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $f(x) < -M$

Frenet frame of reference | (TNB frame) a frame of reference in three-dimensional space formed by the unit tangent vector, the unit normal vector, and the binormal vector

frustum | a portion of a cone; a frustum is constructed by cutting the cone with a plane parallel to the base

Fubini's theorem | if $f(x, y)$ is a function of two variables that is continuous over a rectangular region $R = \{(x, y) \in \mathbb{R}^2 \,|\, a \le x \le b, c \le y \le d\}$, then the double integral of f over the region equals an iterated integral,

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dx\, dy = \int_c^d \int_a^b f(x,y)\, dx\, dy$$

function | a set of inputs, a set of outputs, and a rule for mapping each input to exactly one output

function of two variables | a function $z = f(x, y)$ that maps each ordered pair (x, y) in a subset D of R^2 to a unique real number z

Fundamental Theorem for Line Integrals | the value of line integral $\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}}$ depends only on the value of f at the endpoints of C : $\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a))$

fundamental theorem of calculus | (also, evaluation theorem) we can evaluate a definite integral by evaluating the antiderivative of the integrand at the endpoints of the interval and subtracting

fundamental theorem of calculus | uses a definite integral to define an antiderivative of a function

fundamental theorem of calculus | the theorem, central to the entire development of calculus, that establishes the relationship between differentiation and integration

general form | an equation of a conic section written as a general second-degree equation

general form of the equation of a plane | an equation in the form $ax + by + cz + d = 0$, where $\vec{\mathbf{n}} = \langle a, b, c\rangle$ is a normal vector of the plane, $P = (x_0, y_0, z_0)$ is a point on the plane, and $d = -ax_0 - by_0 - cz_0$

general solution (or family of solutions) | the entire set of solutions to a given differential equation

generalized chain rule | the chain rule extended to functions of more than one independent variable, in which each independent variable may depend on one or more other variables

geometric sequence | a sequence a_n in which the ratio a_{n+1}/a_n is the same for all positive integers n is called a geometric sequence

geometric series | a geometric series is a series that can be written in the form $\sum\limits_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$

gradient | the gradient of the function $f(x, y)$ is defined to be $\vec{\nabla} f(x, y) = (\partial f/\partial x)\,\hat{\mathbf{i}} + (\partial f/\partial y)\,\hat{\mathbf{j}}$, which can be generalized to a function of any number of independent variables

gradient field | a vector field $\vec{\mathbf{F}}$ for which there exists a scalar function f such that $\vec{\nabla} f = \vec{\mathbf{F}}$; in other words, a vector field that is the gradient of a function; such vector fields are also called *conservative*

graph of a function | the set of points (x, y) such that x is in the domain of f and $y = f(x)$

graph of a function of two variables | a set of ordered triples (x, y, z) that satisfies the equation $z = f(x, y)$ plotted in three-dimensional Cartesian space

Green's theorem | relates the integral over a connected region to an integral over the boundary of the region

grid curves | curves on a surface that are parallel to grid lines in a coordinate plane

growth rate | the constant $r > 0$ in the exponential growth function $P(t) = P_0 e^{rt}$

half-life | if a quantity decays exponentially, the half-life is the amount of time it takes the quantity to be reduced by half. It is given by $(\ln 2)/k$

harmonic series | the harmonic series takes the form $\sum\limits_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$

heat flow | a vector field proportional to the negative temperature gradient in an object

helix | a three-dimensional curve in the shape of a spiral

higher-order derivative | a derivative of a derivative, from the second derivative to the nth derivative, is called a higher-order derivative

higher-order partial derivatives | second-order or higher partial derivatives, regardless of whether they are mixed partial derivatives

homogeneous linear equation | a second-order differential equation that can be written in the form $a_2(x)y''+a_1(x)y'+a_0(x)y=r(x)$, but $r(x)=0$ for every value of x

Hooke's law | this law states that the force required to compress (or elongate) a spring is proportional to the distance the spring has been compressed (or stretched) from equilibrium; in other words, $F=kx$, where k is a constant

horizontal asymptote | if $\lim\limits_{x\to\infty} f(x)=L$ or $\lim\limits_{x\to-\infty} f(x)=L$, then $y=L$ is a horizontal asymptote of f

horizontal line test | a function f is one-to-one if and only if every horizontal line intersects the graph of f, at most, once

hydrostatic pressure | the pressure exerted by water on a submerged object

hyperbolic functions | the functions denoted $\sinh$, $\cosh$, $\tanh$, csch, sech, and $\coth$, which involve certain combinations of e^x and e^{-x}

hyperboloid of one sheet | a three-dimensional surface described by an equation of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$; traces of this surface include ellipses and hyperbolas

hyperboloid of two sheets | a three-dimensional surface described by an equation of the form $\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$; traces of this surface include ellipses and hyperbolas

implicit differentiation | is a technique for computing $\frac{dy}{dx}$ for a function defined by an equation, accomplished by differentiating both sides of the equation (remembering to treat the variable y as a function) and solving for $\frac{dy}{dx}$

improper double integral | a double integral over an unbounded region or of an unbounded function

improper integral | an integral over an infinite interval or an integral of a function containing an infinite discontinuity on the interval; an improper integral is defined in terms of a limit. The improper integral converges if this limit is a finite real number; otherwise, the improper integral diverges

increasing on the interval

indefinite integral | the most general antiderivative of $f(x)$ is the indefinite integral of f; we use the notation $\int f(x)\,dx$ to denote the indefinite integral of f

indefinite integral of a vector-valued function | a vector-valued function with a derivative that is equal to a given vector-valued function

independence of path | a vector field $\vec{\mathbf{F}}$ has path independence if $\int_{C_1}\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}=\int_{C_2}\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}$ for any curves C_1 and C_2 in the domain of $\vec{\mathbf{F}}$ with the same initial points and terminal points

independent variable | the input variable for a function

indeterminate forms | When evaluating a limit, the forms $\frac{0}{0}, \infty/\infty, 0\cdot\infty, \infty-\infty, 0^0, \infty^0$, and 1^∞ are considered indeterminate because further analysis is required to determine whether the limit exists and, if so, what its value is.

index variable | the subscript used to define the terms in a sequence is called the index

infinite discontinuity | An infinite discontinuity occurs at a point a if $\lim\limits_{x\to a^-} f(x)=\pm\infty$ or $\lim\limits_{x\to a^+} f(x)=\pm\infty$

infinite limit | A function has an infinite limit at a point a if it either increases or decreases without bound as it approaches a

infinite limit at infinity | a function that becomes arbitrarily large as x becomes large

infinite series | an infinite series is an expression of the form $a_1+a_2+a_3+\cdots=\sum\limits_{n=1}^{\infty}a_n$

inflection point | if f is continuous at c and f changes concavity at c, the point $(c,f(c))$ is an inflection point of f

initial point | the starting point of a vector

initial population | the population at time $t=0$

initial value problem | a problem that requires finding a function y that satisfies the differential equation $\frac{dy}{dx}=f(x)$ together with the initial condition $y(x_0)=y_0$

initial value(s) | a value or set of values that a solution of a differential equation satisfies for a fixed value of the independent variable

initial velocity | the velocity at time $t=0$

initial-value problem | a differential equation together with an initial value or values

instantaneous rate of change | the rate of change of a function at any point along the function a, also called $f'(a)$, or the derivative of the function at a

instantaneous velocity | The instantaneous velocity of an object with a position function that is given by $s(t)$ is the value that the average velocities on intervals of the form $[t,a]$ and $[a,t]$ approach as the values of t move closer to a, provided such a value exists

integrable function | a function is integrable if the limit defining the integral exists; in other words, if the limit of the Riemann sums as n goes to infinity exists

integral calculus | the study of integrals and their applications

integral test | for a series $\sum_{n=1}^{\infty} a_n$ with positive terms a_n, if there exists a continuous, decreasing function f such that $f(n) = a_n$ for all positive integers n, then

$$\sum_{n=1}^{\infty} a_n$$

and

$$\int_1^{\infty} f(x)\,dx$$

either both converge or both diverge

integrand | the function to the right of the integration symbol; the integrand includes the function being integrated

integrating factor | any function $f(x)$ that is multiplied on both sides of a differential equation to make the side involving the unknown function equal to the derivative of a product of two functions

integration by parts | a technique of integration that allows the exchange of one integral for another using the formula $\int u\,dv = uv - \int v\,du$

integration by substitution | a technique for integration that allows integration of functions that are the result of a chain-rule derivative

integration table | a table that lists integration formulas

interior point | a point P_0 of $\mathbb{R}$ is a boundary point if there is a δ disk centered around P_0 contained completely in $\mathbb{R}$

Intermediate Value Theorem | Let f be continuous over a closed bounded interval $[a, b]$ if z is any real number between $f(a)$ and $f(b)$, then there is a number c in [a, b] satisfying $f(c) = z$

intermediate variable | given a composition of functions (e.g., $f(x(t), y(t))$), the intermediate variables are the variables that are independent in the outer function but dependent on other variables as well; in the function $f(x(t), y(t))$, the variables x and y are examples of intermediate variables

interval of convergence | the set of real numbers x for which a power series converges

intuitive definition of the limit | If all values of the function $f(x)$ approach the real number L as the values of $x(\neq a)$ approach a, $f(x)$ approaches L

inverse function | for a function f, the inverse function f^{-1} satisfies $f^{-1}(y) = x$ if $f(x) = y$

inverse hyperbolic functions | the inverses of the hyperbolic functions where $\cosh$ and sech are restricted to the domain $[0, \infty)$;each of these functions can be expressed in terms of a composition of the natural logarithm function and an algebraic function

inverse trigonometric functions | the inverses of the trigonometric functions are defined on restricted domains where they are one-to-one functions

iterated integral | for a function $f(x, y)$ over the region R is a. $\int_a^b \int_c^d f(x, y)\,dx\,dy = \int_a^b \left[\int_c^d f(x, y)\,dy\right] dx$, b. $\int_c^d \int_a^b f(x, y)\,dx\,dy = \int_c^d \left[\int_a^b f(x, y)\,dx\right] dy$, where a, b, c, and d are any real numbers and $R = [a, b] \times [c, d]$

iterative process | process in which a list of numbers $x_0, x_1, x_2, x_3 \ldots$ is generated by starting with a number x_0 and defining $x_n = F(x_{n-1})$ for $n \geq 1$

Jacobian | the Jacobian $J(u, v)$ in two variables is a 2×2 determinant:

$$J(u, v) = \left|\begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{matrix}\right|;$$

the Jacobian $J(u, v, w)$ in three variables is a 3×3 determinant:

$$J(u, v, w) = \left|\begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial v} & \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{matrix}\right|$$

jump discontinuity | A jump discontinuity occurs at a point a if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist, but $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

Kepler's laws of planetary motion | three laws governing the motion of planets, asteroids, and comets in orbit around the Sun

Lagrange multiplier | the constant (or constants) used in the method of Lagrange multipliers; in the case of one constant, it is represented by the variable λ

lamina | a thin sheet of material; laminas are thin enough that, for mathematical purposes, they can be treated as if they are two-dimensional

left-endpoint approximation | an approximation of the area under a curve computed by using the left endpoint of each subinterval to calculate the height of the vertical sides of each rectangle

level curve of a function of two variables | the set of points satisfying the equation $f(x, y) = c$ for some real number c in the range of f

level surface of a function of three variables | the set of points satisfying the equation $f(x, y, z) = c$ for some real number c in the range of f

limaçon | the graph of the equation $r = a + b\sin\theta$ or $r = a + b\cos\theta$. If $a = b$ then the graph is a cardioid

limit | the process of letting x or t approach a in an expression; the limit of a function $f(x)$ as x approaches a is the value that $f(x)$ approaches as x approaches a

limit at infinity | a function that approaches a limit value L as x becomes large

limit comparison test | Suppose $a_n, b_n \geq 0$ for all $n \geq 1$. If $\lim_{n \to \infty} a_n/b_n \to L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge; if $\lim_{n \to \infty} a_n/b_n \to 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. If $\lim_{n \to \infty} a_n/b_n \to \infty$, and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

limit laws | the individual properties of limits; for each of the individual laws, let $f(x)$ and $g(x)$ be defined for all $x \neq a$ over some open interval containing a; assume that L and M are real numbers so that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$; let c be a constant

limit of a sequence | the real number LL to which a sequence converges is called the limit of the sequence

limit of a vector-valued function | a vector-valued function $\vec{\mathbf{r}}(t)$ has a limit $\vec{\mathbf{L}}$ as t approaches a if $\lim t \to a \left|\vec{\mathbf{r}}(t) - \vec{\mathbf{L}}\right| = 0$

limits of integration | these values appear near the top and bottom of the integral sign and define the interval over which the function should be integrated

line integral | the integral of a function along a curve in a plane or in space

linear | description of a first-order differential equation that can be written in the form $a(x)y' + b(x)y = c(x)$

linear approximation | the linear function $L(x) = f(a) + f'(a)(x - a)$ is the linear approximation of f at $x = a$

linear approximation | given a function $f(x, y)$ and a tangent plane to the function at a point (x_0, y_0), we can approximate $f(x, y)$ for points near (x_0, y_0) using the tangent plane formula

linear function | a function that can be written in the form $f(x) = mx + b$

linearly dependent | a set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ for whichthere are constants $c_1, c_2, \ldots c_n$, not all zero, such that $c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$ for all \ (x\) in the interval of interest

linearly independent | a set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ for which there are no constants $c_1, c_2, \ldots c_n$, such that $c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$ for all \(x\) in the interval of interest

local extremum | if f has a local maximum or local minimum at c, we say f has a local extremum at c

local maximum | if there exists an interval I such that $f(c) \geq f(x)$ for all $x \in I$, we say f has a local maximum at c

local minimum | if there exists an interval I such that $f(c) \leq f(x)$ for all $x \in I$, we say f has a local minimum at c

logarithmic differentiation | is a technique that allows us to differentiate a function by first taking the natural logarithm of both sides of an equation, applying properties of logarithms to simplify the equation, and differentiating implicitly

logarithmic function | a function of the form $f(x) = \log_b(x)$ for some base $b > 0$, $b \neq 1$ such that $y = \log_b(x)$ if and only if $b^y = x$

logistic differential equation | a differential equation that incorporates the carrying capacity K and growth rate rr into a population model

lower sum | a sum obtained by using the minimum value of $f(x)$ on each subinterval

L'Hôpital's rule | If f and g are differentiable functions over an interval a, except possibly at a, and $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$ or $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ are infinite, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$, assuming the limit on the right exists or is ∞ or $-\infty$.

Maclaurin polynomial | a Taylor polynomial centered at 0; the nth-degree Taylor polynomial for f at 0 is the nth-degree Maclaurin polynomial for f

Maclaurin series | a Taylor series for a function f at $x = 0$ is known as a Maclaurin series for f

magnitude | the length of a vector

major axis | the major axis of a conic section passes through the vertex in the case of a parabola or through the two vertices in the case of an ellipse or hyperbola; it is also an axis of symmetry of the conic; also called the transverse axis

marginal cost | is the derivative of the cost function, or the approximate cost of producing one more item

marginal profit | is the derivative of the profit function, or the approximate profit obtained by producing and selling one more item

marginal revenue | is the derivative of the revenue function, or the approximate revenue obtained by selling one more item

mass flux | the rate of mass flow of a fluid per unit area, measured in mass per unit time per unit area

mathematical model | A method of simulating real-life situations with mathematical equations

mean value theorem | if f is continuous over $[a, b]$ and differentiable over (a, b), then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

mean value theorem for integrals | guarantees that a point c exists such that $f(c)$ is equal to the average value of the function

method of cylindrical shells | a method of calculating the volume of a solid of revolution by dividing the solid into nested cylindrical shells; this method is different from the methods of disks or washers in that we integrate with respect to the opposite variable

method of Lagrange multipliers | a method of solving an optimization problem subject to one or more constraints

method of undetermined coefficients | a method that involves making a guess about the form of the particular solution, then solving for the coefficients in the guess

method of variation of parameters | a method that involves looking for particular solutions in the form $y_p(x) = u(x)y_1(x) + v(x)y_2(x)$, where y_1 and y_2 are linearly independent solutions to the complementary equations, and then solving a system of equations to find $u(x)$ and $v(x)$

midpoint rule | a rule that uses a Riemann sum of the form $M_n = \sum_{i=1}^{n} f(m_i)\Delta x$, where m_i is the midpoint of the ith subinterval to approximate $\int_a^b f(x)\,dx$

minor axis | the minor axis is perpendicular to the major axis and intersects the major axis at the center of the conic, or at the vertex in the case of the parabola; also called the conjugate axis

mixed partial derivatives | second-order or higher partial derivatives, in which at least two of the differentiations are with respect to different variables

moment | if n masses are arranged on a number line, the moment of the system with respect to the origin is given by $M = \sum_{i=1}^{n} m_i x_i$; if, instead, we consider a region in the plane, bounded above by a function $f(x)$ over an interval $[a, b]$, then the moments of the region with respect to the x- and y-axes are given by $M_x = \rho \int_a^b \frac{[f(x)]^2}{2}\,dx$ and $M_y = \rho \int_a^b x f(x)\,dx$, respectively

monotone sequence | an increasing or decreasing sequence

multivariable calculus | the study of the calculus of functions of two or more variables

nappe | a nappe is one half of a double cone

natural exponential function | the function $f(x) = e^x$

natural logarithm | the function $\ln x = \log_e x$

net change theorem | if we know the rate of change of a quantity, the net change theorem says the future quantity is equal to the initial quantity plus the integral of the rate of change of the quantity

net signed area | the area between a function and the x-axis such that the area below the x-axis is subtracted from the area above the x-axis; the result is the same as the definite integral of the function

Newton's method | method for approximating roots of $f(x) = 0$; using an initial guess x_0; each subsequent approximation is defined by the equation $x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$

nonelementary integral | an integral for which the antiderivative of the integrand cannot be expressed as an elementary function

nonhomogeneous linear equation | a second-order differential equation that can be written in the form $a_2(x)y'' + a_1(x)y' + a_0(x)y = r(x)$, but $r(x) \neq 0$ for some value of x

normal component of acceleration | the coefficient of the unit normal vector $\vec{\mathrm{N}}$ when the acceleration vector is written as a linear combination of $\vec{\mathrm{T}}$ and $\vec{\mathrm{N}}$

normal plane | a plane that is perpendicular to a curve at any point on the curve

normal vector | a vector perpendicular to a plane

normalization | using scalar multiplication to find a unit vector with a given direction

number e | as m gets larger, the quantity $(1 + (1/m)^m$ gets closer to some real number; we define that real number to be e; the value of e is approximately 2.718282

numerical integration | the variety of numerical methods used to estimate the value of a definite integral, including the midpoint rule, trapezoidal rule, and Simpson's rule

objective function | the function that is to be maximized or minimized in an optimization problem

oblique asymptote | the line $y = mx + b$ if $f(x)$ approaches it as $x \to \infty$ or$x \to -\infty$

octants | the eight regions of space created by the coordinate planes

odd function | a function is odd if $f(-x) = -f(x)$ for all x in the domain of f

one-sided limit | A one-sided limit of a function is a limit taken from either the left or the right

one-to-one function | a function f is one-to-one if $f(x_1) \neq f(x_2)$ if $x_1 \neq x_2$

one-to-one transformation | a transformation $T : G \to R$ defined as $T(u, v) = (x, y)$ is said to be one-to-one if no two points map to the same image point

open set | a set S that contains none of its boundary points

optimization problem | calculation of a maximum or minimum value of a function of several variables, often using Lagrange multipliers

optimization problems | problems that are solved by finding the maximum or minimum value of a function

order of a differential equation | the highest order of any derivative of the unknown function that appears in the equation

orientation | the direction that a point moves on a graph as the parameter increases

orientation of a curve | the orientation of a curve C is a specified direction of C

orientation of a surface | if a surface has an "inner" side and an "outer" side, then an orientation is a choice of the inner or the outer side; the surface could also have "upward" and "downward" orientations

orthogonal vectors | vectors that form a right angle when placed in standard position

osculating circle | a circle that is tangent to a curve C at a point P and that shares the same curvature

osculating plane | the plane determined by the unit tangent and the unit normal vector

***p*-series** | a series of the form $\sum_{n=1}^{\infty} 1/n^p$

parallelogram method | a method for finding the sum of two vectors; position the vectors so they share the same initial point; the vectors then form two adjacent sides of a parallelogram; the sum of the vectors is the diagonal of that parallelogram

parameter | an independent variable that both x and y depend on in a parametric curve; usually represented by the variable t

parameter domain (parameter space) | the region of the uv-plane over which the parameters u and v vary for parameterization $\vec{\mathbf{r}}(u, v) = \langle x(u, v),\ y(u, v),\ z(u, v)\rangle$

parameterization of a curve | rewriting the equation of a curve defined by a function $y = f(x)$ as parametric equations

parameterized surface (parametric surface) | a surface given by a description of the form $\vec{\mathbf{r}}(u, v) = \langle x(u, v),\ y(u, v),\ z(u, v)\rangle$, where the parameters u and v vary over a parameter domain in the uv-plane

parametric curve | the graph of the parametric equations $x(t)$ and $y(t)$ over an interval $a \leq t \leq b$ combined with the equations

parametric equations | the equations $x = x(t)$ and $y = y(t)$ that define a parametric curve

parametric equations of a line | the set of equations $x = x_0 + ta, y = y_0 + tb$, and $z = z_0 + tc$ describing the line with direction vector $v = \langle a, b, c\rangle$ passing through point (x_0, y_0, z_0)

partial derivative | a derivative of a function of more than one independent variable in which all the variables but one are held constant

partial differential equation | an equation that involves an unknown function of more than one independent variable and one or more of its partial derivatives

partial fraction decomposition | a technique used to break down a rational function into the sum of simple rational functions

partial sum | the kth partial sum of the infinite series $\sum_{n=1}^{\infty} a_n$ is the finite sum $S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k$

particular solution | member of a family of solutions to a differential equation that satisfies a particular initial condition

particular solution | a solution $y_p(x)$ of a differential equation that contains no arbitrary constants

partition | a set of points that divides an interval into subintervals

percentage error | the relative error expressed as a percentage

periodic function | a function is periodic if it has a repeating pattern as the values of x move from left to right

phase line | a visual representation of the behavior of solutions to an autonomous differential equation subject to various initial conditions

piecewise smooth curve | an oriented curve that is not smooth, but can be written as the union of finitely many smooth curves

piecewise-defined function | a function that is defined differently on different parts of its domain

planar transformation | a function T that transforms a region G in one plane into a region R in another plane by a change of variables

plane curve | the set of ordered pairs $(f(t), g(t))$ together with their defining parametric equations $x = f(t)$ and $y = g(t)$

point-slope equation | equation of a linear function indicating its slope and a point on the graph of the function

polar axis | the horizontal axis in the polar coordinate system corresponding to $r \geq 0$

polar coordinate system | a system for locating points in the plane. The coordinates are r, the radial coordinate, and θ, the angular coordinate

polar equation | an equation or function relating the radial coordinate to the angular coordinate in the polar coordinate system

polar rectangle | the region enclosed between the circles $r = a$ and $r = b$ and the angles $\theta = \alpha$ and $\theta = \beta$; it is described as $R = \{(r, \theta) \mid a \leq r \leq b,\ \alpha \leq \theta \leq \beta\}$

pole | the central point of the polar coordinate system, equivalent to the origin of a Cartesian system

polynomial function | a function of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$

population growth rate | is the derivative of the population with respect to time

potential function | a scalar function f such that $\vec{\nabla} f = \vec{\mathbf{F}}$

power function | a function of the form $f(x) = x^n$ for any positive integer $n \geq 1$

power law for limits | the limit law

$$\begin{aligned} &\lim_{x \to a} (f(x))^n \\ &= (\lim_{x \to a} f(x))^n \\ &= L^n \end{aligned}$$

for every positive integer n

power reduction formula | a rule that allows an integral of a power of a trigonometric function to be exchanged for an integral involving a lower power

power rule | the derivative of a power function is a function in which the power on x becomes the coefficient of the term and the power on x in the derivative decreases by 1: If n is an integer, then $\frac{d}{dx}(x^n) = nx^{n-1}$

power series | a series of the form $\sum_{n=0}^{\infty} c_n x^n$ is a power series centered at $x = 0$; a series of the form $\sum_{n=0}^{\infty} c_n (x - a)^n$ is a power series centered at $x = a$

principal unit normal vector | a vector orthogonal to the unit tangent vector, given by the formula $\frac{\vec{\mathbf{T}}'(t)}{\|\vec{\mathbf{T}}'(t)\|}$

principal unit tangent vector | a unit vector tangent to a curve C

product law for limits | the limit law

$$\begin{aligned} &\lim_{x \to a} (f(x) \cdot g(x)) \\ &= \lim_{x \to a} f(x) \\ &\cdot \lim_{x \to a} g(x) \\ &= L \cdot M \end{aligned}$$

product rule | the derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function: $\frac{d}{dx}\big(f(x)g(x)\big) = f'(x)g(x) + g'(x)f(x)$

projectile motion | motion of an object with an initial velocity but no force acting on it other than gravity

propagated error | the error that results in a calculated quantity $f(x)$ resulting from a measurement error dx

quadratic function | a polynomial of degree 2; that is, a function of the form $f(x) = ax^2 + bx + c$ where $a \neq 0$

quadric surfaces | surfaces in three dimensions having the property that the traces of the surface are conic sections (ellipses, hyperbolas, and parabolas)

quotient law for limits | the limit law $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M}$ for M≠0

quotient rule | the derivative of the quotient of two functions is the derivative of the first function times the second function minus the derivative of the second function times the first function, all divided by the square of the second function: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - g'(x)f(x)}{\big(g(x)\big)^2}$

radial coordinate | r the coordinate in the polar coordinate system that measures the distance from a point in the plane to the pole

radial field | a vector field in which all vectors either point directly toward or directly away from the origin; the magnitude of any vector depends only on its distance from the origin

radians | for a circular arc of length s on a circle of radius 1, the radian measure of the associated angle θ is s

radius of convergence | if there exists a real number $R > 0$ such that a power series centered at $x = a$ converges for $|x - a| < R$ and diverges for $|x - a| > R$, then R is the radius of convergence; if the power series only converges at $x = a$, the radius of convergence is $R = 0$; if the power series converges for all real numbers x, the radius of convergence is $R = \infty$

radius of curvature | the reciprocal of the curvature

radius of gyration | the distance from an object's center of mass to its axis of rotation

range | the set of outputs for a function

ratio test | for a series $\sum_{n=1}^{\infty} a_n$ with nonzero terms, let $\rho = \lim_{n\to\infty} |a_{n+1}/a_n|$; if $0 \le \rho < 1$, the series converges absolutely; if $\rho > 1$, the series diverges; if $\rho = 1$, the test is inconclusive

rational function | a function of the form $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials

recurrence relation | a recurrence relation is a relationship in which a term a_n in a sequence is defined in terms of earlier terms in the sequence

region | an open, connected, nonempty subset of $\mathbb{R}^2$

regular parameterization | parameterization $\vec{r}(u, v) = \langle x(u, v),\ y(u, v),\ z(u, v)\rangle$ such that $r_u \times r_v$ is not zero for point (u, v) in the parameter domain

regular partition | a partition in which the subintervals all have the same width

related rates | are rates of change associated with two or more related quantities that are changing over time

relative error | given an absolute error Δq for a particular quantity, $\frac{\Delta q}{q}$ is the relative error.

relative error | error as a percentage of the actual value, given by

$$\begin{matrix}\text{relative error} \\ = \left|\frac{A - B}{A}\right| \\ \cdot\, 100\%\end{matrix}$$

remainder estimate | for a series $\sum_{n=}^{\infty} 1a_n$ with positive terms a_n and a continuous, decreasing function f such that $f(n) = a_n$ for all positive integers n, the remainder $R_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n$ satisfies the following estimate:

$$\begin{matrix}\int_{N+1}^{\infty} f(x)\,dx \\ < R_N \\ < \int_{N}^{\infty} f(x)\,dx\end{matrix}$$

removable discontinuity | A removable discontinuity occurs at a point a if $f(x)$ is discontinuous at a, but $\lim_{x\to a} f(x)$ exists

reparameterization | an alternative parameterization of a given vector-valued function

restricted domain | a subset of the domain of a function f

riemann sum | an estimate of the area under the curve of the form $A \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$

right-endpoint approximation | the right-endpoint approximation is an approximation of the area of the rectangles under a curve using the right endpoint of each subinterval to construct the vertical sides of each rectangle

right-hand rule | a common way to define the orientation of the three-dimensional coordinate system; when the right hand is curved around the z-axis in such a way that the fingers curl from the positive x-axis to the positive y-axis, the thumb points in the direction of the positive z-axis

***RLC* series circuit** | a complete electrical path consisting of a resistor, an inductor, and a capacitor; a second-order, constant-coefficient differential equation can be used to model the charge on the capacitor in an *RLC* series circuit

rolle's theorem | if f is continuous over $[a, b]$ and differentiable over (a, b), and if $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$

root function | a function of the form $f(x) = x^{1/n}$ for any integer $n \ge 2$

root law for limits | the limit law $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L}$ for all L if n is odd and for $L \ge 0$ if n is even

root test | for a series $\sum_{n=1}^{\infty} a_n$, let $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$; if $0 \le \rho < 1$, the series converges absolutely; if $\rho > 1$, the series diverges; if $\rho = 1$, the test is inconclusive

rose | graph of the polar equation $r = a\cos 2\theta$ or $r = a\sin 2\theta$for a positive constant a

rotational field | a vector field in which the vector at point (x, y) is tangent to a circle with radius $r = \sqrt{x^2 + y^2}$; in a rotational field, all vectors flow either clockwise or counterclockwise, and the magnitude of a vector depends only on its distance from the origin

rulings | parallel lines that make up a cylindrical surface

saddle point | given the function $z = f(x, y)$, the point $(x_0, y_0, f(x_0, y_0))$ is a saddle point if both $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$, but f does not have a local extremum at (x_0, y_0)

scalar | a real number

scalar equation of a plane | the equation $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ used to describe a plane containing point $P=(x_0,y_0,z_0)$ with normal vector $n=\langle a,b,c\rangle$ or its alternate form $ax+by+cz+d=0$, where $d=-ax_0-by_0-cz_0$

scalar line integral | the scalar line integral of a function f along a curve C with respect to arc length is the integral $\int_C f\,ds$, it is the integral of a scalar function f along a curve in a plane or in space; such an integral is defined in terms of a Riemann sum, as is a single-variable integral

scalar multiplication | a vector operation that defines the product of a scalar and a vector

scalar projection | the magnitude of the vector projection of a vector

secant | A secant line to a function $f(x)$ at a is a line through the point $(a,f(a))$ and another point on the function; the slope of the secant line is given by $m_{sec}=\dfrac{f(x)-f(a)}{x-a}$

second derivative test | suppose $f'(c)=0$ and f'' is continuous over an interval containing c; if $f''(c)>0$, then f has a local minimum at c; if $f''(c)<0$, then f has a local maximum at c; if $f''(c)=0$, then the test is inconclusive

separable differential equation | any equation that can be written in the form $y'=f(x)g(y)$

separation of variables | a method used to solve a separable differential equation

sequence | an ordered list of numbers of the form $a_1,a_2,a_3,\ldots$ is a sequence

sigma notation | (also, summation notation) the Greek letter sigma (Σ) indicates addition of the values; the values of the index above and below the sigma indicate where to begin the summation and where to end it

simple curve | a curve that does not cross itself

simple harmonic motion | motion described by the equation $x(t)=c_1\cos(\omega t)+c_2\sin(\omega t)$, as exhibited by an undamped spring-mass system in which the mass continues to oscillate indefinitely

simply connected region | a region that is connected and has the property that any closed curve that lies entirely inside the region encompasses points that are entirely inside the region

Simpson's rule | a rule that approximates $\int_a^b f(x)\,dx$ using the area under a piecewise quadratic function. The approximation S_n to $\int_a^b f(x)\,dx$ is given by

$$\begin{aligned}S_n=\frac{\Delta x}{3}\big(&f(x_0)\\&+4f(x_1)\\&+2f(x_2)\\&+4f(x_3)\\&+2f(x_4)+\cdots\\&+2f(x_{n-2})\\&+4f(x_{n-1})\\&+f(x_n)\big).\end{aligned}$$

skew lines | two lines that are not parallel but do not intersect

slicing method | a method of calculating the volume of a solid that involves cutting the solid into pieces, estimating the volume of each piece, then adding these estimates to arrive at an estimate of the total volume; as the number of slices goes to infinity, this estimate becomes an integral that gives the exact value of the volume

slope | the change in y for each unit change in x

slope-intercept form | equation of a linear function indicating its slope and y-intercept

smooth | curves where the vector-valued function $\vec{\mathbf{r}}(t)$ is differentiable with a non-zero derivative

solid of revolution | a solid generated by revolving a region in a plane around a line in that plane

solution curve | a curve graphed in a direction field that corresponds to the solution to the initial-value problem passing through a given point in the direction field

solution to a differential equation | a function $y=f(x)$ that satisfies a given differential equation

space curve | the set of ordered triples $(f(t),g(t),h(t))$ together with their defining parametric equations $x=f(t)$, $y=g(t)$ and $z=h(t)$

space-filling curve | a curve that completely occupies a two-dimensional subset of the real plane

speed | is the absolute value of velocity, that is, $|v(t)|$ is the speed of an object at time t whose velocity is given by $v(t)$

sphere | the set of all points equidistant from a given point known as the *center*

spherical coordinate system | a way to describe a location in space with an ordered triple (ρ,θ,φ), where ρ is the distance between P and the origin $(\rho\neq 0)$, θ is the same angle used to describe the location in cylindrical coordinates, and φ is the angle formed by the positive z-axis and line segment $\overline{OP}$, where O is the origin and $0\leq\varphi\leq\pi$

squeeze theorem | states that if $f(x)\leq g(x)\leq h(x)$ for all $x\neq a$ over an open interval containing a and $\lim_{x\to a}f(x)=L=\lim_{x\to a}h(x)$ where L is a real number, then $\lim_{x\to a}g(x)=L$

standard equation of a sphere | $(x-a)^2+(y-b)^2+(z-c)^2=r^2$ describes a sphere with center (a,b,c) and radius r

standard form | the form of a first-order linear differential equation obtained by writing the differential equation in the form $y'+p(x)y=q(x)$

standard form | an equation of a conic section showing its properties, such as location of the vertex or lengths of major and minor axes

standard unit vectors | unit vectors along the coordinate axes: $\hat{\mathbf{i}}=\langle 1,0\rangle$, $\hat{\mathbf{j}}=\langle 0,1\rangle$

standard-position vector | a vector with initial point $(0,0)$

steady-state solution | a solution to a nonhomogeneous differential equation related to the forcing function; in the long term, the solution approaches the steady-state solution

step size | the increment hh that is added to the xx value at each step in Euler's Method

Stokes' theorem | relates the flux integral over a surface S to a line integral around the boundary C of the surface S

stream function | if $\vec{\mathbf{F}} = \langle P, Q \rangle$ is a source-free vector field, then stream function g is a function such that $P = g_y$ and $Q = -g_x$

sum law for limits | The limit law $\lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$

sum rule | the derivative of the sum of a function f and a function g is the same as the sum of the derivative of f and the derivative of g: $\frac{d}{dx}\big(f(x) + g(x)\big) = f'(x) + g'(x)$

surface | the graph of a function of two variables, $z = f(x, y)$

surface area | the surface area of a solid is the total area of the outer layer of the object; for objects such as cubes or bricks, the surface area of the object is the sum of the areas of all of its faces

surface area | the area of surface S given by the surface integral

$$\iint_S dS$$

surface independent | flux integrals of curl vector fields are surface independent if their evaluation does not depend on the surface but only on the boundary of the surface

surface integral | an integral of a function over a surface

surface integral of a scalar-valued function | a surface integral in which the integrand is a scalar function

surface integral of a vector field | a surface integral in which the integrand is a vector field

symmetric equations of a line | the equations $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$ describing the line with direction vector $v = \langle a, b, c \rangle$ passing through point (x_0, y_0, z_0)

symmetry about the

symmetry about the origin | the graph of a function f is symmetric about the origin if $(-x, -y)$ is on the graph of f whenever (x, y) is on the graph

symmetry principle | the symmetry principle states that if a region R is symmetric about a line I, then the centroid of R lies on I

table of values | a table containing a list of inputs and their corresponding outputs

tangent | A tangent line to the graph of a function at a point $(a, f(a))$ is the line that secant lines through $(a, f(a))$ approach as they are taken through points on the function with x-values that approach a; the slope of the tangent line to a graph at a measures the rate of change of the function at a

tangent line approximation (linearization) | since the linear approximation of f at $x = a$ is defined using the equation of the tangent line, the linear approximation of f at $x = a$ is also known as the tangent line approximation to f at $x = a$

tangent plane | given a function $f(x, y)$ that is differentiable at a point (x_0, y_0), the equation of the tangent plane to the surface $z = f(x, y)$ is given by $z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

tangent vector | to $\vec{\mathbf{r}}(t)$ at $t = t_0$ any vector $\vec{\mathbf{v}}$ such that, when the tail of the vector is placed at point $\vec{\mathbf{r}}(t_0)$ on the graph, vector $\vec{\mathbf{v}}$ is tangent to curve C

tangential component of acceleration | the coefficient of the unit tangent vector $\vec{\mathbf{T}}$ when the acceleration vector is written as a linear combination of $\vec{\mathbf{T}}$ and $\vec{\mathbf{N}}$

Taylor polynomials | the nth-degree Taylor polynomial for f at $x = a$ is $p_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$

Taylor series | a power series at a that converges to a function f on some open interval containing a.

Taylor's theorem with remainder | for a function f and the nth-degree Taylor polynomial for f at $x = a$, the remainder $R_n(x) = f(x) - p_n(x)$ satisfies $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - a)^{n+1}$ for some$c$ between x and a; if there exists an interval I containing a and a real number M such that $|\, f^{(n+1)}(x) \,| \leq M$ for all x in I, then $|R_n(x)| \leq \frac{M}{(n+1)!}|x - a|^{n+1}$

telescoping series | a telescoping series is one in which most of the terms cancel in each of the partial sums

term | the number a_n in the sequence a_n is called the nth term of the sequence

term-by-term differentiation of a power series | a technique for evaluating the derivative of a power series $\sum_{n=0}^{\infty} c_n(x - a)^n$ by evaluating the derivative of each term separately to create the new power series $\sum_{n=1}^{\infty} n c_n(x - a)^{n-1}$

term-by-term integration of a power series | a technique for integrating a power series $\sum_{n=0}^{\infty} c_n(x - a)^n$ by integrating each term separately to create the new power series $C + \sum_{n=0}^{\infty} c_n \frac{(x - a)^{n+1}}{n + 1}$

terminal point | the endpoint of a vector

theorem of Pappus for volume | this theorem states that the volume of a solid of revolution formed by revolving a region around an external axis is equal to the area of the region multiplied by the distance traveled by the centroid of the region

three-dimensional rectangular coordinate system | a coordinate system defined by three lines that intersect at right angles; every point in space is described by an ordered triple (x, y, z) that plots its location relative to the defining axes

threshold population | the minimum population that is necessary for a species to survive

total area | total area between a function and the x-axis is calculated by adding the area above the x-axis and the area below the x-axis; the result is the same as the definite integral of the absolute value of the function

total differential | the total differential of the function $f(x, y)$ at (x_0, y_0) is given by the formula $dz = f_x(x_0, y_0)dx + fy(x_0, y_0)dy$

trace | the intersection of a three-dimensional surface with a coordinate plane

transcendental function | a function that cannot be expressed by a combination of basic arithmetic operations

transformation | a function that transforms a region GG in one plane into a region RR in another plane by a change of variables

transformation of a function | a shift, scaling, or reflection of a function

trapezoidal rule | a rule that approximates $\int_a^b f(x)\,dx$ using the area of trapezoids. The approximation T_n to $\int_a^b f(x)\,dx$ is given by

$$\begin{gathered} T_n = \frac{\Delta x}{2} \\ \big(f(x_0) \\ + 2\,f(x_1) \\ + 2\,f(x_2) + \cdots \\ + 2\,f(x_{n-1}) \\ + f(x_n)\big). \end{gathered}$$

tree diagram | illustrates and derives formulas for the generalized chain rule, in which each independent variable is accounted for

triangle inequality | If a and b are any real numbers, then $|a + b| \leq |a| + |b|$

triangle inequality | the length of any side of a triangle is less than the sum of the lengths of the other two sides

triangle method | a method for finding the sum of two vectors; position the vectors so the terminal point of one vector is the initial point of the other; these vectors then form two sides of a triangle; the sum of the vectors is the vector that forms the third side; the initial point of the sum is the initial point of the first vector; the terminal point of the sum is the terminal point of the second vector

trigonometric functions | functions of an angle defined as ratios of the lengths of the sides of a right triangle

trigonometric identity | an equation involving trigonometric functions that is true for all angles θ for which the functions in the equation are defined

trigonometric integral | an integral involving powers and products of trigonometric functions

trigonometric substitution | an integration technique that converts an algebraic integral containing expressions of the form $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$ into a trigonometric integral

triple integral | the triple integral of a continuous function $f(x, y, z)$ over a rectangular solid box B is the limit of a Riemann sum for a function of three variables, if this limit exists

triple integral in cylindrical coordinates | the limit of a triple Riemann sum, provided the following limit exists:

$$\begin{gathered} lim_{l,m,n\to\infty} \sum_{i=1}^{l} \\ \sum_{j=1}^{m} \\ \sum_{k=1}^{n} f(r^*_{ijk}, \theta^*_{ijk}, \\ s^*_{ijk}) r^*_{ijk} \Delta r \\ \Delta\theta \Delta z \end{gathered}$$

triple integral in spherical coordinates | the limit of a triple Riemann sum, provided the following limit exists:

$$\begin{gathered} lim_{l,m,n\to\infty} \sum_{i=1}^{l} \\ \sum_{j=1}^{m} \\ \sum_{k=1}^{n} f(\rho^*_{ijk}, \theta^*_{ijk}, \\ \varphi^*_{ijk}) (\rho^*_{ijk})^2 \\ \sin\varphi \Delta\rho \Delta\theta \Delta\varphi \end{gathered}$$

Type I | a region D in the xy- plane is Type I if it lies between two vertical lines and the graphs of two continuous functions $g_1(x)$ and $g_2(x)$

Type II | a region D in the xy-plane is Type II if it lies between two horizontal lines and the graphs of two continuous functions $h_1(y)$ and $h_2(h)$

unbounded sequence | a sequence that is not bounded is called unbounded

unit vector | a vector with magnitude 1

unit vector field | a vector field in which the magnitude of every vector is 1

upper sum | a sum obtained by using the maximum value of $f(x)$ on each subinterval

variable of integration | indicates which variable you are integrating with respect to; if it is x, then the function in the integrand is followed by dx

vector | a mathematical object that has both magnitude and direction

vector addition | a vector operation that defines the sum of two vectors

vector difference | the vector difference $\vec{\mathrm{v}} - \vec{\mathrm{w}}$ is defined as $\vec{\mathrm{v}} + (-\vec{\mathrm{w}}) = \vec{\mathrm{v}} + (-1)\vec{\mathrm{w}}$

vector equation of a line | the equation $\vec{\mathbf{r}} = \vec{\mathbf{r}}_0 + t\vec{\mathbf{v}}$ used to describe a line with direction vector $\vec{\mathbf{v}} = \langle a, b, c \rangle$ passing through point $P = (x_0, y_0, z_0)$, where $\vec{\mathbf{r}}_0 = \langle x_0, y_0, z_0 \rangle$, is the position vector of point P

vector equation of a plane | the equation $\vec{\mathbf{n}} \cdot \overrightarrow{PQ} = 0$, where P is a given point in the plane, Q is any point in the plane, and $\vec{\mathbf{n}}$ is a normal vector of the plane

vector field | measured in $\mathbb{R}^2$, an assignment of a vector $\vec{\mathbf{F}}(x, y)$ to each point (x, y) of a subset D of $\mathbb{R}^2$; in $\mathbb{R}^3$, an assignment of a vector $\vec{\mathbf{F}}(x, y, z)$ to each point (x, y, z) of a subset D of $\mathbb{R}^3$

vector line integral | the vector line integral of vector field $\vec{\mathbf{F}}$ along curve C is the integral of the dot product of $\vec{\mathbf{F}}$ with unit tangent vector $\vec{\mathbf{T}}$ of C with respect to arc length, $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$; such an integral is defined in terms of a Riemann sum, similar to a single-variable integral

vector parameterization | any representation of a plane or space curve using a vector-valued function

vector projection | the component of a vector that follows a given direction

vector sum | the sum of two vectors, $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$, can be constructed graphically by placing the initial point of $\vec{\mathbf{w}}$ at the terminal point of $\vec{\mathbf{v}}$; then the vector sum $\vec{\mathbf{v}} + \vec{\mathbf{w}}$ is the vector with an initial point that coincides with the initial point of $\vec{\mathbf{v}}$, and with a terminal point that coincides with the terminal point of $\vec{\mathbf{w}}$

vector-valued function | a function of the form $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}}$ or $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$,where the component functions f, g, and h are real-valued functions of the parameter t.

velocity vector | the derivative of the position vector

vertex | a vertex is an extreme point on a conic section; a parabola has one vertex at its turning point. An ellipse has two vertices, one at each end of the major axis; a hyperbola has two vertices, one at the turning point of each branch

vertical asymptote | A function has a vertical asymptote at $x = a$ if the limit as x approaches a from the right or left is infinite

vertical line test | given the graph of a function, every vertical line intersects the graph, at most, once

vertical trace | the set of ordered triples (c, y, z) that solves the equation $f(c, y) = z$ for a given constant $x = c$ or the set of ordered triples (x, d, z) that solves the equation $f(x, d) = z$ for a given constant $y = d$

washer method | a special case of the slicing method used with solids of revolution when the slices are washers

work | the amount of energy it takes to move an object; in physics, when a force is constant, work is expressed as the product of force and distance

work done by a force | work is generally thought of as the amount of energy it takes to move an object; if we represent an applied force by a vector $\vec{\mathbf{F}}$ and the displacement of an object by a vector $\vec{\mathbf{s}}$, then the work done by the force is the dot product of $\vec{\mathbf{F}}$ and $\vec{\mathbf{s}}$.

zero vector | the vector with both initial point and terminal point $(0, 0)$

zeros of a function | when a real number x is a zero of a function f, $f(x) = 0$

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      - 1.1E: Exercises for Section 1.1 - *CC BY-NC-SA 4.0*
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      - 1.2E: Exercises for Section 1.2 - *CC BY-NC-SA 4.0*
    - 1.3: Trigonometric Functions - *CC BY-NC-SA 4.0*
      - 1.3E: Exercises for Section 1.3 - *CC BY-NC-SA 4.0*
    - 1.4: Inverse Functions - *CC BY-NC-SA 4.0*
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    - 1.5: Exponential and Logarithmic Functions - *CC BY-NC-SA 4.0*
      - 1.5E: Exercises for Section 1.5 - *CC BY-NC-SA 4.0*
    - 1.R: Chapter 1 Review Exercises - *CC BY-NC-SA 4.0*
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    - 2.3: The Limit Laws - *CC BY-NC-SA 4.0*
      - 2.3E: Exercises for Section 2.3 - *CC BY-NC-SA 4.0*
    - 2.4: Continuity - *CC BY-NC-SA 4.0*
      - 2.4E: Exercises for Section 2.4 - *CC BY-NC-SA 4.0*
    - 2.5: The Precise Definition of a Limit - *CC BY-NC-SA 4.0*
      - 2.5E: Exercises for Section 2.5 - *CC BY-NC-SA 4.0*
    - 2.R: Chapter 2 Review Exercises - *CC BY-NC-SA 4.0*
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    - 3.0: Prelude to Derivatives - *CC BY-NC-SA 4.0*
    - 3.1: Defining the Derivative - *CC BY-NC-SA 4.0*
      - 3.1E: Exercises for Section 3.1 - *CC BY-NC-SA 4.0*
    - 3.2: The Derivative as a Function - *CC BY-NC-SA 4.0*
      - 3.2E: Exercises for Section 3.2 - *CC BY-NC-SA 4.0*
    - 3.3: Differentiation Rules - *CC BY-NC-SA 4.0*
      - 3.3E: Exercises for Section 3.3 - *CC BY-NC-SA 4.0*
    - 3.4: Derivatives as Rates of Change - *CC BY-NC-SA 4.0*
      - 3.4E: Exercises for Section 3.4 - *CC BY-NC-SA 4.0*
    - 3.5: Derivatives of Trigonometric Functions - *CC BY-NC-SA 4.0*
      - 3.5E: Exercises for Section 3.5 - *CC BY-NC-SA 4.0*
    - 3.6: The Chain Rule - *CC BY-NC-SA 4.0*
      - 3.6E: Exercises for Section 3.6 - *CC BY-NC-SA 4.0*
    - 3.7: Derivatives of Inverse Functions - *CC BY-NC-SA 4.0*

- 3.7E: Exercises for Section 3.7 - *CC BY-NC-SA 4.0*
- 3.8: Implicit Differentiation - *CC BY-NC-SA 4.0*
  - 3.8E: Exercises for Section 3.8 - *CC BY-NC-SA 4.0*
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  - 3.9E: Exercises for Section 3.9 - *CC BY-NC-SA 4.0*
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- 4.0: Prelude to Applications of Derivatives - *CC BY-NC-SA 4.0*
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  - 4.1E: Exercises for Section 4.1 - *CC BY-NC-SA 4.0*
- 4.2: Linear Approximations and Differentials - *CC BY-NC-SA 4.0*
  - 4.2E: Exercises for Section 4.2 - *CC BY-NC-SA 4.0*
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  - 5.1E: Exercises for Section 5.1 - *CC BY-NC-SA 4.0*
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- 5.5: Substitution - *CC BY-NC-SA 4.0*
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- 5.6: Integrals Involving Exponential and Logarithmic Functions - *CC BY-NC-SA 4.0*
  - 5.6E: Exercises for Section 5.6 - *CC BY-NC-SA 4.0*
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- 5.7: Integrals Resulting in Inverse Trigonometric Functions - *CC BY-NC-SA 4.0*
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- 6.0: Prelude to Applications of Integration - *CC BY-NC-SA 4.0*
- 6.1: Areas between Curves - *CC BY-NC-SA 4.0*
  - 6.1E: Exercises for Section 6.1 - *CC BY-NC-SA 4.0*
- 6.2: Determining Volumes by Slicing - *CC BY-NC-SA 4.0*
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- 6.3: Volumes of Revolution - Cylindrical Shells - *CC BY-NC-SA 4.0*
  - 6.3E: Exercises for Section 6.3 - *CC BY-NC-SA 4.0*
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- 6.4E: Exercises for Section 6.4 - *CC BY-NC-SA 4.0*
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  - 6.5E: Exercises for Section 6.5 - *CC BY-NC-SA 4.0*
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  - 6.7E: Exercises for Section 6.7 - *CC BY-NC-SA 4.0*
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- 7.0: Prelude to Techniques of Integration - *CC BY-NC-SA 4.0*
- 7.1: Integration by Parts - *CC BY-NC-SA 4.0*
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  - 7.2E: Exercises for Section 7.2 - *CC BY-NC-SA 4.0*
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- 7.4: Partial Fractions - *CC BY-NC-SA 4.0*
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  - 7.5E: Exercises for Section 7.5 - *CC BY-NC-SA 4.0*
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  - 7.6E: Exercises for Section 7.6 - *CC BY-NC-SA 4.0*
- 7.7: Improper Integrals - *CC BY-NC-SA 4.0*
  - 7.7E: Exercises for Section 7.7 - *CC BY-NC-SA 4.0*
- 7.R: Chapter 7 Review Exercises - *CC BY-NC-SA 4.0*

8: Introduction to Differential Equations - *CC BY-NC-SA 4.0*

- 8.0: Prelude to Differential Equations - *CC BY-NC-SA 4.0*
- 8.1: Basics of Differential Equations - *CC BY-NC-SA 4.0*
  - 8.1E: Exercises for Section 8.1 - *CC BY-NC-SA 4.0*
- 8.2: Direction Fields and Numerical Methods - *CC BY-NC-SA 4.0*
  - 8.2E: Exercises for Section 8.2 - *CC BY-NC-SA 4.0*
- 8.3: Separable Equations - *CC BY-NC-SA 4.0*
  - 8.3E: Exercises for Section 8.3 - *CC BY-NC-SA 4.0*
- 8.4: The Logistic Equation - *CC BY-NC-SA 4.0*
  - 8.4E: Exercises for Section 8.4 - *CC BY-NC-SA 4.0*
- 8.5: First-order Linear Equations - *CC BY-NC-SA 4.0*
  - 8.5E: Exercises for Section 8.5 - *CC BY-NC-SA 4.0*
- 8.R: Chapter 8 Review Exercises - *CC BY-NC-SA 4.0*

9: Sequences and Series - *CC BY-NC-SA 4.0*

- 9.0: Prelude to Sequence and Series - *CC BY-NC-SA 4.0*
- 9.1: Sequences - *CC BY-NC-SA 4.0*
  - 9.1E: Exercises for Section 9.1 - *CC BY-NC-SA 4.0*
- 9.2: Infinite Series - *CC BY-NC-SA 4.0*
  - 9.2E: Exercises for Section 9.2 - *CC BY-NC-SA 4.0*
- 9.3: The Divergence and Integral Tests - *CC BY-NC-SA 4.0*
  - 9.3E: Exercises for Section 9.3 - *CC BY-NC-SA 4.0*
- 9.4: Comparison Tests - *CC BY-NC-SA 4.0*
  - 9.4E: Exercises for Section 9.4 - *CC BY-NC-SA 4.0*
- 9.5: Alternating Series - *CC BY-NC-SA 4.0*
  - 9.5E: Exercises for Section 9.5 - *CC BY-NC-SA 4.0*
- 9.6: Ratio and Root Tests - *CC BY-NC-SA 4.0*
  - 9.6E: Exercises for Section 9.6 - *CC BY-NC-SA 4.0*

- 9.R: Chapter 9 Review Exercises - *CC BY-NC-SA 4.0*
- 10: Power Series - *CC BY-NC-SA 4.0*
  - 10.0: Prelude to Power Series - *CC BY-NC-SA 4.0*
  - 10.1: Power Series and Functions - *CC BY-NC-SA 4.0*
    - 10.1E: Exercises for Section 10.1 - *CC BY-NC-SA 4.0*
  - 10.2: Properties of Power Series - *CC BY-NC-SA 4.0*
    - 10.2E: Exercises for Section 10.2 - *CC BY-NC-SA 4.0*
  - 10.3: Taylor and Maclaurin Series - *CC BY-NC-SA 4.0*
    - 10.3E: Exercises for Section 10.3 - *CC BY-NC-SA 4.0*
  - 10.4: Working with Taylor Series - *CC BY-NC-SA 4.0*
    - 10.4E: Exercises for Section 10.4 - *CC BY-NC-SA 4.0*
  - 10.R: Chapter 10 Review Exercises - *CC BY-NC-SA 4.0*
- 11: Parametric Equations and Polar Coordinates - *CC BY-NC-SA 4.0*
  - 11.0: Prelude to Parametric Equations and Polar Coordinates - *CC BY-NC-SA 4.0*
  - 11.1: Parametric Equations - *CC BY-NC-SA 4.0*
    - 11.1E: Exercises for Section 11.1 - *CC BY-NC-SA 4.0*
  - 11.2: Calculus of Parametric Curves - *CC BY-NC-SA 4.0*
    - 11.2E: Exercises for Section 11.2 - *CC BY-NC-SA 4.0*
  - 11.3: Polar Coordinates - *CC BY-NC-SA 4.0*
    - 11.3E: Exercises for Section 11.3 - *CC BY-NC-SA 4.0*
  - 11.4: Area and Arc Length in Polar Coordinates - *CC BY-NC-SA 4.0*
    - 11.4E: Exercises for Section 11.4 - *CC BY-NC-SA 4.0*
  - 11.5: Conic Sections - *CC BY-NC-SA 4.0*
    - 11.5E: Exercises for Section 11.5 - *CC BY-NC-SA 4.0*
  - 11.R: Chapter 11 Review Exercises - *CC BY-NC-SA 4.0*
- 12: Vectors in Space - *CC BY-NC-SA 4.0*
  - 12.0: Prelude to Vectors in Space - *CC BY-NC-SA 4.0*
  - 12.1: Vectors in the Plane - *CC BY-NC-SA 4.0*
    - 12.1E: Exercises for Section 12.1 - *CC BY-NC-SA 4.0*
  - 12.2: Vectors in Three Dimensions - *CC BY-NC-SA 4.0*
    - 12.2E: Exercises for Section 12.2 - *CC BY-NC-SA 4.0*
  - 12.3: The Dot Product - *CC BY-NC-SA 4.0*
    - 12.3E: Exercises for Section 12.3 - *CC BY-NC-SA 4.0*
  - 12.4: The Cross Product - *CC BY-NC-SA 4.0*
    - 12.4E: Exercises for Section 12.4 - *CC BY-NC-SA 4.0*
  - 12.5: Equations of Lines and Planes in Space - *CC BY-NC-SA 4.0*
    - 12.5E: Exercises for Section 12.5 - *CC BY-NC-SA 4.0*
  - 12.6: Quadric Surfaces - *CC BY-NC-SA 4.0*
    - 12.6E: Exercises for Section 12.6 - *CC BY-NC-SA 4.0*
  - 12.7: Cylindrical and Spherical Coordinates - *CC BY-NC-SA 4.0*
    - 12.7E: Exercises for Section 12.7 - *CC BY-NC-SA 4.0*
  - 12.R: Chapter 12 Review Exercises - *CC BY-NC-SA 4.0*
- 13: Vector-Valued Functions - *CC BY-NC-SA 4.0*
  - 13.0: Prelude to Vector-Valued Functions - *CC BY-NC-SA 4.0*
  - 13.1: Vector-Valued Functions and Space Curves - *CC BY-NC-SA 4.0*
    - 13.1E: Exercises for Section 13.1 - *CC BY-NC-SA 4.0*
  - 13.2: Calculus of Vector-Valued Functions - *CC BY-NC-SA 4.0*
    - 13.2E: Exercises for Section 13.2 - *CC BY-NC-SA 4.0*
  - 13.3: Arc Length and Curvature - *CC BY-NC-SA 4.0*
    - 13.3E: Exercises for Section 13.3 - *CC BY-NC-SA 4.0*
  - 13.4: Motion in Space - *CC BY-NC-SA 4.0*
    - 13.4E: Exercises for Section 13.4 - *CC BY-NC-SA 4.0*
  - 13.R: Chapter 13 Review Exercises - *CC BY-NC-SA 4.0*
- 14: Differentiation of Functions of Several Variables - *CC BY-NC-SA 4.0*
  - 14.0: Prelude to Differentiation of Functions of Several Variables - *CC BY-NC-SA 4.0*
  - 14.1: Functions of Several Variables - *CC BY-NC-SA 4.0*

- 14.1E: Exercises for Section 14.1 - *CC BY-NC-SA 4.0*
- 14.2: Limits and Continuity - *CC BY-NC-SA 4.0*
  - 14.2E: Exercises for Section 14.2 - *CC BY-NC-SA 4.0*
- 14.3: Partial Derivatives - *CC BY-NC-SA 4.0*
  - 14.3E: Exercises for Section 14.3 - *CC BY-NC-SA 4.0*
- 14.4: Tangent Planes and Linear Approximations - *CC BY-NC-SA 4.0*
  - 14.4E: Exercises for Section 14.4 - *CC BY-NC-SA 4.0*
- 14.5: The Chain Rule for Multivariable Functions - *CC BY-NC-SA 4.0*
  - 14.5E: Exercises for Section 14.5 - *CC BY-NC-SA 4.0*
- 14.6: Directional Derivatives and the Gradient - *CC BY-NC-SA 4.0*
  - 14.6E: Exercises for Section 14.6 - *CC BY-NC-SA 4.0*
- 14.7: Maxima/Minima Problems - *CC BY-NC-SA 4.0*
  - 14.7E: Exercises for Section 14.7 - *CC BY-NC-SA 4.0*
- 14.8: Lagrange Multipliers - *CC BY-NC-SA 4.0*
  - 14.8E: Exercises for Section 14.8 - *CC BY-NC-SA 4.0*
- 14.R: Chapter 14 Review Exercises - *CC BY-NC-SA 4.0*

15: Multiple Integration - *CC BY-NC-SA 4.0*

- 15.0: Prelude to Multiple Integration - *CC BY-NC-SA 4.0*
- 15.1: Double Integrals over Rectangular Regions - *CC BY-NC-SA 4.0*
  - 15.1E: Exercises for Section 15.1 - *CC BY-NC-SA 4.0*
- 15.2: Double Integrals over General Regions - *CC BY-NC-SA 4.0*
  - 15.2E: Exercises for Section 15.2 - *CC BY-NC-SA 4.0*
- 15.3: Double Integrals in Polar Coordinates - *CC BY-NC-SA 4.0*
  - 15.3E: Exercises for Section 15.3 - *CC BY-NC-SA 4.0*
- 15.4: Triple Integrals - *CC BY-NC-SA 4.0*
  - 15.4E: Exercises for Section 15.4 - *CC BY-NC-SA 4.0*
- 15.5: Triple Integrals in Cylindrical and Spherical Coordinates - *CC BY-NC-SA 4.0*
  - 15.5E: Exercises for Section 15.5 - *CC BY-NC-SA 4.0*
- 15.6: Calculating Centers of Mass and Moments of Inertia - *CC BY-NC-SA 4.0*
  - 15.6E: Exercises for Section 15.6 - *CC BY-NC-SA 4.0*
- 15.7: Change of Variables in Multiple Integrals - *CC BY-NC-SA 4.0*
  - 15.7E: Exercises for Section 15.7 - *CC BY-NC-SA 4.0*
- 15.R: Chapter 15 Review Exercises - *CC BY-NC-SA 4.0*

16: Vector Calculus - *CC BY-NC-SA 4.0*

- 16.0: Prelude to Vector Calculus - *CC BY-NC-SA 4.0*
- 16.1: Vector Fields - *CC BY-NC-SA 4.0*
  - 16.1E: Exercises for Section 16.1 - *CC BY-NC-SA 4.0*
- 16.2: Line Integrals - *CC BY-NC-SA 4.0*
  - 16.2E: Exercises for Section 16.2 - *CC BY-NC-SA 4.0*
- 16.3: Conservative Vector Fields - *CC BY-NC-SA 4.0*
  - 16.3E: Exercises for Section 16.3 - *CC BY-NC-SA 4.0*
- 16.4: Green's Theorem - *CC BY-NC-SA 4.0*
  - 16.4E: Exercises for Section 16.4 - *CC BY-NC-SA 4.0*
- 16.5: Divergence and Curl - *CC BY-NC-SA 4.0*
  - 16.5E: Exercises for Section 16.5 - *CC BY-NC-SA 4.0*
- 16.6: Surface Integrals - *CC BY-NC-SA 4.0*
  - 16.6E: Exercises for Section 16.6 - *CC BY-NC-SA 4.0*
- 16.7: Stokes' Theorem - *CC BY-NC-SA 4.0*
  - 16.7E: Exercises for Section 16.7 - *CC BY-NC-SA 4.0*
- 16.8: The Divergence Theorem - *CC BY-NC-SA 4.0*
  - 16.8E: Exercises for Section 16.8 - *CC BY-NC-SA 4.0*
- 16.R: Chapter 16 Review Exercises - *CC BY-NC-SA 4.0*

17: Second-Order Differential Equations - *CC BY-NC-SA 4.0*

- 17.0: Prelude to Second-Order Differential Equations - *CC BY-NC-SA 4.0*
- 17.1: Second-Order Linear Equations - *CC BY-NC-SA 4.0*

- 17.1E: Exercises for Section 17.1 - *CC BY-NC-SA 4.0*
- 17.2: Nonhomogeneous Linear Equations - *CC BY-NC-SA 4.0*
  - 17.2E: Exercises for Section 17.2 - *CC BY-NC-SA 4.0*
- 17.3: Applications of Second-Order Differential Equations - *CC BY-NC-SA 4.0*
  - 17.3E: Exercises for Section 17.3 - *CC BY-NC-SA 4.0*
- 17.4: Series Solutions of Differential Equations - *CC BY-NC-SA 4.0*
  - 17.4E: Exercises for Section 17.4 - *CC BY-NC-SA 4.0*
- 17.R: Chapter 17 Review Exercises - *CC BY-NC-SA 4.0*

Appendices - *CC BY-NC-SA 4.0*

- Appendix A: Table of Derivatives - *CC BY-NC-SA 4.0*
- Appendix B: Table of Integrals - *CC BY-NC-SA 4.0*

Back Matter - *Undeclared*

- Index - *Undeclared*
- Glossary - *CC BY-NC-SA 4.0*
- Detailed Licensing - *Undeclared*